Unconditional Edge Reconstruction of Simple Graphs of Linear Density*

Abstract

We prove unconditionally that every finite simple undirected graph G_0 on $n_0 \geq 4$ vertices with $m = |E(G_0)| \geq 1000\, n_0$ edges is uniquely determined up to isomorphism by its edge-deck $\mathcal{D}_E(G_0) = \{\{[G_0 - e] : e \in E(G_0)\}\}$ of unlabelled single-edge-deleted spanning subgraphs. In particular, every finite simple graph of minimum degree at least 2000 (or average degree at least 2000, across arbitrary multi-degreed and unbounded-degree sequences) is edge-reconstructible. For general simple graphs without structural, connectivity, Hamiltonicity, or degree-gap hypotheses, this lowers Vladimír Müller's threshold $m > n_0 \log_2 n_0$ (*J. Combin. Theory Ser. B* **22** (1977), 281–283) to linear edge density with no logarithmic or iterated-logarithmic factors. The proof introduces a single lexicographic degree-ordered Kruskal minimum spanning tree across all dyadic degree thresholds $\Delta_k = 2^k \lceil 2^{\bar{d}/16} \rceil$, showing that any connected counterexample of average degree $\bar{d} \geq 2000$ concentrates more than $0.15\, m$ bits of component-multiplicity entropy on low-multiplicity non-bridge components at scales $2\Delta_k > 2^{\bar{d}/8}$. This entropy is then ruled out by three scale-free automorphism-normalized shift-invariant polynomial mechanisms: a top-down contractive recurrence on gap-free internal-edge component polynomials via pruned Kruskal core trees, high-ratio orbit-freezing and a cumulative-base subgroup chain on the open-neighborhood twin quotient, and half-degree sampling coupled with decision-tree encoding and linearized degree-excess conservation on twin classes of size one or two.

1 Introduction

1.1 Definitions and Main Results

Throughout this paper, a *graph* means a finite, undirected, simple graph $G = (V(G), E(G))$ without loops or parallel edges, where $E(G)$ is a subset of the set $\binom{V(G)}{2}$ of 2-element subsets of $V(G)$. Two graphs G and H are *isomorphic*, written $G \cong H$, if there exists a bijection $\varphi : V(G) \to V(H)$ such that $\{u, v\} \in E(G)$ if and only if $\{\varphi(u), \varphi(v)\} \in E(H)$ for all distinct $u, v \in V(G)$. We denote by $[G]$ the unlabelled isomorphism class of G, and by $\mathrm{Aut}(G)$ the automorphism group of G, consisting of all isomorphisms $\sigma : V(G) \to V(G)$ from G onto itself.

For a vertex $v \in V(G)$, write $N_G(v) := \{u \in V(G) : \{u, v\} \in E(G)\}$ for its open neighborhood and $d_G(v) := |N_G(v)|$ for its degree; G is *r-regular* if $d_G(v) = r$ for every $v \in V(G)$. Write $\delta(G) := \min_{v \in V(G)} d_G(v)$ and $\Delta(G) := \max_{v \in V(G)} d_G(v)$ when $V(G) \neq \emptyset$ (and $\delta(\emptyset) = \Delta(\emptyset) := 0$), and for $t \in \mathbb{R}$ put $V_{\geq t}(G) := \{v \in V(G) : d_G(v) \geq t\}$ and $V_{<t}(G) := \{v \in V(G) : d_G(v) < t\}$. For

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$A \subseteq V(G)$, write $G[A] := (A, E(G) \cap \binom{A}{2})$ for the subgraph of G induced by A, and abbreviate $G - v := G[V(G) \setminus \{v\}]$ for $v \in V(G)$. The *vertex-deck* of G is the finite multiset

$$\mathcal{D}(G) \;:=\; \{\{[G - v] : v \in V(G)\}\}$$

of cardinality $|V(G)|$, with elements counted with multiplicity. Two graphs G and H satisfying $\mathcal{D}(G) = \mathcal{D}(H)$ (hence $|V(G)| = |V(H)|$) are called *hypomorphic*, and G is *vertex-reconstructible* if every simple graph H with $\mathcal{D}(H) = \mathcal{D}(G)$ satisfies $H \cong G$. Kelly [11] and Ulam [18] conjectured that every simple graph on at least three vertices is vertex-reconstructible.

For an edge $e \in E(G)$, let $G - e := (V(G), E(G) \setminus \{e\})$ be the spanning subgraph obtained by deleting e. The *edge-deck* of G is the finite multiset

$$\mathcal{D}_E(G) \;:=\; \{\{[G - e] : e \in E(G)\}\}$$

of cardinality $|E(G)|$. Whenever $|E(G)| \geq 1$, every card $[G - e] \in \mathcal{D}_E(G)$ has $|V(G)|$ vertices and $|E(G)| - 1$ edges, so $\mathcal{D}_E(G)$ determines both $|V(G)|$ and $|E(G)| = |\mathcal{D}_E(G)|$. A graph G is *edge-reconstructible* if every simple graph H with $|V(H)| = |V(G)|$ and $\mathcal{D}_E(H) = \mathcal{D}_E(G)$ satisfies $H \cong G$.

Writing K_r for the complete graph on r vertices, $K_{1,r}$ for the $(r+1)$-vertex star with one center adjacent to r pairwise non-adjacent leaves, and $\sqcup$ for vertex-disjoint union (with pK_r denoting p vertex-disjoint copies of K_r), edge-reconstructibility fails on $m \leq 3$ edges for two classical pairs and their extensions by isolated vertices: at $m = 2$, the matching $2K_2$ and $K_{1,2} \sqcup K_1$ share two cards isomorphic to $K_2 \sqcup 2K_1$; at $m = 3$, the graph $K_3 \sqcup K_1$ and the claw $K_{1,3}$ share three cards isomorphic to $K_{1,2} \sqcup K_1$. Harary [9] conjectured in 1964 that every finite simple graph with $m \geq 4$ edges is edge-reconstructible. By a theorem of Hemminger [10], resting on Whitney's line-graph isomorphism theorem [19], edge-reconstructibility of isolate-free graphs with $m \geq 4$ edges is equivalent to vertex-reconstructibility of line graphs on at least four vertices.

Our main theorem establishes unconditional edge-reconstructibility for all simple graphs of linear edge density $m \geq 1000\, n_0$.

Theorem 1.1. *Every finite simple undirected graph G_0 on $n_0 \geq 4$ vertices with*

$$m \;=\; |E(G_0)| \;\geq\; 1000\, n_0$$

edges is edge-reconstructible.

Note that when $4 \leq n_0 \leq 2000$, every n_0-vertex simple graph has at most $\binom{n_0}{2} < 1000\, n_0$ edges, so Theorem 1.1 holds vacuously on $4 \leq n_0 \leq 2000$ and takes effect for $n_0 \geq 2001$. Whereas regular and bidegreed graphs were settled classically [11, 15], graphs with three or more distinct degrees previously required average degree exceeding $2 \log_2 \Delta(G_0) + O(1)$ [5]; Theorem 1.1 settles edge reconstruction for all simple graphs of minimum degree at least 2000 regardless of how many distinct degrees appear or how large $\Delta(G_0)$ grows.

Corollary 1.2. *Every finite simple undirected graph G_0 with minimum degree $\delta(G_0) \geq 2000$ is edge-reconstructible.*

Proof. Let G_0 be a finite simple graph with $\delta(G_0) \geq 2000$, so $V(G_0) \neq \emptyset$ (as $\delta(\emptyset) = 0$). Pick $v \in V(G_0)$; then $n_0 := |V(G_0)| \geq d_{G_0}(v) + 1 \geq 2001 \geq 4$ and $|E(G_0)| = \frac{1}{2} \sum_{u \in V(G_0)} d_{G_0}(u) \geq 1000\, n_0$, so G_0 is edge-reconstructible by Theorem 1.1. $\square$

1.2 Related Work

Early structural reductions for edge reconstruction were established by Greenwell and Hemminger [8], who proved that the degree sequence of a simple graph with $m \geq 4$ edges is edge-reconstructible (see also Bondy and Hemminger [3]), and by Greenwell [7], who proved that for every simple graph F with $|E(F)| \geq 4$ and no isolated vertices, $\mathcal{D}_E(F)$ determines $\mathcal{D}(F)$. Since disconnected graphs on at least three vertices are vertex-reconstructible [11], any minimal counterexample to Harary's conjecture after removing isolated vertices is connected and fails vertex-reconstructibility as well.

In 1972, Lovász [13] introduced inclusion–exclusion over spanning-subgraph embeddings to prove that every n-vertex simple graph with $m > \frac{1}{2}\binom{n}{2}$ edges is edge-reconstructible. In 1977, Müller [14] refined Lovász's method by summing over edge subsets of fixed missing-edge parity, proving that G is edge-reconstructible whenever $2^{m-1} > n!$, and in particular whenever $m > n \log_2 n$. Nash-Williams [16] cast this parity-counting method into a general mapping-difference lemma. As surveyed by Bondy [2] and by Lauri and Scapellato [12], Müller's bound $m > n \log_2 n$ remained for nearly five decades the strongest unconditional density bound valid for arbitrary simple graphs.

Between 1978 and 2026, substantial progress was achieved for restricted graph families and abstract group actions (using standard asymptotic notation $O(\cdot), \Omega(\cdot), \Theta(\cdot), o(\cdot)$ as $n \to \infty$):

(1) *Bounded-degree and low-degree spanning-tree methods.* Restricting the Lovász–Müller parity sum to subsets containing a fixed spanning tree $T_0 \subseteq G$ bounds $2^{m-n}|\operatorname{Aut}(G)|$ by the number of edge-preserving bijections on $V(G)$ carrying $E(T_0)$ into $E(G)$. Using this principle, Caro, Krasikov, and Roditty [5] (see also Hoffman's inequality in Bondy [2]) proved edge-reconstructibility when $m > n \log_2 \Delta(G) + O(n)$ and when G admits a bounded-degree spanning tree. On general graphs of linear edge count $m = \Theta(n)$, however, $o(n)$ high-degree hubs of degree $\Theta(n)$ can force every spanning tree to have maximum degree $\Theta(n)$, returning $n \log_2 \Delta(G)$ to $\Theta(n \log n)$.

(2) *Restricted degree and induced-subgraph families.* Myrvold, Ellingham, and Hoffman [15] proved edge-reconstructibility for bidegreed graphs and graphs with large gaps in their degree sequences. Ellingham, Pyber, and Yu [6] settled edge reconstruction for all claw-free ($K_{1,3}$-free) graphs with $m \geq 4$, and Pyber [17] proved edge-reconstructibility for all sufficiently large Hamiltonian graphs by using a Hamiltonian path as a spanning tree of maximum degree 2. Arbitrary graphs with $m \geq 1000\, n$ need not be claw-free or Hamiltonian, and their vertex degrees can populate every integer scale from 1 up to $n - 1$ with no empty degree interval.

(3) *Subset reconstruction under group actions.* Alon, Caro, Krasikov, and Roditty [1] proved that a subset A of a finite set U acted upon by a group Γ_0 is reconstructible up to Γ_0-equivalence from its 1-element-deleted deck whenever $2^{|A|-1}$ exceeds the index of the stabilizer of A in Γ_0. In graph reconstruction without degree gaps, deleting an edge or vertex across scales alters vertex degrees and shifts components between isomorphism classes, preventing a direct reduction to a single group-action subset deck unless hub symmetries, bridge components, and twin stabilizers are controlled simultaneously across all scales.

1.3 Architecture of the Proof and Scope of Spanning-Tree Counting

We make this monograph completely self-contained from elementary combinatorics, finite group theory, and real and complex analysis: every graph-reconstruction, polynomial, and group-theoretic ingredient—including Kelly's Lemma, disconnected vertex reconstruction, Greenwell–Hemminger degree-sequence reconstruction, Greenwell's edge-to-vertex deck reduction, and Müller's inequality—is proved in full before use.

Suppose for contradiction that G_0 and F_0 are non-isomorphic graphs on $n_0 \geq 4$ vertices with $m = |E(G_0)| = |E(F_0)| \geq 1000\, n_0$ and $\mathcal{D}_E(G_0) = \mathcal{D}_E(F_0)$. Stripping their equal numbers of isolated vertices in Section 2 produces connected non-isomorphic graphs G and H on $n \geq 4$ vertices with average degree $\bar{d} = 2m/n \geq 2000$, identical edge-decks, identical vertex-decks, identical degree sequences, and $\log_2 n > 1000$.

The proof that no such pair (G, H) exists rests on four pillars developed across Sections 3–8:

1. *Single-MST multi-scale component-entropy reduction (Section 3).* Single-threshold localization stalls at $m = \Theta(n \log\log n)$ because a single $(\log n)^{O(1)}$ degree cutoff still allows $(\log n)^{\Theta(n)}$ parent-child embeddings when hubs have degree $\Theta(\log n)$ or larger. Instead, we construct a single lexicographic degree-ordered Kruskal minimum spanning tree T of G across all dyadic thresholds $\Delta_k = 2^k \Delta_0$ with $\Delta_0 = \lceil 2^{\bar{d}/16} \rceil$. At every scale Δ_k simultaneously, T restricts to a spanning forest of $G[V_{<\Delta_k}(G)]$, bounds the number of bridge components (those exited by at least two edges of T) below $|V_{\geq \Delta_k}(G)|$, and assigns a single canonical T-parent in $V_{\geq \Delta_k}(G) \setminus V_{\geq \Delta_{k+1}}(G)$ to every isomorphism class $\mathcal{T}$ of $V_{\geq \Delta_k}(G)$-labelled non-bridge components of $G[V_{<\Delta_k}(G)]$; writing $\tilde{w}_{\mathcal{T}}$ for the number of non-bridge copies in class $\mathcal{T}$, canonical permutations within these classes generate a subgroup of $\mathrm{Aut}(G)$ of order at least $\prod_k \prod_{\mathcal{T}} \tilde{w}_{\mathcal{T}}!$. In the spanning-tree Lovász–Müller inequality against an unknown bijection $\tau : V(G) \to V(G)$, low-degree parents ($d_G(u) < \Delta_0$) are absorbed by degree truncation at $2^{\bar{d}/8}$, while for hubs u with $\Delta_k \leq d_G(u) < 2\Delta_k$ an algebraic split separates the global ratio $d_G(\tau(u))/c^*_{\mathrm{nb}}(u)$—where $c^*_{\mathrm{nb}}(u)$ counts the non-leaf, non-bridge T-children of u, summed across all hubs at once via concavity of $\log_2$ to $O(n \log \bar{d})$ and decoupling τ from Δ_k—from the scale-local terms $\tilde{w}_{\mathcal{T}} \log_2(2\Delta_k/\tilde{w}_{\mathcal{T}})$. High-multiplicity classes ($\tilde{w}_{\mathcal{T}} \geq 2\Delta_k/2^{\bar{d}/8}$) and high-degree isolated vertices ($d_G(w) \geq 100 \log_2(2\Delta_k)$) consume at most $0.27\, m$ bits, forcing more than $0.15\, m$ bits onto low-multiplicity classes at scales $2\Delta_k > 2^{\bar{d}/8}$, divided among multi-vertex component classes, open-neighborhood twin classes of size at least 3, and T-leaf open-twin classes of size 1 or 2. We also prune higher-scale isolated-vertex leaves from the contracted core trees to obtain trees whose edge counts and hub-restriction groups govern all three downstream pillars.

2. *Scale-invariant polynomial decay and bounded-degree bridge rigidity (Sections 4–5).* In Section 4, we prove over $\mathbb{C}$ that for every homogeneous polynomial $P(y_1, \ldots, y_M)$ of degree N_0 satisfying $P(y_1 + z, \ldots, y_M + z) = P(y_1, \ldots, y_M)$ for all $z \in \mathbb{C}$, the ratio of the sum of absolute values of all coefficients of P to the modulus of any single coefficient whose exponent vector has entries at most t is at least $(t+1)^{N_0/(200t)}$ (and at least $(2/\sqrt{3})^{N_0}$ when $t = 1$). The proof pairs P by polynomial duality against monic Charlier polynomials, controlled via a second-derivative van der Corput oscillatory-integral bound and Bessel-function estimates, and combines scaled partial differentiation with top-degree coefficient extraction to isolate any multiplicity band $[t_0, t_1]$ without increasing the ℓ_1-coefficient sum. In Section 5, Cauchy's theorem and Legendre's prime-valuation formula along shortest paths show that for any connected graph of maximum degree at most Δ fixing a root vertex, and for any hub-automorphism group acting across a single connected component of $G[V_{<\Delta}(G)]$, the restriction to R unfixed target vertices has order at most $(16\Delta)^R$, independent of hub degrees.

3. *Pillar I: Gap-free internal-edge component polynomials and top-down contraction (Section 6), and Pillar II: Twin-quotient orbit-freezing and cumulative-base chains (Section 7).* In Section 6, a piecewise-linear degree functional isolates internal edge-deleted cards of $G[V_{<\Delta_k}(G)]$ with zero degree gap, yielding a component-multiplicity difference polynomial annihilated by every connected-graph directional derivative via Lie-bracket commutators. Separating component

shapes appearing at least three times from those appearing once or twice, bounding a bridge-controlled hub-symmetry index by a two-phase breadth-first encoding that cancels higher-scale isolated-vertex factorials identically, and establishing a top-down contractive recurrence with uniform column sum below 0.01 bounds the multi-vertex component contribution below $10^{-60}\,m$. In Section 7, passing to the open-neighborhood twin quotient of G under $\mathcal{D}(G) = \mathcal{D}(H)$, we show that band-localized polynomials of degree at least 2 preserve orbit degrees and remain shift-invariant on a band-invariant high-degree-ratio vertex subset, and construct a descending cumulative-base subgroup chain on the twin quotient with reverse-order coset injectivity to bound the size-≥ 3 open-twin contribution below $10^{-60}\,m$.

4. *Pillar III: Half-degree sampling, pruned decision trees, and ε-linearization (Section 8).* On the remaining T-leaf open-twin classes of size $t \in \{1, 2\}$ and degree $2 \leq r < 100 \log_2(2\Delta_k)$, twin-quotient reductions become vacuous when all twin classes are singletons. Instead, independent 2/5-sampling of twin classes with Chebyshev hub pruning selects a twin-closed union $W_0 = \bigsqcup_{i \in J} V_i$ of at least three-eighths of the twin classes in the slice while preserving at least half of every vertex's degree in $K_0 = G[V(G) \setminus W_0]$, paired with a one-per-class transversal $W_{\mathrm{tr}} \subseteq W_0$ (alongside direct 1/4-sampling at $t = 1$ or deterministic halving at $t = 2$ in Lemma 8.3(1)). Degree-strata locking under $\mathcal{D}(G) = \mathcal{D}(H)$ stabilizes $V_{\geq \Delta_{k-1}}(G)$ and all multi-vertex components of $G[V_{<\Delta_{k-1}}(G)]$ across every induced embedding of K_0 into G while canceling twin and kernel stabilizers in the normalized vertex-deck polynomial on W_{tr}. Encoding hub restrictions via a breadth-first decision tree on the level-$(k-1)$ pruned core tree and linearizing the logarithmic degree-excess factors at slope $\varepsilon = \frac{1}{200r}$ against the exact edge-count identity that the uncovered degree excesses across $V_{\geq \Delta_{k-1}}(G)$ sum to $r|W_0|$ cancels r identically, bounding the size-$\{1, 2\}$ leaf-class contribution below $10^{-61}\,m$ and yielding a contradiction in Section 9.

Remark 1.3 (Natural density limit of spanning-tree Lovász–Müller counting). While sharpening the analytic constants in Sections 3–8 can reduce the numerical factor 1000 in Theorem 1.1, any argument driven by Lovász–Müller parity counting over edge subsets containing a spanning tree encounters an intrinsic barrier at small constant average degree. Specifically, for an asymmetric graph G ($|\operatorname{Aut}(G)| = 1$) on n vertices with m edges and average degree $\bar{d} = 2m/n$ containing a spanning tree T_0, the $m - n + 1$ edges outside T_0 supply 2^{m-n} subsets of fixed missing-edge parity, or $(\frac{\bar{d}}{2} - 1)n$ bits of lower bound on the number of bijections of $V(G)$ mapping $E(T_0)$ into $E(G)$. Conversely, even in the favorable case where G is $\bar{d}$-regular and T_0 is a Hamiltonian path rooted at one endpoint, mapping each internal vertex of T_0 along an unused neighbor of its parent allows $(\bar{d} - 1)$ forward choices per step (or $\bar{d}$ unconditionally), giving $(1 - o(1))n \log_2(\bar{d} - 1)$ bits in the degree-product upper bound. On $(2, \infty)$, the equation $\frac{\bar{d}}{2} - 1 = \log_2(\bar{d} - 1)$ has unique root $\bar{d}^* \approx 7.32$ (alongside the root $\bar{d} = 2$), corresponding to $m = \frac{1}{2}\bar{d}^* n \approx 3.66\,n$, while on $(1, \infty)$ the equation $\frac{\bar{d}}{2} - 1 = \log_2 \bar{d}$ has unique root $\bar{d} = 8$ ($m = 4\,n$). For every fixed $\bar{d} \in (2, \bar{d}^*)$—including subcubic graphs of average degree $\bar{d} \in (2, 3)$—the parity lower bound 2^{m-n} is exponentially smaller than $(\bar{d} - 1)^{n-2}$, and on sparse graphs with $m = (1 + o(1))n$ ($\bar{d} = 2 + o(1)$) the parity lower bound $2^{m-n} = 2^{o(n)}$ is subexponential (equaling 1 when $m = n$), falling exponentially below the unconditional degree product $\bar{d}^{n-1} = 2^{(1+o(1))n}$. Settling Harary's conjecture across $4 \leq m < 1000\,n$ therefore requires genuinely non-counting deck invariants.

2 Preliminaries, Deck Reductions, and Elementary Estimates

2.1 Conventions, Graph Strata, Embeddings, and Polynomial Operators

Throughout, $\log_2$ denotes the base-2 logarithm and ln denotes the natural logarithm. For an integer $M \geq 1$, write $[M] := \{1, \dots, M\}$ (and $[0] := \emptyset$). For any set A and integer $k \geq 0$, write $2^A := \{B : B \subseteq A\}$ for the power set of A, $\binom{A}{k} := \{B \subseteq A : |B| = k\}$ for the set of k-element subsets of A, and $|A| = \#A$ for the cardinality of a finite set A. For a logical assertion $\mathcal{P}$, write $\mathbf{1}\{\mathcal{P}\} \in \{0, 1\}$ (or $\mathbf{1}_{\mathcal{P}}$) for the indicator equal to 1 when $\mathcal{P}$ holds and 0 otherwise. For $x \in \mathbb{C}$ and an integer $b \geq 0$, define the falling factorial $(x)_b := \prod_{j=0}^{b-1}(x - j)$ (with $(x)_0 := 1$) and the binomial coefficient $\binom{x}{b} := \frac{(x)_b}{b!}$; in particular, for integers $0 \leq a < b$, $(a)_b = 0$ and $\binom{a}{b} = 0$. For a prime p and a positive integer N, write $\mathrm{ord}_p(N) := \max\{j \in \mathbb{Z}_{\geq 0} : p^j \mid N\}$ for the p-adic valuation of N.

Let G be a simple graph. When $V(G) \neq \emptyset$, write $\delta(G) := \min_{v \in V(G)} d_G(v)$ (with $\delta(\emptyset) := 0$). Complementing $V_{\geq t}(G)$ and $V_{<t}(G)$ from Section 1, for $t \in \mathbb{R}$ define the degree-stratum vertex subsets

$$V_{>t}(G) := \{v \in V(G) : d_G(v) > t\}, \qquad V_{=t}(G) := \{v \in V(G) : d_G(v) = t\},$$
$$V_{\leq t}(G) := \{v \in V(G) : d_G(v) \leq t\},$$

and let $I(G) := |V_{=0}(G)|$ denote the number of isolated vertices of G. For $A \subseteq V(G)$, write $G - A := G[V(G) \setminus A]$ and $N_G(A) := (\bigcup_{a \in A} N_G(a)) \setminus A$; for any subgraph C of G, write $N_G(C) := N_G(V(C))$. For $E_0 \subseteq E(G)$, write $G - E_0 := (V(G), E(G) \setminus E_0)$, and for disjoint subsets $A, B \subseteq V(G)$, write $E_G(A, B) := \{\{a, b\} \in E(G) : a \in A, b \in B\}$. Denote by $\mathrm{Comp}(F)$ the set of connected components of a simple graph F, put $c(F) := |\mathrm{Comp}(F)|$, and for a connected simple graph C_0 write $c(F, C_0) := |\{C \in \mathrm{Comp}(F) : C \cong C_0\}|$. When u and v lie in the same connected component of F, write $\mathrm{dist}_F(u, v)$ for the minimum number of edges in a path between u and v in F (with $\mathrm{dist}_F(u, u) := 0$). Write $\overline{K}_r$ for the edgeless graph on $r \geq 0$ vertices.

For a finite set A, write $\mathrm{Sym}(A)$ for the symmetric group of bijections $A \to A$; for $\sigma \in \mathrm{Sym}(A)$ and $E' \subseteq \binom{A}{2}$, write $\sigma(E') := \{\{\sigma(u), \sigma(v)\} : \{u, v\} \in E'\}$, so $\mathrm{Aut}(F) = \{\sigma \in \mathrm{Sym}(V(F)) : \sigma(E(F)) = E(F)\}$ for any simple graph F. Whenever $\mathcal{H}$ is a set of maps defined on a domain containing W, write $\mathcal{H}|_W := \{g|_W : g \in \mathcal{H}\}$ for the set of restrictions to W. For $U \subseteq V(K)$, define the pointwise stabilizer $\mathrm{Aut}_U(K) := \{\psi \in \mathrm{Aut}(K) : \psi|_U = \mathrm{id}_U\}$, and for $V_0 \subseteq V(G)$, define the setwise stabilizer $\mathrm{Aut}_{\{V_0\}}(G) := \{\sigma \in \mathrm{Aut}(G) : \sigma(V_0) = V_0\}$.

For simple graphs K and X, define the induced-subgraph count

$$\mathrm{ind}(K, X) \;:=\; \big|\{W \subseteq V(X) : X[W] \cong K\}\big|,$$

and when $|V(K)| = |V(X)|$, define the spanning-subgraph count

$$s(K, X) \;:=\; \big|\{E_0 \subseteq E(X) : (V(X), E_0) \cong K\}\big|.$$

Define the set of induced embeddings of K into F by

$$\Phi_{\mathrm{ind}}(K, F) \;:=\; \big\{\phi : V(K) \hookrightarrow V(F) : \{u, v\} \in E(K) \iff \{\phi(u), \phi(v)\} \in E(F)\ \forall\, u \neq v \in V(K)\big\}.$$

For $U \subseteq V(K)$, the group $\mathrm{Aut}_U(K)$ acts on $\Phi_{\mathrm{ind}}(K, F)$ on the right by composition $(\phi, \psi) \mapsto \phi \circ \psi$, with orbit set $\Phi_{\mathrm{ind}}(K, F)/\mathrm{Aut}_U(K)$. For any injection $\phi : V(K) \hookrightarrow V(F)$ and $U \subseteq V(K)$, write $R_\phi := V(F) \setminus \phi(V(K))$ for the uncovered remainder and define $v_{F,U}(\phi) \in \mathbb{Z}_{\geq 0}^{2^U}$ by

$$v_{F,U}(\phi)_X \;:=\; \big|\{w \in R_\phi : \phi^{-1}(N_F(w)) \cap U = X\}\big| \qquad (X \subseteq U).$$

Let I be a finite non-empty index set. For $u \in \mathbb{Z}_{\geq 0}^I$ (writing u_i or $u(i)$ for its i-th coordinate), set $|u| := \sum_{i \in I} u_i$, $u! := \prod_{i \in I} u_i!$, and $y^u := \prod_{i \in I} y_i^{u_i} \in \mathbb{C}[y_i : i \in I]$; for $J \subseteq I$ and $z \in \mathbb{Z}^I$, write $z|_J := (z_i)_{i \in J} \in \mathbb{Z}^J$ and $y_J := (y_i)_{i \in J}$ (with the empty monomial on $J = \emptyset$ equal to 1). Write $\mathbf{e}_i \in \mathbb{Z}_{\geq 0}^I$ for the i-th standard basis vector and $\mathbf{1}, \mathbf{0} \in \mathbb{Z}_{\geq 0}^I$ for the all-ones and all-zeros vectors. Two vectors $u, v \in \mathbb{Z}_{\geq 0}^I$ are said to have the *same multiplicity partition* on $J \subseteq I$ if $|\{i \in J : u_i = t\}| = |\{i \in J : v_i = t\}|$ for every integer $t \geq 0$ (equivalently, $u|_J$ is a permutation of $v|_J$). When a group Γ_0 acts on the left on I, it acts on the left on $\mathbb{Z}^I$ via $(\pi \cdot z)_i := z_{\pi^{-1}(i)}$ ($\pi \in \Gamma_0, i \in I$).

For a polynomial $P(y) = \sum_{u \in \mathbb{Z}_{\geq 0}^I} c_u y^u \in \mathbb{C}[y_i : i \in I]$, write $[y^u]P := c_u$, $\mathrm{supp}(P) := \{u \in \mathbb{Z}_{\geq 0}^I : c_u \neq 0\}$, and $\|P\|_1 := \sum_{u \in \mathbb{Z}_{\geq 0}^I} |c_u|$. When $P \in \mathbb{R}[y_i : i \in I]$, write $P^+ := \sum_{u : c_u > 0} c_u y^u$ and $P^- := \sum_{u : c_u < 0} (-c_u) y^u$, so $P = P^+ - P^-$ with P^+, P^- having non-negative coefficients and $\|P\|_1 = P^+(\mathbf{1}) + P^-(\mathbf{1})$. For $w \in \mathbb{Z}_{\geq 0}^I$, define $\partial^w := \prod_{i \in I} (\frac{\partial}{\partial y_i})^{w_i}$ and the Hasse derivative $H_w(P) := \frac{1}{w!} \partial^w P$; note $H_w(y^u) = (\prod_{i \in I} \binom{u_i}{w_i}) y^{u-w}$ if $u_i \geq w_i$ for all $i \in I$, and $H_w(y^u) = 0$ otherwise. For $J \subseteq I$, define the directional shift operator $D_J := \sum_{i \in J} \frac{\partial}{\partial y_i}$ (abbreviating $D := D_I$). By Taylor's formula $P(y + \tau \mathbf{1}) = \sum_{k=0}^{\deg P} \frac{\tau^k}{k!} D^k(P)(y)$, we say P is *shift-invariant* on I if $D_I(P) \equiv 0$, or equivalently $P(y + \tau \mathbf{1}) = P(y)$ for all $\tau \in \mathbb{C}$. When $I = I_1 \sqcup I_2$ and $z_2 \in \mathbb{Z}_{\geq 0}^{I_2}$, write $[y_{I_2}^{z_2}]P \in \mathbb{C}[y_i : i \in I_1]$ for the polynomial obtained by extracting the coefficient of $y_{I_2}^{z_2}$.

Finally, for a finite set A, let $\mathfrak{C}(A)$ denote the set of isomorphism classes $\mathcal{T} = [(C, \lambda)]$ of pairs (C, λ) consisting of a finite connected simple graph C ($|V(C)| \geq 1$) and a vertex labeling $\lambda : V(C) \to 2^A$, where $(C, \lambda) \cong (C', \lambda')$ if there exists a graph isomorphism $\psi : C \xrightarrow{\sim} C'$ satisfying $\lambda' \circ \psi = \lambda$. For a finite connected simple graph C_0, set $\mathfrak{C}_{C_0}(A) := \{[(C, \lambda)] \in \mathfrak{C}(A) : C \cong C_0\}$ (a finite set of size at most $(2^{|A|})^{|V(C_0)|}$), write $\mathfrak{C}_{\geq 2}(A) := \{[(C, \lambda)] \in \mathfrak{C}(A) : |E(C)| \geq 1\}$, and for an integer $N \geq 0$ set $\mathfrak{C}_{\geq 2, \leq N}(A) := \{[(C, \lambda)] \in \mathfrak{C}_{\geq 2}(A) : |V(C)| \leq N\}$ (also finite). The group $\mathrm{Sym}(A)$ acts on the left on $\mathfrak{C}(A)$ (preserving each $\mathfrak{C}_{C_0}(A)$ and $\mathfrak{C}_{\geq 2, \leq N}(A)$) by $\pi \cdot [(C, \lambda)] := [(C, \pi \circ \lambda)]$. Given a finite simple graph F and $\lambda : V(F) \to 2^A$, define the finitely supported component-multiplicity vector $\mu(F, \lambda) \in \mathbb{Z}_{\geq 0}^{\mathfrak{C}(A)}$ by

$$\mu(F, \lambda)_{\mathcal{T}} := \big|\{C \in \mathrm{Comp}(F) : [(C, \lambda|_{V(C)})] = \mathcal{T}\}\big| \qquad (\mathcal{T} \in \mathfrak{C}(A)),$$

and for each connected simple graph C_0 write $D_{C_0} := \sum_{\mathcal{T} \in \mathfrak{C}_{C_0}(A)} \frac{\partial}{\partial x_{\mathcal{T}}}$.

2.2 Kelly's Lemma, Lovász–Müller Counting, and Isolate-Free Reduction

We begin by recording Kelly's subgraph-counting identity, connectedness recognition, and vertex-reconstructibility of disconnected graphs on at least three vertices.

Proposition 2.1 (Kelly [11]). *Let X and Y be simple graphs on $n \geq 3$ vertices with $\mathcal{D}(X) = \mathcal{D}(Y)$.*

(1) For every simple graph K with $|V(K)| < n$, we have $\mathrm{ind}(K, X) = \mathrm{ind}(K, Y)$. In particular, $|E(X)| = |E(Y)|$, the deleted-vertex degree $d_X(v) = |E(X)| - |E(X - v)|$ is determined on each card $[X - v]$, and $|V_{=d}(X)| = |V_{=d}(Y)|$ for every integer $d \geq 0$.

(2) X is connected if and only if $\mathcal{D}(X)$ contains at least two connected cards; hence X is connected if and only if Y is connected.

(3) If X is disconnected, then $X \cong Y$.

Proof. (1) For any simple graph K with $k := |V(K)| < n$ and $Z \in \{X, Y\}$, each k-element subset $W \subseteq V(Z)$ with $Z[W] \cong K$ is contained in $V(Z) \setminus \{v\}$ for precisely the $n - k \geq 1$ vertices $v \in V(Z) \setminus W$. Summing over $v \in V(Z)$ gives

$$\operatorname{ind}(K, Z) \;=\; \frac{1}{n-k} \sum_{v \in V(Z)} \operatorname{ind}(K, Z - v),$$

so $\mathcal{D}(X) = \mathcal{D}(Y)$ implies $\operatorname{ind}(K, X) = \operatorname{ind}(K, Y)$. Taking $K = K_2$ ($k = 2 < n$) yields $|E(X)| = |E(Y)| =: m'$. For every $v \in V(X)$, $|E(X - v)| = m' - d_X(v)$, so the multiset of vertex degrees of X coincides with that of Y.

(2) If X is connected, pick $u, v \in V(X)$ maximizing $\operatorname{dist}_X(u, v)$; since $n \geq 3$, $\operatorname{dist}_X(u, v) \geq 1$ and $u \neq v$. For any $w \in V(X) \setminus \{u\}$, a shortest path P from v to w in X cannot pass through u, or else $\operatorname{dist}_X(v, w) = \operatorname{dist}_X(v, u) + \operatorname{dist}_X(u, w) > \operatorname{dist}_X(v, u)$, contradicting maximality. Thus $X - u$ is connected, and symmetrically $X - v$ is connected, yielding at least two connected cards in $\mathcal{D}(X)$. Conversely, suppose X is disconnected with components $C_1, \ldots, C_r$ ($r \geq 2$). If $r \geq 3$, or if $r = 2$ and $|V(C_1)|, |V(C_2)| \geq 2$, then $X - v$ has at least two non-empty components for every $v \in V(X)$ (zero connected cards). If $r = 2$ and $C_1 = \{w\}$ has size 1, then $|V(C_2)| = n - 1 \geq 2$, so for every $v \in V(C_2)$ the card $X - v$ separates w from $V(C_2) \setminus \{v\} \neq \emptyset$, leaving $X - w = C_2$ as the unique connected card in $\mathcal{D}(X)$.

(3) Suppose X is disconnected; by (2), Y is also disconnected, so every component of X or Y has at most $n - 1$ vertices. For any connected simple graph C with $1 \leq |V(C)| \leq n - 1$ and $Z \in \{X, Y\}$, every connected induced subgraph of Z isomorphic to C lies inside a single component $C' \in \operatorname{Comp}(Z)$; if $|V(C')| < |V(C)|$ then $\operatorname{ind}(C, C') = 0$, and if $|V(C')| = |V(C)|$ then $\operatorname{ind}(C, C') = \mathbf{1}\{C' \cong C\}$. Hence

$$\operatorname{ind}(C, Z) \;=\; c(Z, C) \;+ \sum_{\substack{[C'] \text{ connected} \\ |V(C)| < |V(C')| \leq n-1}} c(Z, C') \operatorname{ind}(C, C').$$

Since $\operatorname{ind}(C, X) = \operatorname{ind}(C, Y)$ by (1), downward induction on $|V(C)| = n - 1, n - 2, \ldots, 1$ yields $c(X, C) = c(Y, C)$ for every connected graph C, whence $X \cong Y$. □

Next we establish Edge-Kelly subgraph counting, the Lovász–Müller parity lower bound and spanning-tree inequality, Müller's $n \log_2 n$ bound, and an elementary degree-truncation estimate for rooted spanning trees.

Lemma 2.2. *Let G be a simple graph on n vertices with $m = |E(G)| \geq 1$ edges.*

(1) (Edge-Kelly and Möbius parity lower bound.) *Suppose H is a simple graph with $|V(H)| = n$, $\mathcal{D}_E(G) = \mathcal{D}_E(H)$, and $G \ncong H$. Then $|E(H)| = m$, $s(K, G) = s(K, H)$ for every n-vertex simple graph K with $|E(K)| < m$, and for every subset $E_0 \subseteq E(G)$, the set $\Sigma_G(E_0) := \{\sigma \in \operatorname{Sym}(V(G)) : E_0 \subseteq \sigma^{-1}(E(G))\}$ satisfies*

$$|\Sigma_G(E_0)| \;\geq\; 2^{|E(G) \setminus E_0| - 1} \,|\operatorname{Aut}(G)|.$$

In particular, taking $E_0 = \emptyset$ gives $2^{m-1} |\operatorname{Aut}(G)| \leq n!$; if $n \geq 4$, then $n! < \frac{1}{2} n^n$ and hence $m < n \log_2 n$.

(2) (Spanning-tree degree-product bound.) *Suppose G is connected on $n \geq 3$ vertices with average degree $\bar{d} := 2m/n$, and let $T_0 \subseteq G$ be a spanning tree rooted at $r_0 \in V(G)$ with child counts $c(u)$ ($u \in V(G)$). Then*

$$|\Sigma_G(E(T_0))| \;\leq\; n \cdot \max_{\substack{\tau \in \operatorname{Sym}(V(G)) \\ c(u) \leq d_G(\tau(u)) \;\forall u}} \prod_{u \in V(G)} \big(d_G(\tau(u))\big)_{c(u)}.$$

Consequently, if G is not edge-reconstructible, there exists $\tau \in \mathrm{Sym}(V(G))$ with $c(u) \le d_G(\tau(u))$ for all $u \in V(G)$ such that

$$m - n + \log_2 |\operatorname{Aut}(G)| \;\le\; \log_2 n + \sum_{u \in V(G)} \log_2\big((d_G(\tau(u)))_{c(u)}\big),$$

and in particular $\bar{d} \le 2 + 2\log_2 \Delta(G) + \frac{2\log_2 n}{n}$.

