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An Introduction to Combinatorial Topology through Simplices, Order Complexes and Shellability

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A tutti i miei affetti, grazie.

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Introduction

The last century witnessed significant developments across various areas of mathematics, including its very foundations. Among these areas, algebraic topology emerged as a central field of research, motivating the introduction of category theory and providing new algebraic tools.

Historically, algebraic topology served as a point of contact between combinatorics and topology, and was initially referred to as *combinatorial topology*. An early idea in this subject was to find invariants of topological spaces by investigating their decomposition into simpler building blocks called *simplices*. These building blocks could then be organized into a discrete, combinatorial structure.

Over time, this combinatorial viewpoint gradually faded in favor of pure and more abstract algebraic structures. Consequently, the modern name of the field was adopted.

Nevertheless, the combinatorial approach remains highly relevant in modern topology. Indeed, it is possible not only to reduce topological spaces to discrete structures, but also to translate discrete problems into topological ones. A prime example of this interaction is *poset topology*, a theory that merges order theory with topology, pioneered by Anders Björner and Michelle L. Wachs [Bjö80, BW97, Wac06].

The purpose of this thesis is to provide an introduction to these concepts, treating them first separately, and then exploring the deep translations between topology and order theory.

After reviewing the necessary prerequisites in Chapter 1, we formally define *simplicial complexes* in Chapter 2. Here, we highlight the relationship between the geometric intuition and the discrete data formalized as an *abstract* simplicial complex.

In Chapter 3, we introduce one of the earliest algebraic invariants for topological spaces that admit a decomposition into simplices, namely *homology modules*. We construct these modules starting from the notions of boundary and orientation of a simplicial complex. Intuitively, the n-th homology module encodes the n-dimensional holes in a space. We then state the fundamental result establishing that homology modules are homotopy invariants. Additionally, we provide a characterization of the homologies of a wedge of spheres.

In Chapter 4, we present the notion of *order complexes*, which allows us to transform partially ordered sets into simplicial complexes, alongside the notion of *face posets*, which performs the inverse operation. We then show that the order complex of the face poset is homeomorphic to the space defined by the original complex. Our presentation of these topics mainly follows the approach of Wachs [Wac06, Chapter 1].

In Chapter 5, we finally study a major application of these translations: the theory of *shellability*, which formalizes the idea of gluing simplicial complexes together nicely. This part is based on the theory developed in [Wac06, Chapter 3]. We state and prove the theorem by Björner and Wachs as in [BW96, Theorem 4.1], which establishes that a shellable simplicial complex has the homotopy type of a wedge of spheres.

We then introduce edge-labelings and EL-shellability for posets. These tools provide a computational method to prove shellability using partially ordered sets and order complexes. The original contribution of this text lies in its presentation path: our goal is to unify all the objects and concepts introduced in the previous chapters into a single concluding framework, showing how an EL-shellable poset ultimately yields a shellable order complex.

CHAPTER 1

Preliminaries

Throughout this work, we assume the reader is familiar with the standard foundational concepts of general topology, basic homotopy theory, linear algebra, commutative algebra, and basic order theory, including the notions of topological spaces and vector spaces, continuous and linear maps, rings, modules, and partially ordered sets. For a comprehensive background on these subjects, we refer the reader to [Mun00], [BGS24], [DP02] and [Sta11].

In this preliminary section, we briefly recall several foundational notions without examining their full technical details. Specifically, we review the concepts of *homotopy* and *homotopy equivalence*, as well as the theory of *freely generated modules*, which will provide the necessary algebraic machinery for handling *homology modules* in the next chapters.

Finally, we review the concept of *partially ordered sets* (for short, posets). Together with the simplicial complexes that will be explored in the first chapter, posets constitute the core structural objects of this work.

1.1 Homotopy Theory

Definition 1.1.1 (Homotopy)**.** Let X and Y be topological spaces. A *homotopy* is a continuous map $H\colon X \times [0,1] \to Y$.

We say that H is a homotopy from a continuous map $f\colon X \to Y$ to another continuous map $g\colon X \to Y$ if:

$$H(x,0) = f(x) \quad \text{and} \quad H(x,1) = g(x)$$

for all $x \in X$.

We say f and g are *homotopic* if there exists a homotopy from f to g. In this case, we write $f \simeq g$.

