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A benchmark-type generalization of the Sugeno integral with applications in bibliometrics

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Abstract

We propose a new generalization of the classical Sugeno integral motivated by the Hirsch, Woeginger, and other geometrically-inspired indices of scientific impact. The new integral adapts to the rank-size curve better as it allows for putting more emphasis on highly-valued items and/or the tail of the distribution (level measure). We study its fundamental properties and give the conditions guaranteeing the fulfillment of subadditivity as well as the Jensen, Liapunov, Hardy, Markov, and Paley-Zygmund type inequalities. We discuss its applications in scientometrics.

Keywords: Scientometric indices; H-index; Sugeno integral; Subadditivity; Jensen's inequality; Monotone measure

1. Introduction

The pioneering concepts of nonadditive integrals were introduced by Choquet [11] and Sugeno [37]. They serve as tools for modeling non-deterministic problems in various fields like cooperative game theory, risk theory, decision theory, economics, and scientometrics (see, e.g., [17, 20, 26, 33, 38, 40]). Theoretical investigations of the integrals and their generalizations have been pursued by many researchers, e.g., [1, 12, 20, 27, 28]. A general overview on nonadditive measurement and nonadditive integration theory is presented, among others, in the monographs by Wang and Klir [43], Grabisch [19], and Beliakov, James, and Wu [3].

Motivational problem. A scholar who has published N papers in total can be formally described by an infinite vector $\mathbf{x} = (x_1, x_2, \ldots)$, called a *scientific record*, such that $x_1 \geqslant x_2 \geqslant \ldots$, where $x_i \in \{0, 1, 2, \ldots\}$ and $x_1 \geqslant 1$. A positive value of x_i gives the number of

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citations to the i-th most cited publication, whereas $x_i = 0$ either denotes a paper with zero citations or a nonexisting paper.

Numerous approaches for evaluating the impact of a researcher's publications are known in the literature. Probably the most famous one is the h-index proposed by Hirsch in [21]

$$\mathrm{H}(\mathbf{x}) = \max\{k \in [N] \colon x_k \geqslant k\} = \max_{k\in[N]}\{k \wedge x_k\}, \tag{1}$$

where $a \wedge b = \min\{a, b\}$ and $[N] = \{1, \ldots, N\}$. The total number of citations, called the C-index, is another ubiquitous measure

$$\mathrm{C}(\mathbf{x}) = \sum_{i=1}^{N} x_i. \tag{2}$$

Generalizing impact indices so that they can better adapt to different citation curves (e.g., when a comparison between scientific fields is required) has a long history in the scientometrics literature; see [13, 18] for some recent overviews. For instance, [41] proposed $\mathrm{H}_\alpha(\mathbf{x}) = \max\{k \in [N] \colon x_k \geqslant \alpha k\}$, which is equivalent to the h-index on a scaled version of the inputs. In [30], the authors introduce upper- and lower-h-indices that aim to complement the h-index by looking beyond the H^2 citations of the H most cited papers.

In this paper, we are particularly interested in looking at the bibliometric indices from the perspective of the theory of nonadditive integrals. Torra and Narukawa showed in [39] that the h-index is a particular case of some Sugeno integral [37], while the C-index corresponds to some Choquet integral [11]. Namely, let μ be the counting measure on $\mathbb{N}$, i.e., $\mu(A) = |A|$ is the number of elements in the set $A \in 2^{\mathbb{N}}$. Furthermore, assume that $f(j) = x_j$ and let $\mu(\{f \geqslant t\}) = \mu(\{j \in \mathbb{N} \colon f(j) \geqslant t\}) = \mu(\{j \in \mathbb{N} \colon x_j \geqslant t\})$ denote the number of papers with at least t citations. In such a case, h-index is the Sugeno integral of f w.r.t. μ

$$\mathrm{H}(\mathbf{x}) = \max_{t\geqslant 0}\{t \wedge \mu(\{f \geqslant t\})\} = (S)\int_{\mathbb{N}} f \,\mathrm{d}\mu \tag{3}$$

and C-index coincides with the Choquet integral w.r.t. μ

$$\mathrm{C}(\mathbf{x}) = \int_0^\infty \mu(\{f \geqslant t\})\,\mathrm{d}t = (C)\int_{\mathbb{N}} f \,\mathrm{d}\mu. \tag{4}$$

But there is more. In [2], applications of the Choquet integral in journal ranking were studied. The papers [7, 23] relate an index of H_α type to the Sugeno integral. More generally, [17] considers the indices in the framework of universal integrals [28]. These results were extended in [36]. Most recently, [6] generalizes [23] and [30] by introducing upper- and lower-Sugeno integrals.

A new integral. Furthermore, Woeginger in [44] introduced the w-index. It is given by

$$\mathrm{W}(\mathbf{x}) = \max\{k \in [N] \colon x_i \geqslant k - i + 1 \text{ for all } i \in [k]\}. \tag{5}$$

In Section 2 we shall note that

$$\mathrm{W}(\mathbf{x}) = \max\left\{a \geqslant 0 \colon |\{j \colon x_j \geqslant t\}| \geqslant \max\{a-t, 0\} \text{ for all } t\right\}, \tag{6}$$

but also

$$\mathrm{H}(\mathbf{x}) = \max\left\{a \geqslant 0 \colon |\{j \colon x_j \geqslant t\}| \geqslant a\mathbb{1}_{[0,a)}(t) \text{ for all } t\right\}, \tag{7}$$

and, by replacing $|\{j \colon x_j \geqslant t\}|$ and $\max\{a-t,0\}$ or $a\mathbb{1}_{[0,a)}(t)$ with, respectively, $\mu(\{f \geqslant t\})$ and $r(t,k)$, we propose a new functional of the form

$$\int\limits_X f \,\mathrm{d}(\mu, r) = \max\{a \geqslant 0 \colon \mu(\{f \geqslant t\}) \geqslant r(t,a) \text{ for all } t\}.$$

It will turn out below that the above formula extends, among others, the h-index, the w-index, and the geometric indices introduced in [16]. Despite the generality of the proposal, we will prove that it enjoys a number of very attractive properties. Additionally, we will relate it to the extensions of [16] discussed in [14] (effort-dominating functions which were later rediscovered in [24]) and [15].

Aim and structure of the paper. In Section 2 we formally define the new benchmark integral and present some basic properties and related geometric intuitions. We show that it does not reduce to the Sugeno integral, the Pan-integral, nor to the pseudo-integral. In Section 3 we address some of the desired properties of integrals suggested in [6]. Moreover, we provide the necessary and sufficient conditions for the validity of Jensen type inequalities and discuss them in the case of several particular families of benchmark functions. Additionally, we present some analogs of Liapunov's, Hardy's, Markov's, and Paley-Zygmund's integral inequalities. In Section 4 we consider the subadditivity property (its technical proof is postponed to Appendix), which is very important in many applications that we discuss in Section 5, e.g., scientometrics, and generating new monotone measures. We conclude the paper in Section 6.

2. New nonlinear operator

Let (X, Σ) be a measurable space and $\bar{y} \in (0, \infty]$. Hereafter $\bar{\mathbb{R}}_+ = [0,\infty]$ and $\mathbb{R}_+ = [0, \infty)$. By $\mathbf{F}_Y$ we denote the set of all Σ-measurable functions $f \colon X \to Y$, where $Y = [0, \bar{y}]$ or

$Y = [0, \bar{y})$. Consider the set $\mathbf{M}$ of all *monotone measures* (i.e., the set functions $\mu\colon \Sigma \to [0, \infty]$ satisfying $\mu(C) \leqslant \mu(D)$ whenever $C \subseteq D$ for $C, D \in \Sigma$ with $\mu(\emptyset) = 0$) such that $\mu(X) \leqslant \bar{y}$.

Given $f, g \in \mathbf{F}_Y$ and $\mu \in \mathbf{M}$, we say that g *dominates* f w.r.t. μ and write $f \leqslant_\mu g$, if $\mu(\{f \geqslant t\}) \leqslant \mu(\{g \geqslant t\})$ for all $t \in Y$, where $\{f \geqslant t\} = \{x \in X : f(x) \geqslant t\}$ is the *t-level set*. Hereafter, $f =_\mu g$ means $\mu(\{f \geqslant t\}) = \mu(\{g \geqslant t\})$ for any t. In the sequel, $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$, and $[k] = \{1, 2, \ldots, k\}$ with $k \in \mathbb{N} = \{1, 2, \ldots\}$, while $\mathbb{1}_D$ denotes the indicator function of a set D.

Let us now introduce a new operator that plays the main role in this paper.

Definition 2.1. The *benchmark integral* of $f \in \mathbf{F}_Y$ on a set $A \in \Sigma$ w.r.t. $\mu \in \mathbf{M}$ and r is given by

$$\int_A f \,\mathrm{d}(\mu, r) = \sup\{a \in [0, \mu(X)) \colon \mu(A \cap \{f \geqslant t\}) \geqslant r(t, a) \text{ for all } t \in Y\}, \tag{8}$$

where $r\colon Y \times [0, \mu(X)) \to \mathbb{R}_+$ is a *benchmark function* satisfying the conditions:

(C1) $x \mapsto r(x, a)$ is nonincreasing and $a \mapsto r(x, a)$ is nondecreasing for any fixed, respectively, a and x,

(C2) $r(0, a) = r(0^+, a) = a$ for every a, where $r(0^+, a) = \lim_{x \to 0^+} r(x, a)$,

(C3) for any a, $r(x, a) > 0$ if $x < a$, and $r(x, a) = 0$ if $x > a$.

