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Constrained ordered weighted averaging aggregation with multiple comonotone constraints

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Abstract

The constrained ordered weighted averaging (OWA) aggregation problem arises when we aim to maximize or minimize a convex combination of order statistics under linear inequality constraints that act on the variables with respect to their original sources. The standalone approach to optimizing the OWA under constraints is to consider all permutations of the inputs, which becomes quickly infeasible when there are more than a few variables, however in certain cases we can take advantage of the relationships amongst the constraints and the corresponding solution structures. For example, we can consider a land-use allocation satisfaction problem with an auxiliary aim of balancing land-types, whereby the response curves for each species are non-decreasing with respect to the land-types. This results in comonotone constraints, which allow us to drastically reduce the complexity of the problem.

In this paper, we show that if we have an arbitrary number of constraints that are comonotone (i.e., they share the same ordering permutation of the coefficients), then the optimal solution occurs for decreasing components of the solution. After investigating the form of the solution in some special cases and providing theoretical results that shed light on the form of the solution,

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we detail practical approaches to solving and give real-world examples.

Keywords: Multiple criteria evaluation; Ordered weighted averaging; Constrained OWA aggregation; Ecology; Work allocation

1. Introduction

Yager in [24] introduced the so called ordered weighted averaging operators (OWA for short) which have useful modeling applications in e.g., decision making, portfolio optimization and risk analysis (see [7], [15], [21] and [24]-[27] among others). It is also worth mentioning that OWA operators are important examples of data aggregation functions, (see, e.g., [3], [9], [10]-[12]). While the optimization of OWA weights in the context of data-fitting has been well studied (see [1], [8], [13], [22], or the very recent survey given in paper [16]), another problem that arises is the need to optimize an OWA-based objective function with respect to linear constraints, which was first posed in [23], referred to as the constrained OWA aggregation problem. The solution therein involved mixed integer linear programming, which may not be computationally feasible in most real-world contexts.

In [4], the authors developed an analytical expression of the optimal solution for maximizing the objective function in the case of a single constraint and equal coefficients, which was generalized in [5] to the case of arbitrary coefficients. The idea works in the case of minimization too, as it was proved in [6]. In all these papers the idea was to construct the dual of a linear program such that the optimal value of this dual problem coincides with the optimal value of the initial constraint OWA aggregation problem. As we mentioned in our previous work too, the dual seems to be a useful tool in order to obtain the solution of such problems, see for example papers [19] and [20], where the case of decreasing weights is investigated. Other papers where optimization of various types of OWA are investigated are, for example [17], [18], [26]. The goal of the present paper is to advance the study of the constrained OWA aggregation problem, by showing that in the case of an arbitrary number of comonotone constraints, the problem can be expressed by means of a tractable linear program.

The paper is organized as follows. In Section 2 we present the basic theory relevant to the constrained OWA aggregation problem as well as some general results when we have multiple comonotone constraints. More precisely, the optimal solution of such problems can be deduced from the optimal solution of some linear programs where the components of the variable are decreasing. In Section 3, first we briefly recall the very recent results from papers [5] and [6], where the case of a single constraint was investigated. Then, we describe the constraint OWA aggregation problem in the case when we have 2 comonotone constraints to give insight into the form of the solution. In Section 4, we reformulate the LP problem to improve computation performance in the case of large datasets with multiple comonotone constraints and provide some application examples in ecology and work allocation. The paper ends with conclusions summarizing the present work and discussing some issues regarding the case when we have mixed constraints.

2. The constrained OWA aggregation problem with comonotone constraints

Suppose we have positive weights $w_1, \ldots, w_n$ such that $w_1 + \cdots + w_n = 1$ and define a mapping $F : \mathbb{R}^n \to \mathbb{R}$,

$$F(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i y_i,$$

where y_i is the i-th largest element of the sample $x_1, \ldots, x_n$. Then consider a matrix A of type (m, n) with real entries and a vector $b \in \mathbb{R}^m$. A constrained maximum OWA aggregation problem corresponding to the above data, is the problem (see [23])

$$\max F(x_1, \ldots, x_n) \text{ subject to } Ax \leq b,\ x \geq 0. \tag{1}$$

In this paper, we will investigate two problems where the only particularity is that the coefficients in the constraints can be rearranged to satisfy certain monotonicity properties. The maximization problem is

$$\left\{ \begin{array}{c} \max F(x_1, \ldots, x_n)\ , \\ \alpha_{i1} x_1 + \cdots + \alpha_{in} x_n \leq 1, i = \overline{1, m} \\ x \geq 0, \alpha_{ij} > 0, \end{array} \right. \tag{2}$$

and, there exists a permutation $\sigma \in S_n$ such that

$$\alpha_{i\sigma_1} \leq \alpha_{i\sigma_2} \leq \cdots \leq \alpha_{i\sigma_n}, i = \overline{1, m}. \tag{3}$$

Here S_n denotes the set of all permutations of $\{1, \ldots, n\}$ and for some $\sigma \in S_n$, we use the notation σ_k for the value $\sigma(k)$, for any $k \in \{1, \ldots, n\}$. From now on, we will say that the constraints in problem (2) are comonotone whenever condition (3) is satisfied. The minimization problem is

$$\left\{\begin{array}{c}\min F(x_1,\ldots,x_n)\ ,\\ \alpha_{i1}x_1+\cdots+\alpha_{in}x_n\geq 1, i=\overline{1,m}\\ x\geq 0,\alpha_{ij}>0,\end{array}\right. \tag{4}$$

and, again, there exists $\sigma\in S_n$ such that

$$\alpha_{i\sigma_1}\geq\alpha_{i\sigma_2}\geq\cdots\geq\alpha_{i\sigma_n}, i=\overline{1,m}. \tag{5}$$

We easily notice that condition (3) is satisfied if and only if condition (5) is satisfied (of course, for a different permutation), which means that in problem (4) too, the constraints are comonotone.

We need a property that was used in [5] too. If $a_1\leq\cdots\leq a_n$ and $b_1\geq\cdots\geq b_n$ then for any $\sigma\in S_n$ we have

$$a_1b_1+\cdots+a_nb_n\leq a_1b_{\sigma_1}+\cdots+a_nb_{\sigma_n}. \tag{6}$$

Now, suppose that $(x_1^*,\ldots,x_n^*)$ is a solution of (2) in the special case when $0<\alpha_{i1}\leq\cdots\leq\alpha_{in}$, $i=\overline{1,m}$. Obviously, we have $F\left(x_1^*,\ldots,x_n^*\right)=F\left(y_1^*,\ldots,y_n^*\right)$, where for any $i\in\{1,\ldots,n\}$, y_i^* is the i-th largest element between $x_1^*,\ldots,x_n^*$. On the other hand, taking into account the inequality (6), it follows that

$$\begin{array}{rcl} & & \alpha_{i1}y_1^*+\cdots+\alpha_{in}y_n^*\\ & \leq & \alpha_{i1}x_1^*+\cdots+\alpha_{in}x_n^*\leq 1,\\ i & = & \overline{1,m}.\end{array}$$

Therefore, we have

$$\left\{\begin{array}{c}F\left(x_1^*,\ldots,x_n^*\right)\\ \leq\max w_1x_1+\cdots+w_nx_n\ ,\\ \alpha_{i1}x_1+\cdots+\alpha_{in}x_n\leq 1, i=\overline{1,m},\\ x_1\geq\cdots\geq x_n\geq 0.\end{array}\right.$$

But one can easily check that the converse inequality also holds since $(y_1^*,\ldots,y_n^*)$ is a feasible solution for problem (2). Therefore, any solution of problem

$$\left\{\begin{array}{c}\max w_1x_1+\cdots+w_nx_n\ ,\\ \alpha_{i1}x_1+\cdots+\alpha_{in}x_n\leq 1, i=\overline{1,m}\\ x_1\geq\cdots\geq x_n\geq 0,\end{array}\right. \tag{7}$$

is a solution of problem (2) too. Actually, any solution of (2) is obtained by permuting the components of a solution of problem (7). In the case of a single constraint, in paper [5] the result was even stronger because generalizing with respect to any permutation, the two problems had the same solution. In our case it is possible that a permuted solution of problem (7) would not satisfy one of the constraints of problem (2). This is so because in the case of a single constraint it is not hard at all to prove that the solution must satisfy the constraint with equality. Therefore, in general, the solution set of problem (7) is included in the solution set of problem (2) and this is the first main result of this contribution.

Theorem 1. *Consider problems (2) and (7) (of course with the same coefficients α_{ij} in both problems) in the special case when $0 < \alpha_{i1} \leq \cdots \leq \alpha_{in}$, $i = \overline{1,m}$ (that is, we can take σ the identity permutation). Then any solution of problem (7) is a solution of problem (2). In addition, both problems have nonempty sets of solutions.*

Proof. Obviously, what remains to be proved is that problem (7) has a nonempty solution set. But this is immediate since the feasible set of problem (7) is compact in $\mathbb{R}^n$ and the objective function is linear. ■

What is more, there exists an important case when both problems have exactly the same solution set.

