Naive Set Theory: Solutions Manual

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Contents

2 Misc. Exersizes

2.1 Logical Operators

Exercise 2.1. *We show that $\wedge$, $\vee$ are commutative. We do this via truth table.*

Proof. Consider the logical sentences p, q and the following truth table:

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

Thus, $p \vee q \iff q \vee p$

Proof. Consider the logical sentences p and q and the following truth table:

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

Thus, $p \wedge q \iff q \wedge p$.

2.2 Logical Rules

<u>Exercise</u> 2.2. *We show that $p \vee p \iff p$ and $p \wedge p \iff p$, via truth table.*

Proof. Consider the logical sentence p and the following truth table:

p	$p \vee p$
T	T
F	F

Thus, $p \vee p \iff p$

Proof. Consider the logical sentence p and the following truth table:

p	$p \wedge p$
T	T
F	F

Thus, $p \iff p \wedge p$.

<u>Exercise</u> 2.3. *We show that $p \Rightarrow p \vee q$ via truth table.*

Proof. Consider the logical sentences p and q and the following truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Thus, if p is true, $p \vee q$ is true, i.e. $p \Rightarrow p \vee q$.

3 Unordered Pairs

Exercise 3.1. *Consider the sets* $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, *etc.; Consider all the pairs such as* $\{\emptyset, \{\emptyset\}\}$, *formed by two of them; consider the pairs formed by any such two pairs, or else mixed pairs formed by a singleton and any pair; and proceed ad infinitum. Are all the sets obtained in this way distinct from one another?*

Yes, all sets obtained in this way are distinct.

Proof. By the axiom of pairing, we can form A and B as given, and by the axiom of specification, they are the only such sets which contain their elements. Suppose, to the contrary, that $A = B$.

If $A = B$, then for all $a \in A$, $a \in B$, and for all $b \in B$, $b \in A$. Now, either A and B are singletons, or they are pairs.

- *A and B are singletons:* Say, $A = \{a\}$, and $B = \{b\}$. Thus, by the *Axiom of Extension*, we have $a = b$. Implying that A and B were formed in the same way.

- *A and B are pairs:* Let $A = \{a, a'\}$, $B = \{b, b'\}$. Without loss of generality, suppose that $a = b$, and $a' = b'$. If this is the case, then by the *Axiom of Extension*, again, we have that a, b were obtained in the same fashion, as well as a', b'. But in this case, A and B were constructed in the same manner.

Either way, we have a contradiction. Thus, A and B are formed distinctly.

4 Unions and Intersections

Throughout these exercises we consider a arbitrary non-empty collection $\mathcal{C}$, of sets. For simplicity we will assume that $\mathcal{C} = \{A, B, C\}$ for arbitrary non-empty sets (unless specified) A, B, C and will make liberal use of the notation $\{x : S(x)\}$, in-place of the more precises yet cumbersome notation $\{x \in \bigcup \mathcal{C} : S(x)\}$.

<u>Exercise</u> 4.1. *We show the following:*

$$A \cup \emptyset = A \tag{1}$$
$$A \cup B = B \cup A \tag{2}$$
$$A \cup (B \cup C) = (A \cup B) \cup C \tag{3}$$
$$A \cup A = A \tag{4}$$
$$A \subset B \iff A \cup B = B \tag{5}$$

Proof. ($\iff$) Since $\emptyset$ has no elements, for all $a \in \bigcup \mathcal{C}$, $a \notin \emptyset$ and we have,

$$\begin{aligned} x \in A \cup \emptyset = \{x : x \in A \vee x \in \emptyset\} &\iff x \in A \vee x \in \emptyset = \{\} \\ &\iff x \in A \end{aligned}$$

Thus, $A \cup \emptyset = A$

Proof. ($\iff$)

$$A \cup B = \{x : x \in A \vee x \in B\} = \{x : x \in B \vee x \in A\} = B \cup A$$

Proof. ($\iff$)

$$\begin{aligned} x \in A \cup (B \cup C) &\iff x \in A \vee x \in B \cup C \\ &\iff x \in A \vee x \in B \vee x \in C \\ &\iff x \in \{x : x \in A \vee x \in B\} \vee x \in C \\ &\iff x \in A \cup B \vee x \in C \\ &\iff x \in (A \cup B) \cup C \end{aligned}$$

Proof. ($\iff$)

$$A \cup A = \{x : x \in A \vee x \in A\} = \{x : x \in A\} = A$$

Proof.
($\Rightarrow$) We note that $A \subset B$ iff for all $x \in A$, $x \in B$. Thus, we have that

$$x \in A \cup B = \{x : x \in A \vee x \in B\} \Rightarrow x \in A \vee x \in B \Rightarrow x \in B$$

by the comments above. And likewise,

$$x \in B \Rightarrow x \in B \vee x \in A \Rightarrow x \in \{x : x \in A \vee x \in B\} \Rightarrow x \in A \cup B$$

That is, $A \cup B = B$.
($\Leftarrow$) Suppose $A \cup B = B$. Then,

$$x \in A \Rightarrow x \in \{x : x \in A \cup x \in B\} = A \cup B = B$$

That is, $A \subset B$.

Exercise 4.2. *We show the following:*

$$\begin{align} A \cap \emptyset &= \emptyset \tag{1} \\ A \cap B &= B \cap A \tag{2} \\ A \cap (B \cap C) &= (A \cap B) \cap C \tag{3} \\ A \cap A &= A \tag{4} \\ A \subset B \iff A \cap B &= A \tag{5} \\ & \tag{6} \end{align}$$

Proof. (1) Suppose that $A \neq \emptyset$, then

($\Rightarrow$)

$$\begin{aligned} x \in A \cap \emptyset = \{x : x \in A \wedge x \in \emptyset\} &\Rightarrow x \in \emptyset \wedge x \in A \\ &\Rightarrow x \in \emptyset \end{aligned}$$

($\Leftarrow$)

$$\begin{aligned} &(\emptyset \subset A) \wedge (\emptyset \subset \emptyset) \\ &\Rightarrow \emptyset \subset A \cap \emptyset \end{aligned}$$

Proof. (2) ($\iff$)

$$A \cap B = \{x : x \in A \wedge x \in B\} = \{x : x \in B \wedge x \in A\} = B \cap A$$

Proof. (3) ($\iff$)

$$\begin{aligned} x \in A \cap (B \cap C) &\iff x \in A \wedge x \in B \cap C = \{x : x \in B \wedge x \in C\} \\ &\iff x \in A \wedge x \in B \wedge x \in C \\ &\iff x \in \{x : x \in A \wedge x \in B\} \wedge x \in C \\ &\iff x \in A \cap B \wedge x \in C \\ &\iff x \in (A \cap B) \cap C \end{aligned}$$

Proof. (4) ($\iff$)

$$A \cap A = \{x : x \in A \wedge x \in A\} = \{x : x \in A\} = A$$

Proof. (5)
($\Rightarrow$) Suppose that $A, B \neq \emptyset$. We note that $A \subset B$ iff for all $x \in A$, $x \in B$; then,

$$x \in A \Rightarrow x \in B \Rightarrow x \in A \wedge x \in B \Rightarrow x \in A \cap B$$

And likewise,

$$x \in A \cap B = \{x : x \in A \wedge x \in B\} \Rightarrow x \in A \wedge x \in B \Rightarrow x \in A$$

That is, $A \cap B = A$.
($\Leftarrow$)Suppose that $A \cap B = A$. Then,

$$x \in A \Rightarrow x \in A \cap B = \{x : x \in A \wedge x \in B\} \Rightarrow x \in A \wedge x \in B \Rightarrow x \in B$$

That is, $A \subset B$.

Exercise 4.3. *We wish show that* $(A \cap B) \cup C = A \cap (B \cup C) \iff C \subset A$*:*
We first show (1) in the following, noting that (2) was proved in the text;
From page fifteen,

$$A \cap (B \cup B) = (A \cap B) \cup (A \cap C) \tag{1}$$

and,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{2}$$

Proof. ($\Rightarrow$)

Suppose that not all of A, B, C are the emptyset. Then,

$$x \in A \cap (B \cup C) \Rightarrow x \in A \wedge x \in B \cup C$$

Now, either $x \in B$ or $x \in C$:

$$x \in B \wedge x \in A \Rightarrow x \in A \cap B \Rightarrow (A \cap B) \cup (A \cap C)$$

$$x \in C \wedge x \in A \Rightarrow x \in A \cap C \Rightarrow (A \cap B) \cup (A \cap C)$$

Either way, $x \in (A \cap B) \cup (A \cap C)$. To prove the reverse inclusion, we note that similar logic holds; either $x \in A \cap B$ or $x \in A \cap C$.

Now, we show that $(A \cap B) \cup C = A \cap (B \cup C) \iff C \subset A$:

Proof.
$(\Leftarrow)$

Suppose that $C \subset A$. Then, for all $x \in C$, $x \in A$; By *Exercise 4.1.5* and the proof above, we have the following:

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C) = A \cap (B \cup C) \tag{3}$$

$(\Rightarrow)$
Suppose that $(A \cap B) \cup C = A \cap (B \cup C)$. Then,

$$\begin{aligned} x \in C \Rightarrow x &\in \{x : x \in A \cap B \vee x \in C\} \\ &= (A \cap B) \cup C \\ &= A \cap (B \cup C) \end{aligned}$$

by the assumption. Thus,

$$\begin{aligned} x \in C &\Rightarrow x \in A \cap (B \cup C) \\ &\Rightarrow x \in A \wedge x \in B \cup C \\ &\Rightarrow x \in A \end{aligned}$$

That is, $C \subset A$.

5 Complements and Powers

Throughout these exercises we assume that "all sets are subsets of one and the same set $\mathcal{E}$, and that all complements (unless otherwise specified) are formed relative to that $\mathcal{E}$" (Halmos). Likewise, we will often use the more convenient, yet less accurate notation $\{x : S(x)\}$ in place of $\{x \in \mathcal{E} : S(x)\}$

<u>Exercise</u> 5.1. *We prove the following:*

$$A - B = A \cap B^c \tag{1}$$

$$A \subset B \iff A - B = \emptyset \tag{2}$$

$$A - (A - B) = A \cap B \tag{3}$$

$$A \cap (A - C) = (A \cap B) - (A \cap C) \tag{4}$$

$$A \cap B \subset (A \cap C) \cup (B \cap C^c) \tag{5}$$

$$(A \cup C) \cap (B \cup C^c) \subset A \cup B \tag{6}$$

Proof. (1) ($\iff$)

$$\begin{aligned} A - B = \{x \in \mathcal{E} : x \in A \wedge x \notin B\} \\ \iff x \in A \wedge x \notin B \\ \iff x \in A \wedge x \in B^c \\ \iff x \in A \cap B^c \end{aligned}$$

Proof. (2)

($\Rightarrow$) Suppose that $A - B \neq \emptyset$, but $A \subset B$; since $A \subset B$, if $x \in A$, $x \in B$. Thus, if $x \in A - B = \{x \in \mathcal{E} : x \in A \wedge x \notin B\}$, $x \notin B$. However $x \in A$, which implies $x \in B$: A contradiction.

($\Leftarrow$) Suppose that $A - B = \emptyset$ and that there exists some $x \in A$ which is not in B, i.e. $A \not\subset B$. However this is a contradiction since

$$x \in A \wedge x \notin B \Rightarrow x \in \{x \in \mathcal{E} : x \in A \wedge x \notin B\} = A - B$$

and yet $A - B = \emptyset$.

Proof. (3) ($\iff$)

By the proof above,

$$\begin{aligned} A-(A-B) = A-(A\cap B^c) = \{x\in\mathcal{E} : x\in A\wedge x\notin A\cap B^c\} \\ \iff x\in A\wedge x\notin A\cap B^c \\ \iff x\in A\wedge x\notin B^c \\ \iff x\in A\wedge x\in B \\ \iff x\in\{x\in\mathcal{E}: x\in A\wedge x\in B\} = A\cap B \end{aligned}$$

Proof. (4) ($\iff$)

$$\begin{aligned} x\in A\cap(B-C) &\iff x\in A\wedge x\in(B-C) \\ &\iff x\in A\wedge x\in\{x\in\mathcal{E}: x\in B\wedge x\notin C\} \\ &\iff x\in A\wedge x\in B\wedge x\notin C \\ &\iff x\notin\{x\in\mathcal{E}: x\in A\wedge x\in C\}\wedge x\in A\cap B \\ &\iff x\notin A\cap C\wedge x\in A\cap B \\ &\iff x\in\{x\in\mathcal{E}: x\notin A\cap C\wedge x\in A\cap B\} \\ &\iff x\in(A\cap B)-(A\cap C) \end{aligned}$$

Proof. (5) ($\Rightarrow$)

Consider $C\subset\bigcup\mathcal{C}$, as stated above.

$$\begin{aligned} x\in A\cap B &\Rightarrow x\in\{x\in\mathcal{E}: x\in A\wedge x\in B\} \\ &\Rightarrow x\in A\wedge x\in B \qquad (1) \\ &\Rightarrow (x\in A\wedge x\in B)\wedge(x\in C\vee x\in C^c) \end{aligned}$$

$$x\in C\wedge(1)\Rightarrow x\in A\cap C\wedge x\in B\cap C \qquad (2)$$

$$x\in C^c\wedge(1)\Rightarrow x\in A\cap C^c\wedge x\in B\cap C^c \qquad (3)$$

Thus, either way, $x\in(A\cap C)$ or $x\in(B\cap C^c)$, i.e. $x\in(A\cap C)\cup(A\cap C^c)$

Proof. (6) ($\Rightarrow$)

$$x \in (A \cup C) \cap (B \cup C^c) \Rightarrow x \in A \cup C \wedge x \in B \cup C^c$$
$$\Rightarrow (x \in A \vee x \in C) \wedge (x \in B \vee x \in C^c) \quad (1)$$

We note that either $x \in C$, or $x \in C^c$:

$$x \in C \wedge (1) \Rightarrow x \notin C^c \wedge (x \in B \vee x \in A)$$
$$\Rightarrow x \in A \cup B \quad (2)$$

$$x \in C^c \wedge (1) \Rightarrow x \notin C \wedge (x \in A \vee x \in B)$$
$$\Rightarrow x \in A \cup B \quad (3)$$

Either way, $x \in A \cup B$.