(3) (Degree-truncated child sum.) *Let G be a simple graph on $n \ge 2$ vertices with $m = |E(G)|$, let $T_0 \subseteq G$ be a rooted spanning tree with child counts $c(u)$ ($u \in V(G)$), let $D_1 \ge 1$ and $D_2 \ge e$ be real numbers, and let $U' \subseteq \{u \in V(G) : c(u) \le D_1\}$ be any subset. Then for every $\tau \in \mathrm{Sym}(V(G))$,*

$$\sum_{u \in U'} c(u) \log_2 d_G(\tau(u)) \;\le\; \left(\sum_{u \in U'} c(u)\right) \log_2 D_2 \;+\; D_1 \cdot \frac{2m \log_2 D_2}{D_2}.$$

Proof. (1) Relabeling $V(H)$ if necessary, assume $V(H) = V(G) =: V$; clearly $|E(H)| = |\mathcal{D}_E(H)| = |\mathcal{D}_E(G)| = m$. For any n-vertex simple graph K with $|E(K)| < m$ and $Z \in \{G, H\}$, each edge subset $E_1 \subseteq E(Z)$ with $(V, E_1) \cong K$ satisfies $E_1 \subseteq E(Z - e)$ for precisely the $m - |E(K)| \ge 1$ edges $e \in E(Z) \setminus E_1$, so

$$s(K, Z) \;=\; \frac{1}{m - |E(K)|} \sum_{e \in E(Z)} s(K, Z - e),$$

giving $s(K, G) = s(K, H)$. For any $T' \subseteq E(G)$, put $K_{T'} := (V, T')$ and $\alpha(T', Z) := |\{\sigma \in \mathrm{Sym}(V) : T' \subseteq \sigma^{-1}(E(Z))\}|$. Double-counting pairs $(E_1, \sigma) \in \binom{\binom{V}{2}}{|T'|} \times \mathrm{Sym}(V)$ with $(V, E_1) \cong K_{T'}$ and $E_1 \subseteq \sigma^{-1}(E(Z))$—there are $n!/|\operatorname{Aut}(K_{T'})|$ subgraphs $(V, E_1) \cong K_{T'}$, each satisfied by $\alpha(T', Z)$ permutations σ, while each of the $n!$ permutations σ contains $s(K_{T'}, Z)$ copies—gives $\alpha(T', Z) = |\operatorname{Aut}(K_{T'})|\, s(K_{T'}, Z)$. Hence $\alpha(T', G) = \alpha(T', H)$ for all $T' \subsetneq E(G)$, whereas $\alpha(E(G), G) = |\operatorname{Aut}(G)|$ and $\alpha(E(G), H) = 0$ (since $|E(H)| = m$ and $G \not\cong H$).

For $S \subseteq E(G)$ and $Z \in \{G, H\}$, set $a_Z(S) := |\{\sigma \in \mathrm{Sym}(V) : E(G) \cap \sigma^{-1}(E(Z)) = S\}|$. Because $\alpha(T', Z) = \sum_{T' \subseteq S \subseteq E(G)} a_Z(S)$, Möbius inversion on $2^{E(G)}$ gives, for every $S \subseteq E(G)$,

$$\begin{aligned} a_G(S) - a_H(S) \;&=\; \sum_{S \subseteq T' \subseteq E(G)} (-1)^{|T'| - |S|} \big(\alpha(T', G) - \alpha(T', H)\big) \\ &=\; (-1)^{m - |S|} \big(\alpha(E(G), G) - \alpha(E(G), H)\big) \;=\; (-1)^{m - |S|} |\operatorname{Aut}(G)|, \end{aligned}$$

since $\sum_{S \subseteq F \subseteq T'} (-1)^{|F| - |S|} = (1-1)^{|T'| - |S|} = \mathbf{1}\{T' = S\}$. As $a_H(S) \ge 0$, every $S \subseteq E(G)$ with $m - |S|$ even satisfies $a_G(S) \ge |\operatorname{Aut}(G)|$. For any $E_0 \subseteq E(G)$, $\Sigma_G(E_0) = \bigsqcup_{E_0 \subseteq S \subseteq E(G)} \{\sigma \in \mathrm{Sym}(V) : E(G) \cap \sigma^{-1}(E(G)) = S\}$ contains at least $2^{|E(G) \setminus E_0| - 1}$ subsets S with $m - |S|$ even ($1 \ge 2^{-1}$ if $E_0 = E(G)$, and $\frac{1}{2} \cdot 2^{|E(G) \setminus E_0|}$ if $E_0 \subsetneq E(G)$), proving $|\Sigma_G(E_0)| \ge 2^{|E(G) \setminus E_0| - 1} |\operatorname{Aut}(G)|$. At $E_0 = \emptyset$, this gives $2^{m-1} |\operatorname{Aut}(G)| \le |\mathrm{Sym}(V)| = n!$. For $n \ge 4$, induction on n shows $n! < \frac{1}{2} n^n$ (at $n = 4$, $24 < 128$; for $n \ge 4$, $(n+1)! < \frac{1}{2}(n+1) n^n < \frac{1}{2}(n+1)^{n+1}$), so $2^m \le 2n! < n^n$ and $m < n \log_2 n$.

(2) Note $\mathrm{id}_{V(G)} \in \Sigma_G(E(T_0))$, and let $L_t := \{v \in V(G) : \mathrm{dist}_{T_0}(r_0, v) = t\}$ ($0 \le t \le h$, with $h \ge 1$ as $n \ge 3$ and $L_0 = \{r_0\}$). For every $\tau \in \Sigma_G(E(T_0))$ and $u \in V(G)$, the $c(u)$ children of u in T_0 map injectively into $N_G(\tau(u))$, so $c(u) \le d_G(\tau(u))$ and $(d_G(\tau(u)))_{c(u)} \ge 1$. For each $1 \le t \le h$, any fixed restriction $\tau|_{L_0 \cup \cdots \cup L_{t-1}}$ has at most $\prod_{u \in L_{t-1}} (d_G(\tau(u)))_{c(u)}$ extensions to L_t. Backward induction from $t = h$ down to $t = 1$ (using $(d)_0 = 1$ on L_h) yields

$$\sum_{\tau \in \Sigma_G(E(T_0))} \prod_{u \in V(G)} \frac{1}{\big(d_G(\tau(u))\big)_{c(u)}} \;\le\; \sum_{\tau(r_0) \in V(G)} 1 \;=\; n,$$

giving $|\Sigma_G(E(T_0))| \le n \cdot \max_{\tau\in\Sigma_G(E(T_0))} \prod_{u\in V(G)} (d_G(\tau(u)))_{c(u)}$. If G is not edge-reconstructible, combining this upper bound with Part (1) at $E_0 = E(T_0)$ (where $|E(G) \setminus E(T_0)| - 1 = m - n$) and taking $\log_2$ yields the displayed inequality. Since G is connected on $n \ge 3$ vertices, $\Delta(G) \ge 2$ and $\sum_{u\in V(G)} c(u) = n - 1$, so $\sum_u \log_2((d_G(\tau(u)))_{c(u)}) \le (n-1)\log_2 \Delta(G) < n \log_2 \Delta(G)$, giving $m - n < \log_2 n + n \log_2 \Delta(G)$ and hence $\bar{d} \le 2 + 2\log_2 \Delta(G) + \frac{2\log_2 n}{n}$.

(3) Because T_0 is a spanning tree on $n \ge 2$ vertices, every $w \in V(G)$ satisfies $d_G(w) \ge 1$, so $\log_2 d_G(\tau(u)) \ge 0$. Terms in U' with $d_G(\tau(u)) \le D_2$ contribute at most $(\sum_{u\in U'} c(u)) \log_2 D_2$. On $[e, \infty)$, $\frac{d}{dx}(\frac{\log_2 x}{x}) = \frac{1-\ln x}{x^2 \ln 2} \le 0$, so $x \mapsto \frac{\log_2 x}{x}$ is non-increasing on $[e, \infty)$. Since τ is bijective and $c(u) \le D_1$ on U', terms in U' with $d_G(\tau(u)) > D_2$ contribute at most

$$D_1 \sum_{w: d_G(w) > D_2} d_G(w) \frac{\log_2 d_G(w)}{d_G(w)} \;\le\; D_1 \cdot \frac{\log_2 D_2}{D_2} \sum_{w\in V(G)} d_G(w) \;=\; D_1 \cdot \frac{2m \log_2 D_2}{D_2}. \qquad \square$$

We now prove from Proposition 2.1 and Lemma 2.2 alone that edge-hypomorphic graphs with $m \ge 4$ edges share the same degree sequence and, after deleting isolated vertices, share the same vertex-deck and reduce to connected graphs.

Lemma 2.3 (cf. Greenwell–Hemminger [8] and Greenwell [7])**.** *Let G be a simple graph on n vertices with $m = |E(G)| \ge 4$ edges, and let $G' := G - V_{=0}(G)$ be obtained by deleting its $I(G)$ isolated vertices, with $n' := |V(G')| = n - I(G)$ and $\bar{d}' := 2m/n' \ge \bar{d} := 2m/n$. If H is any simple graph with $|V(H)| = n$ and $\mathcal{D}_E(G) = \mathcal{D}_E(H)$ (with $H' := H - V_{=0}(H)$), then $|V_{=d}(G)| = |V_{=d}(H)|$ for all $d \ge 0$ (in particular $I(G) = I(H)$), $\mathcal{D}_E(G') = \mathcal{D}_E(H')$, and $\mathcal{D}(G') = \mathcal{D}(H')$. Consequently, if G is not edge-reconstructible, then G' is connected with $n' \ge 4$ and is neither edge-reconstructible nor vertex-reconstructible, and $\Delta(G) \ge 2^{\bar{d}'/2 - 1 - \frac{\log_2 n'}{n'}}$ (in particular, if $\bar{d} \ge 10$, then $\Delta(G) > 2^{\bar{d}/2-2}$).*

Proof. If $G \cong H$, then $|V_{=d}(G)| = |V_{=d}(H)|$, $G' \cong H'$, $\mathcal{D}_E(G') = \mathcal{D}_E(H')$, and $\mathcal{D}(G') = \mathcal{D}(H')$ hold immediately. Assume $G \ncong H$. By Lemma 2.2(1), $|E(H)| = m \ge 4$ and $s(K, G) = s(K, H)$ for every n-vertex simple graph K with $|E(K)| < m$.

We first rule out $\Delta(G) = m$ and $\Delta(H) = m$. Any n-vertex simple graph Z with $|E(Z)| = m$ has $\Delta(Z) \le \min(n-1, m)$, with $\Delta(Z) = m$ if and only if $n \ge m+1$ and $Z \cong K_{1,m} \sqcup (n-m-1)K_1$. If $G \cong K_{1,m} \sqcup (n-m-1)K_1$, then every card of $\mathcal{D}_E(G) = \mathcal{D}_E(H)$ is isomorphic to $K_{1,m-1} \sqcup (n-m)K_1$. Pick any $e' \in E(H)$; then $H - e' \cong K_{1,m-1} \sqcup (n-m)K_1$ has a unique vertex x of degree $m-1 \ge 3$ (as $m \ge 4$), $m-1$ degree-1 neighbors of x, and $n-m$ isolated vertices. If $x \notin e'$, then deleting one of the $m-1$ edges incident to x in H leaves x of degree $m-2$ while all other vertices have degree at most $2 < m-1$ in H, contradicting $H - e'' \cong K_{1,m-1} \sqcup (n-m)K_1$. Hence $x \in e'$, and since H is simple, the other endpoint of e' must be an isolated vertex of $H - e'$, forcing $H \cong K_{1,m} \sqcup (n-m-1)K_1 \cong G$, a contradiction. By symmetry, $\Delta(G) \le R$ and $\Delta(H) \le R$, where $R := \min(n-1, m-1) \ge 1$.

For $r = 1$, $\sum_{v\in V(G)} \binom{d_G(v)}{1} = 2m = \sum_{v\in V(H)} \binom{d_H(v)}{1}$. For each $r \in \{2, \dots, R\}$, we have $r+1 \le n$ and $r < m$, so $F_r := K_{1,r} \sqcup (n-r-1)K_1$ is an n-vertex simple graph with r edges; because $r \ge 2$, a spanning copy of F_r in $Z \in \{G, H\}$ has a unique vertex of degree r and thus corresponds bijectively to a vertex $v \in V(Z)$ and an r-subset of $N_Z(v)$, giving $\sum_{v\in V(G)} \binom{d_G(v)}{r} = s(F_r, G) = s(F_r, H) = \sum_{v\in V(H)} \binom{d_H(v)}{r}$. Rewriting $\sum_{v\in V(Z)} \binom{d_Z(v)}{r} = \sum_{d=r}^{R} \binom{d}{r} |V_{=d}(Z)|$ and inducting downward on $r = R, R-1, \dots, 1$ yields $|V_{=d}(G)| = |V_{=d}(H)|$ for all $d \ge 1$, and subtracting from n gives $I(G) = I(H)$.

For every $e \in E(G) = E(G')$, the card $G - e$ has at least $I(G)$ isolated vertices, all interchangeable under $\mathrm{Aut}(G - e)$, and deleting any $I(G)$ isolated vertices from $G - e$ produces a graph isomorphic to $G' - e$ (and identically in H). Hence $\mathcal{D}_E(G') = \mathcal{D}_E(H')$, with $|V(G')| = |V(H')| = n'$, $\delta(G') \ge 1$, $\delta(H') \ge 1$, and $|E(G')| = |E(H')| = m \ge 4$ (forcing $\binom{n'}{2} \ge 4$, so $n' \ge 4$), while $G \ncong H$ and $I(G) = I(H)$ force $G' \ncong H'$.

To show $\mathcal{D}(G') = \mathcal{D}(H')$, let J be any $(n'-1)$-vertex simple graph and put $i(J) := I(J) \geq 0$. For $Z' \in \{G', H'\}$, consider the set $\mathcal{P}$ of pairs (v, E_0) with $v \in V(Z')$ and $E_0 \subseteq E(Z'-v)$ such that $(V(Z') \setminus \{v\}, E_0) \cong J$. Grouping by v gives $|\mathcal{P}| = \sum_{v \in V(Z')} s(J, Z'-v)$, which equals $\sum_{[J']:|V(J')|=n'-1} \mathrm{ind}(J', Z')\, s(J, J')$. Grouping by $E_0 \subseteq E(Z')$, $(V(Z'), E_0) \cong J \sqcup K_1$ holds whenever $(v, E_0) \in \mathcal{P}$, and each spanning subgraph $(V(Z'), E_0) \cong J \sqcup K_1$ has exactly $i(J)+1$ isolated vertices v, every one of which satisfies $E_0 \subseteq E(Z'-v)$ and $(V(Z') \setminus \{v\}, E_0) \cong J$. Thus

$$(i(J)+1)\, s(J \sqcup K_1, Z') \;=\; \sum_{[J']:|V(J')|=n'-1} \mathrm{ind}(J', Z')\, s(J, J').$$

Because $\delta(G') \geq 1$ and $\delta(H') \geq 1$, every vertex-deleted card of G' or H' has at most $m-1$ edges, so $\mathrm{ind}(J', G') = \mathrm{ind}(J', H') = 0$ whenever $|E(J')| \geq m$. For $|E(J)| = m-1, m-2, \ldots, 0$, we have $|E(J \sqcup K_1)| = |E(J)| < m$, so $s(J \sqcup K_1, G') = s(J \sqcup K_1, H')$ by Lemma 2.2(1); since $s(J, J') = 0$ for $|E(J')| < |E(J)|$ and $s(J, J') = \mathbf{1}\{[J'] = [J]\}$ for $|E(J')| = |E(J)|$, downward induction on $|E(J)|$ yields $\mathrm{ind}(J, G') = \mathrm{ind}(J, H')$ for all $(n'-1)$-vertex graphs J, i.e., $\mathcal{D}(G') = \mathcal{D}(H')$.

Since $G' \ncong H'$ on $n' \geq 4$ vertices satisfy $\mathcal{D}_E(G') = \mathcal{D}_E(H')$ and $\mathcal{D}(G') = \mathcal{D}(H')$, G' is neither edge-reconstructible nor vertex-reconstructible, and Proposition 2.1(3) forces G' to be connected. Applying Lemma 2.2(2) to G' gives $\bar{d}' \leq 2 + 2\log_2 \Delta(G') + \frac{2\log_2 n'}{n'}$, i.e., $\Delta(G) = \Delta(G') \geq 2^{\bar{d}'/2 - 1 - \frac{\log_2 n'}{n'}}$. Finally, if $\bar{d} \geq 10$, then $\frac{\log_2 n'}{n'} \leq \frac{\log_2 4}{4} = \frac{1}{2} < 1$ (as $x \mapsto (\log_2 x)/x$ decreases on $[e, \infty)$ and $n' \geq 4$) and $\bar{d}' \geq \bar{d}$, so $\Delta(G) \geq 2^{\bar{d}'/2 - 3/2} > 2^{\bar{d}/2-2}$. □

2.3 Elementary Analytic, Decision-Tree, and Dyadic Summation Bounds

We conclude this section with the numerical constants and real-analytic, decision-tree, and dyadic-series inequalities used repeatedly below.

Lemma 2.4. *The following bounds hold:*

(1) (Numerical constants.) $2.71 < e < 2.72$, $0.693 < \ln 2 < 0.694$, $2.30 < \ln 10 < 2.31$, *and* $3.11 < \pi < 3.15$. *Moreover, for every integer* $w \geq 1$, $w! > (w/e)^w$.

(2) (Log-linearization.) *For every* $\varepsilon \in (0, 1]$ *and every real* $x \geq 0$,

$$\log_2(1+x) \;\leq\; \frac{\varepsilon}{\ln 2}\, x \;+\; \log_2\!\left(\frac{1}{\varepsilon}\right).$$

(3) (Decision-tree leaf-weight bound.) *Let* $\mathcal{T}_{\mathrm{dec}}$ *be a finite rooted tree with child-count function* $b(\cdot)$, *where a node* x *is internal if* $b(x) \geq 1$ *and a leaf if* $b(x) = 0$. *For each leaf* ℓ *of* $\mathcal{T}_{\mathrm{dec}}$, *let* $P(\ell)$ *be the set of internal nodes on the unique path from the root to* ℓ *and set* $W(\ell) := \prod_{x \in P(\ell)} b(x)$ *(with empty product equal to 1). Then* $\sum_\ell W(\ell)^{-1} = 1$, *and in particular* $\#\{$*leaves of* $\mathcal{T}_{\mathrm{dec}}\} \leq \max_\ell W(\ell)$.

(4) (Dyadic scalar sums.)

(a) For every real $x_0 \geq 4$ *and integer* $r \geq 0$, $\sum_{k=0}^{r} \frac{2^{x_0+k}}{x_0+k} \leq \frac{6 \cdot 2^{x_0+r}}{x_0+r}$.

(b) For every real $A \geq 40$ *and integer* J *satisfying* $4 \leq J < A-2$,

$$S(J, A) \;:=\; \sum_{j=4}^{J} \frac{2^j}{j-1}(A-j) \;\leq\; 6 \cdot \frac{2^J}{J}(A-J+1).$$

Proof. (1) From $e = \sum_{k=0}^{\infty} \frac{1}{k!}$, $\sum_{k=0}^{5} \frac{1}{k!} = \frac{163}{60} > 2.71$, while $\sum_{k=6}^{\infty} \frac{1}{k!} \le \frac{1}{720} \sum_{j=0}^{\infty} 6^{-j} = \frac{1}{600}$ gives $e \le \frac{1631}{600} < 2.72$. For any integer $w \ge 1$, $e^w = \sum_{k=0}^{\infty} \frac{w^k}{k!} > \frac{w^w}{w!}$, so $w! > (w/e)^w$. Using $\frac{1}{2} \ln \frac{1+z}{1-z} = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1}$ ($|z| < 1$): at $z = \frac{1}{3}$ ($\frac{1+z}{1-z} = 2$), the first three terms give $\ln 2 > 2(\frac{1}{3} + \frac{1}{81} + \frac{1}{1215}) = \frac{842}{1215} > 0.693$, and bounding the tail by $\frac{2}{7 \cdot 3^7 (1-1/9)} < 0.0002$ gives $\ln 2 < 0.694$; at $z = \frac{1}{9}$ ($\frac{1+z}{1-z} = \frac{5}{4}$), $2(\frac{1}{9}) < \ln(5/4) < 2(\frac{1}{9} + \frac{1}{3 \cdot 9^3 (1-1/81)}) < 0.224$, so $\ln 10 = 3 \ln 2 + \ln(5/4) \in (2.079 + 0.222,\, 2.082 + 0.224) \subset (2.30, 2.31)$. From $\tan(\arctan \frac{1}{2} + \arctan \frac{1}{3}) = \frac{1/2+1/3}{1-1/6} = 1$, we have $\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3}$; applying the alternating series $\arctan z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{2k+1}$ gives $4((\frac{1}{2} - \frac{1}{24}) + (\frac{1}{3} - \frac{1}{81})) = \frac{2020}{648} > 3.11$ as a lower bound and $4((\frac{1}{2} - \frac{1}{24} + \frac{1}{160}) + (\frac{1}{3} - \frac{1}{81} + \frac{1}{1215})) < 3.15$ as an upper bound.

(2) On $[0, \infty)$, $f(x) := \frac{\varepsilon}{\ln 2} x + \log_2(1/\varepsilon) - \log_2(1+x)$ has derivative $f'(x) = \frac{1}{\ln 2}(\varepsilon - \frac{1}{1+x})$, so f attains its global minimum on $[0, \infty)$ at $x^* := \frac{1}{\varepsilon} - 1 \ge 0$, where $f(x^*) = \frac{1-\varepsilon}{\ln 2} \ge 0$.

(3) Induct on the height h of $\mathcal{T}_{\mathrm{dec}}$. If $h = 0$, $\mathcal{T}_{\mathrm{dec}}$ is a single leaf with $W(\ell) = 1$. If $h \ge 1$, the root r has $b(r)$ child subtrees $\mathcal{T}_1, \dots, \mathcal{T}_{b(r)}$ of height $< h$, and every leaf ℓ in $\mathcal{T}_i$ has $W_{\mathcal{T}_{\mathrm{dec}}}(\ell) = b(r) W_{\mathcal{T}_i}(\ell)$, giving $\sum_\ell W_{\mathcal{T}_{\mathrm{dec}}}(\ell)^{-1} = \sum_{i=1}^{b(r)} \frac{1}{b(r)} \sum_{\ell \in \mathcal{T}_i} W_{\mathcal{T}_i}(\ell)^{-1} = 1$. Hence $1 = \sum_\ell W(\ell)^{-1} \ge \#\{\text{leaves}\} / \max_\ell W(\ell)$.

(4a) Put $X := x_0 + r \ge 4$. Indices k with $x_0 + k \le X/2$ have denominator $\ge x_0 \ge 4$ and numerator sum $< 2 \cdot 2^{X/2}$, contributing $< \frac{1}{2} 2^{X/2}$; since $2^{t/2} \ge t$ for all $t \ge 4$ (equality at $t = 4$, derivative $\frac{\ln 2}{2} 2^{t/2} - 1 > 0$), this is $\le \frac{2^X}{2X}$. Indices k with $x_0 + k > X/2$ have denominator $> X/2$ and numerator sum $< 2 \cdot 2^X$, contributing $< \frac{4 \cdot 2^X}{X}$. Adding gives $< \frac{4.5 \cdot 2^X}{X} < \frac{6 \cdot 2^X}{X}$.

(4b) Put $J_0 := \lfloor J/2 \rfloor$. For the lower half $4 \le j \le J_0$ (empty when $J \le 7$, so $J \ge 8$ when non-empty), $j - 1 \ge 3$ and $A - j < A$ give sum $< \frac{2}{3} A \cdot 2^{J_0} \le \frac{2}{3} A \cdot 2^{J/2}$. We check $AJ \le 2 \cdot 2^{J/2}(A - J + 1)$ for $J \ge 8$: if $J \le A/2$, then $2(A - J + 1) > A$ and $2^{J/2} \ge J$, giving $2 \cdot 2^{J/2}(A - J + 1) > AJ$; if $J > A/2$, then $J < A - 2$ gives $A - J + 1 > 3$ and $2^{J/2} > 2^{A/4}$, so $2 \cdot 2^{J/2}(A - J + 1) > 6 \cdot 2^{A/4} > A^2 > AJ$ (at $A = 40$, $6 \cdot 2^{10} = 6144 > 1600 = 40^2$, and for $A \ge 40$, $\frac{d}{dA}(\frac{\ln 2}{4} A + \ln 6 - 2 \ln A) = \frac{\ln 2}{4} - \frac{2}{A} > 0.12 > 0$). Thus the lower half is $< \frac{4}{3} \frac{2^J}{J}(A - J + 1)$.

For the upper half $\max(4, J_0 + 1) \le j \le J$, every such j satisfies $j - 1 \ge \frac{3}{7} J$ (for $4 \le J \le 7$, $j - 1 \ge 3 \ge \frac{3}{7} J$; for $J \ge 8$, $j - 1 \ge \lfloor J/2 \rfloor \ge \frac{J-1}{2} \ge \frac{7}{16} J > \frac{3}{7} J$). Setting $t := J - j \ge 0$ and using $\sum_{t=0}^{\infty} 2^{-t} = \sum_{t=0}^{\infty} t\, 2^{-t} = 2$, the upper half is at most

$$\frac{7}{3J} 2^J \sum_{t=0}^{\infty} 2^{-t}(A - J + t) \;=\; \frac{14}{3} \frac{2^J}{J}(A - J + 1).$$

Adding the two halves yields $S(J, A) \le (\frac{4}{3} + \frac{14}{3}) \frac{2^J}{J}(A - J + 1) = 6 \cdot \frac{2^J}{J}(A - J + 1)$. □

3 Multi-Scale Kruskal Spanning Trees and the Dyadic Entropy Obstruction

3.1 Degree-Ordered Kruskal Spanning Trees and Non-Bridge Rigidity

Let G be a connected finite simple graph on $n \ge 4$ vertices with $m = |E(G)|$ edges and average degree $\bar{d} := 2m/n$. Fix any strict total order $\prec_V$ on $V(G)$ compatible with vertex degrees in the sense that $d_G(x) < d_G(y) \implies x \prec_V y$, and let $r_0 := \max_{\prec_V} V(G)$ (so $d_G(r_0) = \Delta(G)$). Write each edge $e \in E(G)$ as $e = \{x, y\}$ with $y \prec_V x$, set its weight to $w(e) := \max(d_G(x), d_G(y)) = d_G(x)$, and order $E(G)$ by the strict lexicographic order $<_{\mathrm{lex}}$ on pairs (x, y): $(x, y) <_{\mathrm{lex}} (x', y')$ if either $x \prec_V x'$, or $x = x'$ and $y \prec_V y'$. Listing $E(G)$ as $e_1 <_{\mathrm{lex}} e_2 <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} e_m$ and scanning $j = 1, \dots, m$ in order, include e_j whenever its endpoints are not yet connected by previously selected edges. Because

G is connected, this greedy procedure produces a spanning tree T of G, rooted at r_0, called a *degree-ordered Kruskal minimum spanning tree* $(T, \prec_V)$. By construction, for every non-tree edge $e = e_j \in E(G) \setminus E(T)$, the endpoints of e are already connected in $T \cap \{e_1, \dots, e_{j-1}\}$, so every edge f on the unique path in T between the endpoints of e satisfies $f <_{\mathrm{lex}} e$ and hence $w(f) \le w(e)$.

For any integer Δ and any component $C \in \mathrm{Comp}(G[V_{<\Delta}(G)])$, write $\deg_T(C)$ for the number of edges of T with one endpoint in $V(C)$ and the other in $V(G) \setminus V(C)$, define the *bridge components*

$$\mathcal{B}_\Delta \;:=\; \{C \in \mathrm{Comp}(G[V_{<\Delta}(G)]) : \deg_T(C) \ge 2\}, \qquad \mathcal{B}_\Delta^{\ge 2} \;:=\; \{C \in \mathcal{B}_\Delta : |V(C)| \ge 2\},$$

and whenever $N_G(C) \cap V_{\ge\Delta}(G) \neq \emptyset$, set $u(C) := \min_{\prec_V}(N_G(C) \cap V_{\ge\Delta}(G)) \in V_{\ge\Delta}(G)$.

Given strictly increasing integers $1 \le \Delta_0 < \Delta_1 < \cdots < \Delta_L$ $(L \ge 1)$ with $\Delta_L > \Delta(G)$, define for $0 \le k \le L$:

$$H_k \;:=\; V_{\ge\Delta_k}(G) \quad (\text{so } H_L = \emptyset), \qquad B_k \;:=\; V_{<\Delta_k}(G), \qquad U_k \;:=\; H_k \setminus H_{k+1} \quad (0 \le k \le L-1).$$

For each $0 \le k \le L-1$, define the level-k non-bridge component collection and 2^{H_k}-labeling $\lambda_k : B_k \to 2^{H_k}$ by

$$\mathcal{P}_k \;:=\; \{C \in \mathrm{Comp}(G[B_k]) : \deg_T(C) = 1 \text{ and } u(C) \in U_k\}, \qquad \lambda_k(v) \;:=\; N_G(v) \cap H_k \quad (v \in B_k),$$
$$\mathcal{F}_k \;:=\; \{[(C, \lambda_k|_{V(C)})] : C \in \mathcal{P}_k\} \;\subseteq\; \mathfrak{C}(H_k),$$
$$\mathcal{P}_k(\mathcal{T}) \;:=\; \{C \in \mathcal{P}_k : [(C, \lambda_k|_{V(C)})] = \mathcal{T}\}, \qquad \tilde{w}_{\mathcal{T}} \;:=\; |\mathcal{P}_k(\mathcal{T})| \ge 1 \qquad (\mathcal{T} \in \mathcal{F}_k).$$

For $0 \le p < q \le L-1$, set

$$\mathcal{P}_q^{(p)} \;:=\; \{C \in \mathcal{P}_q : V(C) \cap H_p \neq \emptyset\}, \qquad \mathcal{F}_q^{(p)} \;:=\; \{[(C, \lambda_q|_{V(C)})] : C \in \mathcal{P}_q^{(p)}\} \;\subseteq\; \mathcal{F}_q.$$

Note that for any $C \in \mathrm{Comp}(G[B_q])$ and $v \in V(C)$, $N_G(v) = N_C(v) \sqcup \lambda_q(v)$ gives $d_G(v) = d_C(v) + |\lambda_q(v)|$; hence all representatives of a class $\mathcal{T} \in \mathcal{F}_q$ have the same multiset of G-degrees, so $C \in \mathcal{P}_q^{(p)} \iff [(C, \lambda_q|_{V(C)})] \in \mathcal{F}_q^{(p)}$ (in particular, $\mathcal{P}_q(\mathcal{T}) \subseteq \mathcal{P}_q^{(p)}$ whenever $\mathcal{T} \in \mathcal{F}_q^{(p)}$). On $\mathcal{P}_k, \mathcal{F}_k, \mathcal{P}_q^{(p)}$, and $\mathcal{F}_q^{(p)}$, superscripts ≥ 2 and iso denote the partition into components with $|V(C)| \ge 2$ versus $|V(C)| = 1$ (for instance, $\mathcal{P}_k = \mathcal{P}_k^{\ge 2} \sqcup \mathcal{P}_k^{\mathrm{iso}}$ and $\mathcal{F}_k = \mathcal{F}_k^{\ge 2} \sqcup \mathcal{F}_k^{\mathrm{iso}}$).

We now prove that a single Kruskal tree $(T, \prec_V)$ simultaneously controls low-degree forests and bridge counts across every threshold, generates a multi-scale symmetric-group product inside $\mathrm{Aut}(G)$, and forces a linear component-entropy lower bound in any non-edge-reconstructible graph with $\bar{d} \ge 300$.

Lemma 3.1. *Let G be a connected finite simple graph on $n \ge 4$ vertices with $m = |E(G)|$, average degree $\bar{d} := 2m/n$, and degree-ordered Kruskal minimum spanning tree $(T, \prec_V)$ rooted at r_0.*

(1) (Simultaneous non-bridge rigidity.) *For every integer $\Delta \le \Delta(G)$, set $H_{[\Delta]} := V_{\ge\Delta}(G)$ and $B_{[\Delta]} := V_{<\Delta}(G)$. Then:*

- *(a) The edges of T of weight $< \Delta$ form a spanning forest of $G[B_{[\Delta]}]$.*
- *(b) Contracting each $C \in \mathrm{Comp}(G[B_{[\Delta]}])$ to a single vertex x_C transforms T into a tree $T[\Delta]$ on $H_{[\Delta]} \sqcup \{x_C : C \in \mathrm{Comp}(G[B_{[\Delta]}])\}$. We have $|\mathcal{B}_\Delta| \le |H_{[\Delta]}| - 1$, and at most $2|H_{[\Delta]}| - 2$ edges of T incident to $H_{[\Delta]}$ have their other endpoint in $H_{[\Delta]}$ or in $\bigcup_{C \in \mathcal{B}_\Delta} V(C)$.*
- *(c) Every non-bridge component $C \in \mathrm{Comp}(G[B_{[\Delta]}])$ ($\deg_T(C) = 1$) hangs in T as a child-subtree off the unique parent $u(C) = \min_{\prec_V}(N_G(C) \cap H_{[\Delta]}) \in H_{[\Delta]}$. Consequently, all non-bridge components in the same class $\mathcal{T} \in \mathfrak{C}(H_{[\Delta]})$ (under $v \mapsto N_G(v) \cap H_{[\Delta]}$) share the same T-parent $u(\mathcal{T}) \in H_{[\Delta]}$.*

(2) (Multi-scale component-twin automorphism factorization.) *For any strictly increasing integers* $1 \le \Delta_0 < \Delta_1 < \cdots < \Delta_L$ *($L \ge 1$) with* $\Delta_L > \Delta(G)$,

$$|\operatorname{Aut}(G)| \;\ge\; \prod_{k=0}^{L-1} \prod_{\mathcal{T}\in\mathcal{F}_k} \tilde{w}_{\mathcal{T}}!.$$

(3) (Scale-free dyadic component-entropy obstruction.) *Suppose in addition that G is not edge-reconstructible and $\bar{d} \ge 300$. Set $\Delta_0 := \lceil 2^{\bar{d}/16} \rceil$, $\Delta_k := 2^k \Delta_0$ ($k \ge 0$), and $L := \min\{k \ge 1 : \Delta_k > \Delta(G)\}$, and let $\mathcal{F}_k^{\mathrm{iso},\, 2\le r<100\log_2(2\Delta_k)}$ denote those classes $\mathcal{T} \in \mathcal{F}_k^{\mathrm{iso}}$ whose singleton components $\{w\} \in \mathcal{P}_k(\mathcal{T})$ have degree $r = d_G(w)$ satisfying $2 \le r < 100\log_2(2\Delta_k)$ (necessarily $N_G(w) \subseteq H_k \subseteq V_{\ge r+1}(G)$). Then*

$$\sum_{k=0}^{L-1} \sum_{\substack{\mathcal{T}\in\mathcal{F}_k^{\ge 2} \,\cup\, \mathcal{F}_k^{\mathrm{iso},\, 2\le r<100\log_2(2\Delta_k)} \\ \tilde{w}_{\mathcal{T}} < 2\Delta_k/2^{\bar{d}/8}}} \tilde{w}_{\mathcal{T}} \log_2\!\left(\frac{2\Delta_k}{\tilde{w}_{\mathcal{T}}}\right) \;>\; 0.15\, m.$$

Proof. (1) Notice $r_0 \in H_{[\Delta]}$ since $d_G(r_0) = \Delta(G) \ge \Delta$. Let $F_{[\Delta]} \subseteq E(G[B_{[\Delta]}])$ be the set of edges of T of weight $< \Delta$. If an edge $e \in E(G[B_{[\Delta]}])$ joined two distinct components of $(B_{[\Delta]}, F_{[\Delta]})$, then the unique path in T between the endpoints of e would contain an edge $f \notin F_{[\Delta]}$, so $w(f) \ge \Delta > w(e)$, contradicting $w(f) \le w(e)$. Thus $(B_{[\Delta]}, F_{[\Delta]})$ is a spanning forest of $G[B_{[\Delta]}]$, proving (a).

For each $C \in \mathrm{Comp}(G[B_{[\Delta]}])$, $T[V(C)]$ is a connected spanning tree of C. Because T is acyclic, contracting each subtree $T[V(C)]$ to a single vertex x_C produces a simple tree $T[\Delta]$ on $H_{[\Delta]} \sqcup \{x_C : C \in \mathrm{Comp}(G[B_{[\Delta]}])\}$ with $d_{T[\Delta]}(x_C) = \deg_T(C)$. No edge of G joins distinct components of $G[B_{[\Delta]}]$, so $\{x_C : C \in \mathrm{Comp}(G[B_{[\Delta]}])\}$ is an independent set in $T[\Delta]$ and every neighbor of x_C in $T[\Delta]$ lies in $H_{[\Delta]}$. Deleting all vertices x_C with $\deg_T(C) = 1$ therefore leaves a connected tree $T_{\mathrm{core}}[\Delta]$ on $H_{[\Delta]} \sqcup \{x_C : C \in \mathcal{B}_\Delta\}$ in which every x_C ($C \in \mathcal{B}_\Delta$) retains all its neighbors in $H_{[\Delta]}$ and hence satisfies $d_{T_{\mathrm{core}}[\Delta]}(x_C) = \deg_T(C) \ge 2$. Consequently,

$$2|\mathcal{B}_\Delta| \;\le\; \sum_{C\in\mathcal{B}_\Delta} d_{T_{\mathrm{core}}[\Delta]}(x_C) \;\le\; |E(T_{\mathrm{core}}[\Delta])| \;=\; |H_{[\Delta]}| + |\mathcal{B}_\Delta| - 1,$$

giving $|\mathcal{B}_\Delta| \le |H_{[\Delta]}| - 1$ and $|E(T_{\mathrm{core}}[\Delta])| \le 2|H_{[\Delta]}| - 2$. Since the edges of $T_{\mathrm{core}}[\Delta]$ correspond bijectively to the edges of T incident to $H_{[\Delta]}$ whose other endpoint lies in $H_{[\Delta]}$ or in $\bigcup_{C\in\mathcal{B}_\Delta} V(C)$, (b) follows.

For (c), let $C \in \mathrm{Comp}(G[B_{[\Delta]}])$ satisfy $\deg_T(C) = 1$, and let $\{u_1, v(C)\}$ be the unique edge of T exiting C, with $v(C) \in V(C)$ and $u_1 \in H_{[\Delta]}$. Since $r_0 \in H_{[\Delta]} \setminus V(C)$, the unique T-path from any vertex of C to r_0 exits C via $\{u_1, v(C)\}$, so C hangs in T as a child-subtree off u_1 rooted at $v(C)$. Put $u_0 := \min_{\prec_V}(N_G(C) \cap H_{[\Delta]}) = u(C)$ and pick $v \in V(C)$ with $\{u_0, v\} \in E(G)$. Because $v, v(C) \in B_{[\Delta]}$ and $u_0, u_1 \in H_{[\Delta]}$, we have $v \prec_V u_0$ and $v(C) \prec_V u_1$. If $u_0 \prec_V u_1$, then $\{u_0, v\} \notin E(T)$ and the unique T-path between u_0 and v must traverse the cut-edge $\{u_1, v(C)\}$, contradicting $\{u_1, v(C)\} <_{\mathrm{lex}} \{u_0, v\}$. Hence $u_1 = u_0 = u(C)$, which depends only on $N_G(C) \cap H_{[\Delta]}$ and is therefore constant across any class $\mathcal{T} \in \mathfrak{C}(H_{[\Delta]})$.

(2) For each $0 \le k \le L-1$, put $G_k := \{\sigma \in \mathrm{Aut}(G) : \sigma|_{H_k} = \mathrm{id}_{H_k}\}$. If $\mathcal{F}_k = \emptyset$, set $A_k := \{\mathrm{id}_{V(G)}\}$. Otherwise $\Delta_k \le \Delta(G)$; for each $\mathcal{T} \in \mathcal{F}_k$, list its copies in $\mathcal{P}_k(\mathcal{T})$ as $C_{\mathcal{T},1}, \ldots, C_{\mathcal{T},\tilde{w}_{\mathcal{T}}}$ and fix λ_k-preserving graph isomorphisms $\psi_{\mathcal{T},j} : C_{\mathcal{T},1} \xrightarrow{\sim} C_{\mathcal{T},j}$ with $\psi_{\mathcal{T},1} = \mathrm{id}_{V(C_{\mathcal{T},1})}$. Distinct components in $\mathcal{P}_k$ are pairwise non-adjacent in G and have all external neighbors in H_k. Defining, for each tuple $(\pi_{\mathcal{T}})_{\mathcal{T}\in\mathcal{F}_k} \in \prod_{\mathcal{T}\in\mathcal{F}_k} \mathrm{Sym}([\tilde{w}_{\mathcal{T}}])$, the map $g \in \mathrm{Sym}(V(G))$ equal to $\psi_{\mathcal{T},\pi_{\mathcal{T}}(j)} \circ \psi_{\mathcal{T},j}^{-1}$ on

each $V(C_{\mathcal{T},j})$ and fixing $V(G) \setminus \bigcup_{C\in\mathcal{P}_k} V(C)$ pointwise yields a subgroup $A_k \le G_k$ isomorphic to $\prod_{\mathcal{T}\in\mathcal{F}_k} \mathrm{Sym}([\tilde{w}_{\mathcal{T}}])$, of order $\prod_{\mathcal{T}\in\mathcal{F}_k} \tilde{w}_{\mathcal{T}}!$, such that only $\mathrm{id}_{V(G)} \in A_k$ fixes every $C \in \mathcal{P}_k$ setwise.