Theorem 2. *Consider problems (2) and (7) in the special case when $\alpha_{i1} < \alpha_{i2} < \cdots < \alpha_{in}$, $i = \overline{1,m}$. Then both problems have the same solution set which in addition is nonempty.*

Proof. We already explained why the solution sets are nonempty. We also know that any solution of problem (7) is a solution of problem (2). It only remains to prove that any solution of problem (2) is a solution of problem (7) too. By way of contradiction, suppose that $(x_1^*, \ldots, x_n^*)$ is a solution of problem (2) but not a solution of problem (7). It means that there exist $j_1, j_2 \in \{1, \ldots, n\}$, $j_1 < j_2$, such that $x_{j_1}^* < x_{j_2}^*$. We start by constructing $(t_1^*, \ldots, t_n^*)$, where $t_j^* = x_j^*$ for any $i \in \{1, \ldots, n\} \setminus \{j_1, j_2\}$, $t_{j_1}^* = x_{j_2}^*$ and $t_{j_2}^* = x_{j_1}^*$. By simple calculation we obtain

$$\sum_{j=1}^{n} \alpha_{ij} x_j^* - \sum_{j=1}^{n} \alpha_{ij} t_j^* = (\alpha_{j_1} - \alpha_{j_2}) \left(x_{j_1}^* - x_{j_2}^* \right) > 0,\ i = \overline{1,m}.$$

If the components of $(t_1^*, \ldots, t_n^*)$ are not in decreasing order then we continue to switch places for components that obstruct this property. After a finite number of iterations we will arrive to a vector, say $(y_1^*, \ldots, y_n^*)$ which has its components in decreasing order. At every iteration we will obtain the same type of inequalities as above. By transitivity, we get that

$$\sum_{j=1}^{n} \alpha_{ij} x_j^* - \sum_{j=1}^{n} \alpha_{ij} y_j^* > 0,\ i = \overline{1,m}.$$

Thus,

$$\sum_{j=1}^{n} \alpha_{ij} y_j^* < 1,\ i = \overline{1,m}.$$

However, obviously $(y_1^*, \ldots, y_n^*)$ is a permutation of $(x_1^*, \ldots, x_n^*)$ and this implies that $F(x_1^*, \ldots, x_n^*) = F(y_1^*, \ldots, y_n^*)$. This, together with the fact that $(y_1^*, \ldots, y_n^*)$ satisfies the constraints of problem (7), means that $(y_1^*, \ldots, y_n^*)$ is a solution of problem (7). Now, we can choose $\varepsilon > 0$ sufficiently small, such that $\sum_{j=1}^{n} \alpha_{ij} \left(y_j^* + \varepsilon\right) < 1,\ i = \overline{1,m}$. Therefore, $(y_1^* + \varepsilon, \ldots, y_n^* + \varepsilon)$ is a feasible solution for problem (7). Now, clearly we have $F(y_1^*, \ldots, y_n^*) < F(y_1^* + \varepsilon, \ldots, y_n^* + \varepsilon)$ and this contradicts the fact that $(y_1^*, \ldots, y_n^*)$ is a solution of problem (7). In conclusion, we obtain that $(x_1^*, \ldots, x_n^*)$ is a solution of problem (7) too. The proof is complete. ∎

Now, we can easily extend the conclusions of the last two theorems to the general form of problem (2).

Theorem 3. *Consider problem (2) and suppose that $\sigma \in S_n$ is such that $0 < \alpha_{i\sigma_1} \le \alpha_{i\sigma_2} \le \cdots \le \alpha_{i\sigma_n}$, $i = \overline{1,m}$. Furthermore, consider problem*

$$\begin{cases} \max w_1 x_1 + \cdots + w_n x_n\ , \\ \alpha_{i\sigma_1} x_1 + \cdots + \alpha_{i\sigma_n} x_n \le 1, i = \overline{1,m} \\ x_1 \ge \cdots \ge x_n \ge 0. \end{cases} \tag{8}$$

Then both problems (2) and (8) have nonempty solutions sets. Moreover, if $(x_1^*, \ldots, x_n^*)$ is a solution of problem (8) then $(x^*_{\sigma_1^{-1}}, \ldots, x^*_{\sigma_n^{-1}})$ is a solution of problem (2) (here, σ^{-1} denotes the inverse of σ). In the more restrictive case when $0 < \alpha_{i\sigma_1} < \alpha_{i\sigma_2} < \cdots < \alpha_{i\sigma_n}$, $i = \overline{1,m}$, $(x_1^*, \ldots, x_n^*)$ is a solution of problem (8) if and only if $(x^*_{\sigma_1^{-1}}, \ldots, x^*_{\sigma_n^{-1}})$ is a solution of problem (2).

From Theorem 3, it follows that in order to solve problem (2) in the case when the coefficients can be rearranged in nondecreasing order by the same permutation in each constraint (i. e., the constraints are comonotone), then it suffices to solve the linear programming problem (8).

In the case of the minimization problem (4) we obtain similar results. The proofs are very much the same as in the case of the maximization problem. This time we will use the fact that if $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$, then $\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i b_{\tau_i}$ for any $\tau \in S_n$. Following this, all the reasoning is the same as for the maximization problem.

Theorem 4. *Consider problem (4) and suppose that $\sigma \in S_n$ is such that $\alpha_{i\sigma_1} \geq \alpha_{i\sigma_2} \geq \cdots \geq \alpha_{i\sigma_n} > 0$, $i = \overline{1,m}$. Furthermore, consider problem*

$$\begin{cases} \min w_1 x_1 + \cdots + w_n x_n \ , \\ \alpha_{i\sigma_1} x_1 + \cdots + \alpha_{i\sigma_n} x_n \geq 1, i = \overline{1,m}, \\ x_1 \geq \cdots \geq x_n \geq 0. \end{cases} \tag{9}$$

Then both problems (4) and (9) have nonempty solutions sets. Moreover, if $(x_1^*, \ldots, x_n^*)$ is a solution of problem (9) then $(x^*_{\sigma_1^{-1}}, \ldots, x^*_{\sigma_n^{-1}})$ is a solution of problem (2) (here, σ^{-1} denotes the inverse of σ). In the more restrictive case when $0 < \alpha_{i\sigma_1} < \alpha_{i\sigma_2} < \cdots < \alpha_{i\sigma_n}$, $i = \overline{1,m}$, $(x_1^*, \ldots, x_n^*)$ is a solution of problem (9) if and only if $(x^*_{\sigma_1^{-1}}, \ldots, x^*_{\sigma_n^{-1}})$ is a solution of problem (4).

Proof. As we said earlier, the reasoning is almost identical to the case of the maximization problem. There is only one point where the proof differs, namely, the non-emptiness of the solution sets, as this time the feasible set is unbounded. However, one can easily notice that the solution set of problem (9) is actually the projection of the null vector onto the feasible set with respect to the norm $(x_1, \ldots, x_n) = w_1 |x_1| + \cdots + w_n |x_n|$. As this feasible set is closed and convex in $\mathbb{R}^n$, it is well-known that this projection set is nonempty. Therefore, the solution set of problem (9) is nonempty. This easily implies that the solution set of problem (4) is nonempty too. ∎

3. Analytical solution for special cases

In this section, first we recall the results obtained in [5] and [6] where the case of a single constraint is investigated. Then we extend the results to the case when we have two comonotone constraints. In this case too, we will obtain an analytical representation of the solution and this representation depends on an index from a derived linear program.

3.1. Constrained OWA with a single constraint

Let us return for a moment to the general form of a constrained OWA aggregation problem given in (1). A difficult task is to find an exact analytical solution to this problem. Yager used a method based on mixed integer linear programming, which employs the use of auxiliary variables and therefore, may not always be feasible. In the special case where we have the single constraint $x_1+\cdots+x_n=1$, the first analytical solution for the constrained OWA aggregation problem is given in paper [4]. This result has been generalized recently in paper [5] where the coefficients in the constraint are arbitrary. This problem can be formulated as

$$\max F(x_1,\ldots,x_n) \text{ subject to } \alpha_1 x_1+\cdots+\alpha_n x_n \le 1,\ x\ge 0. \tag{10}$$

Let us recall this result in the case when we can provide a nontrivial solution (these cases were solved in Propositions 1-2 in [5]). In what follows, S_n denotes the set of permutations of the set $\{1,\ldots,n\}$.