Observe that (C2) and (C3) imply that $r(x, 0) = 0$ for $x \in Y$, which ensures the well-definiteness of functional (8). Moreover, (C1) and (C2) imply that $0 \leqslant r(x, a) \leqslant r(0, a) = a < \mu(X)$ for any a, x.

For fixed Y and $\mu(X)$, denote by $\mathcal{B}$ the class of all benchmark functions. Let $\mathcal{B}_{cx}$ and $\mathcal{B}_c$, designate the families of all functions from $\mathcal{B}$ which are, respectively, convex and concave with respect to the first argument.

By construction, functional (8) quantifies the extent to which a given function meets (in the sense of being bounded from below) a given prototypical, adaptive benchmark index, hence the name. Intuitively, the benchmark integral is the greatest value of the second coordinate of the benchmark function, for which the function $t \mapsto r(t, a)$ is the closest to $t \mapsto \mu(\{f \geqslant t\})$; compare Fig. 1 that features some examples that we discuss next.

Example 2.2. Let $(f, \mu) \in \mathbf{F}_Y \times \mathbf{M}$. Noteworthy examples of benchmark functions include:

R1. $r_S(x, a) = a\mathbb{1}_{[0,a)}(x)$ or $r_S(x, a) = a\mathbb{1}_{[0,a]}(x)$. One can easily check that

$$\int_A f \,\mathrm{d}(\mu, r_S) = (S)\int_A f \,\mathrm{d}\mu,$$

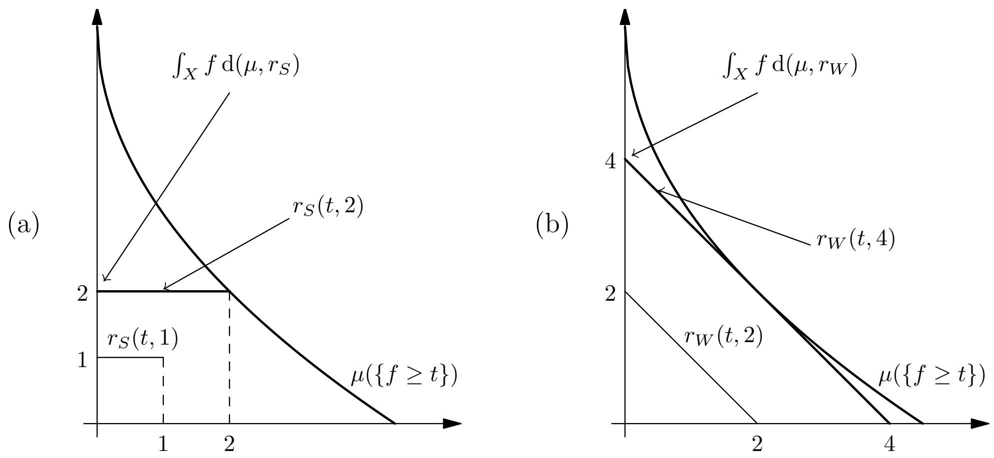

Figure 1: A graphical interpretation of the benchmark integral with the benchmark function: (a) r_S and (b) r_W; see Example 2.2.

where $(S)\int_A f\,\mathrm{d}\mu$ is the Sugeno integral; see [37] and Fig. 1 (a). Since $r(x,a) \leqslant a\mathbb{1}_{[0,a]}(x)$ for any x, a and $r \in \mathcal{B}$, we have

$$(S)\int\limits_A f\,\mathrm{d}\mu = \min_{r\in\mathcal{B}} \int\limits_A f\,\mathrm{d}(\mu, r).$$

R2. $r_W(x,a) = (a-x)_+$, where $a_+ = a \vee 0$. This integral is motivated by Woeginger's index [44]; compare Fig. 1 (b). It is easily seen that

$$\max_{r\in\mathcal{B}_c} \int\limits_A f\,\mathrm{d}(\mu, r) = \int\limits_A f\,\mathrm{d}(\mu, r_W) = \min_{r\in\mathcal{B}_{cx}} \int\limits_A f\,\mathrm{d}(\mu, r).$$

R3. $r_{GG}(x,a) = (a^p - x^p)_+^{1/p}$, $p > 0$, proposed for $p \geqslant 1$ in [16], where $a_+^p = (a \vee 0)^p$, is the geometric index.

R4. $r(x,a) = a(1 - x/a)_+^p$, $a, p > 0$, which are nonsymmetric functions.

R5. $r(x,a) = (p/(p-1) \cdot (a-x)_+) \wedge a$, $p > 1$, which are trapezoidal functions.

R6. $r_b(x,a) = (a + bx)\mathbb{1}_{[0,a]}(x)$, where $-1 \leqslant b \leqslant 0$, which are trapezoidal type II functions; note that

$$\int\limits_A f\,\mathrm{d}(\mu, r_b) = \min_{r\in\mathcal{B}_c(a(1+b))} \int\limits_A f\,\mathrm{d}(\mu, r),$$

where $\mathcal{B}_c(d)$ denotes the family of all functions from $\mathcal{B}_c$ such that $r(a,a) = d$ for all a.

R7. $r(x,a) = \left(a - \sqrt{\frac{x^3}{2a-x}}\right)_+$, which is cissoid type function.

For $a \in [0,\infty)$ and $Y = [0,\infty)$, denote the *generalized inverse of* $Y \ni x \mapsto r(x,a)$ with

$$r^{-1}(y,a) = \inf\{x \in [0,\infty) \colon y \geqslant r(x,a)\}.$$

Thanks to the following result, the new functional can be considered a generalization of the well-known bibliometric impact indices.

Proposition 2.3. *Let $X = \mathbb{N}$, $\Sigma = 2^{\mathbb{N}}$, $Y = [0, \infty)$, μ be the counting measure, (x_i) be a scientific record such that there exists N for which $x_N > x_{N+1} = 0$. Assume that $r(x, a)$ is a right-continuous function of x on Y. Then for $f(j) = x_j$*

$$\int_{\mathbb{N}} f \,\mathrm{d}(\mu, r) = \sup \{a \in [0, N] \colon x_i \geqslant r^{-1}(i-1, a) \textit{ for all } i \in [N]\}. \tag{9}$$

Proof. Let $S = \{i \in [N] \colon x_i > x_{i+1}\} \cup \{0\}$. Then S is of the form $\{k_1, k_2, \ldots, k_m\}$ with $k_1 > k_2 > \ldots > k_{m-1} > k_m = 0$ for some $m \geqslant 2$.[1] Since the function $t \mapsto \mu(\{f \geqslant t\})$ is a left-continuous step function and $x \mapsto r(x, a)$ is nonincreasing and right-continuous, we have

$$\int_{\mathbb{N}} f \,\mathrm{d}(\mu, r) = \sup \{a \colon \mu(\{j \colon x_j \geqslant x_{k_i}^+\}) \geqslant r(x_{k_i}, a) \text{ for all } k_i \in S\},$$

where $x_0 = 0$ by convention. Note that $\mu(\{j \colon x_j \geqslant x_{k_i}^+\}) = \mu(\{j \colon x_j \geqslant x_{k_{i+1}}\}) = k_{i+1}$ for any $i \in [m-1]$, and $\mu(\{j \colon x_j \geqslant x_{k_i}^+\}) = k_1$ for $i = m$. Hence

$$\int_{\mathbb{N}} f \,\mathrm{d}(\mu, r) = \sup \{a \colon k_{i+1} \geqslant r(x_{k_i}, a) \text{ for all } i \in [m-1] \text{ and } k_1 \geqslant r(0, a)\}.$$

For fixed a, t, y, due to the right-continuity of $x \mapsto r(x, a)$, $y \geqslant r(t, a)$ if and only if $t \geqslant r^{-1}(y, a)$. Moreover, $y \mapsto r^{-1}(y, a)$ is nonincreasing. Thus

$$\int_{\mathbb{N}} f \,\mathrm{d}(\mu, r) = \sup \{a \in [0, k_1] \colon x_{k_i} \geqslant r^{-1}(k_{i+1}, a) \text{ for all } i \in [m-1]\}.$$

Observe that $k_1 = N$. We now need to show that

$$\int_{\mathbb{N}} f \,\mathrm{d}(\mu, r) = \sup \{a \in [0, N] \colon x_i \geqslant r^{-1}(i-1, a) \text{ for all } i \in [N]\}. \tag{10}$$

Clearly, $x_{k_i} = x_j = x_{k_{i+1}+1}$ for any $i \in [m-1]$ and any $k_{i+1} + 1 \leqslant j \leqslant k_i$. Due to the nonincreasingness of r^{-1} with respect to the first argument, for all $i \in [m-1]$ and all $j \in \{k_{i+1}+1, \ldots, k_i\}$, we have $x_j = x_{k_i} \geqslant r^{-1}(k_{i+1}, a) \geqslant r^{-1}(j-1, a)$. In consequence, $x_l \geqslant r^{-1}(l-1, a)$ for any $l \in [N]$, which completes the proof. $\square$

[1] For example, if $\mathbf{x} = (5, 3, 3, 3, 2, 1, 1, 0, \ldots)$, then $S = \{7, 5, 4, 1, 0\}$ with $k_1 = 7$, $k_2 = 5$, $k_3 = 4$, $k_4 = 1$, and $k_5 = 0$.

By establishing equivalence between (3) and (7) as well as (5) and (6), Proposition 2.3 shows that our new integral is indeed a generalization of the Hirsch and Woeginger indices. In view of the aforementioned proposition, this integral w.r.t. r_{GG} is also a generalization of the geometric indices. The other benchmark functions listed in Example 2.2 lead to new measures that can be used in bibliometric practice.

