P1) Se tienen dos osciladores acoplados:

$$\hat{H} = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{m\omega^2}{2}(x_1^2 + x_2^2) + C x_1 x_2$$

Encuentre el espectro de autoenergías.

Sol: Pasamos a coordenadas del centro de masa:

$$x = x_1 - x_2 \qquad p = \frac{p_1 - p_2}{2} \qquad \rightarrow \qquad x_1 = \frac{x}{2} + y \qquad p_1 = p + \frac{q}{2}$$

$$y = \frac{x_1 + x_2}{2} \qquad q = p_1 + p_2 \qquad x_2 = -\frac{x}{2} + y \qquad p_2 = -p + \frac{q}{2}$$

el hamiltoniano queda:

$$\hat{H} = \frac{1}{2m}\left(p^2 + \frac{q^2}{4} + \cancel{\frac{pq}{2}} + \cancel{\frac{qp}{2}} + p^2 + \frac{q^2}{4} - \cancel{\frac{pq}{2}} - \cancel{\frac{pq}{2}}\right) + \frac{m\omega^2}{2}\left(\frac{x^2}{4} + y^2 + \cancel{xy} + \frac{x^2}{4} + y^2 - \cancel{xy}\right)$$

$$+ C\left(y^2 - \frac{x^2}{4}\right)$$

$$= \frac{p^2}{m} + \frac{m x^2}{4}\left(\omega^2 - \frac{C}{m}\right) + \frac{q^2}{4m} + m y^2\left(\omega^2 + \frac{C}{m}\right)$$

$$= \hat{H}_{x,p} + \hat{H}_{y,P}$$

frecuencias: $\Omega_x^2 = \omega^2 - \frac{C}{m}$ $\qquad \Omega_y^2 = \omega^2 + \frac{C}{m}$

autoenergías: $\hbar\Omega_x(n_x + \frac{1}{2})$ $\qquad \hbar\Omega_y(n_y + \frac{1}{2})$

$$\Rightarrow \boxed{E_{n_x n_y} = \hbar\left[\Omega_x\left(n_x + \tfrac{1}{2}\right) + \Omega_y\left(n_y + \tfrac{1}{2}\right)\right]}$$

P1 a) Encuentre una expresión para el valor de expectación de x^4 en el estado n.

Sol: Tenemos que: $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$

$$\rightarrow \langle x^4 \rangle_n = \langle n|\hat{x}^4|n\rangle = \int dx\, \phi_n^*(x)\,\hat{x}^4\,\phi_n(x) =$$

no son 0 los con dos $a^\dagger$ y dos a

$$= \left(\frac{\hbar}{2m\omega}\right)^2 \langle n|(a^\dagger a^\dagger a a + a^\dagger a a^\dagger a + a^\dagger a a a^\dagger + a a a^\dagger a^\dagger + a a^\dagger a^\dagger a + a a^\dagger a a^\dagger)|n\rangle =$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2 \left[n(n-1) + n^2 + n(n+1) + (n+1)(n+2) + n(n+1) + (n+1)^2\right] =$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2 \left[n^2 \cancel{-n} + n^2 + n^2 \cancel{+n} + n^2 + 3n + 2 + n^2 + n + n^2 + 2n + 1\right] =$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2 \left[6n^2 + 6n + 3\right] =$$

$$\boxed{\langle x^4 \rangle_n = \left(\frac{\hbar}{2m\omega}\right)^2 \cdot 3\left[2n^2 + 2n + 1\right]}$$

b) Comprobar que para un oscilador armónico en el nivel n:

$$\Delta x\, \Delta p = \hbar\, \frac{2n+1}{2}$$

Sol: $\langle x \rangle = \langle n|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|(a^\dagger + a)|n\rangle = 0$

$\langle p \rangle = \langle n|\hat{p}|n\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\langle n|(a^\dagger - a)|n\rangle = 0$

$\langle x^2 \rangle = \langle n|\hat{x}^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|(a^\dagger a^\dagger + aa + a^\dagger a + a a^\dagger)|n\rangle =$

