[40 × 6]

P1 Obtener la ec. de Schrödinger en espacio de momentum usando el formalismo de Dirac.

Sol:

$$\left(\frac{\hat{p}^2}{2m}+\hat{V}\right)|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle \qquad /\langle p|$$

$$\langle p|\frac{p^2}{2m}|\psi\rangle + \langle p|\hat{V}|\psi\rangle = \langle p|i\hbar\frac{\partial}{\partial t}|\psi\rangle$$

$$\frac{p^2}{2m}\langle p|\psi\rangle + \int\langle p|\hat{V}|p'\rangle dp'\langle p'|\psi\rangle = i\hbar\frac{\partial}{\partial t}\langle p|\psi\rangle$$

$$\frac{p^2}{2m}\tilde{\psi}(p) + \int_{-\infty}^{\infty}\tilde{V}(p,p')\,dp'\,\tilde{\psi}(p') = i\hbar\frac{\partial}{\partial t}\tilde{\psi}(p)$$

Examinamos $\tilde{V}$:

$$\tilde{V}(p,p') = \langle p|\hat{V}|p'\rangle = \int dx\int dx'\underbrace{\langle p|x\rangle}_{\frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}}\langle x|\hat{V}|x'\rangle\underbrace{\langle x'|p'\rangle}_{\frac{1}{\sqrt{2\pi\hbar}}e^{-ip'x'/\hbar}} =$$

$$= \frac{1}{2\pi\hbar}\int dx\int dx'\,e^{ipx/\hbar}e^{-ip'x'/\hbar}\underbrace{V(x,x')}_{V(x)\delta(x-x')} =$$

$$= \frac{1}{2\pi\hbar}\int dx\,e^{-i(p'-p)x/\hbar}\,V(x) = \frac{1}{\sqrt{2\pi\hbar}}\tilde{V}(p-p')$$

$$\Rightarrow \boxed{\frac{p^2}{2m}\tilde{\psi} + \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}\tilde{V}(p-p')\,dp'\,\tilde{\psi}(p') = i\hbar\frac{\partial\tilde{\psi}}{\partial t}}$$

P2 Sea $\hat{V} = \lambda |\alpha\rangle\langle\alpha|$, con $\langle x|\alpha\rangle = f(x)$. Calcule $\langle x'|\hat{V}|x\rangle$ y $\langle k|\hat{V}|k'\rangle$.

Sol:

- $\langle x'|\hat{V}|x\rangle = \langle x'|\lambda|\alpha\rangle\langle\alpha|x\rangle = \lambda\langle x'|\alpha\rangle\langle\alpha|x\rangle =$
$$= \lambda f(x') f^*(x)$$

- $\langle k|\hat{V}|k'\rangle = \langle k|\lambda|\alpha\rangle\langle\alpha|k'\rangle =$

$$= \lambda \int dx\, \underbrace{\langle k|x\rangle}_{\frac{1}{\sqrt{2\pi}}e^{ikx}}\langle x|\alpha\rangle\langle\alpha|k'\rangle =$$

$$= \lambda \int \frac{dx}{\sqrt{2\pi}} e^{ikx} f(x) \int \underbrace{\langle\alpha|x\rangle}_{f^*(x)} \underbrace{\langle x|k'\rangle}_{\frac{1}{\sqrt{2\pi}}e^{-ik'x}} dx =$$

$$= \lambda \tilde{f}(k) \tilde{f}^*(k')$$

P3 a) A, B operadores que cumplen: $[A,B]=c$

Demostrar que $[A, e^B] = c\,e^B$

Sol: Primero demostraremos que $[A, B^n] = nc\,B^{n-1}$

Usamos que: $[A, BC] = [A,B]C + B[A,C]$

$$\rightarrow [A, B^n] = [A, B^{n-1}B] = [A, B^{n-1}]B + B^{n-1}\underbrace{[A,B]}_{c} =$$

$$= [A, B^{n-2}B]B + B^{n-1}c =$$

$$= [A, B^{n-2}]B^2 + \underbrace{B^{n-2}\underbrace{[A,B]}_{c}B}_{B^{n-1}c} + B^{n-1}c =$$

$$\vdots$$

$$= n\,B^{n-1}c$$

Ahora recordamos que:

$$e^B = \sum_{k=0}^{\infty} \frac{1}{k!} B^k$$

$$\rightarrow [A, e^B] = [A, \sum_{k=0}^{\infty} \frac{1}{k!} B^k] = \sum_{k=0}^{\infty} \frac{1}{k!} [A, B^k] = \sum_{k=0}^{\infty} \frac{1}{k!} k c B^{k-1} =$$

$$= c \sum_{k=1}^{\infty} \frac{1}{(k-1)!} B^{k-1} = c \underbrace{\sum_{k=0}^{\infty} \frac{1}{k!} B^k}_{e^B} =$$

$$\boxed{[A, e^B] = c\,e^B}$$

b) Demostrar que $e^{\hat{L}}\hat{a}e^{-\hat{L}} = \hat{a} + [\hat{L}, \hat{a}] + \frac{1}{2!}[\hat{L}, [\hat{L}, \hat{a}]] + \frac{1}{3!}[\hat{L}, [\hat{L}, [\hat{L}, \hat{a}]]] + \ldots$