<u>Exercise</u> 5.2. *We show that* $\mathscr{P}(E) \cap \mathscr{P}(F) = \mathscr{P}(E \cap F)$*:*

Proof. Let E, F be sets such that E, $F \subset \mathcal{E}$; via *Axiom of Powers*, $\mathscr{P}(E)$, $\mathscr{P}(F)$; specifically, from the discussion in Halmos[1], $\mathscr{P}(E) = \{X : X \subset E\}$, and $\mathscr{P}(F) = \{X : X \subset F\}$. Then, we have the following:

$$\begin{aligned}
X \in \mathscr{P}(E) \cap \mathscr{P}(F) &\iff X \in \mathscr{P}(E) \wedge X \in \mathscr{P}(F) \\
&\iff X \in \{Y : Y \subset E \wedge Y \subset F\} \\
&\iff X \subset E \wedge X \subset F \\
&\iff \forall x (x \in X \Rightarrow (x \in E \wedge x \in F)) \\
&\iff \forall x \in X, x \in E \cap F \\
&\iff X = \{x \in \mathcal{E} : x \in X\} \subset E \cap F \\
&\iff X \in \{Y : Y \subset E \cap F\} \\
&= \mathscr{P}(E \cap F)
\end{aligned}$$

<u>Exercise</u> 5.3. *We show that* $\mathscr{P}(E) \cup \mathscr{P}(F) \subset \mathscr{P}(E \cup F)$*:*

Proof. Let E, F be given as in the discussion of proof *5.2.0.* We first claim that if A, B are sets, then $(x \subset A \vee x \subset B \Rightarrow x \subset A \cup B)$. Indeed; if $x \subset A$, then $x \subset A \cup B$ and likewise, $x \subset B$ implies $x \subset B \cup A$. With this in mind, we have the following:

$$\begin{aligned}
X \in \mathscr{P}(E) \cup \mathscr{P}(F) &= \{X : X \in \mathscr{P}(E) \vee X \in \mathscr{P}(F)\} \\
&\iff X \in \mathscr{P}(E) \vee X \in \mathscr{P}(F) \\
&\iff X \in \{Y : Y \subset E\} \vee X \in \{Y : Y \subset F\} \\
&\iff X \in \{Y : Y \subset E \vee Y \subset F\} \\
&\Rightarrow X \in \{Y : Y \subset E \cup F\} \\
&= \mathscr{P}(E \cup F)
\end{aligned}$$

[1] Halmos, P. "*Naive Set Theory*". pg 19

Exercise 5.4. *We show that* $\bigcap_{X\in\mathcal{C}} \mathscr{P}(X) = \mathscr{P}(\bigcap_{X\in\mathcal{C}} X)$*:*

Proof. Let $\mathcal{C}$ be a non-empty collection of sets.

($\Rightarrow$)

$$\begin{aligned} z \in \bigcap_{X\in\mathcal{C}} \mathscr{P}(X) &\Rightarrow (\forall X \in \mathcal{C})(z \in \mathscr{P}(X)) \\ &\Rightarrow (\forall X \in \mathcal{C})(\{z\} \subset \mathscr{P}(X)) \\ &\Rightarrow (\forall X \in \mathcal{C})(z \in X) \\ &\Rightarrow z \in \bigcap_{X\in\mathcal{C}} X \\ &\Rightarrow \{z\} \subset \bigcap_{X\in\mathcal{C}} X \\ &\Rightarrow z \in \mathscr{P}(\bigcap_{X\in\mathcal{C}} X) \end{aligned}$$

($\Leftarrow$)

$$\begin{aligned} z \in \mathscr{P}(\bigcap_{X\in\mathcal{C}} X) &\Rightarrow z \in \{Y : Y \subset \bigcap_{X\in\mathcal{C}} X\} \\ &\Rightarrow z \subset \bigcap_{X\in\mathcal{C}} X \\ &\Rightarrow (\forall X \in \mathcal{C})(z \subset X) \\ &\Rightarrow (\forall X \in \mathcal{C})(\forall v \in z)(v \in X) \\ &\Rightarrow (\forall X \in \mathcal{C})(\forall v \in z)(\{v\} \subset \mathscr{P}(X)) \\ &\Rightarrow (\forall X \in \mathcal{C})(\forall v \in z)(v \in \mathscr{P}(X)) \\ &\Rightarrow (\forall X \in \mathcal{C})(z \in \mathscr{P}(X)) \\ &\Rightarrow z \in \bigcap_{X\in\mathcal{C}} \mathscr{P}(X) \end{aligned}$$

And this completes the proof.

Exercise 5.5. *We show that* $\bigcup_{X\in\mathcal{C}} \mathscr{P}(X) \subset \mathscr{P}(\bigcup_{X\in\mathcal{C}} X)$*:*

Proof. Let $\mathcal{C}$ be a collection of sets. A common fact used in the following proof is that if $x \in \mathscr{P}(X)$, then $x \subset X$ for any set X. This is indeed true, since

$\mathscr{P}(X) = \{Y : Y \subset X\}$. With this in mind, we have

$$\begin{aligned} z \in \bigcup_{X \in \mathcal{C}} \mathscr{P}(X) &\Rightarrow (\exists Y \in \mathcal{C})(z \in \mathscr{P}(Y)) \\ &\Rightarrow z \subset Y \in \mathcal{C} \\ &\Rightarrow z \subset \{x : x \in X (\forall X \in \mathcal{C})\} = \bigcup_{X \in \mathcal{C}} X \\ &\Rightarrow (\forall s \in z)(s \in \bigcup_{X \in \mathcal{C}} X) \\ &\Rightarrow (\forall s \in z)(s \in \mathscr{P}(\bigcup_{X \in \mathcal{C}} X)) \\ &\Rightarrow z \in \mathscr{P}(\bigcup_{X \in \mathcal{C}} X) \end{aligned}$$

Which completes the proof.

Exercise 5.6. *We show that* $\bigcap_{X \in \mathscr{P}(E)} X = \emptyset$*:*

Proof. Let E be a set and consider $\mathscr{P}(E) = \{Y : Y \subset E\}$. We note that this is non-empty because $\emptyset \in \mathscr{P}(E)$, since $\emptyset$ is a subset of every set. We also note that the statement $(x \in \emptyset \Rightarrow x \in \bigcap_{X \in \mathscr{P}(E)} X)$ is vacuously satisfied since $\emptyset$ has no elements. Thus, we only show the forward inclusion:

($\Rightarrow$)

$$\begin{aligned} x \in \bigcap_{X \in \mathscr{P}(E)} X &\Rightarrow (\forall X \in \mathscr{P}(E))(x \in X) \\ &\Rightarrow x \in \emptyset \in \mathscr{P}(E) \end{aligned}$$

And this completes the proof.

Exercise 5.7. *We show that* $(E \subset F \Rightarrow \mathscr{P}(E) \subset \mathscr{P}(F))$*:*

Proof. Suppose that $E \subset F$ for sets E, F. Then, we have the following:

$$\begin{aligned} e \in \mathscr{P}(E) &\Rightarrow e \subset E \\ &\Rightarrow (\forall x \in e)(x \in E) \\ &\Rightarrow (\forall x \in e)(x \in F) && \text{(by Assumption)} \\ &\Rightarrow e \subset F \\ &\Rightarrow e \in \mathscr{P}(F) && \text{(by Definition)} \end{aligned}$$

Which completes the proof.

Exercise 5.8. *We show that* $E = \bigcup \mathscr{P}(E) = \bigcup_{X \in \mathscr{P}(E)} X$*:*

Proof. Let E be a set. Then, we have the following:

(⇐)

$$\begin{aligned} x \in \bigcup \mathscr{P}(E) &\Rightarrow (\exists Y \in \mathscr{P}(E))(x \in Y) \\ &\Rightarrow x \in Y \subset E \\ &\Rightarrow x \in E \end{aligned}$$

(⇒)

$$\begin{aligned} x \in E &\Rightarrow \{x\} \in \mathscr{P}(E) \\ &\Rightarrow x \in \{x\} \in \mathscr{P}(E) \\ &\Rightarrow x \in \bigcup \{X : X \in \mathscr{P}(E)\} \\ &= \bigcup \mathscr{P}(E) \end{aligned}$$

Which completes the proof.

<u>Exercise</u> 5.9. *We show that* $\mathscr{P}\left(\bigcup E\right)$ *is a set that includes* E*, typically, as a proper subset.*

Proof. Indeed, $\mathscr{P}\left(\bigcup E\right) = \mathscr{P}(E) = \{X \in E : X \subset E\}$. If E is non-empty, then it is clear that $E \subsetneq \mathscr{P}(E)$. However, if $E = \emptyset$, then $\mathscr{P}(E) = \emptyset$ and $E = \mathscr{P}(E)$.

6 Ordered Pairs

Throughout this section, we will consider A, B, C, X, Y to be any sets, and will use the notation $\{x : S(x)\}$ in place of the more accurate, yet cumbersome notation $\{x \in \bigcup \mathcal{C} : S(x)\}$.

Exercise 6.1. *We show the following:*

$$(A \cup B) \times X = (A \times X) \cup (B \times X) \tag{1}$$
$$(A \cap B) \times (X \cap Y) = (A \times X) \cap (B \times Y) \tag{2}$$
$$(A - B) \times X = (A \times X) - (B \times X) \tag{3}$$
$$(A = \emptyset \vee B = \emptyset) \iff (A \times B = \emptyset) \tag{4}$$
$$(A \subset X) \wedge (B \subset Y) \iff (A \times X \subset X \times Y) \tag{5}$$

Proof. (1)

($\Rightarrow$)

$$\begin{aligned}
(x,y) \in (A \cup B) \times X &\iff (x,y) \in \{(a,b) : a \in A \cup B \wedge b \in X\} \\
&\iff x \in A \cup B \wedge y \in X \\
&\iff (x \in A \vee x \in B) \wedge y \in X \\
\text{(by Distributivity)} &\iff (x \in A \wedge y \in X) \vee (x \in B \wedge y \in X) \\
&\Rightarrow (x \in A \cup X \wedge y \in A \cup X) \vee (x \in B \cup X \wedge y \in B \cup X) \\
\text{(by Remarks in Text)} &\Rightarrow \{x, \{x,y\}\} \subset \mathscr{P}(A \cup X) \vee \{x, \{x,y\}\} \subset \mathscr{P}(B \cup X) \\
&\iff (x,y) \in \mathscr{P}(\mathscr{P}(A \cup X)) \vee (x,y) \in \mathscr{P}(\mathscr{P}(B \cup X)) \\
\text{(by Remarks in Text)} &\iff (x,y) \in (A \times X) \cup (B \times X)
\end{aligned}$$

($\Leftarrow$)

$$\begin{aligned}
(x,y) \in (A \times X) \cup (B \times X) &\iff (x,y) \in (A \times X) \vee (x,y) \in (B \times X) \\
\text{(by Definition)} &\Rightarrow (x \in A \wedge y \in X) \vee (x \in B \wedge y \in X) \\
&\iff (x \in A \vee x \in B) \wedge y \in X \\
&\iff x \in A \cup B \wedge y \in X \\
&\Rightarrow x \in (A \cup B) \cup X \wedge y \in (A \cup B) \cup X \\
&\Rightarrow \{\{x\}, \{x,y\}\} \subset \mathscr{P}((A \cup B) \cup X) \\
&\iff \{\{x\}, \{x,y\}\} \in \mathscr{P}(\mathscr{P}((A \cup B) \cup X)) \\
&\Rightarrow (x,y) \in (A \cup B) \times X
\end{aligned}$$

Which completes the proof.

Proof. (2)

($\Rightarrow$)

$$
\begin{aligned}
(c,d) \in (A \cap B) \times (X \cap Y) &\iff c \in A \cap B \wedge d \in X \cap Y \\
&\iff (c \in A \wedge c \in B) \wedge (d \in X \wedge d \in Y) \\
\text{(by Commutativity)} \quad &\iff (c \in A \wedge d \in X) \wedge (c \in B \wedge d \in Y) \\
&\Rightarrow (c \in A \cup X \wedge d \in A \cup X) \wedge (c \in B \cup Y \wedge d \in B \cup Y) \\
&\Rightarrow \{\{c\}, \{c,d\}\} \subset \mathscr{P}(A \cup X) \wedge \{\{c\}, \{c,d\}\} \subset \mathscr{P}(B \cup Y) \\
&\iff (c,d) \in \mathscr{P}(\mathscr{P}(A \cup X)) \wedge (c,d) \in \mathscr{P}(\mathscr{P}(B \cup X)) \\
&\iff (c,d) \in A \times X \wedge (c,d) \in B \times X \\
&\iff (c,d) \in (A \times X) \cap (B \times X)
\end{aligned}
$$

($\Leftarrow$)

$$
\begin{aligned}
(c,d) \in (A \times X) \cap (B \times X) &\iff (c,d) \in (A \times X) \wedge (c,d) \in (B \times X) \\
&\iff (c \in A \wedge d \in X) \wedge (c \in B \wedge d \in X) \\
\text{(by Commutativity)} \quad &\iff (c \in A \wedge c \in B) \wedge d \in X \\
&\iff (c \in (A \cap B)) \wedge d \in X \\
&\Rightarrow c \in (A \cap B) \cup X \wedge d \in (A \cap B) \cup X \\
&\Rightarrow \{\{c\}, \{c,d\}\} \subset \mathscr{P}((A \cap B) \cup X) \\
&\Rightarrow (c,d) \in \mathscr{P}(\mathscr{P}((A \cap B) \cup X)) \\
&\iff (c,d) \in (A \cap B) \times X
\end{aligned}
$$

Which completes the proof.

