# AUXILIAR 3

**P1)** Probabilidad de encontrar un $\bar{e}$ de un átomo de H dentro del protón.

$$\phi_{100} = \sqrt{\frac{1}{\pi a_0^3}}\, e^{-r/a_0}$$

con $a_0 \approx 5.29 \cdot 10^{-11}$ m , $r_p \approx 0.88 \cdot 10^{-15}$ m

$$\Rightarrow \quad \phi_{100} \approx \sqrt{\frac{1}{\pi a_0^3}}$$

La probabilidad:

$$P = 4\pi \int_0^{r_p} |\phi_{100}|^2 r^2 dr = \frac{1}{\pi a_0^3} \cdot \frac{4}{3}\pi r_p^3 = \frac{4}{3}\frac{r_p^3}{a_0^3}$$

$$\Rightarrow \quad P \sim \frac{r_p^3}{a_0^3} \sim (2 \cdot 10^{-5})^3 \boxed{\sim 10^{-14}}$$

**P2)** a) Estados ligados para $V(x) = -\frac{\hbar^2 \lambda}{2ma}\delta(x)$

Ec. Shr.: $$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\phi + V\phi = E\phi \quad \rightarrow E = -|E|$$

$$\frac{d^2}{dx^2}\phi - \underbrace{\frac{2m|E|}{\hbar^2}}_{k^2}\phi = V\phi \cdot \frac{2m}{\hbar^2}$$

$$\phi'' - k^2\phi = V\phi$$

Para $x < 0$: $\phi = Ae^{+kx}$

$x > 0$: $\phi = Be^{-kx}$

Continuidad: $A = B$

Derivada: $$\left.\frac{d\phi}{dx}\right|_{0^+} - \left.\frac{d\phi}{dx}\right|_{0^-} = \int_{0^-}^{0^+} dx \frac{d^2\phi}{dx^2} = \int_{0^-}^{0^+} dx\,(V(x) - E)\phi(x)\, d\ldots$$

$$= \int_{0^-}^{0^+}\left(-\frac{\lambda}{a}\delta(x)\right)\phi(x)\,dx =$$

$$= -\frac{\lambda}{a}\phi(0) =$$

$$2\kappa = \frac{\lambda}{a} \Rightarrow \kappa = \frac{\lambda}{2a}$$

$$\frac{2m|E|}{\hbar^2} = \frac{\lambda^2}{(2a)^2} \Rightarrow \boxed{|E| = \frac{\lambda^2\hbar^2}{8ma^2}}$$

Normalizamos:

$$1 = A^2\left[\int_{-\infty}^{0} e^{+2\kappa x}\,dx + \int_{0}^{\infty} e^{-2\kappa x}\,dx\right] = A^2\left[\left.\frac{e^{2\kappa x}}{2\kappa}\right|_{-\infty}^{0} - \left.\frac{e^{-2\kappa x}}{2\kappa}\right|_{0}^{\infty}\right] =$$

$$= \frac{A^2}{2\kappa}\left[2\right] = \frac{A^2}{\kappa} \Rightarrow A = \sqrt{\kappa}$$

$$\Rightarrow A = \left(\frac{2m|E|}{\hbar^2}\right)^{1/4} = \left(\frac{2m}{\hbar^2}\frac{\lambda^2\hbar^2}{8ma^2}\right)^{1/4}$$

$$\rightarrow \boxed{A = \left(\frac{\lambda^2}{4a^2}\right)^{1/4}}$$

b) Estados de Scattering:

$$\phi'' + k^2\phi = \frac{2m}{\hbar^2}V\phi \qquad \text{con } k^2 = \frac{2mE}{\hbar^2}$$

Para $x<0$: $\phi = Ae^{ikx} + Be^{-ikx}$

$x>0$: $\phi = Ce^{ikx}$

Continuidad: $A + B = C$

Derivada: $$ikAe^{ikx}\Big|_{0^+} - ikBe^{-ikx}\Big|_{0^+} - ikCe^{ikx}\Big|_{0^-} = -\frac{\lambda}{a}C$$

$$ikA - ikB - ikC = -\frac{\lambda C}{a}$$

Buscamos el coeficiente de reflexión y trasmisión:

$$R = \frac{|B|^2}{|A|^2}, \quad T = \frac{|C|^2}{|A|^2}$$

Para R: Usamos que $C = A + B$:

$$ik(\cancel{A} - B - \cancel{A} - B) = -\frac{\lambda}{a}(A+B)$$

$$\cancel{-}2ikB = \cancel{-}\frac{\lambda}{a}(A+B)$$

$$B\left(2ik - \frac{\lambda}{a}\right) = \frac{\lambda}{a}A \Rightarrow \frac{B}{A} = \frac{\lambda/a}{2ik - \frac{\lambda}{a}} = \frac{1}{2ik\frac{a}{\lambda} - 1} =$$

$$= \frac{1}{2i\sqrt{\frac{2mE}{\hbar^2}}\frac{a}{\lambda} - 1}$$

$$\Rightarrow \boxed{R = \frac{1}{1 + \frac{8mE}{\hbar^2}\frac{a^2}{\lambda^2}}}$$