Sol: Vamos a definir el operador

$$\hat{b} = e^{s\hat{L}}\hat{a}e^{-s\hat{L}}$$

derivamos:

$$\frac{d\hat{b}}{ds} = \hat{L}e^{s\hat{L}}\hat{a}e^{-s\hat{L}} - e^{s\hat{L}}\hat{a}e^{-s\hat{L}}\hat{L} = [\hat{L}, \hat{b}]$$

$$\frac{d^2\hat{b}}{ds^2} = \frac{d}{ds}[\hat{L}, \hat{b}] = \cancel{[\underbrace{\tfrac{d}{ds}\hat{L}}_{0}, \hat{b}]} + [\hat{L}, \frac{d\hat{b}}{ds}] =$$

$$= [\hat{L}, [\hat{L}, \hat{b}]]$$

$$\frac{d^3\hat{b}}{ds^3} = [\hat{L}, [\hat{L}, [\hat{L}, \hat{b}]]]$$

$$\vdots$$

Ahora vamos a expandir $\hat{b}(s=1) = e^{\hat{L}}\hat{a}e^{-\hat{L}}$ en torno a $s=0$:

$$\hat{b}(s=1) = \underbrace{\hat{b}(s=0)}_{\hat{a}} + \underbrace{\left.\frac{d\hat{b}}{ds}\right|_{s=0}}_{[\hat{L},\hat{a}]} + \frac{1}{2!}\underbrace{\left.\frac{d^2\hat{b}}{ds^2}\right|_{s=0}}_{[\hat{L},[\hat{L},\hat{a}]]} + \dots$$

$$\rightarrow \boxed{e^{\hat{L}}\hat{a}e^{-\hat{L}} = \hat{a} + [\hat{L},\hat{a}] + \frac{1}{2!}[\hat{L},[\hat{L},\hat{a}]] + \dots}$$

P4 Sean A, B y C que satisfacen $AB - BA = iC$.
Demostrar que $\Delta A \, \Delta B \geq \langle C \rangle / 2$

Sol: Definimos los operadores

$$a = A - \langle A \rangle \quad , \quad b = B - \langle B \rangle$$

que también son hermíticos. Sacamos el conmutador:

$$[a, b] = [A - \langle A \rangle, B - \langle B \rangle] =$$
$$= [A, B] - \underbrace{[A, \cancel{\langle B \rangle}]}_{0} - \underbrace{[\langle \cancel{A} \rangle, B]}_{0} + \underbrace{[\langle \cancel{A} \rangle, \langle B \rangle]}_{0} =$$

$$[a, b] = [A, B]$$

Queremos usar

$$\Delta A = \langle A - \langle A \rangle \rangle = \langle a \rangle$$
$$\Delta B = \langle B - \langle B \rangle \rangle = \langle b \rangle$$

$$\rightarrow \langle a^2 \rangle = \langle (A - \langle A \rangle)^2 \rangle = \langle A^2 - 2A\langle A \rangle + \langle A \rangle^2 \rangle =$$
$$= \langle A^2 \rangle - 2\langle A \rangle^2 + \langle A \rangle^2 = \langle A^2 \rangle - \langle A \rangle^2$$
$$= (\Delta A)^2$$

$$\rightarrow \langle b^2 \rangle = (\Delta B)^2$$

Usamos la desigualdad de Cauchy-Schwarz:

$$|\langle a\, b \rangle|^2 \leq \underbrace{\langle a^2 \rangle}_{(\Delta A)^2} \underbrace{\langle b^2 \rangle}_{(\Delta B)^2}$$