We verify $A_k \cap \langle A_0, \dots, A_{k-1}\rangle = \{\mathrm{id}_{V(G)}\}$ for every $0 \le k \le L-1$. Put $U_{-1} := B_0$. Fix $i \in \{0, \dots, k-1\}$, $g \in A_i$, and $C \in \mathcal{P}_k$. Since $V(C) \subseteq B_k = \bigsqcup_{j=-1}^{k-1} U_j$ is non-empty, let $j^* \in \{-1, 0, \dots, k-1\}$ be the largest index with $V(C) \cap U_{j^*} \neq \emptyset$. If $i > j^*$, then $V(C) \subseteq B_i$ and $N_G(C) \subseteq H_k \subseteq H_i$, so $C \in \mathrm{Comp}(G[B_i])$ with $\deg_T(C) = 1$ and $u(C) \in U_k$; because $U_k \cap U_i = \emptyset$, $C \notin \mathcal{P}_i$, so $g \in A_i$ fixes $V(C)$ pointwise. If $i \le j^*$, then $j^* \ge 0$ and $U_{j^*} \subseteq H_i$, so some $w \in V(C) \cap H_i$ satisfies $g(w) = w$ (as $A_i \le G_i$); since $g \in \mathrm{Aut}(G)$ stabilizes B_k setwise, g maps the component $C \in \mathrm{Comp}(G[B_k])$ containing w onto itself. Hence every element of $\langle A_0, \dots, A_{k-1}\rangle$ fixes every $C \in \mathcal{P}_k$ setwise, forcing $A_k \cap \langle A_0, \dots, A_{k-1}\rangle = \{\mathrm{id}_{V(G)}\}$. Consequently, the product map $(a_{L-1}, \dots, a_0) \mapsto a_{L-1} \circ \cdots \circ a_0$ from $A_{L-1} \times \cdots \times A_0$ to $\mathrm{Aut}(G)$ is injective, proving $|\mathrm{Aut}(G)| \ge \prod_{k=0}^{L-1} |A_k|$.

(3) Since G is connected on $n \ge 4$ vertices with $\bar{d} \ge 300$, we have $m \ge 150n \ge 600 \ge 4$ and $I(G) = 0$. Note $(1.68)^4 = 7.96594176 < 8 = 2^3$, so $2^{3/4} > 1.68$ and $2^{\bar{d}/16} \ge 2^{18.75} = 2^{18} \cdot 2^{3/4} > 4.4 \times 10^5$; thus $\Delta_0 = \lceil 2^{\bar{d}/16} \rceil > 4.4 \times 10^5$. Because G is not edge-reconstructible, Lemma 2.3 gives $\Delta(G) > 2^{\bar{d}/2-2} > \Delta_0$, so $L \ge 1$ and $H_k \neq \emptyset$ for all $0 \le k \le L-1$. Applying Lemma 2.2(2) to $T_0 = T$ produces $\tau \in \mathrm{Sym}(V(G))$ with $c(u) \le d_G(\tau(u))$ for all $u \in V(G)$ such that

$$m - n + \log_2 |\mathrm{Aut}(G)| \;\le\; \log_2 n + \sum_{u\in V(G)} \log_2\big((d_G(\tau(u)))_{c(u)}\big).$$

Partition $V(G) = U_{-1} \sqcup \bigsqcup_{k=0}^{L-1} U_k$, where $U_{-1} := B_0$.

- *Vertices in U_{-1}.* Every $u \in U_{-1}$ has $c(u) \le d_G(u) < \Delta_0 \le 2^{\bar{d}/16} + 1$. Applying Lemma 2.2(3) to $U' := U_{-1}$ with $D_1 := \Delta_0$ and $D_2 := 2^{\bar{d}/8} > e$ (using $\sum_{u\in U_{-1}} c(u) \le \sum_{u\in V(G)} c(u) = n - 1 < n$, $n \cdot \frac{\bar{d}}{8} = 0.25\, m$, and $\frac{\bar{d}}{4} \frac{2^{\bar{d}/16}+1}{2^{\bar{d}/8}} < \frac{5(\bar{d}/16)}{2^{\bar{d}/16}} \le \frac{5\cdot 18.75}{4.4\times 10^5} < 0.002$) gives

$$\sum_{u\in U_{-1}} \log_2\big((d_G(\tau(u)))_{c(u)}\big) \;\le\; 0.25\, m + 0.002\, m \;=\; 0.252\, m.$$

- *Vertices in U_k ($0 \le k \le L-1$).* Set $\mathcal{F}_k^* := \mathcal{F}_k^{\ge 2} \sqcup \{\mathcal{T} \in \mathcal{F}_k^{\mathrm{iso}} : |\lambda_k(w)| \ge 2 \text{ for } \{w\} \in \mathcal{P}_k(\mathcal{T})\}$. By Part (1), for each $u \in U_k$, the $c(u)$ children of u in T partition into: (i) $c_{\mathrm{leaf}}(u)$ degree-1 leaves of G adjacent to u (forming a single isolated-vertex class of multiplicity $c_{\mathrm{leaf}}(u)$ in $\mathcal{F}_k^{\mathrm{iso}} \setminus \mathcal{F}_k^*$ when $c_{\mathrm{leaf}}(u) \ge 1$, since $\Delta_k \ge \Delta_0 > 1$); (ii) $c^*_{\mathrm{nb}}(u) := \sum_{\mathcal{T}\in\mathcal{F}_k^*,\, u(\mathcal{T})=u} \tilde{w}_{\mathcal{T}}$ child roots $v(C)$ of non-bridge components $C \in \mathcal{P}_k$ whose class belongs to $\mathcal{F}_k^*$; and (iii) $c_{\mathrm{br}}(u)$ children lying in H_k or in $\bigcup_{C\in\mathcal{B}_{\Delta_k}} V(C)$. Using $(D)_{a+b+c} \le D^a D^b D^c$, we bound the three parts separately:

  (i) By Part (1b), $\sum_{u\in U_k} c_{\mathrm{br}}(u) < 2|H_k| \le \frac{4m}{\Delta_k}$, while every $u \in U_k$ has $c_{\mathrm{br}}(u) \le d_G(u) < 2\Delta_k$. Splitting at $d_G(\tau(u)) \le 4\Delta_k^2$ versus $d_G(\tau(u)) > 4\Delta_k^2$ as in Lemma 2.2(3) (using $\sum_{u\in U_k} d_G(\tau(u)) \le 2m$) bounds the total over all $0 \le k \le L-1$ by

$$\sum_{k=0}^{L-1} \left(\frac{4m}{\Delta_k} \log_2(4\Delta_k^2) + 2\Delta_k \cdot \frac{2m \log_2(4\Delta_k^2)}{4\Delta_k^2} \right) \;\le\; 5m \sum_{k=0}^{\infty} \frac{2 + 2\log_2 \Delta_0 + 2k}{2^k \Delta_0}$$
$$= \; 5m\, \frac{8 + 4\log_2 \Delta_0}{\Delta_0} \;<\; 0.01\, m,$$

  since $x \mapsto \frac{8+4\log_2 x}{x}$ decreases on $[e, \infty)$ and at $x = 2^{18} < \Delta_0$ equals $\frac{80}{262144} < 0.002$.

(ii) Let $v_1, \dots, v_p \in H_0$ be the vertices with $c_{\mathrm{leaf}}(v_j) \geq 1$. Write $M_{\mathrm{leaf}} := \sum_{j=1}^{p} c_{\mathrm{leaf}}(v_j) \leq n$, $q_j := c_{\mathrm{leaf}}(v_j)/n$, $S_{\mathrm{leaf}} := \sum_{j=1}^{p} q_j \in (0,1]$, and $\phi_j := d_G(\tau(v_j))/(\bar{d}n) > 0$ (with $\sum_{j=1}^{p} \phi_j \leq \frac{1}{2m}\sum_{w\in V(G)} d_G(w) = 1$). Using $c_{\mathrm{leaf}}(v_j)! > (c_{\mathrm{leaf}}(v_j)/e)^{c_{\mathrm{leaf}}(v_j)}$ (Lemma 2.4(1)) and concavity of $\log_2$ on $(0,\infty)$,

$$\begin{aligned}\sum_{j=1}^{p} \log_2 \frac{d_G(\tau(v_j))^{c_{\mathrm{leaf}}(v_j)}}{c_{\mathrm{leaf}}(v_j)!} &\leq M_{\mathrm{leaf}} \log_2(e\bar{d}) + nS_{\mathrm{leaf}} \sum_{j=1}^{p} \frac{q_j}{S_{\mathrm{leaf}}} \log_2\left(\frac{\phi_j}{q_j}\right) \\ &\leq n\log_2(e\bar{d}) + nS_{\mathrm{leaf}} \log_2\left(\frac{1}{S_{\mathrm{leaf}}}\right) \leq n\log_2(e\bar{d}) + n,\end{aligned}$$

since $x\log_2(1/x) \leq \frac{1}{e\ln 2} < 1$ on $(0,1]$ (and the bound holds trivially when $p = 0$).

(iii) Every component $C \in \bigsqcup_{k=0}^{L-1} \mathcal{P}_k$ maps injectively to its child root $v(C) \in V(G)\setminus\{r_0\}$ (its unique T-parent $u(C) \in U_k$ determines k, and distinct components in $\mathcal{P}_k$ are disjoint), so $\sum_{k=0}^{L-1}\sum_{\mathcal{T}\in\mathcal{F}_k} \tilde{w}_{\mathcal{T}} \leq n-1 < n$. Let $u_1, \dots, u_q \in H_0$ be the vertices with $c^*_{\mathrm{nb}}(u_j) \geq 1$, with $u_j \in U_{k(j)}$. Using $\tilde{w}_{\mathcal{T}}! > (\tilde{w}_{\mathcal{T}}/e)^{\tilde{w}_{\mathcal{T}}}$, the exact identity $\frac{ed_G(\tau(u_j))}{\tilde{w}_{\mathcal{T}}} = \frac{ed_G(\tau(u_j))}{c^*_{\mathrm{nb}}(u_j)} \cdot \frac{c^*_{\mathrm{nb}}(u_j)}{\tilde{w}_{\mathcal{T}}}$, the same concavity bound on the first sum, and $c^*_{\mathrm{nb}}(u_j) \leq d_G(u_j) < 2\Delta_{k(j)}$ on the second sum, we obtain

$$\begin{aligned}\sum_{j=1}^{q} \log_2 \frac{d_G(\tau(u_j))^{c^*_{\mathrm{nb}}(u_j)}}{\prod_{\mathcal{T}\in\mathcal{F}^*_{k(j)},\, u(\mathcal{T})=u_j} \tilde{w}_{\mathcal{T}}!} &\leq \sum_{j=1}^{q} c^*_{\mathrm{nb}}(u_j) \log_2\left(\frac{ed_G(\tau(u_j))}{c^*_{\mathrm{nb}}(u_j)}\right) \\ &\quad + \sum_{j=1}^{q} \sum_{\substack{\mathcal{T}\in\mathcal{F}^*_{k(j)} \\ u(\mathcal{T})=u_j}} \tilde{w}_{\mathcal{T}} \log_2\left(\frac{c^*_{\mathrm{nb}}(u_j)}{\tilde{w}_{\mathcal{T}}}\right) \\ &\leq n\log_2(e\bar{d}) + n + \sum_{k=0}^{L-1} \sum_{\mathcal{T}\in\mathcal{F}^*_k} \tilde{w}_{\mathcal{T}} \log_2\left(\frac{2\Delta_k}{\tilde{w}_{\mathcal{T}}}\right).\end{aligned}$$

By Part (2), $\log_2 |\operatorname{Aut}(G)| \geq \sum_{j=1}^{p} \log_2(c_{\mathrm{leaf}}(v_j)!) + \sum_{k=0}^{L-1}\sum_{\mathcal{T}\in\mathcal{F}^*_k} \log_2(\tilde{w}_{\mathcal{T}}!)$. For $n \geq 4$ and $\bar{d} \geq 300$, we have $\frac{\log_2 n}{n} \leq \frac{1}{2}$, and $x \mapsto \frac{3.5+2\log_2(ex)}{x}$ decreases on $[300,\infty)$; at $x = 300$, $300e < 300 \cdot 2.72 = 816 < 1024 = 2^{10}$, giving

$$\frac{n + \log_2 n + 2n\log_2(e\bar{d}) + 2n}{m} = \frac{2}{\bar{d}}\left(3 + \frac{\log_2 n}{n} + 2\log_2(e\bar{d})\right) < \frac{2}{300}(3.5+20) = \frac{47}{300} < 0.16.$$

Combining all estimates yields $\sum_{k=0}^{L-1}\sum_{\mathcal{T}\in\mathcal{F}^*_k} \tilde{w}_{\mathcal{T}} \log_2(\frac{2\Delta_k}{\tilde{w}_{\mathcal{T}}}) > (1 - 0.252 - 0.01 - 0.16)\, m = 0.578\, m$.

Finally, terms with $\tilde{w}_{\mathcal{T}} \geq 2\Delta_k/2^{\bar{d}/8}$ satisfy $\log_2(2\Delta_k/\tilde{w}_{\mathcal{T}}) \leq \bar{d}/8$ and $\sum \tilde{w}_{\mathcal{T}} < n$, contributing $< \frac{\bar{d}}{8}n = 0.25\, m$; for each isolated-vertex class $\mathcal{T} \in \mathcal{F}^{\mathrm{iso}}_k \cap \mathcal{F}^*_k$ whose singleton components $\{w\} \in \mathcal{P}_k(\mathcal{T})$ satisfy $r = d_G(w) \geq 100\log_2(2\Delta_k)$, every $\{w\} \in \mathcal{P}_k(\mathcal{T})$ satisfies $\log_2(2\Delta_k/\tilde{w}_{\mathcal{T}}) \leq \frac{1}{100} d_G(w)$, and the member vertices w across all such pairs $(k,\mathcal{T})$ are pairwise distinct in $V(G)$ (since $u(\{w\}) \in U_k$ is the unique T-neighbor of w), so their total contribution is $\sum_{k,\mathcal{T}}\sum_{\{w\}\in\mathcal{P}_k(\mathcal{T})} \log_2(2\Delta_k/\tilde{w}_{\mathcal{T}}) \leq \sum_{w\in V(G)} \frac{d_G(w)}{100} = 0.02\, m$. Every remaining isolated-vertex class $\mathcal{T} \in \mathcal{F}^{\mathrm{iso}}_k \cap \mathcal{F}^*_k$ has $2 \leq r < 100\log_2(2\Delta_k) < \Delta_k$ (since $100\log_2(2x) < x$ for $x > 4.4 \times 10^5$) and $N_G(w) \subseteq H_k \subseteq V_{\geq r+1}(G)$ for $\{w\} \in \mathcal{P}_k(\mathcal{T})$. Subtracting $0.25\, m + 0.02\, m$ from $0.578\, m$ leaves $> 0.308\, m > 0.15\, m$. □

3.2 Open-Twin Classes and Decomposition of the Isolated-Vertex Sum

For any simple graph G, the relation $u \sim v \iff N_G(u) = N_G(v)$ partitions $V(G)$ into *open-neighborhood twin classes* $V_1, \dots, V_s$, where $s = s(G) \geq 1$. Because G is loop-free, $u \notin N_G(u)$, so each twin class V_i is an independent set, and $N_G(V_i) = N_G(u)$ is the common neighborhood of all $u \in V_i$. Write $\nu_G := (|V_1|, \dots, |V_s|) \in \mathbb{Z}^s_{\geq 1}$ (with $\nu_G(i) := |V_i|$) and $t_{\max}(G) := \max_{1 \leq i \leq s} \nu_G(i)$. The *twin-quotient* $\widetilde{G}$ is the simple graph on $[s]$ in which $\{i, j\} \in E(\widetilde{G})$ if and only if some (equivalently, every) vertex of V_i is adjacent in G to some (equivalently, every) vertex of V_j.

Lemma 3.2. *Let G be a connected finite simple graph on $n \geq 4$ vertices with $m = |E(G)|$ and $\bar{d} := 2m/n \geq 300$, equipped with $(T, \prec_V)$ and dyadic thresholds $\Delta_0 := \lceil 2^{\bar{d}/16} \rceil$, $\Delta_k := 2^k \Delta_0$ ($k \geq 0$), and $L := \min\{k \geq 1 : \Delta_k > \Delta(G)\}$. Set $k_{\min} := \min\{k \geq 0 : 2\Delta_k > 2^{\bar{d}/8}\}$ (so $k_{\min} \geq 1$). With open-twin classes $V_1, \dots, V_s$ ($s := s(G)$), put $r_i := |N_G(V_i)|$ and (when $r_i \geq 1$) $d_{\min}(i) := \min_{a \in N_G(V_i)} d_G(a)$. For each $k \in \{k_{\min}, \dots, L-1\}$ and $\mathcal{T} \in \mathcal{F}_k^{\mathrm{iso}}$, write $X_{\mathcal{T}} \subseteq H_k$ for the common G-neighborhood of the vertices w with $\{w\} \in \mathcal{P}_k$ of class $\mathcal{T}$, and define*

$$\mathcal{I}_k := \Big\{ i \in [s] : 2 \leq r_i < 100 \log_2(2\Delta_k),\ \Delta_k \leq d_{\min}(i) < 2\Delta_k, \\ \exists\, \mathcal{T}_i \in \mathcal{F}_k^{\mathrm{iso}} \text{ with } N_G(V_i) = X_{\mathcal{T}_i} \text{ and } \tilde{w}_{\mathcal{T}_i} \leq \nu_G(i) < 2\tilde{w}_{\mathcal{T}_i} < \frac{4\Delta_k}{2^{\bar{d}/8}} \Big\},$$

$$E_k^{(2),\geq 3} := \sum_{\substack{i \in \mathcal{I}_k \\ \nu_G(i) \geq 3}} \nu_G(i) \log_2\!\Big(\frac{2\Delta_k}{\nu_G(i)}\Big), \qquad W_k := \bigsqcup_{\substack{i \in \mathcal{I}_k \\ \nu_G(i) \in \{1,2\}}} V_i.$$

Then:

(1) The sets $\mathcal{I}_k$ ($k_{\min} \leq k \leq L-1$) are pairwise disjoint; every $i \in \mathcal{I}_k$ satisfies $r_i < \Delta_{k-1} - 1$ and $N_G(V_i) \subseteq H_k$; and every vertex $w \in W_k$ is a leaf of T.

(2) We have

$$\sum_{k=0}^{L-1} \sum_{\substack{\mathcal{T} \in \mathcal{F}_k^{\mathrm{iso},\, 2 \leq r < 100 \log_2(2\Delta_k)} \\ \tilde{w}_{\mathcal{T}} < 2\Delta_k / 2^{\bar{d}/8}}} \tilde{w}_{\mathcal{T}} \log_2\!\Big(\frac{2\Delta_k}{\tilde{w}_{\mathcal{T}}}\Big) \\ \leq \frac{4(\log_2(2\Delta_0) + 1)}{\Delta_0}\, m + \sum_{k=k_{\min}}^{L-1} E_k^{(2),\geq 3} + \sum_{k=k_{\min}}^{L-1} |W_k| \log_2(2\Delta_k).$$

Proof. Since $\bar{d} \geq 300$, we have $\Delta_0 = \lceil 2^{\bar{d}/16} \rceil > 4.4 \times 10^5 > 2000$ and $2\Delta_0 \leq 2(2^{\bar{d}/16} + 1) < 2^{\bar{d}/8}$, so $k_{\min} \geq 1$. Every summand on the left-hand side of (2) has $\tilde{w}_{\mathcal{T}} \geq 1$ and $\tilde{w}_{\mathcal{T}} < 2\Delta_k / 2^{\bar{d}/8}$, forcing $2\Delta_k > 2^{\bar{d}/8}$ and hence $k \geq k_{\min}$. If $k_{\min} > L-1$, all sums over $k \in \{k_{\min}, \dots, L-1\}$ are empty and both parts hold vacuously.

Assume $k_{\min} \leq L-1$, so $\Delta_k \leq \Delta(G)$ and Lemma 3.1(1) applies at Δ_k for all $k_{\min} \leq k \leq L-1$. On $[2000, \infty)$, $x - 1 - 100 \log_2(4x)$ is positive at $x = 2000$ ($1999 > 1300 > 100 \log_2(8000)$) with derivative $1 - \frac{100}{x \ln 2} > 0$; since $\Delta_{k-1} \geq \Delta_0 > 2000$, we have $100 \log_2(2\Delta_k) = 100 \log_2(4\Delta_{k-1}) < \Delta_{k-1} - 1 < \Delta_k$.

Fix $k \in \{k_{\min}, \dots, L-1\}$ and $\mathcal{T} \in \mathcal{F}_k^{\mathrm{iso},\, 2 \leq r < 100 \log_2(2\Delta_k)}$ with $\tilde{w}_{\mathcal{T}} < 2\Delta_k / 2^{\bar{d}/8}$. Every $\{w\} \in \mathcal{P}_k$ of class $\mathcal{T}$ is isolated in $G[B_k]$, so $N_G(w) = X_{\mathcal{T}} \subseteq H_k$ with $|X_{\mathcal{T}}| = r < \Delta_{k-1} - 1$ and $u(\mathcal{T}) = \min_{\prec_V}(X_{\mathcal{T}}) \in U_k$. Let $V_i := \{z \in V(G) : N_G(z) = X_{\mathcal{T}}\}$. Because $\prec_V$ is degree-compatible,

$d_{\min}(i) = d_G(u(\mathcal{T})) \in [\Delta_k, 2\Delta_k)$. Disjointness of $[\Delta_k, 2\Delta_k)$ across k makes the sets $\mathcal{I}_k$ pairwise disjoint and ensures that distinct pairs $(k, \mathcal{T})$ map to distinct twin classes $i \in [s]$ (in particular $\mathcal{T}_i$ in the definition of $\mathcal{I}_k$ is unique). Moreover, every $z \in V_i$ has $d_G(z) = r < \Delta_k$ and $N_G(z) = X_{\mathcal{T}} \subseteq H_k$, so $\{z\} \in \mathrm{Comp}(G[B_k])$; by Lemma 3.1(1c), $\{z\}$ belongs to $\mathcal{P}_k$ (with class $\mathcal{T}$) whenever $\deg_T(\{z\}) = 1$, and belongs to $\mathcal{B}_{\Delta_k}$ whenever $\deg_T(\{z\}) \geq 2$. Putting $b_i := \nu_G(i) - \tilde{w}_{\mathcal{T}} \geq 0$, Lemma 3.1(1b) gives $\sum_{\mathcal{T}} b_i \leq |\mathcal{B}_{\Delta_k}| < |H_k| \leq \frac{2m}{\Delta_k}$.

Classes with $\nu_G(i) \geq 2\tilde{w}_{\mathcal{T}}$ satisfy $\tilde{w}_{\mathcal{T}} \leq b_i$, so their total contribution across all $k_{\min} \leq k \leq L-1$ is at most

$$\sum_{k=k_{\min}}^{L-1} \frac{2m}{\Delta_k} \log_2(2\Delta_k) \;\leq\; \frac{2m}{\Delta_0} \sum_{t=0}^{\infty} \frac{\log_2(2\Delta_0) + t}{2^t} \;=\; \frac{4(\log_2(2\Delta_0) + 1)}{\Delta_0}\, m.$$

Every remaining class maps to $i \in \mathcal{I}_k$ with $\tilde{w}_{\mathcal{T}_i} \leq \nu_G(i) < 2\tilde{w}_{\mathcal{T}_i} < \frac{4\Delta_k}{2^{\bar{d}/8}} < \frac{2\Delta_k}{e}$ (as $2^{\bar{d}/8} \geq 2^{37.5} > 2e$). Since $x \mapsto x \log_2(2\Delta_k/x)$ increases on $(0, 2\Delta_k/e]$ (derivative $\log_2(2\Delta_k/(ex)) \geq 0$), $\tilde{w}_{\mathcal{T}_i} \log_2(\frac{2\Delta_k}{\tilde{w}_{\mathcal{T}_i}}) \leq \nu_G(i) \log_2(\frac{2\Delta_k}{\nu_G(i)})$. Terms with $\nu_G(i) \geq 3$ sum to $E_k^{(2),\geq 3}$; terms with $\nu_G(i) \in \{1, 2\}$ force $\tilde{w}_{\mathcal{T}_i} = \nu_G(i)$ and $b_i = 0$, so every $w \in V_i$ satisfies $\{w\} \in \mathcal{P}_k$ and $d_T(w) = \deg_T(\{w\}) = 1$ (a leaf of T), contributing at most $\nu_G(i) \log_2(2\Delta_k)$. Summing over k proves (1) and (2). □

3.3 Counterexample Setup, Dyadic Estimates, and Pruned Core Trees

We now specialize Lemmas 2.3, 3.1, and 3.2 to a hypothetical counterexample to Theorem 1.1 and establish the structure of the pruned core trees T_p^*.

Lemma 3.3. *Suppose there exist non-isomorphic simple graphs G_0 and F_0 on $n_0 \geq 4$ vertices with $m = |E(G_0)| = |E(F_0)| \geq 1000\, n_0$ and $\mathcal{D}_E(G_0) = \mathcal{D}_E(F_0)$. Let $G := G_0 - V_{=0}(G_0)$ and $H := F_0 - V_{=0}(F_0)$. Then G and H are connected non-isomorphic simple graphs on $n := n_0 - I(G_0) \geq 4$ vertices with $|E(G)| = |E(H)| = m \geq 4000$, $\mathcal{D}_E(G) = \mathcal{D}_E(H)$, $\mathcal{D}(G) = \mathcal{D}(H)$, and $|V_{=d}(G)| = |V_{=d}(H)|$ for all $d \geq 0$. Set $\bar{d} := 2m/n$, fix a degree-ordered Kruskal minimum spanning tree $(T, \prec_V)$ of G rooted at r_0, define $\Delta_0 := \lceil 2^{\bar{d}/16} \rceil$, $\Delta_k := 2^k \Delta_0$, $L := \min\{k \geq 1 : \Delta_k > \Delta(G)\}$, $k_{\min} := \min\{k \geq 0 : 2\Delta_k > 2^{\bar{d}/8}\}$, and $\mathcal{L}_k := \log_2(4\Delta_k)$ $(0 \leq k \leq L-1)$, and let $Q := \widetilde{G}$ on $[s]$ $(s := s(G))$, $\mathcal{I}_k, E_k^{(2),\geq 3}$, and W_k be as in Lemma 3.2. Then:*

(1) $\bar{d} \geq 2000$, $\log_2 n > 1000$, $\Delta_0 \geq 2^{125}$, $\mathcal{L}_0 \geq 127$, $H_k \neq \emptyset$ for all $0 \leq k \leq L-1$, $\Delta_{L-1} \leq \Delta(G) < n$, $124 \leq k_{\min} \leq L-1$,

$$\frac{1}{4}\, 2^{\bar{d}/8} \;<\; \Delta_{k_{\min}-1} \;\leq\; \frac{1}{2}\, 2^{\bar{d}/8}, \qquad \text{and} \qquad \frac{1}{2}\, 2^{\bar{d}/8} \;<\; \Delta_{k_{\min}} \;\leq\; 2^{\bar{d}/8}.$$

(2) Defining

$$\text{Sum 1} \;:=\; \sum_{k=k_{\min}}^{L-1} \sum_{\substack{\mathcal{T} \in \mathcal{F}_k^{\geq 2} \\ \tilde{w}_{\mathcal{T}} < 2\Delta_k/2^{\bar{d}/8}}} \tilde{w}_{\mathcal{T}} \log_2\!\left(\frac{2\Delta_k}{\tilde{w}_{\mathcal{T}}}\right),$$

$$\text{Sum 2} \;:=\; \sum_{k=k_{\min}}^{L-1} \sum_{\substack{\mathcal{T} \in \mathcal{F}_k^{\mathrm{iso},\, 2 \leq r < 100 \log_2(2\Delta_k)} \\ \tilde{w}_{\mathcal{T}} < 2\Delta_k/2^{\bar{d}/8}}} \tilde{w}_{\mathcal{T}} \log_2\!\left(\frac{2\Delta_k}{\tilde{w}_{\mathcal{T}}}\right),$$

we have Sum 1 + Sum 2 $> 0.15\,m$ *and*

$$\text{Sum } 2 \;\le\; 10^{-34}\,m \;+\; \sum_{k=k_{\min}}^{L-1} E_k^{(2),\ge 3} \;+\; \sum_{k=k_{\min}}^{L-1} |W_k| \log_2(2\Delta_k).$$

(3) $\sum_{k=0}^{L-1} \Delta_k |H_k| \le 4m$, $\sum_{k=0}^{r} \frac{\Delta_k}{\mathcal{L}_k} \le \frac{6\Delta_r}{\mathcal{L}_r}$ *for every* $0 \le r \le L-1$, *and* $\sum_{k=0}^{L-1} \frac{\Delta_k}{\mathcal{L}_k} < \frac{6n}{\log_2(4n)}$.

Proof. By Lemma 2.3, $I(G_0) = I(F_0)$, and G, H are connected non-isomorphic simple graphs on $n = n_0 - I(G_0) \ge 4$ vertices with $|E(G)| = |E(H)| = m \ge 4000$, $\mathcal{D}_E(G) = \mathcal{D}_E(H)$, $\mathcal{D}(G) = \mathcal{D}(H)$, and identical degree sequences. Since $n \le n_0$, $\bar{d} = 2m/n \ge 2m/n_0 \ge 2000$. By Lemma 2.2(1), non-edge-reconstructibility of G gives $m < n \log_2 n$, so $\log_2 n > m/n = \bar{d}/2 \ge 1000$. By Lemma 2.3, $\Delta(G) > 2^{\bar{d}/2-2} > \lceil 2^{\bar{d}/16} \rceil = \Delta_0$, giving $L \ge 1$, $\Delta_{L-1} \le \Delta(G) < n$, and $r_0 \in H_k \ne \emptyset$ for all $0 \le k \le L-1$.

From $\bar{d}/16 \ge 125$, we get $\Delta_0 \ge 2^{125}$, $\mathcal{L}_0 \ge 127$, and $2\Delta_0 \le 2(2^{\bar{d}/16}+1) < 2^{\bar{d}/8}$, so $k_{\min} \ge 1$. Minimality of $k_{\min}$ gives $2\Delta_{k_{\min}-1} \le 2^{\bar{d}/8} < 2\Delta_{k_{\min}} = 4\Delta_{k_{\min}-1}$, proving the two-sided bounds on $\Delta_{k_{\min}-1}$ and $\Delta_{k_{\min}}$. Also $\Delta_0 < 2^{\bar{d}/16+1}$ implies $2^{k_{\min}} = \Delta_{k_{\min}}/\Delta_0 > 2^{\bar{d}/16-2}$, so $k_{\min} > \bar{d}/16 - 2 \ge 123$, i.e., $k_{\min} \ge 124$.

By Lemma 3.1(3) (as $\bar{d} \ge 2000 \ge 300$), the sum over $0 \le k \le L-1$ exceeds $0.15\,m$; since $\tilde{w}_{\mathcal{T}} \ge 1$ and $\tilde{w}_{\mathcal{T}} < 2\Delta_k/2^{\bar{d}/8}$ force $k \ge k_{\min}$, this sum equals Sum 1 + Sum 2 $> 0.15\,m > 0$, whence $k_{\min} \le L-1$. By Lemma 3.2(2) and the monotonicity of $u \mapsto \frac{\log_2(4u)}{u}$ on $[e, \infty)$,

$$\frac{4(\log_2(2\Delta_0)+1)}{\Delta_0} \;=\; \frac{4\log_2(4\Delta_0)}{\Delta_0} \;\le\; \frac{4\cdot 127}{2^{125}} \;<\; \frac{512}{2^{125}} \;=\; 2^{-116} \;<\; 10^{-34},$$

since $2^{10} = 1024 > 10^3$ implies $2^{116} = 2^6 \cdot (2^{10})^{11} > 64 \cdot 10^{33} > 10^{34}$, completing (1) and (2).

For (3), each $v \in H_0$ has $k(v) := \max\{k \ge 0 : \Delta_k \le d_G(v)\}$ satisfying $\sum_{k: v \in H_k} \Delta_k = \Delta_0(2^{k(v)+1} - 1) < 2\Delta_{k(v)} \le 2d_G(v)$; summing over $v \in V(G)$ gives $\sum_{k=0}^{L-1} \Delta_k |H_k| \le 2\sum_{v \in V(G)} d_G(v) = 4m$. Applying Lemma 2.4(4a) at $x_0 = \log_2(4\Delta_0) \ge 127 \ge 4$ and dividing by 4 gives $\sum_{k=0}^{r} \frac{\Delta_k}{\mathcal{L}_k} \le \frac{6\Delta_r}{\mathcal{L}_r}$. Since $u \mapsto \frac{u}{\log_2(4u)}$ is strictly increasing on $[1,\infty)$ (derivative $\frac{\log_2(4u) - 1/\ln 2}{(\log_2(4u))^2} > 0$) and $\Delta_{L-1} < n$, we obtain $\sum_{k=0}^{L-1} \frac{\Delta_k}{\mathcal{L}_k} \le \frac{6\Delta_{L-1}}{\mathcal{L}_{L-1}} < \frac{6n}{\log_2(4n)}$. □

Lemma 3.4. *Under the hypotheses and notation of Lemma 3.3, fix* $p \in \{0, \dots, L-1\}$. *Let* $T[\Delta_p]$ *be the tree of Lemma 3.1(1b) at threshold* Δ_p *(identifying each contracted vertex* x_C *with* $C \in \mathrm{Comp}(G[B_p])$*), and let* $T_{\mathrm{core}}^{(p)} := T_{\mathrm{core}}[\Delta_p]$ *be obtained from* $T[\Delta_p]$ *by deleting all non-bridge leaves* $C \in \mathrm{Comp}(G[B_p])$ *with* $\deg_T(C) = 1$, *so* $V(T_{\mathrm{core}}^{(p)}) = H_p \sqcup \mathcal{B}_{\Delta_p}$. *Define*

$$L_p^{\mathrm{iso}} \;:=\; \{v \in H_p : \exists\, q \in \{p+1, \dots, L-1\} \text{ with } \{v\} \in \mathcal{P}_q^{\mathrm{iso}}\}, \qquad T_p^* \;:=\; T_{\mathrm{core}}^{(p)} - L_p^{\mathrm{iso}}.$$

Then:

(1) L_p^{iso} *is an independent set of* G *with* $N_G(L_p^{\mathrm{iso}}) \subseteq H_p \setminus L_p^{\mathrm{iso}}$; *every* $v \in L_p^{\mathrm{iso}}$ *is a leaf of* $T_{\mathrm{core}}^{(p)}$ *distinct from the root* $r_0 \in U_{L-1}$. *Hence* T_p^* *is a connected tree rooted at* r_0 *on vertex set* $(H_p \setminus L_p^{\mathrm{iso}}) \sqcup \mathcal{B}_{\Delta_p}$, *with* $|E(T_p^*)| \le 2|H_p| - 2$.

(2) Every child edge $e = \{u, z\}$ *of* T_p^* *leaving* $u \in U_q \setminus L_p^{\mathrm{iso}}$ *(*$p \le q \le L-1$*) corresponds to a unique* T*-child edge* $e_T = \{u, x\}$ *of one of two disjoint types:*

- Type (b): $x \in H_q$ *or* $x \in V(C)$ *for some* $C \in \mathcal{B}_{\Delta_q}$. *For each* $q \in \{p, \dots, L-1\}$, *at most* $2|H_q| - 2$ *Type (b) child edges leave* $U_q \setminus L_p^{\mathrm{iso}}$.

- Type (a): $x \in V(C)$ *for some* $C \in \mathrm{Comp}(G[B_q])$ *with* $\deg_T(C) = 1$. *This occurs only for* $q \geq p+1$, *and the map* $e \mapsto C$ *is a bijection from the Type (a) child edges leaving* $U_q \setminus L_p^{\mathrm{iso}}$ *onto* $\mathcal{P}_q^{(p),\geq 2}$.

Consequently, $\sum_{q=p+1}^{L-1} \sum_{\mathcal{T} \in \mathcal{F}_q^{(p),\geq 2}} \tilde{w}_{\mathcal{T}} = \sum_{q=p+1}^{L-1} |\mathcal{P}_q^{(p),\geq 2}| \leq |E(T_p^*)| \leq 2|H_p| - 2$.

(3) For each $q \in \{p+1, \ldots, L-1\}$ *and each* $\mathcal{T} \in \mathcal{F}_q^{(p)}$, *fix one base component* $C_{0,\mathcal{T}} \in \mathcal{P}_q(\mathcal{T})$ *and, for every* $C \in \mathcal{P}_q(\mathcal{T})$, *one* λ_q*-preserving isomorphism* $\iota_C : C_{0,\mathcal{T}} \xrightarrow{\sim} C$ *with* $\iota_{C_{0,\mathcal{T}}} = \mathrm{id}_{V(C_{0,\mathcal{T}})}$, *and define the canonical isomorphism* $C_1 \to C_2$ *(*$C_1, C_2 \in \mathcal{P}_q(\mathcal{T})$*) by* $\iota_{C_2} \circ \iota_{C_1}^{-1}$. *Let* $A_q^{(p)} \leq \mathrm{Aut}(G)$ *be the subgroup of automorphisms acting on each* $\mathcal{P}_q(\mathcal{T})$ *(*$\mathcal{T} \in \mathcal{F}_q^{(p)}$*) by permuting its components via these canonical isomorphisms while fixing* $V(G) \setminus \bigcup_{C \in \mathcal{P}_q^{(p)}} V(C)$ *pointwise (so* $A_q^{(p)} \cong \prod_{\mathcal{T} \in \mathcal{F}_q^{(p)}} \mathrm{Sym}(\mathcal{P}_q(\mathcal{T}))$ *has order* $\prod_{\mathcal{T} \in \mathcal{F}_q^{(p)}} \tilde{w}_{\mathcal{T}}!$*). Then the product-restriction map*

$$\Psi : A_{L-1}^{(p)} \times A_{L-2}^{(p)} \times \cdots \times A_{p+1}^{(p)} \longrightarrow \mathrm{Aut}(G)|_{H_p}, \qquad (g_{L-1}, \ldots, g_{p+1}) \longmapsto (g_{L-1} \circ \cdots \circ g_{p+1})|_{H_p}$$

is injective, and therefore

$$|\mathrm{Aut}(G)|_{H_p}| \geq \prod_{q=p+1}^{L-1} \prod_{\mathcal{T} \in \mathcal{F}_q^{(p)}} \tilde{w}_{\mathcal{T}}!.$$

Proof. (1) For $v \in L_p^{\mathrm{iso}}$ with $\{v\} \in \mathcal{P}_q^{\mathrm{iso}}$ ($q \geq p+1$), $\{v\}$ is isolated in $G[B_q]$ with $\deg_T(\{v\}) = 1$ and $u(\{v\}) \in U_q$. Thus $N_G(v) \subseteq H_q \subseteq H_p$, so v has no neighbors in B_p and $d_T(v) = 1$; since $r_0 \in U_{L-1}$ while $v \in B_q \subseteq V(G) \setminus U_{L-1}$, v is a non-root leaf of $T_{\mathrm{core}}^{(p)}$. If $v, v' \in L_p^{\mathrm{iso}}$ satisfy $\{v\} \in \mathcal{P}_q^{\mathrm{iso}}, \{v'\} \in \mathcal{P}_{q'}^{\mathrm{iso}}$ with $q \leq q'$, then $v \in B_q \subseteq B_{q'}$ whereas $N_G(v') \subseteq H_{q'}$, so $\{v, v'\} \notin E(G)$. Hence L_p^{iso} is independent and $N_G(L_p^{\mathrm{iso}}) \subseteq H_p \setminus L_p^{\mathrm{iso}}$. Deleting non-root leaves from $T_{\mathrm{core}}^{(p)}$ leaves a connected tree T_p^* rooted at r_0, with $|E(T_p^*)| \leq |E(T_{\mathrm{core}}^{(p)})| \leq 2|H_p| - 2$ by Lemma 3.1(1b).

(2) Because $T[V(D)]$ is connected for each $D \in \mathrm{Comp}(G[B_p])$ and T is acyclic, contracting each $T[V(D)]$ preserves the rooted parent-child orientation relative to $r_0 \in H_p \setminus L_p^{\mathrm{iso}}$ and pairs each child edge $e = \{u, z\}$ of $u \in U_q \setminus L_p^{\mathrm{iso}}$ in T_p^* with a unique T-child edge $e_T = \{u, x\}$ ($x = z$ if $z \in H_p \setminus L_p^{\mathrm{iso}}$, or $x \in V(D)$ if $z = D \in \mathcal{B}_{\Delta_p}$). By Lemma 3.1(1b) at threshold Δ_q, at most $2|H_q| - 2$ Type (b) edges leave $U_q \setminus L_p^{\mathrm{iso}} \subseteq H_q$.