Remark 1.1.2. A homotopy intuitively represents the continuous deformation of one function into another over time. The parameter space $[0, 1]$ provides symmetry to this operation: a homotopy H from f to g naturally yields a reverse homotopy H' from g to f by simply inverting the time parameter, specifically by setting $H'(x, t) = H(x, 1 - t)$.

Definition 1.1.3 (Homotopy equivalence)**.** Let X and Y be two topological spaces. We say that a continuous map $f\colon X \to Y$ is a *homotopy equivalence* if there exists a continuous map $g\colon Y \to X$ such that:

$$g \circ f \simeq \mathrm{id}_X \quad \text{and} \quad f \circ g \simeq \mathrm{id}_Y .$$

If such a map f exists, we say that X and Y are *homotopy equivalent* (or that they share the same *homotopy type*), denoted by $X \simeq Y$.

Remark 1.1.4. Homotopy equivalence is a strictly *weaker* topological condition than homeomorphism. While a homeomorphism perfectly preserves the exact topological structure and local dimension of a space (intuitively, it is a continuous stretching and bending without tearing), a homotopy equivalence only preserves the global "shape" or fundamental holes of the space, *allowing its dimension to collapse or expand.*

A classic example is the 1-dimensional unit circle S^1 and a 2-dimensional flat annulus (i.e., a thickened ring), illustrated by Figure 1.1. These two spaces are not homeomorphic, since removing any two distinct points from the circle breaks it into two separate connected components, whereas removing any two points from the annulus leaves the space as a single connected piece. However, they are naturally homotopy equivalent, as the annulus can be continuously deformed onto its central circular skeleton.

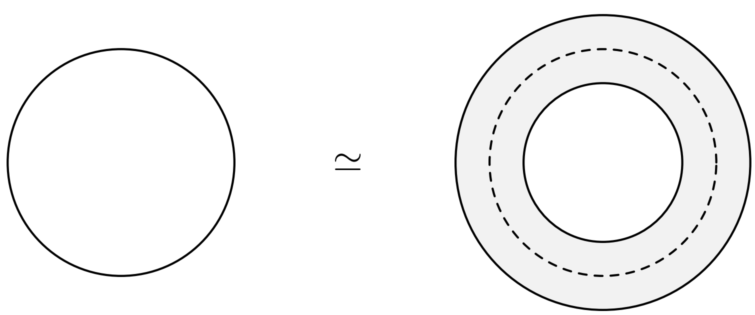


Figure 1.1: The circle S^1 (left) and the annulus (right) are homotopy equivalent, since the annulus can continuously retract onto the dashed line.

Definition 1.1.5 (Contractible space)**.** A topological space X is said to be *contractible* if it is homotopy equivalent to a point.

An exceptional role in topology is played by particular continuous maps that are homotopic to the identity map $\mathrm{id}_X \colon X \to X$:

Definition 1.1.6 (Deformation retraction)**.** Let X be a topological space and let $Y \subseteq X$ be a subspace. A *deformation retraction* of X onto Y is a homotopy $H \colon X \times [0,1] \to X$ such that:

$$\begin{aligned} H(x,0) &= x \quad \text{for all } x \in X, \\ H(y,t) &= y \quad \text{for all } y \in Y \text{ and } t \in [0,1], \\ H(x,1) &\in Y \quad \text{for all } x \in X. \end{aligned}$$

In this case, Y is said to be a *deformation retract* of X.

Proposition 1.1.7. Let X be a topological space and let $Y \subseteq X$ be a subspace. If Y is a deformation retract of X, then $X \simeq Y$.

Proof. The deformation retraction H explicitly provides the homotopy making the standard inclusion $Y \hookrightarrow X$ and the final retraction map $x \mapsto H(x,1)$ homotopy inverses. $\square$

1.2 Freely Generated Modules

Definition 1.2.1 (Free module generated by a set)**.** Let R be a commutative ring and let S be an arbitrary set. The *free R-module generated by S*, denoted by $R\langle S\rangle$, is defined as the set of all formal finite linear combinations of elements of S with coefficients in R.

Formally, it is the module of all functions $f \colon S \to R$ such that $f(s) = 0$ for all but finitely many $s \in S$. An element of $R\langle S\rangle$ is conventionally written as a formal sum:

$$\sum_{s \in S} r_s s,$$

where $r_s \in R$ and only finitely many coefficients r_s are non-zero. The module operations are defined naturally by component-wise addition and scalar multiplication.