$= \frac{\hbar}{2m\omega}[n + (n+1)] = \frac{\hbar}{2m\omega}[2n+1]$

$\langle p^2 \rangle = -\frac{m\hbar\omega}{2}(-2n-1) = \frac{m\hbar\omega}{2}(2n+1)$

$$\Rightarrow \boxed{\Delta x\, \Delta p = \frac{\hbar}{2}(2n+1)}$$

P3) En $t=0$: $\psi(0,x) = A\phi_0(x) + B\phi_1(x) + C\phi_3(x)$

Encuentre $\langle H\rangle$, $\langle P^2/2m\rangle$, $\langle \frac{m\omega^2}{2}x^2\rangle$, $\langle p\rangle$, $\langle x\rangle$

Sol: El hamiltoniano: $\hat{H} = \frac{1}{2m}\hat{p}_x^2 + \frac{1}{2}m\omega^2\hat{x}^2$

Sabemos que: $\hat{H}|n\rangle = \hbar\omega(\hat{N}+\frac{1}{2})|n\rangle = \hbar\omega(n+\frac{1}{2})|n\rangle$

$$\rightarrow \hat{H}|\psi\rangle = \hbar\omega\left[A\frac{1}{2}|0\rangle + \frac{3}{2}B|1\rangle + \frac{7}{2}C|3\rangle\right]$$

El valor de expectación:

$$\langle H\rangle = \langle\psi|\hat{H}|\psi\rangle = \boxed{\hbar\omega\left[|A|^2\frac{1}{2} + |B|^2\frac{3}{2} + |C|^2\frac{7}{2}\right] = \langle H\rangle}$$

donde $|A|^2+|B|^2+|C|^2 = 1$

Ahora usamos que $\hat{P} = \sqrt{\frac{1}{m\hbar\omega}}\,\hat{p} = \frac{i}{\sqrt{2}}(a^+ - a) \rightarrow \hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}(a^+ - a)$

$$\rightarrow \frac{\hat{p}^2}{2m} = \frac{1}{2\cancel{m}}\cdot\frac{-m\hbar\omega}{2}(a^+ - a)^2 = -\frac{\hbar\omega}{4}\left(a^{+2} + a^2 - aa^+ - a^+a\right)$$

La función de onda en función del tiempo:

$$|\psi(t)\rangle = Ae^{-i\omega t/2}|0\rangle + Be^{-3i\omega t/2}|1\rangle + Ce^{-7i\omega t/2}|3\rangle$$

Vemos los distintos componentes de $\frac{\hat{p}^2}{2m}$:

$$a^{+2}|\psi(t)\rangle = \sqrt{2}Ae^{-i\omega t/2}|2\rangle + \sqrt{6}Be^{-3i\omega t/2}|3\rangle + \sqrt{20}Ce^{-7i\omega t/2}|5\rangle$$

$$a^2|\psi(t)\rangle = \sqrt{6}Ce^{-7i\omega t/2}|1\rangle$$

$$a^+a|\psi(t)\rangle = Be^{-3i\omega t/2}|1\rangle + 3Ce^{-7i\omega t/2}|3\rangle$$

$$aa^+|\psi(t)\rangle = Ae^{-i\omega t/2}|0\rangle + 2Be^{-3i\omega t/2}|1\rangle + 4Ce^{-7i\omega t/2}|3\rangle$$

$$\rightarrow \langle\frac{p^2}{2m}\rangle = \langle\psi(t)|\frac{\hat{p}^2}{2m}|\psi(t)\rangle =$$

$$= -\frac{\hbar\omega}{4}\Big[\sqrt{6}C^*Be^{4i\omega t/2} + \sqrt{6}B^*Ce^{-4i\omega t/2} - |B|^2 - 3|C|^2$$

$$- |A|^2 - 2|B|^2 - 4|C|^2\Big] =$$

$$= \boxed{\frac{\hbar\omega}{4}\left[2\left(|B|^2+3|C|^2\right)+1-2\sqrt{6}\,|B^*C|\cos(2\omega t+\phi)\right]=\left\langle \frac{\hat{p}^2}{2m}\right\rangle}$$