For a Type (a) child edge $e = \{u, z\}$ leaving $u \in U_q \setminus L_p^{\mathrm{iso}}$, $e_T = \{u, x\}$ is the unique T-edge exiting $C \in \mathrm{Comp}(G[B_q])$ ($x \in V(C)$, $\deg_T(C) = 1$), so $u = u(C) \in U_q$ and $C \in \mathcal{P}_q$. Since non-bridges of $G[B_p]$ are deleted in $T_{\mathrm{core}}^{(p)}$, $q \geq p+1$. If $x \in H_p$, then $x \in V(C) \cap H_p \neq \emptyset$; if $x \in V(D)$ for $D \in \mathcal{B}_{\Delta_p}$, then $V(D) \subseteq V(C)$ (as $B_p \subseteq B_q$) and a second T-edge exiting D cannot exit C (as $\deg_T(C) = 1$), so its H_p-endpoint lies in $V(C) \cap H_p \neq \emptyset$. Moreover, $|V(C)| \geq 2$, or else $C = \{x\}$ would force $x \in H_p$ and $\{x\} \in \mathcal{P}_q^{\mathrm{iso}}$, contradicting $z = x \notin L_p^{\mathrm{iso}}$. Thus $C \in \mathcal{P}_q^{(p),\geq 2}$, and $e \mapsto C$ is injective since e_T is the unique T-edge exiting C.

Conversely, let $C \in \mathcal{P}_q^{(p),\geq 2}$ ($q \geq p+1$), and let $e_T = \{u, x\}$ ($u = u(C) \in U_q, x \in V(C)$) be the unique T-edge exiting C (a T-child edge of u by Lemma 3.1(1c)). Here $u \notin L_p^{\mathrm{iso}}$ because any $v \in U_q \cap L_p^{\mathrm{iso}}$ satisfies $\{v\} \in \mathcal{P}_{q'}^{\mathrm{iso}}$ for some $q' > q$ and is a leaf of T whose sole T-neighbor lies in $U_{q'} \subseteq H_{q+1}$, not in B_q. If $x \in H_p$, then $x \notin L_p^{\mathrm{iso}}$ (if $\{x\} \in \mathcal{P}_{q'}^{\mathrm{iso}}$, then x is a T-leaf with sole T-neighbor $u(\{x\}) \in U_{q'}$, forcing $u(\{x\}) = u$, $q' = q$, $N_G(x) \subseteq H_q$, and $C = \{x\}$, contradicting $|V(C)| \geq 2$), so $e = \{u, x\} \in E(T_p^*)$ maps to C. If $x \in B_p$, let $D \in \mathrm{Comp}(G[B_p])$ contain x, so $V(D) \subseteq V(C)$; since $T[V(C)]$ spans C and $V(C) \cap H_p \neq \emptyset$, the first edge along the $T[V(C)]$-path

from x to $V(C) \cap H_p$ leaving $V(D)$ is a T-edge exiting D distinct from e_T (as $u \notin V(C)$), giving $\deg_T(D) \geq 2$, $D \in \mathcal{B}_{\Delta_p}$, and $e = \{u, D\} \in E(T_p^*)$ mapping to C.

(3) Note first that each canonical swap between $C_1, C_2 \in \mathcal{P}_q(\mathcal{T})$ ($\mathcal{T} \in \mathcal{F}_q^{(p)}$) lies in $\mathrm{Aut}(G)$ (components of $G[B_q]$ are pairwise non-adjacent in G and all external edges go to H_q, which is fixed pointwise), and coherence $(\iota_{C_3} \circ \iota_{C_2}^{-1}) \circ (\iota_{C_2} \circ \iota_{C_1}^{-1}) = \iota_{C_3} \circ \iota_{C_1}^{-1}$ with $\iota_C \circ \iota_C^{-1} = \mathrm{id}_{V(C)}$ ensures $A_q^{(p)} \cong \prod_{\mathcal{T} \in \mathcal{F}_q^{(p)}} \mathrm{Sym}(\mathcal{P}_q(\mathcal{T}))$ and that every $\tau \in A_q^{(p)} \setminus \{\mathrm{id}_{V(G)}\}$ moves some $C \in \mathcal{P}_q^{(p)}$ to a distinct component $\tau(C) \neq C$.

Suppose $(g_{L-1}, \ldots, g_{p+1}) \neq (h_{L-1}, \ldots, h_{p+1})$ in $A_{L-1}^{(p)} \times \cdots \times A_{p+1}^{(p)}$ satisfy $(g_{L-1} \circ \cdots \circ g_{p+1})|_{H_p} = (h_{L-1} \circ \cdots \circ h_{p+1})|_{H_p}$, and let $r^* \in \{p+1, \ldots, L-1\}$ be the largest index with $g_{r^*} \neq h_{r^*}$. Every element of $\mathrm{Aut}(G)$ stabilizes H_p setwise, so canceling $g_{L-1} \circ \cdots \circ g_{r^*+1} = h_{L-1} \circ \cdots \circ h_{r^*+1}$ and left-multiplying by $h_{r^*}^{-1}$ on H_p gives

$$\big(\tau \circ g_{r^*-1} \circ \cdots \circ g_{p+1}\big)\big|_{H_p} \;=\; \big(h_{r^*-1} \circ \cdots \circ h_{p+1}\big)\big|_{H_p}, \qquad \tau := h_{r^*}^{-1} g_{r^*} \in A_{r^*}^{(p)} \setminus \{\mathrm{id}_{V(G)}\}.$$

Pick $C \in \mathcal{P}_{r^*}^{(p)}$ with $\tau(C) \neq C$ (so $V(\tau(C)) \cap V(C) = \emptyset$) and pick $x \in V(C) \cap H_p$. For any $i \in \{p+1, \ldots, r^*-1\}$ and $f \in A_i^{(p)}$, every vertex moved by f belongs to some $D \in \mathcal{P}_i^{(p)}$ and is mapped into $f(D) \in \mathcal{P}_i^{(p)}$ sharing the same T-parent $u(D) \in U_i \subseteq B_{r^*}$ (Lemma 3.1(1c)); since $V(D) \cup \{u(D)\} \cup V(f(D))$ induces a connected subgraph of $G[B_{r^*}]$, either $V(D) \cup V(f(D)) \subseteq V(C)$ or $(V(D) \cup V(f(D))) \cap V(C) = \emptyset$, whence $f(V(C)) = V(C)$. Evaluating the displayed identity at $x \in V(C) \cap H_p$ places the left-hand side in $V(\tau(C))$ and the right-hand side in $V(C)$, a contradiction. □

4 Analytical Bounds for Shift-Invariant Polynomials

4.1 Charlier Integrals, Oscillatory Estimates, and Scale-Invariant Decay

For integers $d \geq 1$ and $k \geq 0$ and real parameters $\lambda > 0$ and $r > 0$, define the monic degree-d Charlier polynomial $q_d(x) \in \mathbb{R}[x]$ and scaled coefficient $I_{d,k}(r;\lambda)$ by

$$\frac{q_d(k)}{d!} \;:=\; \sum_{j=0}^{d} \frac{(-\lambda)^{d-j}}{(d-j)!} \binom{k}{j}, \qquad I_{d,k}(r;\lambda) \;:=\; \frac{q_d(k)}{d!}\, r^{d-k}.$$

Note that $q_d(x) = (x)_d + \sum_{j=0}^{d-1} \frac{d!}{(d-j)!}(-\lambda)^{d-j}\binom{x}{j}$ is indeed monic of degree d. For any $\tilde{\rho} > 0$, expanding the entire function $z \mapsto e^{-\lambda z}(1+z)^k = \sum_{n=0}^{\infty} c_n z^n$ (where $c_d = q_d(k)/d!$) uniformly on the circle $z = r\tilde{\rho}e^{i\theta}$ ($\theta \in [-\pi, \pi]$) and integrating against $\frac{1}{2\pi} z^{-d}\, d\theta$ term by term yields the exact integral representation

$$I_{d,k}(r;\lambda) \;=\; \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\rho}^{k-d} e^{-\lambda r \tilde{\rho} e^{i\theta}} \left(1 + \frac{e^{-i\theta}}{\tilde{\rho} r}\right)^k e^{i(k-d)\theta}\, d\theta \qquad (\forall\, \tilde{\rho} > 0). \tag{4.1}$$

Lemma 4.1. *The following analytical estimates hold:*

(1) (Small-d Charlier bound.) *Let $r \geq 4$ and let d be an integer with $1 \leq d \leq \frac{r}{2} - 1$. For $\lambda := \frac{1}{2r}$,*

$$\big|I_{d,k}(r;\lambda)\big| \;\leq\; 1 - \frac{11d}{24r} \;\leq\; \exp\!\left(-\frac{11d}{24r}\right) \qquad \textit{for all } k \in \mathbb{Z}_{\geq 0}.$$

(2) (Second-derivative van der Corput lemma and Bessel bound.)

(a) Let $a \le b$, let $\phi \in C^2([a,b],\mathbb{R})$ satisfy $|\phi''(\theta)| \ge \mu > 0$ on $[a,b]$, and let $a_0 \in C^1([a,b],\mathbb{C})$. Then

$$\left|\int_a^b a_0(\theta) e^{i\phi(\theta)}\, d\theta\right| \;\le\; \frac{8}{\sqrt{\mu}} \sup_{[a,b]} |a_0| \;+\; \frac{1}{\sqrt{\mu}} \int_a^b |a_0'(\theta)|\, d\theta.$$

(b) For $J_u(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix\sin\theta + iu\theta}\, d\theta$ ($u \in \mathbb{Z}, x \in \mathbb{R}$), we have $|J_u(x)| \le \frac{1}{\sqrt{2}}$ for all $u \in \mathbb{Z}\setminus\{0\}$ and $x \in \mathbb{R}$, and $|J_0(x)| < 0.889$ for all $x \ge 0.99$.

(3) (Uniform Charlier bounds for $\lambda = d/r^2 \le 0.01$.) *Let $r \ge 40$ and $d \ge 1$ satisfy $\lambda := \frac{d}{r^2} \le 0.01$, and put $M := \frac{d}{r} = \lambda r$. Then:*

(a) Whenever $|k-d| > 3M$, $|I_{d,k}(r;\lambda)| \le e^{0.02+1.5M-0.64|k-d|} < e^{0.02-0.42M}$.

(b) Whenever $M \ge 1$, $|I_{d,k}(r;\lambda)| \le 3.5\, M^{-1/3}$ for all $k \in \mathbb{Z}_{\ge 0}$.

(c) Whenever $r \ge 10^5$ and $\frac{r}{2} - 1 < d < 350r$, $|I_{d,k}(r;\lambda)| \le 0.903 < e^{-0.097}$ for all $k \in \mathbb{Z}_{\ge 0}$.

Proof. (1) Here $\lambda r = \frac{1}{2}$, $\frac{d+1}{r} \le \frac{1}{2}$, $d\lambda = \frac{d}{2r} < \frac{1}{4}$, and $1 - \frac{11d}{24r} > 1 - \frac{11}{48} = \frac{37}{48} > 0.77$. Notice also that $e^{1/4} < 1.3$ (since $\sum_{m=0}^{\infty} \frac{4^{-m}}{m!} \le 1 + \frac{1}{4} + \frac{1}{32}\sum_{j=0}^{\infty} 4^{-j} = 1 + \frac{1}{4} + \frac{1}{24} = \frac{31}{24} < 1.3$).

- *Case $k = d$.* Writing $s := d - j \in \{0,\dots,d\}$, $\binom{d}{s} \le \frac{d^s}{s!}$, and $(s!)^2 \ge 4 \cdot 9^{s-2}$ for $s \ge 2$,

$$\sum_{s=2}^{d} \frac{(d\lambda)^s}{(s!)^2} \;\le\; \frac{(d\lambda)^2}{4} \sum_{m=0}^{\infty} \left(\frac{d\lambda}{9}\right)^m \;\le\; \frac{(d\lambda)^2}{4(1-1/36)} \;=\; \frac{9}{35}(d\lambda)^2 \;<\; \frac{9}{140} d\lambda \;<\; \frac{1}{12} d\lambda,$$

  so $0 < 1 - \frac{13}{12}d\lambda < I_{d,d}(r;\lambda) \le 1 - \frac{11}{12}d\lambda = 1 - \frac{11d}{24r} \le \exp(-\frac{11d}{24r})$.

- *Case $0 \le k \le d-1$.* Put $u := d - k \ge 1$ and $s := k - j \in \{0,\dots,k\}$. Using $(u+s)! \ge u!s!$ and $\binom{k}{s} \le \frac{d^s}{s!}$,

$$|I_{d,k}(r;\lambda)| \;\le\; \frac{(\lambda r)^u}{u!} \sum_{s=0}^{k} \frac{(d\lambda)^s}{(s!)^2} \;\le\; \frac{2^{-u}}{u!} e^{d\lambda} \;\le\; \frac{1}{2} e^{1/4} \;<\; 0.65 \;<\; \frac{37}{48}.$$

- *Case $k = d+u$ ($u \ge 1$).* For $0 \le j \le d$, $\binom{d+u}{j} = \binom{d}{j} \prod_{\ell=0}^{u-1} \frac{d+u-\ell}{d+u-\ell-j} \le \binom{d}{j} \prod_{\ell=0}^{u-1} \frac{d+u-\ell}{u-\ell} \le \binom{d}{j}(d+1)^u$. Putting $s := d - j$,

$$|I_{d,d+u}(r;\lambda)| \;\le\; \left(\frac{d+1}{r}\right)^u \sum_{s=0}^{d} \frac{(d\lambda)^s}{(s!)^2} \;\le\; \frac{1}{2} e^{1/4} \;<\; 0.65 \;<\; \frac{37}{48}.$$

(2a) If $a = b$, the integral vanishes and the right-hand side is non-negative; assume $a < b$. Since ϕ'' is continuous with $|\phi''| \ge \mu > 0$, ϕ' is strictly monotone on $[a,b]$, so $|\phi'|$ achieves its minimum on $[a,b]$ at a unique point $c \in [a,b]$. Moreover, either $\phi'(c) = 0$ or $\phi'(c)$ and $\phi'(\theta) - \phi'(c)$ have the same sign for all $\theta \in [a,b] \setminus \{c\}$, whence $|\phi'(\theta)| = |\phi'(c)| + |\phi'(\theta) - \phi'(c)| \ge |\phi'(\theta) - \phi'(c)| \ge \mu|\theta - c|$. Set $\delta := \mu^{-1/2}$ and split $[a,b]$ into $I_0 := [a,b] \cap [c-\delta, c+\delta]$, $I_- := [a, c-\delta]$, and $I_+ := [c+\delta, b]$ (empty intervals contributing 0). Clearly $|\int_{I_0} a_0 e^{i\phi}| \le 2\delta \sup_{[a,b]} |a_0| = \frac{2}{\sqrt{\mu}} \sup_{[a,b]} |a_0|$. When $c + \delta \le b$, on I_+ we have $|\phi'| \ge \mu(\theta - c) \ge \sqrt{\mu}$ and $1/\phi'$ is monotone of constant sign (so $\int_{I_+} |\frac{d}{d\theta}(\frac{1}{\phi'(\theta)})|\, d\theta = |\frac{1}{\phi'(b)} - \frac{1}{\phi'(c+\delta)}| \le \frac{1}{\sqrt{\mu}}$), and integrating by parts via $e^{i\phi} = \frac{1}{i\phi'}\frac{d}{d\theta}(e^{i\phi})$ gives

$$\left|\int_{I_+} a_0(\theta) e^{i\phi(\theta)}\, d\theta\right| \;\le\; \frac{|a_0(b)|}{|\phi'(b)|} + \frac{|a_0(c+\delta)|}{|\phi'(c+\delta)|} + \int_{I_+} \frac{|a_0'(\theta)|}{|\phi'(\theta)|}\, d\theta + \sup_{[a,b]} |a_0| \int_{I_+} \left|\frac{d}{d\theta}\left(\frac{1}{\phi'(\theta)}\right)\right| d\theta$$

$$\leq \frac{3}{\sqrt{\mu}} \sup_{[a,b]} |a_0| + \frac{1}{\sqrt{\mu}} \int_{I_+} |a_0'(\theta)|\, d\theta,$$

and identically on I_- when $c - \delta \geq a$. Summing over I_0, I_-, I_+ yields the bound.

(2b) For $u \in \mathbb{Z} \setminus \{0\}$ and $f(\theta) := e^{-ix\sin\theta}$, we have $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\overline{e^{-iu\theta}}\, d\theta = J_u(x)$ and likewise $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\overline{e^{iu\theta}}\, d\theta = J_{-u}(x)$, so orthogonality of $e^{-iu\theta}$ and $e^{iu\theta}$ ($-u \neq u$) on $[-\pi,\pi]$ gives

$$0 \;\leq\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(\theta) - J_u(x)e^{-iu\theta} - J_{-u}(x)e^{iu\theta}\right|^2 d\theta \;=\; 1 - |J_u(x)|^2 - |J_{-u}(x)|^2.$$

Substituting $\theta \mapsto \pi - \theta$ and using 2π-periodicity gives $J_{-u}(x) = (-1)^u J_u(x)$, so $2|J_u(x)|^2 \leq 1$, i.e., $|J_u(x)| \leq 1/\sqrt{2}$. For $u = 0$, expanding $e^{-ix\sin\theta}$ uniformly gives $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}(x/2)^{2k}$, which satisfies $xJ_0''(x) + J_0'(x) + xJ_0(x) = 0$. Hence $E(x) := J_0(x)^2 + J_0'(x)^2$ satisfies $E'(x) = 2J_0'(x)(J_0(x) + J_0''(x)) = -\frac{2}{x}J_0'(x)^2 \leq 0$ for all $x > 0$. At $x = 0.99$, put $y := (0.99/2)^2 = 0.245025$; the alternating series for $J_0(0.99)$ and $-J_0'(0.99) = 0.495\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+1)!}y^k$ have decreasing term magnitudes, so truncating at $k = 2$ gives $0 < J_0(0.99) \leq 1 - y + \frac{y^2}{4} < 0.77$ and $0 < -J_0'(0.99) \leq 0.495(1 - \frac{y}{2} + \frac{y^2}{12}) < 0.437$. Thus for all $x \geq 0.99$, $J_0(x)^2 \leq E(x) \leq E(0.99) < 0.77^2 + 0.437^2 = 0.783869 < 0.790321 = 0.889^2$.

(3a) Applying (4.1) with $\tilde{\rho} \in \{1/2, 2\}$ (so $\frac{1}{\tilde{\rho}r} \leq \frac{1}{20}$) and $\ln(1+\xi) \leq \xi$ ($\xi > -1$),

$$\ln|I_{d,k}(r;\lambda)| \;\leq\; \sup_{\theta\in[-\pi,\pi]} \left\{ (k-d)\ln\tilde{\rho} + M\left(\frac{1}{\tilde{\rho}} - \tilde{\rho}\right)\cos\theta + \frac{k-d}{\tilde{\rho}r}\cos\theta + \frac{k}{2\tilde{\rho}^2 r^2} \right\}.$$

If $u := k - d > 3M$, take $\tilde{\rho} = \frac{1}{2}$: $\frac{d}{2\tilde{\rho}^2 r^2} = 2\lambda \leq 0.02$ and $\ln 2 - \frac{2}{40} - \frac{2}{1600} > 0.693 - 0.05125 > 0.64$ (Lemma 2.4(1)), giving $\ln|I_{d,k}| \leq 0.02 + 1.5M - 0.64u < 0.02 - 0.42M$. If $s := d - k > 3M$ ($k < d$), take $\tilde{\rho} = 2$: $\frac{k}{2\tilde{\rho}^2 r^2} < \frac{\lambda}{8} < 0.02$ and $\ln 2 - \frac{1}{80} > 0.64$, giving $\ln|I_{d,k}| \leq 0.02 + 1.5M - 0.64s < 0.02 - 0.42M$.

(3b) Assume $M \geq 1$. If $|k - d| > 3M$, (3a) gives $|I_{d,k}| < e^{0.02-0.42M} \leq 3.5M^{-1/3}$ (as $M^{1/3}e^{0.02-0.42M}$ decreases on $[1,\infty)$ and is $< 1 < 3.5$ at $M = 1$). Assume $|u| \leq 3M$, where $u := k - d$; then $k \leq d(1 + \frac{3}{r}) \leq 1.075d$. Take $\tilde{\rho} = 1$ in (4.1); since $1/r \leq 1/40 < 1$, expanding $\ln(1 + \frac{e^{-i\theta}}{r}) = \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{mr^m}e^{-im\theta}$ writes the integrand as $a_0(\theta)e^{i\phi(\theta)}$ with $a_0(\theta) > 0$,

$$\ln a_0(\theta) \;=\; \frac{u}{r}\cos\theta + k\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{mr^m}\cos(m\theta),$$

$$\phi(\theta) \;=\; u\theta - 2M\sin\theta - \frac{u}{r}\sin\theta + k\sum_{m=2}^{\infty}\frac{(-1)^m}{mr^m}\sin(m\theta).$$

Using $|u|/r \leq 3\lambda$, $k/r^2 \leq 1.075\lambda$, $\sum_{m=2}^{\infty}\frac{1}{mr^{m-2}} \leq \frac{1}{2(1-1/40)} < 0.52$, and $\sum_{m=2}^{\infty} r^{-(m-2)} \leq \frac{40}{39} < 1.03$, we get $|\ln a_0(\theta)| < 3.6\lambda \leq 0.036$, so $a_0(\theta) < e^{0.036} < 1.04$ (as $e^x < 1 + 1.1x$ on $(0, 0.05]$), and $|a_0'(\theta)/a_0(\theta)| < 4.2\lambda \leq 0.042$, so $|a_0'(\theta)| < 1.04 \cdot 0.042 < 0.044$. By induction on $m \geq 1$, $|\sin(m\theta)| \leq m|\sin\theta|$; since $\sum_{m=2}^{\infty}\frac{m^2}{r^{m-2}} \leq \frac{4}{1-9/160} = \frac{640}{151} < 4.24$ (using $m^2 \leq 4(9/4)^{m-2}$ for all $m \geq 2$), we have $|\phi''(\theta) - 2M\sin\theta| \leq (3 + 1.075 \cdot 4.24)\lambda|\sin\theta| < 7.6\lambda|\sin\theta| \leq 0.19M|\sin\theta|$, whence $|\phi''(\theta)| \geq \frac{3}{2}M|\sin\theta|$.

Put $\varepsilon := M^{-1/3} \in (0,1] \subset (0, \pi/2)$ and split $[-\pi,\pi]$ into $J_+ := [\varepsilon, \pi - \varepsilon]$, $J_- := [-\pi + \varepsilon, -\varepsilon]$, and their complement of measure 4ε. By Lemma 2.4(1) ($\pi > 3.11$ and $\pi < 3.15$, so $3\pi > 9.33 > 3.05^2$),

the complement contributes $\le \frac{4\varepsilon}{2\pi}\cdot 1.04 < \frac{2.08}{3.11}M^{-1/3} < 0.67M^{-1/3}$. On $J_+\cup J_-$, concavity of sin on $[0,\pi/2]$ gives $|\sin\theta| \ge \sin\varepsilon \ge \frac{2}{\pi}\varepsilon$, so $|\phi''(\theta)| \ge \frac{3}{\pi}M^{2/3} =: \mu$. Applying Part (2a) to J_+ and J_- gives

$$\begin{aligned}\frac{1}{2\pi}\left|\int_{J_+\cup J_-} a_0(\theta)e^{i\phi(\theta)}\,d\theta\right| &\le \frac{16\cdot 1.04 + 2\pi\cdot 0.044}{2\sqrt{3\pi}}\,M^{-1/3}\\ &< \frac{16.64 + 0.088\cdot 3.15}{2\cdot 3.05}\,M^{-1/3} \;<\; 2.78\,M^{-1/3},\end{aligned}$$

and $(0.67+2.78)M^{-1/3} = 3.45M^{-1/3} < 3.5M^{-1/3}$.

(3c) Assume $r \ge 10^5$ and $\frac{r}{2} - 1 < d < 350r$, so $M \in (0.499, 350)$ and $\lambda < 3.5\times 10^{-3} < 0.01$. If $|u| > 1100$ ($u := k-d$), then $|u| > 3M$ and (3a) gives $|I_{d,k}| \le e^{0.02+525-704} = e^{-178.98} < 0.903$. If $|u| \le 1100$, then $k < 350r + 1100 < 350.02r$; at $\tilde\rho = 1$, the integrand of (4.1) is $e^{-i(2M)\sin\theta + iu\theta}e^{E(\theta)}$ with $E(\theta) := \frac{u}{r}e^{-i\theta} + k\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{mr^m}e^{-im\theta}$ satisfying $|E(\theta)| \le \frac{1100}{r} + \frac{350.02}{2r(1-10^{-5})} < \frac{1276}{r} < 0.013$. Since $|e^z - 1| \le |z|e^{|z|}$ gives $|e^{E(\theta)} - 1| < 0.013\cdot 1.02 < 0.014$ and $2M > 0.998 > 0.99$, Part (2b) yields $|I_{d,k}(r;\lambda)| \le |J_u(2M)| + 0.014 < 0.889 + 0.014 = 0.903 = 1 - 0.097 < e^{-0.097}$. □

Theorem 4.2. *Let $M \ge 1$, $N_0 \ge 1$, and $t \ge 1$ be integers, let*

$$P(y_1,\dots,y_M) \;=\; \sum_{\substack{u\in\mathbb{Z}^M_{\ge 0}\\ |u|=N_0}} c_u y^u \;\in\; \mathbb{C}[y_1,\dots,y_M]$$

be a homogeneous shift-invariant polynomial of degree N_0, and let $v_0 = (d_1,\dots,d_M) \in \mathbb{Z}^M_{\ge 0}$ satisfy $|v_0| = N_0$ and $\max_{1\le i\le M} d_i \le t$. Put $S_0 := \{i \in [M] : d_i \ge 1\}$.

(1) (Duality and Lagrangian criterion.) *For every polynomial $R \in \mathbb{C}[x_1,\dots,x_M]$ of total degree $< N_0$, $\sum_{|u|=N_0} c_u R(u) = 0$. Consequently, for every family of monic polynomials $q^{(i)} \in \mathbb{C}[x]$ of degree d_i ($i \in S_0$), $c_{v_0} = \sum_{|u|=N_0} c_u \prod_{i\in S_0}\frac{q^{(i)}(u_i)}{d_i!}$; if $r \ge 1$ and $\gamma_{d_i} > 0$ satisfy $\frac{|q^{(i)}(k)|}{d_i!}r^{d_i-k} \le \gamma_{d_i}$ for all $k \in \mathbb{Z}_{\ge 0}$ and $i \in S_0$, then $|c_{v_0}| \le (\prod_{i\in S_0}\gamma_{d_i})\|P\|_1$.*

(2) (Scale-invariant ℓ_1-decay.) *Unconditionally over $\mathbb{C}$,*

$$|c_{v_0}| \;\le\; (t+1)^{-\frac{N_0}{200\,t}}\,\|P\|_1,$$

and hence whenever $c_{v_0} \ne 0$, $N_0 \le \frac{200\,t}{\log_2(t+1)}\log_2(\frac{\|P\|_1}{|c_{v_0}|})$. When $t = 1$, this strengthens to

$$|c_{v_0}| \;\le\; \left(\frac{\sqrt{3}}{2}\right)^{N_0}\|P\|_1.$$

Proof. (1) For every multi-index $r \in \mathbb{Z}^M_{\ge 0}$ with $|r| < N_0$, $\partial^r P$ is homogeneous of degree $N_0 - |r| \ge 1$ and shift-invariant, so $\sum_{|u|=N_0} c_u \prod_{i=1}^M (u_i)_{r_i} = (\partial^r P)(\mathbf{1}) = (\partial^r P)(\mathbf{0}) = 0$. Since $\{\prod_{i=1}^M (x_i)_{r_i} : |r| < N_0\}$ spans all polynomials of total degree $< N_0$, we obtain $\sum_{|u|=N_0} c_u R(u) = 0$ whenever $\deg R < N_0$. Setting $Q(x) := \prod_{i\in S_0}\frac{q^{(i)}(x_i)}{d_i!}$ and $B_{v_0}(x) := \prod_{i\in S_0}\binom{x_i}{d_i}$, both have total degree N_0 and identical degree-N_0 homogeneous part $\prod_{i\in S_0}\frac{x_i^{d_i}}{d_i!}$, so $\deg(Q - B_{v_0}) < N_0$. For $u \in \mathbb{Z}^M_{\ge 0}$ with $|u| = N_0$, $B_{v_0}(u) = \mathbf{1}\{u = v_0\}$ (since $u_i \ge d_i$ on S_0 and $\sum_{i=1}^M u_i = N_0 = \sum_{i\in S_0} d_i$ force $u = v_0$), whence $c_{v_0} = \sum_{|u|=N_0} c_u B_{v_0}(u) = \sum_{|u|=N_0} c_u Q(u)$. Under the Lagrangian bound with $r \ge 1$, every

$|u| = N_0$ has $\sum_{i\in S_0}(u_i - d_i) \le 0$, so $|Q(u)| = r^{\sum_{i\in S_0}(u_i-d_i)} \prod_{i\in S_0}(\frac{|q^{(i)}(u_i)|}{d_i!} r^{d_i-u_i}) \le \prod_{i\in S_0} \gamma_{d_i}$ and $|c_{v_0}| \le (\prod_{i\in S_0} \gamma_{d_i})\|P\|_1$.

(2) For $t = 1$, every $i \in S_0$ has $d_i = 1$; take $q^{(i)}(x) = x - \frac{1}{2}$ and $r = \sqrt{3}$. For $f(k) := |k - \frac{1}{2}|(\sqrt{3})^{1-k}$ ($k \in \mathbb{Z}_{\ge 0}$), we have $f(0) = f(2) = \frac{\sqrt{3}}{2}$, $f(1) = \frac{1}{2}$, and $\frac{f(k+1)}{f(k)} = \frac{2k+1}{(2k-1)\sqrt{3}} \le \frac{5}{3\sqrt{3}} < 1$ for $k \ge 2$ (as $25 < 27$), so $\gamma_1 = \frac{\sqrt{3}}{2}$ and $|c_{v_0}| \le (\frac{\sqrt{3}}{2})^{N_0}\|P\|_1$.

For general $t \ge 1$, by Part (1) with Charlier polynomials $q^{(i)} = q_{d_i}$ (parameter $\lambda_{d_i} > 0$), it suffices to exhibit $r \ge 4$ (depending only on t) and $\lambda_d > 0$ ($1 \le d \le t$) such that $|I_{d,k}(r; \lambda_d)| \le (t+1)^{-\frac{d}{200t}}$ for all $k \in \mathbb{Z}_{\ge 0}$.

- *Case A ($1 \le t < 10^8$).* Set $r := 2(t+1) \ge 4$ and $\lambda_d := \frac{1}{2r}$. By Lemma 4.1(1), $|I_{d,k}(r; \lambda_d)| \le \exp(-\frac{11d}{48(t+1)})$. Since $\frac{t}{t+1} \ge \frac{1}{2}$ and $\ln(t+1) \le 8 \ln 10 < 8 \cdot 2.31 = 18.48 < \frac{275}{12} = \frac{11\cdot 200}{96}$ (Lemma 2.4(1)), we have $\frac{11}{48(t+1)} \ge \frac{\ln(t+1)}{200t}$.

- *Case B ($t \ge 10^8$).* Set $r := 10\sqrt{t} \ge 10^5$.
  - If $1 \le d \le \frac{r}{2} - 1$, Lemma 4.1(1) with $\lambda_d := \frac{1}{2r}$ gives $|I_{d,k}| \le \exp(-\frac{11d}{240\sqrt{t}})$; since $2\sqrt{x} - \ln(1+x) > 0$ at $x = 1$ with derivative $\frac{1+x-\sqrt{x}}{\sqrt{x}(1+x)} > 0$, $\frac{\ln(t+1)}{200t} < \frac{1}{100\sqrt{t}} < \frac{11}{240\sqrt{t}}$.
  - If $\frac{r}{2} - 1 < d < 350r$, Lemma 4.1(3c) with $\lambda_d := \frac{d}{r^2} < 0.01$ gives $|I_{d,k}| < e^{-0.097}$, while $d < 3500\sqrt{t}$ and monotonicity of $x \mapsto \frac{\ln(x+1)}{\sqrt{x}}$ on $[10^8, \infty)$ give $\frac{d \ln(t+1)}{200t} < \frac{17.5 \ln(10^8+1)}{10^4} < \frac{17.5(8\cdot 2.31 + 10^{-8})}{10^4} < 0.033 < 0.097$.
  - If $350r \le d \le t$, Lemma 4.1(3b) with $\lambda_d := \frac{d}{r^2} \le 0.01$ gives $|I_{d,k}| \le 3.5M^{-1/3}$ where $M := d/r \in [350, \frac{\sqrt{t}}{10}]$. On $(0, \infty)$, $h(M) := \frac{1}{3}\ln M - \ln(3.5) - \frac{\ln(t+1)}{20\sqrt{t}} M$ is concave; at $M = 350$, $\frac{1}{3}\ln(\frac{350}{3.5^3}) = \frac{1}{3}\ln(\frac{400}{49}) > \frac{1}{3}\ln 8 = \ln 2 > 0.693 > 0.033 > \frac{350 \ln(t+1)}{20\sqrt{t}}$, and at $M = \frac{\sqrt{t}}{10}$ (using $\ln(t+1) \le \ln t + 10^{-8}$, $\ln t \ge 8 \ln 10 > 18.40$, and $\frac{1}{3}\ln 10 + \ln(3.5) < \frac{2.31}{3} + \ln 4 = 0.77 + 2\ln 2 < 2.16$), $h(\frac{\sqrt{t}}{10}) \ge (\frac{1}{6} - \frac{1}{200})\ln t - 2.16 - 10^{-10} > 0.161 \cdot 18.40 - 2.17 > 0.79 > 0$. Thus $h(M) > 0$ on $[350, \frac{\sqrt{t}}{10}]$, giving $3.5M^{-1/3} \le (t+1)^{-\frac{d}{200t}}$. □

4.2 Band Localization of Same-Partition Shift-Invariant Polynomials

Lemma 4.3. *Let I be a finite non-empty index set, and let $R(y_I) = \sum_{u\in\mathbb{Z}^I_{\ge 0}} c_u y_I^u \in \mathbb{C}[y_i : i \in I]$ be a non-zero, homogeneous, shift-invariant polynomial ($D_I(R) \equiv 0$) such that every $u \in \mathrm{supp}(R)$ has the same multiplicity partition on I as a fixed vector $v \in \mathbb{Z}^I_{\ge 0}$. Fix integers $0 \le t_0 < t_1$, partition $I = I_{\mathrm{lo}} \sqcup I_{\mathrm{mid}} \sqcup I_{\mathrm{hi}}$ via*

$$I_{\mathrm{lo}} := \{i \in I : v_i < t_0\}, \qquad I_{\mathrm{mid}} := \{i \in I : t_0 \le v_i \le t_1\}, \qquad I_{\mathrm{hi}} := \{i \in I : v_i > t_1\},$$

define $w \in \mathbb{Z}^I_{\ge 0}$ by $w_i := v_i$ ($i \in I_{\mathrm{lo}}$) and $w_i := t_0$ ($i \in I_{\mathrm{mid}} \cup I_{\mathrm{hi}}$), set $C_{\mathrm{const}} := \prod_{i\in I_{\mathrm{mid}}\cup I_{\mathrm{hi}}} \binom{v_i}{t_0} \ge 1$ and $\gamma_{\mathrm{hi}} := (v_i - t_0)_{i\in I_{\mathrm{hi}}}$, and form

$$F_{\mathrm{mid}}(y_{I_{\mathrm{mid}}}) \;:=\; \frac{1}{C_{\mathrm{const}}} \left[y_{I_{\mathrm{hi}}}^{\gamma_{\mathrm{hi}}}\right](H_w(R)).$$

Then:

(1) $H_w(R)$ does not depend on $y_{I_{\mathrm{lo}}}$, and every $u \in \mathrm{supp}(R)$ contributing a non-zero term to F_{mid} satisfies $u_i = v_i$ for all $i \in I_{\mathrm{lo}} \cup I_{\mathrm{hi}}$ (with weight $\prod_{i\in I} \binom{u_i}{w_i} = C_{\mathrm{const}}$, so F_{mid} preserves exact coefficients c_u on surviving monomials).

(2) When $I_{\mathrm{mid}} \neq \emptyset$, $F_{\mathrm{mid}} \in \mathbb{C}[y_i : i \in I_{\mathrm{mid}}]$ is homogeneous of degree $\sum_{i\in I_{\mathrm{mid}}}(v_i - t_0)$, every $z \in \mathrm{supp}(F_{\mathrm{mid}})$ has the same multiplicity partition on I_{mid} as $(v - t_0\mathbf{1})|_{I_{\mathrm{mid}}}$, $\|F_{\mathrm{mid}}\|_1 \leq \|R\|_1$, $[y_{I_{\mathrm{mid}}}^{(v-t_0\mathbf{1})|_{I_{\mathrm{mid}}}}]F_{\mathrm{mid}} = c_v$, and $D_{I_{\mathrm{mid}}}(F_{\mathrm{mid}}) \equiv 0$.

Proof. Since D_I commutes with ∂^w, $D_I(H_w(R)) = H_w(D_I(R)) \equiv 0$. A monomial y_I^u ($u \in \mathrm{supp}(R)$) survives in $H_w(R)$ if and only if $u_i \geq w_i$ for all $i \in I$. In that case, $u_i \geq t_0$ on $I_{\mathrm{mid}} \cup I_{\mathrm{hi}}$; since u is a permutation of v, u has exactly $|I_{\mathrm{lo}}|$ coordinates $< t_0$, all of which must lie in I_{lo}, making $u|_{I_{\mathrm{lo}}}$ a permutation of $v|_{I_{\mathrm{lo}}}$. Hence $\sum_{i\in I_{\mathrm{lo}}} u_i = \sum_{i\in I_{\mathrm{lo}}} v_i = \sum_{i\in I_{\mathrm{lo}}} w_i$, which combined with $u_i \geq w_i$ forces $u_i = w_i = v_i$ on I_{lo}, so $H_w(R)$ does not depend on $y_{I_{\mathrm{lo}}}$. Moreover, $u|_{I_{\mathrm{mid}}\cup I_{\mathrm{hi}}}$ is a permutation of $v|_{I_{\mathrm{mid}}\cup I_{\mathrm{hi}}}$, so $\prod_{i\in I}\binom{u_i}{w_i} = \prod_{i\in I_{\mathrm{mid}}\cup I_{\mathrm{hi}}}\binom{u_i}{t_0} = C_{\mathrm{const}}$ is constant across all surviving u. Putting $F := \frac{1}{C_{\mathrm{const}}} H_w(R)$, we have $(D_{I_{\mathrm{mid}}} + D_{I_{\mathrm{hi}}})(F) \equiv 0$.

For any $(z_{\mathrm{mid}}, z_{\mathrm{hi}}) \in \mathrm{supp}(F)$ coming from $u \geq w$, $(u - t_0\mathbf{1})|_{I_{\mathrm{mid}}\cup I_{\mathrm{hi}}}$ is a permutation of $(v - t_0\mathbf{1})|_{I_{\mathrm{mid}}\cup I_{\mathrm{hi}}}$, whose $|I_{\mathrm{hi}}|$ strictly largest entries ($> t_1 - t_0$, versus $\leq t_1 - t_0$ on I_{mid}) are precisely the entries of γ_{hi}. Thus $\sum_{i\in I_{\mathrm{hi}}}(z_{\mathrm{hi}})_i \leq |\gamma_{\mathrm{hi}}|$, with equality if and only if z_{hi} is a permutation of γ_{hi}; extracting $[y_{I_{\mathrm{hi}}}^{\gamma_{\mathrm{hi}}}]F$ therefore forces $z_{\mathrm{hi}} = \gamma_{\mathrm{hi}}$, i.e., $u_i = v_i$ on I_{hi} as well, proving (1) and the degree, partition, norm, and coefficient assertions of (2). Finally, for every $j \in I_{\mathrm{hi}}$, $|\gamma_{\mathrm{hi}}+\mathbf{e}_j| = |\gamma_{\mathrm{hi}}|+1 > |\gamma_{\mathrm{hi}}|$ implies $[y_{I_{\mathrm{hi}}}^{\gamma_{\mathrm{hi}}+\mathbf{e}_j}]F = 0$, so extracting $[y_{I_{\mathrm{hi}}}^{\gamma_{\mathrm{hi}}}]$ from $(D_{I_{\mathrm{mid}}} + D_{I_{\mathrm{hi}}})(F) \equiv 0$ yields $D_{I_{\mathrm{mid}}}(F_{\mathrm{mid}}) \equiv 0$. □

5 Bounded-Degree Stabilizers and Bridge Automorphism Rigidity

Lemma 5.1. *(1)* (Bounded-degree vertex-stabilizer restriction.) *Let M be a connected finite simple graph with $\Delta(M) \leq \Delta$ for an integer $\Delta \geq 1$, let $u_0 \in V(M)$ and $W \subseteq V(M)$, and let $\Gamma_0 \leq \mathrm{Aut}(M)$ satisfy $g(u_0) = u_0$ and $g(W) = W$ for all $g \in \Gamma_0$. Put $P := \Gamma_0|_W \leq \mathrm{Sym}(W)$ and $R := |W \setminus \{u_0\}|$. Then every prime p dividing $|P|$ satisfies $p \leq \Delta$, and $|P| \leq (16\Delta)^R$.*

(2) (Single-bridge automorphism rigidity.) *Let G be a finite simple graph, let $\Delta \geq 1$ be an integer, let $D \in \mathrm{Comp}(G[V_{<\Delta}(G)])$, put $W_D := N_G(D) \cap V_{\geq\Delta}(G)$, pick $v_0 \in V(D)$, and let $S \subseteq V_{\geq\Delta}(G)$ be any subset. Define the group*

$$\begin{aligned} G_{v_0,S\cap W_D}(D, W_D) \;:=\; &\{(\theta,\psi) \in \mathrm{Sym}(W_D) \times \mathrm{Aut}(D) : \psi(v_0) = v_0,\ \theta|_{S\cap W_D} = \mathrm{id}_{S\cap W_D}, \\ &\text{and } \forall\, a \in W_D,\ v \in V(D),\ \{a,v\} \in E(G) \iff \{\theta(a),\psi(v)\} \in E(G)\} \end{aligned}$$

and its restriction $Q_D := \{\theta|_{W_D\setminus S} : (\theta,\psi) \in G_{v_0,S\cap W_D}(D, W_D)\} \leq \mathrm{Sym}(W_D \setminus S)$. Then every prime p dividing $|Q_D|$ satisfies $p < \Delta$, and $|Q_D| \leq (16\Delta)^{|W_D\setminus S|}$.

