On the Relationship Between Symmetric Maxitive, Minitive, and Modular Aggregation Operators

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Abstract

In this paper the relationship between symmetric minitive, maxitive, and modular aggregation operators is considered. It is shown that the intersection between any two of the three discussed classes is the same. Moreover, the intersection is explicitly characterized.

It turns out that the intersection contains families of aggregation operators such as OWMax, OWMin, and many generalizations of the widely-known Hirsch's h-index, often applied in scientific quality control.

Keywords: Aggregation operators; OWMax; OMA; OWA; Hirsch's h-index; scientometrics.

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1. Introduction

Aggregation operators consist of functions used to combine multiple numeric values into a single one, in some way representative of the whole input. They may be applied in many areas of human activity, e.g. in statistics, engineering, operational research, quality control, image processing, pattern recognition, webometrics, and scientometrics. For example, in scientific quality control at an individual level, we are often interested in assessing authors of scholarly publications by means of the number of citations received by each of his/her papers or by using some other measures of their quality, see e.g. [11].

From now on let $\mathbb{I} = [a, b]$ denote any closed interval of the extended real line, $\bar{\mathbb{R}} = [-\infty, \infty]$. Note that in many practical applications we set $\mathbb{I} = [0, 1]$ or $\mathbb{I} = [0, \infty]$, cf. [9] and [1], respectively. The set of all vectors of arbitrary length with elements in $\mathbb{I}$, i.e. $\bigcup_{n=1}^{\infty} \mathbb{I}^n$, is denoted by $\mathbb{I}^{1,2,\dots}$.

Let $\mathcal{E}(\mathbb{I})$ denote the set of all *aggregation operators* in $\mathbb{I}^{1,2,\dots}$, i.e. $\mathcal{E}(\mathbb{I}) = \{\mathsf{F} : \mathbb{I}^{1,2,\dots} \to \mathbb{I}\}$. Please note that the aggregation (averaging) functions, cf. [1, 14, 15, 13], most commonly appearing in the literature, form a particular subclass of $\mathcal{E}(\mathbb{I})$. We require each such function F to be nondecreasing in each variable and to fulfill two boundary conditions: $\mathsf{F}(a, a, \dots, a) = a$ and $\mathsf{F}(b, b, \dots, b) = b$. Also observe that typically the aggregation (averaging) functions are considered for a fixed-length input vectors (for other approaches see e.g. [3, 4, 8, 12, 19, 21]).

In this paper, however, we focus our attention on aggregation operators that are only nondecreasing (in each variable) and, additionally, symmetric (i.e. which do not depend on the order of elements' presentation). The boundary conditions are omitted in our framework, as in some applications they are too restrictive [9, 8]. Even though, note each aggregation operator F is a function into $\mathbb{I}$, therefore there is an implicit assumption that $\inf \mathsf{F} \geq a$, and $\sup \mathsf{F} \leq b$.

Definition 1. We say that $\mathsf{F} \in \mathcal{E}(\mathbb{I})$ is *symmetric*, denoted $\mathsf{F} \in \mathcal{P}_{(\mathrm{sym})}$, if

$$(\forall n \in \mathbb{N})\ (\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)\ \mathbf{x} \cong \mathbf{y} \Longrightarrow \mathsf{F}(\mathbf{x}) = \mathsf{F}(\mathbf{y}),$$

where $\mathbf{x} \cong \mathbf{y}$ if and only if there exists a permutation σ of $[n] := \{1, 2, \dots, n\}$ such that $\mathbf{x} = (y_{\sigma(1)}, \dots, y_{\sigma(n)})$

Definition 2. We say that $\mathsf{F} \in \mathcal{E}(\mathbb{I})$ is *nondecreasing*, denoted $\mathsf{F} \in \mathcal{P}_{(\mathrm{nd})}$, if

$$(\forall n \in \mathbb{N})\ (\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)\ \mathbf{x} \leq \mathbf{y} \Longrightarrow \mathsf{F}(\mathbf{x}) \leq \mathsf{F}(\mathbf{y}),$$

where $\mathbf{x} \leq \mathbf{y}$ if and only if $(\forall i \in [n])\ x_i \leq y_i$.

In the theory of aggregation we are often interested in aggregation operators which fulfill a number of desirable properties. Among most basic ones we may find e.g. maxitivity, minitivity, additivity, see [13], or modularity [20, 17]. In this paper we study the relationship between the symmetrized versions of these properties.

1.1. Notational Convention

If not stated otherwise explicitly we assume that $n, m \in \mathbb{N}$. Arithmetic and lattice operations on vectors of the same length, e.g. $+$, $-$, $\vee$ (maximum), $\wedge$ (minimum), are always performed element-wise. Let $x_{(i)}$ denote the ith order statistic of $\mathbf{x} \in \mathbb{I}^n$.

For each $\mathbf{x} \in \mathbb{I}^n$ and $\mathbf{y} \in \mathbb{I}^m$, $(\mathbf{x}, \mathbf{y})$ denotes the concatenation of the vectors, i.e. $(x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{I}^{n+m}$. A vector $(x, x, \ldots, x) \in \mathbb{I}^n$ is denoted briefly by $(n * x)$.

If $\mathsf{f}, \mathsf{g} : \mathbb{I} \to \mathbb{I}$, then $\mathsf{f} \preceq \mathsf{g}$ (g dominates f) if and only if $(\forall x \in \mathbb{I})\ \mathsf{f}(x) \leq \mathsf{g}(x)$.

Additionally, $\mathbf{1}$ denotes the indicator function.

In the next section we present and characterize three very interesting classes of symmetric aggregation operators, with which we are concerned in this paper.

2. Symmetric Maxitive, Minitive, and Modular Aggregation Operators

2.1. Definitions

Let us first recall the notion of a triangle of functions [10, 7]:

Definition 3. A *triangle of functions* is a sequence $\triangle = (\mathsf{f}_{i,n} : \mathbb{I} \to \mathbb{I})_{i \in [n], n \in \mathbb{N}}$.

Note that such an object is similar to a triangle of coefficients, $(c_{i,n} \in \bar{\mathbb{R}})_{i \in [n], n \in \mathbb{N}}$, considered e.g. in [4, 8, 19].

Quasi-S- and quasi-L-statistics were introduced in [10].

Definition 4. A *quasi-S-statistic* generated by a triangle of functions $\triangle$ is a function $\mathsf{qS}_\triangle \in \mathcal{E}(\mathbb{I})$ defined for any $(x_1, \ldots, x_n) \in \mathbb{I}^{1,2,\cdots}$ as $\mathsf{qS}_\triangle(\mathbf{x}) = \bigvee_{i=1}^n \mathsf{f}_{i,n}(x_{(n-i+1)})$.

Quasi-S-statistics generalize the well-known OWMax operators [5] for which we have $\mathsf{f}_{i,n}(x) = x \wedge c_{i,n}$, $c_{i,n} \in \mathbb{I}$, and $(\forall n)\ \bigvee_{i=1}^n c_{i,n} = b$.

Definition 5. Let $\triangle = (\mathsf{f}_{i,n})_{i \in [n], n \in \mathbb{N}}$ be a triangle of functions such that $(\forall n)\ \sum_{i=1}^n \inf \mathsf{f}_{i,n} \geq a$, and $\sum_{i=1}^n \sup \mathsf{f}_{i,n} \leq b$. Then the *quasi-L-statistic* generated by $\triangle$ is a function $\mathsf{qL}_\triangle \in \mathcal{E}(\mathbb{I})$ defined for any $(x_1, \ldots, x_n) \in \mathbb{I}^{1,2,\ldots}$ as $\mathsf{qL}_\triangle(\mathbf{x}) = \sum_{i=1}^n \mathsf{f}_{i,n}(x_{(n-i+1)})$.

Note that e.g. the condition $(\forall n)\ \sum_{i=1}^{n} \inf \mathsf{f}_{i,n} \geq a$ is important for $a < 0$. The class of quasi-L-statistics includes the OWA operators [26] (for $0 \in \mathbb{I}$) for which it holds $\mathsf{f}_{i,n}(x) = c_{i,n}x$, $c_{i,n} \in [0,1]$, $(\forall n)\ \sum_{i=1}^{n} c_{i,n} = 1$, and the OMA operators [20] with $(\forall n)\ \sum_{i=1}^{n} \mathsf{f}_{i,n} = \mathrm{id}$.

Let us introduce another interesting class of aggregation operators.

Definition 6. A *quasi-I-statistic* generated by a triangle of functions $\triangle$ is an aggregation operator $\mathsf{qI}_\triangle$, for which we have $\mathsf{qI}_\triangle(\mathbf{x}) = \bigwedge_{i=1}^{n} \mathsf{f}_{i,n}(x_{(n-i+1)})$, where $(x_1, \ldots, x_n) \in \mathbb{I}^{1,2,\ldots}$.

This class of functions generalizes the OWMin operators [5], for which we have $\mathsf{f}_{i,n}(x) = x \vee c_{i,n}$, where $c_{i,n} \in \mathbb{I}$, and $(\forall n)\ \bigwedge_{i=1}^{n} c_{i,n} = a$. However, observe that for each OWMax operator there exists an equivalent OWMin operator, and inversely [13].

The name L-statistics (linear combination of order statistics or, sometimes, linear combination of a function of order statistics) probably first appeared in [2] in the field of probability. Moreover, please note that $\vee$ and $\wedge$ denotes the maximum (Supremum) and, respectively, the minimum (Infimum) operator, hence the other names.