$B^*C=|B^*C|e^{i\phi}$

Usando que: $\langle H\rangle = \left\langle \frac{\hat{p}^2}{2m}\right\rangle + \left\langle \frac{m\omega^2}{2}x^2\right\rangle$

$\rightarrow \left\langle \frac{m\omega^2}{2}x^2\right\rangle = \langle H\rangle - \left\langle \frac{\hat{p}^2}{2m}\right\rangle$ } ya obtenido

Ahora $\langle p\rangle$:

$$\langle p\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\langle\psi(t)|(a^+-a)|\psi(t)\rangle =$$

$$= i\sqrt{\frac{m\hbar\omega}{2}}\langle\psi(t)|\Big[Ae^{-i\omega t/2}|1\rangle+\sqrt{2}\,Be^{-3i\omega t/2}|2\rangle+\sqrt{4}\,Ce^{-7i\omega t/2}|4\rangle$$
$$-Be^{-3i\omega t/2}|0\rangle-\sqrt{3}\,Ce^{-7i\omega t/2}|2\rangle\Big]=$$

$$= i\sqrt{\frac{m\hbar\omega}{2}}\left[B^*Ae^{i\omega t}-A^*Be^{-i\omega t}\right] = \boxed{-\sqrt{2m\hbar\omega}\,|A^*B|\,\mathrm{sen}(\omega t+\alpha)=\langle p\rangle}$$

$A^*B=|A^*B|e^{i\alpha}$

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[B^*Ae^{i\omega t}+A^*Be^{-i\omega t}\right] = \boxed{\sqrt{\frac{2\hbar}{m\omega}}\,|A^*B|\cos(\omega t+\alpha)=\langle x\rangle}$$

$\hat{x}=\sqrt{\frac{\hbar}{2m\omega}}(a^++a)$

P4| Se dice que un estado coherente de un oscilador armónico cumple: $a|\lambda\rangle = \lambda|\lambda\rangle$, λ puede ser complejo

a) Probar que $|\lambda\rangle = e^{-|\lambda|^2/2} e^{\lambda a^+}|0\rangle$ es un estado coherente norm.

Sol: $a|\lambda\rangle = e^{-|\lambda|^2/2} a e^{\lambda a^+}|0\rangle = e^{-|\lambda|^2/2}[a, e^{\lambda a^+}]|0\rangle$

$a|0\rangle = 0$

evaluamos el conmutador:

$$[a, e^{\lambda a^+}] = [a, \sum_{n=0}^{\infty}\frac{1}{n!}(\lambda a^+)^n] = \sum_{n=0}^{\infty}\frac{1}{n!}\lambda^n[a,(a^+)^n] =$$

$$= \sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\sum_{k=1}^{n}(a^+)^{k-1}\underbrace{[a,a^+]}_{1}(a^+)^{n-k} = \sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\sum_{k=1}^{n}(a^+)^{n-1} = \sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\cdot n\,(a^+)^{n-1} =$$

$$= \sum_{n=1}^{\infty}\frac{1}{(n-1)!}\lambda^n(a^+)^{n-1} = \lambda\sum_{n=0}^{\infty}\frac{1}{n!}(\lambda a^+)^n = \lambda e^{\lambda a^+}$$

entonces:

$$a|\lambda\rangle = e^{-|\lambda|^2/2}\lambda e^{\lambda a^+}|0\rangle = \lambda|\lambda\rangle \quad ✓$$