Remark 2.4. A recent paper [24] somewhat rediscovers the idea of the effort-dominating functions [14]. Namely, let $\mathbf{x} = (x_k)_{k\geqslant 1}$ be a scientific record such that $x_n > x_{n+1} = 0$ for some $n \in \mathbb{N}$ and

$$\mathrm{gH}(\mathbf{x}) = \bigvee_{a=1}^{x_1} \{a \wedge \max\{q \in [n]\colon x_k \geqslant g(a,k) \text{ for all } k \leqslant q,\ k \in \mathbb{N}\}\}$$

be the generalized h-index introduced in [24, p. 849]), where $g\colon \mathbb{N}\times\mathbb{N} \to \mathbb{R}_+$ is the reference function (i.e., $a \mapsto g(a,k)$ is nondecreasing and $k \mapsto g(a,k)$ is nonincreasing). For a given benchmark function g, the operator gH and our new benchmark integral are different, though. For instance, consider the scientific record $\mathbf{x} = (5, 3, 1, 1, 0, \ldots)$ with the benchmark function $r_{GG}(x, a) = (a^2 - x^2)_+^{1/2}$. Then $\mathrm{gH}(\mathbf{x}) = (1 \wedge 4) \vee (2 \wedge 4) \vee (3 \wedge 4) \vee (4 \wedge 1) = 3$, while, by Proposition 2.3, $\int_{\mathbb{N}} f \,\mathrm{d}(\mu, r_{GG}) > 3.1$ for $f(j) = x_j$.

Remark 2.5. Formula (9) resembles the notion of effort-dominating functions defined in [14]. It has been shown therein that each such function is symmetric minitive, i.e., for a fixed, right-continuous with respect to the first coordinate benchmark function r, there exist $f_{1,N}, \ldots, f_{N,N}$ $(N = \max\{j\colon x_j > 0\})$ such that

$$\int_{\mathbb{N}} f \,\mathrm{d}(\mu, r) = \bigwedge_{i=1}^{N} f_{i,N}(x_i).$$

For example, by [14, Example 5], the geometric indices $(r = r_{GG})$ [16] are given by

$$\int_{\mathbb{N}} f \,\mathrm{d}(\mu, r_{GG}) = \mathrm{R}_{GG}^{(p)}(\mathbf{x}) = \begin{cases} \bigwedge_{i=1}^{n} ((n \wedge x_i) \vee (i-1)) & \text{if } p = \infty, \\ \bigwedge_{i=1}^{n} \left(\left(n \wedge \sqrt[p]{x_i^p + (i-1)^p}\right) \vee (i-1)\right) & \text{if } p \in [1, \infty). \end{cases}$$

Remark 2.6. Two other generalizations of the geometric indices $\mathrm{R}_{GG}^{(p)}$ have been proposed in the literature. Graph-based monotone integrals [15] extend the notion of not only the Sugeno, but also the Shilkret integral. The proposal is very broad and its properties are yet to be studied in detail.

It is natural to call functional (8) the integral because it has the following desired properties (see Boczek et al. [6, Definition 2.2]).

Lemma 2.7. *Let $A \in \Sigma$, $f, g \in \mathbf{F}_Y$, $r \in \mathcal{B}$ and $\mu, \nu \in \mathbf{M}$.*

(P1) If $f\mathbb{1}_A \leqslant_\mu g\mathbb{1}_A$, then $\int_A f \,\mathrm{d}(\mu, r) \leqslant \int_A g \,\mathrm{d}(\mu, r)$.

(P2) If $f\mathbb{1}_A =_\mu g\mathbb{1}_A$, then $\int_A f \,\mathrm{d}(\mu, r) = \int_A g \,\mathrm{d}(\mu, r)$.

(P3) If $\mu(A) \leqslant \nu(A)$, then $\int_A f \,\mathrm{d}(\mu, r) \leqslant \int_A f \,\mathrm{d}(\nu, r)$.

(P4) $\int_A f \,\mathrm{d}(\mu, r) = \int_X f\mathbb{1}_A \,\mathrm{d}(\mu, r)$.

(P5) $\int_X c\mathbb{1}_A \,\mathrm{d}(\mu, r) = c \wedge \mu(A)$, where $c \geqslant 0$.

Proof. The properties (P1)–(P4) follow immediately from the monotonicity of μ. We give the proof of the property (P5). Define

$$B(f) = \{a \in [0, \mu(X)) \colon \mu(\{f \geqslant t\}) \geqslant r(t, a) \text{ for all } t \in Y\}.$$

Let $g = c\mathbb{1}_A$. Observe that $\mu(\{g \geqslant t\}) = \mu(X)$ for $t = 0$, $\mu(\{g \geqslant t\}) = \mu(A)$ for $0 < t \leqslant c$, and $\mu(\{g \geqslant t\}) = 0$ for $t > c$.

Let $\mu(A) \geqslant c$. Then, by (C1) and (C2), $r(t, c) \leqslant r(0, c) = c \leqslant \mu(A) \leqslant \mu(\{g \geqslant t\})$ for all $t \leqslant c$, and by (C3), $r(t, c) = 0 \leqslant \mu(\{g \geqslant t\})$ for $t > c$. This implies that $c \in B(g)$. We show that $c = \sup B(g) = \int_X g \,\mathrm{d}(\mu, r)$. Suppose on the contrary that there exists an $a \in B(g)$ such that $a > c$. For $\hat{t} = (a + c)/2$, $r(\hat{t}, a) > 0 = \mu(\{g \geqslant \hat{t}\})$, by $c < \hat{t} < a$ and (C3). Hence, $a \notin B(g)$, a contradiction. Therefore, $c = \sup B(g)$.

Let $\mu(A) < c$. From (C1) and (C2) it follows that $\mu(\{g \geqslant t\}) \geqslant \mu(A) = r(0, \mu(A)) \geqslant r(t, \mu(A))$ for $0 \leqslant t \leqslant c$. For $t > c > \mu(A)$ we have $r(t, \mu(A)) = 0 \leqslant \mu(\{g \geqslant t\})$, by (C3). Hence, $\mu(A) \in B(g)$. We claim that $\mu(A) = \sup B(g)$. Suppose there exists $a > \mu(A)$ and $a \in B(g)$. Since $a = r(0, a) > \mu(A)$, the right-hand side continuity of $x \mapsto r(x, a)$ at point 0 implies the existence of $\widetilde{t} \in (0, c)$ such that $r(\widetilde{t}, a) > \mu(A) = \mu(\{g \geqslant \widetilde{t}\})$, so $a \notin B(g)$, a contradiction. Thus, the proof is complete. $\square$

We have shown that the benchmark integral generalizes the Sugeno integral (see Example 2.2). Additionally, from property (P5) and [43, Theorem 10.5 (5)] it follows that integral (8) does not coincide with the Pan-integral (see [43, Section 10.2]). Integral (8) is also not identical with the pseudo-integral w.r.t. the $\oplus$-measure [1, Definition 2.7] as we do not assume that the monotone measure μ is additive w.r.t. a pseudo-addition operator $\oplus$.

Let $Y = [0, \bar{y}]$ and $\circ \colon Y \times [0, \mu(X)] \to [0, \infty]$ be a nondecreasing binary function in each coordinate. The following generalization of the Sugeno integral

$$\int_{\circ, A} f \,\mathrm{d}\mu = \sup_{t \in Y} \{t \circ \mu(A \cap \{f \geqslant t\})\}, \tag{11}$$

is studied in the literature (see [4]). If $0 \circ b = 0$, we have $\int_{\circ,A} f \,\mathrm{d}\mu = \int_{\circ,X} f \mathbb{1}_A \,\mathrm{d}\mu$, so operator (11) satisfies (P4). Moreover, if $0 \circ b = 0$ for all b, operator (11) has the property (P5), i.e., $\int_{\circ,X} c\mathbb{1}_A \,\mathrm{d}\mu = c \wedge \mu(A)$ if and only if $\circ = \wedge$. To see it, we calculate (11) with $f = c\mathbb{1}_A$, i.e.,

$$\int_{\circ,X} c\mathbb{1}_A \,\mathrm{d}\mu = (0 \circ \mu(X)) \vee \sup_{t\in(0,c]} \{t \circ \mu(A)\} = c \circ \mu(A),$$

where, by convention, $\sup_{\emptyset}\{\cdot\} = 0$. To sum up, for $0 \circ b = 0$ for all b, operators (8) with $r = r_S$ and (11) are equal for any f, μ if and only if $\circ = \wedge$.

We say that $\mu \in \mathbf{M}$ is *minitive* if $\mu(A \cap B) = \mu(A) \wedge \mu(B)$ for all $A, B \in \Sigma$. Recall that $f, g \in \mathbf{F}_Y$ are *comonotone* on D if $(f(x) - f(y))(g(x) - g(y)) \geqslant 0$ for all $x, y \in D$. If $f, g \in \mathbf{F}_Y$ are comonotone on D, then for any $t \in Y$ either $D \cap \{f \geqslant t\} \subset D \cap \{g \geqslant t\}$ or $D \cap \{g \geqslant t\} \subset D \cap \{f \geqslant t\}$. Integral (8) shares the following properties with the Sugeno integral (see [43]).

Lemma 2.8. *Let $r \in \mathcal{B}$, $A, B \in \Sigma$, $\mu \in \mathbf{M}$ and $f, g \in \mathbf{F}_Y$.*

(a) *If $\mu(A) = 0$, then $\int_A f \,\mathrm{d}(\mu, r) = 0$.*

(b) *If $A \subset B$, then $\int_A f \,\mathrm{d}(\mu, r) \leqslant \int_B f \,\mathrm{d}(\mu, r)$.*

(c) $\int_A (f \vee g) \,\mathrm{d}(\mu, r) \geqslant \int_A f \,\mathrm{d}(\mu, r) \vee \int_A g \,\mathrm{d}(\mu, r)$.