It may easily be shown that the restriction of quasi-S-statistics and quasi-I-statistics to $\mathbb{I}^n$ (for any n) generalizes Sugeno integrals (cf. [13]) of $\mathbf{x} \in \mathbb{I}^n$ with respect to any monotonic symmetric set function $\xi : 2^{[n]} \to \mathbb{I}$. Additionally, it should be noted that e.g. for $\mathbb{I} = [0, \infty]$ the restriction of quasi-L-statistics to $\mathbb{I}^n$ (for any n) generalizes Choquet integrals (cf. [13]) of $\mathbf{x} \in \mathbb{I}^n$ with respect to any symmetric capacity $\mu : 2^{[n]} \to \mathbb{I}$.

2.2. Monotonicity

Obviously, we have $\mathsf{qS}_\triangle, \mathsf{qL}_\triangle, \mathsf{qI}_\triangle \in \mathcal{P}_{(\mathrm{sym})}$ for any $\triangle$. Let us check when each introduced function is nondecreasing. Additionally, the three lemmas below state that, without loss of generality, each triangle of functions may in such case have some simplified form.

Lemma 7. *Let $\mathbb{I} = [a, b]$ and $\triangle = (\mathsf{f}_{i,n})_{i\in[n],n\in\mathbb{N}}$. Then $\mathsf{qS}_\triangle \in \mathcal{P}_{(\mathrm{nd})}$ if and only if there exists $\nabla = (\mathsf{g}_{i,n})_{i\in[n],n\in\mathbb{N}}$ satisfying the following conditions:*

(i) $(\forall n)\ (\forall i \in [n])$ $\mathsf{g}_{i,n}$ *is nondecreasing,*

(ii) $(\forall n)\ (\forall i \in [n])$ $\mathsf{g}_{i,n}(a) = \mathsf{g}_{n,n}(a)$*,*

(iii) $(\forall n)$ $\mathsf{g}_{1,n} \preceq \cdots \preceq \mathsf{g}_{n,n}$*,*

such that $\mathsf{qS}_\triangle = \mathsf{qS}_\triangledown$.

Proof. ($\Longrightarrow$) Let us fix n. Let $d_n := \mathsf{qS}_\triangle(n * a) = \bigvee_{i=1}^n \mathsf{f}_{i,n}(a)$. Therefore, as $\mathsf{qS}_\triangle \in \mathcal{P}_{(\mathrm{nd})}$, for all $\mathbf{x} \in \mathbb{I}^n$ it holds $\mathsf{qS}_\triangle(\mathbf{x}) \geq d_n \geq a$. As a consequence,

$$\mathsf{qS}_\triangle(\mathbf{x}) = \bigvee_{i=1}^n \mathsf{f}_{i,n}(x_{(n-i+1)}) = \bigvee_{i=1}^n \left(\mathsf{f}_{i,n}(x_{(n-i+1)}) \vee d_n\right).$$

Note that, as $\mathsf{qS}_\triangle$ is nondecreasing, we have $(\forall \mathbf{x} \in \mathbb{I}^n)$ $(\forall i \in [n])$ $\mathsf{qS}_\triangle(\mathbf{x}) \geq \mathsf{qS}_\triangle(i * x_{(n-i+1)}, (n-i) * a)$ because $(x_{(n)}, \ldots, x_{(1)}) \geq (i * x_{(n-i+1)}, (n-i) * a)$. We therefore have $\mathsf{qS}_\triangle(\mathbf{x}) \geq \mathsf{f}_{j,n}(x_{(n-i+1)})$, where $1 \leq j \leq i \leq n$. However, by definition, for each $\mathbf{x}$ there exists $k \in [n]$ for which $\mathsf{qS}_\triangle(\mathbf{x}) = \mathsf{f}_{k,n}(x_{(n-k+1)})$. Thus,

$$\begin{array}{rcl} \mathsf{qS}_\triangle(\mathbf{x}) & = & \mathsf{qS}_\triangle(1 * x_{(n)}, (n-1) * a) \\ & \vee & \mathsf{qS}_\triangle(2 * x_{(n-1)}, (n-2) * a) \\ & \cdots & \\ & \vee & \mathsf{qS}_\triangle(n * x_{(1)}, (n-n) * a). \end{array}$$

This implies

$$\mathsf{qS}_\triangle(\mathbf{x}) = \bigvee_{i=1}^n \left(\bigvee_{j=1}^i \mathsf{f}_{j,n}(x_{(n-i+1)}) \vee d_n \right).$$

We may set $\mathsf{g}_{i,n}(x) := \bigvee_{j=1}^i \mathsf{f}_{j,n}(x) \vee d_n$ for all $i \in [n]$. We see that $\mathsf{g}_{1,n} \preceq \cdots \preceq \mathsf{g}_{n,n}$ and $\mathsf{g}_{1,n}(a) = \cdots = \mathsf{g}_{n,n}(a) = d_n$.

Let us show that each $\mathsf{g}_{i,n}$ is nondecreasing. Assume otherwise. Let there exist i and $a \leq x < y \leq b$ such that $\mathsf{g}_{i,n}(x) > \mathsf{g}_{i,n}(y)$. We have $\mathsf{qS}_\triangledown(i*x, (n-i)*a) = \mathsf{g}_{i,n}(x) > \mathsf{qS}_\triangledown(i*y, (n-i)*a) = \mathsf{g}_{i,n}(y)$, a contradiction.

($\Longleftarrow$) Trivial. $\square$

In the next lemma we assume that $a = 0$, because otherwise the resulting form of a triangle of functions becomes very complicated. In this case for each $\triangle = (\mathsf{f}_{i,n})_{i \in [n], n \in \mathbb{N}}$ we have, by Definition 5, $(\forall n)$ $\sum_{i=1}^n \inf \mathsf{f}_{i,n} \geq 0$.

Lemma 8. *Let* $\mathbb{I} = [0, b]$ *and* $\triangle = (\mathsf{f}_{i,n})_{i \in [n], n \in \mathbb{N}}$ *such* $\sum_{i=1}^n \sup \mathsf{f}_{i,n} \leq b$. *Then* $\mathsf{qL}_\triangle \in \mathcal{P}_{(\mathrm{nd})}$ *if and only if there exists* $\triangledown = (\mathsf{g}_{i,n})_{i \in [n], n \in \mathbb{N}}$ *satisfying the following conditions:*

(i) $(\forall n)$ $(\forall i \in [n])$ $\mathsf{g}_{i,n}$ *is nondecreasing,*

(ii) $(\forall n)$ $\sum_{i=1}^{n} \mathsf{g}_{i,n}(b) \le b$,

(iii) $(\forall n)$ $(\forall i > 1)$ $\mathsf{g}_{i,n}(0) = 0$,

such that $\mathsf{qL}_\triangle = \mathsf{qL}_\nabla$.

Proof. $(\Longrightarrow)$ We may obviously set $\mathsf{g}_{1,1} := \mathsf{f}_{1,1}$ as $0 \preceq \mathsf{f}_{1,1} \preceq b$ and $\mathsf{f}_{1,1}$ is nondecreasing.

Fix $n > 1$. Let $d_n := \mathsf{qL}_\triangle(n * 0)$ and set $\mathsf{g}_{1,n}(x) := \mathsf{qL}_\triangle(x, (n-1) * 0)$. Thus, $\mathsf{g}_{1,n}(0) = d_n \ge 0$. Moreover, let us set $(\forall i > 1)$ $\mathsf{g}_{i,n}(0) = 0$.

Consider any $\mathbf{x} \in \mathbb{I}^n$. We have:

$$\mathsf{qL}_\nabla(x_{(n)}, x_{(n-1)}, (n-2) * 0) \quad = \quad \mathsf{g}_{1,n}(x_{(n)}) + \mathsf{g}_{2,n}(x_{(n-1)}),$$

and therefore:

$$\begin{aligned}
\mathsf{g}_{2,n}(x_{(n-1)}) &= \mathsf{qL}_\triangle(x_{(n)}, x_{(n-1)}, (n-2) * 0) - \mathsf{g}_{1,n}(x_{(n)}) \\
&= \mathsf{qL}_\triangle(x_{(n)}, x_{(n-1)}, (n-2) * 0) - \mathsf{qL}_\triangle(x_{(n)}, (n-1) * 0) \\
&= \mathsf{f}_{2,n}(x_{(n-1)}) - \mathsf{f}_{2,n}(0) \ge 0.
\end{aligned}$$

By considering consecutive elements of $\mathbf{x}$ (in a nonincreasing order) we get the following:

$$\begin{aligned}
\mathsf{g}_{1,n}(x) &= \mathsf{qL}_\triangle(x, (n-1) * 0) = \mathsf{f}_{1,n}(x) + \sum_{i=2}^{n} \mathsf{f}_{i,n}(0), \\
\mathsf{g}_{2,n}(x) &= \mathsf{f}_{2,n}(x) - \mathsf{f}_{2,n}(0), \\
&\cdots \\
\mathsf{g}_{n,n}(x) &= \mathsf{f}_{n,n}(x) - \mathsf{f}_{n,n}(0),
\end{aligned}$$

which gives $\mathsf{qL}_\triangle = \mathsf{qL}_\nabla$, and $\sum_{i=1}^{n} \mathsf{g}_{i,n}(b) \le b$.

We will show that each $\mathsf{g}_{i,n}$ is nondecreasing. Assume otherwise. Let there exist i and $0 \le x < y \le b$ such that $\mathsf{g}_{i,n}(x) > \mathsf{g}_{i,n}(y)$. We have $\mathsf{qL}_\nabla((i-1) * y, x, (n-i) * 0) - \mathsf{qL}_\nabla(i * y, (n-i) * 0) = \mathsf{g}_{i,n}(x) - \mathsf{g}_{i,n}(y) > 0$, a contradiction, because $\mathsf{qL}_\triangle$ is nondecreasing.