Proof. (3)

($\Rightarrow$)

$$
\begin{aligned}
(c,d) \in (A - B) \times X &\iff c \in (A - B) \wedge d \in X \\
&\iff (c \in A \wedge c \notin B) \wedge d \in X \\
&\iff (c \in A \wedge d \in X) \wedge (c \notin B \wedge d \in X) \\
\text{(since } c \notin B) \quad &\Rightarrow (c,d) \in A \times X \wedge (c,d) \notin B \times X \\
&\iff (c,d) \in (A \times X) - (B \times X)
\end{aligned}
$$

(⇐)

$$\begin{aligned}
(c,d) \in (A \times X) - (B \times X) &\iff (c,d) \in (A \times X) \wedge (c,d) \notin (B \times X) \\
&\Rightarrow (c \in A \wedge d \in X) \wedge (c \notin B \vee d \notin X) \\
\text{(since } (d \in X \wedge d \notin X, \rightarrow\leftarrow)) \quad &\Rightarrow (c \in A \wedge d \in X) \wedge (c \notin B) \\
&\iff (c \in A \wedge c \notin B) \wedge d \in X \\
&\iff c \in (A - B) \wedge d \in X \\
&\Rightarrow \{\{c\}, \{c,d\}\} \subset \mathscr{P}((A-B) \cup X) \\
&\iff \{\{c\}, \{c,d\}\} \in \mathscr{P}(\mathscr{P}((A-B) \cup X)) \\
&\iff (c,d) \in (A-B) \times X
\end{aligned}$$

Which completes the proof.

Proof. (4)

(⇒) Suppose, without loss of generality, that $A = \emptyset$. Then, A contains no elements. Consider $A \times B = \{(a,b) : a \in A \wedge b \in B\}$. Then, since there are no $a \in A$, $\{(a,b) : x \in A \wedge x \in B\} = \emptyset$.

(⇐) Suppose to the contrary that $A \neq \emptyset$ and $B \neq \emptyset$, but $A \times X = \emptyset$. Since $A \neq \emptyset$ and $B \neq \emptyset$, there exists some $a \in A$ and $b \in B$. And so, $\{\{a\}, \{a,b\}\} \subset \mathscr{P}(A \cup B)$ and further we have $(a,b) \in A \times B$. However, this is a contradiction since we assumed $A \times B = \emptyset$.

Proof. (5)

(⇒) We show that if $A \subset X$ and $B \subset Y$, then $A \times B \subset X \times Y$. Since $A \subset X$, for all $a \in A$, $a \in X$ and likewise for B.

$$\begin{aligned}
(c,d) \in A \times B &\iff c \in A \wedge d \in B \\
\text{(by Assumption)} \quad &\Rightarrow c \in X \wedge d \in Y \\
&\Rightarrow c \in X \cup Y \wedge d \in X \cup Y \\
&\Rightarrow \{\{c\}, \{c,d\}\} \subset \mathscr{P}(X \cup Y) \\
&\Rightarrow (c,d) \in \mathscr{P}(\mathscr{P}(X \cup Y)) \\
&\iff (c,d) \in X \times Y
\end{aligned}$$

($\Leftarrow$) Suppose that $A \times X \neq \emptyset$. And that$A \times B \subset X \times Y$. We show that $A \subset X$ and $B \subset Y$.

$$\begin{aligned} a \in A \wedge b \in B &\Rightarrow \{\{a\}, \{a, b\}\} \in \mathscr{P}(\mathscr{P}(A \cup B)) \\ &\iff (a, b) \in A \times B \\ &\Rightarrow (a, b) \in X \times Y \\ &\Rightarrow (a, b) \in \{(x, y) : x \in X \wedge y \in Y\} \\ &\Rightarrow a \in X \wedge b \in Y \end{aligned}$$

Which completes the proof.

7 Relations

Exercise 7.1. *We show that for each of **symmetric**, **reflexive**, and **transitive** properties, there exists a relation that does not have that property but does have the other two.*

Proof. Let X be a set. We first show that if $X = \emptyset$, then any relation in X is an equivalence relation:

> *Lemma.* Let $X = \emptyset$. Then, by a previous theorem, we have $X \times X = \emptyset$. And so, if R is a relation in X, $R = \emptyset$. However, R is an equivalence relation, since all statements are vacuously satisfied.

Likewise, suppose that $X = \{a\}$; i.e. X is a singleton. We claim, also that any relation in X is an equivalence relation:

> *Lemma.* Let $X = \{a\}$. Then, by the comments of the previous section, we have $X \times X = \{(a,a)\}$. Thus, since R is the set $\{(a,b) : a \in X \wedge b \in X\}$ it follows that $R = \{(a,a)\}$ for any relation.[2] It is easy to verify that the transitive condition is vacuously satisfied for R. And, it is symmetric, since $(a,a) = (a,a)$. And, clearly, R is reflexive. Thus, R is an equivalence relation.

We also show that for $X = \{a,b\}$ each of which is distinct, there does not exist a relation, which is symmetric, reflexive, but not transitive:

> *Lemma.* Let $X = \{a,b\}$. Then, the relation $\mathcal{R}$ in question must contain $\{(a,a),(b,b)\} = I$. However, it is easy to verify that this is an equivalence relation.[3] Thus, we must add an element of $X \times X$ which maintains its symmetry, yet is not transitive. However, this is not possible since either way (a,b), (b,a) must be in $\mathcal{R}$ and so we would have $\mathcal{R} = \{(a,a),(b,b),(a,b),(b,a)\}$.And, this is an equivalence relation since $(x,x) \in \mathcal{R}$ for all $x \in X$ and, by the comments above, $\mathcal{R}$ is symmetric, *as well as* transitive.[4]

Thus, let $X = \{a,b,c....\}$, for distinct elements. We first show the relational properties which include reflexivity:

[2] see Halmos pg. 27
[3] see the above lemma
[4] this is left to the reader

Let $\mathcal{R}$ be a relation in X which is reflexive. Then, we must have $I = \{(x,x) : x \in X\} \subset \mathcal{R}$, by definition. Thus, we show a representation of $\mathcal{R}$ which is transitive, but not symmetric, and symmetric but not transitive:

- Consider $\mathcal{R} = I \cup \{(a,b),(b,a),(b,c),(c,b)\}$. Then, this relation is symmetric, since $(x,x) = (x,x)$ for all $x \in \mathcal{R}$ and $\{(a,b),(b,a),(b,c),(c,b)\} \subset \mathcal{R}$. However, $\mathcal{R}$ is not transitive, since $(a,b) \in \mathcal{R}$ and $(b,c) \in \mathcal{R}$ and yet $(a,c) \notin \mathcal{R}$.

- Consider $\mathcal{R} = I \cup \{(a,b)\}$. Then, $\mathcal{R}$ is transitive; note, we only need to check (a,b), (a,a), (b,b) for transitivity, since they are the only ordered pairs which share a first, or second coordinate, where only (a,a), or (a,b) come first. Correctly, $(a,b) \in \mathcal{R}$ and $(b,b) \in \mathcal{R}$ implies $(a,b) \in \mathcal{R}$; $(a,a) \in \mathcal{R}$ and $(a,b) \in \mathcal{R}$ implies $(a,b) \in \mathcal{R}$. However, $\mathcal{R}$ is not symmetric: $(a,b) \in \mathcal{R}$, however, $(b,a) \notin \mathcal{R}$.

Finally, we show the relation in X which is not reflexive. Let X be given as above:

- Consider $\mathcal{R} = \{(a,b),(b,a),(a,a),(b,b)\} \subset X \times X$. This is symmetric since $(x,x) = (x,x)$ for all $x \in X$ and $\{(a,b),(b,a)\} \subset \mathcal{R}$. It is, also, reflexive since $(a,b) \in \mathcal{R}$ and $(b,a) \in \mathcal{R}$ implies $(a,a) \in \mathcal{R}$; likewise, $(b,a) \in \mathcal{R}$ and $(a,b) \in \mathcal{R}$ implies $(b,b) \in \mathcal{R}$.[5] However, $\mathcal{R}$ is not reflexive since $c \in X$, yet $(c,c) \notin \mathcal{R}$.

<u>Exercise</u> 7.2. *We show that the set of all equivalence classes, $X/\mathcal{R}$, is indeed a set.*

Proof. Let $\mathcal{R}$ be an equivalence relation on a set X. We show that $X/\mathcal{R} \subset \mathscr{P}(X)$:

$$\begin{aligned} z \in X/\mathcal{R} &\Rightarrow (\exists x \in X)(z = \{x/\mathcal{R}\} = \{y \in X : (x,y) \in \mathcal{R}\} \subset X) \\ &\Rightarrow z \in \mathscr{P}(X) \\ &\Rightarrow \{z\} \subset X \end{aligned}$$

Which completes the proof.

[5]the others follow

8 Functions

Exercise 8.1. *We show the special cases in which the projections are 1-1.*

Proof. Without loss of generality, we only consider $g : X \times Y \to Y$, the range projection.

- We first note that if X, Y have a pair or more of distinct elements, then g is not 1-1:

  ***Lemma* 8.1.** *Suppose X, Y, have more than a pair of distinct elements, then g is not 1-1: To see this, let $\{a, b\} \subset X$, and $\{c, d\} \subset Y$, then $\{(a, c), (a, d), (b, c), (b, d)\} \subset X \times Y$. If we suppose g is one-to-one, then $g((a, c)) = g((b, d))$ implies $(a, c) = (b, d)$. But this is a contradiction, since $a = b$ and yet a, b are distinct.*

- Next, if $X = \{x\}$ and $Y = \{y\}$, then we claim g is 1-1.

  ***Lemma* 8.2.** *Let $X = \{x\}$ and $Y = \{y\}$; then, $X \times Y = \{(x, y), (y, x)\}$. Upon inspection, we see that g is 1-1: if $(x, y) \neq (y, x)$, then $y \neq x$ and $y = g((x, y)) \neq x = g((y, x)$. We note that this covers the case in which $x = y$.*

- Next, we show that if either X or Y is a singleton and the other is an arbitrary set, g is one-to-one.

  ***Lemma* 8.3.** *Suppose, without loss of generality, that X is the singleton. Then either $Y = \emptyset$ or $Y \neq \emptyset$. If $Y = \emptyset$, then $X \times Y = \emptyset$, by previous exercise; which is trivially 1-1. Else, we assume Y has more than a pair of distinct elements. Since $X = \{x\}$ for some x, it follows that $\{(a, b) : a \in X \wedge b \in Y\} = \{(x, b) : b \in Y\}$, which means g is 1-1.*

- Lastly, if $X = Y = \emptyset$, then all conditions are vacuously satisfied, and so g is 1-1.

Exercise 8.2. *We show that $Y^{\emptyset}$ has exactly one element, namely $\emptyset$, whether Y is empty or not; i.e. $Y^{\emptyset} = \{\emptyset\}$.*

Proof. Suppose that Y is a set. Consider $f = Y^{\emptyset}$. From page thirty, and by a previous theorem, we have $f \subset \mathscr{P}(\emptyset \times Y) = \mathscr{P}(\emptyset) = \{\emptyset\}$, where f is a set of elements which are ordered pairs. From the comments in section seven, $\emptyset$ is a set of ordered pairs. It follows that $f = \{\emptyset\}$.

Exercise 8.3. *We show that if $X \neq \emptyset$, then $\emptyset^X$ is empty; i.e. $\emptyset^X = \emptyset$.*

Proof. Suppose that there is some function $f \in \emptyset^X$. Then, for all $x \in X$, there exists some $b \in \emptyset$ such that $(x, b) \in f$. However, this is not true, since $\emptyset$ contains no elements. Thus, $\emptyset^X = \emptyset$.

9 Families

<u>**Exercise**</u> **9.1.** *After the comments on page thirty-five, we formulate and prove a generalized version of the commutative law for unions.*

***Proposition* 9.1.** *Let K be an arbitrary, non-empty set, and let $p \in \mathrm{Aut}(K^K)$ be an element of the set of one-to-one and onto functions from K to K. We note that in the current notation this means that p is the family and $\{p_i\}$ denotes the values of p evaluating some $i \in K$. Let $\{A_k\}$ be a family of functions with domain K. We claim that $\bigcup_{k\in K} A_k = \bigcup_{i\in K} A_{p_i}$.*

Proof.
($\Rightarrow$)

$$\begin{aligned} x \in \bigcup_{k\in K} A_k &\Rightarrow (\exists k_0 \in K)(x \in A_{k_0}) \\ \text{(since } p \text{ is onto)} \quad &\Rightarrow (x \in A_{k_0}) \wedge (\exists s \in K)(k_0 = p_s) \\ &\Rightarrow x \in A_{k_0} = A_{p_s} \\ &\Rightarrow x \in \bigcup_{i\in K} A_{p_i} \end{aligned}$$

($\Leftarrow$)

$$\begin{aligned} x \in \bigcup_{i\in K} A_{p_i} &\Rightarrow (\exists s \in K)(x \in A_{p_s}) \\ \text{(since } p_s \in K \text{ and onto)} \quad &\Rightarrow (x \in A_{p_s}) \wedge (\exists k_0 \in K)(p_s = k_0) \\ &\Rightarrow x \in A_{p_s} = A_{k_0} \\ &\Rightarrow x \in \bigcup_{k\in K} A_k \end{aligned}$$

Which completes the proof.