Remark 1.2.2. The set S itself is naturally embedded into $R\langle S\rangle$ by identifying each $s \in S$ with the indicator function $e_s \colon S \to R$, defined as:

$$e_s(x) = \begin{cases} 1 & \text{if } x = s, \\ 0 & \text{if } x \neq s, \end{cases}$$

for any $x \in S$. Through this identification, the set S forms a canonical basis for the free module.

Example 1.2.3. Let $S = \{1, x, x^2, x^3, \ldots\}$ be the countably infinite set of formal powers of a variable x. The free R-module generated by S coincides with the polynomial ring $R[x]$ viewed as a module over R.

The algebraic importance of free modules is captured by their universal property, which essentially guarantees that to define a module homomorphism starting from a free module, it is sufficient to specify the images of its generators.

Theorem 1.2.4 (Universal property of free modules)**.** Let R be a commutative ring and S a set. Let $R\langle S\rangle$ be the free R-module generated by S, with the natural inclusion map $\iota : S \hookrightarrow R\langle S\rangle$.

For any R-module M and any set-theoretic function $\phi\colon S \to M$, there exists a *unique* R-module homomorphism $\overline{\phi}\colon R\langle S\rangle \to M$ such that the following diagram commutes, meaning $\overline{\phi} \circ \iota = \phi$:

$$\begin{tikzcd} S \arrow[r, hook, "\iota"] \arrow[dr, "\phi"'] & R\langle S\rangle \arrow[d, dashed, "\overline{\phi}"] \\ & M \end{tikzcd}$$

Algebraically, this unique homomorphism is explicitly given by extending ϕ linearly:

$$\overline{\phi}\left(\sum_{s\in S} r_s s\right) = \sum_{s\in S} r_s \phi(s).$$

1.3 Partially Ordered Sets

Definition 1.3.1 (Partially ordered set)**.** A *partially ordered set* is a pair $(P, \leq)$, where P is a set and $\leq$ is a binary relation on P, called a *partial order*, satisfying the following three axioms for all $x, y, z \in P$:

(i.) *Reflexivity:* $x \leq x$;

(ii.) *Antisymmetry:* if $x \leq y$ and $y \leq x$, then $x = y$;

(iii.) *Transitivity:* if $x \leq y$ and $y \leq z$, then $x \leq z$.

Two elements $x, y \in P$ are said to be *comparable* if either $x \leq y$ or $y \leq x$; otherwise, they are *incomparable*.

A partial order is said to be *total* (or *linear*) if every pair of elements in P is comparable.

Given two elements x, $y \in P$ with $x \leq y$, we denote with $[x, y]$ the set of all intermediate elements:

$$[x, y] = \{z \in P \mid x \leq z \leq y\}.$$

Remark 1.3.2. Partial orders can be restricted, yielding subposets: if P is a poset and $Q \subseteq P$ is an arbitrary subset, then Q naturally inherits a poset structure from P by simply restricting the ordering relation to the elements of Q.

Example 1.3.3. Given any fixed set S, its power set $\mathcal{P}(S)$, consisting of all subsets of S, forms a poset where the partial order is naturally given by the standard set inclusion $\subseteq$.

Convention. Let $(P, \leq)$ be a partially ordered set. Omitting the equality bar, we employ the symbol $<$ to denote the relation given by:

$$x < y \iff x \leq y \text{ and } x \neq y.$$

Conversely, just as the relation $<$ may be derived from $\leq$, one may construct the partial order $\leq$ from a given relation $<$ by setting:

$$x \leq y \iff x < y \text{ or } x = y.$$

In this latter scenario, the relation $<$ is required to satisfy the following two axioms for all $x, y, z \in P$:

(i.) *Irreflexivity:* $x \not< x$;

(ii.) *Transitivity:* if $x < y$ and $y < z$, then $x < z$.

Any relation $<$ satisfying these axioms is also inherently asymmetric: if $x < y$, then $y \not< x$.

Convention. We employ the symbols $\geq$ and $>$ to write $x \geq y$ whenever $y \leq x$, and $x > y$ whenever $y < x$.

Definition 1.3.4 (Dual order)**.** Let $(P, \leq)$ be a partially ordered set. The *dual* (or *opposite*) order on P is the binary relation, denoted by $\leq^{\mathrm{op}}$, defined by reversing the original order:

$$x \leq^{\mathrm{op}} y \iff y \leq x$$

for all $x, y \in P$. The set P equipped with this reversed relation is called the *dual poset* and is conventionally denoted by P^{op}.