$(\Longleftarrow)$ Trivial. ∎

Lemma 9. *Let* $\mathbb{I} = [a, b]$ *and* $\triangle = (\mathsf{f}_{i,n})_{i \in [n], n \in \mathbb{N}}$. *Then* $\mathsf{ql}_\triangle \in \mathcal{P}_{(\mathrm{nd})}$ *if and only if there exists* $\nabla = (\mathsf{g}_{i,n})_{i \in [n], n \in \mathbb{N}}$ *satisfying the following conditions:*

(i) $(\forall n)$ $(\forall i \in [n])$ $\mathsf{g}_{i,n}$ *is nondecreasing,*

(ii) $(\forall n)$ $(\forall i \in [n])$ $\mathsf{g}_{i,n}(b) = \mathsf{g}_{1,n}(b)$,

(iii) $(\forall n)$ $\mathsf{g}_{1,n} \preceq \cdots \preceq \mathsf{g}_{n,n}$,

such that $\mathsf{ql}_\triangle = \mathsf{ql}_\triangledown$.

Proof. ($\Longrightarrow$) Let us fix n. Let $e_n := \mathsf{ql}_\triangle(n * b) = \bigwedge_{i=1}^n \mathsf{f}_{i,n}(b)$. Therefore, as $\mathsf{ql}_\triangle \in \mathcal{P}_{(\mathrm{nd})}$, for all $\mathbf{x} \in \mathbb{I}^n$ it holds $\mathsf{ql}_\triangle(\mathbf{x}) \le e_n \le b$. As a consequence,

$$\mathsf{ql}_\triangle(\mathbf{x}) = \bigwedge_{i=1}^{n} \mathsf{f}_{i,n}(x_{(n-i+1)}) = \bigwedge_{i=1}^{n} \left(\mathsf{f}_{i,n}(x_{(n-i+1)}) \wedge e_n\right).$$

Please note that, as $\mathsf{ql}_\triangle$ is nondecreasing, we have $(\forall \mathbf{x} \in \mathbb{I}^n)$ $(\forall i \in [n])$ $\mathsf{ql}_\triangle(\mathbf{x}) \le \mathsf{ql}_\triangle((i-1) * b, (n-i+1) * x_{(n-1+1)})$ because $(x_{(n)}, \ldots, x_{(1)}) \le ((i-1) * b, (n-i+1) * x_{(n-i+1)})$. We therefore have $\mathsf{ql}_\triangle(\mathbf{x}) \le \mathsf{f}_{j,n}(x_{(n-i+1)})$, where $1 \le i \le j \le n$. However, by definition, for each $\mathbf{x}$ there exists $k \in [n]$ for which $\mathsf{ql}_\triangle(\mathbf{x}) = \mathsf{f}_{k,n}(x_{(n-k+1)})$. Thus,

$$\begin{aligned} \mathsf{ql}_\triangle(\mathbf{x}) &= \mathsf{ql}_\triangle((n-1) * b, 1 * x_{(1)}) \\ &\wedge \mathsf{ql}_\triangle((n-2) * b, 2 * x_{(2)}) \\ &\cdots \\ &\wedge \mathsf{ql}_\triangle((n-n) * b, n * x_{(n)}). \end{aligned}$$

Consequently,

$$\mathsf{ql}_\triangle(\mathbf{x}) = \bigwedge_{i=1}^{n} \left(\bigwedge_{j=i}^{n} \mathsf{f}_{j,n}(x_{(n-i+1)}) \wedge e_n \right).$$

Therefore we may set $\mathsf{g}_{i,n}(x) := \bigwedge_{j=i}^n \mathsf{f}_{j,n}(x) \wedge e_n$ for all $i \in [n]$. We see that $\mathsf{g}_{1,n} \preceq \cdots \preceq \mathsf{g}_{n,n}$ and $\mathsf{g}_{1,n}(b) = \cdots = \mathsf{g}_{n,n}(b) = e_n$.

We will show that each $\mathsf{g}_{i,n}$ is nondecreasing. Assume otherwise. Let there exist i and $a \le x < y \le b$ such that $\mathsf{g}_{i,n}(x) > \mathsf{g}_{i,n}(y)$. We have $\mathsf{ql}_\triangledown((n-i)*b, i*x) = \mathsf{g}_{i,n}(x) > \mathsf{ql}_\triangledown((n-i)*b, i*y) = \mathsf{g}_{i,n}(y)$, a contradiction.

($\Longleftarrow$) Trivial. □

2.3. Characterizations

Let us introduce the two following vector operations ([10], see also [13]). The symmetric maximum is defined for any $\mathbf{x}, \mathbf{y} \in \mathbb{I}^n$ as $\mathbf{x} \overset{S}{\vee} \mathbf{y} = (x_{(1)} \vee y_{(1)}, \ldots, x_{(n)} \vee y_{(n)})$, and the symmetric minimum as $\mathbf{x} \overset{S}{\wedge} \mathbf{y} = (x_{(1)} \wedge y_{(1)}, \ldots, x_{(n)} \wedge y_{(n)})$. These operations are called symmetric because for any $\mathbf{u} \cong \mathbf{x}$ and $\mathbf{v} \cong \mathbf{y}$ we have $\mathbf{x} \overset{S}{\vee} \mathbf{y} = \mathbf{u} \overset{S}{\vee} \mathbf{v}$ and $\mathbf{x} \overset{S}{\wedge} \mathbf{y} = \mathbf{u} \overset{S}{\wedge} \mathbf{v}$.

Definition 10. Let $\mathsf{F} \in \mathcal{E}(\mathbb{I})$. Then F is *symmetric maxitive* (denoted $\mathsf{F} \in \mathcal{P}_{(\text{smax})}$) whenever $(\forall n)$ $(\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)$ $\mathsf{F}(\mathbf{x} \overset{S}{\vee} \mathbf{y}) = \mathsf{F}(\mathbf{x}) \vee \mathsf{F}(\mathbf{y})$.

Definition 11. Let $\mathsf{F} \in \mathcal{E}(\mathbb{I})$. Then F is *symmetric minitive* (denoted $\mathsf{F} \in \mathcal{P}_{(\text{smin})}$) if $(\forall n)$ $(\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)$ $\mathsf{F}(\mathbf{x} \overset{S}{\wedge} \mathbf{y}) = \mathsf{F}(\mathbf{x}) \wedge \mathsf{F}(\mathbf{y})$.

Modularity was discussed in [1, 20, 17]. Let us propose its symmetrized version.

Definition 12. Let $\mathsf{F} \in \mathcal{E}(\mathbb{I})$. Then F is *symmetric modular* (denoted $\mathsf{F} \in \mathcal{P}_{(\text{smod})}$) whenever $(\forall n)$ $(\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)$ $\mathsf{F}(\mathbf{x} \overset{S}{\vee} \mathbf{y}) + \mathsf{F}(\mathbf{x} \overset{S}{\wedge} \mathbf{y}) = \mathsf{F}(\mathbf{x}) + \mathsf{F}(\mathbf{y})$.

It may be shown easily that $\mathcal{P}_{(\text{smax})}, \mathcal{P}_{(\text{smin})}, \mathcal{P}_{(\text{smod})} \subseteq \mathcal{P}_{(\text{sym})} \cap \mathcal{P}_{(\text{nd})}$. Moreover, each symmetric modular aggregation operator is also symmetric additive (cf. [10, 13]).

Let us now present characterizations of nondecreasing quasi-S-, quasi-I-, and quasi-L-statistics. The following proposition (without proof) was stated in [10].

Proposition 13. *Let* $\mathbb{I} = [a, b]$ *and* $\mathsf{F} \in \mathcal{E}(\mathbb{I})$. *Then* $\mathsf{F} \in \mathcal{P}_{(\text{smax})}$ *if and only if* F *is a nondecreasing quasi-S-statistic.*

Proof. $(\Longrightarrow)$ Fix n and let $\mathbf{x} \in \mathbb{I}^n$. We have $\mathsf{F}(\mathbf{x}) = \mathsf{F}((n * x_{(1)}, 0 * a) \vee ((n-1) * x_{(2)}, 1 * a) \vee \cdots \vee (1 * x_{(n)}, (n-1) * a)) = \bigvee_{i=1}^n \mathsf{F}(i * x_{(n-i+1)}, (n-i) * a)$, and therefore may set $\mathsf{f}_{i,n}(x) = \mathsf{F}(i * x, (n-i) * a)$. Note that $(\forall i \in [i])$ $\mathsf{f}_{i,n}(a) = \mathsf{f}_{n,n}(a)$ and, as F is nondecreasing, each $\mathsf{f}_{i,n}$ is also nondecreasing. Also, $\mathsf{f}_{1,n} \preceq \cdots \preceq \mathsf{f}_{n,n}$ (cf. Lemma 7).