(d)

$$\int_A (f \wedge g) \,\mathrm{d}(\mu, r) \leqslant \int_A f \,\mathrm{d}(\mu, r) \wedge \int_A g \,\mathrm{d}(\mu, r) \tag{12}$$

and the equality is attained if μ is minitive or if f and g are comonotone on A.

(e) *Let $\mu(X) = \infty$ and $c > 0$. If $r(x, a) \leqslant r(x + c, a + c)$ for all $a \geqslant 0$ and $0 \leqslant x \leqslant a$, then for $h \in \mathbf{F}_{\bar{\mathbb{R}}_+}$*

$$\int_X (h + c) \,\mathrm{d}(\mu, r) \leqslant \int_X h \,\mathrm{d}(\mu, r) + c. \tag{13}$$

Equality occurs in (13) if $r(x, a) = r(x+c, a+c)$ for all $a \geqslant 0$ and $0 \leqslant x \leqslant a$. Moreover, the inequality is reversed if $r(x, a) \geqslant r(x + c, a + c)$ for all $a \geqslant 0$ and $0 \leqslant x \leqslant a$.

(f) $\int_A f \,\mathrm{d}(\mu, r) \leqslant \mu(A) \wedge \operatorname{ess\,sup}_\mu(f_{|A})$, *where* $\operatorname{ess\,sup}_\mu(f_{|A}) = \sup\{c \colon \mu(A \cap \{f \geqslant c\}) > 0\}$. *The inequality becomes the equality if the function $t \mapsto \mu(A \cap \{f \geqslant t\})$ is concave and the function $x \mapsto r(x, a)$ is convex for any a.*

Proof. Put $B_A(f) = \{a \in [0, \mu(X)) \colon \mu(A \cap \{f \geqslant t\}) \geqslant r(t, a)$ for all $t \in Y\}$. Obviously, $\mu(A) = 0$ implies that $\mu(A \cap \{f \geqslant t\}) = 0$ for any $t \in Y$ and that $\mu(A \cap \{f \geqslant x\}) < r(x, a)$ for $0 \leqslant x < a$, $a > 0$. By (C3), $B_A(f) = \{0\}$.

The inequalities in the properties (b)–(d) follow directly from (P1) and (P4). If monotone measure μ is minitive, then

$$\mu(A \cap \{f \wedge g \geqslant t\}) = \mu(A \cap \{f \geqslant t\}) \wedge \mu(A \cap \{g \geqslant t\}) \tag{14}$$

as $\{f \wedge g \geqslant t\} = \{f \geqslant t\} \cap \{g \geqslant t\}$. Hence $B_A(f \wedge g) = B_A(f) \cap B_A(g)$, by the definition of $B_A(f)$. Since $\sup(C \cap D) = \sup C \wedge \sup D$ for intervals C, D with $0 \in C, D$, we have the equality in (12). The same conclusion can be drawn for comonotone functions f, g on A because then (14) also holds. This completes the proof of the property (d).

Let $c > 0$. With the help of (C3) and $\mu(X) = \infty$,

$$\begin{aligned} \int_X (f + c)\,\mathrm{d}(\mu, r) &= \sup\left\{a \geqslant 0\colon \mu(\{f + c \geqslant t\}) \geqslant r(t, a) \text{ for all } t \geqslant 0\right\} \\ &= \sup\left\{a \geqslant c\colon \mu(\{f \geqslant t - c\}) \geqslant r(t, a) \text{ for all } t \in [c, a]\right\} \\ &= c + \sup\left\{b \geqslant 0\colon \mu(\{f \geqslant s\}) \geqslant r(s + c, b + c) \text{ for all } s \in [0, b]\right\}. \end{aligned}$$

As $r(s, b) \leqslant r(s + c, b + c)$ for $b \geqslant 0$ and $0 \leqslant s \leqslant b$, we have

$$\begin{aligned} \int_X (f + c)\,\mathrm{d}(\mu, r) &\leqslant c + \sup\left\{b \geqslant 0\colon \mu(\{f \geqslant s\}) \geqslant r(s, b) \text{ for all } s \in [0, b]\right\} \\ &= c + \int_X f\,\mathrm{d}(\mu, r), \end{aligned}$$

by (C3). The remaining cases can be checked similarly. The proof of (e) is complete.

Observe that $\mu(A \cap \{f \geqslant t\}) = 0 < r(t, a)$ provided $\operatorname{ess\,sup}_\mu(f_{|A}) < t < a$ (see (C3)). Hence $a \in B_A(f)$ if and only if $a \leqslant \operatorname{ess\,sup}_\mu(f_{|A})$ and $a = r(0, a) \leqslant \mu(A \cap \{f \geqslant 0\}) = \mu(A)$, which completes the proof of part (f). $\square$

Open problem. An open question is whether the conditions of Lemma 2.8 (e) are the necessary ones.

In the case of the Sugeno integral, it is well-known that the following identity

$$\int_X (f \vee g)\,\mathrm{d}(\mu, r_S) = \int_X f\,\mathrm{d}(\mu, r_S) \vee \int_X g\,\mathrm{d}(\mu, r_S)$$

holds for any comonotone functions f, g on X (see [33, p. 1341]). We show that if $r \in \mathcal{B}$ is such that for any x, $a \mapsto r(x, a)$ is right continuous and that for any $a > 0$ there exists a positive number $c < a$ satisfying the condition $r(c, a) < a$, then integral (8) does not have this property, i.e., there exist comonotone functions f, g on X for which

$$\int_X (f \vee g)\,\mathrm{d}(\mu, r) > \int_X f\,\mathrm{d}(\mu, r) \vee \int_X g\,\mathrm{d}(\mu, r). \tag{15}$$

Let A be an arbitrary set such that $\mu(A) < \mu(X)$, $f = \mu(X)\mathbb{1}_A$ and $g = c\mathbb{1}_X$, in which $c \in (0, \mu(A))$ is such that $r(c, \mu(A)) < \mu(A)$. Since $c < \mu(A) < \mu(X)$, we have $\int_X f\,\mathrm{d}(\mu, r) = \mu(A)$ and $\int_X g\,\mathrm{d}(\mu, r) = c$, by the property (P5). Clearly, $\mu(\{f \vee g \geqslant t\}) = \mu(A)$ for $c < t \leqslant \mu(X)$ and $\mu(\{f \vee g \geqslant t\}) = \mu(X)$ for $t \leqslant c$. By the inequality $r(c, \mu(A)) < \mu(A)$ and the right continuity of $a \mapsto r(x, a)$, there exists an $\varepsilon \in (0, \mu(X) - \mu(A))$ such that $\mu(\{f \vee g \geqslant t\}) \geqslant r(t, \mu(A) + \varepsilon)$ for $t \in \{0, c, \mu(A)\}$. Hence, for any t, $\mu(\{f \vee g \geqslant t\}) \geqslant r(t, \mu(A) + \varepsilon)$ by (C1) and (C3), and therefore $\int_X (f \vee g)\,\mathrm{d}(\mu, r) > \mu(A) = \mu(A) \vee c$. The proof of (15) is complete.

3. Integral inequalities

*3.1. **Jensen type inequality***

The majority of inequalities for integrals with respect to additive measures as well as for the Sugeno type integrals is a consequence of Jensen's inequality (see [5, 27, 29]). We show that the Jensen inequality holds also for the benchmark integral.

Throughout this subsection we assume that $Y = [0, \bar{y}]$ and $\mu(X) = \bar{y}$. Let $Y_0 = [0, \bar{y})$, and $H\colon Y \to Y$ be a nondecreasing function such that $H(Y_0) = Y_0$. Note that the latter implies the continuity of H (see [10, Corollary 2.18]).

Theorem 3.1. *Let $r_1, r_2 \in \mathcal{B}$. If the function H fulfills the Lipschitz type condition*

$$r_1(H(x), H(y)) \leqslant r_2(x, y), \quad x \in Y,\, y \in Y_0, \tag{16}$$

then for any $A \in \Sigma$ and any $f \in \mathbf{F}_Y$ the following Jensen type inequality

$$\int_A H(f)\,\mathrm{d}(\mu, r_1) \geqslant H\Big(\int_A f\,\mathrm{d}(\mu, r_2)\Big) \tag{17}$$

is satisfied.

Proof. Since $H(Y_0) = Y_0$ and H is nondecreasing, we have $H(Y) = Y$. By $\mu(X) = \bar{y}$, we get

$$\begin{aligned}\int_A H(f)\,\mathrm{d}(\mu, r_1) &= \sup\{a \in Y_0\colon \mu(A \cap \{H(f) \geqslant H(t)\}) \geqslant r_1(H(t), a) \text{ for all } t \in Y\} \\ &\geqslant \sup\{a \in Y_0\colon \mu(A \cap \{f \geqslant t\}) \geqslant r_1(H(t), a) \text{ for all } t \in Y\}.\end{aligned}$$

Since $H(Y_0) = Y_0$, we have

$$\int_A H(f)\,\mathrm{d}(\mu, r_1) \geqslant \sup\{H(a)\colon \mu(A \cap \{f \geqslant t\}) \geqslant r_1(H(t), H(a)) \text{ for all } t \in Y,\ a \in Y_0\}. \tag{18}$$

Set

$$B_1 = \{a \in Y_0\colon \mu(A \cap \{f \geqslant t\}) \geqslant r_1(H(t), H(a)) \text{ for all } t \in Y\}. \tag{19}$$

From (18) we have

$$\int_A H(f)\,\mathrm{d}(\mu, r_1) \geqslant \sup H(B_1), \tag{20}$$

where $H(C) = \{H(c)\colon c \in C\}$. The monotonicity and the continuity of H implies that $\sup H(B_1) = H(\sup B_1)$, while (16) leads to $B_2 \subset B_1$, in which

$$B_2 = \{a \in Y_0\colon \mu(A \cap \{f \geqslant t\}) \geqslant r_2(t, a) \text{ for all } t \in Y\}. \tag{21}$$

Hence, by (20),

$$\int_A H(f)\,\mathrm{d}(\mu, r_1) \geqslant H(\sup B_1) \geqslant H(\sup B_2) = H\Big(\int_A f\,\mathrm{d}(\mu, r_2)\Big).$$

$\square$

Theorem 3.1 remains valid for $Y = [0, \bar{y})$ provided that $\int_A f\,\mathrm{d}(\mu, r_2) < \mu(X) = \bar{y}$.