To illustrate the previous proof, consider $K = \{0, 1\}$ and $p_i = \begin{cases} 1 & i = 0 \\ 0 & i = 1 \end{cases}$; it is easy to show that p is 1-1 and onto. From the statement above, we have

$$\bigcup_{k\in K} A_k = A_0 \cup A_1 \tag{1}$$

$$\bigcup_{i\in K} A_{p_i} = A_{p_0} \cup A_{p_1} = A_1 \cup A_0 \tag{2}$$

<u>Exercise</u> 9.2. *We show that if both* $\{A_i\}$ *and* $\{B_i\}$ *are families of sets, with domains* I, J, *then*

$$\left(\bigcup_i A_i\right) \cap \left(\bigcup_j B_j\right) = \bigcup_{i,j}(A_i \cap B_j) \tag{1}$$

$$\left(\bigcap_i A_i\right) \cup \left(\bigcap_j B_j\right) = \bigcap_{i,j}(A_i \cup B_j) \tag{2}$$

Where the symbols such as $\bigcap_{i,j}$ *mean* $\bigcap_{(i,j)\in I\times J}$

Proof. (1)

$(\iff)$

$$\begin{aligned}
x \in \left(\bigcup_i A_i\right) \cap \left(\bigcup_j B_j\right) &\iff x \in \left(\bigcup_i A_i\right) \wedge x \in \left(\bigcup_j B_j\right) \\
&\iff (\exists i_0 \in I)(x \in A_{i_0}) \wedge (\exists j_0 \in J)(x \in B_{j_0}) \\
&\iff (\exists (i_0, j_0) \in I \times J)(x \in (A_{i_0} \cap B_{j_0})) \\
&\iff x \in \bigcup_{i,j}(A_i \cap B_j)
\end{aligned}$$

Proof. (2)

$(\iff)$

$$\begin{aligned}
x \in \left(\bigcap_i A_i\right) \cup \left(\bigcap_j B_j\right) &\iff x \in \left(\bigcap_i A_i\right) \vee x \in \left(\bigcap_j B_j\right) \\
&\iff (\forall i \in I)(x \in A_i) \vee (\forall j \in J)(x \in B_j) \\
&\iff (\forall i \in I)(\forall j \in J)(x \in A_i \vee x \in B_j) \\
&\iff (\forall i \in I)(\forall j \in J)(x \in A_i \cup x \in B_j) \\
&\iff (\forall (i, j) \in I \times J)(x \in (A_i \cup B_j)) \\
&\iff x \in \bigcap_{(i,j)\in I\times J}(A_i \cup B_j)
\end{aligned}$$

<u>Exercise</u> 9.3. *Let $\{A_i\}$, $\{B_j\}$ be families of sets with domains $I \neq \emptyset$, $J \neq \emptyset$, respectively. We show that*

$$\left(\bigcup_i A_i\right) \times \left(\bigcup_j B_j\right) = \bigcup_{i,j}(A_i \times B_j) \tag{1}$$

$$\left(\bigcap_i A_i\right) \times \left(\bigcap_j B_j\right) = \bigcap_{i,j}(A_i \times B_j) \tag{2}$$

Proof. (1)

$(\iff)$

$$\begin{aligned}
(a,b) \in \left(\bigcup_i A_i\right) \times \left(\bigcup_j B_j\right) &\iff a \in \bigcup_i A_i \wedge b \in \bigcup_j B_j \\
&\iff (\exists i_0 \in I)(\exists j_0 \in J)(a \in A_{i_0} \wedge b \in B_{j_0}) \\
&\iff (\exists (i_0, j_0) \in I \times J)((a,b) \in A_{i_0} \times B_{j_0}) \\
&\iff (a,b) \in \bigcup_{(i,j) \in I \times J} (A_i \times B_j)
\end{aligned}$$

Proof. (2) Suppose that for each $i \in I$, $A_i \neq \emptyset$ and for each $j \in J$, $B_j \neq \emptyset$.

$(\iff)$

$$\begin{aligned}
(a,b) \in \left(\bigcap_i A_i\right) \times \left(\bigcap_j B_j\right) &\iff a \in \bigcap_i A_i \wedge b \in \bigcap_j B_j \\
&\iff (\forall i \in I)(a \in A_i) \wedge (\forall j \in J)(b \in B_j) \\
&\iff (\forall i \in I)(\forall j \in J)(a \in A_i \wedge b \in B_j) \\
&\iff (\forall i \in I)(\forall j \in J)((a,b) \in A_i \times B_j) \\
&\iff (\forall (i,j) \in I \times J)((a,b) \in A_i \times B_j) \\
&\iff (a,b) \in \bigcap_{i,j} A_i \times B_j
\end{aligned}$$

<u>Exercise</u> 9.4. *Suppose that for each $i \in I$, X_i is non-empty. Consider the family of sets $\{X_i\}$ with domain I. We show that $\bigcap_i X_i \subset X_j \subset \bigcup_i X_i$ for each index $j \in I$.*

Proof. Let $\{X_i\}$ be stated as above, and let $j \in I$ be arbitrary.

$$\begin{aligned} x \in \bigcap_i X_i &\Rightarrow (\forall i \in I)(x \in X_i) \\ &\Rightarrow x \in X_j \quad (\text{since } j \in I) \end{aligned}$$

Which proves the first inclusion. For the second, suppose $j \in I$ is arbitrary, then we have the following:

$$\begin{aligned} x \in X_j &\Rightarrow x \in \bigcup_j X_j \\ &\Rightarrow x \in \Big(\bigcup_j X_j \cup \bigcup_{z \in I-J} X_z\Big) \\ &\Rightarrow x \in \bigcup_i X_i \end{aligned}$$

Which prove the second inclusion.

<u>Exercise</u> 9.5. *We show the minimality condition of intersection and unions: (1) That is, if $X_j \subset Y$ for each index j and some set Y, then $\bigcup_i X_i \subset Y$ and that it is unique; (2) Likewise, if $Y \subset \bigcap_j X_j$ for each index j and some set Y, then $Y \subset \bigcap_i X_i$ and it is unique.*

Proof. (1) Suppose that $\{X_i\}$ is a family of sets with domain I, such that $j \in I$, and let Y be a set. Then, we have the following:

$$\begin{aligned} (\forall j)(X_j \subset Y) &\Rightarrow \bigcup_j X_j \subset Y \\ &\Rightarrow \bigcup_i X_i \subset Y \end{aligned}$$

Proof. (2) Suppose that $\{X_i\}$ is a family of sets with domain I, such that $j \in I$. And let Y be a set. Then, we have the following:

$$Y \subset \bigcap_j X_j \Rightarrow Y \subset \bigcap_i X_i$$

10 Inverses and Composites

Exercise 10.1. *Suppose that $f : X \to Y$. Let $\{A_i\}$ be a family of subsets of X. Define $f' : \mathscr{P}(X) \to \mathscr{P}(Y)$, as the mapping from each A_i to the image subset, $f'(A_i)$, as stated in the text. We claim that*

$$f'\Big(\bigcup_i A_i\Big) = \bigcup_i f'(A_i)$$

Proof. After the comments on page thirty-four, we note that $A : I \to \mathscr{P}(X)$, for some index set $I \neq \emptyset$. Then, we have the following:

($\Rightarrow$)

$$\begin{aligned} x \in \bigcup_i f'(A_i) &\Rightarrow (\exists i_0 \in I)(x \in f'(A_{i_0})) \\ \text{(since } A_{i_0} \subset \bigcup_i A_i) \quad &\Rightarrow x \in f'\Big(\bigcup_i A_i\Big) \end{aligned}$$

($\Leftarrow$)

$$\begin{aligned} x \in f'\Big(\bigcup_i A_i\Big) &\Rightarrow (\exists i_0 \in I)(x \in f'(A_{i_0})) \\ &\Rightarrow x \in \Big(\bigcup_{z \in I - \{i_0\}} f'(A_z)\Big) \cup f'(A_{i_0}) = \bigcup_i f'(A_i) \end{aligned}$$

And the result follows.

Exercise 10.2. *We show that, in general,*

$$f'\Big(\bigcap_i A_i\Big) = \bigcap_i f'(A_i)$$

does not hold.

The following proof was suggested by David. H..

Proof. Consider $X = \{a, b, c\}$ and $Y = \{d, e\}$ and suppose all the elements are distinct. Then define $f' : \mathscr{P}(X) \to \mathscr{P}(Y)$ as the following:

$$\begin{aligned} f'(\emptyset) &= \emptyset \\ f'(\{a\}) &= \{d\} \\ f'(\{b\}) &= \{e\} \\ f'(\{c\}) &= \{d\} \\ f'(\{a,b\}) &= \{d,e\} \\ f'(\{b,c\}) &= \{d\} \\ f'(\{a,c\}) &= \{e\} \\ f'(\{a,b,c\}) &= \emptyset \end{aligned}$$

Then, let $\{A_i\}$ be a family of subsets of X with domain $I = 2$, such that $A_0 = \{a\}$ and $A_1 = \{a, c\}$. Then, we have the following:

$$\begin{aligned} f'\Big(\bigcap_i A_i\Big) &= f'(\{a\}) = \{d\} \\ \bigcap_i f'(A_i) &= f'(A_0) \cap f'(A_1) = \{d\} \cap \{e\} = \emptyset \end{aligned}$$

Which completes the proof.

Exercise 10.3. *Let $f : X \to Y$. We show that f maps X onto Y iff for every non-empty subset B of Y, $f^{-1}(B) \neq \emptyset$.*

Proof.
($\Rightarrow$) If f maps X onto Y, then for every $y \in Y$, there exists an $x \in X$ such that $f(x) = y$. Since this is the case, if B is a non-empty subset of Y, then for any $b \in B$, $b \in Y$. And we have that there exists an $x_0 \in X$ such that $f(x_0) = b$; therefore, $f^{-1}(B) = \{x \in X : f(x) \in B\} \neq \emptyset$.

($\Leftarrow$) Suppose that $f^{-1}(B) \neq \emptyset$ for any $B \subset Y$. In particular, for each $y \in Y$, $\{y\} \subset Y$ and $f^{-1}(\{y\}) = \{x \in X : f(x) = y\} \neq \emptyset$. The result follows immediately.

Exercise 10.4. *In the context of page forty-one, $R :=$ "son", $S :=$ "brother", what do R^{-1}, S^{-1}, RS and $R^{-1}S^{-1}$ mean?*

Proof. (R^{-1}): From the comments on page forty-one, xRy implies that that x is son-related to y. That is, y is the father of x. Thus, $yR^{-1}x$ means that y has the father relation to x.

(S^{-1}): Likewise, xSy means that x bears the brother-relation to y. That is, x is the brother of y. Thus, $yS^{-1}x$ implies y is the brother of x.

($T = R \circ S$): Also, by the previous comments, if xTz, then there exists y such that xSy and yRz. Thus, x is the brother of y and z is the son to y. That is, x is the uncle of y.

($T = R^{-1} \circ S^{-1}$): Likewise, if xTz, then there exists a y such that $xS^{-1}y$ and $yR^{-1}z$. Thus, x is the brother of y and y is the father to z. Thus, x is the uncle of z.

<u>Exercise</u> 10.5. *Assume that for each of the following $f : X \to Y$. We show the following:*

1. *If $g : Y \to Z$ such that $g \circ f$ is the identity on X, then f is 1-1 and g maps Y onto X.*
2. *$f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subset X$ iff f is 1-1.*
3. *$f(X - A) \subset Y - f(A)$ for all subsets A of X iff f is 1-1.*
4. *$Y - f(A) \subset f(X - A)$ for all subsets A of X iff f maps X onto Y.*

Proof. (1): Suppose that $g : Y \to Z$ such that $g \circ f$ is the identity on X. Since $g \circ f$ is the identity on X, for all $x \in X$, $g(f(x)) = x$. Thus, if $f(x_0) = f(x_1)$, then $x_0 = g(f(x_0)) = g(f(x_1)) = x_1$ and f is 1-1. To show that g maps Y onto X let $x \in X$, then $x = g(f(x))$ where $f(x) \in Y$. Thus, g maps Y onto X.

Proof. (2):

($\Rightarrow$) By assumption, we have that $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subset X$. Suppose that $f(x) = f(y)$ for some $x, y \in X$. If $x = y$, then we are done. So, suppose that $x \neq y$. Letting $A = \{x\}, B = \{y\}$, we have that $f(A \cap B) = \emptyset \neq \{f(x) = f(y)\} = f(A) \cap f(B)$, a contradiction.

($\Leftarrow$) Suppose that f is 1-1. If $y \in f(A \cap B)$, then $y = f(x)$ for some $x \in A \cap B$. Thus, $x \in A$ and $x \in B$. This implies that $f(x) = y \in f(A)$ and $f(x) = y \in f(B)$. Thus, $y \in f(A) \cap f(B)$.

Now take $y \in f(A) \cap f(B)$. Then $y \in f(A)$ so there is some $a \in A$ with $f(a) = y$. Also, $y \in f(B)$, so there is some $b \in B$ such that $y = f(b)$. Now $f(a) = y = f(b)$, so f being 1-1 now implies $a = b$. Thus, $a \in A \cap B$ and $f(a) = y \in f(A \cap B)$.

Proof. (3):

($\Rightarrow$) Suppose that $f(x) = f(y)$ for some $x, y \in X$. Let $A = \{x, y\}$. Then, $f(X - \{x, y\}) \subset Y - f(\{x, y\}) = Y - \{f(x) = f(y)\} = Y - \{f(x)\}$.By the comments on page thirty-nine, we have that

$$\begin{aligned} &f^{-1}f((X - A)) \subset f^{-1}(Y - \{f(x)\}) \\ &\Rightarrow X - A = X - f^{-1}(\{f(x)\}) \\ &\Rightarrow X - \{x, y\} = X - \{x\} \end{aligned}$$

It follows that $\{x, y\} = \{x\}$ i.e. $y = x$.

($\Leftarrow$) Suppose that f is 1-1. And let $y \in f(X - A)$ for any non-empty subset $A \subset X$. Then, there exists an $x_0 \in X - A$ such that $f(x_0) = y$. Suppose to the contrary that $y \notin Y - f(A)$. Then, $f(x_0) = y \in (Y - f(A))^c = f(A)$. This implies that there exists some $a \in A$ such that $f(x_0) = y = f(a)$. However, this is a contradiction since f being 1-1 implies $a = x_0 \in A$.

Proof. (4):

($\Rightarrow$) Let $y \in Y$ and consider an arbitrary subset A of X. Now, either $y \in f(A)$ or it is not. If it is, we are done. So suppose that it is not. Then, $y \in Y - f(A) \subset f(X - A)$ and so, there exists some $x_0 \in X - A$ such that $y = f(x_0)$; and the result follows.