$(\Longleftarrow)$ Fix n and let $\mathbf{x}, \mathbf{y} \in \mathbb{I}^n$. Let $\triangle$ be such that $\mathsf{qS}_\triangle \in \mathcal{P}_{(\text{nd})}$ (of the form given in Lemma 7). We have $\mathsf{qS}_\triangle(\mathbf{x} \overset{S}{\vee} \mathbf{y}) = \bigvee_{i=1}^n \mathsf{f}_{i,n}(x_{(n-i+1)} \vee y_{(n-i+1)}) = \bigvee_{i=1}^n \left(\mathsf{f}_{i,n}(x_{(n-i+1)}) \vee \mathsf{f}_{i,n}(y_{(n-i+1)})\right) = \bigvee_{i=1}^n \mathsf{f}_{i,n}(x_{(n-i+1)}) \vee \bigvee_{i=1}^n \mathsf{f}_{i,n}(y_{(n-i+1)}) = \mathsf{qS}_\triangle(\mathbf{x}) \vee \mathsf{qS}_\triangle(\mathbf{y})$, which completes the proof. $\square$

Proposition 14. *Let* $\mathbb{I} = [a, b]$ *and* $\mathsf{F} \in \mathcal{E}(\mathbb{I})$. *Then* $\mathsf{F} \in \mathcal{P}_{(\text{smin})}$ *if and only if* F *is a nondecreasing quasi-I-statistic.*

Proof. $(\Longrightarrow)$ Fix n and let $\mathbf{x} \in \mathbb{I}^n$. We have $\mathsf{F}(\mathbf{x}) = \mathsf{F}((0*b, n*x_{(n)}) \wedge (1*b, (n-1)*x_{(n-1)}) \wedge \cdots \wedge ((n-1)*b, 1*x_{(1)})) = \bigwedge_{i=1}^n \mathsf{F}((i-1)*b, (n-i+1)*x_{(n-i+1)})$, and therefore we may set $\mathsf{f}_{i,n}(x) = \mathsf{F}((i-1) * b, (n-i+1) * x_{(n-i+1)})$. Note that $(\forall i \in [i])$ $\mathsf{f}_{i,n}(b) = \mathsf{f}_{n,n}(b)$ and, as F is nondecreasing, each $\mathsf{f}_{i,n}$ is also nondecreasing. Also, $\mathsf{f}_{1,n} \preceq \cdots \preceq \mathsf{f}_{n,n}$ (cf. Lemma 9).

($\Longleftarrow$) Trivial (analogous to the above proof). ☐

OMA operators were introduced in [20]. However, no proof was given for their characterization (the authors discussed modular [1] operators and then made their symmetrization). Also note that OMAs are idempotent and were originally discussed in the case $\mathbb{I} = [0, 1]$. Our result is thus more general.

Proposition 15. *Let $\mathbb{I} = [0, b]$ and $\mathsf{F} \in \mathcal{E}(\mathbb{I})$. Then $\mathsf{F} \in \mathcal{P}_{(\mathrm{smod})}$ if and only if F is a nondecreasing quasi-L-statistic.*

Proof. ($\Longrightarrow$) It is easily seen that if $\mathsf{F} \in \mathcal{P}_{(\mathrm{smod})}$, then the following inclusion-exclusion-like principle holds, i.e. for any given n and all $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{I}^n$ we have

$$\begin{aligned} \mathsf{F}(\mathbf{x}_1 \overset{S}{\vee} \cdots \overset{S}{\vee} \mathbf{x}_k) \;=\; & \sum_{1 \le i_1 \le k} \mathsf{F}(\mathbf{x}_{i_1}) - \sum_{1 \le i_1 < i_2 \le k} \mathsf{F}(\mathbf{x}_{i_1} \overset{S}{\wedge} \mathbf{x}_{i_2}) \\ & + \sum_{1 \le i_1 < i_2 < i_3 \le k} \mathsf{F}(\mathbf{x}_{i_1} \overset{S}{\wedge} \mathbf{x}_{i_2} \overset{S}{\wedge} \mathbf{x}_{i_3}) - \ldots \\ & + (-1)^{k-1} \mathsf{F}(\mathbf{x}_1 \overset{S}{\wedge} \cdots \overset{S}{\wedge} \mathbf{x}_k). \end{aligned}$$

Let $\mathbf{x} \in \mathbb{I}^n$. Set $\mathbf{x}^{[i,j]} := (i * x_{(n-j+1)}, (n-i) * 0)$, for $i, j \in [n]$. We have:

$$\mathbf{x} \simeq \left(\mathbf{x}^{[1,1]} \overset{S}{\vee} \cdots \overset{S}{\vee} \mathbf{x}^{[n,n]}\right).$$

For any $i, i', j, j' \in [n]$ it holds $\left(\mathbf{x}^{[i,j]} \overset{S}{\wedge} \mathbf{x}^{[i',j']}\right) \cong \mathbf{x}^{[i \wedge i', j \vee j']}$. Therefore,

$$\begin{aligned} \mathsf{F}(\mathbf{x}) \;=\; & \sum_{1 \le i_1 \le n} \mathsf{F}(\mathbf{x}^{[i_1,i_1]}) - \sum_{1 \le i_1 < i_2 \le n} \mathsf{F}(\mathbf{x}^{[i_1,i_2]}) \\ & + \sum_{1 \le i_1 < i_2 < i_3 \le n} \mathsf{F}(\mathbf{x}^{[i_1,i_3]}) - \ldots \\ & + (-1)^{n-1} \mathsf{F}(\mathbf{x}^{[1,n]}) \\ =\; & \sum_{i=1}^{n} \mathsf{f}_{i,n}(x_{(n-1+1)}), \end{aligned}$$

where

$$\begin{aligned}
\mathsf{f}_{1,n}(x) &= \mathsf{F}(1 * x_{(n)}, (n-1) * 0), \\
\mathsf{f}_{2,n}(x) &= \mathsf{F}(2 * x_{(n-1)}, (n-2) * 0) - \mathsf{F}(1 * x_{(n-1)}, (n-1) * 0), \\
\mathsf{f}_{3,n}(x) &= \quad \mathsf{F}(3 * x_{(n-2)}, (n-3) * 0) - \mathsf{F}(1 * x_{(n-2)}, (n-1) * 0) \\
&\quad -\mathsf{F}(2 * x_{(n-2)}, (n-2) * 0) + \mathsf{F}(1 * x_{(n-2)}, (n-1) * 0) \\
&= \mathsf{F}(3 * x_{(n-2)}, (n-3) * 0) - \mathsf{F}(2 * x_{(n-2)}, (n-2) * 0), \\
&\ldots \\
\mathsf{f}_{n,n}(x) &= \mathsf{F}(n * x_{(1)}, 0 * 0) - \mathsf{F}((n-1) * x_{(1)}, 1 * 0),
\end{aligned}$$

and hence F is a quasi-L-statistic. Note that as F is nondecreasing then each $\mathsf{f}_{i,n}$ is nondecreasing. We also have $(\forall i > 1)$ $\mathsf{f}_{i,n}(0) = 0$, and $\sum_{i=1}^{n} \mathsf{f}_{i,n}(b) \leq b$ (cf. Lemma 8).

($\Longleftarrow$) Fix n and let $\mathbf{x}, \mathbf{y} \in \mathbb{I}^n$. Let $\triangle$ be such that $\mathsf{qL}_\triangle \in \mathcal{P}_{(\mathrm{nd})}$ (of the form given in Lemma 8). We have $\mathsf{qL}_\triangle(\mathbf{x} \overset{S}{\vee} \mathbf{y}) + \mathsf{qL}_\triangle(\mathbf{x} \overset{S}{\wedge} \mathbf{y}) = \sum_{i=1}^{n} \mathsf{f}_{i,n}(x_{(n-i+1)} \vee y_{(n-i+1)}) + \sum_{i=1}^{n} \mathsf{f}_{i,n}(x_{(n-i+1)} \wedge y_{(n-i+1)}) = \sum_{i=1}^{n} \big(\mathsf{f}_{i,n}(x_{(n-i+1)}) + \mathsf{f}_{i,n}(y_{(n-i+1)})\big) = \mathsf{qL}_\triangle(\mathbf{x}) + \mathsf{qL}_\triangle(\mathbf{y})$. □

Let us explore the relationship between the three presented classes of aggregation operators.

3. On the Relationship Between the Three Classes

Let us first find for which quasi-S-statistic there exists an equivalent quasi-L-statistic. The following result was presented in [10] (for $\mathbb{I} = [0, \infty]$), however, only sketch of its proof was given there.

Proposition 16. *Let $\mathbb{I} = [0, b]$ and $\triangle = (\mathsf{f}_{i,n})_{i \in [n], n \in \mathbb{N}}$ such that $(\forall n)$ $\mathsf{f}_{1,n} \preceq \cdots \preceq \mathsf{f}_{n,n}$, $(\forall i \in [n])$ $\mathsf{f}_{i,n}$ is nondecreasing, and $\mathsf{f}_{i,n}(0) = \mathsf{f}_{n,n}(0) \geq 0$. Then $\mathsf{qS}_\triangle$ is a quasi-L-statistic if and only if $(\forall n)$ $(\forall i \in [n])$ $\mathsf{f}_{i,n}(x) = \mathsf{w}_n(x) \wedge c_{i,n}$ for some nondecreasing functions $\mathsf{w}_1, \mathsf{w}_2, \dots : \mathbb{I} \to \mathbb{I}$ and a triangle of coefficients $(c_{i,n})_{i \in [n], n \in \mathbb{N}}$ such that $0 \leq \mathsf{w}_n(0) \leq c_{1,n} \leq \cdots \leq c_{n,n} \leq b$.*

Proof. ($\Longleftarrow$) Let us fix n. Let $\mathsf{u}(\mathbf{x}) = \bigvee_{i=1}^{n} \mathsf{w}_n(x_{(n-i+1)}) \wedge c_{i,n}$ for $\mathbf{x} \in \mathbb{I}^n$, where w_n is a nondecreasing function, and $\mathsf{w}_n(0) \leq c_{1,n} \leq \cdots \leq c_{n,n}$. Obviously, u is nondecreasing and $\mathsf{u}(\mathbf{x}) \geq \mathsf{w}_n(0)$.