Theorem 5. *Consider problem (10). Then:*

(i) if there exists $i_0\in\{1,\ldots,n\}$ such that $\alpha_{i_0}\le 0$, then F is unbounded on the feasible set and its supremum over the feasible set is ∞;

(ii) if $\alpha_i>0$, $i\in\{1,\ldots,n\}$, then taking (any) $\sigma\in S_n$ with the property that $\alpha_{\sigma_1}\le\alpha_{\sigma_2}\le\cdots\le\alpha_{\sigma_n}$, and $k^*\in\{1,\ldots,n\}$, such that

$$\frac{w_1+\cdots+w_{k^*}}{\alpha_{\sigma_1}+\cdots+\alpha_{\sigma_{k^*}}}=\max\left\{\frac{w_1+\cdots+w_k}{\alpha_{\sigma_1}+\cdots+\alpha_{\sigma_k}}:k\in\{1,\ldots,n\}\right\},$$

then $(x_1^*,\ldots,x_n^*)$ is an optimal solution of problem (10), where

$$\begin{aligned} x^*_{\sigma_1} &= \cdots = x^*_{\sigma_{k^*}} = \frac{1}{\alpha_{\sigma_1}+\cdots+\alpha_{\sigma_{k^*}}},\\ x^*_{\sigma_{k^*+1}} &= \cdots = x^*_{\sigma_n}=0. \end{aligned}$$

In particular, if $0<\alpha_1\le\alpha_2\le\cdots\le\alpha_n$, and $k^*\in\{1,\ldots,n\}$ is such that

$$\frac{w_1+\cdots+w_{k^*}}{\alpha_{\sigma_1}+\cdots+\alpha_{\sigma_{k^*}}}=\max\left\{\frac{w_1+\cdots+w_k}{\alpha_{\sigma_1}+\cdots+\alpha_{\sigma_k}}:k\in\{1,\ldots,n\}\right\},$$

then $(x_1^*,\ldots,x_n^*)$ is a solution of (10), where

$$\begin{aligned} x^*_1 &= \cdots = x^*_{k^*} = \frac{1}{\alpha_1+\cdots+\alpha_{k^*}},\\ x^*_{k^*+1} &= \cdots = x^*_n = 0. \end{aligned}$$

Although, in paper [24] the maximum of the OWA operators is investigated, optimizing with respect to the minimum is also an important problem. Here, the general form is

$$\min F(x_1,\dots,x_n) \text{ subject to } Ax \le b,\, x \ge 0.$$

In the case of a single constraint, the most interesting problem is

$$\min F(x_1,\dots,x_n) \text{ subject to } \alpha_1 x_1 + \dots + \alpha_n x_n \ge 1,\, x \ge 0. \tag{11}$$

In the remaining cases one can easily prove that the problem is trivial with the solution $(0,\dots,0)$. For problem (11), an analogue of Theorem 5 can be found in the recent paper [6].

Theorem 6. *(see [6], Theorem 2) Consider problem (11). If $\alpha_i > 0$, $i \in \{1,\dots,n\}$, then taking $\sigma \in S_n$ with the property that $\alpha_{\sigma_1} \ge \alpha_{\sigma_2} \ge \dots \ge \alpha_{\sigma_n}$, and $k^* \in \{1,\dots,n\}$, such that*

$$\frac{w_1+\dots+w_{k^*}}{\alpha_{\sigma_1}+\dots+\alpha_{\sigma_{k^*}}} = \min\left\{\frac{w_1+\dots+w_k}{\alpha_{\sigma_1}+\dots+\alpha_{\sigma_k}} : k \in \{1,\dots,n\}\right\},$$

then $(x_1^*,\dots,x_n^*)$ is an optimal solution of problem (11), where

$$\begin{aligned} x_{\sigma_1}^* &= \dots = x_{\sigma_{k^*}}^* = \frac{1}{\alpha_{\sigma_1}+\dots+\alpha_{\sigma_{k^*}}}, \\ x_{\sigma_{k^*+1}}^* &= \dots = x_{\sigma_n}^* = 0. \end{aligned}$$

In particular, if $\alpha_1 \ge \alpha_2 \ge \dots \ge \alpha_n$, and $k^* \in \{1,\dots,n\}$ is such that

$$\frac{w_1+\dots+w_{k^*}}{\alpha_1+\dots+\alpha_{k^*}} = \min\left\{\frac{w_1+\dots+w_k}{\alpha_1+\dots+\alpha_k} : k \in \{1,\dots,n\}\right\},$$

then $(x_1^*,\dots,x_n^*)$ is a solution of (11), where

$$\begin{aligned} x_1^* &= \dots = x_{k^*}^* = \frac{1}{\alpha_1+\dots+\alpha_{k^*}}, \\ x_{k^*+1}^* &= \dots = x_n^* = 0. \end{aligned}$$

3.2. Constrained OWA aggregation with two comonotone constraints

In Section 2, considering the case of comonotone coefficients, we proved that a solution of problem (2) can be obtained by solving a linear program. The same is true in the case of the minimization problem. In the next section we will present an algorithm that finds this solution faster than the classical methods available on various mathematical packages. Still, it is important to have an indication of how the solution depends on the weights and coefficients when studying the stability of the solution set. In the case of one constraint, the problem is completely solved (see Theorems 5 and 6). In this section we will investigate the case of two constraints. We will see that the problem is more complex in this case. Besides the weights and coefficients, the solution also depends on some indices corresponding to some constraints in the dual problem. An interesting special case is when in one constraint all coefficients are equal and in the second constraint the coefficients are arbitrary. The method we will use here may not be easily generalizable to the case of more than two constraints.

In what follows, we are interested in the study of the problem

$$\begin{cases} \max F(x_1, \ldots, x_n) \ , \\ \alpha_1 x_1 + \cdots + \alpha_n x_n \leq 1, \\ \beta_1 x_1 + \cdots + \beta_n x_n \leq 1, \\ x \geq 0. \end{cases} \tag{12}$$

First, we consider the case when $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ and $0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$. By Theorem 1 it follows that any solution of problem

$$\begin{cases} \max w_1 x_1 + \cdots + w_n x_n \ , \\ \alpha_1 x_1 + \cdots + \alpha_n x_n \leq 1, \\ \beta_1 x_1 + \cdots + \beta_n x_n \leq 1 \\ x_1 \geq \cdots \geq x_n \geq 0, \end{cases} \tag{13}$$

is a solution of problem (12). What is more, both problems have nonempty solution sets. It means that it suffices to investigate problem (13).

For that, first, we need the dual of problem (13). This dual problem is

$$\begin{cases} \min t_1 + t_2, \\ \alpha_1 t_1 + \beta_1 t_2 - t_3 \geq \omega_1, \\ \alpha_2 t_1 + \beta_2 t_2 + t_3 - t_4 \geq \omega_2, \\ \alpha_3 t_1 + \beta_3 t_2 + t_4 - t_5 \geq \omega_3, \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{n-1} t_1 + \beta_{n-1} t_2 + t_n - t_{n+1} \geq \omega_{n-1}, \\ \alpha_n t_1 + \beta_n t_2 + t_{n+1} \geq \omega_n \\ t_1 \geq 0, t_2 \geq 0, \cdots, t_{n+1} \geq 0. \end{cases} \tag{14}$$

If we find a solution for this problem then we can use it to find a solution of problem (13). Let us consider now problem

$$\begin{cases} \min t_1 + t_2, \\ \alpha_1 t_1 + \beta_1 t_2 \geq w_1, \\ (\alpha_1 + \alpha_2) t_1 + (\beta_1 + \beta_2) t_2 \geq w_1 + w_2, \\ \cdot \\ \cdot \\ \left(\sum\limits_{i=1}^{k} \alpha_i\right) \cdot t_1 + \left(\sum\limits_{i=1}^{k} \beta_i\right) \cdot t_2 \geq \sum\limits_{i=1}^{k} w_i, \\ \cdot \\ \cdot \\ \left(\sum\limits_{i=1}^{n} \alpha_i\right) \cdot t_1 + \left(\sum\limits_{i=1}^{n} \beta_i\right) \cdot t_2 \geq \sum\limits_{i=1}^{n} w_i, \\ t_1 \geq 0, t_2 \geq 0. \end{cases} \tag{15}$$

Suppose that $\left(t_1^*, t_2^*, \ldots, t_{n+1}^*\right)$ is a solution of problem (14). It is immediate that (t_1^*, t_2^*) is a feasible solution for problem (15). Indeed, if we sum up the first k constraints in problem (14), $k = \overline{1,n}$, we get

$$\left(\sum_{i=1}^{k} \alpha_i\right) \cdot t_1^* + \left(\sum_{i=1}^{k} \beta_i\right) \cdot t_2^* - t_{k+2}^* \geq \sum_{i=1}^{k} w_i, k = \overline{1, n-1}$$

and

$$\left(\sum_{i=1}^{n} \alpha_i\right) \cdot t_1^* + \left(\sum_{i=1}^{n} \beta_i\right) \cdot t_2^* \geq \sum_{i=1}^{n} w_i.$$