Observe that, under our established notation, the condition $x \geq y$ corresponds precisely to $x \leq^{\mathrm{op}} y$ in the dual partial order.

Definition 1.3.5 (Extremal elements)**.** Let $(P, \leq)$ be a poset and let $x \in P$.

- x is the *maximum* (or *top* element) of P if $y \leq x$ for all $y \in P$. If it exists, it is unique and will be denoted by $\hat{1}$.

- x is the *minimum* (or *bottom* element) of P if $x \leq y$ for all $y \in P$. If it exists, it is unique and will be denoted by $\hat{0}$.
- x is a *maximal element* of P if there is no element $y \in P$ such that $x < y$.
- x is a *minimal element* of P if there is no element $y \in P$ such that $y < x$.

Remark 1.3.6. A maximum is always a maximal element, but a poset can possess multiple maximal elements without having a global maximum. An analogous distinction holds for minimums and minimal elements.

Definition 1.3.7 (Chains)**.** A subset C of a partially ordered set P is called a *chain* if every pair of elements in C is comparable. Equivalently, C is a chain if it forms a totally ordered set under the induced order from P.

If a chain C is finite, it can be written by indexing its elements in increasing strict order:

$$C = \{x_0 < x_1 < \cdots < x_n\}.$$

The *length* of such a finite chain C, denoted by $\ell(C)$, is defined as the number of strict relations, which is exactly one less than the number of its elements:

$$\ell(C) = |C| - 1 = n.$$

By assuming the Axiom of Choice, one can prove the following equivalent assertion:

Theorem 1.3.8 (Zermelo)**.** *Every set X admits a well-ordering, meaning there exists a total order on X such that every non-empty subset contains a bottom element.*

For a detailed discussion on Theorem 1.3.8 and its relationship with the Axiom of Choice, we refer the reader to [Mun00, Chapter 1] or [Hal98, Chapters 14–17].

Definition 1.3.9 (Covering relation)**.** Let $(P, \leq)$ be a partially ordered set. An element $y \in P$ is said to *cover* an element $x \in P$, denoted by $x \lessdot y$, if $x < y$ and there exists no element $z \in P$ such that $x < z < y$.

Remark 1.3.10. For finite posets, the covering relation and the strict partial order mutually determine each other: $x < y$ if and only if there exists a finite sequence of intermediate elements such that

$$x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = y.$$

Intuitively, the covering relation acts as the irreducible, “step-by-step” skeleton of the poset. By stripping away all redundant transitive information, it leaves only the atomic relations, from which the full partial order can always be reconstructed.

In arbitrary infinite posets, this step-by-step reconstruction frequently breaks down. The canonical counterexamples are dense linear orders, such as the rational numbers $\mathbb{Q}$ or the real numbers $\mathbb{R}$ equipped with their standard inequalities.

Definition 1.3.11 (Saturated finite chain)**.** A finite chain $x_0 < x_1 < \cdots < x_n$ in a poset P is said to be *saturated* if every relation in the sequence is a covering relation, that is, $x_i \lessdot x_{i+1}$ for all $0 \leq i < n$.

Definition 1.3.12 (Hasse diagram)**.** A *Hasse diagram* is a graphical representation of a finite partially ordered set $(P, \leq)$. Each element of the poset is represented as a vertex in the plane, and a line segment is drawn upward from x to y if and only if y covers x (i.e., if $x \lessdot y$).

Example 1.3.13. Consider the set $S = \{1, 2, 3\}$ and its power set $\mathcal{P}(S)$ ordered by the standard inclusion $\subseteq$. The empty set $\emptyset$ acts as the global minimum, while S acts as the global maximum. Its Hasse diagram forms the edge skeleton of a three-dimensional cube, as illustrated by Figure 1.2.

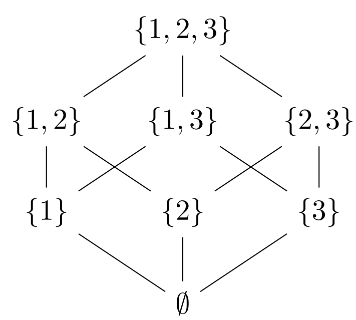


Figure 1.2: Hasse diagram of $(\mathcal{P}(S), \subseteq)$ for $S = \{1, 2, 3\}$.