($\Leftarrow$) Let $A \subset X$ and suppose that $y \in Y - f(A)$. Since f maps X onto Y, there exists some $x \in X$ such that $y = f(x) \in Y - f(A)$. Thus, $x \notin A$ and so $x \in A^c = X - A$. It follows from he definition that $y = f(x) \in f(X - A)$ and the result follows.

11 Numbers

<u>Exercise</u> 11.1. *Let $\{A_i\}$ be a family of non-empty successor sets for some non-empty domain I. We show that*

$$\bigcap_i A_i$$

is a successor set.

Proof. For all $i \in I$, A_i is a successor set, and so $0 \in A_i$. This implies that $0 \in \bigcap_i A_i$. Likewise, suppose that $n \in \bigcap_i A_i$; then $n \in A_i$ for all $i \in I$ and so $n^+ \in A_i$ for each $i \in I$; from the definition of intersection $n^+ \in \bigcap_i A_i$. This completes the proof.

12 Peano Axioms

Exercise 12.1. *We prove that if $n \in \omega$ then $n \neq n^+$.*

Proof. Suppose, to the contarary, that $n = n^+$ for some $n \in \omega$. Then, from the axiom of extension we have that $n^+ \subset n$. However, then we would have that $n^+ = n \cup \{n\} \subset n$ implying that $n \in n$ which is a contradiction.

Exercise 12.2. *Let $n \in \omega$. We show that if $n \neq 0$, then $n = m^+$ for some $m \in \omega$ — We note that this show any element $n \in \omega$ has a predecessor.*

Proof. We proceed by mathematical induction: Let S be that subset of ω for which if $n \neq 0$, then $n = m^+$ for some $m \in \omega$. We note that $1 \in S$, since $1 = 0^+$ and $0 \in \omega$. Suppose that for some $n \in \omega$, $n = m^+$ for some $m \in \omega$; indeed, if $n^+ = m$ for some $m \in \omega$ we have $(n^+)^+ = m^+ = z$ for some $z \in \omega$ by the induction hypothesis. This completes the proof.

Exercise 12.3. *We prove that ω is transitive. That is, if $x \in y$ and $y \in \omega$, then $x \in \omega$.*

Proof. We proceed by mathematical induction. Let S be the set of all $z \in \omega$ such that if $x \in z$, then $x \in \omega$. We note that $\{\} = 0 \in S$, vacuously. Thus, let $y \in S$. We show that $y^+ \in S$. That is, let x be any element in y^+: $x \in y^+ = y \cup \{y\}$. Then, either $x \in y$ or $x = y$. In the former cases, we that $x \in \omega$, by assumption; in the later case we have that $y = x \in S$. Thus, $\omega = S$ and so ω is transitive.

Exercise 12.4. *We show the following: if E is non-empty subset of some natural number, then there exists an element $k \in E$ such that $k \in m$ whenever m is an element of E and distinct from k — This says that any finite set has a lower bound, namely a greatest lower bound.*

Proof. We proceed by mathematical induction. Let S be the set of $z \in \omega$ such that the above holds. We note that $\{\} = 0 \in S$, vacuously. Suppose that $n \in S$ for some $n \in \omega$; we show that $n^+ \in S$. Suppose that E is a non-empty subset of $n^+ = n \cup \{n\}$. If E is a singleton, then we are done; the above is vacuously satisfied. We note that if $E = n$, then we are done, by the induction hypothesis. Thus, there exists some $k \in E$, as specified above. We claim that this k, so specified, will work even if $n \in E$; Indeed from a previous proof, we know that no element is equal to its successor and so it follows that for any $a \in E$, $a \in n$. In particular, this is true for k. Therefore, $S = \omega$.

13 Arithmetic

In many of these exercises, we use induction. As pointed out in the *Axiom of Substitution*, and the conversation I had with S. Bleiler, and later sections, it is still valid to start induction at some $x \in \omega$; we have used this fact a couple times.

Exercise 13.1. *Prove that if $m < n$ for some natural numbers m, n, then $m + k < n + k$ for any $k \in \omega$.*

Proof. We proceed by mathematical induction. Suppose that for some natural numbers m, n, $m < n$. Let S be the set of all $z \in \omega$ such that $m + z < n + z$. We show that $S = \omega$. Indeed; we note that since $m + 0 = m$ and $n + 0 = n$, it follows that $m = m + 0 < n + 0 = n$, and so, $0 \in S$. Now, suppose that $k \in S$ for some $k \in \omega$. We show that $k^+ \in S$.

Note, $s_n(k^+) = (s_n(k))^+ = s_n(k) \cup \{s_n(k)\}$, and by the definition of $<$, and the induction hypothesis, we have that $\{s_n(k)\} \subset s_m(k) \subset (s_m(k) \cup \{s_m(k)\}) = s_m(k^+)$. This implies that $s_n(k^+) = s_n(k) \cup \{s_n(k)\} \subset s_m(k) \cup \{s_m(k)\} = s_m(k^+)$. That is, $S = \omega$ ∎

Exercise 13.2. *Prove that if $m < n$ for some $m, n \in \omega$, and $k \neq 0$, then $m \cdot k < n \cdot k$.*

Proof. We proceed by mathematical induction. Suppose that $m < n$ for some $m, n \in \omega$. Let S be the set of $z \in \omega$, which satisfy the following statement: if $z \neq 0$, then $m \cdot z < n \cdot z$. We note that $1 \in S$. Suppose that $k \in S$ for some $k \in \omega$; we show that $k^+ \in S$. Indeed: by definition of product, we have the following:

$$\begin{aligned}
p_m(k^+) &= p_m(k) + m \\
&= s_m(p_m(k)) && (1)\\
&< s_m(p_n(k)) && (2)\\
&= p_n(k) + m \\
&< p_n(k) + n && (3)\\
&= p_n(k^+)
\end{aligned}$$

We only note that (1),(2), and (3) follow from the inductive hypothesis and the previous exercise. Therefore, $S = \omega$. ∎

***Lemma* 13.1.** *For any $n \in \omega$, n is transitive.*

Proof. Suppose that $E = n \in \omega$ and $y \in E$. Since $y \in E$, y is a proper subset of E. Now, if $x \in y$, then x is a proper subset of y and it follows directly that $x \subset y \subset E$. That is, x is a proper subset of E. That is $x \in E$. ∎

***Lemma* 13.2.** *If $m, n \in \omega$ such that $m \neq n$, then $m \in n$ iff $m \subset n$*

Proof. Let m, n be stated as above.

($\Rightarrow$)

Suppose that $m \in n$. We show that $m \subset n$: let $x \in m$, then by the lemma above, $x \in n$. Thus, $m \subset n$.

($\Leftarrow$)

Suppose that $m \subset n$. If $m = n$, then we are done. Otherwise, $m \neq n$. If this is the case, then $n \in m$ cannot happen (for them m would be a subset of one of its elements), and therefor $m \in n$; this is by the comparability of and $x, y \in \omega$.

Exercise 13.3. *We prove that if E is a non-empty set of natural numbers, then there exists an element $k \in E$ such that $k \leq m$ for all $m \in E$. — This statement says that any non-empty subset in ω has a minimum.*

Proof. Let $E \subset \omega$ such that $E \neq \emptyset$. Since E is non-empty, we can form the intersection of elements in E. Consider

$$z = \bigcap_{x \in E} x$$

It follows directly from the formulation of z that $z \subset x$, for all $x \in E$: $z \leq x$. Thus, it is only necessary to prove that $z \in E$. From the previous lemma we see that since $z \in x$ for all $x \in E$, and $x \in E$, that $z \in E$. The result follows.

Exercise 13.4. *A set E is called finite if it is equivalent to some natural number; otherwise E is infinite. We use this definition to prove that ω is infinite.*

Proof. Suppose, to the contrary that ω is finite. Then, there exists a 1-1 and onto function, f, from ω to n for some $n \in \omega$: $f \subset \omega \times n$. Since f is 1-1 and onto, for ever $y \in n$, there exists a unique $x \in \omega$ such that $(x, y) \in f$. Let g be the union of all such x, which is an element of ω. Then, $g^+ = g \cup \{g\}$ is in ω but not in g. Since f is a function, we must have $(g^+, y_1) \in f$ for some $y_1 \in n$; likewise since f is onto, there exists some $x \in g$ such that $(x, y_1) \in f$. From here we see that we must have $(g^+, y_1) = (x, y_1)$, implying $g^+ = x$; which is a contradiction since $g^+ \notin g$, whereas $x \in g$.

Exercise 13.5. *If A is a finite set, then A is not equivalent to a proper subset of A. The contrastive of this statement is that if A is equivalent to a proper subset of A, then A is not finite (infinite). We use this definition to show that ω is infinite.*

Proof. We show that $\omega \sim A$, where $A = \omega - \{0\}$. First, it is clear that A is a non-empty proper subset of ω. Consider the map $f \subset \omega \times \omega$, given by $f(n) = n^+$. First, we show that f is indeed a function, and then that it is 1-1 and onto:

- *Well-Defined*: Suppose that $x = y$ for some $x, y \in \omega$. From a previous theorem, we have $x^+ = y^+$. This shows that f is well-defined.

- *Everywhere-Defined*: This follows directly from the definition of ω as being a successor set. This shows that f is everywhere defined.
- *1-1*: Suppose that $n^+ = m^+$. Again, from a previous exercise (page 46), we have $n = m$. This shows that f is 1-1.
- *Onto*: Suppose that $n \in A = \omega - \{0\}$. Then no $n \in A$ is 0. Thus, by a previous theorem, we have that $n = m^+ = f(m)$ for some $m \in \omega$. This implies that f is onto.

Exercise 13.6. *We prove that union of a finite set of finite sets is finite.*

Proof. We proceed by mathematical induction. Let S be a finite set, such that $\#(S) = n$, for some $n \in \omega$. Let B be the set of all $N \in \omega$ such that $\bigcup S$ is finite.

Note $0 \in B$ since, if $\#(S) = 0$, then $S = \{\}$ and $\bigcup\{\} = \{\}$ (which is finite). Next, suppose that $k \in B$: that is, if S is a finite set such that $\#(S) = k$, then $\bigcup S$ is finite. We show that $k^+ \in B$. Suppose that $\#(S) = k^+$. Then, there exists some 1-1 and onto function f i.e. $f : k^+ \leftrightarrow S$. And we have the following:

$$\bigcup S = \bigcup_{i \in k^+} f(i) = \Big(\bigcup_{i \in k} f(i)\Big) \cup f(k)$$

From the comments on page 53, and the induction hypothesis, we see that $k^+ \in B$. Thus, $B = \omega$.

Exercise 13.7. *We prove the following: If E is finite, then $\mathscr{P}(E)$ is finite and, moreover, $\#(\mathscr{P}(E)) = 2^{\#(E)}$.*

Proof. It is only necessary to show that if $\#(E) = n$ for some $n \in \omega$, then $\#\mathscr{P}(E) = 2^{\#(E)}$; we conclude that $2^{\#(E)} \in \omega$ by a previous exercise. We proceed by mathematical induction.

Let S be the set of all $N \in \omega$ such that $\#(E) = N$ and $\#\mathscr{P}(E) = 2^{\#(E)}$. Note that $0 \in S$, since $\#(E) = 0$ implies $E = \{\}$ and $\#(\mathscr{P}(E)) = \#(\{\emptyset, \{\emptyset\}\}) = 2 = 2^0$. Next, assuming that $k \in S$, we show that $k^+ \in S$:

Suppose that $\#(E) = k^+ = k+1$, and consider the set $E^* = E - \{0\}$. Note, $\#(E^*) = k$. We note that either 0 is in a subset of E, or it is not. If it is not, then we are forming subsets of E^*. From the inductive hypothesis, there are $2^{\#(E^*)} = 2^k$ subsets of E^*. By similar logic, there are 2^k subsets which include 0. It follows that there are $2^k + 2^k = 2 \cdot 2^k = 2^{k+1} = 2^{\#(E)}$ subsets of E. Thus, $k + 1 = k^+ \in S$. That is, $S = \omega$.

Exercise 13.8. *If E is a non-empty finite set of natural numbers, then there exists and element $k \in E$ such that $m \leq k$ for all $m \in E$ — This statement says that for any finite set E, non-empty, of natural numbers, there is an upper bound, actually a maximum.*

Proof. Let E be a finite, non-empty, subset of natural numbers. We proceed by mathematical induction. Let S be the set of all $N \in \omega$ such that $\#(E) = N$ and there exists a $k \in E$ such that the above holds.

Note that $1 \in S$; $1 = \#(E)$ implies $E = n$ for some $n \in \omega$, and clearly $n \leq n$. Next, assuming that $k \in S$, for some $k \geq 1$, we show that $k^+ \in S$:

Suppose that $\#(E) = k^+ = k + 1$. Fix some $x \in E$, and consider $E^* = E - \{x\}$. We note that $\#(E^*) = k$, and so, it follows from the induction hypothesis that there exists some $g \in E^*$ such that $m \leq g$, for all $m \in E^*$. By the comparability of elements in ω, we must have that either $x \in g$, or $g \in x$; we only note that $x \neq g$, since otherwise we would have $\#(E) = k$. In the former case, if $x \in g$, then $x < g$, and in particular $x \leq g$. In the later case, supposing that $g \in x$, we have $g < x$; and, again, in particular $g \leq x$. But then $m \leq g$ and $g \leq x$ for all $m \in E = E^* \cup \{x\}$, implying $m \leq x$ for all $m \in E$ by the transitivity of $\leq$. This shows that $k^+ \in S$, and so $S = \omega$.