This implies that indeed (t_1^*, t_2^*) is a feasible solution for problem (15). Now, suppose that $(\bar{t}_1, \bar{t}_2)$ is a solution of problem (15). If we replace (t_1, t_2) with $(\bar{t}_1, \bar{t}_2)$ in the constraints of problem (14) and if we consider the special case of equality in these constraints, after simple calculations, this system of equations gives the solution

$$\begin{cases} t_1^* = \bar{t}_1, t_2^* = \bar{t}_2, \\ t_k^* = \left(\sum\limits_{i=1}^{k-2} \alpha_i\right) \cdot \bar{t}_1 + \left(\sum\limits_{i=1}^{k-2} \beta_i\right) \cdot \bar{t}_2 - \sum\limits_{i=1}^{k-2} w_i, k = \overline{3, n+1}. \end{cases} \tag{16}$$

As $(\bar{t}_1, \bar{t}_2)$ is feasible for problem (15), it is immediate that $\left(t_1^*, \ldots, t_{n+1}^*\right)$ is feasible for problem (14). Summarizing, considering only the first two components of the feasible solutions of problems (14) and (15), we obtain the same sets. As the objective functions of both problems depend only on the first two components, and have the same values, it means that for both problems we have the same minimal value. We will see a little later that this minimum value is finite in our setting. Moreover, having the solution of one problem leads to a simple construction for the solution of the second problem.

In order to find a solution for problem (15), we need to investigate a problem given as

$$\begin{cases} \min t_1 + t_2, \\ a_k \cdot t_1 + b_k \cdot t_2 \geq 1, \, k = \overline{1, n}, \\ t_1 \geq 0, t_2 \geq 0. \end{cases} \tag{17}$$

It will suffice to consider only the case when $a_k > 0$ and $b_k > 0$, for all $k \in \{1, \ldots, n\}$. If we denote

$$a_k = \frac{\sum\limits_{i=1}^{k} \alpha_i}{\sum\limits_{i=1}^{k} w_i} \text{ and } b_k = \frac{\sum\limits_{i=1}^{k} \beta_i}{\sum\limits_{i=1}^{k} w_i}, \, k = \overline{1, n}. \tag{18}$$

problem (15) becomes exactly problem (17). Therefore, solving problem (17) will result in solving problem (15) as well. To continue our investigation, we need some concepts that are well-known in linear programming. Let us denote with $\mathbf{C}_k$ constraint number k of problem (17), $k = \overline{1, n}$. Then, we denote with U the feasible region of problem (15). Next, for some $k \in \{1, \ldots, n\}$ let $I_k = \{1, \ldots, n\} \setminus \{k\}$ and

$$U_k = \{(t_1, t_2) \in [0, \infty) \times [0, \infty) : a_i \cdot t_1 + b_i \cdot t_2 \geq 1, i \in I_k\}.$$

In other words, U_k is the feasible region of any optimization problem which keeps all the constraints from problem (17) except for constraint $\mathbf{C}_k$. The constraint $\mathbf{C}_k$ is called redundant if $U = U_k$ as the solution set of any optimization problem with feasible region U coincides with the solution set of the optimization problem that has the same objective function and all the constraints except for $\mathbf{C}_k$. This means that constraint $\mathbf{C}_k$ can be removed when solving the given problem. The constraint $\mathbf{C}_k$ is called strongly redundant if it is redundant and

$$a_k t_1 + b_k t_2 > 1, \text{ for all } (t_1, t_2) \in U.$$

Therefore, $\mathbf{C}_k$ is strongly redundant if and only if the segment which corresponds to the solutions of the equation $a_k t_1 + b_k t_2 = 1$, $t_1 \geq 0$, $t_2 \geq 0$, does not intersect U. A redundant constraint which is not strongly redundant is called weakly redundant. The constraint $\mathbf{C}_k$ is called binding if there exists at least one optimal point which satisfies this constraint with equality. It means, that the segment corresponding to the equation $a_k t_1 + b_k t_2 = 1$, $t_1 \geq 0$, $t_2 \geq 0$, contains an optimal point of the problem. Note that it is possible for a weakly redundant constraint to be binding as well. All these concepts were discussed with respect to our problem (17) but of course they can be defined accordingly for any kind of optimization problem.

We need the following auxiliary result.

Lemma 7. *Consider problem (17) in the case when* $a_k > 0$ *and* $b_k > 0$*, for all* $k \in \{1, \ldots, n\}$*. Suppose that* $k_1, k_2 \in \{1, \ldots, n\}$ *are such that* $a_{k_1} \geq b_{k_1}$ *and* $a_{k_2} \leq b_{k_2}$*. If* (t_1^*, t_2^*) *is a solution of the system*

$$\begin{cases} a_{k_1} t_1 + b_{k_1} t_2 = 1, \\ a_{k_2} t_1 + b_{k_2} t_2 = 1, \end{cases}$$

and if (t_1^*, t_2^*) *is feasible for problem (17), then* (t_1^*, t_2^*) *is an optimal solution for problem (17).*

Proof. Let f be the objective function, that is, $f(t_1, t_2) = t_1 + t_2$. In what follows, as usual, we denote with $[(a, b), (c, d)]$ the closed segment with endpoints (a, b) and (c, d) in $\mathbb{R}^2$. We notice that $(t_1^*, t_2^*) \in [(1/a_1, 0), (0, 1/b_1)] \cap [(1/a_2, 0), (0, 1/b_2)]$. Then, we have $f(1/a_1, 0) \leq f(0, 1/b_1)$ which by the linearity of f implies that $f(1/a_1, 0) \leq f(t_1, t_2) \leq f(0, 1/b_1)$, for all $(t_1, t_2) \in [(1/a_1, 0), (0, 1/b_1)]$. In particular, we get $f(t_1^*, t_2^*) \leq f(0, 1/b_1)$. Again, by the linearity of f it follows that $f(t_1^*, t_2^*) \leq f(t_1, t_2)$, for all $(t_1, t_2) \in [(t_1^*, t_2^*), (0, 1/b_1)]$. By similar reasoning we get that $f(t_1^*, t_2^*) \leq f(t_1, t_2)$, for all $(t_1, t_2) \in [(t_1^*, t_2^*), (0, 1/a_2)]$. Now, let us choose (t_1, t_2) arbitrary in the feasible region. It is clear that either $[(0,0), (t_1, t_2)] \cap [(t_1^*, t_2^*), (0, 1/b_1)] \neq \emptyset$, or $[(0,0), (t_1, t_2)] \cap [(t_1^*, t_2^*), (0, 1/a_2)] \neq \emptyset$. Without any loss of generality (the other case has identical reasoning) suppose that $[(0,0), (t_1, t_2)] \cap [(t_1^*, t_2^*), (0, 1/b_1)] \neq \emptyset$ and let (u_1, u_2) be the intersection point. As $t_1 \geq 0$ and $t_2 \geq 0$, it is immediate that $f(u_1, u_2) \leq f(t_1, t_2)$. On the other hand, we also have $f(u_1, u_2) \geq f(t_1^*, t_2^*)$, hence, we get $f(t_1^*, t_2^*) \leq f(t_1, t_2)$. In conclusion, (t_1^*, t_2^*) is an optimal point of problem (17). ■

Now, we can approach a formula to compute the solution of problem (12) in the case when $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ and $0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$.

Let $a_{k^*} = \min\{a_1, \ldots, a_n\}$ and $b_{k^{**}} = \min\{b_1, \ldots, b_n\}$, where $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are given in (18). In order to obtain the solution of problem (12), we need a discussion split in four cases which are not necessarily disjoint.

Case 1) $a_{k^*} \geq b_{k^*}$. In this case, taking into account Theorem 5, first, we get that $(x_1^*, \ldots, x_n^*)$ is a solution of (10), where

$$\begin{aligned} x_1^* &= \cdots = x_{k^*}^* = \frac{1}{\alpha_1 + \cdots + \alpha_{k^*}}, \\ x_{k^*+1}^* &= \cdots = x_n^* = 0. \end{aligned}$$

But as $a_{k^*} \geq b_{k^*}$, we immediately get that $(x_1^*, \ldots, x_n^*)$ is a feasible solution for problem (12). As the feasible region of problem (12) is included in the feasible region of problem (10), we get that $(x_1^*, \ldots, x_n^*)$ is a solution of problem (12). In addition, $\frac{w_1 + \cdots + w_{k^*}}{\alpha_1 + \cdots + \alpha_{k^*}}$ is the optimal value of problem (12).