Sacamos el lado izquierdo:

$$|\langle ab \rangle|^2 = \langle ab \rangle^* \langle ab \rangle = \langle ba \rangle \langle ab \rangle =$$

$$= \langle \tfrac{1}{2}(ba - ab) + \tfrac{1}{2}(ba + ab) \rangle \langle \tfrac{1}{2}(ab - ba) + \tfrac{1}{2}(ab + ba) \rangle =$$

$$= \frac{1}{2} \langle [b, a] + (ba + ab) \rangle \frac{1}{2} \langle [a, b] + (ab + ba) \rangle =$$

$$= \frac{1}{4} \langle [B,A] + (ba+ab) \rangle \langle [A,B] + (ab+ba) \rangle$$

$$\geq \frac{1}{4} \langle [B,A] \rangle \langle [A,B] \rangle = \frac{1}{4} (-i) \langle C \rangle (i) \langle C \rangle =$$

$$= \frac{1}{4} \langle C \rangle^2$$

Sacando la raiz:

$$\boxed{\Delta A \; \Delta B \geq \frac{1}{2} \langle C \rangle}$$

P6 Se tiene $\hat{H} = \varepsilon\left(|a\rangle\langle a| - |b\rangle\langle b| + |a\rangle\langle b| + |b\rangle\langle a|\right)$

Demuestre que $\hat{H}$ es hermítico y encuentre sus autoenergías y autovectores.

Sol: Sacamos $\hat{H}^\dagger$:

$$\langle\alpha|\hat{H}^\dagger|\beta\rangle = \left(\langle\beta|\hat{H}|\alpha\rangle\right)^* =$$

$$= \varepsilon\left(\langle\beta|a\rangle\langle a|\alpha\rangle - \langle\beta|b\rangle\langle b|\alpha\rangle + \langle\beta|a\rangle\langle b|\alpha\rangle + \langle\beta|b\rangle\langle a|\alpha\rangle\right)^*$$

$$= \varepsilon\left(\langle\alpha|a\rangle\langle a|\beta\rangle - \langle\alpha|b\rangle\langle b|\beta\rangle + \langle\alpha|b\rangle\langle a|\beta\rangle + \langle\alpha|a\rangle\langle b|\beta\rangle\right) =$$

$$= \varepsilon\langle\alpha|\hat{H}|\beta\rangle \Rightarrow \boxed{\hat{H}^\dagger = \hat{H}}$$

Ahora sacamos las autoenergías:

$$\hat{H}|\phi\rangle = E|\phi\rangle$$

$$|a\rangle\langle a|\phi\rangle - |b\rangle\langle b|\phi\rangle + |a\rangle\langle b|\phi\rangle + |b\rangle\langle a|\phi\rangle = \frac{E}{\varepsilon}|\phi\rangle \qquad (*)$$

Dado que $\langle a|a\rangle = \langle b|b\rangle = 1$ y $\langle a|b\rangle = \langle b|a\rangle = 0$:

$$\langle a|\hat{H}|\phi\rangle = E\langle a|\phi\rangle \rightarrow \langle a|\phi\rangle + \langle b|\phi\rangle = \frac{E}{\varepsilon}\langle a|\phi\rangle$$

$$\langle b|\hat{H}|\phi\rangle = E\langle b|\phi\rangle \rightarrow \langle a|\phi\rangle - \langle b|\phi\rangle = \frac{E}{\varepsilon}\langle b|\phi\rangle$$

En forma matricial:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \langle a|\phi\rangle \\ \langle b|\phi\rangle \end{pmatrix} = \lambda\begin{pmatrix} \langle a|\phi\rangle \\ \langle b|\phi\rangle \end{pmatrix}$$

$$\rightarrow \det\begin{pmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} = -(1-\lambda)(1+\lambda) - 1 = \lambda^2 - 2 = 0 \Rightarrow \boxed{\frac{E}{\varepsilon} = \lambda = \pm\sqrt{2}}$$

Los vectores propios:

$\underline{\lambda = +\sqrt{2}}$: $\langle a|\phi\rangle + \langle b|\phi\rangle = \sqrt{2}\langle a|\phi\rangle \rightarrow \langle b|\phi\rangle = (\sqrt{2}-1)\langle a|\phi\rangle$

Remplazando en $(*)$:

$$\left(\cancel{|a\rangle} - \sqrt{2}|b\rangle + |b\rangle + \sqrt{2}|a\rangle - \cancel{|a\rangle} + |b\rangle\right) = \sqrt{2}|\phi_+\rangle$$

$$\Rightarrow \boxed{|\phi_+\rangle = |a\rangle + (\sqrt{2}-1)|b\rangle}$$

$\underline{\lambda = -\sqrt{2}}$. $\langle b|\phi\rangle = (-\sqrt{2}-1)\langle a|\phi\rangle$

$\rightarrow \cancel{|a\rangle} + \sqrt{2}\,|b\rangle + |b\rangle - \sqrt{2}\,|a\rangle - \cancel{|a\rangle} + |b\rangle = -\sqrt{2}\,|\phi_-\rangle$

$$\Rightarrow \boxed{|\phi_-\rangle = |a\rangle - (\sqrt{2}+1)\,|b\rangle}$$