14 Order

Exercise 14.1. *We express the conditions of antisymmetry and totality for a relation R by means of equations involving R and its inverse — This exercise illustrates that we can determine a partial, or total order, by looking the the co(domains), of a relation.*

Proof. From the comments on page 54 and the notation in section 10, we adopt the usual assumptions and notation. That is, assume that R is a relation in X. And, we have the following:
Antisymmetry:

$$\begin{aligned} & xRy \wedge yRx \Rightarrow x = y \\ \text{iff} \quad & (R(x) = y \wedge R^{-1}(x) = y) \Rightarrow R(x) = R^{-1}(y) \end{aligned}$$

Totality:

$$(\forall x, y \in X)(xRy \vee yRx) \iff (\forall x, y \in X)(R(x) = y \vee R^{-1}(x) = y)$$

15 Axiom of Choice

Exercise 15.1. *We prove that every relation includes a function with the same domain.*

Proof. Suppose that X and Y are sets and let R be a relation from X into Y. By definition, $R \subset X \times Y$, and so if X or Y is empty, the $X \times Y$ is empty and we are done. Otherwise, there exists at least an element in each of X, Y. Let X_R be the subset of X given by the first coordinates of R. Claim: there exists some function $f : X_R \to Y$.

Let $y \in Y$ be arbitrary and consider $T_y = \{x \in X_R : xRy\}$. Let I be the set of $z \in Y$ such that $T_z \neq \emptyset$. Then, define $T = \bigcup_{y \in I}\{T_y\} = \{T_i\}$. Assuming the axiom of choice, and by the comments on page sixty, $\{T_i\}$ is a family of non-empty sets indexed by I and so, there exists a family $\{t_i\}$, $i \in I$ such that $t_i \in T_i$ for each $i \in I$. Since R is a relation on X_R, so defined, it follows that $f = \{(t_i, b) : i \in I \wedge b = R(t_i)\}$ is a candidate function.

Indeed; we verify that f is well-defined and everywhere defined:

- *Well-Defined*: Suppose that $t_i = t_j$ for some $i, j \in I$. By construction, and the previous supposition, we have that $T_{t_i} = \{x \in X_R : xRt_i\} = \{x \in X_R : xRt_j\} = T_{t_j}$. Thus, $(t_i, R(t_i)) = (t_j, R(t_j)) = f(t_i) = f(t_j)$.

- *Every-Where Defined*: This is clear by construction.

Thus, f is a function.

16 Zorn's Lemma

Exercise 16.1. *We show that Zorn's Lemma is equivalent to the axiom of choice.*

Proof. Let X be a set and define $\mathcal{F}$ to be the following set of functions:

$$f \in \mathcal{F} \iff \begin{cases} \operatorname{dom} f \subset \mathscr{P}(X) \\ \operatorname{Im} f \subset X \\ (\forall A \in \operatorname{dom} f)(f(A) \in A) \end{cases}$$

Next, consider the relation $\leq$, in X, given by the following: for all $f_1, f_2 \in \mathcal{F}$, $f_1 \leq f_2$ iff f_2 is an extension of f_1 i.e. $f_1 \subset f_2$. As the comments on page 65 point out, $\leq$ is a partial ordering. We note that $(F, \leq)$ is a partially ordered set. Assuming *Zorn's Lemma*, we show that there exists some maximal element $f \in \mathcal{F}$:

$\mathcal{F}$ is already partially ordered. So, let $\mathscr{X}$ be a chain in $\mathcal{F}$. Note that $\mathscr{X}$ is a collection of ordered pairs, satisfying the above criterion. We claim that $\bigcup \mathscr{X}$ is the upper bound for $\mathscr{X}$; we only need to show that $\bigcup \mathscr{X}$ is a function:

- *Function*: Suppose that $(a, x), (a, y) \in \bigcup \mathscr{X}$, then $(a, x) \in f$ for some $f \in \mathscr{X}$ and, likewise, $(a, y) \in g$ for some $g \in \mathscr{X}$. Then, since $\mathscr{X}$ is a chain, either $f \leq g$, $f = g$, $g \leq f$. Without loss of generality, we have $(a, x), (a, y) \in g$. Since g is a function, we have $x = y$. The fact that $\bigcup \mathscr{X}$ is every-where defined is obvious.

- The rest of the properties follow by definition.

By *Zorn's Lemma*, there exists a maximal element in $\mathcal{F}$, call it f. To complete the proof, we show that $\operatorname{dom} f = \mathscr{P}(X) - \{\emptyset\}$:

- Suppose that $\operatorname{dom} f \neq \mathscr{P}(X) - \{\emptyset\}$. Then, there exists some $a \in P(X)$ such that $a \notin \operatorname{dom} f$. Let $g = f \cup \{(a, b)\}$, for any $b \in X$. It is clear that $g \in \mathcal{F}$ and that $g \geq f$. However, this contradicts the minimality of f.

We note that using the *Axiom of Choice* to prove *Zorn's Lemma* was shown in text. This completes the proof.

Exercise 16.2. *We prove that the following is equivalent to the axiom of choice: Every partially ordered set has a maximal chain — This is the Hausdorff Maximality Principal (HMP).*

Proof. We show that *HMP* is equivalent to *Zorn's Lemma* and use the fact that *Zorn's Lemma* is equivalent to the *Axiom of Choice* to conclude.

- (⇒) Suppose the *HMP*. Let $(X, \leq)$ be the partially ordered set, and A the maximal chain. By the assumptions of *Zorn's Lemma*, A has an upper-bound, call it a. Now, $a \in A$, since otherwise, we could form $B = A \cup \{a\}$, which is again a totally ordered set, containing A; for all $s \in B$, the elements of A are comparably, likewise, $s \leq a$. To complete the proof in this direction, we note that if $z \in X$ is any element such that $z \geq a$, then z is an upper-bound and must be in A, by the previous discussion. This implies that $a \geq z$ for all $z \in X$. Specifically, since A is partially ordered, we have $a \leq z$ and $z \leq a$ implying $a = z$. Thus, a is maximal.

- ($\Leftarrow$) Suppose *Zorn's Lemma.* Let $(X, \leq)$ be a partially ordered set. Consider $\mathcal{F}$ be the set of all totally ordered subsets of X. The inclusion order on $\mathcal{F}$ is a partial order. Note that if S is a maximal element in $\mathcal{F}$, then it will be a maximal chain in X. We show that any chain A in $\mathcal{F}$ has an upper-bound:
  - (The following was suggested by this example): Indeed, consider $A' = \bigcup_{E \in A} E$. By definition, A' contains all elements of A. We only show that $A' \in \mathcal{F}$: To show this take x and y, elements of A'. Then there are sets E_x and E_y, members of A, with $x \in E_x$ and $y \in E_y$. Since A is partially ordered by inclusion, $E_x \subset E_y$ or $E_y \subset E_x$, that is, either $x, y \in E_y$ or $x, y \in E_x$. Since both E_x and E_y are linearly ordered in X then either $x \leq y$ or $y \leq x$. This shows that A' is a totally ordered set, and so a chain in X i.e. $A' \in \mathcal{F}$.

This completes the proof.

Exercise 16.3. *We show that the following statements are equivalent to the axiom of choice: (i) Every partially ordered set has a maximal chain — This is the Hausdorff Maximality Principal (ii) Every chain in a partially ordered set is included in some maximal chain. (iii) Every partially ordered set in which each chain has a least upper bound has a maximal element.*

Proof. To do so, we show that $i \Rightarrow ii \Rightarrow iii \Rightarrow ZL \Rightarrow i$, noting that *Zorn's Lemma* is equivalent to the axiom of choice. Throughout these exercises, we assume that $(X, \leq)$ is the partially ordered set in discussion.

- (i $\Rightarrow$ ii) Suppose that every partially ordered set has a maximal chain. Since $(X, \leq)$ is partially ordered, there exists some maximal chain $A \subset X$. We note that for chains, inclusion is the partial order. Thus, by definition, for all other chains $A' \subset X$, we must have $A' \subset A$, since otherwise, $A \subsetneq A'$, contradicting the maximality of A.

- (ii $\Rightarrow$ iii) Suppose that every chain in a partially ordered set is included in some maximal chain. Let $(X, \leq)$ be such a partially ordered set. Then, there exists some maximal chain A such that $A' \subset A$ for all other chains $A' \subset X$. We show that each chain A' of X has a least upper bound:
  - Let G be the set of all upper-bounds of some chain A'. Then, we have have $A' \leq g$ for all $g \in G$. Likewise $G \cap A' = A'$ implying A' is the least upper-bound of A'.

From the antecedent in (iii), we see that A has a maximal element in A, call it a. We also note that a is maximal in X: If not, then we can define $T = A \cup \{a\}$, which is a totally ordered set in which $A \subset T$; contradicting the maximality of A.

- (iii $\Rightarrow$ *Zorn's Lemma*) Suppose that every partially ordered set in which each chain has a least upper bound has a maximal element. Let X be such a set. Then for any chain $A \subset X$, A has a l.u.b. call it δ_A. In particular, δ_A is an upper-bound. By assumption X has a maximal element.
- (*Zorn's Lemma* $\Rightarrow$ i) This was proved in the previous exercise.

17 Well Ordering

Exercise 17.1. *A subset A of a partially ordered set X is cofinal in X in case for each element $x \in X$, there exists an element $a \in A$ such that $x \leq a$. We prove that every totally ordered set has a cofinal well ordered subset.*

Proof. Let $(X, \leq)$ be a non-empty totally ordered set. If X has a maximum element, then we are done. Thus, suppose that X has no maximum. Let $\mathcal{F}$ be the set of all well ordered subsets of X with the following order: for all $f, g \in \mathcal{F}$, $f \preceq g$ iff f is an initial segment of g. We seek to apply *Zorn's Lemma*.

- From the previous result, $\preceq$ is a partial order on $\mathcal{F}$.
- Suppose that Z is chain in $\mathcal{F}$. We show that Z has an upper bound in $\mathcal{F}$. Consider $\bigcup Z$. It is clear that $Z \preceq \bigcup Z$. We show that $\bigcup Z$ is well ordered; Indeed, these are the comments on page 67.

We have satisfied the assumptions in *Zorn's Lemma*, and so it follows that there exists some maximal element $M \in \mathcal{F}$. We claim that M is a well ordered cofinal subset of X.

First, it is a subset. Second, it is well ordered by the definition of $\mathcal{F}$. Next, if $x \in X$, then for any $m \in M$, either $m \leq x$, $x \leq m$ or $x = m$. But, since M is maximal, if $m \leq x$, then $x \in M$; otherwise, we could form $N = M \cup \{x\}$ which is a contradiction to the maximality of M. Thus, M is the well ordered cofinal subset of X.

Exercise 17.2. *Does any such condition apply to partially ordered sets?*

Proof.

Exercise 17.3. *We prove that a totally ordered set is well ordered iff the set of strict predecessor of each element is well ordered.*

Proof.
($\Rightarrow$) Suppose that we have a well ordered set $(X, \leq)$. Note that the set of strict predecessors of some $x \in X$ forms a subset. The rest follows from the definition.
($\Leftarrow$) Suppose that $(X, \leq)$ is a partially ordered set and that for any $x \in X$ the strict initial segment $s(x)$ is well ordered. Let J be a subset of X. Pick some $j_0 \in J$ and consider $s(j_0) = \{x \in X : x < j_0\}$. If $s(j_0) = \emptyset$, j_0 is the least element, and we are done. Otherwise, $s(j_0) \neq \emptyset$. But since $s(j_0)$ is well ordered, we have $J \cap s(j_0) \subset s(j_0)$ and $J \cap s(j_0) \subset J$. Implying that $J \cap s(j_0)$ has a least element in J, call it z. We note that z is the least element for J; Otherwise, there is some $x \in J$, $x \neq z$ such that $x < z < j$ for all $j \in J$ and in particular for j_0. This implies that $x \in s(j_0)$. However, then we must have $x \in J \cap s(j_0)$ and so $x = z$; a contradiction.

<u>Exercise</u> 17.4. *We prove that the well ordering theorem implies the Axiom of Choice.*

Proof. The following proof was adapted from this website:

Let C be a collection of nonempty sets. Then $\bigcup_{S\in C} S$ is a set. By the well-ordering principle, $\bigcup_{S\in C} S$ is well-ordered under some relation $<$. Since each S is a nonempty subset of $\bigcup_{S\in C} S$, each S has a least member m_S with respect to the relation $<$. Define $f : C \to \bigcup_{S\in C} S$ by $f(S) = m_S$. We show that f is a choice function:

- *Well-Defined*: Suppose that $(a, x), (a, y) \in f$. Then, there exists some $S_1 \in C$ such that $x \in S_1$. Likewise, there exists some $S_2 \in C$ such that $y \in S_2$. Without loss of generality, it follows that $x, y \in S_2$, and so $x = y = m_{S_2}$.
- *Everywhere Defined*: This follows fromt he discussion above.

Therefore, f is a choice function. Hence, the axiom of choice holds.

<u>Exercise</u> 17.5. *We prove that if R is a partial order in a set X, then there exists a total order S in X such that $R \subset S$; in other words, every partial order can be extended to a total order without enlarging the domain.*

Proof. Let $(R, \leq)$ be a partially ordered subset of X. Let C be the set of all A in R of partial orders, with respect to $\leq$. Let $(C, \subset)$ be the partial ordered set[6]. Let $\mathscr{X}$ be the collection of all chains in C. Note that for all $y \in \mathscr{X}$ $y \leq \bigcup_{y\in\mathscr{X}} y$. Thus, every chain has an upper-bound. By *Zorn's Lemma*, C has a maximal element. Denote this by c. Claim: c is a total order in X:

- Since $c \in C$, c is a partial order in X. Since it is maximal, for all $c' \in C$, we have $c' \subset c$. Thus, for all $x, y \in X$, either $(x, y), (y, x) \in c$. Either way, we must have $x \leq y$, $y \leq x$ or $x = y$.

[6]This has been shown other exercises.

18 Transfinite Recursion

Exercise 18.1. *We prove that a similarity preserves $\prec$ (in the same sense in which the definition demands the preservation of $\leq$).*

Proof. Let $(X, \preceq)$, $(Y, \leq)$ be two partially ordered sets and that there exists some function $f : X \to Y$, 1-1 and onto. f has the property that for all $a, b \in X$ with $a \preceq b$ iff $f(a) \leq f(b)$, for all $f(a), f(b) \in Y$.