Case 2) $b_{k^{**}} \geq a_{k^{**}}$. By similar reasoning as above, we obtain $(x_1^*, \ldots, x_n^*)$ as a solution of (12), where

$$\begin{aligned} x_1^* &= \cdots = x_{k^{**}}^* = \frac{1}{\beta_1 + \cdots + \beta_{k^{**}}}, \\ x_{k^{**}+1}^* &= \cdots = x_n^* = 0. \end{aligned}$$

In addition, $\frac{w_1 + \cdots + w_{k^{**}}}{\beta_1 + \cdots + \beta_{k^{**}}}$ is the optimal value of problem (12).

Case 3) Suppose that in problem (17) there exists a binding constraint $\mathbf{C}_{k_0}$ such that $a_{k_0} = b_{k_0}$. Obviously, in this case the minimum value for the objective function of problem (17) coincides with the minimum value for the objective function of problem

$$\left\{\begin{array}{c} t_1 + t_2 \to \min\,, \\ a_{k_0} t_1 + a_{k_0} t_2 \geq 1, \\ t_1 \geq 0,\, t_2 \geq 0. \end{array}\right. \tag{19}$$

The dual of the above problem is

$$\left\{\begin{array}{c} x \to \max\,, \\ a_{k_0} x \leq 1, \\ x \geq 0. \end{array}\right.$$

The solution of this problem is

$$x^* = \frac{1}{a_{k_0}} = \frac{w_1 + \cdots + w_{k_0}}{\alpha_1 + \cdots + \alpha_{k_0}}.$$

We observe that $(x_1^*, \ldots, x_n^*)$, where

$$\begin{aligned} x_1^* &= \cdots = x_{k_0}^*, \\ &= \frac{1}{\alpha_1 + \cdots + \alpha_{k_0}}, \\ x_{k_0+1}^* &= \cdots = x_n^* = 0, \end{aligned}$$

is a feasible solution for problem (12) (here, it is important that $a_{k_0} = b_{k_0}$). Let us prove that actually $(x_1^*, \ldots, x_n^*)$ is an optimal solution for problem (12). As already mentioned earlier, the minimum value for the objective function in the dual problem of (12), that is, problem (14), coincides with the minimum value of the objective function in problem (19). This minimum value is $\frac{w_1+\cdots+w_{k_0}}{\alpha_1+\cdots+\alpha_{k_0}}$. From the duality theorem, it means that for any feasible solution of problem (12), $x = (x_1, \ldots, x_n)$ we have

$$F(x_1, \ldots, x_n) \leq \frac{w_1 + \cdots + w_{k_0}}{\alpha_1 + \cdots + \alpha_{k_0}}.$$

On the other hand, we have

$$F(x_1^*, \ldots, x_n^*) = \frac{w_1 + \cdots + w_{k_0}}{\alpha_1 + \cdots + \alpha_{k_0}}.$$

Hence, as $(x_1^*, \ldots, x_n^*)$ belongs to the feasible region of problem (12), it follows that $(x_1^*, \ldots, x_n^*)$ is an optimal solution of problem (12) and $\frac{w_1+\cdots+w_{k_0}}{\alpha_1+\cdots+\alpha_{k_0}}$ is the optimal value of problem (12).

Case 4) This case is the more complex one. Therefore, we will apply it only when any of the previous three cases is not applicable. Hence, in this case we have that $a_{k^*} < b_{k^*}$, $a_{k^{**}} > b_{k^{**}}$ and, there exists an optimal solution of problem (17), (t_1^*, t_2^*), for which there does not exist a constraint $\mathbf{C}_k$, $k \in \{1, \ldots, n\}$, with $a_k = b_k$, such that (t_1^*, t_2^*) satisfies $\mathbf{C}_k$ with equality. In this case it is not hard at all to prove that there are two constraints $\mathbf{C}_{k_1}$ and $\mathbf{C}_{k_2}$ such that $a_{k_1} > b_{k_1}$, $a_{k_2} < b_{k_2}$ and such that (t_1^*, t_2^*) is the unique solution of the system

$$\begin{cases} a_{k_1} t_1 + b_{k_1} t_2 = 1, \\ a_{k_2} t_1 + b_{k_2} t_2 = 1. \end{cases}$$

This implies that (t_1^*, t_2^*) is the unique solution of the problem

$$\begin{cases} t_1 + t_2 \to \min, \\ a_{k_1} t_1 + b_{k_1} t_2 \geq 1, \\ a_{k_2} t_1 + b_{k_2} t_2 \geq 1, \\ t_1 \geq 0,\ t_2 \geq 0. \end{cases} \tag{20}$$

Without any loss of generality suppose that $k_1 < k_2$. By simple calculation, using (18), we have

$$t_1^* = \frac{\left(\sum_{i=1}^{k_1} w_i\right) \cdot \left(\sum_{i=k_1+1}^{k_2} \beta_i\right) - \left(\sum_{i=k_1+1}^{k_2} w_i\right) \cdot \left(\sum_{i=1}^{k_1} \beta_i\right)}{\left(\sum_{i=1}^{k_1} \alpha_i\right) \cdot \left(\sum_{i=k_1+1}^{k_2} \beta_i\right) - \left(\sum_{i=k_1+1}^{k_2} \alpha_i\right) \cdot \left(\sum_{i=1}^{k_1} \beta_i\right)} \tag{21}$$

and

$$t_2^* = \frac{\left(\sum_{i=k_1+1}^{k_2} w_i\right) \cdot \left(\sum_{i=1}^{k_1} \alpha_i\right) - \left(\sum_{i=1}^{k_1} w_i\right) \cdot \left(\sum_{i=k_1+1}^{k_2} \alpha_i\right)}{\left(\sum_{i=1}^{k_1} \alpha_i\right) \cdot \left(\sum_{i=k_1+1}^{k_2} \beta_i\right) - \left(\sum_{i=k_1+1}^{k_2} \alpha_i\right) \cdot \left(\sum_{i=1}^{k_1} \beta_i\right)}. \tag{22}$$

This results in the optimal value of problem (20) being

$$t_1^* + t_2^* = \frac{\left(\sum_{i=1}^{k_1} w_i\right) \cdot \left(\sum_{i=k_1+1}^{k_2} (\beta_i - \alpha_i)\right) + \left(\sum_{i=k_1+1}^{k_2} w_i\right) \cdot \left(\sum_{i=1}^{k_1} (\alpha_i - \beta_i)\right)}{\left(\sum_{i=1}^{k_1} \alpha_i\right) \cdot \left(\sum_{i=k_1+1}^{k_2} \beta_i\right) - \left(\sum_{i=k_1+1}^{k_2} \alpha_i\right) \cdot \left(\sum_{i=1}^{k_1} \beta_i\right)}. \tag{23}$$

The optimal value from above coincides with the optimal value of problem

$$\begin{cases} t_1 + t_2 \to \min\,, \\ \left(\sum_{i=1}^{k_1} \alpha_i\right) t_1 + \left(\sum_{i=1}^{k_1} \beta_i\right) t_2 - t_3 \geq \sum_{i=1}^{k_1} w_i, \\ \left(\sum_{i=k_1+1}^{k_2} \alpha_i\right) t_1 + \left(\sum_{i=k_1+1}^{k_2} \beta_i\right) t_2 + t_3 \geq \sum_{i=k_1+1}^{k_2} w_i, \\ t_1 \geq 0, t_2 \geq 0, t_3 \geq 0. \end{cases} \tag{24}$$

Indeed, these two problems are special cases of problems (14) and (15). The dual of this later problem is

$$\begin{cases} \left(\sum_{i=1}^{k_1} w_i\right) u_+ \left(\sum_{i=k_1+1}^{k_2} w_i\right) v \to \max\,, \\ \left(\sum_{i=1}^{k_1} \alpha_i\right) u + \left(\sum_{i=k_1+1}^{k_2} \alpha_i\right) v \leq 1, \\ \left(\sum_{i=1}^{k_1} \beta_i\right) u + \left(\sum_{i=k_1+1}^{k_2} \beta_i\right) v \leq 1, \\ u \geq v \geq 0. \end{cases} \tag{25}$$

Recall, we assumed that $a_{k_1} > b_{k_1}$ and $a_{k_2} < b_{k_2}$. This implies that $\sum_{i=1}^{k_1} \alpha_i > \sum_{i=1}^{k_1} \beta_i$ and $\sum_{i=1}^{k_2} \alpha_i < \sum_{i=1}^{k_2} \beta_i$. It also means that $\sum_{i=k_1+1}^{k_2} \alpha_i < \sum_{i=k_1+1}^{k_2} \beta_i$. The solution of the system

$$\begin{cases} \left(\sum_{i=1}^{k_1} \alpha_i\right) u + \left(\sum_{i=k_1+1}^{k_2} \alpha_i\right) v = 1, \\ \left(\sum_{i=1}^{k_1} \beta_i\right) u + \left(\sum_{i=k_1+1}^{k_2} \beta_i\right) v = 1, \end{cases}$$

is

$$\begin{aligned} u^* &= \frac{\sum_{i=k_1+1}^{k_2} (\beta_i - \alpha_i)}{\left(\sum_{i=1}^{k_1} \alpha_i\right) \cdot \left(\sum_{i=k_1+1}^{k_2} \beta_i\right) - \left(\sum_{i=1}^{k_1} \beta_i\right) \cdot \left(\sum_{i=k_1+1}^{k_2} \alpha_i\right)}, \\ v^* &= \frac{\sum_{i=1}^{k_1} (\alpha_i - \beta_i)}{\left(\sum_{i=1}^{k_1} \alpha_i\right) \cdot \left(\sum_{i=k_1+1}^{k_2} \beta_i\right) - \left(\sum_{i=1}^{k_1} \beta_i\right) \cdot \left(\sum_{i=k_1+1}^{k_2} \alpha_i\right)}. \end{aligned}$$