Ahora veremos si esta normalizado:

$$\langle\lambda|\lambda\rangle = \langle 0|e^{-|\lambda|^2/2}e^{\lambda^* a}e^{-|\lambda|^2/2}e^{\lambda a^+}|0\rangle = e^{-|\lambda|^2}\langle 0|e^{\lambda^* a}e^{\lambda a^+}|0\rangle =$$

$$= e^{-|\lambda|^2}\sum_{n,m}\frac{1}{n!m!}(\lambda^*)^n\lambda^m\langle 0|a^n\underbrace{(a^+)^m|0\rangle}_{\sqrt{m!}|m\rangle} =$$

$$= e^{-|\lambda|^2}\sum_{n,m}\frac{\sqrt{n!}\sqrt{m!}}{n!m!}(\lambda^*)^n\lambda^m\underbrace{\langle n|m\rangle}_{\delta_{n,m}} = e^{-|\lambda|^2}\sum_n\frac{1}{n!}[|\lambda|^2]^n = e^{-|\lambda|^2}e^{+|\lambda|^2} = 1$$

luego si esta normalizada.

b) Escriba $|\lambda\rangle = \sum_{n=0}^{\infty}f(n)|n\rangle$. Muestre que $|f(n)|^2$ es una distribución de Poisson: $f(k,\lambda) = e^{-\lambda}\lambda^k/k!$

Sol: $|\lambda\rangle = \sum_{n=0}^{\infty}|n\rangle\langle n|\lambda\rangle$

$\langle n|\lambda\rangle \to f(n)$

$$\to \langle n|\lambda\rangle = \langle n|e^{-|\lambda|^2/2}e^{\lambda a^+}|0\rangle = e^{-|\lambda|^2/2}\langle n|\sum_{m=0}^{\infty}\frac{1}{m!}(\lambda a^+)^m|0\rangle =$$

$$= e^{-|\lambda|^2/2}\sum_{m=0}^{\infty}\frac{\lambda^m}{m!}\underbrace{\langle n|(a^+)^m|0\rangle}_{\sqrt{m!}\langle n|m\rangle} = e^{-|\lambda|^2/2}\frac{1}{\sqrt{n!}}\lambda^n$$

$\Rightarrow \boxed{|f(n)|^2 = e^{-|\lambda|^2}(|\lambda|^2)^n/n!}$ que es una distribución de Poisson.

d) Muestre que se puede obtener un estado coherente aplicando una traslación $e^{-ipl/\hbar}$ al estado fundamental. l es la distancia de desplazamiento

Sol: $a\, e^{-ipl/\hbar}|0\rangle = [a, e^{-ipl/\hbar}]|0\rangle$

Vemos el conmutador:

$$[a, e^{-ipl/\hbar}] = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{il}{\hbar}\right)^n [a, p^n] = \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{-il}{\hbar}\right)^n \sum_{k=1}^{n} p^{k-1}[a,p]\,p^{n-k}$$

Vemos el conmutador $[a,p]$:

$$[a,p] = i\sqrt{\frac{m\hbar\omega}{2}}\,\underbrace{[a,(a^\dagger - a)]}_{1} = i\,\frac{m\hbar\omega}{2}$$

$$\rightarrow [a, e^{-ipl/\hbar}] = \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{-il}{\hbar}\right)^n \sum_{k=1}^{n} p^{n-1} \cdot i\sqrt{\frac{m\hbar\omega}{2}} = \sum_{n=1}^{\infty} i\sqrt{\frac{m\hbar\omega}{2}}\,\frac{1}{(n-1)!}\left(\frac{-ilp}{\hbar}\right)^{n-1}\cdot\left(\frac{-il}{\hbar}\right) =$$

$$= l\sqrt{\frac{m\omega}{2\hbar}} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-ilp}{\hbar}\right)^n = l\sqrt{\frac{m\omega}{2\hbar}}\, e^{-ipl/\hbar}$$

remplazando:

$$\underbrace{a\, e^{-ipl/\hbar}|0\rangle}_{\text{autovector}} = \underbrace{l\sqrt{\frac{m\omega}{2\hbar}}}_{\text{autovalor}}\, \underbrace{e^{-ipl/\hbar}|0\rangle}_{\text{autovector}}$$