P1) Evalúe $[x(t), x(0)]$ en el cuadro de Heisenberg para una partícula libre.

Sol: Para una partícula libre: $\hat{H} = \frac{\hat{p}^2}{2m}$

Las ecuaciones de movimiento:

$$\frac{d\hat{p}}{dt} = \frac{1}{i\hbar}[\hat{p}, \hat{H}] = -\frac{\partial \hat{H}}{\partial \hat{x}}$$

$$\frac{d\hat{x}}{dt} = \frac{1}{i\hbar}[\hat{x}, \hat{H}] = \frac{\partial \hat{H}}{\partial \hat{p}}$$

$\rightarrow \frac{d\hat{p}}{dt} = 0 \Rightarrow p(t) = p(0)$

$\rightarrow \frac{d\hat{x}}{dt} = \frac{\hat{p}(t)}{m} = \frac{p(0)}{m} \Rightarrow x(t) = \frac{p(0)}{m}t + x(0)$

entonces:

$$[x(t), x(0)] = \left[\frac{p(0)}{m}t + x(0), x(0)\right] = \frac{t}{m}[p(0), x(0)] = \boxed{-\frac{i\hbar t}{m}}$$

P2) Considere el hamiltoniano en 3D:

$$\hat{H} = \frac{\vec{p}^2}{2m} + V(x)$$

calculando $[\vec{x}\cdot\vec{p}, H]$, obtenga:

$$\frac{d}{dt}\langle \vec{x}\cdot\vec{p}\rangle = \left\langle \frac{p^2}{2m}\right\rangle - \langle \vec{x}\cdot\vec{\nabla} V\rangle$$

Sol: Sacamos el conmutador:

$$[\vec{x}\cdot\vec{p}, H] = \left[\sum_i x_i p_i, H\right] = \sum_i \Big(x_i \underbrace{[p_i, H]}_{-i\hbar\frac{\partial H}{\partial x_i}} + \underbrace{[x_i, H]}_{i\hbar\frac{\partial H}{\partial p_i}} p_i\Big) =$$

$$= \sum_i \left[-i\hbar x_i \frac{\partial V}{\partial x_i} + i\hbar \frac{p_i}{m} p_i\right] = \frac{i\hbar}{m}\vec{p}^2 - i\hbar \vec{x}\cdot\vec{\nabla} V$$

Usamos: $\quad i\hbar \frac{d\hat{A}}{dt} = [\hat{A}, \hat{H}] + i\hbar \frac{\partial \hat{A}}{\partial t}$

$$\rightarrow \frac{d(\vec{x}\cdot\vec{p})}{dt} = \frac{1}{i\hbar}[\vec{x}\cdot\vec{p}, H] = \frac{\vec{p}^2}{m} - \vec{x}\cdot\vec{\nabla}V$$

$$\Rightarrow \boxed{\frac{d}{dt}\langle\vec{x}\cdot\vec{p}\rangle = \left\langle\frac{\vec{p}^2}{m}\right\rangle - \langle\vec{x}\cdot\vec{\nabla}V\rangle}$$

P3) Se busca resolver el oscilador armónico en el cuadro de Heisenberg.

a) Encuentre $X(t)$ y $P(t)$

b) Evaluando el elemento $\langle n' | \hat{X}(t) | n \rangle$, encuentre el espectro de energía.

Sol: a) $\hat{H} = \frac{\hat{P}^2}{2m} + \frac{m}{2}\omega_0^2 \hat{X}^2$

De las ecuaciones de movimiento:

(1) $\underbrace{\frac{d\hat{X}}{dt} = \frac{\hat{P}}{m}}$, (2) $\frac{d\hat{P}}{dt} = -m\omega_0^2 \hat{X}^2$

$$\frac{d^2\hat{X}}{dt^2} = \frac{1}{m}\frac{d\hat{P}}{dt} = -\omega_0^2 \hat{X}^2 \Rightarrow \hat{X}(t) = \hat{A}\cos(\omega_0 t) + \hat{B}\,\mathrm{sen}(\omega_0 t)$$

$$\hat{X}(0) = \hat{X}_0$$

derivando $\hat{X}$:

$$\left.\frac{d\hat{X}}{dt}\right|_{t=0} = \frac{\hat{P}(0)}{m} = \frac{\hat{P}_0}{m} = \omega_0 \hat{B} \Rightarrow \hat{B} = \frac{\hat{P}_0}{m\omega_0}$$

entonces: $$\boxed{\hat{X}(t) = \hat{X}_0 \cos(\omega_0 t) + \frac{\hat{P}_0}{m\omega_0}\mathrm{sen}(\omega_0 t)} \quad (3)$$

Usando (1):

$$\hat{P}(t) = -m\omega_0 \hat{A}\,\mathrm{sen}(\omega_0 t) + m\omega_0 \hat{B}\cos(\omega_0 t)$$

$$\boxed{\hat{P}(t) = -m\omega_0 \hat{X}_0\,\mathrm{sen}(\omega_0 t) + \hat{P}_0 \cos(\omega_0 t)} \quad (4)$$

b) el elemento de matriz $\langle n'|\hat{X}(t)|n\rangle$:

$$\langle n'|\hat{X}(t)|n\rangle = \langle n'|e^{\frac{i\hat{H}t}{\hbar}}\hat{X}(0)\,e^{-\frac{i\hat{H}t}{\hbar}}|n\rangle = e^{i(E_{n'}-E_n)t/\hbar}\langle n'|\hat{X}_0|n\rangle$$

usando (3):

$$\langle n'|\hat{X}(t)|n\rangle = \langle n'|\hat{X}_0\cos(\omega_0 t)|n\rangle + \langle n'|\frac{\hat{P}_0}{m\omega_0}\operatorname{sen}(\omega_0 t)|n\rangle =$$

$$= \frac{1}{2}\Big[e^{i\omega_0 t}\big(\langle n'|\hat{X}_0|n\rangle - i\langle n'|\frac{\hat{P}_0}{m\omega_0}|n\rangle\big)$$

$$+ e^{-i\omega_0 t}\big(\langle n'|\hat{X}_0|n\rangle + i\langle n'|\frac{\hat{P}_0}{m\omega_0}|n\rangle\big)\Big]$$