Consider any $\mathbf{x} \in \mathbb{I}^n$. Moreover, let $u_i = \mathsf{u}(x_{(n)}, \ldots, x_{(n-i+1)}, (n-i) * 0)$ for $i \in [n]$. It is clear that $u_n = \mathsf{qS}_\triangle|_{\mathbb{I}^n}(\mathbf{x})$. We have the following.

$$\begin{aligned}
u_1 &= \mathsf{w}_n(x_{(n)}) \wedge c_{1,n}, \\
u_2 &= \mathsf{w}_n(x_{(n)}) \wedge c_{1,n} \\
&\quad + \mathbf{1}_{\left(-\infty, \mathsf{w}_n(x_{(n-1)})\right]}(c_{1,n}) \left(\left(\mathsf{w}_n(x_{(n-1)}) \wedge c_{2,n} \right) - c_{1,n} \right), \\
u_3 &= \mathsf{w}_n(x_{(n)}) \wedge c_{1,n} \\
&\quad + \mathbf{1}_{\left(-\infty, \mathsf{w}_n(x_{(n-1)})\right]}(c_{1,n}) \left(\left(\mathsf{w}_n(x_{(n-1)}) \wedge c_{2,n} \right) - c_{1,n} \right) \\
&\quad + \mathbf{1}_{\left(-\infty, \mathsf{w}_n(x_{(n-2)})\right]}(c_{2,n}) \left(\left(\mathsf{w}_n(x_{(n-2)}) \wedge c_{3,n} \right) - c_{2,n} \right), \\
&\vdots \\
u_n &= \mathsf{w}_n(x_{(n)}) \wedge c_{1,n} \\
&\quad + \sum_{i=1}^{n-1} \mathbf{1}_{\left(-\infty, \mathsf{w}_n(x_{(n-i)})\right]}(c_{i,n}) \left(\left(\mathsf{w}_n(x_{(n-i)}) \wedge c_{i+1,n} \right) - c_{i,n} \right).
\end{aligned}$$

Now if we generalize the above discussion to any n, we conclude that a quasi-S-statistic $\mathsf{qS}_\triangle$ determined by the expression on the right side of the theorem's statement is equivalent to a quasi-L-statistic $\mathsf{qL}_\triangledown$, for which $\triangledown = (\mathsf{g}_{i,n})_{i\in[n],n\in\mathbb{N}}$ if defined as follows.

$$\mathsf{g}_{i,n}(x) \quad = \quad \mathbf{1}_{(-\infty, \mathsf{w}_n(x)]}(c_{i-1,n}) \left((\mathsf{w}_n(x) \wedge c_{i,n}) - c_{i-1,n} \right),$$

where $n = 1, 2, \ldots$, $i \in [n]$, and, for brevity, $c_{0,n} := 0$. Also note that $\mathsf{g}_{i,n} \succeq 0$.

($\Longrightarrow$) Without loss of generality (cf. Lemma 8), let $\triangledown = (\mathsf{g}_{i,n} : i \leq n)$ be such that $(\forall n)$ $(\forall i \in [n])$ $\mathsf{g}_{i,n}$ is nondecreasing, $\mathsf{g}_{1,n} \succeq 0$, $\mathsf{g}_{i,n}(0) = 0$ if $i > 1$, and $\sum_{j=1}^n \mathsf{g}_{j,n}(b) \leq b$.

Fix $n > 1$, otherwise we obviously set $\mathsf{g}_{1,n} := \mathsf{f}_{1,n}$. Let $d_n := \mathsf{qS}_\triangle(n * 0) = \mathsf{f}_{1,n}(0) = \cdots = \mathsf{f}_{n,n}(0)$. We are going to determine $\triangledown$ such that $\mathsf{qS}_\triangle = \mathsf{qL}_\triangledown$, i.e. to find for which $\mathsf{f}_{1,n}, \ldots, \mathsf{f}_{n,n}$, $\mathsf{g}_{1,n}, \ldots, \mathsf{g}_{n,n}$ the equality

$$\bigvee_{i=1}^{n} \mathsf{f}_{i,n}(x_{(n-i+1)}) = \sum_{i=1}^{n} \mathsf{g}_{i,n}(x_{(n-i+1)})$$

holds for all $\mathbf{x} \in \mathbb{I}^n$.

We have $\mathsf{g}_{1,n}(0) = d_n$ because $\mathsf{qS}_\triangle(n * 0) = \mathsf{qL}_\triangledown(n * 0) = d_n$.

Now consider any $\mathbf{x} \in \mathbb{I}^n$. We have:

$$\begin{aligned}
\mathsf{qS}_{\triangle}(x_{(n)}, (n-1)*0) &= \mathsf{f}_{1,n}(x_{(n)}) \vee \bigvee_{i=2}^{n} \mathsf{f}_{i,n}(0) \\
&= \mathsf{f}_{1,n}(x_{(n)}), \\
\mathsf{qL}_{\triangledown}(x_{(n)}, (n-1)*0) &= \mathsf{g}_{1,n}(x_{(n)}) + \sum_{i=2}^{n} \mathsf{g}_{i,n}(0) \\
&= \mathsf{g}_{1,n}(x_{(n)}),
\end{aligned}$$

therefore we must set $\mathsf{g}_{1,n} := \mathsf{f}_{1,n}$ (however we do not know the possible form of $\mathsf{f}_{1,n}$ yet).

Next,

$$\begin{aligned}
\mathsf{qS}_{\triangle}(x_{(n)}, x_{(n-1)}, (n-2)*0) &= \mathsf{f}_{1,n}(x_{(n)}) \vee \mathsf{f}_{2,n}(x_{(n-1)}), \\
\mathsf{qL}_{\triangledown}(x_{(n)}, x_{(n-1)}, (n-2)*0) &= \mathsf{g}_{1,n}(x_{(n)}) + \mathsf{g}_{2,n}(x_{(n-1)}).
\end{aligned}$$

We therefore look for all solutions $(\mathsf{f}_{1,n}, \mathsf{f}_{2,n})$ to a functional equation

$$\mathsf{f}_{1,n}(x_{(n)}) \vee \mathsf{f}_{2,n}(x_{(n-1)}) = \mathsf{g}_{1,n}(x_{(n)}) + \mathsf{g}_{2,n}(x_{(n-1)}),$$

which is equivalent to

$$\mathsf{f}_{1,n}(x_{(n)}) \vee \mathsf{f}_{2,n}(x_{(n-1)}) = \mathsf{f}_{1,n}(x_{(n)}) + \mathsf{g}_{2,n}(x_{(n-1)}). \tag{1}$$

Recall that $\mathsf{f}_{1,n} \preceq \mathsf{f}_{2,n}$. Let $y_1 := \inf\{y : \mathsf{f}_{1,n}(y) < \mathsf{f}_{2,n}(y)\}$ (if it does not exist we obviously have $\mathsf{f}_{2,n} = \mathsf{f}_{1,n}$ and $\mathsf{g}_{2,n} \equiv 0$).

Note that if $x_{(n-1)} < y_1$, then

$$\mathsf{qS}_{\triangle}(x_{(n)}, x_{(n-1)}, (n-2)*0) = \mathsf{f}_{1,n}(x_{(n)}),$$

and $\mathsf{g}_{2,n}(x_{(n-1)}) = 0$.

As $\mathsf{g}_{2,n}$ may be discontinuous at y_1, we shall consider 2 cases.

(i) $\mathsf{g}_{2,n}(y_1) > 0$, or

(ii) $\mathsf{g}_{2,n}(y_1) = 0$.

Please bear in mind that $(\forall z > y_1)\ \mathsf{g}_{2,n}(z) > 0$, as $\mathsf{g}_{2,n}$ is nondecreasing.

(i) If $x_{(n)} \geq x_{(n-1)} \geq y_1$, then (1) may be written as

$$\mathsf{f}_{2,n}(x_{(n-1)}) = \mathsf{f}_{1,n}(x_{(n)}) + \mathsf{g}_{2,n}(x_{(n-1)}).$$

This is because $\mathsf{g}_{2,n}(x_{(n-1)}) > 0$ implies that

$$\mathsf{f}_{1,n}(y_1) \le \mathsf{f}_{1,n}(x_{(n)}) \le \mathsf{f}_{1,n}(\infty) < \mathsf{f}_{2,n}(y_1) \le \mathsf{f}_{2,n}(x_{(n-1)}).$$

We therefore have $\mathsf{g}_{2,n}(x_{(n-1)}) = \mathsf{f}_{2,n}(x_{(n-1)}) - \mathsf{f}_{1,n}(x_{(n)})$ and, as $\mathsf{g}_{2,n}$ may not be dependent on $x_{(n)}$, it must hold $(\forall z > y_1)$ $\mathsf{f}_{1,n}(z) = \mathsf{f}_{1,n}(y_1)$.

Thus $\mathsf{f}_{1,n}$ must be of the following form:

$$\mathsf{f}_{1,n}(x) = \begin{cases} \mathsf{f}_{2,n}(x) & \text{for } x < y_1, \\ c_{1,n} & \text{for } x \ge y_1, \end{cases} \tag{2}$$

for some $\mathsf{f}_{2,n}(y_1) > c_{1,n} \ge \mathsf{f}_{2,n}(y_1^-)$, where $\mathsf{f}_{2,n}(y_1^-) = \lim_{x \to y_1^-} \mathsf{f}_{2,n}(x)$ (limit as x approaches y_1 from the left).