The hypotheses imply that the denominators in the expressions of u^* and v^* are strictly greater than 0. From here, it is immediate that $u^* > v^* > 0$. Hence, (u^*, v^*) belongs to the feasible region of problem (25). Let us prove that (u^*, v^*) is actually the optimal solution of problem (25). As problem (25) is the dual of problem (24) and the optimal value of this later problem coincides with the optimal value of problem (20), it suffices to prove that the objective function of problem (25) applied in (u^*, v^*) will give us the optimal value of problem (20), that is, the value of $t_1^* + t_2^*$ in (23). The objective function of problem (25) applied in (u^*, v^*) gives

$$
\begin{aligned}
& \left(\sum_{i=1}^{k_1} w_i\right) u^*_+ \left(\sum_{i=k_1+1}^{k_2} w_i\right) v^* \\
= & \frac{\left(\sum_{i=1}^{k_1} w_i\right) \cdot \left(\sum_{i=k_1+1}^{k_2} (\beta_i - \alpha_i)\right) + \left(\sum_{i=k_1+1}^{k_2} w_i\right) \cdot \left(\sum_{i=1}^{k_1} (\alpha_i - \beta_i)\right)}{\left(\sum_{i=1}^{k_1} \alpha_i\right) \cdot \left(\sum_{i=k_1+1}^{k_2} \beta_i\right) - \left(\sum_{i=1}^{k_1} \beta_i\right) \cdot \left(\sum_{i=k_1+1}^{k_2} \alpha_i\right)} \\
= & t_1^* + t_2^*.
\end{aligned}
$$

It necessarily follows that (u^*, v^*) is an optimal solution for problem (25).

Now, we observe that $x = (x_1^*, \ldots, x_n^*)$, where

$$
\begin{aligned}
x_1^* &= x_2^* = \cdots = x_{k_1}^* = u^*, \\
x_{k_1+1}^* &= \cdots = x_{k_2}^* = v^*, \\
x_{k_2+1}^* &= \cdots = x_n^* = 0,
\end{aligned}
$$

is a feasible solution for problem (12). Moreover, the optimal value of this problem is $t_1^* + t_2^*$. Hence, reasoning as in the previous Case 3), we get that $(x_1^*, \ldots, x_n^*)$ is an optimal solution of problem (12). In addition, the optimal value of this problem is $t_1^* + t_2^*$.

The above discussion covers all possible cases for problem (12). It can be summarized in the following theorem.

Theorem 8. *Consider problem (12) in the special case when* $0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$ *and* $0 < \beta_1 \le \beta_2 \le \cdots \le \beta_n$*. Then consider problem (17) where* a_k *and* b_k *are obtained using the substitutions given in (18),* $k = \overline{1, n}$*. Furthermore, take* $l = \arg\min\{a_k : k \in \{1, \ldots, n\}\}$ *and* $p = \arg\min\{b_k : k \in \{1, \ldots, n\}\}$*. We have the following cases (not necessarily* *distinct but covering all possible scenarios) in obtaining the optimal solution and the optimal value of problem (12).*

i) If $a_l \geq b_l$ then $(x_1^*, \dots, x_n^*)$ is a solution of problem (12), where

$$\begin{aligned} x_1^* &= \cdots = x_l^* = \frac{1}{\alpha_1 + \cdots + \alpha_l}, \\ x_{l+1}^* &= \cdots = x_n^* = 0. \end{aligned}$$

In addition, $\frac{w_1 + \cdots + w_l}{\alpha_1 + \cdots + \alpha_l}$ is the optimal value of problem (12).

ii) If $a_p \leq b_p$ then $(x_1^*, \dots, x_n^*)$ is a solution of problem (12), where

$$\begin{aligned} x_1^* &= \cdots = x_p^* = \frac{1}{\beta_1 + \cdots + \beta_p}, \\ x_{p+1}^* &= \cdots = x_n^* = 0. \end{aligned}$$

In addition, $\frac{w_1 + \cdots + w_p}{\beta_1 + \cdots + \beta_p}$ is the optimal value of problem (12).

iii) If in problem (17) there exists a binding constraint $\mathbf{C}_{k_0}$ such that $a_{k_0} = b_{k_0}$ then $(x_1^*, \dots, x_n^*)$ is a solution of problem (12), where

$$\begin{aligned} x_1^* &= \cdots = x_{k_0}^*, \\ &= \frac{1}{\alpha_1 + \cdots + \alpha_{k_0}}, \\ x_{k_0+1}^* &= \cdots = x_n^* = 0. \end{aligned}$$

In addition, $\frac{w_1 + \cdots + w_{k_0}}{\alpha_1 + \cdots + \alpha_{k_0}}$ is the optimal value of problem (12).

iv) If in problem (17) the optimal solution satisfies with equality the constraints $\mathbf{C}_{k_1}$ and $\mathbf{C}_{k_2}$, where $k_1 < k_2$ and $(a_{k_1} - b_{k_1}) \cdot (a_{k_2} - b_{k_2}) < 0$, then $(x_1^*, \dots, x_n^*)$ is a solution of problem (12), where

$$\begin{aligned} x_i^* &= \frac{\sum\limits_{i=k_1+1}^{k_2} (\beta_i - \alpha_i)}{\left(\sum\limits_{i=1}^{k_1} \alpha_i\right) \cdot \left(\sum\limits_{i=k_1+1}^{k_2} \beta_i\right) - \left(\sum\limits_{i=1}^{k_1} \beta_i\right) \cdot \left(\sum\limits_{i=k_1+1}^{k_2} \alpha_i\right)},\ i = \overline{1, k_1} \\ x_i^* &= \frac{\sum\limits_{i=1}^{k_1} (\alpha_i - \beta_i)}{\left(\sum\limits_{i=1}^{k_1} \alpha_i\right) \cdot \left(\sum\limits_{i=k_1+1}^{k_2} \beta_i\right) - \left(\sum\limits_{i=1}^{k_1} \beta_i\right) \cdot \left(\sum\limits_{i=k_1+1}^{k_2} \alpha_i\right)},\ i = \overline{k_1 + 1, k_2}, \\ x_{k_2+1}^* &= \cdots = x_n^* = 0. \end{aligned}$$

In addition, the optimal value of problem (12) is equal to the value of $t_1^* + t_2^*$ *computed in (23).*

Next, we generalize the result in Theorem 8. This time, the monotonicity assumption for the coefficients in the constraints does not necessarily hold. But, we suppose that there exists a permutation $\sigma \in S_n$ such that $0 < \alpha_{\sigma_1} \leq \alpha_{\sigma_2} \leq \cdots \leq \alpha_{\sigma_n}$ and $0 < \beta_{\sigma_1} \leq \beta_{\sigma_2} \leq \cdots \leq \beta_{\sigma_n}$. In such cases, the solution of problem (12) will be easily obtained by using an auxiliary problem which satisfies the hypotheses in Theorem 8.

Theorem 9. *Consider problem (12) and suppose that there exists a permutation* $\sigma \in S_n$ *such that* $0 < \alpha_{\sigma_1} \leq \alpha_{\sigma_2} \leq \cdots \leq \alpha_{\sigma_n}$ *and* $0 < \beta_{\sigma_1} \leq \beta_{\sigma_2} \leq \cdots \leq \beta_{\sigma_n}$. *Then consider the problem with the same objective function as in problem (12) and with the constraints*

$$\begin{cases} \alpha_{\sigma_1} x_1 + \alpha_{\sigma_2} x_2 + \cdots + \alpha_{\sigma_n} x_n \leq 1, \\ \beta_{\sigma_1} x_1 + \beta_{\sigma_2} x_2 + \cdots + \beta_{\sigma_n} x_n \leq 1, \\ x \geq 0. \end{cases}$$

If $(u_1^*, \ldots, u_n^*)$ *is a solution of this auxiliary problem, then* $(x_1^*, \ldots, x_n^*)$ *is a solution of problem (12), where* $x_i^* = u_{\sigma_i^{-1}}^*$, $i = \overline{1, n}$. *Here,* σ^{-1} *denotes the inverse of* σ.