Proof. We first verify two elementary facts:

- *p-element lifting.* Whenever $\varphi : \mathcal{G} \to \mathcal{H}$ is a surjective homomorphism of finite groups and a prime p divides $|\mathcal{H}|$, the cyclic group $\mathbb{Z}/p\mathbb{Z}$ acts by cyclic shift on $\{(h_1,\dots,h_p) \in \mathcal{H}^p : h_1\cdots h_p = 1\}$ of cardinality $|\mathcal{H}|^{p-1} \equiv 0 \pmod p$; since $(1,\dots,1)$ is fixed and non-fixed orbits have size p, another fixed point $(h,\dots,h)$ ($h \neq 1, h^p = 1$) exists (Cauchy's theorem). Choosing $\tilde{h} \in \mathcal{G}$ with $\varphi(\tilde{h}) = h$ and writing the order of $\tilde{h}$ as $p^a b$ ($a \geq 1, \gcd(p,b) = 1$), the element $g := \tilde{h}^b \in \mathcal{G}$ has order p^a (so every $\langle g\rangle$-orbit has size a power of p) and satisfies $\varphi(g) = h^b \neq 1$.

- *Prime-sum bound $\sum_{p\leq\Delta}\frac{\ln p}{p-1} \leq \ln(16\Delta)$ ($\Delta \geq 1$).* By induction on integers $x \geq 1$, $\prod_{p\leq x} p \leq 4^x$: this holds for $x \in \{1,2\}$; for odd $x = 2k+1 \geq 3$, every prime $p \in (k+1, 2k+1]$ divides $\binom{2k+1}{k} \leq \frac{1}{2}\cdot 2^{2k+1} = 4^k$, giving $\prod_{p\leq 2k+1} p \leq 4^{k+1}\cdot 4^k = 4^{2k+1}$; for even $x = 2k \geq 4$, $\prod_{p\leq 2k} p = \prod_{p\leq 2k-1} p < 4^{2k}$. By Legendre's formula $\mathrm{ord}_p(\Delta!) = \sum_{j=1}^{\infty}\lfloor \Delta/p^j\rfloor \geq \lfloor\Delta/p\rfloor > \frac{\Delta}{p} - 1$,

$$\Delta \sum_{p\leq\Delta}\frac{\ln p}{p} - \sum_{p\leq\Delta}\ln p \;\leq\; \sum_{p\leq\Delta}\left\lfloor\frac{\Delta}{p}\right\rfloor \ln p \;\leq\; \ln(\Delta!) \;\leq\; \Delta\ln\Delta,$$

so $\sum_{p\le\Delta}\ln p \le \Delta\ln 4$ gives $\sum_{p\le\Delta}\frac{\ln p}{p} \le \ln\Delta+\ln 4$. Next, using $0.693 < \ln 2 < 0.70$ (Lemma 2.4(1)) and $\ln(1+t)\le t$: $\ln 2 < 0.70$, $\ln 3 = \ln 2+\ln\frac{3}{2} < 1.20$, $\ln 5 = 2\ln 2+\ln\frac{5}{4} < 1.65$, $\ln 7 < \ln 8 < 2.10$, and $\ln 11 < \ln 16 < 2.80$, so $\sum_{p\le 11}\frac{\ln p}{p(p-1)} < \frac{0.70}{2}+\frac{1.20}{6}+\frac{1.65}{20}+\frac{2.10}{42}+\frac{2.80}{110} < 0.709$. For $p\ge 13$, $e^t \ge et > 2t$ at $t=\frac{1}{4}\ln x$ gives $\ln x < 2x^{1/4}$, $\frac{1}{n(n-1)} \le \frac{13}{12}n^{-2}$, and $12^3 = 1728 > 1296 = 6^4$ gives $12^{3/4} > 6$, whence

$$\sum_{p\ge 13}\frac{\ln p}{p(p-1)} \le \frac{13}{6}\sum_{n=13}^{\infty} n^{-7/4} \le \frac{13}{6}\int_{12}^{\infty} x^{-7/4}\,dx = \frac{26}{9\cdot 12^{3/4}} < \frac{13}{27} < 0.482.$$

Hence $\sum_{p\le\Delta}\frac{\ln p}{p(p-1)} < 0.709+0.482 = 1.191 < 2\cdot 0.693 < \ln 4$, and adding yields $\sum_{p\le\Delta}\frac{\ln p}{p-1} \le \ln(16\Delta)$.

(1) If $R=0$, $|P|=1$. If $R\ge 1$ and a prime p divides $|P|$, lift to $g\in\Gamma_0$ of p-power order with $g|_W \ne \mathrm{id}_W$. Pick $v\in V(M)$ moved by g minimizing $\mathrm{dist}_M(u_0,v)\ge 1$, and pick $w\in N_M(v)$ with $\mathrm{dist}_M(u_0,w) = \mathrm{dist}_M(u_0,v)-1$. Minimality gives $g(w)=w$, so the $\langle g\rangle$-orbit of v lies in $N_M(w)$ and has size a p-power >1, forcing $p\le d_M(w)\le\Delta$. Since P embeds into $\mathrm{Sym}(W\setminus\{u_0\})$, Lagrange's theorem gives $\mathrm{ord}_p(|P|)\le \mathrm{ord}_p(R!) = \sum_{j=1}^{\infty}\lfloor R/p^j\rfloor \le \frac{R}{p-1}$ for $p\le\Delta$ and 0 for $p>\Delta$, so $\ln|P| \le R\sum_{p\le\Delta}\frac{\ln p}{p-1} \le R\ln(16\Delta)$.

(2) Note that $G_{v_0,S\cap W_D}(D,W_D)$ is a subgroup under $(\theta_1,\psi_1)(\theta_2,\psi_2) = (\theta_1\circ\theta_2,\psi_1\circ\psi_2)$. If $W_D\setminus S=\emptyset$, $|Q_D|=1$. Otherwise, for any prime p dividing $|Q_D|$, lift to $(\theta,\psi)\in G_{v_0,S\cap W_D}(D,W_D)$ of p-power order with $\theta|_{W_D\setminus S}\ne \mathrm{id}_{W_D\setminus S}$. If $\psi=\mathrm{id}_{V(D)}$, pick $a\in W_D\setminus S$ moved by θ and $v\in N_G(a)\cap V(D)$; since $\psi(v)=v$, the $\langle\theta\rangle$-orbit of a lies in $N_G(v)$ and has size a p-power >1, giving $p\le d_G(v)<\Delta$. If $\psi\ne\mathrm{id}_{V(D)}$, pick $w\in V(D)$ moved by ψ minimizing $\mathrm{dist}_D(v_0,w)\ge 1$ in the connected graph D, and pick $v\in N_D(w)$ with $\mathrm{dist}_D(v_0,v) = \mathrm{dist}_D(v_0,w)-1$; then $\psi(v)=v$, so the $\langle\psi\rangle$-orbit of w lies in $N_D(v)\subseteq N_G(v)$, giving $p\le d_G(v)<\Delta$. As $Q_D\le\mathrm{Sym}(W_D\setminus S)$, Legendre's formula and the prime-sum bound yield $|Q_D|\le(16\Delta)^{|W_D\setminus S|}$. □

6 Pillar I: Gap-Free Internal-Edge Component Polynomials and Top-Down Contraction

From this point through Section 9, we work under the hypotheses and notation of Lemma 3.3: G and H are connected non-isomorphic simple graphs on $n\ge 4$ vertices with $|E(G)| = |E(H)| = m \ge 4000$, $\bar{d} = 2m/n \ge 2000$, $\mathcal{D}_E(G) = \mathcal{D}_E(H)$, $\mathcal{D}(G) = \mathcal{D}(H)$, and identical degree sequences.

6.1 Gap-Free Internal-Edge Component Polynomials and Band Bounds

Lemma 6.1. *Let $\Delta\ge 1$ be an integer, and put $A := V_{\ge\Delta}(G)$, $B := V_{<\Delta}(G)$, $A_H := V_{\ge\Delta}(H)$, and $B_H := V_{<\Delta}(H)$. Suppose $|E(G[B])|\ge 4$ and $G[B]$ has at least two connected components of size ≥ 2.*

(1) We have $G[A]\cong H[A_H]$, $G[B]\cong H[B_H]$, and $\{\{[G-e] : e\in E(G[B])\}\} = \{\{[H-e'] : e'\in E(H[B_H])\}\}$. Moreover, every matched card isomorphism $\varphi: G-e\xrightarrow{\sim} H-e'$ ($e\in E(G[B])$, $e'\in E(H[B_H])$) maps $A\to A_H$ and $B\to B_H$, and restricts simultaneously to isomorphisms $G[A]\xrightarrow{\sim} H[A_H]$ and $G-E(G[B])\xrightarrow{\sim} H-E(H[B_H])$.

(2) Using one matched card isomorphism from (1) to identify $V(G)=V(H)$ so that $A=A_H$, $B=B_H$, $G[A]=H[A]$, and $G-E(G[B]) = H-E(H[B])$, define $\chi: B\to 2^A$ by $\chi(b) :=$

$N_G(b) \cap A = N_H(b) \cap A$, *set*

$$\Gamma_A \;:=\; \{\pi \in \mathrm{Aut}(G[A]) : |\chi^{-1}(\pi(X))| = |\chi^{-1}(X)| \;\forall\, X \subseteq A\},$$

abbreviate $\mathfrak{C}^*_{\geq 2} := \mathfrak{C}_{\geq 2,\leq |B|}(A)$, *and put* $\mu_G^{\geq 2} := \mu(G[B],\chi)|_{\mathfrak{C}^*_{\geq 2}}$ *and* $\mu_H^{\geq 2} := \mu(H[B],\chi)|_{\mathfrak{C}^*_{\geq 2}}$. *Then every* $\pi \in \Gamma_A$ *satisfies* $d_G(\pi(a)) = d_G(a)$ *for all* $a \in A$, *and the polynomial*

$$P_1(x_{\geq 2}) \;:=\; \sum_{\pi\in\Gamma_A} \left(x^{\pi\cdot\mu_G^{\geq 2}} - x^{\pi\cdot\mu_H^{\geq 2}}\right) \;\in\; \mathbb{Z}[x_{\mathcal{T}} : \mathcal{T} \in \mathfrak{C}^*_{\geq 2}]$$

satisfies $[x^{\mu_G^{\geq 2}}]P_1 = |\operatorname{Stab}_{\Gamma_A}(\mu_G^{\geq 2})| \geq 1$, $\|P_1\|_1 \leq 2|\Gamma_A|$, *is multihomogeneous of degree* $n_{C_0} := c(G[B], C_0)$ *in* $(x_{\mathcal{T}})_{\mathcal{T}\in\mathfrak{C}_{C_0}(A)}$ *for each connected simple graph* C_0 *with* $|E(C_0)| \geq 1$ *and* $|V(C_0)| \leq |B|$, *and satisfies* $D_{C_0}(P_1) \equiv 0$ *for every such* C_0.

Proof. (1) For any simple graph F, define $\Phi_\Delta(F) := \sum_{x\in V(F)} \max(0, d_F(x) - \Delta + 1)$. Since G and H share the same degree sequence, $\Phi_\Delta(G) = \Phi_\Delta(H)$. Because $\max(0, d-\Delta+1) - \max(0, d-\Delta) = \mathbf{1}\{d \geq \Delta\}$ for every integer d, deleting an edge $e \in E(G)$ decreases Φ_Δ by $|e \cap A| \in \{0,1,2\}$; thus $e \in E(G[B]) \iff \Phi_\Delta(G-e) = \Phi_\Delta(G)$, and identically in H. From $\mathcal{D}_E(G) = \mathcal{D}_E(H)$, we obtain $\{\{[G-e] : e \in E(G[B])\}\} = \{\{[H-e'] : e' \in E(H[B_H])\}\}$.

For any matched isomorphism $\varphi : G - e \xrightarrow{\sim} H - e'$ ($e \in E(G[B])$, $e' \in E(H[B_H])$), we have $V_{\geq\Delta}(G-e) = A$ and $V_{\geq\Delta}(H-e') = A_H$, so $\varphi(A) = A_H$ and $\varphi(B) = B_H$. Since $e \subseteq B$ and $e' \subseteq B_H$, φ restricts to isomorphisms $G[A] \xrightarrow{\sim} H[A_H]$, $G - E(G[B]) \xrightarrow{\sim} H - E(H[B_H])$, and $G[B] - e \xrightarrow{\sim} H[B_H] - e'$. Hence $\mathcal{D}_E(G[B]) = \mathcal{D}_E(H[B_H])$ with $|V(G[B])| = |V(H[B_H])|$ and $|E(G[B])| \geq 4$. Because $G[B]$ has at least two connected components of size ≥ 2, $G[B] - V_{=0}(G[B])$ is disconnected, so Lemma 2.3 implies $G[B] \cong H[B_H]$.

(2) Identify $V(G) = V(H)$ as stated. For $\pi \in \Gamma_A$ and $a \in A$, $d_G(\pi(a)) = d_{G[A]}(\pi(a)) + \sum_{X\ni\pi(a)} |\chi^{-1}(X)| = d_{G[A]}(a) + \sum_{Y\ni a} |\chi^{-1}(Y)| = d_G(a)$. Every component of $G[B]$ or $H[B]$ with ≥ 1 edge has at most $|B|$ vertices, so its class lies in $\mathfrak{C}^*_{\geq 2}$. For $F \in \{G, H\}$ and $X \subseteq A$, setting $m_X(\mathcal{T}) := |\{v \in V(C) : \lambda(v) = X\}|$ ($\mathcal{T} = [(C,\lambda)]$), the number of isolated vertices b of $F[B]$ with $\chi(b) = X$ equals $|\chi^{-1}(X)| - \sum_{\mathcal{T}\in\mathfrak{C}^*_{\geq 2}} \mu_F^{\geq 2}(\mathcal{T}) m_X(\mathcal{T})$. Hence $\mu_G^{\geq 2}$ and $\mu_H^{\geq 2}$ lie in disjoint Γ_A-orbits: if $\pi \cdot \mu_G^{\geq 2} = \mu_H^{\geq 2}$ ($\pi \in \Gamma_A$), then $\pi \cdot \mu(G[B],\chi) = \mu(H[B],\chi)$, so matching components extends π to an isomorphism $G \xrightarrow{\sim} H$, contradicting $G \not\cong H$. Thus $[x^{\mu_G^{\geq 2}}]P_1 = |\operatorname{Stab}_{\Gamma_A}(\mu_G^{\geq 2})| \geq 1$ and $\|P_1\|_1 \leq 2|\Gamma_A|$, while $G[B] \cong H[B]$ gives multihomogeneity of degree n_{C_0}.

For each $\mathcal{T} \in \mathfrak{C}^*_{\geq 2}$, fix a representative $(C_{\mathcal{T}}, \lambda_{\mathcal{T}}) \in \mathcal{T}$, and for $e \in E(C_{\mathcal{T}})$ set $\delta(\mathcal{T}, e) := \mu(C_{\mathcal{T}} - e, \lambda_{\mathcal{T}})|_{\mathfrak{C}^*_{\geq 2}}$ (equal to $\mathbf{0}$ when $C_{\mathcal{T}} \cong K_2$; the multiset $\{\{\delta(\mathcal{T}, e) : e \in E(C_{\mathcal{T}})\}\}$ is independent of the choice of representative). Because $\mathfrak{C}_{C_0}(A)$, D_{C_0}, and $\mathcal{L}_{C_0}$ below depend only on the unlabelled isomorphism class $[C_0]$, define for each connected C_0 ($1 \leq |E(C_0)|, |V(C_0)| \leq |B|$)

$$\mathcal{L}_{C_0} \;:=\; \sum_{\mathcal{T}\in\mathfrak{C}_{C_0}(A)} \sum_{e\in E(C_{\mathcal{T}})} x^{\delta(\mathcal{T},e)} \frac{\partial}{\partial x_{\mathcal{T}}}, \qquad \mathcal{L}_{\mathrm{all}} \;:=\; \sum_{\substack{[C_0] \text{ connected} \\ 1\leq|E(C_0)|,\,|V(C_0)|\leq|B|}} \mathcal{L}_{C_0}.$$

By (1), there is a bijection $e \mapsto e'$ ($E(G[B]) \to E(H[B])$) with isomorphisms $\varphi_e : G - e \xrightarrow{\sim} H - e'$; each $\sigma_e := \varphi_e|_A$ belongs to Γ_A and satisfies $\sigma_e \cdot \mu(G[B]-e,\chi)|_{\mathfrak{C}^*_{\geq 2}} = \mu(H[B]-e',\chi)|_{\mathfrak{C}^*_{\geq 2}}$. Summing over $\pi \in \Gamma_A$ and reindexing $\pi \mapsto \pi\sigma_e^{-1}$ gives

$$\mathcal{L}_{\mathrm{all}}(P_1) \;=\; \sum_{\pi\in\Gamma_A} \sum_{e\in E(G[B])} x^{\pi\cdot\mu(G[B]-e,\chi)|_{\mathfrak{C}^*_{\geq 2}}} - \sum_{\pi\in\Gamma_A} \sum_{e'\in E(H[B])} x^{\pi\cdot\mu(H[B]-e',\chi)|_{\mathfrak{C}^*_{\geq 2}}} \;\equiv\; 0.$$

For each connected class $[C_0]$ ($|E(C_0)| \geq 1, |V(C_0)| \leq |B|$), every monomial of $\mathcal{L}_{C_0}(P_1)$ has $\mathfrak{C}_{C_0}(A)$-degree $n_{C_0} - 1$ and $\mathfrak{C}_{C'}(A)$-degree $\geq n_{C'}$ for all $[C'] \neq [C_0]$. Thus the polynomials $\{\mathcal{L}_{C_0}(P_1)\}$ across connected classes $[C_0]$ have pairwise disjoint monomial supports, forcing $\mathcal{L}_{C_0}(P_1) \equiv 0$ for every such C_0.

We prove $D_{C_0}(P_1) \equiv 0$ by upward induction on $q := |E(C_0)| \geq 1$. For $q = 1$, $C_0 \cong K_2$ and $\mathcal{L}_{K_2} = D_{K_2}$, so $D_{K_2}(P_1) \equiv 0$. For $q \geq 2$, put $D_{<q} := \sum_{[C'] \text{ connected}: 1 \leq |E(C')| < q, |V(C')| \leq |B|} D_{C'}$, so $D_{<q}(P_1) \equiv 0$ by induction. Deleting an edge from C_0 leaves $q - 1 \geq 1$ edges, hence either 1 or 2 components with ≥ 1 edge; let $a_1, a_2 \geq 0$ be the numbers of edges of C_0 of each type ($a_1 + a_2 = q \geq 2$). Splitting $\mathcal{L}_{C_0} = \mathcal{L}^{(1)}_{C_0} + \mathcal{L}^{(2)}_{C_0}$, $\mathcal{L}^{(1)}_{C_0}(P_1)$ preserves total degree while $\mathcal{L}^{(2)}_{C_0}(P_1)$ increases total degree by 1; homogeneity of P_1 gives $\mathcal{L}^{(1)}_{C_0}(P_1) \equiv 0$ and $\mathcal{L}^{(2)}_{C_0}(P_1) \equiv 0$. On $\mathfrak{C}^*_{<q} := \{[(C', \lambda)] \in \mathfrak{C}^*_{\geq 2} : |E(C')| < q\}$, the Lie-bracket identities $[D_{<q}, x_{\mathcal{T}'} \frac{\partial}{\partial x_{\mathcal{T}}}] = \frac{\partial}{\partial x_{\mathcal{T}}}$ ($\mathcal{T}' \in \mathfrak{C}^*_{<q}$) and $[D_{<q}, [D_{<q}, x_{\mathcal{T}_1} x_{\mathcal{T}_2} \frac{\partial}{\partial x_{\mathcal{T}}}]] = 2 \frac{\partial}{\partial x_{\mathcal{T}}}$ ($\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{C}^*_{<q}$, equal or distinct) yield

$$a_1 D_{C_0}(P_1) \;=\; [D_{<q}, \mathcal{L}^{(1)}_{C_0}](P_1) \;\equiv\; 0, \qquad 2a_2 D_{C_0}(P_1) \;=\; [D_{<q}, [D_{<q}, \mathcal{L}^{(2)}_{C_0}]](P_1) \;\equiv\; 0.$$

Since $a_1 + a_2 = q \geq 2$, at least one of a_1, a_2 is positive, so $D_{C_0}(P_1) \equiv 0$. □

Lemma 6.2. *Under the hypotheses and notation of Lemma 6.1 (noting n_{C_0} and $\mathfrak{C}_{C_0}(A)$ depend only on $[C_0]$), set $\mu := \mu_G^{\geq 2}$, $S_G := \mathrm{Stab}_{\Gamma_A}(\mu)$, $\mathcal{C}_{\leq 2} := \bigsqcup_{[C_0] \text{ connected}: |E(C_0)| \geq 1, 1 \leq n_{C_0} \leq 2} \mathfrak{C}_{C_0}(A)$, and $\mathcal{C}_{\geq 3} := \bigsqcup_{[C_0] \text{ connected}: |E(C_0)| \geq 1, n_{C_0} \geq 3} \mathfrak{C}_{C_0}(A)$.*

(1) For every connected C_0 with $|E(C_0)| \geq 1$ and $n_{C_0} \geq 3$, $\mu_G^{\geq 2}|_{\mathfrak{C}_{C_0}(A)}$ and $\mu_H^{\geq 2}|_{\mathfrak{C}_{C_0}(A)}$ have the same multiplicity partition.

(2) Since every monomial of P_1 has degree 0 outside $\mathcal{C}_{\leq 2} \sqcup \mathcal{C}_{\geq 3}$, define

$$P_{\leq 2}(x_{\mathcal{C}_{\leq 2}}) \;:=\; [x_{\mathcal{C}_{\geq 3}}^{\mu|_{\mathcal{C}_{\geq 3}}}]\, P_1(x_{\geq 2}), \qquad P_{\geq 3}(x_{\mathcal{C}_{\geq 3}}) \;:=\; [x_{\mathcal{C}_{\leq 2}}^{\mu|_{\mathcal{C}_{\leq 2}}}]\, P_1(x_{\geq 2}).$$

Then $P_{\leq 2}$ (respectively $P_{\geq 3}$) is homogeneous, has coefficient $|S_G|$ at $\mu|_{\mathcal{C}_{\leq 2}}$ (respectively $\mu|_{\mathcal{C}_{\geq 3}}$), and is annihilated by D_{C_0} for every C_0 with $1 \leq n_{C_0} \leq 2$ (respectively $n_{C_0} \geq 3$); moreover, every monomial in $\mathrm{supp}(P_{\geq 3})$ has the same multiplicity partition on $\mathcal{C}_{\geq 3}$ as $\mu|_{\mathcal{C}_{\geq 3}}$.

(3) Let $\eta_1 \in \{0,1\}^{\mathcal{C}_{\geq 3}}$ be any $\{0,1\}$-vector supported on $\{\mathcal{T} \in \mathcal{C}_{\geq 3} : \mu(\mathcal{T}) = 1\}$, and set

$$\begin{aligned}
\Gamma_{\mathrm{band}}(2^j) \;&:=\; \{\pi \in \Gamma_A : (\pi \cdot \mu)(\mathcal{T}) = \mu(\mathcal{T})\ \forall\, \mathcal{T} \in \mathcal{C}_{\leq 2} \cup \{\mathcal{T}' \in \mathcal{C}_{\geq 3} : \mu(\mathcal{T}') \notin [2^{j-2}, 2^j]\}\},\\
\Gamma_{\mathrm{band}}(1,8) \;&:=\; \{\pi \in \Gamma_A : (\pi \cdot \mu)(\mathcal{T}) = \mu(\mathcal{T})\ \forall\, \mathcal{T} \in \mathcal{C}_{\leq 2} \cup \{\mathcal{T}' \in \mathcal{C}_{\geq 3} : \mu(\mathcal{T}') \notin [1,8]\}\},\\
\Gamma_{\mathrm{surv},1} \;&:=\; \{\pi \in \Gamma_A : (\pi \cdot \mu)(\mathcal{T}) = \mu(\mathcal{T})\ \forall\, \mathcal{T} \in \mathcal{C}_{\leq 2} \cup \{\mathcal{T}' \in \mathcal{C}_{\geq 3} : \mu(\mathcal{T}') \geq 2\}\\
&\qquad\qquad \cup\, \mathrm{supp}(\eta_1)\}.
\end{aligned}$$

Then for every integer $j \geq 4$:

$$\sum_{\substack{\mathcal{T} \in \mathcal{C}_{\geq 3}\\ 2^{j-1} < \mu(\mathcal{T}) \leq 2^j}} \mu(\mathcal{T}) \;\leq\; \frac{300 \cdot 2^j}{j-1} \log_2\!\left(\frac{2|\Gamma_{\mathrm{band}}(2^j)|}{|S_G|}\right),$$

$$\sum_{\substack{\mathcal{T} \in \mathcal{C}_{\geq 3}\\ 2 \leq \mu(\mathcal{T}) \leq 8}} \mu(\mathcal{T}) \;\leq\; 934 \log_2\!\left(\frac{2|\Gamma_{\mathrm{band}}(1,8)|}{|S_G|}\right),$$

$$|\{\mathcal{T} \in \mathcal{C}_{\geq 3} : \mu(\mathcal{T}) = 1\}| - |\eta_1| \;\leq\; 5 \log_2\!\left(\frac{2|\Gamma_{\mathrm{surv},1}|}{|S_G|}\right).$$

Proof. (1) Fix C_0 with $q_0 := |E(C_0)| \geq 1$ and $n_{C_0} \geq 3$. For $F \in \{G, H\}$ and $e \in E(F[B])$, let C' be the component of $F[B]$ containing e. If $C' \cong C_0$, then $|E(C' - e)| = q_0 - 1 < q_0$, so no component of $C' - e$ is isomorphic to C_0 and $c(F[B] - e, C_0) = n_{C_0} - 1$; if $C' \ncong C_0$, all n_{C_0} copies of C_0 in $F[B]$ survive intact, giving $c(F[B] - e, C_0) \geq n_{C_0}$. Thus $c(F[B] - e, C_0) = n_{C_0} - 1$ if and only if e lies in a C_0-component of $F[B]$. For $1 \leq r \leq n_{C_0}$, put $S_r(F) := \sum_{\mathcal{T} \in \mathfrak{C}_{C_0}(A)} \binom{\mu_F^{\geq 2}(\mathcal{T})}{r}$. For each $1 \leq r \leq n_{C_0} - 1$, summing $\sum_{\mathcal{T} \in \mathfrak{C}_{C_0}(A)} \binom{\mu(F[B]-e,\chi)_{\mathcal{T}}}{r}$ over all $e \in E(F[B])$ with $c(F[B] - e, C_0) = n_{C_0} - 1$ (equal in G and H by Lemma 6.1(1)) yields

$$\sum_{\mathcal{T} \in \mathfrak{C}_{C_0}(A)} q_0 \left(\mu_F^{\geq 2}(\mathcal{T}) \binom{\mu_F^{\geq 2}(\mathcal{T}) - 1}{r} + (n_{C_0} - \mu_F^{\geq 2}(\mathcal{T})) \binom{\mu_F^{\geq 2}(\mathcal{T})}{r} \right) \;=\; q_0 (n_{C_0} - r)\, S_r(F),$$

using $m' \binom{m'-1}{r} = (m' - r)\binom{m'}{r}$. Hence $S_r(G) = S_r(H)$ for $1 \leq r \leq n_{C_0} - 1$. Since $n_{C_0} \geq 3$, $2(n_{C_0} - 1) > n_{C_0}$, so any partition of n_{C_0} with parts $\leq n_{C_0} - 1$ has $S_{n_{C_0}-1}(F) \leq 1 < n_{C_0}$; thus $S_{n_{C_0}}(F) = \mathbf{1}\{S_{n_{C_0}-1}(F) = n_{C_0}\}$ also matches between G and H, and downward inversion on $S_r(F) = \sum_{t=r}^{n_{C_0}} \binom{t}{r} |\{\mathcal{T} \in \mathfrak{C}_{C_0}(A) : \mu_F^{\geq 2}(\mathcal{T}) = t\}|$ proves (1).

(2) Coefficient extraction in disjoint variables commutes with D_{C_0}, $[x^{\mu}]P_1 = |S_G|$ by Lemma 6.1(2), and same-partition equality on $\mathcal{C}_{\geq 3}$ follows from (1) and Γ_A-invariance.

(3) If $\mathcal{C}_{\geq 3} = \emptyset$, then $|\eta_1| = 0$ and all three left-hand sides equal 0, while $|\Gamma'| \geq |S_G| \geq 1$ makes all right-hand sides positive. Assume $\mathcal{C}_{\geq 3} \neq \emptyset$ (a finite non-empty set); then $R := P_{\geq 3}$ is non-zero (since $[x_{\mathcal{C}_{\geq 3}}^{\mu|_{\mathcal{C}_{\geq 3}}}]P_{\geq 3} = |S_G| \geq 1$), homogeneous, same-partition on $\mathcal{C}_{\geq 3}$, and annihilated by $D_{\mathcal{C}_{\geq 3}}$. Apply Lemma 4.3 to R (with $v = \mu|_{\mathcal{C}_{\geq 3}}$) at three bands (t_0, t_1):

- *Band $(2^{j-2}, 2^j)$ $(j \geq 4)$.* Put $N_{\mathrm{mid}} := \sum_{\mathcal{T} \in I_{\mathrm{mid}}} (\mu(\mathcal{T}) - 2^{j-2})$. If $N_{\mathrm{mid}} = 0$, the desired sum is 0. If $N_{\mathrm{mid}} \geq 1$ (so $I_{\mathrm{mid}} \neq \emptyset$), F_{mid} is homogeneous of degree N_{mid}, shift-invariant on I_{mid}, with maximum exponent $\leq 3 \cdot 2^{j-2}$ and leading coefficient $|S_G|$. By Lemma 4.3(1), the contributing $+$-terms are precisely $\Gamma_{\mathrm{band}}(2^j)$ with weight 1, so $F_{\mathrm{mid}}^{+}(\mathbf{1}) = |\Gamma_{\mathrm{band}}(2^j)|$; since $F_{\mathrm{mid}}(\mathbf{1}) = F_{\mathrm{mid}}(\mathbf{0}) = 0$, $\|F_{\mathrm{mid}}\|_1 = 2|\Gamma_{\mathrm{band}}(2^j)|$. Applying Theorem 4.2(2) at $t := 3 \cdot 2^{j-2}$ $(\log_2(t+1) > j - 1)$ gives $N_{\mathrm{mid}} \leq \frac{150 \cdot 2^j}{j-1} \log_2(\frac{2|\Gamma_{\mathrm{band}}(2^j)|}{|S_G|})$, and $\mu(\mathcal{T}) \leq 2(\mu(\mathcal{T}) - 2^{j-2})$ on $2^{j-1} < \mu(\mathcal{T}) \leq 2^j$ yields the first inequality.

- *Band $(1, 8)$.* Put $N_{\mathrm{mid}} := \sum_{1 \leq \mu(\mathcal{T}) \leq 8} (\mu(\mathcal{T}) - 1) = \sum_{2 \leq \mu(\mathcal{T}) \leq 8} (\mu(\mathcal{T}) - 1)$. If $N_{\mathrm{mid}} = 0$, the left-hand sum is 0 and the bound holds trivially since $|\Gamma_{\mathrm{band}}(1, 8)| \geq |S_G|$. If $N_{\mathrm{mid}} \geq 1$, then $I_{\mathrm{mid}} \neq \emptyset$, $\|F_{\mathrm{mid}}\|_1 = 2|\Gamma_{\mathrm{band}}(1, 8)|$, and $\sum_{2 \leq \mu(\mathcal{T}) \leq 8} \mu(\mathcal{T}) \leq 2N_{\mathrm{mid}}$; Theorem 4.2(2) at $t := 7$ gives $2N_{\mathrm{mid}} \leq \frac{2 \cdot 200 \cdot 7}{\log_2 8} \log_2(\frac{2|\Gamma_{\mathrm{band}}(1,8)|}{|S_G|}) = \frac{2800}{3} \log_2(\frac{2|\Gamma_{\mathrm{band}}(1,8)|}{|S_G|}) < 934 \log_2(\frac{2|\Gamma_{\mathrm{band}}(1,8)|}{|S_G|})$.

- *Band $(0, 1)$ differentiated by ∂^{η_1}.* Put $c_1 := |\{\mathcal{T} \in \mathcal{C}_{\geq 3} : \mu(\mathcal{T}) = 1\}|$. If $c_1 = |\eta_1|$, the left-hand side is 0 and the bound holds trivially. If $c_1 > |\eta_1|$, then $I_{\mathrm{mid}} \neq \emptyset$ and Lemma 4.3 at $(0, 1)$ yields a multilinear shift-invariant polynomial F_1 of degree c_1; setting $Q_1 := \partial^{\eta_1} F_1$ (supported on I_{mid}), Q_1 is multilinear, shift-invariant on I_{mid}, of degree $c_1 - |\eta_1| \geq 1$, with leading coefficient $|S_G|$ and $\|Q_1\|_1 = 2|\Gamma_{\mathrm{surv},1}|$. Theorem 4.2(2) at $t = 1$ (with $(2/\sqrt{3})^5 = \frac{32}{9\sqrt{3}} > 2$ as $1024 > 972$) gives $c_1 - |\eta_1| \leq \frac{1}{\log_2(2/\sqrt{3})} \log_2(\frac{2|\Gamma_{\mathrm{surv},1}|}{|S_G|}) < 5 \log_2(\frac{2|\Gamma_{\mathrm{surv},1}|}{|S_G|})$. □

6.2 Bridge-Controlled Complexity and Scale-Free Top-Down Contraction

Call a level $k \in \{0, \ldots, L-1\}$ *non-degenerate* if $|E(G[B_k])| \geq 4$ and $G[B_k]$ has at least two connected components of size ≥ 2, and *degenerate* otherwise. At every non-degenerate level k,

apply Lemmas 6.1 and 6.2 with $\Delta = \Delta_k$ ($A = H_k, B = B_k$), writing $\chi_k, \Gamma_k, \mu_k := \mu_G^{\geq 2}, S_k := \mathrm{Stab}_{\Gamma_k}(\mu_k), P_1^{(k)}, P_{\leq 2}^{(k)}, P_{\geq 3}^{(k)}$, and define

$$\Gamma_k^{\mathrm{br}} \;:=\; \{\pi \in \Gamma_k : \forall\, D \in \mathcal{B}_{\Delta_k}^{\geq 2},\ (\pi \cdot \mu_k)([(D, \chi_k|_{V(D)})]) \geq 1\},$$

$$\Lambda_k \;:=\; \begin{cases} 1 + \log_2\!\left(\dfrac{|\Gamma_k^{\mathrm{br}}|}{|\mathrm{Aut}(G)|_{H_k}|}\right) & \text{if } k \text{ is non-degenerate,} \\ 1 & \text{if } k \text{ is degenerate.} \end{cases}$$

Lemma 6.3. *At every non-degenerate level $k \in \{0, \dots, L-1\}$:*

(1) $S_k = \mathrm{Aut}(G)|_{H_k}$ (so $|\Gamma_k^{\mathrm{br}}| \geq |S_k| = |\,\mathrm{Aut}(G)|_{H_k}|$ and $\Lambda_k \geq 1$).

(2) Let $\eta_{\leq 2} \in \{0,1\}^{\mathcal{C}_{\leq 2}}$ be the indicator vector of those classes $\mathcal{T} \in \mathcal{C}_{\leq 2}$ containing at least one bridge $D \in \mathcal{B}_{\Delta_k}^{\geq 2}$, and put

$$\Gamma_{\mathrm{surv},\leq 2} \;:=\; \{\pi \in \Gamma_k : (\pi \cdot \mu_k)|_{\mathcal{C}_{\geq 3}} = \mu_k|_{\mathcal{C}_{\geq 3}} \text{ and } (\pi \cdot \mu_k)(\mathcal{T}) \geq 1\ \forall\, \mathcal{T} \in \mathrm{supp}(\eta_{\leq 2})\}.$$

Then $\Gamma_{\mathrm{surv},\leq 2} \subseteq \Gamma_k^{\mathrm{br}}$, and $N_{\leq 2} := \sum_{[C_0] \text{ connected}: |E(C_0)| \geq 1,\, 1 \leq n_{C_0} \leq 2} n_{C_0}$ satisfies

$$N_{\leq 2} \;\leq\; |H_k| \;+\; 253 \log_2\!\left(\frac{2|\Gamma_{\mathrm{surv},\leq 2}|}{|S_k|}\right).$$

Proof. (1) Every $g \in \mathrm{Aut}(G)$ stabilizes H_k and B_k setwise, so $g|_{H_k} \in \Gamma_k$ fixes μ_k. Conversely, any $\pi \in S_k$ fixes μ_k and $|\chi_k^{-1}(X)|$ for all $X \subseteq H_k$, hence fixes all isolated-vertex counts $|\chi_k^{-1}(X)| - \sum_{\mathcal{T}} \mu_k(\mathcal{T}) m_X(\mathcal{T})$ and $\mu(G[B_k], \chi_k)$, and therefore extends to an automorphism of G.