When working with ordered structures, it is often necessary to compare finite sequences formed by elements of a given set, such as chains. The *lexicographic order* provides a natural way to extend a partial order from individual elements to sequences of varying lengths, generalizing the standard alphabetical order found in a dictionary.

Definition 1.3.14 (Lexicographic Order)**.** Let $(P, \leq)$ be a partially ordered set and P^* be the set of all finite sequences of elements from P. Let $\alpha = (a_1, a_2, \ldots, a_m)$ and $\beta = (b_1, b_2, \ldots, b_n)$ be two finite sequences in P^*. We say that α is *strictly lexicographically smaller* than β, denoted as $\alpha <_L \beta$, if one of the following two conditions holds:

(i.) There exists an index $k \geq 1$, with $k \leq \min(m, n)$, such that $a_i = b_i$ for all $1 \leq i < k$, and $a_k < b_k$ strictly in P.

(ii.) $m < n$ and $a_i = b_i$ for all $1 \leq i \leq m$. In other words, the sequence α is a proper prefix of β.

Remark 1.3.15. If the base poset $(P, \leq)$ is a totally ordered set, then the induced lexicographic order $(P^*, \leq_L)$ is also a total order. However, if P is only partially ordered, then $\leq_L$ remains a partial order on P^*, as sequences diverging at incomparable elements of P will themselves be incomparable under $\leq_L$.

CHAPTER 2

Simplicial Complexes

Most of the foundational tools we will use to investigate topological spaces are rooted in combinatorics. Unsurprisingly, algebraic topology was originally known as *combinatorial topology*, a name that explicitly reflects its historical and conceptual origin: studying the topological features of a space by means of its underlying combinatorial structures.

By employing this framework, we can translate order-theoretic questions into the language of topological invariants, such as *homology modules*, and vice versa. To build the necessary algebraic machinery, this chapter is dedicated to defining and establishing the fundamental properties of geometric and abstract simplicial complexes, which will serve as the primary bridge between posets and topology in the next chapters.

From a historical perspective, the very term *simplex* (plural *simplices*) was introduced precisely to emphasize that these geometric objects represent the "simplest possible polyhedra" in any given dimension. They were conceived as the elemental, indivisible "building blocks" from which more intricate global spaces could be easily assembled and managed computationally.

2.1 A Geometric Introduction

Definition 2.1.1 (n-dimensional simplex)**.** An n-dimensional simplex (or simply, *n-simplex*) is the convex hull of $n+1$ points in general position in a Euclidean space V.

In other words, given $n+1$ points P_0, ..., P_n in general position in V, the n-simplex generated by these points is the set:

$$\left\{ \sum_{i=0}^{n} t_i P_i \in V \;\middle|\; \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\},$$

where the sum $\sum_{i=0}^{n} t_i P_i$ represents a convex combination of the given points. For any point within the simplex, the uniquely determined coefficients t_0, ..., t_n are called its *barycentric coordinates*.

Since the points must be in general position, this condition implies that $\dim V \geq n$.

In particular, the *standard n-simplex* is defined as the following subspace of $\mathbb{R}^{n+1}$:

$$\Delta^n = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i \right\},$$

which corresponds exactly to the convex hull of the standard basis vectors of $\mathbb{R}^{n+1}$.

A *k-face* of an n-simplex is the convex hull formed by a subset of $k+1$ of its $n+1$ vertices, which itself constitutes a k-simplex. If $k < n$, we say that the face is *proper*. We call *vertices* of the simplex its 0-faces, and *edges* its 1-faces.

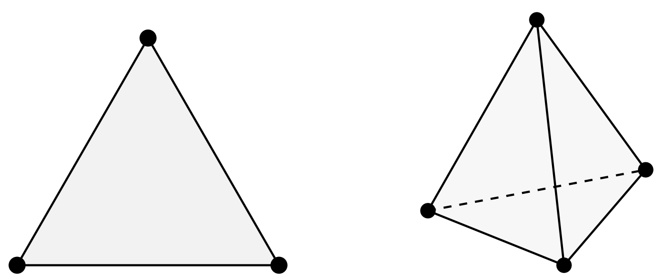


Figure 2.1: 2-simplices coincide precisely with solid triangles (left), whereas 3-simplices coincide precisely with solid tetrahedra (right).

Our aim is to assemble collections of n-simplices to analyze familiar topological spaces. This process is what we classically call "triangulating" a space, a term that historically comes from the practice of cutting two-dimensional surfaces into 2-simplices (i.e., triangles).

