AUXILIAR 4

P1 a) $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t)+V(x)\Psi(x)=i\hbar\frac{\partial\Psi(x,t)}{\partial t}$

Tenemos que, $\tilde{\Psi}(k,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}\Psi(x,t)\,dx$

$$\Psi(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}\tilde{\Psi}(k,t)\,dk$$

remplazamos:

$$-\frac{\hbar^2}{2m}\underbrace{\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}e^{ikx}}_{(ik)^2e^{ikx}}\tilde{\Psi}(k,t)+\int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}V(x)e^{ikx}\tilde{\Psi}(k,t)=i\hbar\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\frac{e^{ikx}\tilde{\Psi}}{\sqrt{2\pi}}dk$$

$$\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}dk\,k^2e^{ikx}\tilde{\Psi}+\int_{-\infty}^{\infty}dk\,V(x)e^{ikx}\tilde{\Psi}=i\hbar\frac{\partial}{\partial t}\int_{-\infty}^{\infty}e^{ikx}\tilde{\Psi}dk$$

$$/\int dx\,e^{-ik'x}$$

$$\frac{\hbar^2}{2m}\int dk\int dx\,k^2e^{i(k-k')x}\tilde{\Psi}+\int dk\int dx\,V(x)e^{i(k-k')x}\tilde{\Psi}=$$

$$i\hbar\frac{\partial}{\partial t}\int dk\int dx\,e^{i(k-k')x}\tilde{\Psi}$$

uso que: $\int_{-\infty}^{\infty}e^{ix(k-k')}dx=2\pi\,\delta(k-k')$

$$2\pi\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}dk\,k^2\tilde{\Psi}(k,t)\delta(k-k')+\int_{-\infty}^{\infty}dk\,\tilde{\Psi}(k,t)\int_{-\infty}^{\infty}dx\,V(x)e^{i(k-k')}=i\hbar\frac{\partial}{\partial t}\int_{-\infty}^{\infty}dk\,\tilde{\Psi}\cdot\delta(k-)\cdot 2\pi$$

$$\rightarrow\boxed{\frac{\hbar^2k'^2}{2m}\tilde{\Psi}+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk\,\tilde{V}(k'-k)\tilde{\Psi}(k,t)=i\hbar\frac{\partial\tilde{\Psi}}{\partial t}}$$

donde:

$$\tilde V(k'-k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(k'-k)x}\, dx\, V(x)$$

Para extender a 3D elevamos al cubo las raíces y cambiar a producto punto.

b) Queremos calcular $\tilde V$ para distintos potenciales

- $V(x) = v_0\, \delta(x)$

$$\tilde V(k) = \frac{v_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, \delta(x)\, dx = \boxed{\frac{v_0}{\sqrt{2\pi}}}$$

- $V(r) = g\, e^{-\mu r}/r$

$$\tilde V(k) = \frac{g}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} e^{-i\vec k\cdot\vec x}\, \frac{e^{-\mu r}}{r}\, d\vec x = \frac{4\pi g}{(2\pi)^{3/2}} \int_0^{\infty} e^{-ikr}\, e^{-\mu r}\, r\, dr =$$

$$= \sqrt{\frac{2^{\cancel{4}}\, \cancel{\pi^2}}{\cancel{2^3}\, \pi^{\cancel{3}}}}\; g \int_0^{\infty} e^{-(ik+\mu)r}\, r\, dr =$$

$$= \sqrt{\frac{2}{\pi}}\, g \left[-\frac{e^{-(ik+\mu)r}\left((ik+\mu)r+1\right)}{(ik+\mu)^2} \right]\Bigg|_0^{\infty} = \boxed{\sqrt{\frac{2}{\pi}}\, g\, \frac{1}{k^2+\mu^2}}$$

- $V(r) = g/r$

$$\tilde V(k) = \int_0^{\infty} e^{-ikr}\, r\, dr\, \sqrt{\frac{2}{\pi}}\, g = \boxed{\sqrt{\frac{2}{\pi}}\, g\, \frac{1}{k^2}}$$

- $V(r) = g\, e^{-r^2/a^2}$

$$\tilde V(k) = \sqrt{\frac{2}{\pi}}\, g \int_0^{\infty} e^{-ikr}\, e^{-r^2/a^2}\, r^2\, dr$$