Suppose, to the contrary, that $a \prec b$ and $f(a) \geq f(b)$. However, this is a contradiction, since, by definition, we must have $b \preceq a$. Thus f preserves strict order.

Exercise 18.2. *We prove the following statement: A 1-1 function, f that maps one partially ordered set onto another is a similarity iff f preserves $<$ (strict order).*

Proof. Throughout this exercise, we assume that $(X, \preceq)$ and $(Y, \leq)$ are partially ordered sets and $f : X \to Y$ is 1-1 and onto.

- ($\Rightarrow$): Suppose that f is a similarity, then the previous exercise prove this conclusion.
- ($\Leftarrow$): Suppose that f is as stated above, such that for any $a, b \in X$ we have $a \prec b$ iff $f(a) < f(b)$. To show that f is a similarity, we show that $a = b$ iff $f(a) = f(b)$:
  - ($\Leftarrow$): Since f is 1-1, then $f(a) = f(b)$ implies $a = b$.
  - ($\Rightarrow$): Suppose, to the contrary, that $a = b$ and, (without loss generality) $f(a) < f(b)$. Since f preserves strict order, we must have $a \prec b$. However, this is a contradiction.

This completes the proof.

Exercise 18.3. *We show that a subset Y of a well ordered set X is, again, well ordered.*

Proof. Let $(X, \leq)$ be a well-ordered set, and $Y \subset X$. If $Y = X$ or $Y = \emptyset$, then Y is clearly a well-order. Thus, suppose that Y is proper subset of X. Suppose that Y was *not* well-ordered. Then, some subset, Z of Y does not have a least element. However, $Z \subset X$, and so by definition of a well-ordered set, it has a least element. A contradiction.

Exercise 18.4. *We prove that each subset of a well-ordered set X is similar either to X, or an initial segment of X.*

Proof. Suppose that $(X, \preceq)$ is a well-ordered set. If $X = \emptyset$, then every subset of X is equivalent to X. So, suppose $X \neq \emptyset$, and let S be a non-empty subset of X. Since $X \subset X$, X has a least element, call it x. Likewise, S has a least element, s. Consider the map f given by $f(s) = x$. And then, since $X - \{x\} \subset X$ it has a least element, call it x'. Likewise, $S - \{s\}$ has a least element, call it s' and define $f(s') = x'$, and so on. We show that f is a sequence of type S in X. Now, either this process terminates, or it does not. If it does, then it terminates at some x^*, s^* i.e. $f(s^*) = x^*$. Claim: $S \cong s(x^*)$.

- The fact that f is a function is clear by previous definition. It only need to be shown that f is 1-1, onto and preserves $\prec$. The fact that f is 1-1 is clear. Thus, if $\bar{x} \prec x^*$, then by construction, $s(\bar{x}) = f(t)$ for some $t \in S$.
- f preserves $\prec$: This follows from the definition of initial segments give on page 65, 78.

Otherwise, the process does not terminate. In this case, the claim is that $S \cong X$. The fact that f is a 1-1 and onto function is shown similarly as before. Likewise for $\prec$.

Exercise 18.5. *We prove that if $(X, \prec)$, $(Y, \leq)$ are well ordered sets, and $X \cong Y$, then the similarity maps the least upper bound (if any) of each subset of X onto the least upper bound of the image of that subset.*

Proof. Let X, Y be stated as above. And suppose that $S \subset X$ has a least upper bound, s^*. This implies that for all $s \in S$, $s \leq s^*$ and for any other upper bound $x \in X$, $x \leq s^*$ implies $x = s^*$. Suppose, to the contrary, that $f(s^*)$ was not the least upper-bound of $f(S)$. Since f is order preserving, we have that $f(s^*)$ is an upper-bound for $f(S)$. Let $\alpha \neq f(s^*)$ be the l.u.b. of $f(S)$ (since f is onto there exists some $x \in X$ such that $\alpha = f(x)$. Thus from here, $\alpha = f(x)$); for all $f(s) \in f(S)$, we have $f(s) \leq f(x)$, and that for any other upper-bound z of $f(S)$, $z \leq f(x)$ implies $z = f(x)$, in particular, this implies that $s^* \geq x$. But then, since f is order-preserving, we have that $f(s) \leq f(x)$ implies that $s \leq x$ for all $s \in S$. But since s^* is the l.u.b. for S, we must have $s^* \leq x$. Since X, and consequently S is well-ordered, it is partially ordered, and so $s^* = x$, a contradiciton.

19 Ordinal Numbers

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20 Sets Of Ordinal Numbers

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21 Ordinal Arithmetic

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22 The Schröder-Bernstein Theorem

Exercise 22.1. *Let X, and Y be sets. Suppose that $f : X \to Y$, is a function, and $g : Y \to X$, is a function. We prove that there exist subsets $A \subset X$, $B \subset Y$ such that $f(A) = B$, and $g(Y - B) = X - A$. (This statement is used to prove the Schröder-Bernstein theorem.)*

Proof. Let X, Y, f and g be stated as above. Throughout this exercise, we will use $f[A]$ instead of the more common, yet less accurate, $f(A)$. We look at four cases:

- f onto, and g is not: In this case, we have $f[X] = Y$, and $g[Y - f(X)] = g[Y - Y] = g[\emptyset] = \{\} = X - X$. Thus, choosing $A = X \subset X$ and $B = \{\} \subset Y$, the result follows.

- g onto, and f is not: In this case, we have $f[\emptyset] = \emptyset$, and $g[Y - \emptyset] = g(Y) = X = X - \{\}$. Thus, choosing $A = \{\} \subset X$, and $B = \{\} \subset Y$, the result follows.

- f onto, g onto: Again, in this case, letting $A = \emptyset \subset X$ we have $f[\emptyset] = \emptyset$, and $g[Y - \emptyset] = g[Y] = X = X - \emptyset$; the result follows.

- f not onto, g not onto: We point out that since neither f nor g is onto, there exists some $\bar{x} \in X$ such that the fiber of $\bar{x}$ under g is empty. Likewise, there exists some $\bar{y} \in Y$ such that the fiber of $\bar{y}$ under f is empty. Let $F_{\{\}}$ be the set of all such $\bar{x} \in X$.

  For any $x \in X$ such that the fiber of x under g is not empty, either $g^{-1}[x] = \{f(x)\}$ or $f(x) \in g^{-1}[x]$, and $g^{-1}[x] \neq \{f(x)\}$. In the first case, let $F_{\{f(x)\}}$ be the set of all $x \in X$ such that $g^{-1}[x] = \{f(x)\}$ and then let F_x be the set of all $x \in X$ for which $f(x) \in g^{-1}[x]$, and $g^{-1}[x] \neq \{f(x)\}$. From the section on relations, we have that X is partitioned by $F_{\{\}}$, F_x and $F_{\{f(x)\}}$. That is, these sets are pairwise disjoint, and their union is X.

  These sets have some interesting properties. Namely $g[Y - f[F_{\{\}}]] = g[f[F_x] \cup f[F_{\{f(x)\}}]]$. This is from the fact that $F_{\{\}}$, F_x, and $F_{\{f(x)\}}$ partition X and that if g is not mapping from to the set $F_{\{\}}$, then it must map to the other two. Likewise, $g[Y - f[F_{\{f(x)\}}]] = g[f[F_x] \cup f[F_{\{\}}]]$.

  The claim is that the subset A of X, so desired is, $F_{\{\}} \cup F_{\{f(x)\}}$. We compute:

$$
\begin{aligned}
g\Big[Y - (f[F_{\{\}}] \cup f[F_{\{f(x)\}}])\Big] &= g\Big[Y - f[F_{\{\}}]\Big] \cup g\Big[Y - f[F_{\{f(x)\}}]\Big] \\
&= g\Big[f[F_x] \cup f[F_{\{f(x)\}}]\Big] \cup g\Big[f[F_x] \cup f[F_{\{\}}]\Big] \\
&= g\Big[f[F_x]\Big] \cup g\Big[f[F_{\{\}}]\Big] \\
&= g\Big[f[F_x]\Big] \cup \emptyset \qquad (1) \\
&= g\Big[f[F_x]\Big] \\
&= F_x
\end{aligned}
$$

Where the second line follows form the comments above, and (1) is from the construction of F. Indeed, from the comments above, $X - (F_{\{\}} \cup F_{\{f(x)\}}) = F_x$.

Exercise 22.2. *We prove the Schröder-Bernstein theorem using the above exercise.*

Proof. Suppose that $X \precsim Y$, and $Y \precsim X$. Then, there exist some $f : X \to Y$ such that f is 1-1 and onto a subset Y' of Y. Likewise, there exists some $g : Y \to X$ which is 1-1 and onto a subset X' of X. We note that if either $X' = X$ or $Y' = Y$, then we are done. So, suppose that $X' \neq X$ and $Y' \neq Y$. Likewise, if either X' or Y' was the empty set, then we are done, since it is trivially true that $X \sim Y$.So, suppose that $X' \subsetneq X$, such that $X' \neq \emptyset$, and $Y' \subsetneq Y$ and $Y' \neq Y$.

So, consider $f|_{X-X'}$ and $g|_{Y-Y'}$. From the previous theorem, there exists a subset $X'' \subset X - X'$ and some $Y'' \subset Y - Y'$ such that $f|_{X-X'}[X''] = Y''$ and $g|_{Y-Y'}[(Y - Y') - Y''] = (X - X') - X''$. We claim that $\emptyset = (X - X') - X''$ i.e. $X'' = X - X'$; if not, then $g|_{Y-Y'}[(Y - Y') - Y''] \neq \emptyset$ and so there is some $g(y) \in X'$ such that $y \in (Y - Y') - Y''$. But then, we have $f(g(y)) \in Y' \not\subset (Y - Y') - Y''$; this implies that $y \in Y'$, a contradiction. Thus, $X'' = X - X'$.

Since this is the case, $f|_{X-X'}[X''] \subset Y'$. But then, $g|_{Y-Y'}[(Y - Y') - f|_{X-X'}[X'']] = g|_{Y-Y'}[Y - Y'] = (X - X') - X'' = \emptyset$. This implies that $Y - Y' = \emptyset$, or in other words, $Y = Y'$. A contradiction.

23 Countable Sets

Exercise 23.1. *We prove that the set of all finite subsets of a countable set is countable.*

Proof. The following proof was suggested by this website:

Let $A(n)$ be the set of subsets of A with no more than n elements. Thus, $A(0) = \{a\}$, $A(1) = A(0) \cup \{\{a\} : a \in A\}$ and for all $n \geq 0$:

$$A(n+1) = \{a(n) \cup a(1) : a(n) \in A(n) \wedge a(1) \in A(1)\}$$

We verify, by induction, that each $A(n)$ is countable. Let A be countable. Then, $A(1)$ is countable, by construction. Suppose $A(n)$ is countable. Then by since the union of countable sets is countable, $A(n+1)$ is also countable.

Denote with A_f as the set of finite subsets of A. It is apparent that every finite subset is in some $A(n)$, and so:

$$A_f = \bigcup_{n \in \mathbb{N}} A(n)$$

And since the union of countable subsets is countable, A_f is countable.

Exercise 23.2. *We prove that if every countable subset of a totally ordered set X is well ordered, then X itself is well ordered.*

Proof. Let $A \subset X$ be an arbitrary countable subset. Then, A is well-ordered, and every subset a of A is well-ordered, so they have least elements. Let $\bigcup_n A_n$ be the union of all countable subsets of X. Since $\bigcup_n A_n$ is the union of well-ordered sets, it is well-ordered. The claim is that $\bigcup_n A_n = X$:

Suppose otherwise; Since $\bigcup_n A_n \subset X$, it follows that $X = \bigcup_n A_n \cup (X - \bigcup_n A_n) = \bigcup_n A_n \cup Y$. Since $\bigcup_n A_n$ is the union of all countable subsets of X, Y must be finite; otherwise, we have missed a countable subset. But, since Y is finite, Y is included in some countable subset of X. Thus,

$$X = \bigcup_n A_n \cup Y = \bigcup_n A_n$$

This contradiction concludes the proof.

24 Cardinal Arithmetic

Exercise 24.1. *We prove that if a, b, c and d are cardinal numbers such that $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.*

Proof. Let a, b, c, d be stated as given. By definition, there exists sets A, B, C, D (without loss of generality, pairwise disjoint) with cardinality a, b, c, d, respectively. Using the stated conditions, we have $A \precsim B$, $C \precsim D$. Thus, there exists $B' \subset B$, and $D' \subset D$ such that $A \sim B'$ and $C \sim D'$ i.e. $\operatorname{card} A = \operatorname{card} B'$ and $\operatorname{card} C = \operatorname{card} D'$. Then, $a + b = b' + d' \leq b + d$.

Exercise 24.2. *We prove that if a, b, c and d are cardinal numbers such that $a \leq b$ and $c \leq d$, then $ac \leq bd$.*

Proof. We proceed in a fashion similar to exercise 24.1. Let a, b, c, d be stated as given. By definition, there exists sets A, B, C, D (without loss of generality, pairwise disjoint) with cardinality a, b, c, d, respectively. Using the stated conditions, we have $A \precsim B$, $C \precsim D$. Thus, there exists $B' \subset B$, and $D' \subset D$ such that $A \sim B'$ and $C \sim D'$ i.e. $\operatorname{card} A = \operatorname{card} B'$ and $\operatorname{card} C = \operatorname{card} D'$. Thus, $ab = \operatorname{card}(A \times B) = \operatorname{card}(B' \times D')$. From a previous exercise and the comments on page 94, we have $B' \times D' \precsim B \times D$, and so it follows that $\operatorname{card}(B' \times D') \leq \operatorname{card}(B \times D)$.

Exercise 24.3. *We prove that if $\{a_i\}$ $(i \in I)$ and $\{b_i\}$ $(i \in I)$ are families of cardinal numbers such that $a_i < b_i$ for each $i \in I$, then $\sum_i a_i \leq \prod_i b_i$.*

Proof. The following proof was suggested to me by Sophie C..