From Theorem 9 we easily obtain the following corollary in which we assume that in the first constraint all coefficients are equal.

Theorem 10. *Consider problem (12) in the special case when* $0 < \alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$. *Let* $\sigma \in S_n$ *be any permutation such that* $0 < \beta_{\sigma_1} \leq \beta_{\sigma_2} \leq \cdots \leq \beta_{\sigma_n}$. *Then consider the problem with the same objective function as in problem (12) and with the constraints*

$$\begin{cases} \alpha x_1 + \alpha x_2 + \cdots + \alpha x_n \leq 1, \\ \beta_{\sigma_1} x_1 + \beta_{\sigma_2} x_2 + \cdots + \beta_{\sigma_n} x_n \leq 1, \\ x \geq 0. \end{cases}$$

If $(u_1^*, \ldots, u_n^*)$ *is a solution of this auxiliary problem, then* $(x_1^*, \ldots, x_n^*)$ *is a solution of problem (12), where* $x_i^* = u_{\sigma_i^{-1}}^*$, $i = \overline{1, n}$. *Here,* σ^{-1} *denotes the inverse of* σ.

Consider now the minimization problem

$$\begin{cases} \min F(x_1, \dots, x_n) \ , \\ \alpha_1 x_1 + \dots + \alpha_n x_n \le 1, \\ \beta_1 x_1 + \dots + \beta_n x_n \le 1, \\ x \ge 0 \end{cases} \tag{26}$$

in the special case when $\alpha_1 \ge \alpha_2 \ge \dots \ge \alpha_n > 0$ and $\beta_1 \ge \beta_2 \ge \dots \ge \beta_n > 0$. Obviously, the reasoning is very much the same as in the case of the maximization problem. The dual has the same objective function but needs to be maximized and the constraints are the same but with reversed inequalities. Therefore, we can easily construct an analogue for (17), which is

$$\begin{cases} \max t_1 + t_2, \\ a_k \cdot t_1 + b_k \cdot t_2 \le 1, \ k = \overline{1, n}, \\ t_1 \ge 0, t_2 \ge 0, \end{cases} \tag{27}$$

where a_k and b_k are given in (18), $i = \overline{1, n}$. Taking into account all these facts, we easily deduce analogue results (therefore, we omit the proofs) for the minimization problem.

Theorem 11. *Consider problem (26) in the special case when $\alpha_1 \ge \alpha_2 \ge \dots \ge \alpha_n > 0$ and $\beta_1 \ge \beta_2 \ge \dots \ge \beta_n > 0$. Then consider problem (27) where a_k and b_k are obtained using the substitutions given in (18), $k = \overline{1, n}$. Furthermore, take $l = \arg\max \{a_k : k \in \{1, \dots, n\}\}$ and $p = \arg\max \{b_k : k \in \{1, \dots, n\}\}$. We have the following cases (not necessarily distinct but covering all possible scenarios) in obtaining the optimal solution and the optimal value of problem (26).*

i) If $a_l \le b_l$ then $(x_1^*, \dots, x_n^*)$ is a solution of problem (26), where

$$\begin{aligned} x_1^* &= \dots = x_l^* = \frac{1}{\alpha_1 + \dots + \alpha_l}, \\ x_{l+1}^* &= \dots = x_n^* = 0. \end{aligned}$$

In addition, $\frac{w_1 + \dots + w_l}{\alpha_1 + \dots + \alpha_l}$ is the optimal value of problem (26).

ii) If $a_p \ge b_p$ then $(x_1^*, \dots, x_n^*)$ is a solution of problem (26), where

$$\begin{aligned} x_1^* &= \dots = x_p^* = \frac{1}{\beta_1 + \dots + \beta_p}, \\ x_{p+1}^* &= \dots = x_n^* = 0. \end{aligned}$$

In addition, $\frac{w_1+\cdots+w_p}{\beta_1+\cdots+\beta_p}$ *is the optimal value of problem (26).*

iii) If in problem (27) there exists a binding constraint $\mathbf{C}_{k_0}$ *such that* $a_{k_0}=b_{k_0}$ *then* $(x_1^*,\ldots,x_n^*)$ *is a solution of problem (26), where*

$$\begin{aligned} x_1^* &= \cdots = x_{k_0}^*, \\ &= \frac{1}{\alpha_1+\cdots+\alpha_{k_0}}, \\ x_{k_0+1}^* &= \cdots = x_n^* = 0. \end{aligned}$$

In addition, $\frac{w_1+\cdots+w_{k_0}}{\alpha_1+\cdots+\alpha_{k_0}}$ *is the optimal value of problem (26).*

iv) If in problem (17) the optimal solution satisfies with equality the constraints $\mathbf{C}_{k_1}$ *and* $\mathbf{C}_{k_2}$*, where* $k_1<k_2$ *and* $(a_{k_1}-b_{k_1})\cdot(a_{k_2}-b_{k_2})<0$*, then* $(x_1^*,\ldots,x_n^*)$ *is a solution of problem (26), where*

$$\begin{aligned} x_i^* &= \frac{\sum\limits_{i=k_1+1}^{k_2}(\beta_i-\alpha_i)}{\left(\sum\limits_{i=1}^{k_1}\alpha_i\right)\cdot\left(\sum\limits_{i=k_1+1}^{k_2}\beta_i\right)-\left(\sum\limits_{i=1}^{k_1}\beta_i\right)\cdot\left(\sum\limits_{i=k_1+1}^{k_2}\alpha_i\right)},\ i=\overline{1,k_1} \\ x_i^* &= \frac{\sum\limits_{i=1}^{k_1}(\alpha_i-\beta_i)}{\left(\sum\limits_{i=1}^{k_1}\alpha_i\right)\cdot\left(\sum\limits_{i=k_1+1}^{k_2}\beta_i\right)-\left(\sum\limits_{i=1}^{k_1}\beta_i\right)\cdot\left(\sum\limits_{i=k_1+1}^{k_2}\alpha_i\right)},\ i=\overline{k_1+1,k_2}, \\ x_{k_2+1}^* &= \cdots = x_n^* = 0. \end{aligned}$$

In addition, the optimal value of problem (26) is equal to the value of $t_1^*+t_2^*$ *computed in (23).*

Theorem 12. *Consider problem (26) and suppose that there exists a permutation* $\sigma\in S_n$ *such that* $\alpha_{\sigma_1}\geq\alpha_{\sigma_2}\geq\cdots\geq\alpha_{\sigma_n}>0$ *and* $\beta_{\sigma_1}\geq\beta_{\sigma_2}\geq\cdots\geq\beta_{\sigma_n}>0$*. Then consider the problem with the same objective function as in problem (26) and with the constraints*

$$\begin{cases} \alpha_{\sigma_1}x_1+\alpha_{\sigma_2}x_2+\cdots+\alpha_{\sigma_n}x_n\geq 1, \\ \beta_{\sigma_1}x_1+\beta_{\sigma_2}x_2+\cdots+\beta_{\sigma_n}x_n\geq 1, \\ x\geq 0. \end{cases}$$

If $(u_1^*,\ldots,u_n^*)$ *is a solution of this auxiliary problem, then* $(x_1^*,\ldots,x_n^*)$ *is a solution of problem (26), where* $x_i^*=u^*_{\sigma_i^{-1}}$*,* $i=\overline{1,n}$*. Here,* σ^{-1} *denotes the inverse of* σ*.*

Theorem 13. *Consider problem (26) in the special case when* $0 < \alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$. *Let* $\sigma \in S_n$ *be any permutation such that* $\beta_{\sigma_1} \geq \beta_{\sigma_2} \geq \cdots \geq \beta_{\sigma_n} > 0$. *Then consider the problem with the same objective function as in problem (26) and with the constraints*

$$\begin{cases} \alpha x_1 + \alpha x_2 + \cdots + \alpha x_n \geq 1, \\ \beta_{\sigma_1} x_1 + \beta_{\sigma_2} x_2 + \cdots + \beta_{\sigma_n} x_n \geq 1, \\ x \geq 0. \end{cases}$$

If $(u_1^*, \ldots, u_n^*)$ *is a solution of this auxiliary problem, then* $(x_1^*, \ldots, x_n^*)$ *is a solution of problem (26), where* $x_i^* = u^*_{\sigma_i^{-1}}$, $i = \overline{1, n}$. *Here,* σ^{-1} *denotes the inverse of* σ.

4. Implementation and applications

Here we show that the size of the constraints array can be reduced by representing the OWA function in terms of the iterated differences of the ordered variables, and then present some example applications in worker allocation and ecology.

4.1. Solving the LP for large n

It turns out that using a generic LP solver on the problem formulated as in Eq. (8) directly makes the process of finding the solution quite time consuming. For solvers where the decision variables are assumed to be non-negative, we can avoid the $n-1$ constraints that induce the ordering on $\mathbf{x}$ by instead expressing the objective and constraints in terms of cumulative sums.