P2 | Resolver $V(x) = v_0\,\delta(x)$ en espacio de momentum

$$\frac{\hbar^2 k^2}{2m}\tilde{\varphi} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, \frac{(-v_0)\tilde{\varphi}}{\sqrt{2\pi}} = -|E|\tilde{\varphi}$$

$$\frac{v_0}{2\pi}\underbrace{\int_{-\infty}^{\infty} dk\, \tilde{\varphi}(k)}_{\alpha} = \left(|E| + \frac{\hbar^2 k^2}{2m}\right)\tilde{\varphi}(k)$$

$$(*)\quad \frac{v_0}{2\pi}\,\alpha \cdot \frac{1}{\left(|E| + \frac{\hbar^2 k^2}{2m}\right)} = \tilde{\varphi}(k) \qquad \Big/ \int_{-\infty}^{\infty} dk\,(\,)$$

$$\alpha\frac{v_0}{2\pi}\underbrace{\int_{-\infty}^{\infty} \frac{dk}{\left(|E| + \frac{\hbar^2 k^2}{2m}\right)}}_{\to\ \left.\frac{\tan\left(\frac{\hbar}{2m\sqrt{E}}k\right)}{\frac{\hbar}{\sqrt{2m}}\sqrt{|E|}}\right|_{-\infty}^{\infty}} = \underbrace{\int_{-\infty}^{\infty} dk\, \tilde{\varphi}(k)}_{\alpha}$$

$$\cancel{\alpha}\,\frac{v_0}{2\cancel{\pi}}\cdot\frac{\sqrt{2m}\cdot\cancel{\pi}}{\hbar\sqrt{|E|}} = \cancel{\alpha}$$

$$\to \sqrt{|E|} = v_0\sqrt{\frac{m}{2}}\Big/\hbar \Rightarrow \boxed{|E| = \frac{v_0^2 m}{2\hbar^2}}$$

La función de onda:

$$(*) \to \tilde{\varphi}(k) = \frac{\alpha v_0}{2\pi}\,\frac{1}{\left(\frac{v_0^2 m}{2\hbar^2} + \frac{\hbar^2 k^2}{2m}\right)} = \boxed{\frac{\alpha\, v_0}{4\pi}\,\frac{1}{\frac{\hbar^2}{m}\left(\frac{v_0^2 m^2}{\hbar^4} + k^2\right)}}$$

La cte. α se saca normalizando.

P3 Para el potencial

$$V(x) = \begin{cases} \infty & x<0,\ x>a \\ 0 & 0 \le x \le a \end{cases}$$

Ec. Sch: $\psi'' + \underbrace{\frac{2mE}{\hbar^2}}_{k^2}\psi = 0 \Rightarrow \psi(x) = A\,\text{sen}(kx)$

Borde: $\psi(a) = 0 = A\,\text{sen}(ka) \Rightarrow ka = n\pi \Rightarrow k_n = \frac{n\pi}{a}$

Normalizando: $1 = A^2 \int_0^a \text{sen}^2\left(\frac{n\pi x}{a}\right) dx \Rightarrow A = \sqrt{\frac{2}{a}}$

$$\Rightarrow \boxed{\psi(x) = \sqrt{\frac{2}{a}}\,\text{sen}\left(\frac{n\pi x}{a}\right)} \quad \text{con} \quad E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$

Ahora los valores de expectación:

- $\langle x \rangle = \int_0^a x|\psi(x)|^2 dx = \frac{2}{a}\int_0^a x\,\text{sen}^2\left(\frac{n\pi x}{a}\right) dx \boxed{= \frac{a}{2}}$

- $\langle x^2 \rangle = \int_0^a x^2|\psi(x)|^2 dx = \frac{2}{a}\int_0^a x^2\,\text{sen}^2\left(\frac{n\pi x}{a}\right) dx \boxed{= a^2\,\frac{2n^2\pi^2 - 3}{6n^2\pi^2}}$

$\rightarrow (\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 \boxed{= a^2\,\frac{n^2\pi^2 - 6}{12n^2\pi^2}}$

Ahora en espacio de momentum:

- $\langle p \rangle = \int_0^a \psi^* \underbrace{\hat{p}}_{-i\hbar\frac{\partial}{\partial x}} \psi\, dx = \int_0^a \sqrt{\frac{2}{a}}\,\text{sen}\left(\frac{n\pi x}{a}\right) \cdot (-i\hbar)\sqrt{\frac{2}{a}} \cdot \frac{n\pi}{a}\cos\left(\frac{n\pi x}{a}\right) =$

$$= \frac{2}{a}(-i\hbar)\frac{n\pi}{a}\int_0^a \text{sen}\left(\frac{n\pi x}{a}\right)\cos\left(\frac{n\pi x}{a}\right) dx \boxed{= 0}$$

$$\bullet \langle p^2\rangle = \int_0^a \psi^* \underbrace{\hat{p}^2}_{-\hbar^2 \frac{\partial^2}{\partial x^2}} \psi \, dx = \frac{2}{a}(-\hbar^2)\int_0^a \operatorname{sen}\left(\frac{n\pi x}{a}\right)\left[\frac{n\pi}{a}\right]^2\left(-\operatorname{sen}\left(\frac{n\pi x}{a}\right)\right)dx =$$

$$= \frac{2\hbar^2}{a}\int_0^a \operatorname{sen}^2\left(\frac{n\pi x}{a}\right)\left[\frac{n\pi}{a}\right]^2 dx =$$

$$= \frac{2\hbar^2}{a}\,\frac{n^2\pi^2}{a^2}\cdot\frac{a}{2} = \boxed{\frac{n^2\hbar^2\pi^2}{a^2}}$$

$$\rightarrow \boxed{(\Delta p)^2 = \frac{n^2\hbar^2\pi^2}{a^2}}$$

$$\rightarrow \Delta p\,\Delta x = \frac{n\hbar\pi}{a}\cdot a\frac{\sqrt{n^2\pi^2-6}}{n\pi\, 2\sqrt{3}} =$$

$$= \frac{\hbar}{2\sqrt{3}}\sqrt{n^2\pi^2-6} \geq \frac{\hbar}{2\sqrt{3}}\underbrace{\sqrt{\pi^2-6}}_{>3} > \frac{\hbar\sqrt{3}}{2\sqrt{3}}$$

$$\Rightarrow \boxed{\Delta p\,\Delta x > \frac{\hbar}{2}}$$

Podemos obtener la función de onda en espacio de momentum:

$$\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}}\int_0^a dx\, e^{-ikx}\psi(x) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{a}}\int_0^a e^{-ikx}\frac{\left(e^{\frac{in\pi x}{a}} - e^{-\frac{in\pi x}{a}}\right)}{2i}dx =$$

$$= \frac{1}{\sqrt{\pi a}\cdot 2i}\int_0^a\left[e^{ix\left(-k+\frac{n\pi}{a}\right)} - e^{-ix\left(k+\frac{n\pi}{a}\right)}\right]dx =$$

$$= \frac{1}{\sqrt{\pi a}\cdot 2i}\cdot\left[\left.\frac{e^{ix\left(\frac{n\pi}{a}-k\right)}}{i\left(\frac{n\pi}{a}-k\right)}\right|_0^a + \left.\frac{e^{-ix\left(\frac{n\pi}{a}+k\right)}}{i\left(\frac{n\pi}{a}+k\right)}\right|_0^a\right] =$$

$$= -\frac{1}{2\sqrt{\pi a}}\left[\frac{e^{in\pi - iak}-1}{\frac{n\pi}{a}-k} + \frac{e^{-in\pi-iak}-1}{\frac{n\pi}{a}+k}\right] =$$

$$= -\frac{1}{2\sqrt{\pi a}}\left[\frac{\frac{n\pi}{a}\left(e^{in\pi}+e^{-in\pi}\right)e^{-iak} - \frac{n\pi}{a} - \frac{n\pi}{a} - \cancel{k} + \cancel{k} + k\left(e^{in\pi}-e^{-in\pi}\right)e^{-iak}}{\frac{n^2\pi^2}{a^2}-k^2}\right] =$$

$$= -\frac{1}{\cancel{2}\sqrt{\pi a}}\cdot\frac{n\pi}{a}\cdot\frac{1}{\frac{n^2\pi^2}{a^2}-k^2}\left[e^{-iak}\Big(\underbrace{\cos(n\pi)}_{(-1)^n}+\overset{0}{\cancel{i\sin(n\pi)}}\Big)\cancel{2}-\cancel{2}\right] =$$

$$\boxed{\tilde{\psi}(k) = \frac{n\sqrt{\pi}}{a^{3/2}}\,\frac{1}{k^2-\frac{n^2\pi^2}{a^2}}\left[e^{-iak}(-1)^n - 1\right]}$$