Note,

$$\sum_i a_i = a_0 + a_1 + \ldots \leq a_0 \times a_1 \times \cdots = \prod_i a_i$$

We proceed by mathematical induction.

- *Initial Case*: $a_0 < b_0$ — By assumption.
- *Hypothesis*: Suppose that $\prod_{0 \leq i \leq n} a_i < \prod_{0 \leq i \leq n} b_i$
- *Step*: $\prod_{0 \leq i \leq n} a_i + a_{n+1} < \prod_{0 \leq i \leq n} b_i \times b_{n+1}$ for $a_{n+1} < b_{n+1}$. Therefore, $\prod_{0 \leq i \leq n+1} a_i < \prod_{0 \leq i \leq n+1} b_i$

From the above, we have $\sum_i a_i \leq \prod_i a_i < \prod_i b_i$. Which completes the proof.

Exercise 24.4. *We prove that if a, b and c are cardinal numbers such that $a \leq b$, then $a^c \leq b^c$.*

Proof. The following proof was suggested by Sophie C.. We take the definition of a^c as multiply a by itself c times. And likewise for b^c. We proceed by mathematical induction:

- *Initial Case*: $a^1 \leq b^1$.
- *Inductive Hypothesis*: Suppose that $\prod_{i \in I} a_i \leq \prod_{i \in I} b_i$ such that $\operatorname{card} I = C$.
- *Step*: If $\prod_{i \in I} a_i \leq \prod_{i \in I} b_i$, then $\prod_{i \in I} a_i \times a \leq \prod_{i \in I} b_i \times b$, since $a \leq b$.

Thus, $\prod_{i \in I} a_i \leq \prod_{i \in I} b_i$ where $\operatorname{card} I = C + 1$. And so, $a^c \leq b^c$

Exercise 24.5. *Prove that if a and b are finite, greater than 1, and if c is infinite, then $a^c = b^c$.*

Proof. The following proof was suggested by Sophie C.. We take the definition of a^c as multiply a by itself c times. And likewise for b^c. If c is countable then, a^c, b^c are countable, and if c is uncountable, then a^c, b^c are uncountable. Thus, $a^c = b^c$

Exercise 24.6. *We prove that if a and b are cardinal numbers at least one of which is infinite, then $a + b = ab$.*

Proof. The following proof was suggest by Sophie C.. Suppose that a, b are given as stated. Without loss of generality, there exists sets A, B such that $\operatorname{card} A = a, \operatorname{card} B = b$, and $A \sim B'$, for some $B' \subset B$. Thus, $a + b = \operatorname{card} A \sim \operatorname{card} B' \cup \operatorname{card} B = \operatorname{card}(A \cup B) \sim \operatorname{card}(B' \cup B) = \operatorname{card} B = b = ab$. Therefore, $a + b = ab$.

Suppose that a and b are both infinite, then as above, we have $a+b = a+a = a$ and $ab = aa = a$. Therefore, $a + b = ab$.

Suppose, without loss of generality, that a, and b are both infinite such that $\operatorname{card} A < \operatorname{card} B$. Then, $a + b = b$ and $ab = b$, thus, $a + b = ab$.

Exercise 24.7. *We prove that if a and b are cardinal numbers such that a is infinite and b is finite, then $a^b = a$.*

Proof. The following proof was suggested by Sophie C.. Suppose that a, b are given, as above. Then, $a^b = \prod_{i \in I} a_i$ where $\operatorname{card} I = b$ and where $a_i = a$, for all $i \in I$. Then, we have the following:

$$
\begin{aligned}
a^b &= \operatorname{card}(A \times A \times \ldots \times A) \quad \text{where } \operatorname{card} A = a \\
&= a \times a \times \ldots \times a \quad b \text{ times} \\
&= a \times a \times \ldots \times a \quad b/2 \text{ times, since } a \times a = a \\
&= a \times a \times \ldots \times a \quad b/4 \text{ times, since } a \times a = a \\
&\vdots \\
&= a
\end{aligned}
$$

This completes the proof.

25 Cardinal Numbers

Exercise 25.1. *We prove that each infinite cardinal number is a limit number.*

Proof. This proof was inspired from this website.

Suppose to the contrary that k is an infinite cardinal which was not a limit number. Then, $k = a^+$ for some ordinal a^+. If a is finite, we are done. Otherwise, we proceed to show that $k = a$.

Not that since k is infinite, $\omega \precsim k$, and in particular, $\omega \precsim a$. Thus, consider $\phi : a \to k$ given by $\phi(0) = a$, and $\phi(m^+) = m$ (if $m \in \omega$), and $\phi(x) = x$ (if $x \in a - \omega$). We show that ϕ is a 1-1 and onto function:

- *Well-Defined*: Since no ordinal is equivalent to its successor, it follows that ϕ is well-defined.
- *Everywhere Defined*: By construction, this is clear.
- *1-1*: If $\phi(x) = \phi(y)$, then in any of the cases, we have $x = y$.
- *Onto*: Let $k_1 \in k$. Then, either $k_1 \in a$, or $k_1 = a$. In either case, there exists a $\ell \in a$ for which $\phi(\ell) = a$.

Thus $k = a$, a contradiction.

Exercise 25.2. *We answer a few questions: If* $\operatorname{card} A = a$*, what is the cardinal number of the set of all 1-1 and onto in* A*? What is the cardinal number of the set of all countably infinite subsets of* A*?*

Proof. If A is infinite, then $\omega \precsim A$ and we can consider the infinitely manly maps f_i, such that $i \in \omega$, given by $f_i(x) = x + 1$ (if $x \in \omega$), and $f_i(x) = x$ (if $x \in A - \omega$). However, if A is finite, then a simple counting argument show that the number is the multiplication of all predecessors of a, except zero.

This answer was suggested by this website: "A has a^ω countably infinite subsets, and there's not much more that you can say unless you know something about the cardinal a. For example, $2 \leq a \leq 2^\omega$, then $a^\omega = 2^\omega$. If cf $a = \omega$, i.e., if a has cofinality ω, then $a^\omega > a$."

Exercise 25.3. *We prove that if* α, β *are ordinal numbers, then* $\operatorname{card}(\alpha + \beta) = \operatorname{card} \alpha + \operatorname{card} \beta$ *and* $\operatorname{card} \alpha\beta = \operatorname{card} \alpha \operatorname{card} \beta$*. We use the ordinal interpretation of the operations on the left side and the cardinal interpretation on the right.*

Proof. The following proof was suggested by Sophie C.. We compute:

- $\operatorname{card} \alpha\beta = \operatorname{card}(\operatorname{ord} A \operatorname{ord} B) = \operatorname{card}(\operatorname{ord}(A \times B)) = \operatorname{card}(A \times B)$
- $(\operatorname{card} \alpha)(\operatorname{card} \beta) = (\operatorname{card}(\operatorname{ord} A))(\operatorname{card}(\operatorname{ord} B)) =$ $\operatorname{card}(A) \operatorname{card}(B) = \operatorname{card}(A \times B)$

Thus, $\operatorname{card} \alpha\beta = (\operatorname{card} \alpha)(\operatorname{card} \beta)$

Proof. The following proof was suggest by Sophie C.. We compute:

- $\operatorname{card}(\alpha + \beta) = \operatorname{card}(\operatorname{ord} A + \operatorname{ord} B) = \operatorname{card}(\operatorname{ord}(A \cup B)) = \operatorname{ord}(A \cup B) = \operatorname{ord}(A) + \operatorname{ord}(B) = \operatorname{card}(\operatorname{ord}(A)) + \operatorname{card}(\operatorname{ord}(B)) = \operatorname{card} \alpha + \operatorname{card} \beta$.

Thus, $\operatorname{card}(\alpha + \beta) = \operatorname{card} \alpha + \operatorname{card} \beta$.

26 Extra Exercises

Many of these exercises were collaborations with other student. They are not marked exercises in the text, however, they are useful.

26.1 Arithmetic

<u>Exercise</u> 26.1. *We show that addition for elements of ω is associative. The following proof was suggested by Aidan H. and David H., and I attribute all credit to them.*

Proof. We proceed by mathematical induction: By the recursion theorem there exists a function $s_m : \omega \to \omega$ such that $s_m(0) = m$ and $s_m(n^+) = (s_m(n))^+$ where $m, n \in \omega$ and $s_m(n) = m + n$.First note $s_{k+m}(n) = (k+m)+n$, and similarly $s_k(m+n) = k + (m+n)$. Consider the set $T = \{n \in \omega \mid s_{k+m}(n) = s_k(m+n)\}$.

Initial case: See that $0 \in T$ since $s_{k+m}(0) = k + m = s_k(m+0) = k + (m+0) = k + m$.

Inductive hypothesis: Let $n \in T$ hold. Then $s_{k+m}(n) = s_k(m+n)$.
This implies $(k+m)+n = k+(m+n)$.
Now $s_{k+m}(n^+) = (k+m)+n^+ = (s_{k+m}(n))^+ = ((k+m)+n)^+$.
Similarly, $s_k((m+n)^+) = k+(m+n)^+ = (s_k(m+n))^+ = (k+(m+n))^+$.

Since $s_{k+m}(n) = s_k(m+n)$ by the inductive hypothesis. We obtain

$$(k+m)+n^+ = ((k+m)+n)^+ = (k+(m+n))^+ = k+(m+n)^+$$

Note that $s_m(n) = m+n$, so $(s_m(n))^+ = (m+n)^+ = m+n^+ = s_m(n^+)$.
Thus, $(k+m)+n^+ = k+(m+n^+)$

Conclusion: Therefore, $n^+ \in T$, and thus $T = \omega$, and thus addition is associative.

<u>Exercise</u> 26.2. *We prove that addition is commutative.*

Proof. The following proof was suggested by Aidan H. and David H., and I attribute all credit to them.

Let $B = \{m \in \omega \mid s_m(n) = s_n(m)\}$.
Initial case: $0 \in B$, since $s_0(n) = s_n(0) = n$.
Inductive hypothesis: Let $b \in B$ hold. That is, $s_b(n) = b+n = n+b = s_n(b)$. Then $s_n(b^+) = (s_n(b))^+ = (n+b)^+$, by definition. Similarly, $s_{b^+}(n) = (s_b(n))^+ = (b+n)^+$. Then, by the inductive hypothesis, $s_{b^+}(n) = (s_b(n))^+ = (s_n(b))^+ = s_n(b^+)$.
Inductive conclusion: $b^+ \in B$. Thus, by the induction principle, $B = \omega$. This

implies $s_m(n) = s_n(m)\ \forall m, n \in \omega$, and thus $m + n = n + m\ \forall m, n \in \omega$, and thus addition is commutative.

Exercise 26.3. *We prove the properties of exponentiation.*

Proof. The following proof was suggested by Aidan H. and David H., and I attribute all credit to them.

First note $e_m : \omega \to \omega$ is a function defined as

$$e_m(0) = 1$$

$$e_m(n^+) = e_m(n) \cdot m \qquad \text{where } e_m(n) = m^n \text{ for } m, n \in \omega$$

I want to show $m^{(n+l)} = m^n \cdot m^l$, that is $e_m(n+l) = e_m(n) \cdot e_m(l)$ for $m, n, l \in \omega$. Define $P_m : \omega \to \omega$ such that $P_m(0) = 0$ and $P_m(n^+) = P_m(n) + m$, and $P_m(n) = m \cdot n$ for $m, n \in \omega$. First I will show $P_1(n) = n\ \forall n \in \omega$ by induction. Consider the set $B = \{n \in \omega \mid P_1(n) = n\}$. Initial case: Note $0 \in B$ since $P_1(0) = 0$ by definition.
Inductive hypothesis: Let $n \in B$ hold, then $P_1(n) = n$. Then $P_1(n^+) = P_1(n) + 1 = n + 1 = n^+$.
Inductive conclusion: Thus $n^+ \in B$ and by principle of mathematical induction, $B = \omega$. Thus, $P_1(n) = n\ \forall n \in \omega$.
Now to show $m^{(n+l)} = m^n \cdot m^l$. Let $F = \{n \in \omega \mid e_m(n + l) = e_m(n) \cdot e_m(l)\}$.
Initial case: Note $0 \in F$ since $e_m(0 + l) = e_m(l)$ and $e_m(0) \cdot e_m(l) = 1 \cdot e_m(l) = P_1(e_m(l)) = e_m(l)$.
Inductive hypothesis: Let $n \in F$ hold, then $e_m(n + l) = e_m(n) \cdot e_m(l)$. Then

$$\begin{aligned} e_m(n^+ + l) &= e_m((n + l)^+) \\ &= e_m(n + l) \cdot m \\ &= e_m(n) \cdot e_m(l) \cdot m \\ &= e_m(n) \cdot m \cdot e_m(l) \\ &= e_m(n^+) \cdot e_m(l) \end{aligned}$$

Inductive conclusion: Thus, $n^+ \in F$, and by the principle of mathematical induction, $F = \omega$. Thus, $m^{(n+l)} = m^n \cdot m^l\ \forall m, n, l \in \omega$.

Exercise 26.4. *We prove the left distributive law.*

Proof. The following proof was suggested by Aidan H. and David H., and I attribute all credit to them.

We proceed by mathematical induction:

Initial case:

$$k \cdot (m + 0) = k \cdot m + k \cdot 0$$

$$k \cdot m = k \cdot m + 0 = k \cdot m$$

Inductive hypothesis: $k \cdot (mn) = k \cdot m + k \cdot n$ Then

$$k \cdot (m + n^+) = (k \cdot (m + n))^+ \text{ by definition of addition}$$

$$= (k \cdot m + k \cdot n)^+ = k \cdot m + (k \cdot n)^+ \text{ by hypothesis and addition}$$

$$= k \cdot m + k \cdot n^+ \text{ by definition of addition}$$

$$\text{Conclusion: } k \cdot (m + n^+) = k \cdot m + k \cdot n^+$$