Para T: Usamos que $B = C - A$

$$ik(A - C + A - C) = -\frac{\lambda}{a}C$$

$$2ik(A - C) = -\frac{\lambda}{a}C$$

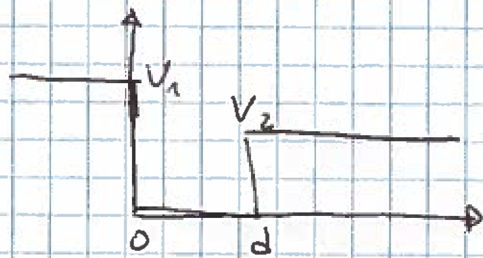
$$\rightarrow C\left(2ik - \frac{\lambda}{a}\right) = 2ikA$$

$$\rightarrow \frac{C}{A} = \frac{2ik}{2ik - \frac{\lambda}{a}} = \frac{1}{1 - \frac{\lambda}{2ika}} = \frac{1}{1 + \frac{i\lambda}{2ka}} =$$

$$= \frac{1}{1 + \frac{i\lambda}{2a\sqrt{\frac{2mE}{\hbar}}}}$$

$$\Rightarrow \boxed{T = \frac{1}{1 + \frac{\lambda^2\hbar^2}{8a^2mE}}}$$

P3) Encontrar espectro de energía y función de onda para:




Tenemos: $\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (E-V_1)\psi = 0 \qquad x<0$

$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + E\psi = 0 \qquad 0<x<a$

$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (E-V_1)\psi = 0$

$\rightarrow \quad \psi'' - k_1^2\psi = 0 \qquad$ con: $\quad k_1^2 = \frac{2m}{\hbar^2}(V_1-E)$

$\psi'' - k^2\psi = 0 \qquad k_2^2 = \frac{2m}{\hbar^2}(V_2-E)$

$\psi'' - k_2^2\psi = 0 \qquad k^2 = -\frac{2m}{\hbar^2}E$

Las soluciones son:

$\psi = A_1 e^{-k_1 x} + B_1 e^{k_1 x} \qquad x<0$

$\psi = A e^{-kx} + B e^{kx} \qquad 0<x<a$

$\psi = A_2 e^{-k_2 x} + B_2 e^{k_2 x} \qquad x>a$

Para $E<V_2$:

$\Rightarrow k_1, k_2$ reales, $k$ imaginario

Queda: $\psi = A\,\text{sen}(Kx+\delta) \qquad 0<x<a \qquad$ donde $K = ik$

De inmediato vemos que

$A_1 = B_2 = 0$

Impondremos continuidad:

$$\psi' = k_1 B_1 e^{k_1 x} \qquad x<0$$

$$\psi' = A K \cos(Kx+\delta) \qquad 0<x<a$$

$$\psi' = -k_2 A_2 e^{-k_2 x} \qquad x>a$$

$$\rightarrow \left.\frac{d\psi/dx}{\psi}\right|_{0^-} = \left.\frac{d\psi/dx}{\psi}\right|_{0^+} \Big\} \quad \frac{k_1 \cancel{B_1}}{\cancel{B_1}} = \frac{\cancel{A}K\cos(\delta)}{\cancel{A}\,\text{sen}(\delta)} \Big\} \quad k_1 = K\cot(\delta)$$

$$\rightarrow \left.\frac{d\psi/dx}{\psi}\right|_{a^-} = \left.\frac{d\psi/dx}{\psi}\right|_{a^+} \Big\} \quad \frac{\cancel{A}K\cos(Ka+\delta)}{A\,\text{sen}(Ka+\delta)} = \frac{-k_2 \cancel{A_2 e^{-k_2 a}}}{\cancel{A_2 e^{-k_2 a}}}$$

$$\Rightarrow -k_2 = K\cot(Ka+\delta)$$

reescribiendo:

$$\cot\delta = \frac{k_1}{K} = \frac{\sqrt{\frac{2m}{\hbar^2}V_1 - K^2}}{K} = \sqrt{\frac{2mV_1}{\hbar^2 K^2} - 1} \qquad (1)$$

$$\cot(Ka+\delta) = -\sqrt{\frac{2mV_2}{\hbar^2 K^2} - 1} \qquad (2)$$

Para despejar:

$$\cot\alpha = \frac{\cos\alpha}{\text{sen}\,\alpha} = \frac{\sqrt{1-\text{sen}^2\alpha}}{\text{sen}\,\alpha}$$

$$\cot^2\alpha = \frac{1-\text{sen}^2\alpha}{\text{sen}^2\alpha}$$

Para (1):

$$\frac{2mV_1}{\hbar^2 K^2} - 1 = \frac{1-\text{sen}^2\delta}{\text{sen}^2\delta}$$

$$\frac{2mV_1}{\hbar^2 K^2}\text{sen}^2\delta - \cancel{\text{sen}^2\delta} = 1 - \cancel{\text{sen}^2\delta}$$

$$\Rightarrow \text{sen}\,\delta = \sqrt{\frac{\hbar^2 K^2}{2mV_1}} \Rightarrow \boxed{\delta = \text{arcsen}\left(\frac{\hbar K}{\sqrt{2mV_1}}\right) + n_1\pi}$$

de forma análoga:

$$\boxed{Ka + \delta = -\text{arcsen}\left(\frac{\hbar K}{\sqrt{2mV_2}}\right) + n_2\pi}$$

con arcsen entre 0 y $\frac{\pi}{2}$

juntando ambas:

$$\boxed{Ka = \underbrace{(n_2 - n_1)}_{n}\pi - \arcsin\left(\frac{\hbar K}{\sqrt{2mV_1}}\right) - \arcsin\left(\frac{\hbar K}{\sqrt{2mV_2}}\right)}$$

Que es el espectro de energía.