Si $E_{n'} > E_n$:

* $\langle n'|\hat{X}_0|n\rangle + i\langle n'|\frac{\hat{P}_0}{m\omega_0}|n\rangle = 0$
* $e^{i(E_{n'}-E_n)t/\hbar} = e^{i\omega_0 t}$

entonces: $E_{n'} = E_n + \hbar\omega_0$

Mientras $\langle n'|\hat{X}_0|n\rangle \neq 0$ se puede seguir la secuencia para energías mayores indefinidamente.

Si $E_{n'} < E_n$:

* $\langle n'|\hat{X}_0|n\rangle - i\langle n'|\frac{\hat{P}_0}{m\omega_0}|n\rangle = 0$
* $e^{i(E_{n'}-E_n)t/\hbar} = e^{-i\omega_0 t}$

entonces: $E_{n'} = E_n - \hbar\omega_0$

No se puede bajar indefinidamente ya que las energías deben ser positivas.

Denotando al estado fundamental como $|0\rangle$:

$$E_n = E_0 + n\hbar\omega_0$$

Al examinar el caso $\langle n|\hat{x}(t)|n\rangle$, se ve que debe ser cero porque $\hat{x}_0$ no tiene dependencia temporal. Por tanto:

$$\langle n|\underbrace{(\hat{x}_0 + i\frac{\hat{p}_0}{m\omega_0})|0\rangle}_{0} = 0$$

$$\Rightarrow \quad (\hat{x}_0 - i\frac{\hat{p}_0}{m\omega_0})(\hat{x}_0 + \frac{i\hat{p}_0}{m\omega_0})|0\rangle = 0$$

$$\left[\hat{x}_0^2 + \frac{\hat{p}_0^2}{m^2\omega_0^2} + x_0\frac{i\hat{p}_0}{m\omega_0} - \frac{i\hat{p}_0}{m\omega_0}\hat{x}_0\right]|0\rangle = 0$$

$$\left[\hat{x}_0^2 + \frac{\hat{p}_0^2}{m^2\omega_0^2} + \frac{i}{m\omega_0}\underbrace{[\hat{x}_0, \hat{p}_0]}_{i\hbar}\right]|0\rangle = 0 \qquad / \cdot \frac{1}{2}m\omega_0^2$$

$$\left[\frac{\hat{p}_0^2}{2m} + \frac{1}{2}m\omega_0^2\hat{x}_0^2 + \frac{1}{2}i\omega_0 \cdot i\hbar\right]|0\rangle = 0$$

$$\left[\hat{H} - \frac{\hbar\omega_0}{2}\right]|0\rangle = 0$$

$$E_0 - \frac{\hbar\omega_0}{2} = 0 \quad \Rightarrow \quad E_0 = \frac{\hbar\omega_0}{2}$$

$$\Rightarrow \boxed{E_n = \hbar\omega_0(n + \tfrac{1}{2})}$$

P3) Considere el hamiltoniano:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m}{2}\omega_0^2 \hat{x}^2 + eE\hat{x}$$

Usando $[\hat{p}(t), \hat{x}(t)] = -i\hbar$ encuentre la ec. de movimiento.
Muestre que $[x(t_1), x(t_2)] \neq 0$ para $t_1 \neq t_2$

Sol: Tenemos:
$$* \ \frac{d\hat{x}}{dt} = \frac{p}{m}$$

$$* \ \frac{d\hat{p}}{dt} = -m\omega_0^2 \hat{x} - eE$$

$$\Rightarrow \frac{d^2\hat{x}}{dt} = -\omega_0^2 \hat{x} - \frac{eE}{m} = -\omega_0^2 \left(x + \frac{eE}{m}\right)$$

$$\Rightarrow \hat{x}(t) = \hat{A}\cos(\omega_0 t) + \hat{B}\,\mathrm{sen}(\omega_0 t) - \frac{eE}{m\omega_0^2}$$

de forma similar a la pregunta anterior:

$$\boxed{\hat{x}(t) = \left(\hat{x}_0 + \frac{eE}{m\omega_0^2}\right)\cos(\omega_0 t) + \frac{\hat{p}_0}{m\omega_0}\mathrm{sen}(\omega_0 t) - \frac{eE}{m\omega_0^2}}$$

Ahora calculamos el conmutador:

$$[\hat{x}(t_1), \hat{x}(t_2)] = \left[\hat{x}_0 \cos(\omega_0 t_1) + \frac{\hat{p}_0}{m\omega_0}\mathrm{sen}(\omega_0 t_1), \hat{x}_0 \cos(\omega_0 t_2) + \frac{\hat{p}_0}{m\omega_0}\mathrm{sen}(\omega_0 t_2)\right] =$$

$$= \cos(\omega_0 t_1)\,\mathrm{sen}(\omega_0 t_2)\frac{1}{m\omega_0}\underbrace{[\hat{x}_0, \hat{p}_0]}_{i\hbar} + \mathrm{sen}(\omega_0 t_1)\cos(\omega_0 t_2)\frac{1}{m\omega_0}\underbrace{[\hat{p}_0, \hat{x}]}_{-i\hbar}$$

$$\boxed{= \frac{i\hbar}{m\omega_0}\mathrm{sen}(\omega_0(t_2 - t_1))}$$