As in the case of the classical Jensen inequality, the equality

$$\int_X H(f)\,\mathrm{d}(\mu, r_1) = H\Big(\int_X f\,\mathrm{d}(\mu, r_2)\Big)$$

holds if $f = c\mathbb{1}_X$, where $0 \leqslant c \leqslant \mu(X)$.

We now show that, under some additional restrictions, the Lipschitz type condition (16) is also necessary to the validity of the Jensen type inequality (17). A benchmark function r is said to be *consistent with the monotone measure* μ if for any $a \in Y_0$ there exist a set $A \in \Sigma$ and a function $g \in \mathbf{F}_Y$ such that for every $t \in Y$

$$\mu(A \cap \{g \geqslant t\}) = r(t, a).$$

Theorem 3.2. *Let $r_1, r_2 \in \mathcal{B}$ and let $H\colon Y \to Y$ be an increasing function. Assume the benchmark function r_2 is consistent with the monotone measure μ and the functions $y \mapsto r_1(x, y)$, $x \in Y$, are left continuous. If the Jensen type inequality (17) is satisfied for any $A \in \Sigma$ and any $f \in \mathbf{F}_Y$, then the Lipschitz condition (16) holds true.*

Proof. Given any $b \in Y_0$, choose such a set A and a function $g \in \mathbf{F}_Y$ that $\mu(A \cap \{g \geqslant t\}) = r_2(t, b)$ for any $t \in Y$. Since $r(0, a) = a$ for all $a \in Y_0$, we have

$$\int_A g \, \mathrm{d}(\mu, r_2) = b. \tag{22}$$

By the definition of the benchmark integral and strict monotonicity of H,

$$\begin{aligned}\int_A H(g) \, \mathrm{d}(\mu, r_1) &= \sup \{a \in Y_0 \colon \mu(A \cap \{g \geqslant H^{-1}(t)\}) \geqslant r_1(t, a) \text{ for all } t \in Y\} \\ &= \sup \{a \in Y_0 \colon r_2(H^{-1}(t), b) \geqslant r_1(t, a) \text{ for all } t \in Y\}. \end{aligned} \tag{23}$$

From (23) and (C1) we conclude that for any $\varepsilon > 0$ and $t \in Y$

$$r_2(H^{-1}(t), b) \geqslant r_1\Big(t, \Big(\int_A H(g) \, \mathrm{d}(\mu, r_1) - \varepsilon\Big)_+\Big).$$

As $y \mapsto r_1(x, y)$ is left continuous and $H(Y) = Y$, we have for any $x \in Y$

$$r_2(x, b) \geqslant r_1\Big(H(x), \int_A H(g) \, \mathrm{d}(\mu, r_1)\Big).$$

By the Jensen inequality (17), (C1) and (22), $r_2(x, b) \geqslant r_1(H(x), H(b))$ for any $x \in Y$. $\square$

Observe that from (16) and (C2) it follows that for all $y \in Y_0$

$$H(y) = r_1(0, H(y)) \leqslant r_2(0, y) = y.$$

Therefore any function satisfying the Jensen type inequality (17) has to also satisfy the condition (J1): $H(x) \leqslant x$, $x \in Y_0$ (see [34]).

We now examine in Examples 3.3–3.8 for which increasing functions $H \colon Y \to Y$ the Lipschitz condition (16) holds with $H(Y) = Y$.

Example 3.3. Define $r_i(x, a) = a \mathbb{1}_{[0,a]}(x)$ for $i = 1, 2$. Then Lipschitz's condition is equivalent to the condition (J1) (see also [27]).

Example 3.4. Set $r_i(x, a) = (a - x)_+$, $i = 1, 2$. Then (16) holds iff $(H(y) - H(x))_+ \leqslant (y - x)_+$ for $x \in Y$, $y \in Y_0$. Because H is an increasing function, this condition is equivalent to the classical Lipschitz condition $|H(x) - H(y)| \leqslant |x - y|$.

Example 3.5. Let $r_i(x, a) = (a^p - x^p)_+^{1/p}$, where $p > 1$, $i = 1, 2$. It is easy to check that the function H satisfies (16) if and only if $x \mapsto H(x)^p - x^p$ is a nonincreasing function.

For instance, $H(x) = x - g(x)$, $x \geqslant 0$, in which g is differentiable such that $g(0) = 0$ and $0 \leqslant g'(x) < 1$ for $x > 0$, fulfills this condition. An example of such a function is $g(x) = \ln(1+x)$.

Example 3.6. Let $r_1(x,a) = (a-x)_+$ and $r_2(x,a) = a\mathbb{1}_{[0,a]}(x)$. Then Lipschitz's condition is valid if and only if for all $x \in Y$ and $y \in Y_0$

$$(H(y) - H(x))_+ \leqslant y\mathbb{1}_{[0,y]}(x). \tag{24}$$

Since H is increasing, (24) holds if $H(y) - H(x) \leqslant y$ for $0 \leqslant x \leqslant y$, $y \in Y_0$. The latter condition is equivalent to (J1), as $H(0) = 0$.

Example 3.7. If $\mu(X) = 1$ and $r_1, r_2 \in \mathcal{B}$ are such that $r_i(x,a) = a(1 - x/a)_+^p$ for $i = 1,2$, $p > 0$ and $a \in (0,1)$, then Lipschitz's condition is of the form

$$H(y)\Big(1 - \frac{H(x)}{H(y)}\Big)_+^p \leqslant y\Big(1 - \frac{x}{y}\Big)_+^p, \qquad 0 \leqslant x \leqslant 1,\, 0 < y < 1, \tag{25}$$

since $H(0) = 0$. Taking the limit as $y \to 1$ gives $(1 - H(x))^p \leqslant (1-x)^p$ for $x \in [0,1]$ as $H\colon [0,1] \to [0,1]$ is a continuous function and $H(1) = 1$. Hence $x \leqslant H(x)$ for all $x \in [0,1]$. Moreover, putting $x = 0$ in (25), we get $H(y) \leqslant y$ for all $0 \leqslant y < 1$. Therefore, $H(x) = x$ is the only function which satisfies inequality (16).

Example 3.8. If $r_1, r_2 \in \mathcal{B}$, $r_i(x,a) = ar_i(x/a, 1)$ for $i = 1,2$, and $r_1(x,1) \leqslant r_2(x,1)$ for arbitrary $x \in Y = \mathbb{R}_+$ and $a \in (0,\infty)$, then the inequality $r_1(H(x), H(y)) \leqslant r_2(x,y)$ holds for $H(x) = cx$, where $0 < c \leqslant 1$. It is worth pointing out that all the above considered benchmark functions (R1 - R6) fulfill the identity $r(x,a) = ar(x/a, 1)$.

We also have the following reverse Jensen type inequality.

Theorem 3.9. *Suppose the function $H\colon Y \to Y$ is increasing, $H(Y) = Y$ and the reverse Lipschitz condition*

$$r_1(H(x), H(y)) \geqslant r_2(x,y), \quad x \in Y,\, y \in Y_0,$$

holds. Then for any $A \in \Sigma$ and any $f \in \mathbf{F}_Y$

$$\int_A H(f)\,\mathrm{d}(\mu, r_1) \leqslant H\Big(\int_A f\,\mathrm{d}(\mu, r_2)\Big). \tag{26}$$

Proof. Arguing as in the proof of Theorem 3.1 it can be shown that

$$\begin{aligned}\int_A H(f)\,\mathrm{d}(\mu, r_1) &= \sup\{H(a)\colon \mu(A \cap \{f \geqslant t\}) \geqslant r_1(H(t), H(a)) \text{ for all } t \in Y,\, a \in Y_0\} \\ &= H(\sup B_1) \leqslant H(\sup B_2),\end{aligned}$$

as H is continuous, where B_i, $i = 1, 2$, are given by (19) and (21). Consequently, inequality (26) holds true. □

Open problem. We are unable to obtain the necessary conditions for the validity of the reverse Jensen type inequality (26) using arguments similar to those in the proof of Theorem 3.2. We leave it as open problem.

We now present two consequences of Jensen's inequality: the Liapunov type inequality and the Hardy type inequality (cf. [22, 35]).

Proposition 3.10. *Let $U, V : Y \to Y$ be increasing functions such that $U(Y) = V(Y) = Y$ and let for all $x \in Y$ and all $y \in Y_0$*

$$r_1(U(x), U(y)) \leqslant r_2(V(x), V(y)).$$

Then for any $A \in \Sigma$ and any $g \in \mathbf{F}_Y$ the Liapunov type inequality

$$U^{-1}\Big(\int\limits_A U(g)\,\mathrm{d}(\mu, r_1)\Big) \geqslant V^{-1}\Big(\int\limits_A V(g)\,\mathrm{d}(\mu, r_2)\Big)$$

holds true.