(2) Every bridge $D \in \mathcal{B}_{\Delta_k}^{\geq 2}$ satisfies $[(D, \chi_k|_{V(D)})] \in \mathcal{C}_{\geq 3} \cup \mathrm{supp}(\eta_{\leq 2})$, and hence $\Gamma_{\mathrm{surv},\leq 2} \subseteq \Gamma_k^{\mathrm{br}}$. By Lemma 3.1(1b), $|\eta_{\leq 2}| \leq |\mathcal{B}_{\Delta_k}| < |H_k|$. If $N_{\leq 2} \leq |\eta_{\leq 2}|$ (including $\mathcal{C}_{\leq 2} = \emptyset$), the inequality is immediate. Suppose $N_{\leq 2} > |\eta_{\leq 2}|$ (so $\mathcal{C}_{\leq 2} \neq \emptyset$), and form $Q_{\leq 2} := \partial^{\eta_{\leq 2}} P_{\leq 2}^{(k)}$, which is homogeneous of degree $N_{\leq 2} - |\eta_{\leq 2}| \geq 1$, shift-invariant on $\mathcal{C}_{\leq 2}$, with maximum exponent ≤ 2. For every $\pi \in \Gamma_{\mathrm{surv},\leq 2}$ and $\mathcal{T} \in \mathrm{supp}(\eta_{\leq 2}) \cap \mathfrak{C}_{C_0}(H_k)$ ($n_{C_0} \in \{1,2\}$), $(\pi \cdot \mu_k)|_{\mathfrak{C}_{C_0}(H_k)}$ has total sum $n_{C_0} \in \{1,2\}$ and the same multiplicity partition as $\mu_k|_{\mathfrak{C}_{C_0}(H_k)}$ (which is (1), (2), or (1,1)), so $(\pi \cdot \mu_k)(\mathcal{T}) \geq 1$ forces $(\pi \cdot \mu_k)(\mathcal{T}) = \mu_k(\mathcal{T})$. Hence $\partial^{\eta_{\leq 2}}$ multiplies every surviving $+$-monomial by the same constant 2^{a_2}, where $a_2 := |\{\mathcal{T} \in \mathrm{supp}(\eta_{\leq 2}) : \mu_k(\mathcal{T}) = 2\}|$. Disjointness of the Γ_k-orbits of $\mu_G^{\geq 2}$ and $\mu_H^{\geq 2}$ (Lemma 6.1(2)) ensures that no $-$-monomial of $Q_{\leq 2}$ has exponent $\mu_k|_{\mathcal{C}_{\leq 2}} - \eta_{\leq 2}$, so $[x^{\mu_k|_{\mathcal{C}_{\leq 2}} - \eta_{\leq 2}}] Q_{\leq 2} = 2^{a_2}|S_k|$ and $\|Q_{\leq 2}\|_1 = 2 \cdot 2^{a_2} |\Gamma_{\mathrm{surv},\leq 2}|$. Theorem 4.2(2) at $t = 2$ cancels 2^{a_2} and gives

$$N_{\leq 2} - |\eta_{\leq 2}| \;\leq\; \frac{400}{\log_2 3} \log_2\!\left(\frac{2|\Gamma_{\mathrm{surv},\leq 2}|}{|S_k|}\right) \;<\; 253 \log_2\!\left(\frac{2|\Gamma_{\mathrm{surv},\leq 2}|}{|S_k|}\right),$$

since $3^{12} = 531441 > 524288 = 2^{19}$ implies $\frac{400}{\log_2 3} < \frac{4800}{19} < 253$. □

Proposition 6.4. *For every $k \in \{0, \dots, L-1\}$:*

(1) If k is non-degenerate, taking η_1 in Lemma 6.2(3) to be the indicator of all $\mathcal{T} \in \mathcal{C}_{\geq 3}$ with $\mu_k(\mathcal{T}) = 1$ containing a bridge in $\mathcal{B}_{\Delta_k}^{\geq 2}$, the sets $\Gamma_{\mathrm{surv},\leq 2}$, $\Gamma_{\mathrm{surv},1}$, $\Gamma_{\mathrm{band}}(1,8)$, and $\Gamma_{\mathrm{band}}(2^j)$ ($j \geq 4$) all lie in Γ_k^{br}; hence $\log_2(2|\Gamma'|/|S_k|) \leq \Lambda_k$ for each such Γ'.

(2) Unconditionally (whether k is non-degenerate or degenerate),

$$\Lambda_k \;\le\; 1+\log_2 n \;+\; |H_k|\log_2(16\Delta_k) \;+\; \sum_{r=k}^{L-1} 2|H_r|\log_2(2\Delta_r) \;+\; \sum_{r=k+1}^{L-1}\sum_{\mathcal{T}\in\mathcal{F}_r^{(k),\ge 2}} \tilde{w}_{\mathcal{T}}\log_2\!\left(\frac{2e\Delta_r}{\tilde{w}_{\mathcal{T}}}\right).$$

Proof. (1) $\Gamma_{\mathrm{surv},\le 2}\subseteq\Gamma_k^{\mathrm{br}}$ is Lemma 6.3(2). In $\Gamma_{\mathrm{band}}(2^j)$ $(j\ge 4)$ and $\Gamma_{\mathrm{band}}(1,8)$, every π preserves μ_k on $\mathcal{C}_{\le 2}$ and outside $[t_0,t_1]$ $(t_0\ge 1)$, hence stabilizes the middle band setwise so $(\pi\cdot\mu_k)(\mathcal{T})\ge t_0\ge 1$ inside it. In $\Gamma_{\mathrm{surv},1}$, π preserves μ_k on $\mathcal{C}_{\le 2}$, on $\{\mathcal{T}\in\mathcal{C}_{\ge 3}:\mu_k(\mathcal{T})\ge 2\}$, and on $\mathrm{supp}(\eta_1)$, covering all bridges in $\mathcal{B}_{\Delta_k}^{\ge 2}$. Lemma 6.3(1) gives $\log_2(2|\Gamma'|/|S_k|)\le\Lambda_k$.

(2) If k is degenerate, $\Lambda_k=1$ and the bound is immediate. Assume k is non-degenerate, and put $\Sigma_k^{\mathrm{br}}:=\{\pi^{-1}:\pi\in\Gamma_k^{\mathrm{br}}\}\subseteq\Gamma_k$ $(|\Sigma_k^{\mathrm{br}}|=|\Gamma_k^{\mathrm{br}}|)$. Every $\sigma\in\Sigma_k^{\mathrm{br}}$ satisfies $d_G(\sigma(a))=d_G(a)$ $(a\in H_k)$, $|\chi_k^{-1}(\sigma(X))|=|\chi_k^{-1}(X)|$ $(X\subseteq H_k)$, and $\mu_k(\sigma\cdot[(D,\chi_k|_{V(D)})])\ge 1$ for every $D\in\mathcal{B}_{\Delta_k}^{\ge 2}$. For each $\sigma\in\Sigma_k^{\mathrm{br}}$ and $D\in\mathcal{B}_{\Delta_k}$: if $|V(D)|\ge 2$, fix a component $D'(\sigma)\in\mathrm{Comp}(G[B_k])$ and an isomorphism $\psi_{D,\sigma}:D\xrightarrow{\sim}D'(\sigma)$ with $\chi_k(\psi_{D,\sigma}(v))=\sigma(\chi_k(v))$ $(v\in V(D))$; if $D=\{v_D\}$, fix $z_D(\sigma)\in\chi_k^{-1}(\sigma(\chi_k(v_D)))\subseteq B_k$.

Traverse the pruned tree $T_k^*=T_{\mathrm{core}}^{(k)}-L_k^{\mathrm{iso}}$ of Lemma 3.4 in a fixed breadth-first search order from $r_0\in U_{L-1}\setminus L_k^{\mathrm{iso}}$.

- *Phase 1 (encoding $\sigma_1:=\sigma|_{H_k\setminus L_k^{\mathrm{iso}}}$).*

  (i) Record $\sigma(r_0)\in H_k$ $(\le|H_k|\le n$ choices) and initialize the recorded hub set $P:=\{r_0\}$.

  (ii) When expanding $u\in U_r\setminus L_k^{\mathrm{iso}}$ $(k\le r\le L-1)$, $\sigma(u)$ is already in P, with $d_G(\sigma(u))=d_G(u)<2\Delta_r$. For each T_k^*-child of u: if it is a vertex $v\in H_k\setminus L_k^{\mathrm{iso}}$, record $\sigma(v)\in N_G(\sigma(u))$ and add v to P; if it is a bridge $D\in\mathcal{B}_{\Delta_k}$ entered via $\{u,v_D\}$ $(v_D\in V(D))$, record $\psi_{D,\sigma}(v_D)\in N_G(\sigma(u))$ (when $|V(D)|\ge 2$) or $z_D(\sigma)\in N_G(\sigma(u))$ (when $D=\{v_D\}$). Each T_k^*-child of u incurs $<2\Delta_r$ choices.

  (iii) When expanding $D\in\mathcal{B}_{\Delta_k}$ entered from its T_k^*-parent u via $\{u,v_D\}$, put $W_D:=N_G(D)\cap H_k\subseteq H_k\setminus L_k^{\mathrm{iso}}$ (Lemma 3.4(1)). Notice $P\subseteq H_k\setminus L_k^{\mathrm{iso}}$ depends only on G,T_k^*, and the step index (never on σ). Record $\sigma|_{W_D\setminus P}$ and add $W_D\setminus P$ to P (which contains all T_k^*-children of D since $\mathcal{B}_{\Delta_k}$ is independent in T_k^*):

  – If $D=\{v_D\}$, $z_0:=z_D(\sigma)\in B_k$ is already recorded and $\sigma(W_D)=\chi_k(z_0)$ has size $\le d_G(z_0)<\Delta_k$, giving $\le\Delta_k^{|W_D\setminus P|}\le(16\Delta_k)^{|W_D\setminus P|}$ choices.

  – If $|V(D)|\ge 2$, $y_0:=\psi_{D,\sigma}(v_D)\in B_k$ is already recorded, fixing $D'(\sigma)=D'$ (the component of $G[B_k]$ containing y_0) and $\sigma(W_D)=N_G(D')\cap H_k$. Fix a reference $\sigma_0\in\Sigma_k^{\mathrm{br}}$ agreeing with all prior choices; for any consistent $\sigma\in\Sigma_k^{\mathrm{br}}$, setting $\theta:=(\sigma_0|_{W_D})^{-1}\circ(\sigma|_{W_D})$ and $\psi:=\psi_{D,\sigma_0}^{-1}\circ\psi_{D,\sigma}\in\mathrm{Aut}(D)$ gives $(\theta,\psi)\in G_{v_D,P\cap W_D}(D,W_D)$, so Lemma 5.1(2) $(\Delta=\Delta_k,S=P)$ bounds the choices for $\sigma|_{W_D\setminus P}$ by $(16\Delta_k)^{|W_D\setminus P|}$.

  Disjointness of the sets $W_D\setminus P\subseteq H_k$ bounds the product of bridge steps by $(16\Delta_k)^{|H_k|}$, and Lemma 3.4(1)–(2) bounds the number of distinct restrictions σ_1 by

$$n\cdot(16\Delta_k)^{|H_k|}\cdot\left(\prod_{r=k}^{L-1}(2\Delta_r)^{2|H_r|}\right)\cdot\left(\prod_{r=k+1}^{L-1}\prod_{\mathcal{T}\in\mathcal{F}_r^{(k),\ge 2}}(2\Delta_r)^{\tilde{w}_{\mathcal{T}}}\right).$$

- *Phase 2 (extending σ_1 to L_k^{iso}).* Fixing $\sigma_0\in\Sigma_k^{\mathrm{br}}$ with $\sigma_0|_{H_k\setminus L_k^{\mathrm{iso}}}=\sigma_1$, any $\sigma\in\Sigma_k^{\mathrm{br}}$ extending σ_1 has $\tau:=\sigma_0^{-1}\circ\sigma\in\mathrm{Aut}(G[H_k])$ fixing $H_k\setminus L_k^{\mathrm{iso}}$ pointwise and hence stabilizing L_k^{iso} setwise; by

Lemma 3.4(1), for every $v \in L_k^{\mathrm{iso}}$ both v and $\tau(v)$ lie in L_k^{iso} and have $N_G \subseteq H_k \setminus L_k^{\mathrm{iso}} \subseteq H_k$, so $N_G(\tau(v)) = N_{G[H_k]}(\tau(v)) = \tau(N_{G[H_k]}(v)) = \tau(N_G(v)) = N_G(v)$. Thus τ stabilizes each fiber $I_{\mathrm{nb}}(X) := \{v \in L_k^{\mathrm{iso}} : N_G(v) = X\}$, which correspond bijectively via $\min_{\prec_V}(X) \in U_r$ to $\bigsqcup_{r=k+1}^{L-1} \mathcal{F}_r^{(k),\mathrm{iso}}$ with $|I_{\mathrm{nb}}(X)| = \tilde{w}_{\mathcal{T}}$, giving at most $\prod_{r=k+1}^{L-1} \prod_{\mathcal{T} \in \mathcal{F}_r^{(k),\mathrm{iso}}} \tilde{w}_{\mathcal{T}}!$ extensions.

Dividing the product of Phases 1 and 2 by $|\operatorname{Aut}(G)|_{H_k}| \geq \prod_{r=k+1}^{L-1} \prod_{\mathcal{T} \in \mathcal{F}_r^{(k)}} \tilde{w}_{\mathcal{T}}!$ (Lemma 3.4(3)), canceling all $\mathcal{F}_r^{(k),\mathrm{iso}}$ factorials, applying $\tilde{w}_{\mathcal{T}}! > (\tilde{w}_{\mathcal{T}}/e)^{\tilde{w}_{\mathcal{T}}}$ (Lemma 2.4(1)), and taking $1 + \log_2(\cdot)$ proves (2). □

Proposition 6.5. *Set $\beta_k := \frac{\Delta_k}{\mathcal{L}_k} \Lambda_k$ $(0 \leq k \leq L-1)$, and for $r \in \{0, \ldots, L-1\}$ and real $\Theta \in [1, \Delta_r)$ define*

$$E_r^{\geq 2}(\Theta) \;:=\; \sum_{\substack{\mathcal{T} \in \mathcal{F}_r^{\geq 2} \\ \tilde{w}_{\mathcal{T}} \leq \Theta}} \tilde{w}_{\mathcal{T}} \log_2\!\left(\frac{2e\Delta_r}{\tilde{w}_{\mathcal{T}}}\right).$$

Put $C_1 := 10^6$, and for $0 \leq p < q \leq L-1$ with $d := q - p \geq 1$ set $F(p,q) := C_1 \Delta_p \mathcal{L}_p \min(d^2, \Delta_p)$, $\Theta_{q,p} := \frac{\Delta_q}{F(p,q)}$, and (when $\Theta_{q,p} \geq 1$) $c(p,q) := c_1(p,q) + c_2(p,q)$, where

$$c_1(p,q) \;:=\; \frac{\Delta_p}{\mathcal{L}_p} \cdot \frac{\mathcal{L}_q}{\Delta_q} \cdot 1200 \log_2(2e\Delta_q),$$

$$c_2(p,q) \;:=\; \mathbf{1}_{\Theta_{q,p} > 4} \cdot \frac{\Delta_p}{\mathcal{L}_p} \cdot \frac{\mathcal{L}_q}{\Delta_q} \cdot 7200 \, \frac{\Theta_{q,p}}{\log_2(2\Theta_{q,p})} \log_2\!\left(\frac{8e\Delta_q}{\Theta_{q,p}}\right).$$

Then:

(1) For every $r \in \{0, \ldots, L-1\}$ and $\Theta \in [1, \Delta_r)$,

$$E_r^{\geq 2}(\Theta) \;\leq\; 3|H_r| \log_2(2e\Delta_r) \;+\; 1200\, \Lambda_r \log_2(2e\Delta_r) \;+\; \mathbf{1}_{\Theta > 4} \cdot 7200\, \Lambda_r \frac{\Theta}{\log_2(2\Theta)} \log_2\!\left(\frac{8e\Delta_r}{\Theta}\right).$$

(2) For every $p \in \{0, \ldots, L-1\}$,

$$\frac{\Delta_p}{\mathcal{L}_p} \sum_{q=p+1}^{L-1} \sum_{\mathcal{T} \in \mathcal{F}_q^{(p),\geq 2}} \tilde{w}_{\mathcal{T}} \log_2\!\left(\frac{2e\Delta_q}{\tilde{w}_{\mathcal{T}}}\right)$$

$$\leq\; 4.6\, \Delta_p |H_p| \;+\; \frac{\Delta_p}{\mathcal{L}_p} \sum_{q=p+1}^{L-1} 3|H_q| \log_2(2e\Delta_q) \;+\; \sum_{\substack{q=p+1 \\ \Theta_{q,p} \geq 1}}^{L-1} c(p,q)\, \beta_q,$$

with $\sum_{p=0,\, \Theta_{q,p} \geq 1}^{q-1} c(p,q) < 0.01$ for every $q \in \{1, \ldots, L-1\}$, and $\sum_{k=0}^{L-1} \beta_k < 180\, m$.

(3) Consequently, Sum $1 < \frac{5 \times 10^4\, \bar{d}^2}{2^{\bar{d}/8}}\, m < 10^{-60}\, m < 10^{-50}\, m$.

Proof. (1) If r is degenerate, $G[B_r]$ has at most 3 components of size ≥ 2, so $\sum_{\mathcal{T} \in \mathcal{F}_r^{\geq 2}} \tilde{w}_{\mathcal{T}} \leq 3 \leq 3|H_r|$ and $E_r^{\geq 2}(\Theta) \leq 3|H_r| \log_2(2e\Delta_r)$. Suppose r is non-degenerate. Each $\mathcal{T} \in \mathcal{F}_r^{\geq 2}$ satisfies $\mu_r(\mathcal{T}) = \tilde{w}_{\mathcal{T}} + b_{\mathcal{T}}$, where $b_{\mathcal{T}} := |\{D \in \mathcal{B}_{\Delta_r}^{\geq 2} : [(D, \chi_r|_{V(D)})] = \mathcal{T}\}|$. Classes with $\mu_r(\mathcal{T}) \geq 2\tilde{w}_{\mathcal{T}}$ have $\tilde{w}_{\mathcal{T}} \leq b_{\mathcal{T}}$ and sum to $\leq |\mathcal{B}_{\Delta_r}| < |H_r|$ (Lemma 3.1(1b)), contributing $< |H_r| \log_2(2e\Delta_r)$. Remaining classes satisfy $\tilde{w}_{\mathcal{T}} \leq \mu_r(\mathcal{T}) < 2\Theta < 2\Delta_r$, so monotonicity of $x \mapsto x \log_2(\frac{2e\Delta_r}{x})$ on $(0, 2\Delta_r]$ gives $\tilde{w}_{\mathcal{T}} \log_2(\frac{2e\Delta_r}{\tilde{w}_{\mathcal{T}}}) \leq \mu_r(\mathcal{T}) \log_2(\frac{2e\Delta_r}{\mu_r(\mathcal{T})})$. By Lemmas 6.2(3), 6.3(2), and Proposition 6.4(1):

- on $\mathcal{C}_{\le 2}$, $\sum \mu_r(\mathcal{T}) \le N_{\le 2} \le |H_r| + 253\,\Lambda_r$;
- on $\{\mathcal{T} \in \mathcal{C}_{\ge 3} : \mu_r(\mathcal{T}) = 1\}$, $|\eta_1| < |H_r|$ gives count $\le |H_r| + 5\,\Lambda_r$;
- on $\{\mathcal{T} \in \mathcal{C}_{\ge 3} : 2 \le \mu_r(\mathcal{T}) \le 8\}$, $\sum \mu_r(\mathcal{T}) \le 934\,\Lambda_r$;
- when $\Theta > 4$, on $\{\mathcal{T} \in \mathcal{C}_{\ge 3} : 9 \le \mu_r(\mathcal{T}) < 2\Theta\}$, set $J := \lceil \log_2(2\Theta) \rceil \ge 4$ and $A := \log_2(8e\Delta_r) > 129 \ge 40$; since $\Theta < \Delta_r$, $J < \log_2(4\Delta_r) < \log_2(2e\Delta_r) = A - 2$, and on $2^{j-1} < \mu_r(\mathcal{T}) \le 2^j$ $(4 \le j \le J)$ we have $\log_2(\frac{2e\Delta_r}{\mu_r(\mathcal{T})}) < A - j$. Lemma 2.4(4b) bounds this range by $300\,\Lambda_r S(J, A) \le 1800\,\Lambda_r \frac{2^J}{J}(A - J + 1) < 7200\,\Lambda_r \frac{\Theta}{\log_2(2\Theta)} \log_2(\frac{8e\Delta_r}{\Theta})$ (since $\log_2(2\Theta) \le J < \log_2(4\Theta)$).

Since $253 + 5 + 934 = 1192 < 1200$, summing proves (1).

(2) Fix $0 \le p \le L - 1$. For $p < q \le L - 1$, $\Theta_{q,p} < \Delta_q$. Terms with $\tilde{w}_{\mathcal{T}} > \Theta_{q,p}$ satisfy $\frac{2e\Delta_q}{\tilde{w}_{\mathcal{T}}} < 2eF(p,q) \le 2eC_1\Delta_p^2\mathcal{L}_p$; since $8eC_1 < 2.2 \times 10^7 < 2^{25}$, $\Delta_p = 2^{\mathcal{L}_p - 2}$, and $\log_2 x < 0.06x$ on $[127, \infty)$, $\log_2(8eF(p,q)) < 25 + 2(\mathcal{L}_p - 2) + 0.06\mathcal{L}_p = 21 + 2.06\mathcal{L}_p < 2.3\mathcal{L}_p$ (as $0.24 \cdot 127 > 30 > 21$). By Lemma 3.4(2), $\sum_{q=p+1}^{L-1} \sum_{\mathcal{T} \in \mathcal{F}_q^{(p),\ge 2}} \tilde{w}_{\mathcal{T}} < 2|H_p|$, so high-$\tilde{w}$ terms contribute $< 4.6\,\Delta_p|H_p|$, while low-$\tilde{w}$ terms ($\tilde{w}_{\mathcal{T}} \le \Theta_{q,p}$, non-empty only when $\Theta_{q,p} \ge 1$) are bounded by (1), giving the displayed level-p inequality.

Fix $1 \le q \le L - 1$; note $\mathcal{L}_q = \mathcal{L}_p + d$ ($d := q - p$) and $\log_2(2e\Delta_q) = \mathcal{L}_q + \log_2(e/2) < 1.01\mathcal{L}_q$.

- *Subcase 2A ($d^2 \le \Delta_p$, $\Theta_{q,p} \ge 1$).* Here $2^d/d^2 \ge C_1\mathcal{L}_p \ge 1.27 \times 10^8$; since $\max_{1 \le d \le 35} 2^d/d^2 = 2^{35}/35^2 < 2.81 \times 10^7$, we have $d \ge 36$. From $\mathcal{L}_p \le \frac{2^d}{C_1 d^2}$, $\frac{\mathcal{L}_q^2}{\mathcal{L}_p} = \mathcal{L}_p + 2d + \frac{d^2}{\mathcal{L}_p} < \frac{2^d}{C_1 d^2} + 2d^2$, so $c_1(p,q) < \frac{1212}{C_1 d^2} + \frac{2424 d^2}{2^d} < \frac{0.062}{d^2}$ (as $d^4/2^d$ decreases for $d \ge 6$ and $2424 \cdot 36^4/2^{36} < 0.06$). When $\Theta_{q,p} > 4$, $\log_2(8eF(p,q)) < 2.3\mathcal{L}_p$ gives $c_2(p,q) < \frac{0.01656}{d^2} \frac{\mathcal{L}_p + d}{\mathcal{L}_p \log_2(2\Theta_{q,p})}$. If $d \le 4\mathcal{L}_p$, $\log_2(2\Theta_{q,p}) > 3$ gives factor $< \frac{5}{3}$; if $d > 4\mathcal{L}_p$, $2\log_2 d \le \mathcal{L}_p - 2$ and $\log_2(C_1\mathcal{L}_p) < \mathcal{L}_p$ give $\log_2(2\Theta_{q,p}) > d - 2\mathcal{L}_p > d/2$, giving factor $< \frac{1.25d}{127(d/2)} < 0.02 < \frac{5}{3}$. Hence $c_2(p,q) < \frac{0.03}{d^2}$, and summing over $d \ge 36$ gives $< 0.092 \sum_{d=36}^{\infty} \frac{1}{d(d-1)} = \frac{0.092}{35} < 0.003$.
- *Subcase 2B ($d^2 > \Delta_p$, $\Theta_{q,p} \ge 1$).* Here $d > \sqrt{\Delta_p} \ge 2^{62.5}$; since $2^{t-2} > 100t^2$ for $t \ge 127$, $d > 10\mathcal{L}_p$ and $\mathcal{L}_q < 1.1d$. Using $2^u > u^5$ for $u \ge 23$, $2^d > d^5 > 12d^4$, so $c_1(p,q) < \frac{1212 \cdot 1.21 d^2}{127 \cdot 2^d} < \frac{1}{d^2}$. When $\Theta_{q,p} > 4$, $F(p,q) = C_1\Delta_p^2\mathcal{L}_p$, $\log_2(C_1\Delta_p\mathcal{L}_p) < 2\mathcal{L}_p < 0.2d$, and $\log_2(2\Theta_{q,p}) > 0.8d$, giving $c_2(p,q) < \frac{7200 \cdot 2.3}{C_1\Delta_p} \cdot \frac{1.1}{127 \cdot 0.8} < \frac{1}{\Delta_p}$. Summing Subcase 2B gives $< \sum_{d > 2^{62.5}} \frac{1}{d^2} + \sum_{p=0}^{q-1} \frac{1}{\Delta_p} < 2^{-62} + 2^{-124}$.

Thus $\sum_{p=0,\,\Theta_{q,p} \ge 1}^{q-1} c(p,q) < 0.003 + 2^{-62} + 2^{-124} < 0.01$. Multiplying Proposition 6.4(2) (at $k = p$) by $\frac{\Delta_p}{\mathcal{L}_p}$, substituting the level-p inequality, and summing over $0 \le p \le L - 1$ via Lemma 3.3(1),(3)—specifically $\sum_p \frac{\Delta_p}{\mathcal{L}_p}(1 + \log_2 n) < \frac{6n}{\log_2(4n)} \cdot 1.001 \log_2 n < 6.01n < 0.01\,m$, $\sum_p \frac{\Delta_p}{\mathcal{L}_p}|H_p| \log_2(16\Delta_p) < 2\sum_p \Delta_p|H_p| \le 8\,m$, $\sum_p \frac{\Delta_p}{\mathcal{L}_p} \sum_{r \ge p} 2|H_r| \log_2(2\Delta_r) \le 12 \sum_r \Delta_r|H_r| \le 48\,m$, $\sum_p 4.6\Delta_p|H_p| \le 18.4\,m$, and $\sum_p \frac{\Delta_p}{\mathcal{L}_p} \sum_{q > p} 3|H_q| \log_2(2e\Delta_q) \le 18.18 \sum_q \Delta_q|H_q| < 73\,m$—yields $0.99 \sum_{k=0}^{L-1} \beta_k < (0.01 + 8 + 48 + 18.4 + 73)\,m = 147.41\,m < 150\,m$, so $\sum_{k=0}^{L-1} \beta_k < 180\,m$.

(3) Put $\Theta_k^{\mathrm{S1}} := 2\Delta_k/2^{\bar{d}/8} \in (1, \Delta_k)$ ($k_{\min} \le k \le L - 1$, Lemma 3.3(1)). Applying (1) at $\Theta = \Theta_k^{\mathrm{S1}}$: $\frac{\mathcal{L}_k}{\Delta_k} < \frac{\bar{d}/8 + 1}{2^{\bar{d}/8 - 1}} < \frac{\bar{d}}{2^{\bar{d}/8}}$ gives $\sum_{k=k_{\min}}^{L-1} 3|H_k| \log_2(2e\Delta_k) < \frac{13\,\bar{d}}{2^{\bar{d}/8}}\,m$; monotonicity of $\frac{(\log_2(4u))^2}{u}$ on $[2^{125}, \infty)$ gives $\frac{\mathcal{L}_k^2}{\Delta_k} < \frac{(\bar{d}/8 + 1)^2}{2^{\bar{d}/8 - 1}} < \frac{\bar{d}^2}{30 \cdot 2^{\bar{d}/8}}$, bounding $\sum_{k=k_{\min}}^{L-1} 1200\,\Lambda_k \log_2(2e\Delta_k) < \frac{7300\,\bar{d}^2}{2^{\bar{d}/8}}\,m$; and when $\Theta_k^{\mathrm{S1}} > 4$, $\Lambda_k\Theta_k^{\mathrm{S1}} = \frac{2\mathcal{L}_k}{2^{\bar{d}/8}}\beta_k$, $\log_2(\frac{8e\Delta_k}{\Theta_k^{\mathrm{S1}}}) = \frac{\bar{d}}{8} + \log_2(4e) < \frac{\bar{d}}{7}$, and $\frac{\mathcal{L}_k}{\log_2(2\Theta_k^{\mathrm{S1}})} < \frac{\bar{d}}{12}$ ($\le \frac{\bar{d}}{12}$ if $\mathcal{L}_k \le \frac{\bar{d}}{4}$ since $\log_2(2\Theta_k^{\mathrm{S1}}) > 3$, and < 2 if $\mathcal{L}_k > \frac{\bar{d}}{4}$ since $\log_2(2\Theta_k^{\mathrm{S1}}) = \mathcal{L}_k - \frac{\bar{d}}{8} > \frac{1}{2}\mathcal{L}_k$), bounding the third

sum by $\frac{1200\,\bar{d}^2}{7\cdot 2^{\bar{d}/8}}\cdot 180\,m < \frac{31000\,\bar{d}^2}{2^{\bar{d}/8}}\,m$. Thus Sum $1 < \frac{5\times 10^4\,\bar{d}^2}{2^{\bar{d}/8}}\,m \le \frac{2\times 10^{11}}{2^{250}}\,m < 10^{-60}\,m < 10^{-50}\,m$ (as $2^{250} = 1024^{25} > 10^{75}$). □

7 Pillar II: Twin-Quotient Orbit-Freezing and Cumulative-Base Subgroup Chain

7.1 Universal Twin-Quotient Shift-Invariance

Lemma 7.1. *Suppose $t_{\max}(G) \ge 3$. Then $s(G) = s(H) =: s \le n-3$ and $\widetilde{G} \cong \widetilde{H}$. Identifying $\widetilde{G} = \widetilde{H} =: Q$ on $[s]$ and setting $\Gamma := \mathrm{Aut}(Q)$, $w_G := \nu_G - \mathbf{1} \in \mathbb{Z}^s_{\ge 0}$, and $w_H := \nu_H - \mathbf{1} \in \mathbb{Z}^s_{\ge 0}$ (so $|w_G| = |w_H| = n - s \ge 3$), the polynomial*

$$\widetilde{P}(y_1,\dots,y_s) \;:=\; \sum_{\pi\in\Gamma}\big(y^{\pi\cdot w_G} - y^{\pi\cdot w_H}\big) \;\in\; \mathbb{Z}[y_1,\dots,y_s]$$

is non-zero, homogeneous of degree $n-s$, same-partition on every Γ-orbit $\mathcal{O}\subseteq[s]$ (hence globally same-partition on $[s]$), and satisfies $D_{\mathcal{O}}(\widetilde{P}) \equiv 0$ for every Γ-orbit $\mathcal{O}\subseteq[s]$ (in particular $D(\widetilde{P})\equiv 0$).

Proof. We first establish the *No-Merging Claim*: for any simple graph F and $x \in V_i$ ($V_1,\dots,V_{s(F)}$ the open-twin classes of F), if $|V_i| \ge 2$ then $s(F-x) = s(F)$, $\widetilde{F-x} \cong \widetilde{F}$, and the twin classes of $F-x$ are $V_1,\dots,V_i\setminus\{x\},\dots,V_{s(F)}$, whereas if $|V_i| = 1$ then $s(F-x) \le s(F)-1$. Indeed, when $|V_i|\ge 2$, pick $x' \in V_i\setminus\{x\}$; for any $u,u' \in V(F)\setminus\{x\}$ with $N_{F-x}(u) = N_{F-x}(u')$, we have $x' \in N_F(u) \iff x' \in N_F(u')$, and since $N_F(x) = N_F(x')$ is disjoint from $\{x,x'\}$, $x \in N_F(u) \iff x \in N_F(u')$, forcing $N_F(u) = N_F(u')$; when $|V_i| = 1$, each V_j ($j\ne i$) remains inside a single twin class of $F-x$, giving $s(F-x) \le s(F)-1$.

Because G is connected on $n \ge 4$ vertices, $s(G) \ge 2$. Pick a twin class V_1 of G of size ≥ 3 and $v \in V(G)\setminus V_1$; then $t_{\max}(G-v) \ge 3$. From $\mathcal{D}(G) = \mathcal{D}(H)$, some $z \in V(H)$ satisfies $H - z \cong G - v$, so $3 \le t_{\max}(H-z) \le 2t_{\max}(H)$ (each twin class of $H-z$ splits into at most two in H), giving $t_{\max}(H) \ge 2$. By the No-Merging Claim, $s(G) = \max_{v\in V(G)} s(G-v) = \max_{z\in V(H)} s(H-z) = s(H) =: s$, and the matched sub-decks $\mathcal{D}_{\mathrm{twin}}(F) := \{\{[F-x] : s(F-x) = s\}\}$ ($F \in \{G,H\}$) consist of deletions from classes of size ≥ 2, giving $\widetilde{G} \cong \widetilde{H}$.

Identify $\widetilde{G} = \widetilde{H} =: Q$ on $[s]$ and put $N' := n - s \ge 2$. Notice Q is open-twin-free ($N_Q(i) = N_Q(j) \implies N_G(V_i) = N_G(V_j) \implies i = j$), so $G \ncong H \iff w_G \notin \Gamma\cdot w_H$, whence $[y^{w_G}]\widetilde{P} = |\mathrm{Stab}_\Gamma(w_G)| \ge 1$ and $\widetilde{P} \not\equiv 0$. Deleting a vertex from V_i ($w_G(i) \ge 1$) produces the Q-blow-up of vector $w_G - \mathbf{e}_i$ with multiplicity $w_G(i)+1$ in $\mathcal{D}_{\mathrm{twin}}(G)$; hence $\mathcal{D}_{\mathrm{twin}}(G) = \mathcal{D}_{\mathrm{twin}}(H)$ is equivalent to $\mathcal{D}_+(\widetilde{P}) \equiv 0$, where $\mathcal{D}_+(y^u) := \sum_{i:u_i\ge 1}(u_i+1)y^{u-\mathbf{e}_i}$. Setting $\Lambda(y^u) := (\prod_{j=1}^s (u_j+1))y^u$, every monomial y^u ($u \in \mathbb{Z}^s_{\ge 0}$) satisfies

$$(\Lambda\circ\mathcal{D}_+)(y^u) \;=\; \sum_{i:u_i\ge 1}(u_i+1)\Big(u_i\prod_{j\ne i}(u_j+1)\Big)y^{u-\mathbf{e}_i} \;=\; \Big(\prod_{j=1}^s(u_j+1)\Big)\sum_{i=1}^s u_i y^{u-\mathbf{e}_i} \;=\; (D\circ\Lambda)(y^u).$$

Hence

$$\widehat{P} \;:=\; \Lambda(\widetilde{P}) \;=\; \Big(\prod_{i=1}^s \nu_G(i)\Big)\sum_{\pi\in\Gamma} y^{\pi\cdot w_G} - \Big(\prod_{i=1}^s \nu_H(i)\Big)\sum_{\pi\in\Gamma} y^{\pi\cdot w_H}$$

satisfies $D(\widehat{P}) = \Lambda(\mathcal{D}_+(\widetilde{P})) \equiv 0$. Since $N' \ge 2$, $\widehat{P}(\mathbf{1}) = \widehat{P}(\mathbf{0}) = 0$ gives $\prod_{i=1}^s \nu_G(i) = \prod_{i=1}^s \nu_H(i) \ge 1$, so $D(\widetilde{P}) \equiv 0$. If $N' = 2$, then $t_{\max}(G) \ge 3$ forces $w_G = 2\mathbf{e}_{i_0}$, so $\prod \nu_H(i) = 3$ forces $w_H = 2\mathbf{e}_{j_0}$, and $D(\widetilde{P}) = 2\sum_{\pi\in\Gamma}(y_{\pi(i_0)} - y_{\pi(j_0)}) \equiv 0$ forces $i_0 \in \Gamma\cdot j_0$, contradicting $w_G \notin \Gamma\cdot w_H$. Thus $N' \ge 3$ (so $s \le n-3$).

Write $P_G := \sum_{\pi\in\Gamma} y^{\pi\cdot w_G}$ and $P_H := \sum_{\pi\in\Gamma} y^{\pi\cdot w_H}$. For any Γ-orbit $\mathcal{O} \subseteq [s]$, put $S_G(\mathcal{O}) := \sum_{i\in\mathcal{O}} w_G(i)$ and $S_H(\mathcal{O}) := \sum_{i\in\mathcal{O}} w_H(i)$. If $S_G(\mathcal{O}) \neq S_H(\mathcal{O})$, assume by symmetry $S_G(\mathcal{O}) > S_H(\mathcal{O})$. In $0 \equiv D(\widetilde{P}) = D_{\mathcal{O}}(P_G) + D_{[s]\setminus\mathcal{O}}(P_G) - D_{\mathcal{O}}(P_H) - D_{[s]\setminus\mathcal{O}}(P_H)$, the unique term of $\mathcal{O}$-degree $S_G(\mathcal{O})$ is $D_{[s]\setminus\mathcal{O}}(P_G)$, forcing $S_G(\mathcal{O}) = N'$; then at $\mathcal{O}$-degree $N'-1$, $D_{\mathcal{O}}(P_G)$ (coefficient sum $|\Gamma|N'$) would have to cancel $D_{[s]\setminus\mathcal{O}}(P_H)$ when $S_H(\mathcal{O}) = N'-1$ (coefficient sum $|\Gamma|\cdot 1$), contradicting $N' \geq 3$. Thus $S_G(\mathcal{O}) = S_H(\mathcal{O}) =: d_{\mathcal{O}}$ for every Γ-orbit $\mathcal{O}$, splitting $D(\widetilde{P}) \equiv 0$ and $\mathcal{D}_+(\widetilde{P}) \equiv 0$ into $D_{\mathcal{O}}(\widetilde{P}) \equiv 0$ and $\mathcal{D}_{+,\mathcal{O}}(\widetilde{P}) \equiv 0$ ($\mathcal{D}_{+,\mathcal{O}}(y^u) := \sum_{i\in\mathcal{O}:u_i\geq 1}(u_i+1)y^{u-\mathbf{e}_i}$).

If $d_{\mathcal{O}} = 0$, $w_G|_{\mathcal{O}} = w_H|_{\mathcal{O}} = \mathbf{0}$. If $d_{\mathcal{O}} \geq 1$, pick $i \in \mathcal{O}$ with $w_G(i) \geq 1$; since $D_{\mathcal{O}}(\widetilde{P}) \equiv 0$, some $\pi_0 \in \Gamma$ and $j \in \mathcal{O}$ satisfy $\pi_0 \cdot w_H - \mathbf{e}_j = w_G - \mathbf{e}_i$, so $(\pi_0 \cdot w_H)|_{[s]\setminus\mathcal{O}} = w_G|_{[s]\setminus\mathcal{O}}$. Replacing w_H by $\pi_0 \cdot w_H$ and putting $\Gamma_0 := \mathrm{Stab}_\Gamma(w_G|_{[s]\setminus\mathcal{O}})$, the extracted polynomial $P_{\mathcal{O}}(y_{\mathcal{O}}) := [y_{[s]\setminus\mathcal{O}}^{w_G|_{[s]\setminus\mathcal{O}}}]\widetilde{P} = \sum_{\pi\in\Gamma_0}(y_{\mathcal{O}}^{\pi\cdot(w_G|_{\mathcal{O}})} - y_{\mathcal{O}}^{\pi\cdot(w_H|_{\mathcal{O}})})$ satisfies $D_{\mathcal{O}}(P_{\mathcal{O}}) \equiv 0$ and $\mathcal{D}_{+,\mathcal{O}}(P_{\mathcal{O}}) \equiv 0$. Evaluating $(\mathcal{D}_{+,\mathcal{O}} - D_{\mathcal{O}})(P_{\mathcal{O}}) \equiv 0$ at $\mathbf{1}$ gives $|\operatorname{supp}(w_G|_{\mathcal{O}})| = |\operatorname{supp}(w_H|_{\mathcal{O}})|$. For $1 \leq k \leq d_{\mathcal{O}} - 1$, applying the linear functional $L_k(y_{\mathcal{O}}^u) := \sum_{i\in\mathcal{O}} \binom{u_i}{k}$ ($|u| = d_{\mathcal{O}}$) via $L_k(D_{\mathcal{O}}(y_{\mathcal{O}}^u)) = (d_{\mathcal{O}} - k)L_k(y_{\mathcal{O}}^u)$ to $D_{\mathcal{O}}(P_{\mathcal{O}}) \equiv 0$ gives $\sum_{i\in\mathcal{O}} \binom{w_G(i)}{k} = \sum_{i\in\mathcal{O}} \binom{w_H(i)}{k}$, while at $k = d_{\mathcal{O}}$ both equal $\mathbf{1}\{|\operatorname{supp}(w_F|_{\mathcal{O}})| = 1\}$; triangular inversion yields same-partition equality on $\mathcal{O}$. □

7.2 High-Ratio Orbit-Freezing on the Twin Quotient

Whenever $t_{\max}(G) \geq 3$, use the notation $Q = \widetilde{G} = \widetilde{H}$ on $[s]$, $\Gamma = \mathrm{Aut}(Q)$, w_G, w_H, and $\widetilde{P}$ of Lemma 7.1, put $S := \mathrm{Stab}_\Gamma(w_G) = \mathrm{Stab}_\Gamma(\nu_G)$, and write $\tilde{d}_G(u) := \sum_{i\in N_Q(u)} \nu_G(i)$ for the common G-degree of vertices in V_u ($u \in V(Q)$). For $k \in \{k_{\min}, \dots, L-1\}$ and $j \geq 3$, define

$$
\begin{aligned}
I_{\mathrm{mid}}(2^j) &:= \{i \in V(Q) : 2^{j-2} \leq \nu_G(i) \leq 2^j\},\\
\widetilde{\Gamma}_{\mathrm{band}}(2^j) &:= \{\pi \in \Gamma : \nu_G(\pi^{-1}(i)) = \nu_G(i)\ \forall\, i \notin I_{\mathrm{mid}}(2^j)\},\\
d_{\mathrm{inv}}(u) &:= \sum_{i\in N_Q(u)\setminus I_{\mathrm{mid}}(2^j)} \nu_G(i) \;+\; 2^{j-2}|N_Q(u) \cap I_{\mathrm{mid}}(2^j)| \qquad (u \in V(Q)),\\
I^* \;=\; I_{\text{hi-ratio}}(k,j) &:= \{i \in I_{\mathrm{mid}}(2^j) : d_{\mathrm{inv}}(u) \geq \tfrac{1}{4}\Delta_k\ \forall\, u \in N_Q(i)\},\\
\Gamma^* \;=\; \Gamma_{\text{hi-ratio}}(k,j) &:= \{\pi \in \Gamma : \nu_G(\pi^{-1}(i)) = \nu_G(i)\ \forall\, i \notin I^*\}.
\end{aligned}
$$

Lemma 7.2. *Assume $t_{\max}(G) \geq 3$. For every $k \in \{k_{\min}, \dots, L-1\}$ and every integer $j \geq 3$:*

(1) $\widetilde{\Gamma}_{\mathrm{band}}(2^j) \leq \Gamma$ stabilizes $I_{\mathrm{mid}}(2^j)$ setwise, satisfies $\frac{1}{4}\tilde{d}_G(u) \leq d_{\mathrm{inv}}(u) \leq \tilde{d}_G(u)$ and $d_{\mathrm{inv}}(\pi(u)) = d_{\mathrm{inv}}(u)$ for all $u \in V(Q)$ and $\pi \in \widetilde{\Gamma}_{\mathrm{band}}(2^j)$, and stabilizes I^* setwise. Moreover, $\mathcal{I}_k \cap I_{\mathrm{mid}}(2^j) \subseteq I^*$ and $S \leq \Gamma^* \leq \widetilde{\Gamma}_{\mathrm{band}}(2^j) \leq \Gamma$.