(ii) If $x_{(n)} \ge x_{(n-1)} > y_1$ (in this case both $\mathsf{f}_{1,n}$ and $\mathsf{f}_{2,n}$ may also be continuous at y_1), then (1) may be written as

$$\mathsf{f}_{2,n}(x_{(n-1)}) = \mathsf{f}_{1,n}(x_{(n)}) + \mathsf{g}_{2,n}(x_{(n-1)}).$$

This is because $\mathsf{g}_{2,n}(x_{(n-1)}) > 0$ and $g_{2,n}(y_1) = 0$ implies that

$$\mathsf{f}_{1,n}(y_1) = \mathsf{f}_{2,n}(y_1) \le \mathsf{f}_{1,n}(x_{(n)}) \le \mathsf{f}_{1,n}(\infty) < \mathsf{f}_{2,n}(x_{(n-1)}).$$

It must therefore hold $(\forall z > y_1)$ $\mathsf{f}_{1,n}(z) = \mathsf{f}_{1,n}(y_1^+) > \mathsf{f}_{1,n}(y_1)$.

Thus $\mathsf{f}_{1,n}$ must be of the following form:

$$\mathsf{f}_{1,n}(x) = \begin{cases} \mathsf{f}_{2,n}(x) & \text{for } x \le y_1, \\ c_{1,n} & \text{for } x > y_1, \end{cases} \tag{3}$$

for some $\mathsf{f}_{2,n}(y_1^+) > c_{1,n} \ge \mathsf{f}_{2,n}(y_1)$, where $\mathsf{f}_{2,n}(y_1^+) = \lim_{x \to y_1^+} \mathsf{f}_{2,n}(x)$.

Note that the both equations (2) and (3) may be written in a simpler form $\mathsf{f}_{1,n}(x) = \mathsf{f}_{2,n}(x) \wedge c_{1,n}$, and, as a consequence, we may set $\mathsf{g}_{2,n}(x) := \mathbf{1}_{(-\infty, \mathsf{f}_{2,n}(x)]}(c_{1,n})\,(\mathsf{f}_{2,n}(x) - c_{1,n})$.

Next we look for all solutions to a functional equation

$$\mathsf{qS}_\triangle(x_{(n)}, x_{(n-1)}, x_{(n-2)}, (n-3)*0) = \mathsf{qL}_\triangle(x_{(n)}, x_{(n-1)}, x_{(n-2)}, (n-3)*0),$$

which is equivalent to

$$\mathsf{f}_{1,n}(x_{(n)}) \vee \mathsf{f}_{2,n}(x_{(n-1)}) \vee \mathsf{f}_{3,n}(x_{(n-2)}) = \mathsf{g}_{1,n}(x_{(n)}) + \mathsf{g}_{2,n}(x_{(n-1)}) + \mathsf{g}_{3,n}(x_{(n-2)}),$$

and, consequently, to

$$\begin{aligned} & (\mathsf{f}_{2,n}(x_{(n)}) \wedge c_{1,n}) \vee \mathsf{f}_{2,n}(x_{(n-1)}) \vee \mathsf{f}_{3,n}(x_{(n-2)}) \\ = & (\mathsf{f}_{2,n}(x_{(n)}) \wedge c_{1,n}) \\ & +\mathbf{1}_{\left(-\infty,\mathsf{f}_{2,n}(x_{(n-1)})\right]}(c_{1,n}) \left(\mathsf{f}_{2,n}(x_{(n-1)}) - c_{1,n}\right) \\ & +\mathsf{g}_{3,n}(x_{(n-2)}). \end{aligned} \tag{4}$$

It holds $\mathsf{f}_{2,n} \preceq \mathsf{f}_{3,n}$. Let us take $y_2 := \inf\{y : \mathsf{f}_{2,n}(y) < \mathsf{f}_{3,n}(y)\}$ (if it exists, otherwise we have $\mathsf{f}_{3,n} = \mathsf{f}_{2,n}$). We certainly have $y_2 \geq y_1$.

Please note that if $x_{(n-2)} < y_2$ then

$$\mathsf{qS}_{\triangle}(x_{(n)}, x_{(n-1)}, x_{(n-2)}, (n-3) * 0) = \mathsf{qS}_{\triangle}(x_{(n)}, x_{(n-1)}, (n-2) * 0),$$

which implies $\mathsf{g}_{3,n}(x_{(n-2)}) = 0$.

As $\mathsf{g}_{3,n}$ may be discontinuous at y_2, we shall consider 2 cases.

(i') $\mathsf{g}_{3,n}(y_2) > 0$, or

(ii') $\mathsf{g}_{3,n}(y_2) = 0$.

Of course, $(\forall z > y_2)$ $\mathsf{g}_{3,n}(z) > 0$, as $\mathsf{g}_{3,n}$ is nondecreasing.

In either case (i') for $x_{(n)} \geq x_{(n-1)} \geq x_{(n-2)} \geq y_1$ or case (ii') for $x_{(n)} \geq x_{(n-1)} \geq x_{(n-2)} > y_1$, (4) may be written as

$$\begin{aligned} \mathsf{f}_{3,n}(x_{(n-2)}) &= c_{1,n} + \mathsf{f}_{2,n}(x_{(n-1)}) - c_{1,n} + \mathsf{g}_{3,n}(x_{(n-2)}) \\ &= \mathsf{f}_{2,n}(x_{(n-1)}) + \mathsf{g}_{3,n}(x_{(n-2)}). \end{aligned}$$

Requiring that $\mathsf{g}_{3,n}$ be independent of $x_{(n-1)}$, we get $\mathsf{f}_{2,n}(x) = \mathsf{f}_{3,n}(x) \wedge c_{2,n}$, for some $c_{2,n} \geq c_{1,n}$.

By applying a similar reasoning to the remaining elements of $\mathbf{x}$, we get $d_n \leq c_{1,1} \leq c_{2,n} \leq \cdots \leq c_{n,n}$, and $\mathsf{f}_{i+1,n}(x) = \mathsf{f}_{i,n}(x) \wedge c_{i,n}$ for $i \in [n-1]$. We may set $\mathsf{w}_n := \mathsf{f}_{n,n}$ and thus, as $\mathsf{f}_{n,n}$ is nondecreasing, the proof is complete. □

Next let us find the relation between quasi-S- and quasi-I-statistics.

Proposition 17. *Let $\mathbb{I} = [a, b]$ and $\triangle = (\mathsf{f}_{i,n})_{i\in[n],n\in\mathbb{N}}$ such that $(\forall n)$ $\mathsf{f}_{1,n} \preceq \cdots \preceq \mathsf{f}_{n,n}$, $(\forall i \in [n])$ $\mathsf{f}_{i,n}$ is nondecreasing, and $\mathsf{f}_{i,n}(a) = \mathsf{f}_{n,n}(a)$. Then $\mathsf{qS}_{\triangle}$ is a quasi-I-statistic if and only if $(\forall n)$ $(\forall i \in [n])$ $\mathsf{f}_{i,n}(x) = \mathsf{w}_n(x) \wedge c_{i,n}$ for some nondecreasing functions $\mathsf{w}_1, \mathsf{w}_2, \cdots : \mathbb{I} \to \mathbb{I}$, and a triangle of coefficients $(c_{i,n})_{i\in[n],n\in\mathbb{N}}$ such that $a \leq \mathsf{w}_n(a) \leq c_{1,n} \leq \cdots \leq c_{n,n} \leq b$.*

Proof. Without loss of generality (see Lemma 9), let $\triangledown = (\mathsf{g}_{i,n})_{i\in[n],n\in\mathbb{N}}$ be such that $(\forall n)$ $(\forall i \in [n])$ $\mathsf{g}_{i,n}$ is nondecreasing, $\mathsf{g}_{i,n}(b) = \mathsf{g}_{1,n}(b)$, and $\mathsf{g}_{1,n} \preceq \cdots \preceq \mathsf{g}_{n,n}$.

($\Longleftarrow$) Let us fix n. Additionally, let $\mathsf{f}_{i,n}$ be such that $\mathsf{f}_{i,n}(x) = \mathsf{w}_n(x) \wedge c_{i,n}$, where $\mathsf{w}_n : \mathbb{I} \to \mathbb{I}$ is nondecreasing and $a \leq \mathsf{w}_n(a) \leq c_{1,n} \leq \cdots \leq c_{n,n} \leq b$. What is more, we temporarily assume (with no loss in generality) that $\mathsf{w}_n(x) := \mathsf{w}_n(x) \wedge c_{n,n}$.

For all $y \in \mathbb{I}$ we have $\mathsf{qS}_\triangle(n * y) = \mathsf{w}_n(y) \wedge c_{n,n}$, and therefore $\mathsf{qS}_\triangle = \mathsf{qI}_\triangledown$ if and only if

$$\mathsf{qS}_\triangle(n * y) = \mathsf{f}_{n,n}(y) = \mathsf{w}_n(y) \wedge c_{n,n} = \mathsf{g}_{1,n}(y).$$

Consider any $\mathbf{x} \in \mathbb{I}^n$. It holds:

$$\begin{aligned} \mathsf{qS}_\triangle(x_{(n)}, (n-1) * a) &= \mathsf{w}_n(x_{(n)}) \wedge c_{1,n} \\ &= \mathsf{g}_{1,n}(x_{(n)}) \wedge \mathsf{g}_{2,n}(a) \\ &= \mathsf{w}_n(x_{(n)}) \wedge \mathsf{g}_{2,n}(a), \end{aligned}$$

and thus $\mathsf{g}_{2,n}(a) = c_{1,n}$.