Proof. The statement follows by applying Theorem 3.1 to $f = V \circ g$ and $H = U \circ V^{-1}$. □

Proposition 3.11. *Let $d > 0$, $p \geqslant 1$, $X = [0, 1] = Y$ and $f \in \mathbf{F}_Y$. We assume that $\mu([0, \alpha]) = \alpha^d$ for all $\alpha \in [0, 1]$ and $r_1(t, a) = ar_1(t/a, 1)$ for $t \in Y$ and $a \in (0, 1)$. If $r_1(x^p, y^p) \leqslant r_2(x, y)$ for $x \in Y$ and $y \in [0, 1)$, then*

$$1 \wedge \Big(\frac{1}{c_p}\int\limits_{[0,1]} f^p\,\mathrm{d}(\mu, r_1)\Big)^{\frac{d}{p+d}} \geqslant \int\limits_{[0,1]} \Big(\frac{F(x)}{x}\Big)^p\,\mathrm{d}(\mu, r_1),$$

where $F(x) = \int_{[0,x]} f\,\mathrm{d}(\mu, r_2)$ and $c_p = \sup\{s(r_1(s, 1))^{p/d} \colon s \in [0, 1]\}$.

Proof. Let $K = \int_{[0,1]} f^p\,\mathrm{d}(\mu, r_1)$. By applying Theorem 3.1 to $H(x) = x^p$, we have

$$K \geqslant (F(1))^p \geqslant (F(x))^p, \quad x \in [0, 1].$$

Dividing both sides by x^p and then integrating w.r.t. μ and r_1 yields

$$\int\limits_{[0,1]} \frac{K}{x^p} \,\mathrm{d}(\mu, r_1) \geqslant \int\limits_{[0,1]} \Big(\frac{F(x)}{x}\Big)^p \,\mathrm{d}(\mu, r_1). \tag{27}$$

Since $\mu(\{x\colon K/x^p \geqslant t\}) = (K/t)^{d/p}$,

$$\begin{aligned}\int\limits_{[0,1]} \frac{K}{x^p} \,\mathrm{d}(\mu, r_1) &= \sup\big\{a \in [0,1)\colon K \geqslant t\big(r_1(t,a)\big)^{p/d} \text{ for all } t \in [0,a]\big\}\\ &= \sup\big\{a \in [0,1)\colon K \geqslant ta^{p/d}\big(r_1(t/a,1)\big)^{p/d} \text{ for all } t \in [0,a]\big\}\\ &= \sup\big\{a \in [0,1)\colon K \geqslant c_p a^{1+p/d}\big\}.\end{aligned}$$

Hence

$$\int\limits_{[0,1]} \frac{K}{x^p} \,\mathrm{d}(\mu, r_1) = 1 \wedge (K/c_p)^{d/(p+d)}. \tag{28}$$

Combining (27) and (28) completes the proof. □

3.2. ***Markov type inequality***

The Markov inequality is a tool to estimate from above the tail of the distribution of a random variable by means of its expected value, used for studying convergence in probability theory, see [25]. For a non-additive operator it has been investigated for the Sugeno integral [8], seminormed fuzzy integral [9] and Choquet integral [42, Theorem 3.1]. We will present a version of this inequality for the benchmark integral, and then provide an upper bound on the benchmark integral using the Choquet integral.

Let $f \in \mathbf{F}_Y$ and $H\colon Y \to Y$ be a nondecreasing function. Set $A^\star = A \cap \{f \geqslant c\}$ for some $c \in Y$. By (P1),

$$\int\limits_A H(f) \,\mathrm{d}(\mu, r) \geqslant \int\limits_A H(f)\mathbb{1}_{A^\star} \,\mathrm{d}(\mu, r) \geqslant \int\limits_A H(c)\mathbb{1}_{A^\star} \,\mathrm{d}(\mu, r).$$

Therefore, by (P4) and (P5),

$$\int\limits_A H(f) \,\mathrm{d}(\mu, r) \geqslant \int\limits_{A^\star} H(c) \,\mathrm{d}(\mu, r) = H(c) \wedge \mu(A \cap \{f \geqslant c\}).$$

If $0 < H(c) \leqslant 1$ and $\mu(A) \leqslant 1$, then the Markov type inequality

$$\mu(A \cap \{f \geqslant c\}) \leqslant \frac{1}{H(c)} \int\limits_A H(f) \,\mathrm{d}(\mu, r)$$

holds true as $a \wedge b \geqslant ab$ for $a, b \in [0,1]$.

By the definition of $\mathtt{I}(f) = \int_A f \,\mathrm{d}(\mu, r)$, if $0 < \mathtt{I}(f) < \infty$, then for any $\varepsilon > 0$

$$\int_0^\infty r(x, (\mathtt{I}(f) - \varepsilon)_+)\,\mathrm{d}x \leqslant \int_0^\infty \mu(A \cap \{f \geqslant t\})\,\mathrm{d}t = (C)\int_A f\,\mathrm{d}\mu,$$

where $(C)\int_A f\,\mathrm{d}\mu$ stands for the Choquet integral [11]. From (C3) it follows that

$$\int_0^{(\mathtt{I}(f)-\varepsilon)_+} r(x, (\mathtt{I}(f) - \varepsilon)_+)\,\mathrm{d}x \leqslant (C)\int_A f\,\mathrm{d}\mu.$$

Since $r(x, (\mathtt{I}(f)-\varepsilon)_+) \leqslant r(0, (\mathtt{I}(f)-\varepsilon)_+) \leqslant \mathtt{I}(f)$ (see the properties (C1) and (C2)), one can derive the limit as $\varepsilon \to 0$ under the integral sign. If we assume that the function $y \mapsto r(x, y)$ is left continuous and $r(x, a) = ar(x/a, 1)$, then after a change of variable we find that

$$(\mathtt{I}(f))^2 \int_0^1 r(t, 1)\,\mathrm{d}t \leqslant (C)\int_A f\,\mathrm{d}\mu. \tag{29}$$

For example, in the case when $r_{GG}(x, a) = (a^p - x^p)_+^{1/p}$ we obtain

$$\Big(\int_A f\,\mathrm{d}(\mu, r_{GG})\Big)^2 \leqslant p(\mathrm{B}(1/p, 1/p+1))^{-1}\,(C)\int_A f\,\mathrm{d}\mu, \tag{30}$$

where $\mathrm{B}(a, b)$ is the beta Euler function (see [16]).

3.3. ***Paley-Zygmund type inequality***

In probability theory, the lower estimate of the tail of the distribution of a random variable using its first two moments is called the Paley-Zygmund inequality [25, Lemma 4.1]. We will provide the lower bound on $\mu(A \cap \{f \geqslant c\mathtt{I}(f)\})$ in terms of the benchmark integral $\mathtt{I}(f)$ for any $c \in (0, 1)$.

By the definition of benchmark integral (8), for any $\varepsilon > 0$ and any $t \in Y$

$$\mu(A \cap \{f \geqslant t\}) \geqslant r(t, (\mathtt{I}(f) - \varepsilon)_+), \tag{31}$$

where $\mathtt{I}(f) = \int_A f\,\mathrm{d}(\mu, r)$ and $0 < \mathtt{I}(f) < \infty$. If $r(x, a) = ar(x/a, 1)$ and $y \mapsto r(x, y)$ is left continuous, then substituting $t = c\mathtt{I}(f)$ with $0 < c < 1$ into (31) we obtain the following Paley-Zygmund type inequality

$$\mu(A \cap \{f \geqslant c\mathtt{I}(f)\}) \geqslant r(c, 1)\mathtt{I}(f).$$

Next we provide the second Paley-Zygmund type inequality. If we additionally assume that $\operatorname{ess\,sup}_\mu(f_{|A}) < \infty$, then for $0 < \beta < \operatorname{ess\,sup}_\mu(f_{|A})$

$$
\begin{aligned}
(C)\int_A f\,\mathrm{d}\mu &= \int_0^\beta \mu(A\cap\{f\geqslant t\})\,\mathrm{d}t + \int_\beta^{\operatorname{ess\,sup}_\mu(f_{|A})} \mu(A\cap\{f\geqslant t\})\,\mathrm{d}t \\
&\leqslant \mu(A\cap\{f>0\})\beta + \big(\operatorname{ess\,sup}_\mu(f_{|A}) - \beta\big)\mu(A\cap\{f\geqslant\beta\}).
\end{aligned}
$$

Substituting $\beta = c\mathtt{I}(f)$ with $0<c<1$ (see Lemma 2.8 (f)), from (29) we get

$$
\mu(A\cap\{f\geqslant c\mathtt{I}(f)\}) \geqslant \frac{\mathtt{I}(f)\int_0^1 r(t,1)\,\mathrm{d}t - c\mu(A\cap\{f>0\})}{\operatorname{ess\,sup}_\mu(f_{|A}) - c\mathtt{I}(f)}\mathtt{I}(f).
$$

4. Subadditivity

In Lemma 2.8, we proved the translativity for the benchmark integral. The aim of this section is to establish, under a suitable set of assumptions, the subadditivity of the benchmark integral for comonotone functions.

By $\mathcal{B}_2$ we denote the class of the functions $r\colon [0,\infty)^2 \to [0,\infty)$, which satisfy the following conditions:

(B1) for any $a\geqslant 0$, the function $0\leqslant x\mapsto r(x,a)$ is nonincreasing, right continuous and $r(0,a)=a$,

(B2) for any $x\geqslant 0$, the function $a\mapsto r(x,a)$ is increasing on $[x,\infty)$ and left continuous on $(0,\infty)$,

(B3) for any a, $r(x,a)>0$ if $x<a$, and $r(x,a)=0$ if $x>a$,

(B4) $r(x,a)\leqslant r(x+d,a+d)$ for $x,a,d\geqslant 0$,

(B5) for any $a>0$, $\sup_{0\leqslant x\leqslant a}\{r(x+d,a+d)-r(x,a)\}\to 0$ as $d\to 0$.