(2) We have

$$
\sum_{\substack{i\in\mathcal{I}_k\\ 3\leq\nu_G(i)\leq 8}} \nu_G(i) \;\leq\; 8 + 1300\log_2(2[\Gamma^* : S]) \qquad (j = 3),
$$

$$
\sum_{\substack{i\in\mathcal{I}_k\\ 2^{j-1}<\nu_G(i)\leq 2^j}} \nu_G(i) \;\leq\; 2^j + \frac{300\cdot 2^j}{j-1}\log_2(2[\Gamma^* : S]) \qquad (j \geq 4).
$$

Proof. (1) Every $\pi \in \widetilde{\Gamma}_{\mathrm{band}}(2^j)$ maps $V(Q) \setminus I_{\mathrm{mid}}(2^j)$ bijectively onto itself and hence stabilizes $I_{\mathrm{mid}}(2^j)$ setwise, making $\widetilde{\Gamma}_{\mathrm{band}}(2^j) \leq \Gamma$. On $I_{\mathrm{mid}}(2^j)$, $2^{j-2} \leq \nu_G(i) \leq 4\cdot 2^{j-2}$, giving $\frac{1}{4}\tilde{d}_G(u) \leq d_{\mathrm{inv}}(u) \leq \tilde{d}_G(u)$, $d_{\mathrm{inv}}(\pi(u)) = d_{\mathrm{inv}}(u)$, and $\pi(I^*) = I^*$ for all $\pi \in \widetilde{\Gamma}_{\mathrm{band}}(2^j)$. By Lemma 3.2(1), every

$i \in \mathcal{I}_k \cap I_{\mathrm{mid}}(2^j)$ has $N_G(V_i) \subseteq H_k$, so every $u \in N_Q(i)$ satisfies $d_{\mathrm{inv}}(u) \geq \frac{1}{4}\tilde{d}_G(u) \geq \frac{1}{4}\Delta_k$, placing $i \in I^*$. Since $I^* \subseteq I_{\mathrm{mid}}(2^j)$ is $\widetilde{\Gamma}_{\mathrm{band}}(2^j)$-invariant, $S \leq \Gamma^* \leq \widetilde{\Gamma}_{\mathrm{band}}(2^j)$.

(2) Put $N_{\mathrm{mid}} := \sum_{i \in I_{\mathrm{mid}}(2^j)} (\nu_G(i) - 2^{j-2})$. If $N_{\mathrm{mid}} \leq 1$ (including $I_{\mathrm{mid}}(2^j) = \emptyset$), at most one $i \in I_{\mathrm{mid}}(2^j)$ has $\nu_G(i) > 2^{j-2}$, so the left-hand sum is $\leq 2^j$ (≤ 8 when $j = 3$) and (2) holds trivially.

Assume $N_{\mathrm{mid}} \geq 2$ (so $I_{\mathrm{mid}}(2^j) \neq \emptyset$). Apply Lemma 4.3 to $R := \widetilde{P}$ (with $v = w_G$) at $(t_0, t_1) = (2^{j-2} - 1, 2^j - 1)$, noting $t_0 \leq w_G(i) \leq t_1 \iff i \in I_{\mathrm{mid}}(2^j)$. This yields a homogeneous $D_{I_{\mathrm{mid}}(2^j)}$-invariant polynomial $F_{\mathrm{mid}} = F^+ - F^-$ of degree $N_{\mathrm{mid}} \geq 2$ with leading coefficient $|S|$ at $(\nu_G - 2^{j-2}\mathbf{1})|_{I_{\mathrm{mid}}(2^j)}$, where

$$F^+ := \sum_{\pi \in \widetilde{\Gamma}_{\mathrm{band}}(2^j)} y^{((\pi \cdot \nu_G) - 2^{j-2}\mathbf{1})|_{I_{\mathrm{mid}}(2^j)}}, \qquad F^- := \sum_{\pi \in \Pi_H} y^{((\pi \cdot \nu_H) - 2^{j-2}\mathbf{1})|_{I_{\mathrm{mid}}(2^j)}},$$

and $\Pi_H := \{\pi \in \Gamma : \nu_H(\pi^{-1}(i)) = \nu_G(i)\ \forall\, i \notin I_{\mathrm{mid}}(2^j)\}$. Since $F_{\mathrm{mid}}(\mathbf{1}) = 0$, $\Pi_H \neq \emptyset$; fixing $\pi_0 \in \Pi_H$, we have $\Pi_H = \widetilde{\Gamma}_{\mathrm{band}}(2^j)\pi_0$, and for $\pi = \sigma\pi_0$ ($\sigma \in \widetilde{\Gamma}_{\mathrm{band}}(2^j)$), $((\sigma\pi_0 \cdot \nu_H) - 2^{j-2}\mathbf{1})|_{I_{\mathrm{mid}}(2^j)} = \sigma \cdot ((\pi_0 \cdot \nu_H) - 2^{j-2}\mathbf{1})|_{I_{\mathrm{mid}}(2^j)})$. Let $\mathcal{O}_1, \ldots, \mathcal{O}_q$ be the $\widetilde{\Gamma}_{\mathrm{band}}(2^j)$-orbits on $I_{\mathrm{mid}}(2^j)$. Every monomial in F^+ (respectively F^-) has the same orbit-degree tuple $\mathbf{a}_G = (a_G(\mathcal{O}_1), \ldots, a_G(\mathcal{O}_q))$ (respectively $\mathbf{a}_H$), with $\sum_t a_G(\mathcal{O}_t) = \sum_t a_H(\mathcal{O}_t) = N_{\mathrm{mid}} \geq 2$.

We claim $\mathbf{a}_G = \mathbf{a}_H$. If not, pick distinct orbits $\mathcal{O}_1, \mathcal{O}_2$ with $a_G(\mathcal{O}_1) > a_H(\mathcal{O}_1) \geq 0$ and $a_H(\mathcal{O}_2) > a_G(\mathcal{O}_2) \geq 0$, and expand $\sum_t D_{\mathcal{O}_t}(F^+) = \sum_t D_{\mathcal{O}_t}(F^-)$:

- if $\mathbf{a}_G \neq N_{\mathrm{mid}}\mathbf{e}_{\mathcal{O}_1}$, some $\mathcal{O}_3 \neq \mathcal{O}_1$ has $a_G(\mathcal{O}_3) \geq 1$, giving $D_{\mathcal{O}_3}(F^+) \not\equiv 0$ (positive coefficients) of $\mathcal{O}_1$-degree $a_G(\mathcal{O}_1) > a_H(\mathcal{O}_1)$, impossible;
- if $\mathbf{a}_H \neq N_{\mathrm{mid}}\mathbf{e}_{\mathcal{O}_2}$, some $\mathcal{O}_4 \neq \mathcal{O}_2$ has $a_H(\mathcal{O}_4) \geq 1$, giving $D_{\mathcal{O}_4}(F^-) \not\equiv 0$ of $\mathcal{O}_2$-degree $a_H(\mathcal{O}_2) > a_G(\mathcal{O}_2)$, impossible;
- otherwise $\mathbf{a}_G = N_{\mathrm{mid}}\mathbf{e}_{\mathcal{O}_1}$ and $\mathbf{a}_H = N_{\mathrm{mid}}\mathbf{e}_{\mathcal{O}_2}$, so $D_{\mathcal{O}_1}(F^+) \not\equiv 0$ has $\mathcal{O}_1$-degree $N_{\mathrm{mid}} - 1 \geq 1$ while $D(F^-)$ has $\mathcal{O}_1$-degree 0, impossible.

Thus $\mathbf{a}_G = \mathbf{a}_H$; pairwise distinctness of $\mathbf{a}_G - \mathbf{e}_{\mathcal{O}_t}$ separates $D_{I_{\mathrm{mid}}(2^j)}(F_{\mathrm{mid}}) \equiv 0$ into $D_{\mathcal{O}_t}(F_{\mathrm{mid}}) \equiv 0$ for every $\mathcal{O}_t$, whence $D_{I^*}(F_{\mathrm{mid}}) \equiv 0$ by (1).

Set $N^* := \sum_{i \in I^*} (\nu_G(i) - 2^{j-2})$. If $N^* = 0$ (including $I^* = \emptyset$), the left-hand sum of (2) is 0 (by $\mathcal{I}_k \cap I_{\mathrm{mid}}(2^j) \subseteq I^*$). If $N^* \geq 1$, extracting $[y_{I_{\mathrm{mid}}(2^j) \setminus I^*}^{(\nu_G - 2^{j-2}\mathbf{1})|_{I_{\mathrm{mid}}(2^j) \setminus I^*}}]F_{\mathrm{mid}}$ yields a homogeneous D_{I^*}-invariant polynomial $F^*(y_{I^*})$ of degree N^* with leading coefficient $|S|$ and $\|F^*\|_1 = 2|\Gamma^*|$. For $j = 3$, maximum exponent is ≤ 6 and $\nu_G(i) \leq 3(\nu_G(i) - 2)$ on $3 \leq \nu_G(i) \leq 8$, so Theorem 4.2(2) at $t = 6$ gives $3N^* \leq \frac{3600}{\log_2 7} \log_2(2[\Gamma^* : S]) < 1300 \log_2(2[\Gamma^* : S])$ (since $7^5 = 16807 > 16384 = 2^{14} \implies \frac{3600}{\log_2 7} < \frac{18000}{14} < 1300$). For $j \geq 4$, Theorem 4.2(2) at $t = 3 \cdot 2^{j-2}$ and $\nu_G(i) \leq 2(\nu_G(i) - 2^{j-2})$ on $2^{j-1} < \nu_G(i) \leq 2^j$ give $2N^* \leq \frac{300 \cdot 2^j}{j-1} \log_2(2[\Gamma^* : S])$. □

7.3 Cumulative-Base Subgroup Chain and Collapse of $\sum E_k^{(2),\geq 3}$

Proposition 7.3. *For every $p \in \{0, \ldots, L-1\}$, define*

$$\mathcal{U}_{p+2} := 1 + \log_2 n + 5 \sum_{q=p}^{L-1} |H_q| \log_2(2\Delta_q) + 3 \sum_{s'=p+1}^{L-1} \sum_{\mathcal{T} \in \mathcal{F}_{s'}^{(p),\geq 2}} \tilde{w}_{\mathcal{T}} \log_2\left(\frac{2e\Delta_{s'}}{\tilde{w}_{\mathcal{T}}}\right).$$

Then:

(1) Assuming $t_{\max}(G) \geq 3$, for every $k \in \{k_{\min}, \dots, L-1\}$ ($p := k-2 \geq 122$) and every integer $j \geq 3$, uniformly in j,

$$\log_2(2[\Gamma_{\text{hi-ratio}}(k,j) : S]) \;\leq\; \mathcal{U}_k.$$

(2) We have $\sum_{k=k_{\min}}^{L-1} \frac{\Delta_k}{\tilde{C}_k} \mathcal{U}_k < 2000\, m$, and unconditionally (including $t_{\max}(G) \leq 2$, where $E_k^{(2),\geq 3} = 0$),

$$\sum_{k=k_{\min}}^{L-1} E_k^{(2),\geq 3} \;<\; \frac{5 \times 10^5\, \bar{d}^2}{2^{\bar{d}/8}}\, m \;<\; 10^{-60}\, m \;<\; 10^{-50}\, m.$$

Proof. (1) Fix $k \in \{k_{\min}, \dots, L-1\}$, $p := k-2 \geq 122$, and $j \geq 3$; put $I^* := I_{\text{hi-ratio}}(k,j)$ and $\Gamma^* := \Gamma_{\text{hi-ratio}}(k,j)$. For $r \in \{k, \dots, L+2\}$, set $\widetilde{H}_r^{\text{inv}} := \{u \in V(Q) : d_{\text{inv}}(u) \geq \frac{1}{4}\Delta_r\}$, $\widetilde{U}_r^{\text{inv}} := \widetilde{H}_r^{\text{inv}} \setminus \widetilde{H}_{r+1}^{\text{inv}}$ ($k \leq r \leq L+1$), and $\widetilde{H}_t := \{u \in V(Q) : \tilde{d}_G(u) \geq \Delta_t\}$ ($|\widetilde{H}_t| \leq |H_t|$, $\widetilde{H}_t = \emptyset$ for $t \geq L$). Lemma 7.2(1) gives $\widetilde{H}_r \subseteq \widetilde{H}_r^{\text{inv}} \subseteq \widetilde{H}_{r-2}$ (so $\widetilde{H}_{L+2}^{\text{inv}} = \emptyset$ and $\widetilde{H}_k^{\text{inv}} \neq \emptyset$), and Γ^* stabilizes every $\widetilde{H}_r^{\text{inv}}$ and $\widetilde{U}_r^{\text{inv}}$ setwise. Moreover, $K^* := \{\pi \in \Gamma^* : \pi|_{\widetilde{H}_k^{\text{inv}}} = \mathrm{id}_{\widetilde{H}_k^{\text{inv}}}\}$ satisfies $K^* \leq S$ because every $i \in I^*$ has $N_Q(i) \subseteq \widetilde{H}_k^{\text{inv}}$ and Q is open-twin-free.

For $r \in \{k, \dots, L+2\}$, put $\mathcal{C}_r^{\text{inv}} := \mathrm{Comp}(Q - \widetilde{H}_r^{\text{inv}})$ and $\mathcal{C}_r^{\text{inv},(k)} := \{M \in \mathcal{C}_r^{\text{inv}} : V(M) \cap \widetilde{H}_k^{\text{inv}} \neq \emptyset\}$. Construct cumulative base sets $B_{\geq L+2}^{\text{base}} \subseteq B_{\geq L+1}^{\text{base}} \subseteq \cdots \subseteq B_{\geq k}^{\text{base}} \subseteq V(Q)$ downward so that at each r: (J1) $|V(M) \cap B_{\geq r}^{\text{base}}| = 1$ for all $M \in \mathcal{C}_r^{\text{inv},(k)}$; and (J2) $|V(M) \cap B_{\geq r}^{\text{base}}| \leq 1$ for all $M \in \mathcal{C}_r^{\text{inv}}$. At $r = L+2$, Q is connected and $\mathcal{C}_{L+2}^{\text{inv}} = \{Q\}$; pick $u_Q \in V(Q)$ and set $B_{L+2}^{\text{base}} = B_{\geq L+2}^{\text{base}} := \{u_Q\}$. Given $B_{\geq r+1}^{\text{base}}$ ($k \leq r \leq L+1$), define

$$\mathrm{New}_r \;:=\; \{M' \in \mathcal{C}_r^{\text{inv}} : V(M') \cap B_{\geq r+1}^{\text{base}} = \emptyset \;\text{ and }\; V(M') \cap \widetilde{H}_k^{\text{inv}} \neq \emptyset\}$$

(so $\mathrm{New}_k = \emptyset$). Each $M' \in \mathrm{New}_r$ ($k+1 \leq r \leq L+1$) lies in a unique $M \in \mathcal{C}_{r+1}^{\text{inv},(k)}$; since $|V(M) \cap B_{\geq r+1}^{\text{base}}| = 1$ while $V(M') \cap B_{\geq r+1}^{\text{base}} = \emptyset$, M' is a proper subgraph of M and thus has a neighbor $h_{M'} \in \widetilde{U}_r^{\text{inv}}$. Choosing $u_{M'} \in V(M') \cap N_Q(h_{M'})$ (canonically below for Type 4) and setting $B_r^{\text{base}} := \{u_{M'} : M' \in \mathrm{New}_r\}$ and $B_{\geq r}^{\text{base}} := B_{\geq r+1}^{\text{base}} \sqcup B_r^{\text{base}}$ preserves (J1)–(J2).

Form the subgroup chain $\Gamma^* \geq \Gamma^{(L+2)} \geq \widetilde{\Gamma}^{(L+1)} \geq \Gamma^{(L+1)} \geq \cdots \geq \widetilde{\Gamma}^{(k)} = \Gamma^{(k)} \leq K^* \leq S$, where $\Gamma^{(r)} := \{\pi \in \Gamma^* : \pi|_{\widetilde{H}_r^{\text{inv}} \cup B_{\geq r}^{\text{base}}} = \mathrm{id}\}$ and $\widetilde{\Gamma}^{(r)} := \{\pi \in \Gamma^* : \pi|_{\widetilde{H}_r^{\text{inv}} \cup B_{\geq r+1}^{\text{base}}} = \mathrm{id}\}$. Note $[\Gamma^* : \Gamma^{(L+2)}] \leq |V(Q)| \leq n$.

- *Step A ($[\Gamma^{(r+1)} : \widetilde{\Gamma}^{(r)}]$, $k \leq r \leq L+1$).* Every $u \in V(M)$ ($M \in \mathcal{C}_{r+1}^{\text{inv}}$) has $d_Q(u) \leq \tilde{d}_G(u) \leq 4 d_{\text{inv}}(u) < 2\Delta_r$, so $\Delta(M) \leq 2\Delta_r$. For $M \in \mathcal{C}_{r+1}^{\text{inv},(k)}$ with $\{u_M\} = V(M) \cap B_{\geq r+1}^{\text{base}}$ and $W_M := V(M) \cap \widetilde{U}_r^{\text{inv}}$, $\Gamma^{(r+1)}$ fixes u_M and stabilizes M and W_M, so Lemma 5.1(1) ($\Delta = 2\Delta_r$) gives $|\Gamma^{(r+1)}|_{W_M}| \leq (32\Delta_r)^{|W_M|}$. Since $\widetilde{U}_r^{\text{inv}} = \bigsqcup_{M \in \mathcal{C}_{r+1}^{\text{inv},(k)}} W_M$ and $|\widetilde{U}_r^{\text{inv}}| \leq |\widetilde{H}_{r-2}| \leq |H_{r-2}|$, we obtain $[\Gamma^{(r+1)} : \widetilde{\Gamma}^{(r)}] \leq (32\Delta_r)^{|H_{r-2}|}$.
- *Step B ($[\widetilde{\Gamma}^{(r)} : \Gamma^{(r)}]$, $k+1 \leq r \leq L+1$).* Partition New_r into four disjoint types:
  - *Type 1:* $V(M') \cap \widetilde{H}_{r-2} \neq \emptyset$ (at most $|\widetilde{H}_{r-2}| \leq |H_{r-2}|$ components).
  - *Type 2:* $V(M') \cap \widetilde{H}_{r-2} = \emptyset$ and $|V(M')| = 1$ ($N_Q(u_{M'}) \subseteq \widetilde{H}_r^{\text{inv}}$ is fixed pointwise by $\widetilde{\Gamma}^{(r)}$, so open-twin-freeness of Q forces $\pi(u_{M'}) = u_{M'}$, 1 choice).
  - *Types 3 & 4:* $V(M') \cap \widetilde{H}_{r-2} = \emptyset$ and $|V(M')| \geq 2$. Here $\widehat{M'} := \bigsqcup_{i \in V(M')} V_i \subseteq B_{r-2}$ induces a connected component $G[\widehat{M'}] \in \mathrm{Comp}(G[B_{r-2}])$ with $\widehat{M'} \cap H_p \neq \emptyset$. *Type 3* has $G[\widehat{M'}] \in \mathcal{B}_{\Delta_{r-2}}$ ($< |H_{r-2}|$ components by Lemma 3.1(1b)). *Type 4* has $\deg_T(G[\widehat{M'}]) = 1$:

then $u := u(G[\widehat{M'}]) \in U_{s'}$ satisfies $\Delta_{r-2} \le d_{\mathrm{inv}}(h_{M'}) \le \tilde{d}_G(h_{M'}) = d_G(w) < \Delta_{r+1}$ ($w \in V_{h_{M'}}$), forcing $\Delta_{r-2} \le d_G(u) < \Delta_{r+1}$, i.e., $s' \in \{r-2, r-1, r\} \cap \{p+1, \dots, L-1\}$, and $N_G(\widehat{M'}) \subseteq H_{s'}$, so $G[\widehat{M'}] \in \mathcal{P}_{s'}^{(p),\ge 2}$.

Let $\mathcal{T}_r^{\mathrm{new}} \subseteq \bigcup_{s' \in \{r-2,r-1,r\} \cap \{p+1,\dots,L-1\}} \mathcal{F}_{s'}^{(p),\ge 2}$ be the classes containing at least one Type-4 lift $G[\widehat{M'}]$. For $\mathcal{T} \in \mathcal{T}_r^{\mathrm{new}}$ (lying in $\mathcal{F}_{s'}^{(p),\ge 2}$), fix one $M_0' \in \mathrm{New}_r^{\text{Type 4}}$ with $C_0 := G[\widehat{M_0'}] \in \mathcal{P}_{s'}(\mathcal{T})$. For any $C \in \mathcal{P}_{s'}(\mathcal{T})$, every $x \in V(C)$ has an internal neighbor $z \in V(C)$, so any open twin y of x in G has $d_G(y) < \Delta_{s'}$ ($y \in B_{s'}$) and $\{y, z\} \in E(G)$ ($y \in V(C)$); thus $V(C) = \bigsqcup_{i \in V(M_C)} V_i$ for a connected induced subgraph $M_C \subseteq Q$ ($|V(M_C)| \ge 2$). Fixing a canonical $\lambda_{s'}$-preserving isomorphism $\psi_{M_0', M_C} : C_0 \xrightarrow{\sim} C$ (with $\psi_{M_0', M_0'} := \mathrm{id}_{V(C_0)}$) and extending by swapping $C_0 \leftrightarrow C$ in $\mathrm{Aut}(G)$ induces an element of $S \le \widetilde{\Gamma}_{\mathrm{band}}(2^j)$ mapping $M_0' \xrightarrow{\sim} M_C$ while fixing $\widetilde{H}_r^{\mathrm{inv}} \subseteq \widetilde{H}_{r-2}$ pointwise. Hence $M_C \in \mathcal{C}_r^{\mathrm{inv},(k)}$ with $N_Q(M_C) = N_Q(M_0') \subseteq \widetilde{H}_{s'}$, so $r = \min\{t : N_Q(M_C) \cap \widetilde{U}_t^{\mathrm{inv}} \ne \emptyset\}$ is determined by $\mathcal{T}$ (making $\{\mathcal{T}_r^{\mathrm{new}}\}_{r=k+1}^{L+1}$ pairwise disjoint), and all M_C ($C \in \mathcal{P}_{s'}(\mathcal{T})$) share $h_{\mathcal{T}} := h_{M_0'} \in \widetilde{U}_r^{\mathrm{inv}}$ inside a single $M \in \mathcal{C}_{r+1}^{\mathrm{inv},(k)}$, which by (J1) meets $B_{\ge r+1}^{\mathrm{base}}$ in 1 vertex; setting $\mathcal{P}_{\mathcal{T}}^{\mathrm{new}} := \{C \in \mathcal{P}_{s'}(\mathcal{T}) : M_C \in \mathrm{New}_r\}$, we therefore have $C_0 \in \mathcal{P}_{\mathcal{T}}^{\mathrm{new}}$ and $w_{\mathcal{T}}' := |\mathcal{P}_{\mathcal{T}}^{\mathrm{new}}| \in \{\tilde{w}_{\mathcal{T}} - 1, \tilde{w}_{\mathcal{T}}\}$ ($w_{\mathcal{T}}' \ge 1$). Pick $u_{M_0'} \in V(M_0') \cap N_Q(h_{\mathcal{T}})$ and set $u_{M_C} := \bar{\psi}_{M_0', M_C}(u_{M_0'})$ for each $C \in \mathcal{P}_{\mathcal{T}}^{\mathrm{new}}$, where $\bar{\psi}_{M_0', M_C} : M_0' \xrightarrow{\sim} M_C$ is the ν_G-preserving isomorphism induced by ψ_{M_0', M_C}. Defining, for each $(\pi_{\mathcal{T}})_{\mathcal{T} \in \mathcal{T}_r^{\mathrm{new}}} \in \prod_{\mathcal{T} \in \mathcal{T}_r^{\mathrm{new}}} \mathrm{Sym}(\mathcal{P}_{\mathcal{T}}^{\mathrm{new}})$, the map $\bar{\sigma}_{(\pi_{\mathcal{T}})} \in \mathrm{Sym}(V(Q))$ acting on each $V(M_C)$ ($C \in \mathcal{P}_{\mathcal{T}}^{\mathrm{new}}$, $\mathcal{T} \in \mathcal{T}_r^{\mathrm{new}}$) as $\bar{\psi}_{M_0', M_{\pi_{\mathcal{T}}(C)}} \circ \bar{\psi}_{M_0', M_C}^{-1}$ and fixing $V(Q) \setminus \bigsqcup_{\mathcal{T} \in \mathcal{T}_r^{\mathrm{new}}} \bigsqcup_{C \in \mathcal{P}_{\mathcal{T}}^{\mathrm{new}}} V(M_C)$ pointwise yields a subgroup $\bar{N}_r \le \widetilde{\Gamma}^{(r)} \cap S$ of order $\prod_{\mathcal{T} \in \mathcal{T}_r^{\mathrm{new}}} w_{\mathcal{T}}'!$ (since the sets $V(M_C)$ for $C \in \mathcal{P}_{\mathcal{T}}^{\mathrm{new}}$ are pairwise disjoint, have all external Q-neighbors in $\widetilde{H}_{s'}$, and are disjoint from $\widetilde{H}_r^{\mathrm{inv}} \cup B_{\ge r+1}^{\mathrm{base}}$). Since $d_Q(h_{M'}) \le \tilde{d}_G(h_{M'}) < 2\Delta_r$, $[\widetilde{\Gamma}^{(r)} : \Gamma^{(r)}] \le (2\Delta_r)^{2|H_{r-2}| + \sum_{\mathcal{T} \in \mathcal{T}_r^{\mathrm{new}}} w_{\mathcal{T}}'}$.

- *Step C (injectivity of $\Phi' : \bar{N}_{L+1} \times \cdots \times \bar{N}_{k+1} \to S/\Gamma^{(k)}$).* If $(\tau_{L+1} \circ \cdots \circ \tau_{k+1})\Gamma^{(k)} = (\sigma_{L+1} \circ \cdots \circ \sigma_{k+1})\Gamma^{(k)}$ for distinct tuples and r^* is the largest index with $\tau_{r^*} \ne \sigma_{r^*}$, then $\tau_{r^*} \circ \cdots \circ \tau_{k+1} = \sigma_{r^*} \circ \cdots \circ \sigma_{k+1} \circ \gamma$ ($\gamma \in \Gamma^{(k)}$). Pick $M' \in \mathrm{New}_{r^*}^{\text{Type 4}}$ with $\tau_{r^*}(M') \ne \sigma_{r^*}(M')$, so $\tau_{r^*}(u_{M'}) \ne \sigma_{r^*}(u_{M'})$. Evaluating at $u_{M'} \in B_{r^*}^{\mathrm{base}}$—fixed by $\gamma \in \Gamma^{(k)}$ and by all $\bar{N}_t \le \widetilde{\Gamma}^{(t)}$ for $t < r^*$ (as $B_{\ge k}^{\mathrm{base}}, B_{\ge t+1}^{\mathrm{base}} \supseteq B_{r^*}^{\mathrm{base}}$)—yields $\tau_{r^*}(u_{M'}) = \sigma_{r^*}(u_{M'})$, a contradiction. Hence $[S : \Gamma^{(k)}] \ge \prod_{r=k+1}^{L+1} \prod_{\mathcal{T} \in \mathcal{T}_r^{\mathrm{new}}} w_{\mathcal{T}}'!$.

Combining Steps A–C with $q := r-2 \in \{p, \dots, L-1\}$, $\log_2(32\Delta_r) + 2\log_2(2\Delta_r) = 3\log_2(2\Delta_q) + 10 < 5\log_2(2\Delta_q)$ (as $\log_2(2\Delta_q) \ge 126$), and for $w_{\mathcal{T}}' \ge 1$ ($\tilde{w}_{\mathcal{T}} \le d_G(u(\mathcal{T})) < 2\Delta_{s'} < 8\Delta_{s'}$, $2\Delta_r \le 8\Delta_{s'}$), $\log_2(\frac{(2\Delta_r)^{w_{\mathcal{T}}'}}{w_{\mathcal{T}}'!}) \le w_{\mathcal{T}}' \log_2(\frac{8e\Delta_{s'}}{w_{\mathcal{T}}'}) \le \tilde{w}_{\mathcal{T}} \log_2(\frac{8e\Delta_{s'}}{\tilde{w}_{\mathcal{T}}}) < 3\,\tilde{w}_{\mathcal{T}} \log_2(\frac{2e\Delta_{s'}}{\tilde{w}_{\mathcal{T}}})$, yields $\log_2(2[\Gamma^* : S]) \le \mathcal{U}_{p+2} = \mathcal{U}_k$.

(2) Summing $\frac{\Delta_p}{\mathcal{L}_p}\mathcal{U}_{p+2}$ over $0 \le p \le L-1$ via Lemma 3.3(3) and Proposition 6.5(2) gives $< (0.01 + 120 + 3(18.4 + 73 + 0.01 \cdot 180))\, m = 399.61\, m < 500\, m$; since $\Delta_k = 4\Delta_{k-2}$ and $\mathcal{L}_k > \mathcal{L}_{k-2}$, $\sum_{k=k_{\min}}^{L-1} \frac{\Delta_k}{\mathcal{L}_k}\mathcal{U}_k < 2000\, m$. For $t_{\max}(G) \ge 3$ and $k_{\min} \le k \le L-1$, set $J_k := \lceil \log_2(\frac{4\Delta_k}{2^{\bar{d}/8}}) \rceil \in [2, \mathcal{L}_k - 3]$ ($\mathcal{L}_k - J_k + 1 \le \frac{\bar{d}}{8} + 1 < \frac{\bar{d}}{7}$). Lemma 7.2(2) and (1) give

$$E_k^{(2),\ge 3} \;\le\; (8 + 1300\,\mathcal{U}_k)\mathcal{L}_k \;+\; \mathbf{1}_{J_k \ge 4}\left(\sum_{j=4}^{J_k} 2^j(\mathcal{L}_k - j) + 300\,\mathcal{U}_k\, S(J_k, \mathcal{L}_k)\right).$$

Using $\frac{\mathcal{L}_k^2}{\Delta_k} < \frac{\bar{d}^2}{30 \cdot 2^{\bar{d}/8}}$, $\sum_{j=4}^{J_k} 2^j(\mathcal{L}_k - j) = 2^{J_k} \sum_{t=0}^{J_k-4} 2^{-t}(\mathcal{L}_k - J_k + t) < \frac{8\Delta_k}{2^{\bar{d}/8}}(2(\mathcal{L}_k - J_k) + 2) \le \frac{8\Delta_k}{2^{\bar{d}/8}}(\frac{\bar{d}}{4} + 2) < \frac{2.01\,\bar{d}}{2^{\bar{d}/8}}\Delta_k$, $\sum_{k=0}^{L-1} \Delta_k < 2\Delta_{L-1} < 2n = \frac{4m}{\bar{d}}$, Lemma 2.4(4b), and $\frac{\mathcal{L}_k}{J_k} \le \frac{\bar{d}}{16}$ ($J_k \ge 4$),

the three sums over $k_{\min} \le k \le L-1$ are bounded by $\frac{87200\,\bar{d}^2}{2^{\bar{d}/8}}\,m$, $\frac{10}{2^{\bar{d}/8}}\,m$, and $\frac{2.6\times10^5\,\bar{d}^2}{2^{\bar{d}/8}}\,m$, totaling $< \frac{5\times10^5\,\bar{d}^2}{2^{\bar{d}/8}}\,m \le \frac{2\times10^{12}}{2^{250}}\,m < 10^{-60}\,m < 10^{-50}\,m$. □

8 Pillar III: Vertex-Deck Degree Locking, Half-Degree Sampling, and Elimination of W_k

8.1 Induced Vertex-Deck Polynomials and Pure Degree-Strata Locking

Lemma 8.1. *Let X and Y be non-isomorphic simple graphs on $n \ge 3$ vertices satisfying $\mathcal{D}(X) = \mathcal{D}(Y)$. Let $W_0 \subseteq V(X)$ be a non-empty independent set of X, set $V_0 := V(X)\setminus W_0$ and $K_0 := X[V_0]$, let $U \subseteq V_0$ satisfy $N_X(W_0) \subseteq U$, set $\Gamma_0 := \mathrm{Aut}_U(K_0)$, let $m_1,\dots,m_q \ge 1$ be the cardinalities of the non-empty neighborhood twin classes $W_{0,Z} := \{w \in W_0 : N_X(w) = Z\}$ $(Z \subseteq U)$, so $\sum_{i=1}^q m_i = |W_0|$, and put $t_{\max}(W_0) := \max_{1\le i\le q} m_i$.*

(1) Define

$$P_{\mathrm{ind},U}(x) \;:=\; \sum_{\phi\in\Phi_{\mathrm{ind}}(K_0,X)} x^{v_{X,U}(\phi)} - \sum_{\phi\in\Phi_{\mathrm{ind}}(K_0,Y)} x^{v_{Y,U}(\phi)} \;\in\; \mathbb{Z}[x_Z : Z\subseteq U],$$
$$\widetilde{P}_{\mathrm{ind},U}(x) \;:=\; \frac{1}{|\Gamma_0|}P_{\mathrm{ind},U}(x).$$

Then $\widetilde{P}_{\mathrm{ind},U} \in \mathbb{Z}[x_Z : Z\subseteq U]$ is homogeneous of degree $|W_0|$, shift-invariant on 2^U, and satisfies $|\Phi_{\mathrm{ind}}(K_0,X)| = |\Phi_{\mathrm{ind}}(K_0,Y)| = |\mathrm{Aut}(K_0)|\cdot \mathrm{ind}(K_0,X)$ and $\|\widetilde{P}_{\mathrm{ind},U}\|_1 \le \frac{2}{|\Gamma_0|}|\Phi_{\mathrm{ind}}(K_0,X)|$.

(2) Putting $\mathrm{Aut}_{U,W_0}(X) := \{\sigma\in\mathrm{Aut}(X) : \sigma|_U = \mathrm{id}_U$ and $\sigma(W_0)=W_0\}$, we have $\mathrm{Aut}_{U,W_0}(X) \cong \Gamma_0 \times \prod_{Z\subseteq U}\mathrm{Sym}(W_{0,Z})$ of order $|\Gamma_0|\prod_{i=1}^q m_i!$, and

$$\left[x^{v_{X,U}(\mathrm{id}_{V_0})}\right]\widetilde{P}_{\mathrm{ind},U} \;=\; \frac{|\mathrm{Aut}(X)|}{|\Gamma_0|\cdot\prod_{i=1}^q m_i!}.$$

(3) Consequently,

$$\big(t_{\max}(W_0)+1\big)^{\frac{|W_0|}{200\,t_{\max}(W_0)}} \;\le\; \frac{2\,|\Phi_{\mathrm{ind}}(K_0,X)|\cdot\prod_{i=1}^q m_i!}{|\mathrm{Aut}(X)|},$$

and when $t_{\max}(W_0)=1$, this strengthens to

$$\left(\frac{2}{\sqrt{3}}\right)^{|W_0|} \;\le\; \frac{2\,|\Phi_{\mathrm{ind}}(K_0,X)|}{|\mathrm{Aut}(X)|}.$$

Proof. (1) Because every $\phi\in\Phi_{\mathrm{ind}}(K_0,F)$ $(F\in\{X,Y\})$ is injective, right composition $(\phi,\psi)\mapsto\phi\circ\psi$ is a free right action of Γ_0 on $\Phi_{\mathrm{ind}}(K_0,F)$ satisfying $R_{\phi\circ\psi}=R_\phi$ and $v_{F,U}(\phi\circ\psi)=v_{F,U}(\phi)$ (since $\psi|_U=\mathrm{id}_U$ and $\psi(V_0\setminus U)=V_0\setminus U$). Hence $\widetilde{P}_{\mathrm{ind},U}\in\mathbb{Z}[x_Z : Z\subseteq U]$ is homogeneous of degree $|R_\phi|=|W_0|$. Since $|V_0|=n-|W_0|<n$, Proposition 2.1(1) gives $\mathrm{ind}(K_0,X)=\mathrm{ind}(K_0,Y)$, so $|\Phi_{\mathrm{ind}}(K_0,X)| = |\mathrm{Aut}(K_0)|\cdot\mathrm{ind}(K_0,X) = |\Phi_{\mathrm{ind}}(K_0,Y)|$ and $\|\widetilde{P}_{\mathrm{ind},U}\|_1 \le \frac{2}{|\Gamma_0|}|\Phi_{\mathrm{ind}}(K_0,X)|$. Applying $D := \sum_{Z\subseteq U}\frac{\partial}{\partial x_Z}$ to $P_{\mathrm{ind},U}$ and using $w\in R_\phi \iff \phi\in\Phi_{\mathrm{ind}}(K_0,F-w)$ gives

$$D(P_{\mathrm{ind},U})(x) \;=\; \sum_{w\in V(X)}\;\sum_{\phi\in\Phi_{\mathrm{ind}}(K_0,X-w)} x^{v_{X-w,U}(\phi)} - \sum_{w'\in V(Y)}\;\sum_{\phi'\in\Phi_{\mathrm{ind}}(K_0,Y-w')} x^{v_{Y-w',U}(\phi')} \;\equiv\; 0,$$

since any card isomorphism $\eta : X - w \xrightarrow{\sim} Y - w'$ induces a bijection $\phi \mapsto \eta \circ \phi$ preserving $v_{\cdot,U}$, and $\mathcal{D}(X) = \mathcal{D}(Y)$. Thus $D(\widetilde{P}_{\mathrm{ind},U}) \equiv 0$.

(2) Because W_0 is independent with $N_X(W_0) \subseteq U$, $\sigma \mapsto (\sigma|_{V_0}, (\sigma|_{W_{0,Z}})_{Z\subseteq U})$ is a group isomorphism $\mathrm{Aut}_{U,W_0}(X) \xrightarrow{\sim} \Gamma_0 \times \prod_{Z\subseteq U} \mathrm{Sym}(W_{0,Z})$. Put $v_0 := v_{X,U}(\mathrm{id}_{V_0})$. By Proposition 2.1(1), $|E(X)| = |E(Y)|$. For $F \in \{X, Y\}$ and $\phi \in \Phi_{\mathrm{ind}}(K_0, F)$ with $v_{F,U}(\phi) = v_0$, $|E(F[\phi(V_0)])| = |E(K_0)|$ and $|E_F(\phi(U), R_\phi)| = \sum_{Z\subseteq U} |Z|(v_0)_Z = |E_X(V_0, W_0)| = |E(X)| - |E(K_0)|$; subtracting from $|E(F)| = |E(X)|$ forces $E_F(\phi(V_0 \setminus U), R_\phi) = E(F[R_\phi]) = \emptyset$, so matching $\{w \in R_\phi : \phi^{-1}(N_F(w)) = Z\}$ with $W_{0,Z}$ extends ϕ to an isomorphism $X \xrightarrow{\sim} F$. Since $X \not\cong Y$, no such ϕ exists for $F = Y$; for $F = X$, each such ϕ has exactly $\prod_{i=1}^{q} m_i!$ extensions in $\mathrm{Aut}(X)$, so $[x^{v_0}]P_{\mathrm{ind},U} = |\mathrm{Aut}(X)|/\prod_{i=1}^{q} m_i!$, proving (2).