If $\mathsf{w}_n(x_{(n)}) \wedge c_{1,n} \geq \mathsf{w}_n(x_{(n-1)}) \wedge c_{2,n}$, that is whenever $\mathsf{w}_n(x_{(n-1)}) \leq c_{1,n}$, we have

$$\mathsf{qS}_\triangle(x_{(n)}, x_{(n-1)}, (n-2) * a) = \mathsf{w}_n(x_{(n)}) \wedge c_{1,n},$$

and otherwise

$$\mathsf{qS}_\triangle(x_{(n)}, x_{(n-1)}, (n-2) * a) = \mathsf{w}_n(x_{(n-1)}) \wedge c_{2,n}.$$

Please note that

$$\begin{aligned} \mathsf{qI}_\triangledown(x_{(n)}, x_{(n-1)}, (n-2) * a) &= \mathsf{g}_{1,n}(x_{(n)}) \wedge \mathsf{g}_{2,n}(x_{(n-1)}) \wedge \mathsf{g}_{3,n}(a) \\ &= \mathsf{w}_n(x_{(n)}) \wedge \mathsf{g}_{2,n}(x_{(n-1)}) \wedge \mathsf{g}_{3,n}(a). \end{aligned}$$

As $\mathsf{qI}_\triangledown(b, (n-1) * y) = \mathsf{g}_{2,n}(y)$ for $\mathsf{w}_n(y) > c_{1,n}$, the above implies that:

$$\mathsf{g}_{2,n}(y) = \begin{cases} c_{1,n} & \text{if } \mathsf{w}_n(y) \leq c_{1,n}, \\ \mathsf{w}_n(y) & \text{otherwise,} \end{cases}$$

and $\mathsf{g}_{3,n}(a) = c_{2,n}$.

By applying similar reasoning for the remaining elements of $\mathbf{x}$ we approach the solution:

$$\mathsf{g}_{i,n}(y) = (\mathsf{w}_n(y) \vee c_{i-1,n}) \wedge c_{n,n},$$

where, for brevity, $c_{0,n} := a$. Hence, $\mathsf{qS}_\triangle$ of the assumed form is indeed equivalent to some quasi-I-statistic.

($\Longrightarrow$) Fix n. Assume that $\mathsf{qS}_\triangle = \mathsf{qI}_\triangledown$. Note for any $x \in \mathbb{I}$ we have:

$$\begin{array}{lllll} \mathsf{qS}_\triangle(n * x, 0 * a) & = & \mathsf{f}_{n,n}(x) & = & \mathsf{g}_{1,n}(x), \\ \mathsf{qS}_\triangle((n-1) * x, 1 * a) & = & \mathsf{f}_{n-1,n}(x) & = & \mathsf{g}_{1,n}(x) \wedge \mathsf{g}_{n,n}(a), \\ & \cdots & & \cdots & \\ \mathsf{qS}_\triangle(1 * x, (n-1) * a) & = & \mathsf{f}_{1,n}(x) & = & \mathsf{g}_{1,n}(x) \wedge \mathsf{g}_{2,n}(a), \end{array}$$

hence we $\mathsf{f}_{i,n}$ must be such that $\mathsf{f}_{i,n}(x) = \mathsf{w}_n(x) \wedge c_{i,n}$ for some nondecreasing function $\mathsf{w}_n : \mathbb{I} \to \mathbb{I}$ and $a \le c_{1,n} \le \cdots \le c_{n,n} \le b$. Clearly, a triangle of functions defined in such way fulfills the assumptions of the proposition and the proof is complete. □

Lastly, we find the relation between quasi-L- and quasi-I-statistics. Again, we assume $\mathbb{I} = [0, b]$.

Proposition 18. *Let $\mathbb{I} = [0, b]$ and $\triangle = (\mathsf{f}_{i,n})_{i\in[n],n\in\mathbb{N}}$ such that $(\forall n)$ $(\forall i \in [n])$ $\mathsf{f}_{i,n}$ is nondecreasing, $\mathsf{f}_{1,n} \succeq 0$, $(\forall j > 1)$ $\mathsf{f}_{i,n}(0) = 0$, and $\sum_{j=1}^{n} \mathsf{f}_{j,n}(b) \le b$. Then $\mathsf{qL}_\triangle$ is a quasi-I-statistic if and only if $(\forall n)$ $(\forall i \in [n])$ $\mathsf{f}_{i,n}(x) = \mathbf{1}_{(-\infty,\mathsf{w}_n(x)]}(c_{i-1,n})\left((\mathsf{w}_n(x) \wedge c_{i,n}) - c_{i-1,n}\right)$ for some nondecreasing functions $\mathsf{w}_1, \mathsf{w}_2, \cdots : \mathbb{I} \to \mathbb{I}$ and a triangle of coefficients $(c_{i,n})_{i\in[n],n\in\mathbb{N}}$ such that $0 \le \mathsf{w}_n(0) \le c_{1,n} \le \cdots \le c_{n,n} \le b$, with convention $c_{0,n} = 0$.*

Proof. Without loss of generality (cf. Lemma 9), let $\triangledown = (\mathsf{g}_{i,n})_{i\in[n],n\in\mathbb{N}}$ be a triangle of functions such that $(\forall n)$ $(\forall i \in [n])$ $\mathsf{g}_{i,n}$ is nondecreasing, $\mathsf{g}_{i,n}(b) = \mathsf{g}_{1,n}(b)$, $\mathsf{g}_{1,n} \preceq \cdots \preceq \mathsf{g}_{n,n}$.

($\Longleftarrow$) Let each $\mathsf{f}_{i,n}$ be of the assumed form. From the proofs of Props. 17, 16 we instantly approach the formula

$$\mathsf{g}_{i,n}(x) = (\mathsf{w}_n(x) \vee c_{i-1,n}) \wedge c_{n,n},$$

where, for brevity, $c_{0,n} := \mathsf{w}_n(0)$, for which we have $\mathsf{qL}_\triangle = \mathsf{qI}_\triangledown$.

($\Longrightarrow$) Fix n. We check when $\mathsf{qL}_\triangle = \mathsf{qI}_\triangledown$.

We have for all $x \in \mathbb{I}$:

$$\begin{aligned}
\mathsf{qL}_{\triangle}(x,(n-1)*0) &= \mathsf{f}_{1,n}(x)\\
&= \mathsf{g}_{1,n}(x) \wedge \mathsf{g}_{2,n}(0),\\
\mathsf{qL}_{\triangle}(b,x,(n-2)*0) &= \mathsf{f}_{1,n}(b) + \mathsf{f}_{2,n}(x)\\
&= \mathsf{g}_{1,n}(b) \wedge \mathsf{g}_{2,n}(x) \wedge \mathsf{g}_{3,n}(0),\\
&= \mathsf{g}_{2,n}(x) \wedge \mathsf{g}_{3,n}(0),\\
\mathsf{qL}_{\triangle}(b,b,x,(n-3)*0) &= \mathsf{f}_{1,n}(b) + \mathsf{f}_{2,n}(b) + \mathsf{f}_{3,n}(x)\\
&= \mathsf{g}_{1,n}(b) \wedge \mathsf{g}_{2,n}(b) \wedge \mathsf{g}_{3,n}(x) \wedge \mathsf{g}_{4,n}(0),\\
&= \mathsf{g}_{3,n}(x) \wedge \mathsf{g}_{4,n}(0),\\
&\dots
\end{aligned}$$

which implies that:

$$\begin{aligned}
\mathsf{f}_{1,n}(x) &= \mathsf{g}_{1,n}(x) \wedge \mathsf{g}_{2,n}(0),\\
\mathsf{f}_{2,n}(x) &= \mathsf{g}_{2,n}(x) \wedge \mathsf{g}_{3,n}(0) - \mathsf{g}_{1,n}(b) \wedge \mathsf{g}_{2,n}(0)\\
&= \mathsf{g}_{2,n}(x) \wedge \mathsf{g}_{3,n}(0) - \mathsf{g}_{2,n}(0),\\
\mathsf{f}_{3,n}(x) &= \mathsf{g}_{3,n}(x) \wedge \mathsf{g}_{4,n}(0) - \mathsf{g}_{1,n}(b) \wedge \mathsf{g}_{2,n}(0) - \mathsf{g}_{2,n}(b) \wedge \mathsf{g}_{3,n}(0) + g_{2,n}(0)\\
&= \mathsf{g}_{3,n}(x) \wedge \mathsf{g}_{4,n}(0) - \mathsf{g}_{3,n}(0),
\end{aligned}$$

and so forth. Note that for all $i \in [n]$ we may write the above as

$$\mathsf{f}_{i,n}(x) = \mathbf{1}_{(-\infty,\mathsf{w}_n(x)]}(c_{i-1,n})\left((\mathsf{w}_n(x) \wedge c_{i,n}) - c_{i-1,n}\right)$$

for some nondecreasing function $\mathsf{w}_n : \mathbb{I} \to \mathbb{I}$ and real constants $c_{0,n} = 0$, $c_{j,n} = \mathsf{g}_{i-1,n}(0)$ $(j \in [n-1])$, $c_{n,n} = \mathsf{g}_{1,n}(b)$, which completes the proof. ☐

We are now ready to present main results of this paper. They follow directly from Props. 16, 17, and 18.

The first theorem is illustrated in Fig. 1.

Theorem 19. *Let* $\mathbb{I} = [0, b]$. *Then* $\mathcal{P}_{(\mathrm{smax})} \cap \mathcal{P}_{(\mathrm{smin})} = \mathcal{P}_{(\mathrm{smax})} \cap \mathcal{P}_{(\mathrm{smod})} = \mathcal{P}_{(\mathrm{smin})} \cap \mathcal{P}_{(\mathrm{smod})} = \mathcal{P}_{(\mathrm{smax})} \cap \mathcal{P}_{(\mathrm{smod})} \cap \mathcal{P}_{(\mathrm{smin})}$.