Obviously $\mathcal{B}_2\subset\mathcal{B}$ if $Y=[0,\bar{y})$ and $\bar{y}=\mu(X)=\infty$. The functions r_S, r_W and r_{GG} with $p\geqslant 1$ satisfy the conditions (B1)–(B5). The property (B4) for r_{GG} follows from the inequality $(x+c)^p - x^p \leqslant (a+c)^p - a^p$, $x\leqslant a$, which is a consequence of the convexity of the function x^p for $p\geqslant 1$. The proof of (B5) for r_{GG} is based on the inequality $(a+b)_+^{1/p}\leqslant a_+^{1/p}+b_+^{1/p}$ for $p\geqslant 1$ and $a,b\in\mathbb{R}$. If some functions r_i fulfill the conditions (B1)–(B5), then any convex combination of them satisfies these conditions as well.

We say that the monotone measure μ is *continuous from below*, if $\mu\big(\bigcup_{n=1}^\infty A_n\big) = \lim_{n\to\infty}\mu(A_n)$ for an arbitrary sequence (A_n) such that $A_n\subset A_{n+1}$, $n\geqslant 1$.

Theorem 4.1. *Let μ be a monotone measure continuous from below and $r\in\mathcal{B}_2$. If $f_1,f_2\in\mathbf{F}_{\mathbb{R}_+}$ are comonotone functions on X, then $\mathtt{I}(f_1+f_2)\leqslant\mathtt{I}(f_1)+\mathtt{I}(f_2)$, where $\mathtt{I}(f)=\int_X f\,\mathrm{d}(\mu,r)$.*

Proof. See Appendix A. $\square$

We show that the functional $\mathtt{I}$ may not be subadditive if (B4) is not valid. In fact, suppose that $r(x+d,a+d)<r(x,a)$ for $x<a$, $d>0$, e.g., $r_{GG}(x,a)=(a^p-x^p)_+^{1/p}$ with $0<p<1$. Let $\mu(X)=\infty$, $g\in\mathbf{F}_{\mathbb{R}_+}$ and $f=c>0$ be a constant function. Clearly, the functions f and g are comonotone on X. By (B3),

$$\begin{aligned}\mathtt{I}(f+g)&=\sup\{a\geqslant 0\colon \mu(\{g\geqslant t-c\})\geqslant r(t,a)\text{ for all } t\in[c,a]\}\\&=c+\sup\{b\geqslant 0\colon \mu(\{g\geqslant s\})\geqslant r(s+c,b+c)\text{ for all } s\in[0,b]\}\\&\geqslant c+\sup\{b\geqslant 0\colon \mu(\{g\geqslant s\})\geqslant r(s,b)\text{ for all } s\in[0,b]\}\\&=\mathtt{I}(f)+\mathtt{I}(g),\end{aligned}$$

and for some functions g the strict inequality holds true.

5. Applications and Impact

There are a few possible areas that could benefit from the introduction of our new benchmark integral.

1. The first area is connected to establishing new elementary inequalities. For instance, the inequality (see (29))

   $$\Big(\int\limits_{[0,1]} x^p\,\mathrm{d}(\mu,r_W)\Big)^2\int\limits_0^1 r_W(t,1)\,\mathrm{d}t\leqslant (C)\int\limits_{[0,1]} x^p\,\mathrm{d}\mu,$$

   where μ is the Lebesgue measure, implies the following elementary inequality

   $$1-p^{\frac{1}{1-p}}+p^{\frac{p}{1-p}}\leqslant\sqrt{\frac{2}{p+1}},\quad p>1.$$

   Such inequalities are used in functional analysis to derive some integral inequalities (see [31]).

2. The second application relates to generating new monotone measures by some $\mu\in\mathbf{M}$ according to the formula

   $$\nu(A)=\int\limits_A f\,\mathrm{d}(\mu,r),\quad A\in\Sigma,$$

   in which the function f plays the role of the Radon-Nikodym derivative (see Lemma 2.8 (a)-(b)).

3. The results of this paper can also be used to establish some relations among the indices (1), (2), (5) as well as some other indices. Thanks to subadditivity, we know that each scientometric index of a group of researchers calculated using the benchmark integral is not greater than the sum of the indices of the group members. Additionally, our Markov type and Paley-Zygmund type inequalities can be applied to evaluate the number C_k of papers which have k citations if the only available information is the index value.

Example 5.1. Let $H\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that $H(x) \leqslant x$ and let $H(\mathbb{R}_+) = \mathbb{R}_+$. Consider two researchers A and B with citation functions $H(f(x))$ and $f(x)$, respectively. From Theorem 3.1 and Example 3.3 it follows that the h-index h_A of A is not less that $H(h_B)$, where h_B denotes the h-index of B. Furthermore, from Example 3.6 we conclude that $h_A \geqslant H(w_B)$, in which w_B stands for the w-index of B.

Example 5.2. Inequality (29) for $r_W(x,a) = (a-x)_+$ implies that the w-index satisfies the condition $\mathrm{W}(\mathbf{x}) \leqslant \sqrt{2\mathrm{C}(\mathbf{x})}$, where $\mathrm{W}(\mathbf{x})$ and $\mathrm{C}(\mathbf{x})$ are given, respectively, by (5) and (4). The corresponding extension to the case of the w_{GG}-index (geometric index) [16] of order $p > 1$ is of the form (see (30))

$$\mathrm{w}_{GG}(\mathbf{x}) \leqslant \left(B(1/p, 1/p+1)\right)^{-1/2} \sqrt{p\mathrm{C}(\mathbf{x})}.$$

6. Conclusion

In this paper, we have introduced the benchmark integral, a new functional with respect to a non-additive measure inspired by scientometric indices such as the geometric index [16] or the w-index [44], which is a generalization of the Sugeno integral. We have presented some properties of the new integral including classical Jensen type inequalities, Markov type inequalities, Paley-Zygmund type inequalities, and subadditivity for comonotonic functions. We also formulate some open problems and indicate that the benchmark integral can be applied to evaluate the achievements of scientists by comparing their citation curves with a benchmark function that is the citation curve of a scientist widely recognised as an authority in the field. Determining such a benchmark function would require extensive empirical studies in different scientific disciplines taking into account the researchers' periods of activity, the number of joint publications and many other factors, well beyond the scope of this work.

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Proof of Theorem 4.1

The proof will be divided into three steps.

STEP 1: We show that the benchmark integral has the following property:

(Z1) if $\mathtt{I}(f) < \infty$, then there exist a decreasing sequence (x_n) and a nondecreasing sequence (y_n) which are convergent to the same $c \in [0, \mathtt{I}(f)]$ and such that $\lim r(y_n, \mathtt{I}(f)) \geqslant \lim \mu(\{f \geqslant x_n\})$.

Hereafter, lim denotes the limit as $n \to \infty$. By the definition of $\mathtt{I}(f)$ and (B3), for each n there exists such a $t_n \in [0, \mathtt{I}(f) + 1/n]$ that

$$r(t_n, \mathtt{I}(f) + 1/n) > \mu(\{f \geqslant t_n\}). \tag{A.1}$$

Without loss of generality we can assume that (t_n) is convergent to some $c \in [0, \mathtt{I}(f)]$. The conditions (B4) and (B1) imply that, for $z_n = (t_n - 1/n)_+$,

$$0 \leqslant r(t_n, \mathtt{I}(f) + 1/n) - r(z_n, \mathtt{I}(f)) \leqslant D(\mathtt{I}(f), 1/n), \tag{A.2}$$

where

$$\begin{aligned} D(a, d) &= \sup_{0 \leqslant x \leqslant a+d} \{r(x, a+d) - r((x-d)_+, a)\} \\ &= \max\Big\{ \sup_{0 \leqslant x \leqslant d} \{r(x, a+d) - r(0, a)\}, \sup_{0 \leqslant x \leqslant a} \{r(x+d, a+d) - r(x, a)\} \Big\}. \end{aligned} \tag{A.3}$$

Combining (B4) and (B2) yields $r(0, a) \leqslant r(x, a+x) \leqslant r(x, a+d)$ for $0 \leqslant x \leqslant d$. Hence, in view of (B1), we obtain

$$0 \leqslant r(x, a+d) - r(0, a) \leqslant r(0, a+d) - a = d. \tag{A.4}$$

By (B5) and (A.3)-(A.4) we have $D(a, d) \to 0$ if $d \to 0$, and as a consequence of (A.2) we see that

$$\lim \left(r(t_n, \mathtt{I}(f) + 1/n) - r(z_n, \mathtt{I}(f))\right) = 0. \tag{A.5}$$

Since t_n tends to c, there is no loss of generality in supposing that either (t_n) is nondecreasing or (t_n) is decreasing. Consider first the case when (t_n) is nondecreasing. Then the sequence (z_n) is nondecreasing and converges to c, so the limit $\lim r(z_n, \mathtt{I}(f))$ exists and is equal to $\lim r(t_n, \mathtt{I}(f) + 1/n)$ (by (A.5)). From this and (A.1), we obtain

$$\lim r(z_n, \mathtt{I}(f)) \geqslant \lim \mu(\{f \geqslant t_n\}). \tag{A.6}$$

Put $y_n = z_n$ and $x_n = 2c - y_n + 1/n$. Since the sequence (x_n) is decreasing, converges to c and $t_n \leqslant c \leqslant x_n$ for every n, the condition (Z1)

$$\lim r(y_n, \mathtt{I}(f)) \geqslant \lim \mu(\{f \geqslant t_n\}) \geqslant \lim \mu(\{f \geqslant x_n\})$$

holds, by (A.6).