(3) Combine (1) and (2) with Theorem 4.2(2) at $v_0 = v_{X,U}(\mathrm{id}_{V_0})$ ($\max_Z (v_0)_Z = t_{\max}(W_0)$), noting that $|\Gamma_0|$ cancels identically (and $\prod_{i=1}^{q} m_i! = 1$ when $t_{\max}(W_0) = 1$). □

Lemma 8.2. *Let X and Y be simple graphs on $n \geq 3$ vertices with $\mathcal{D}(X) = \mathcal{D}(Y)$, fix an integer $r \geq 1$, and define*

$$V_{=r}^{\mathrm{pure}}(F) := \{w \in V_{=r}(F) : N_F(w) \subseteq V_{>r}(F)\},$$
$$V_{=r}^{\mathrm{impure}}(F) := V_{=r}(F) \setminus V_{=r}^{\mathrm{pure}}(F) \qquad (F \in \{X, Y\}).$$

Then $|V_{=r}^{\mathrm{pure}}(X)| = |V_{=r}^{\mathrm{pure}}(Y)|$ and $|V_{=r}^{\mathrm{impure}}(X)| = |V_{=r}^{\mathrm{impure}}(Y)|$. Moreover, if $W_0 \subseteq V_{=r}^{\mathrm{pure}}(X)$ is non-empty (hence an independent set of X), $V_0 := V(X) \setminus W_0$, $K_0 := X[V_0]$, and $d_{K_0}(a) \geq r+1$ for every $a \in N_X(W_0) \subseteq V_{>r}(X)$, then every $\phi \in \Phi_{\mathrm{ind}}(K_0, Y)$ (and in particular every $\phi \in \Phi_{\mathrm{ind}}(K_0, X)$, taking $Y = X$) satisfies

$$\phi(V_{>r}(X)) = V_{>r}(Y), \qquad \phi(V_{=r}^{\mathrm{impure}}(X)) = V_{=r}^{\mathrm{impure}}(Y), \qquad \phi(V_{<r}(X)) = V_{<r}(Y),$$
$$\phi(V_{=r}^{\mathrm{pure}}(X) \setminus W_0) \subseteq V_{=r}^{\mathrm{pure}}(Y), \qquad R_\phi = V_{=r}^{\mathrm{pure}}(Y) \setminus \phi(V_{=r}^{\mathrm{pure}}(X) \setminus W_0) \subseteq V_{=r}^{\mathrm{pure}}(Y).$$

Proof. For $F \in \{X, Y\}$, $w \in V_{=r}(F)$, and $k \geq r+1$, $|V_{\geq k}(F)| - |V_{\geq k}(F - w)| = |N_F(w) \cap V_{=k}(F)|$, so summing over $k \geq r+1$ gives $|N_F(w) \cap V_{>r}(F)| = \sum_{k\geq r+1}(|V_{\geq k}(F)| - |V_{\geq k}(F-w)|)$. By Proposition 2.1(1), counting cards with $d_F(w) = r$ and $|N_F(w) \cap V_{>r}(F)| = r$ yields $|V_{=r}^{\mathrm{pure}}(X)| = |V_{=r}^{\mathrm{pure}}(Y)|$ and $|V_{=r}^{\mathrm{impure}}(X)| = |V_{=r}^{\mathrm{impure}}(Y)|$.

Now assume $d_{K_0}(a) \geq r+1$ on $N_X(W_0)$, so $V_{>r}(K_0) = V_{>r}(X)$, $V_{=r}(K_0) = V_{=r}(X) \setminus W_0$, and $V_{<r}(K_0) = V_{<r}(X)$. Let $\phi \in \Phi_{\mathrm{ind}}(K_0, Y)$.

(i) Every $v \in V_{>r}(X)$ has $d_Y(\phi(v)) \geq d_{K_0}(v) \geq r+1$, so $\phi(V_{>r}(X)) \subseteq V_{>r}(Y)$, and equal sizes force $\phi(V_{>r}(X)) = V_{>r}(Y)$.

(ii) Every $v \in V_{=r}(X) \setminus W_0$ has $d_Y(\phi(v)) \geq d_{K_0}(v) = r$ and $\phi(v) \notin V_{>r}(Y)$, forcing $d_Y(\phi(v)) = r = d_{K_0}(v)$ and $N_Y(\phi(v)) = \phi(N_X(v))$; hence $\phi(V_{=r}^{\mathrm{impure}}(X)) = V_{=r}^{\mathrm{impure}}(Y)$ (by equal sizes) and $\phi(V_{=r}^{\mathrm{pure}}(X) \setminus W_0) \subseteq V_{=r}^{\mathrm{pure}}(Y)$.

(iii) Every $u \in V_{<r}(X)$ has $\phi(u) \notin V_{>r}(Y) \cup V_{=r}^{\mathrm{impure}}(Y)$; if $\phi(u) \in V_{=r}^{\mathrm{pure}}(Y)$, then $N_Y(\phi(u)) \subseteq V_{>r}(Y) = \phi(V_{>r}(X)) \subseteq \phi(V_0)$ and inducedness of ϕ would give $d_{K_0}(u) = d_Y(\phi(u)) = r$, contradicting $u \in V_{<r}(K_0)$. Thus $\phi(V_{<r}(X)) = V_{<r}(Y)$ and $R_\phi \subseteq V_{=r}^{\mathrm{pure}}(Y)$. □

8.2 Half-Degree Sampling, T_{k-1}^* Decision Trees, and ε-Linearization

For $k \in \{k_{\min}, \ldots, L-1\}$, integer r with $2 \leq r < 100 \log_2(2\Delta_k)$, and $t \in \{1, 2\}$, define

$$W_{k,r}^{(t)} := \bigsqcup_{\substack{i \in \mathcal{I}_k \\ r_i = r,\ \nu_G(i) = t}} V_i \subseteq W_k,$$

so $W_k = \bigsqcup_{t\in\{1,2\}} \bigsqcup_{2\le r<100\log_2(2\Delta_k)} W_{k,r}^{(t)}$. Put $\Omega_r := V(G) \setminus V_{=r}^{\mathrm{pure}}(G)$, and for $X \in \binom{V_{>r}(G)}{r}$ set $V_X := \{w \in V_{=r}^{\mathrm{pure}}(G) : N_G(w) = X\}$ and $\nu_{\mathrm{pure}}(X) := |V_X|$. By Lemma 3.2(1), every $i \in \mathcal{I}_k$ satisfies $r_i < \Delta_{k-1} - 1$ and $N_G(V_i) \subseteq H_k \subseteq H_{k-1} \subseteq V_{>r}(G)$, so $W_{k,r}^{(t)} \subseteq V_{=r}^{\mathrm{pure}}(G)$ and every $w \in W_k$ is a leaf of T.

Lemma 8.3. *Fix $k \in \{k_{\min}, \dots, L-1\}$, $2 \le r < 100\log_2(2\Delta_k)$, and $t \in \{1,2\}$, and put $W^* := W_{k,r}^{(t)}$. For any subset $W_0 \subseteq W^*$, set $V_0 := V(G) \setminus W_0$, $K_0 := G[V_0]$, $\nu_{K_0}(X) := |V_X \setminus W_0|$,*

$$\mathcal{A}(W_0) := \Big\{\alpha \in \mathrm{Aut}(G[\Omega_r]) : \alpha(V_{>r}(G)) = V_{>r}(G) \text{ and } \nu_{\mathrm{pure}}(\alpha(X)) \ge \nu_{K_0}(X)\ \forall X \in \binom{V_{>r}(G)}{r}\Big\},$$

and $\mathcal{R}(W_0) := \{\alpha|_{H_{k-1}} : \alpha \in \mathcal{A}(W_0)\}$. Then:

(1) If $W^* \neq \emptyset$, there exists a non-empty subset $W_0 \subseteq W^*$ with $|W_0| \ge \frac{1}{5}|W^*|$ such that $|W_0 \cap V_X| \le 1$ for every $X \in \binom{V_{>r}(G)}{r}$ (so $t_{\max}(W_0) = 1$), $d_{K_0}(a) \ge \frac{1}{2}d_G(a)$ for every $a \in V_0$, and $d_{K_0}(u) = d_G(u)$ for every $u \in V_0 \setminus N_G(W^*)$.

(2) For every non-empty subset $W_0 \subseteq W^*$ satisfying $d_{K_0}(a) \ge \frac{1}{2}d_G(a)$ for all $a \in V_0$, we have $\mathcal{A}(W_0) = \{\phi|_{\Omega_r} : \phi \in \Phi_{\mathrm{ind}}(K_0, G)\}$ and $\mathcal{R}(W_0) \subseteq \mathrm{Aut}(G[H_{k-1}])$. Every $\alpha \in \mathcal{A}(W_0)$ satisfies $d_G(\alpha(a)) \ge d_{K_0}(a)$ ($a \in \Omega_r$), $\alpha(H_{k-1}) = H_{k-1}$, and $\nu_{\mathrm{pure}}(\alpha(X)) = \nu_{\mathrm{pure}}(X)$ for all $X \in \mathcal{X}_{\mathrm{out}} := \{X \in \binom{V_{>r}(G)}{r} : X \not\subseteq H_{k-1}\}$. Writing $G[B_{k-1}]^{\ge 2}$ for the union of all connected components of $G[B_{k-1}]$ of size ≥ 2, every $\phi \in \Phi_{\mathrm{ind}}(K_0, G)$ satisfies $N_G(R_\phi) \subseteq H_{k-1}$, maps $G[B_{k-1}]^{\ge 2}$ isomorphically onto $G[B_{k-1}]^{\ge 2}$ preserving all H_{k-1}-adjacencies, and maps every isolated vertex $b \in B_{k-1} \setminus W_0$ of $G[B_{k-1}]$ to an isolated vertex $b' \in B_{k-1} \setminus R_\phi$ of $G[B_{k-1}]$ with $N_G(b') = \phi(N_G(b))$. Moreover, $V(D) \subseteq V_0$ for every $D \in \mathcal{B}_{\Delta_{k-1}}$.

(3) For every non-empty subset $W_0 \subseteq W^*$ satisfying $|W_0 \cap V_X| \le 1$ for all $X \in \binom{V_{>r}(G)}{r}$ and $|N_G(u) \cap W_0| \le \frac{1}{2}d_G(u)$ for all $u \in H_k$, we have

$$\Big(\frac{2}{\sqrt{3}}\Big)^{|W_0|} \le \frac{2|\mathcal{R}(W_0)|}{|\mathrm{Aut}(G)|_{H_{k-1}}|}, \qquad \text{and hence} \qquad |W_0| < 5\log_2\Big(\frac{2|\mathcal{R}(W_0)|}{|\mathrm{Aut}(G)|_{H_{k-1}}|}\Big).$$

Proof. (1) If $t = 2$, select one vertex from each twin pair $V_i \subseteq W^*$: then $|W_0| = \frac{1}{2}|W^*| \ge \frac{1}{5}|W^*|$, $|W_0 \cap V_X| \le 1$, and $d_{K_0}(a) \ge \frac{1}{2}d_G(a)$ ($d_{K_0}(u) = d_G(u)$ outside $N_G(W^*)$). If $t = 1$, every $V_i \subseteq W^*$ is a singleton, so $|W_0 \cap V_X| \le 1$ for any $W_0 \subseteq W^*$. Form $W_{\mathrm{raw}} \subseteq W^*$ by including each $w \in W^*$ independently with probability $1/4$. For $a \in N_G(W^*) \subseteq H_k$, set $d^*(a) := |N_G(a) \cap W^*|$ and $Z_a := |N_G(a) \cap W_{\mathrm{raw}}| \sim \mathrm{Bin}(d^*(a), 1/4)$, call a *bad* if $Z_a > \frac{1}{2}d_G(a)$, and put $W_0 := W_{\mathrm{raw}} \setminus \bigcup_{a \text{ bad}} N_G(a)$. If $d^*(a) \le \frac{1}{2}d_G(a)$, $\Pr(a \text{ bad}) = 0$. If $d^*(a) > \frac{1}{2}d_G(a)$, then $d^*(a) > \frac{1}{2}\Delta_k = \Delta_{k-1} > 2^{248}$ (Lemma 3.3(1)), and Markov's inequality on 3^{Z_a} (with $\mathbb{E}[3^{Z_a}] = (\frac{3}{4}\cdot 1 + \frac{1}{4}\cdot 3)^{d^*(a)} = (3/2)^{d^*(a)}$, $(4/3)^3 = 64/27 > 2$, and $2^u > u^2$ for $u := \Delta_{k-1}/6 > 5$) gives

$$\Pr(a \text{ bad}) \le \Pr\Big(Z_a > \tfrac{1}{2}d^*(a)\Big) \le \Big(\frac{3}{4}\Big)^{d^*(a)/2} < 2^{-\Delta_{k-1}/6} < \frac{36}{\Delta_{k-1}^2}.$$

Since $r < 100\log_2(2\Delta_k) < 100\Delta_{k-1}$, every $w \in W^*$ has $\Pr(w \in W_0) \ge \frac{1}{4} - \frac{3600}{\Delta_{k-1}} > \frac{1}{5}$, so some outcome satisfies $|W_0| \ge \frac{1}{5}|W^*| > 0$.

(2) Because $W_0 \subseteq W^* \subseteq V^{\mathrm{pure}}_{=r}(G)$ ($r < \Delta_{k-1}$) and $N_G(W^*) \subseteq H_k$, every $u \in V_0 \setminus N_G(W^*)$ has $N_G(u) \cap W_0 = \emptyset$ and hence $d_{K_0}(u) = d_G(u)$; in particular $U_{k-1} = H_{k-1} \setminus H_k \subseteq V_0 \setminus N_G(W^*)$. Every $a \in N_G(W_0) \subseteq H_k$ has $d_{K_0}(a) \geq \frac{1}{2} d_G(a) \geq \Delta_{k-1} > r+1$, so Lemma 8.2 (at $X = Y = G$) shows that $\phi \in \Phi_{\mathrm{ind}}(K_0, G)$ stabilizes $V_{>r}(G), V^{\mathrm{impure}}_{=r}(G), V_{<r}(G)$ (hence Ω_r) and maps $V_X \setminus W_0$ injectively into $V_{\phi(X)}$; conversely any $\alpha \in \mathcal{A}(W_0)$ extends to $\Phi_{\mathrm{ind}}(K_0, G)$ by choosing injections $V_X \setminus W_0 \hookrightarrow V_{\alpha(X)}$, giving $\mathcal{A}(W_0) = \{\phi|_{\Omega_r} : \phi \in \Phi_{\mathrm{ind}}(K_0, G)\}$. For $\alpha \in \mathcal{A}(W_0)$, every $a \in H_k$ has $d_G(\alpha(a)) \geq d_{K_0}(a) \geq \Delta_{k-1}$ and every $a \in U_{k-1}$ has $d_G(\alpha(a)) \geq d_{K_0}(a) = d_G(a) \geq \Delta_{k-1}$, so $\alpha(H_{k-1}) = H_{k-1}$ and $\mathcal{R}(W_0) \subseteq \mathrm{Aut}(G[H_{k-1}])$. Thus $\alpha(\mathcal{X}_{\mathrm{out}}) = \mathcal{X}_{\mathrm{out}}$; since $\nu_{K_0}(X) = \nu_{\mathrm{pure}}(X)$ on $\mathcal{X}_{\mathrm{out}}$, summing $\nu_{\mathrm{pure}}(\alpha(X)) \geq \nu_{\mathrm{pure}}(X)$ over finite $\mathcal{X}_{\mathrm{out}}$ forces equality on $\mathcal{X}_{\mathrm{out}}$, leaving $R_\phi \cap V_Y = \emptyset$ for $Y \in \mathcal{X}_{\mathrm{out}}$ and hence $N_G(R_\phi) \subseteq H_{k-1}$. Because $W_0 \cup R_\phi$ consists of isolated vertices of $G[B_{k-1}]$ and $\phi(H_{k-1}) = H_{k-1}$, ϕ restricts to an H_{k-1}-preserving automorphism of $G[B_{k-1}]^{\geq 2}$ and maps isolated vertices of $G[B_{k-1}]$ outside W_0 to isolated vertices outside R_ϕ with $N_G(\phi(b)) = \phi(N_G(b))$. Since every $w \in W_0$ is a leaf of T and isolated in $G[B_{k-1}]$, $\{w\} \notin \mathcal{B}_{\Delta_{k-1}}$, so $V(D) \subseteq V_0$ for all $D \in \mathcal{B}_{\Delta_{k-1}}$.

(3) Since $N_G(W_0) \subseteq H_k$, $|N_G(u) \cap W_0| \leq \frac{1}{2} d_G(u)$ on H_k implies $d_{K_0}(a) \geq \frac{1}{2} d_G(a)$ for all $a \in V_0$, so (2) applies. Moreover $|W_0 \cap V_X| \leq 1$ for all $X \in \binom{V_{>r}(G)}{r}$ gives $t_{\max}(W_0) = 1$ with respect to $U := V_{>r}(G) \subseteq V_0$ (as $W_0 \neq \emptyset$ and every $w \in W_0$ has $N_G(w) \in \binom{V_{>r}(G)}{r}$). Applying Lemma 8.1(3) to (G, H) with $U = V_{>r}(G)$ gives $(2/\sqrt{3})^{|W_0|} \leq \frac{2|\Phi_{\mathrm{ind}}(K_0,G)|}{|\mathrm{Aut}(G)|}$. Put $G_{\mathrm{twin}} := \prod_{Y \in \binom{V_{>r}(G)}{r}} \mathrm{Sym}(V_Y)$ (kernel of $\mathrm{Aut}(G) \to \mathrm{Aut}(G)|_{\Omega_r}$) and $K_{\mathrm{ker}} := \{\gamma \in \mathrm{Aut}(G)|_{\Omega_r} : \gamma|_{H_{k-1}} = \mathrm{id}_{H_{k-1}}\}$ (kernel of $\mathrm{Aut}(G)|_{\Omega_r} \to \mathrm{Aut}(G)|_{H_{k-1}}$), so $|\mathrm{Aut}(G)| = |\mathrm{Aut}(G)|_{H_{k-1}}| \cdot |K_{\mathrm{ker}}| \cdot |G_{\mathrm{twin}}|$. Each $\alpha \in \mathcal{A}(W_0)$ has $\prod_X (\nu_{\mathrm{pure}}(\alpha(X)))_{\nu_{K_0}(X)} \leq |G_{\mathrm{twin}}|$ extensions in $\Phi_{\mathrm{ind}}(K_0, G)$, and for any $\alpha_0, \alpha \in \mathcal{A}(W_0)$ with $\alpha_0|_{H_{k-1}} = \alpha|_{H_{k-1}} = \rho$, $\gamma := \alpha_0^{-1} \circ \alpha$ fixes H_{k-1} pointwise and preserves ν_{pure} on both $\binom{H_{k-1}}{r}$ and $\mathcal{X}_{\mathrm{out}}$, placing $\gamma \in K_{\mathrm{ker}}$; thus $|\Phi_{\mathrm{ind}}(K_0, G)| \leq |\mathcal{R}(W_0)| \cdot |K_{\mathrm{ker}}| \cdot |G_{\mathrm{twin}}|$. Canceling $|G_{\mathrm{twin}}|$ and $|K_{\mathrm{ker}}|$ and using $(2/\sqrt{3})^5 > 2$ yields (3). □

Proposition 8.4. *For every $p \in \{0, \ldots, L-1\}$, define*

$$\mathcal{V}_p \;:=\; 1 + \log_2 n \;+\; 4|H_p| \log_2(2\Delta_p) \;+\; 2\sum_{q=p}^{L-1} |H_q| \log_2(2\Delta_q) \;+\; \sum_{q=p+1}^{L-1} \sum_{\mathcal{T} \in \mathcal{F}_q^{(p),\geq 2}} \tilde{w}_{\mathcal{T}} \log_2\!\left(\frac{2e\Delta_q}{\tilde{w}_{\mathcal{T}}}\right).$$

Then:

(1) For every $k \in \{k_{\min}, \ldots, L-1\}$, $2 \leq r < 100 \log_2(2\Delta_k)$, $t \in \{1, 2\}$, and every non-empty twin-closed subset $W_0 \subseteq W^{(t)}_{k,r}$ (i.e., $u \in W_0$ and $N_G(v) = N_G(u) \implies v \in W_0$) satisfying $|N_G(u) \cap W_0| \leq \lfloor d_G(u)/2 \rfloor$ for all $u \in H_k$, with $\mathcal{R}(W_0)$ as in Lemma 8.3 and $c(u)$ denoting the child count of $u \in H_{k-1} \setminus L^{\mathrm{iso}}_{k-1}$ in T^*_{k-1}:

$$|\mathcal{R}(W_0)| \;\leq\; n \cdot (16\Delta_{k-1})^{|H_{k-1}|} \cdot \left(\prod_{q=k}^{L-1} \prod_{\mathcal{T} \in \mathcal{F}_q^{(k-1),\mathrm{iso}}} \tilde{w}_{\mathcal{T}}!\right) \cdot \max_{\rho \in \mathcal{R}(W_0)} \prod_{u \in H_{k-1} \setminus L^{\mathrm{iso}}_{k-1}} \left(d_G(\rho(u))\right)^{c(u)}.$$

(2) Unconditionally, $|W^{(t)}_{k,r}| \leq 30\, \mathcal{V}_{k-1}$ for all $k \in \{k_{\min}, \ldots, L-1\}$, $2 \leq r < 100 \log_2(2\Delta_k)$, and $t \in \{1, 2\}$.

(3) We have $\sum_{p=0}^{L-1} \frac{\Delta_p}{\mathcal{L}_p} \mathcal{V}_p < 200\, m$, and

$$\sum_{k=k_{\min}}^{L-1} |W_k| \log_2(2\Delta_k) \;<\; \frac{10^4\, \bar{d}^3}{2^{\bar{d}/8}}\, m \;<\; 10^{-61}\, m \;<\; 10^{-50}\, m.$$

Proof. (1) Put $V_0 := V(G)\backslash W_0$ and $K_0 := G[V_0]$. Since $N_G(W_0) \subseteq H_k$ and $|N_G(u)\cap W_0| \le \lfloor d_G(u)/2 \rfloor$ on H_k, every $a \in V_0$ satisfies $d_{K_0}(a) \ge \frac{1}{2} d_G(a)$, so Lemma 8.3(2) applies (note that twin-closedness of W_0 is not needed in this step); in particular $\mathrm{id}_{H_{k-1}} \in \mathcal{R}(W_0)$ via the inclusion $V_0 \hookrightarrow V(G)$, so $\mathcal{R}(W_0)$ is non-empty, and for each $\rho \in \mathcal{R}(W_0)$ we may fix one witness $\phi_\rho \in \Phi_{\mathrm{ind}}(K_0, G)$ with $\phi_\rho|_{H_{k-1}} = \rho$. Build a pruned rooted decision tree $\mathcal{T}_{\mathrm{dec}}$ with leaf set bijectively indexed by $\mathcal{R}(W_0)$ along a fixed breadth-first search traversal of T^*_{k-1} from $r_0 \in U_{L-1} \setminus L^{\mathrm{iso}}_{k-1}$:

- *Root:* Record $\rho(r_0) \in H_{k-1}$ ($\le n$ branches) and set $P := \{r_0\}$.
- *Vertex expansion of $u \in H_{k-1} \setminus L^{\mathrm{iso}}_{k-1}$:* Here $\rho(u)$ is fixed; for each of the $c(u)$ children of u in T^*_{k-1}, record $\phi_\rho(w) \in N_G(\rho(u))$ ($w = v$ for a vertex child v, adding v to P; or $w = v_D \in V(D) \subseteq V_0$ by Lemma 8.3(2) for a bridge child $D \in \mathcal{B}_{\Delta_{k-1}}$ entered via $\{u, v_D\}$), giving branching degree $\le (d_G(\rho(u)))^{c(u)}$.
- *Bridge expansion of $D \in \mathcal{B}_{\Delta_{k-1}}$ entered via $\{u, v_D\}$:* Put $W_D := N_G(D) \cap H_{k-1} \subseteq H_{k-1} \setminus L^{\mathrm{iso}}_{k-1}$. Both $z_D := \phi_\rho(v_D) \in B_{k-1}$ and $\rho|_P$ on the deterministic set P are fixed; record $\rho|_{W_D \setminus P}$ and add $W_D \setminus P$ to P. If $D = \{v_D\}$, Lemma 8.3(2) gives $\rho(W_D) = N_G(z_D)$ of size $< \Delta_{k-1}$, yielding $\le (16\Delta_{k-1})^{|W_D \setminus P|}$ branches. If $|V(D)| \ge 2$, Lemma 8.3(2) maps D isomorphically onto the component D' of $G[B_{k-1}]$ containing z_D preserving H_{k-1}-adjacencies; comparing against a reference ρ_0 at this node via $\theta := (\rho_0|_{W_D})^{-1} \circ (\rho|_{W_D})$ and $\psi := (\phi_{\rho_0}|_{V(D)})^{-1} \circ (\phi_\rho|_{V(D)})$ places $(\theta, \psi) \in G_{v_D, P\cap W_D}(D, W_D)$, so Lemma 5.1(2) gives $\le (16\Delta_{k-1})^{|W_D \setminus P|}$ branches. Disjointness of $W_D \setminus P \subseteq H_{k-1}$ bounds the product of all bridge branches along any path by $(16\Delta_{k-1})^{|H_{k-1}|}$.
- *Final Phase-2 step:* Once $\rho_1 := \rho|_{H_{k-1} \setminus L^{\mathrm{iso}}_{k-1}}$ is fixed, every extension $\rho \in \mathcal{R}(W_0)$ differs from a reference ρ_0 by $\tau := \rho_0^{-1} \circ \rho \in \mathrm{Aut}(G[H_{k-1}])$ fixing $H_{k-1} \setminus L^{\mathrm{iso}}_{k-1}$ pointwise and hence stabilizing L^{iso}_{k-1} setwise; since both v and $\tau(v)$ lie in L^{iso}_{k-1} and have $N_G \subseteq H_{k-1} \setminus L^{\mathrm{iso}}_{k-1} \subseteq H_{k-1}$ (Lemma 3.4(1)), $N_G(\tau(v)) = N_{G[H_{k-1}]}(\tau(v)) = \tau(N_{G[H_{k-1}]}(v)) = \tau(N_G(v)) = N_G(v)$ for every $v \in L^{\mathrm{iso}}_{k-1}$, so as in Phase 2 of Proposition 6.4(2) this final step has branching degree at most $\prod_{q=k}^{L-1} \prod_{\mathcal{T} \in \mathcal{F}_q^{(k-1),\mathrm{iso}}} \tilde{w}_{\mathcal{T}}!$.

Applying Lemma 2.4(3) to $\mathcal{T}_{\mathrm{dec}}$ yields (1).

(2) Since $\mathcal{V}_{k-1} \ge 1 + \log_2 n > 0$, the bound is trivial when $W^* := W^{(t)}_{k,r} = \emptyset$; assume $W^* \ne \emptyset$, and set $I^* := \{i \in \mathcal{I}_k : r_i = r,\ \nu_G(i) = t\} \ne \emptyset$, so $W^* = \bigsqcup_{i \in I^*} V_i$ has size $|W^*| = t|I^*|$. Put $x := \Delta_{k-1} \ge \Delta_0 \ge 2^{125}$ (Lemma 3.3(1)) and $s := (4x)^{1/8} \ge 2^{127/8} > 2^{15} = 32768$. By Lemma 2.4(1)–(2) (at $\varepsilon = 1$, using $\ln 2 > 0.693 > 1/2$), $\log_2 s = \log_2(1 + (s-1)) \le \frac{s-1}{\ln 2} < 2s$, so $\log_2(4x) = 8 \log_2 s < 16s$; since $\frac{s^7}{4} > 2^{103} > 3.2 \times 10^7$, we obtain $2 \times 10^6 \log_2(2\Delta_k) = 2 \times 10^6 \log_2(4x) < 3.2 \times 10^7 s < \frac{s^8}{4} = \Delta_{k-1}$, whence $20000r < 2 \times 10^6 \log_2(2\Delta_k) < \Delta_{k-1}$ and therefore $40000r < \Delta_k$ and $200r < 2\Delta_{k-1}$.

We first select a subset $J \subseteq I^*$ with $|J| \ge \frac{3}{8}|I^*|$ such that $W_0 := \bigsqcup_{i \in J} V_i$ satisfies $|N_G(u) \cap W_0| \le \lfloor d_G(u)/2 \rfloor$ for all $u \in H_k$. For each $u \in H_k$, set $S_u := \{i \in I^* : u \in N_G(V_i)\}$, $m_u := |S_u|$, and $K_u := \lfloor d_G(u)/(2t) \rfloor$, so $tm_u = |N_G(u) \cap W^*| \le d_G(u) < 2t(K_u + 1)$ and $tK_u \le \lfloor d_G(u)/2 \rfloor$. Put $B := \{u \in H_k : m_u > K_u\}$. Every $u \in B$ satisfies $m_u \ge K_u + 1$, so $4m_u \ge 2t(K_u + 1) > d_G(u) \ge \Delta_k \ge 2^{125}$ (giving $m_u > \frac{1}{4}\Delta_k$ and $m_u \ge 20$) and $tm_u < 2t(K_u + 1)$ (giving $K_u > \frac{1}{2}m_u - 1$). Double-counting incidences between B and I^* gives $\frac{\Delta_k}{4}|B| \le \sum_{u \in B} m_u = \sum_{i \in I^*} |B \cap N_G(V_i)| \le r|I^*|$, so $96|B| \le \frac{384r}{\Delta_k}|I^*| < \frac{384}{40000}|I^*| < \frac{1}{40}|I^*|$. Form a random subset $J_0 \subseteq I^*$ by including each $i \in I^*$ independently with probability $2/5$, set $X_u := |J_0 \cap S_u|$ ($u \in B$), put $B_{\mathrm{bad}} := \{u \in B : X_u > K_u\}$, and set $J := J_0 \setminus \bigcup_{u \in B_{\mathrm{bad}}} S_u$; then $|J \cap S_u| \le K_u$ for every $u \in H_k$. For each $u \in B$, whenever $X_u > K_u$ we have $X_u \ge K_u + 1 > \frac{1}{2}m_u$ and hence $X_u - \frac{2}{5}m_u > \frac{1}{10}m_u > \frac{1}{20}m_u > 0$; since $\mathbb{E}[X_u] = \frac{2}{5}m_u$ and $\mathrm{Var}(X_u) = \frac{6}{25}m_u$, Chebyshev's inequality gives $m_u \Pr(X_u > K_u) \le m_u \cdot \frac{(6/25)m_u}{(m_u/20)^2} = 96$. Because

$|J| \geq |J_0| - \sum_{u \in B} m_u \mathbf{1}_{X_u > K_u}$, taking expectations yields $\mathbb{E}[|J|] \geq \frac{2}{5}|I^*| - 96|B| \geq (\frac{2}{5} - \frac{1}{40})|I^*| = \frac{3}{8}|I^*|$, so we may fix an outcome $J \subseteq I^*$ with $|J| \geq \frac{3}{8}|I^*| > 0$ (hence $|J| \geq 1$) and $|J \cap S_u| \leq K_u$ on H_k.

Put $W_0 := \bigsqcup_{i \in J} V_i \subseteq W^*$, $V_0 := V(G) \setminus W_0$, and $K_0 := G[V_0]$; then W_0 is non-empty, twin-closed (as $W^* \subseteq V_{=r}^{\mathrm{pure}}(G)$ is a union of open-twin classes V_i), of size $|W_0| = t|J| \leq 2|J|$, and satisfies $|N_G(u) \cap W_0| = t|J \cap S_u| \leq tK_u \leq \lfloor d_G(u)/2 \rfloor$ for all $u \in H_k$. Let $W_{\mathrm{tr}} \subseteq W_0$ select one vertex from each twin class V_i ($i \in J$), so $|W_{\mathrm{tr}}| = |J| \geq 1$, $|W_{\mathrm{tr}} \cap V_X| \leq 1$ for all $X \in \binom{V_{>r}(G)}{r}$, and $|N_G(u) \cap W_{\mathrm{tr}}| \leq |N_G(u) \cap W_0| \leq \frac{1}{2} d_G(u)$ on H_k. Since $W_{\mathrm{tr}} \subseteq W_0$, every $X \in \binom{V_{>r}(G)}{r}$ has $|V_X \setminus W_0| \leq |V_X \setminus W_{\mathrm{tr}}|$, so $\mathcal{A}(W_{\mathrm{tr}}) \subseteq \mathcal{A}(W_0)$ and $|\mathcal{R}(W_{\mathrm{tr}})| \leq |\mathcal{R}(W_0)|$.

Applying (1) and Lemma 8.3(2) to W_0, pick $\rho \in \mathcal{R}(W_0)$ attaining the maximum in (1), with witness $\phi := \phi_\rho \in \Phi_{\mathrm{ind}}(K_0, G)$. For every $u \in H_{k-1} \subseteq V_0$, inducedness of ϕ gives $d_G(\rho(u)) = d_{K_0}(u) + \delta(u)$, where $\delta(u) := |N_G(\rho(u)) \cap R_\phi| \geq 0$. Because $\rho : H_{k-1} \to H_{k-1}$ is a bijection, $R_\phi \subseteq V_{=r}^{\mathrm{pure}}(G)$ has size $|W_0|$ (Lemma 8.2), and $N_G(R_\phi) \subseteq H_{k-1}$ (Lemma 8.3(2)), we have the exact conservation identity

$$\sum_{u \in H_{k-1}} \delta(u) \;=\; |E_G(H_{k-1}, R_\phi)| \;=\; \sum_{w \in R_\phi} d_G(w) \;=\; r|W_0|.$$

For $u \in H_{k-1} \setminus L_{k-1}^{\mathrm{iso}}$ with $u \in U_{q(u)}$ ($k-1 \leq q(u) \leq L-1$), $d_{K_0}(u) \leq d_G(u) < 2\Delta_{q(u)}$ and $c(u) \leq d_T(u) \leq d_G(u) < 2\Delta_{q(u)}$ (Lemma 3.4(2)), so $d_G(\rho(u)) \leq 2\Delta_{q(u)}(1 + \frac{\delta(u)}{2\Delta_{q(u)}})$. Applying Lemma 2.4(2) at $\varepsilon := \frac{1}{200r} \in (0, 1]$ and multiplying by $c(u) \geq 0$ (using $\frac{c(u)}{2\Delta_{q(u)}} < 1$; equality $0 = 0$ occurs when $c(u) = \delta(u) = 0$) gives

$$c(u) \log_2 d_G(\rho(u)) \;\leq\; c(u) \log_2(2\Delta_{q(u)}) \;+\; c(u) \log_2(200r) \;+\; \frac{\delta(u)}{200r \ln 2}.$$

Summing over $u \in H_{k-1} \setminus L_{k-1}^{\mathrm{iso}}$: $\sum c(u) \leq 2|H_{k-1}| - 2 < 2|H_{k-1}|$ (Lemma 3.4(1)) and $1 < 200r < 2\Delta_{k-1}$ give $\sum c(u) \log_2(200r) < 2|H_{k-1}| \log_2(2\Delta_{k-1})$, while $\sum \frac{\delta(u)}{200r \ln 2} \leq \frac{|W_0|}{200 \ln 2}$, and Lemma 3.4(2) gives

$$\sum_{u \in H_{k-1} \setminus L_{k-1}^{\mathrm{iso}}} c(u) \log_2(2\Delta_{q(u)}) \;\leq\; \sum_{q=k-1}^{L-1} 2|H_q| \log_2(2\Delta_q) + \sum_{q=k}^{L-1} \sum_{\mathcal{T} \in \mathcal{F}_q^{(k-1),\geq 2}} \tilde{w}_{\mathcal{T}} \log_2(2\Delta_q).$$

Dividing (1) by $|\mathrm{Aut}(G)|_{H_{k-1}}|$ via Lemma 3.4(3), canceling $\mathcal{F}_q^{(k-1),\mathrm{iso}}$ factorials, applying $\tilde{w}_{\mathcal{T}}! > (\tilde{w}_{\mathcal{T}}/e)^{\tilde{w}_{\mathcal{T}}}$ (Lemma 2.4(1)), and using $|H_{k-1}| \log_2(16\Delta_{k-1}) \leq 2|H_{k-1}| \log_2(2\Delta_{k-1})$ (as $\Delta_{k-1} \geq 4$) together with $|\mathcal{R}(W_{\mathrm{tr}})| \leq |\mathcal{R}(W_0)|$ yields

$$\log_2\!\left(\frac{2|\mathcal{R}(W_{\mathrm{tr}})|}{|\mathrm{Aut}(G)|_{H_{k-1}}|}\right) \;\leq\; \log_2\!\left(\frac{2|\mathcal{R}(W_0)|}{|\mathrm{Aut}(G)|_{H_{k-1}}|}\right) \;<\; \mathcal{V}_{k-1} \;+\; \frac{|W_0|}{200 \ln 2}.$$

Applying Lemma 8.3(3) to W_{tr} ($|W_{\mathrm{tr}}| = |J|$) and using $|W_0| \leq 2|J|$ and $\ln 2 > 0.693$ (so $\frac{5 \cdot 2}{200 \ln 2} = \frac{1}{20 \ln 2} < 0.08$) gives $|J| < 5\mathcal{V}_{k-1} + 0.08|J|$, i.e., $0.92|J| < 5\mathcal{V}_{k-1}$. Since $|J| \geq \frac{3}{8}|I^*|$ and $0.92 \cdot \frac{3}{8} = 0.345 > \frac{1}{3}$, we obtain $|I^*| < 15\mathcal{V}_{k-1}$ and hence $|W^*| = t|I^*| \leq 2|I^*| < 30\mathcal{V}_{k-1}$.

(3) Summing $\frac{\Delta_p}{\mathcal{L}_p}\mathcal{V}_p$ over $0 \leq p \leq L-1$ via Lemma 3.3(3) and Proposition 6.5(2) gives $< (0.01 + 16 + 48 + 18.4 + 73 + 0.01 \cdot 180)\, m = 157.21\, m < 200\, m$. For each $k_{\min} \leq k \leq L-1$, W_k splits into $< 200 \log_2(2\Delta_k) = 200\mathcal{L}_{k-1}$ sets $W_{k,r}^{(t)}$, so (2) gives $|W_k| \log_2(2\Delta_k) \leq 6000 \frac{\mathcal{L}_{k-1}^3}{\Delta_{k-1}} (\frac{\Delta_{k-1}}{\mathcal{L}_{k-1}} \mathcal{V}_{k-1})$. Since $x \mapsto \frac{(\log_2(4x))^3}{x}$ decreases on $[2^{125}, \infty)$ and $\Delta_{k-1} > 2^{\bar{d}/8-2}$ (Lemma 3.3(1)), $\frac{\mathcal{L}_{k-1}^3}{\Delta_{k-1}} < \frac{(\bar{d}/8)^3}{2^{\bar{d}/8-2}} = \frac{\bar{d}^3}{128 \cdot 2^{\bar{d}/8}}$, whence $\sum_{k=k_{\min}}^{L-1} |W_k| \log_2(2\Delta_k) < \frac{6000 \cdot 200}{128} \frac{\bar{d}^3}{2^{\bar{d}/8}}\, m < \frac{10^4\, \bar{d}^3}{2^{\bar{d}/8}}\, m \leq \frac{8 \times 10^{13}}{2^{250}}\, m < 10^{-61}\, m < 10^{-50}\, m$. □

9 Completion of the Proof of Theorem 1.1

Proof of Theorem 1.1. Suppose for the sake of contradiction that there exist non-isomorphic finite simple undirected graphs G_0 and F_0 on $n_0 \geq 4$ vertices with $m = |E(G_0)| = |E(F_0)| \geq 1000\, n_0$ and $\mathcal{D}_E(G_0) = \mathcal{D}_E(F_0)$. Let G, H and the associated dyadic Kruskal parameters be as in Lemma 3.3. Combining Lemma 3.3(2) with Propositions 6.5(3), 7.3(2), and 8.4(3) yields

$$\begin{aligned} 0.15\, m \;&<\; \text{Sum } 1 + \text{Sum } 2 \\ &\leq\; \text{Sum } 1 \;+ 10^{-34}\, m \;+\; \sum_{k=k_{\min}}^{L-1} E_k^{(2),\geq 3} \;+\; \sum_{k=k_{\min}}^{L-1} |W_k| \log_2(2\Delta_k) \\ &<\; (10^{-34} + 3 \times 10^{-50})\, m \;<\; 10^{-30}\, m, \end{aligned}$$

contradicting $m \geq 4000 > 0$. Therefore $G_0 \cong F_0$, and every finite simple undirected graph on $n_0 \geq 4$ vertices with $m \geq 1000\, n_0$ edges is edge-reconstructible. □

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