Please note that the above theorem can also be proved in an alternative way, using Props. 13, 14, 15, i.e. the characterizations of the functions. However, these 3 propositions do not specify the form of quasi-S/L-/I-statistics that lie in $\mathcal{P}_{(\mathrm{smax})} \cap \mathcal{P}_{(\mathrm{smod})} \cap \mathcal{P}_{(\mathrm{smin})}$.

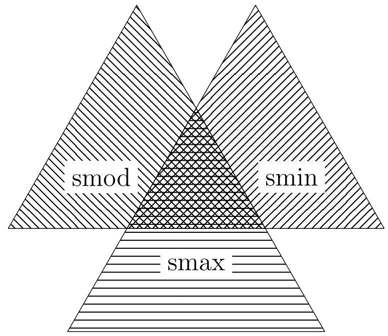

Figure 1: The intersection of the 3 classes of aggregation operators (see Theorem 19).

Alternative proof. Let $\mathsf{F} \in \mathcal{P}_{(\text{smax})} \cap \mathcal{P}_{(\text{smin})}$ and $(\forall n)$ $(\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)$, then we have $\mathsf{F}(\mathbf{x} \overset{S}{\vee} \mathbf{y}) + \mathsf{F}(\mathbf{x} \overset{S}{\wedge} \mathbf{y}) = \mathsf{F}(\mathbf{x}) \vee \mathsf{F}(\mathbf{y}) + \mathsf{F}(\mathbf{x}) \wedge \mathsf{F}(\mathbf{y}) = \mathsf{F}(\mathbf{x}) + \mathsf{F}(\mathbf{y})$, which implies $\mathsf{F} \in \mathcal{P}_{(\text{smod})}$.

If $\mathsf{F} \in \mathcal{P}_{(\text{smax})} \cap \mathcal{P}_{(\text{smod})}$, then $(\forall n)$ $(\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)$ it holds $\mathsf{F}(\mathbf{x} \overset{S}{\wedge} \mathbf{y}) = \mathsf{F}(\mathbf{x}) + \mathsf{F}(\mathbf{y}) - \mathsf{F}(\mathbf{x}) \vee \mathsf{F}(\mathbf{y}) = \mathsf{F}(\mathbf{x}) \wedge \mathsf{F}(\mathbf{y})$, which gives $\mathsf{F} \in \mathcal{P}_{(\text{smin})}$.

On the other hand, whenever $\mathsf{F} \in \mathcal{P}_{(\text{smin})} \cap \mathcal{P}_{(\text{smod})}$, then $(\forall n)$ $(\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)$ it holds $\mathsf{F}(\mathbf{x} \overset{S}{\vee} \mathbf{y}) = \mathsf{F}(\mathbf{x}) + \mathsf{F}(\mathbf{y}) - \mathsf{F}(\mathbf{x}) \wedge \mathsf{F}(\mathbf{y}) = \mathsf{F}(\mathbf{x}) \vee \mathsf{F}(\mathbf{y})$, which implies $\mathsf{F} \in \mathcal{P}_{(\text{smax})}$.

Therefore we have $\mathcal{P}_{(\text{smax})} \cap \mathcal{P}_{(\text{smin})} \subseteq \mathcal{P}_{(\text{smod})}$, $\mathcal{P}_{(\text{smax})} \cap \mathcal{P}_{(\text{smod})} \subseteq \mathcal{P}_{(\text{smin})}$, and $\mathcal{P}_{(\text{smin})} \cap \mathcal{P}_{(\text{smod})} \subseteq \mathcal{P}_{(\text{smax})}$, which gives $\mathcal{P}_{(\text{smax})} \cap \mathcal{P}_{(\text{smin})} = \mathcal{P}_{(\text{smax})} \cap \mathcal{P}_{(\text{smod})} = \mathcal{P}_{(\text{smin})} \cap \mathcal{P}_{(\text{smod})}$ and the proof is complete. □

Theorem 20. *Let $\mathbb{I} = [0, b]$. Given nondecreasing functions $\mathsf{w}_1, \mathsf{w}_2, \dots : \mathbb{I} \to \mathbb{I}$ and a triangle of coefficients $(c_{i,n})_{i \in [n], n \in \mathbb{N}}$ such that $0 \le \mathsf{w}_n(0) \le c_{1,n} \le \dots \le c_{n,n} \le b$, we have for all $(x_1, \dots, x_n) \in \mathbb{I}^{1,2,\dots}$:*

$$\begin{aligned} \bigvee_{i=1}^{n} \mathsf{w}_n(x_{(n-i+1)}) \wedge c_{i,n} &= \bigwedge_{i=1}^{n} (\mathsf{w}_n(x_{(n-i+1)}) \vee c_{i-1,n}) \wedge c_{n,n} \\ &= \sum_{i=1}^{n} \left(\left(\mathsf{w}_n(x_{(n-i+1)}) \vee c_{i-1,n} \right) \wedge c_{i,n} - c_{i-1,n} \right). \end{aligned}$$

with convention $c_{0,n} = 0$.

For example, in case of OWMax and OWMin operators generated by a triangle of coefficients $\triangle = (c_{i,n})_{i \in [n], n \in \mathbb{N}}$ such that $(\forall n)$ $c_{n,n} = b$, we have

$\mathsf{w}_n(x) = x$, where $x \in \mathbb{I}$ (thus, each OWMax and OWMin operator is symmetric minitive, maxitive, and modular). On the other hand, it is easily seen that if $b > 0$, then any OWA operator (which is of course, by definition, modular) is neither maxitive nor minitive.

4. Applications

The discussed aggregation operators have many valuable applications. For example, in the so-called Producers Assessment Problem [9, 8] we focus ourselves on construction of a class of mappings that project the space of arbitrary-sized real-numbered vectors of individual goods' quality measures into a single number that reflects both (a) general quality of the goods and (b) their producer's overall productivity. Among many interesting instances of the Producers Assessment Problem we have the issue of measuring an author's scientific merit by means of e.g. the number of citations received by his/her publications (for other approaches see e.g. [11]). In this case we assume that $\mathbb{I} = [0, \infty]$.

The first quality component may simply be represented by nondecreasing, symmetric aggregations operators. However, the second one needs some additional assumptions. It has been widely accepted that we should require that an output of any new product does not result in a decrease the producer's valuation. This behavior is assured by the following property.

Definition 21. We say that $\mathsf{F} \in \mathcal{E}(\mathbb{I})$ is *arity-monotonic*, denoted $\mathsf{F} \in \mathcal{P}_{(\mathrm{am})}$, if

$$(\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^{1,2,\dots})\ \mathsf{F}(\mathbf{x}) \le \mathsf{F}(\mathbf{x}, \mathbf{y}).$$

An aggregation operator useful in the Producers Assessment Problem (a so-called generalized impact function) is therefore any function in $\mathcal{P}_{(\mathrm{sym})} \cap \mathcal{P}_{(\mathrm{nd})} \cap \mathcal{P}_{(\mathrm{am})}$, see [9, 8], cf. also [7, 24].

Hirsch's h-index [16, 23], is the most widely-known tool that may be applied in this domain. It is an impact function defined as $\mathsf{H}(x_1, \dots, x_n) = \max\{0, 1, \dots, n : x_{(n-i+1)} \ge i\}$, where, for brevity of notation, $x_{(n+1)} := x_{(n)}$. It has been shown in [9] that such an aggregation operator is generated by nondecreasing functions $\mathsf{w}_n(x) = \lfloor x \rfloor$ and the triangle of coefficients for which we have $c_{i,n} = i$, $i \in [n]$.

Scientometricians considered many generalizations of the h-index. For example, the h^α-index [18], where $\alpha \ge 1$, is an impact function defined as

$\mathsf{H}^\alpha(x_1,\ldots,x_n) = \max\{0,1,\ldots,n : x_{(n-i+1)} \geq i^\alpha\}$ (we obtain such aggregation function by setting $(\forall n)$ $\mathsf{w}_n(x) = \lfloor x^{1/\alpha} \rfloor$ and $c_{i,n} = i$, $i \in [n]$). On the other hand, h_β-index [6, 25, 11], where $\beta > 0$, is defined as $\mathsf{H}_\beta(x_1,\ldots,x_n) = \max\{0,1,\ldots,n : x_{(n-i+1)} \geq \beta i^\alpha\}$, which may be obtained by setting $(\forall n)$ $\mathsf{w}_n(x) = \lfloor x/\beta \rfloor$ and $c_{i,n} = i$, $i \in [n]$.

It is easily seen, that for $\mathsf{F} \in \mathcal{P}_{(\mathrm{smax})} \cap \mathcal{P}_{(\mathrm{smin})} \cap \mathcal{P}_{(\mathrm{smod})}$ we have $\mathsf{F} \in \mathcal{P}_{(\mathrm{am})}$ if and only if F is generated by nondecreasing functions such that $\mathsf{w}_1 \preceq \mathsf{w}_2 \preceq \ldots$ and a triangle of coefficients $(c_{i,n})_{i\in[n],n\in\mathbb{N}}$ such that $0 \leq \mathsf{w}_n(0) \leq c_{1,n} \leq \cdots \leq c_{n,n} \leq b$, which fulfills $c_{i,n} \leq c_{i,n+1}$. This result may of course be used to construct new interesting aggregation operators.

5. Conclusions

In this paper we have examined the intersections between three classes of important aggregation operators. It turns out that all functions fulfilling any two chosen properties automatically has the remaining one.

We also observed that many influential aggregation operators used in e.g. scientometrics or webometrics are symmetric minitive, maxitive, and, simultaneously, modular. Also please note that our result serves as their natural, intuitive generalization.

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