We let $\mathbf{w}_j^{(c)} = \sum_{k=1}^{j} w_k$ and $\alpha_{i,j}^{(c)} = \sum_{k=1}^{j} \alpha_{i,k}$ for all i, j and let δ_j denote the differences in $\mathbf{x}$ such that $x_1 = \sum_{j=1}^{n} \delta_j, x_2 = \sum_{j=2}^{n} \delta_j, \ldots, x_n = \delta_n$, or alternatively $\delta_j = x_j - x_{j+1}$ with $x_{n+1} = 0$ by convention. Our fitting problem then becomes,

$$\begin{aligned} \text{Maximize} \quad & \sum_{j=1}^{n} w_j^{(c)} \delta_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} \alpha_{i,j}^{(c)} \delta_j \leq 1, \quad i = 1, \ldots, m \end{aligned}$$

As an indication of the run-time saving, we implemented the above with `lpSolve` in R (100 runs with $n = 10000$, 2 constraints, coefficients generated from a uniform distribution). Solving Eq. (8) with sparse coefficients matrix had a mean of 1.33 s, median of 0.174 s and maximum of 11.13 s, while solving via the iterated differences had a mean of 0.014 s, median of 0.012 s and maximum of 0.092 s.

4.2. Example application in ecology

In ecology we can consider the problem of proportioning land-use according to different categories where each category may affect species differently. For example, in desert areas, regular burning of different regions can encourage species diversity. Let x_j represent the proportion of land with fire-age j and $a_{i,j}$ represent the occurrence of species i in areas with fire-age j. We want to allocate the proportions x_j such that the abundance of each species is above a minimum threshold and at the same time maximize diversity of fire-ages. Our measure for unevenness of fire ages can be given by $\mathsf{OWA}(\mathbf{x})$ with $\mathbf{w} = (1, 1/2, 1/4, 1/8)$. Higher weight allocated to high proportions means that unevenness will be highest in such cases.

Suppose for 3 species we have the following constraints

$$
\begin{aligned}
0.1x_1 + 0.4x_2 + 0.7x_3 + 0.8x_4 &\geq 0.4 \\
0.3x_1 + 0.5x_2 + 0.8x_3 + 0.9x_4 &\geq 0.3 \\
0.2x_1 + 0.6x_2 + 0.7x_3 + 0.8x_4 &\geq 0.6
\end{aligned}
$$

The interpretation of comonotonicity here is that each species occurs more frequently for older fire ages.

We add the additional constraints (also comonotone) to ensure that our x_i values add to 1.

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 &\geq 1 \\
-x_1 - x_2 - x_3 - x_4 &\geq -1
\end{aligned}
$$

In order to minimize the value of our OWA, we can use the LP, reordering the inputs in this case so that $a_{\sigma(1)} \geq a_{\sigma(2)} \geq \cdots \geq a_{\sigma(n)}$ and solve. We obtain an objective of 0.491666 with a vector of $\mathbf{x} = (4/15, 4/15, 4/15, 1/5)$. If we were to ensure that the second species stayed above 0.7 instead, the optimal allocation would be $\mathbf{x} = (0.31, 0.31, 0.31, 0.07)$.

4.3. Example application in work allocation

There may be situations where the ordering induced by the OWA objective induces the same ordering across the constraints, i.e. an OWA with respect to OWA constraints. This reduces to a simpler problem where we do not need to worry about comonotonicity of the constraints.

For example, integrals with respect to non-additive measures have been proposed as suitable for modeling worker output. In the case of the Choquet integral, the fuzzy measure can be interpreted as representing the rate of production for each combination of workers and the Choquet integral output represents the total production with respect to the hours worked by each worker $\mathbf{x}$. Now suppose that we want to minimize the inequality of hours allocated subject to ensuring we meet minimum production targets.

Inequality of hours can be represented by the difference in OWAs, or by a single OWA where the weights are allowed to be negative, see, e.g., [2]. For example the Gini index when $n = 3$ corresponds with a weighting vector $\mathbf{w} = (2/9, 0, -2/9)$ when the inputs are normalized by dividing through by their sum.

In such a case, we would have the following optimization problem.

$$\begin{aligned} \text{Minimize} \quad & \mathsf{OWA}(\mathbf{x}) \\ \text{s.t.} \quad & \mathsf{Ch}(\mathbf{x}) \geq y \\ & \sum_{i=1}^{n} x_i = 1 \end{aligned}$$

In this case, we assume a given number of hours that we allocate to available shifts, and assume any worker could work any of the shifts, however we need to ensure that the production targets are met regardless.

Suppose we have 3 workers and their production per day when working together in teams is modeled by the monotone non-additive measure μ, where $\mu(A, B, C) = 1$ (all three workers together results in 1 unit of production per day), $\mu(A, B) = 0.6$ (workers A and B can produce 0.6 units per day), $\mu(A, C) = 0.5$, $\mu(B, C) = 0.9$, $\mu(A) = 0.5, \mu(B) = \mu(C) = 0.4.$, $\mu(\emptyset) = 0$.

Under the condition that all workers begin at the same time and once someone finishes working, they cannot return, we obtain constraints with respect to the required production y. For example, if the maximum number of hours is allocated to worker A and the minimum to worker C, the production will be

$$\mu(A)x_{(1)} + (\mu(A,B) - \mu(A))x_{(2)} + (\mu(A,B,C) - \mu(A,B))x_{(3)}.$$

Each of the potential orderings will result in the following set of constraints

$$\begin{aligned}
0.5x_{(1)} + 0.1x_{(2)} + 0.4x_{(3)} &\geq y \\
0.5x_{(1)} + 0x_{(2)} + 0.5x_{(3)} &\geq y \\
0.4x_{(1)} + 0.2x_{(2)} + 0.3x_{(3)} &\geq y \\
0.4x_{(1)} + 0.5x_{(2)} + 0.1x_{(3)} &\geq y \\
0.4x_{(1)} + 0.1x_{(2)} + 0.5x_{(3)} &\geq y \\
0.4x_{(1)} + 0.5x_{(2)} + 0.1x_{(3)} &\geq y
\end{aligned}$$

If the production target is less than $1/3$, then clearly (due to idempotency), the optimal shift allocation will be $(1/3, 1/3, 1/3)$, however higher production targets may not allow this to be achieved. If the production target is 0.35, then inequality is minimized with the shift allocation $\mathbf{x} = (1/2, 1/4, 1/4)$, while if the production target is 0.37 then the best allocation is $\mathbf{x} = (0.7, 0.15, 0.15)$ and if the production target is 0.4 then this can only be ensured by allocating all hours to one person.

Of course we could also have an objective function that minimizes cost, e.g. where the person who works the longest gets paid more or less than the other workers.

5. Conclusions

In this paper we investigated the constrained OWA aggregation problem when we have multiple constraints such that the coefficients can be rearranged in nondecreasing order via the same permutation. This includes the case of equality constraints with equal coefficients. Analyzing Theorem 8, we obtained an analytical form of the optimal solution in the case of two constraints. For more general cases, we can observe the following.

Inapplicability for more general problems. By means of a simple counterexample, we can show that the ordering assumption for maximizing the OWA does not apply if we have $\geq$-type constraints (with the same coefficient ordering).

For $\mathbf{w} = (3, 2, 1)$, $\alpha_1 = (0.66, 0.69, 0.92)$ and $\alpha_2 = (0.02, 0.70, 0.98)$ and the constraints

$$0.66x_1 + 0.69x_2 + 0.92x_3 \leq 0.66$$
$$0.02x_1 + 0.70x_2 + 0.98x_3 \geq 0.30$$

then the optimal solution is not achieved for $x_1 \geq x_2 \geq x_3$, which would give $\mathbf{x} = (0.581, 0.172, 0.172)$ but instead for x_2 being the largest value and $\mathbf{x} = (0, 0.957, 0)$. The objective achieved with the former ordering is 0.376 while we obtain 0.478 when only using x_2.

Non-comonotone Constraints. The optimal $\mathbf{x}$ does not necessarily have an ordering that corresponds with one of the inequalities. For example

$$\begin{aligned} 6x_1 + 4x_2 + 2x_3 &\leq 1 \\ 2x_1 + 4x_2 + 6x_3 &\leq 1 \end{aligned}$$

will have an optimal ordering, either starting with 2 (which would have solution $x_2 = 1/4$ if the first objective weight is large enough, or starting with $1,3$ or $3,1$ if the second weight is larger than the first.

Note that neither of these solutions have x_2 in the middle.

However, if the solution with respect to a single constraint is $1/(\sum_{i=1}^{k} a_k)$ and a_k is greater than all other b_k, c_k etc. Then this will be the optimal solution (so there is a chance that we can find it by checking the orderings induced by each of the constraints.

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