Let us now consider the case when (t_n) is decreasing. Then $z_n = (t_n - 1/n)_+ \to c$ and we can choose either a nondecreasing subsequence (z_{n_k}) or a decreasing subsequence (z_{n_k}). In the first case we put $y_k = z_{n_k}$ and $x_k = t_{n_k}$. As $\lim_{k\to\infty} r(z_{n_k}, \mathtt{I}(f))$ exists and is equal to $\lim_{k\to\infty} r(t_{n_k}, \mathtt{I}(f) + 1/n_k)$ (see (A.5)), by (A.1) we have $\lim_{k\to\infty} r(z_{n_k}, \mathtt{I}(f)) \geqslant \lim_{k\to\infty} \mu(\{f \geqslant t_{n_k}\})$, and (Z1) follows. If (z_{n_k}) is decreasing, the limit $\lim_{k\to\infty} r(z_{n_k}, \mathtt{I}(f))$ exists and equals $\lim r(t_{n_k}, \mathtt{I}(f) + 1/n_k)$. Define $y_k = c$ and $x_k = t_{n_k}$. Since $z_{n_k} > c$ for all k, by (B1) we have

$$\lim_{k\to\infty} r(y_k, \mathtt{I}(f)) \geqslant \lim_{k\to\infty} r(z_{n_k}, \mathtt{I}(f)) \geqslant \lim_{k\to\infty} \mu(\{f \geqslant x_k\}),$$

so (Z1) also holds true. The proof of the property (Z1) is complete.

STEP 2: Let $f \in \mathbf{F}_{\mathbb{R}_+}$ and $\mathtt{I}(f) < \infty$. Define $g(x) = c + (\sup f - c)\mathbb{1}_A(x)$, where $A = \{f > c\}$ and c is as in (Z1). Clearly, $f \leqslant g$ and $\mu(\{g \geqslant x\}) = \mu(X)$ for $x \in [0, c]$, $\mu(\{g \geqslant x\}) = \mu(A)$ for $x \in (c, \sup f]$ and $\mu(\{g \geqslant x\}) = 0$, otherwise. Moreover, we have $\mu(A) = \mu(\bigcup_{n=1}^{\infty}\{f \geqslant x_n\}) = \lim \mu(\{f \geqslant x_n\})$. By the definition of $\mathtt{I}(f)$, $\mu(\{f \geqslant x\}) \geqslant r(x, (\mathtt{I}(f) - \varepsilon)_+)$ for all $x \geqslant 0$, $\varepsilon > 0$. Letting $\varepsilon \to 0$, by (B2), we get

$$\mu(\{f \geqslant x\}) \geqslant r(x, \mathtt{I}(f)). \tag{A.7}$$

Since $f \leqslant g$, $\mathtt{I}(f) \leqslant \mathtt{I}(g)$. We show that $\mathtt{I}(f) = \mathtt{I}(g)$. Consider first the case of $c = 0$. By (B1), the continuity from below of μ and (A.7), for a decreasing sequence x_n tending to 0

$$\mathtt{I}(f) = r(0, \mathtt{I}(f)) = \lim r(x_n, \mathtt{I}(f)) \leqslant \lim \mu(\{f \geqslant x_n\}) = \mu(\{f > 0\}). \tag{A.8}$$

Applying (Z1) with the constant sequence $y_n = 0$, we obtain

$$\mathtt{I}(f) = r(0, \mathtt{I}(f)) \geqslant \lim \mu(\{f \geqslant x_n\}) = \mu(\{f > 0\}). \tag{A.9}$$

From (A.8) and (A.9) we conclude that $\mathtt{I}(f) = \mu(\{f > 0\})$. Combining this with the equality $\mu(\{g > 0\}) = \mu(\{f > 0\})$ yields $\mathtt{I}(f) = \mu(\{g > 0\})$. In the same manner (see (A.8)) we get that $\mathtt{I}(g) \leqslant \mu(\{g > 0\})$. Hence $\mathtt{I}(f) \geqslant \mathtt{I}(g)$, and finally $\mathtt{I}(f) = \mathtt{I}(g)$.
We now turn to the case $c > 0$. We know that $\mathtt{I}(f) \leqslant \mathtt{I}(g)$. Suppose that $\mathtt{I}(f) < \mathtt{I}(g)$. Then $\mathtt{I}(f) < b < b + 4d < \mathtt{I}(g)$ for some b and $d \in (0, c)$. By (B4), (B1), (B2) and (Z1),

$$\begin{aligned} r(c+d, b+2d) &\geqslant r(c-d, b) \geqslant \lim r(y_n, b) \\ &\geqslant \lim r(y_n, \mathtt{I}(f)) \geqslant \lim \mu(\{f \geqslant x_n\}) = \mu(\{f > c\}) \\ &= \lim \mu(\{g \geqslant x_n\}) \geqslant \mu(\{g \geqslant c + d\}). \end{aligned} \tag{A.10}$$

Since $c \leqslant \mathtt{I}(f) < b$, we have $c + d < b + 2d$. Applying (B2) gives

$$r(c+d, b+4d) > r(c+d, b+2d). \tag{A.11}$$

From (A.10) and (A.11) we get $r(c+d, b+4d) > \mu(\{g \geqslant c+d\})$, which is in contradiction with $b + 4d < \mathtt{I}(g)$. Therefore $\mathtt{I}(f) = \mathtt{I}(g)$.

STEP 3: We are now in a position to show the subadditivity of the integral $\mathtt{I}$. Let g_i be the functions corresponding to f_i, $i = 1, 2$, defined in the same manner as in step 2, that is, $g_i = c_i + (\sup f_i - c_i)\mathbb{1}_{A_i}$, where $A_i = \{f_i > c_i\} = \{g > c_i\}$ and c_i plays the same role as c in step 1. Without loss of generality we can restrict ourselves to $\mathtt{I}(f_i) < \infty$ for all i. Since $f_i \leqslant g_i$ for all i,

$$\mathtt{I}(f_1 + f_2) \leqslant \mathtt{I}(g_1 + g_2).$$

It remains to prove that $\mathtt{I}(g_1 + g_2) \leqslant \mathtt{I}(g_1) + \mathtt{I}(g_2)$. The comonotonicity of f_1 and f_2 implies that either $A_1 \subset A_2$ or $A_2 \subset A_1$ (see [32, Lemma 6.6]). There is no loss of generality in assuming that $A_2 \subset A_1$. Then $\mu(\{g_1 + g_2 \geqslant x\}) = \mu(A_1)$ for $x \in (c_1 + c_2, \sup f_1 + c_2]$. For an arbitrary $\varepsilon > 0$, by the definition of the integral $\mathtt{I} = \mathtt{I}(g_1 + g_2)$

$$\lim_{x \searrow c_1 + c_2} r(x, (\mathtt{I} - \varepsilon)_+) \leqslant \lim_{x \searrow c_1 + c_2} \mu(\{g_1 + g_2 \geqslant x\}) = \mu(A_1),$$

where $c \searrow a$ means that $c \to a$ and $c > a$, and the last equality follows from the continuity of μ. Combining this with (B1) and (B2) we obtain

$$r(c_1 + c_2, \mathtt{I}) \leqslant \mu(A_1). \tag{A.12}$$

Since $\mathtt{I}(g_1) = \mathtt{I}(f_1)$ and $\lim_{x \searrow c_1} \mu(\{g_1 \geqslant x\}) = \lim_{x \searrow c_1} \mu(\{f_1 \geqslant x\})$, by (Z1),

$$\lim_{x \searrow c_1} \mu(\{g_1 \geqslant x\}) \leqslant \lim r(y_n, \mathtt{I}(g_1)).$$

By the definition of g_1, $\lim_{x \searrow c_1} \mu(\{g_1 \geqslant x\}) = \mu(A_1)$. Hence

$$\mu(A_1) \leqslant \lim r(y_n, \mathtt{I}(g_1)). \tag{A.13}$$

Applying (A.12), (A.13) and (B1) we see that

$$r(c_1 + c_2, \mathtt{I}) \leqslant \lim r(y_n, \mathtt{I}(g_1)) \leqslant r(y_k, \mathtt{I}(g_1)) \tag{A.14}$$

for any k. Since $0 \leqslant c_i \leqslant \mathtt{I}(g_i)$ for all i,

$$\begin{aligned} r(c_1 + c_2, \mathtt{I}) &\leqslant r(y_k + c_2, \mathtt{I}(g_1) + c_2) \\ &\leqslant r(y_k + c_2, \mathtt{I}(g_1) + \mathtt{I}(g_2)) \\ &\leqslant r(c_1 + c_2, \mathtt{I}(g_1) + \mathtt{I}(g_2) + c_1 - y_k), \end{aligned} \tag{A.15}$$

by (A.14), (B4) and (B2). If $\mathtt{I} \leqslant c_1 + c_2$, then $\mathtt{I} \leqslant \mathtt{I}(g_1) + \mathtt{I}(g_2)$, as $c_i \leqslant \mathtt{I}(g_i)$ for all i. If $c_1 + c_2 < \mathtt{I}$, then the monotonicity of $r(c_1 + c_2, \cdot)$ on $[c_1 + c_2, \infty)$ (see (B2)), the inequality $c_1 + c_2 \leqslant \mathtt{I}(g_1) + \mathtt{I}(g_2) + c_1 - y_k$ and (A.15) imply that $\mathtt{I} \leqslant \mathtt{I}(g_1) + \mathtt{I}(g_2) + c_1 - y_k$. Letting $k \to \infty$ yields $\mathtt{I} \leqslant \mathtt{I}(g_1) + \mathtt{I}(g_2)$, which is the desired conclusion.

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