Definition 2.1.2 (Geometric simplicial complex)**.** A *geometric simplicial complex* $\mathcal{K}$ in V is a collection of simplices in V satisfying the following conditions:

(i.) Every face of a simplex from $\mathcal{K}$ is also in $\mathcal{K}$.

(ii.) If two simplices in $\mathcal{K}$ intersect, then their intersection must be a face of each one of them.

(iii.) Its union is equipped with the *weak topology*, meaning a subset $A \subseteq \bigcup_{\sigma \in \mathcal{K}} \sigma$ is closed if and only if its intersection $A \cap \sigma$ is closed in σ for every simplex $\sigma \in \mathcal{K}$.

We call *facets* of $\mathcal{K}$ the simplices within the complex which are not a proper face of any other simplex in $\mathcal{K}$. The *dimension* of $\mathcal{K}$, denoted by $\dim \mathcal{K}$, is defined as the maximum dimension of its simplices; explicitly, $\dim \mathcal{K} = \max_{\sigma \in \mathcal{K}} \dim \sigma$.

Remark 2.1.3. If $\mathcal{K}$ is a *finite* geometric simplicial complex—that is, it consists of finitely many simplices—its weak topology coincides with the usual Euclidean topology. This is due to the fact that its simplices form a finite cover of closed sets: if $A \cap F$ is closed for each simplex $F \in \mathcal{K}$, then $A = \bigcup_{F \in \mathcal{K}} (A \cap F)$ is a finite union of closed sets, which is necessarily closed in the Euclidean topology.

The need for the weak topology is in fact motivated by the infinite case, where an arbitrary union of simplices might *not* be closed in the ambient Euclidean topology: the weak topology ensures that such unions remain closed.

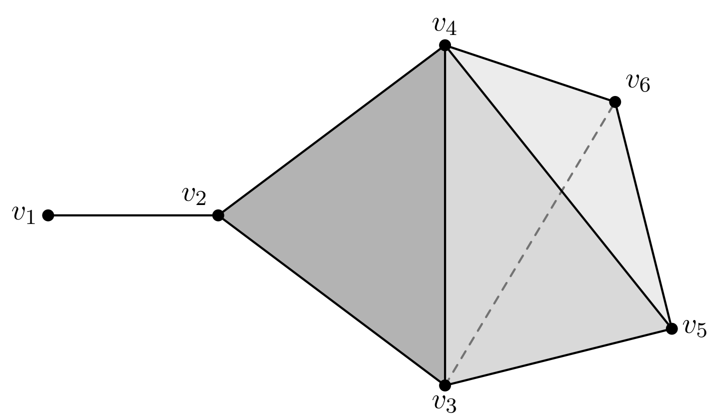


Figure 2.2: A simplicial complex consisting of an edge connecting v_1 and v_2, which meets at vertex v_2 a triangle on the vertices v_2, v_3, and v_4. The triangle shares the edge connecting v_3 and v_4 with a tetrahedron spanned by vertices v_3, v_4, v_5, and v_6. The complex has exactly three facets: the edge $\{v_1, v_2\}$, the triangle $\{v_2, v_3, v_4\}$, and the tetrahedron $\{v_3, v_4, v_5, v_6\}$.

Example 2.1.4 (The circle S^1)**.** For instance, we can view the circle S^1 topologically as a hollow triangle, namely as a simplicial complex consisting of exactly three vertices and three edges. Note that such a structure is distinct from that of a 2-simplex, which instead corresponds to a solid, filled triangle.

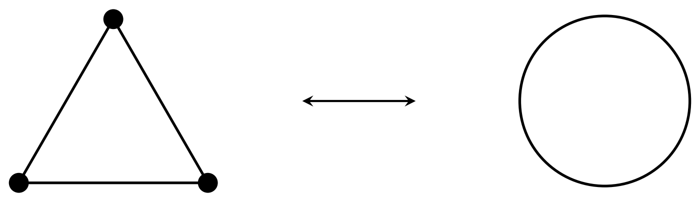


Figure 2.3: The simplicial complex of a circle, constructed by gluing three edges at their vertices.

2.2 Simplicial Complexes via Set Theory

Soon after defining simplicial complexes geometrically, one fundamental property becomes apparent: *every simplex is uniquely determined by its vertices.*

Specifically, if two simplices in a geometric simplicial complex share the exact same set of vertices, they must coincide identically. This rigid correspondence means that the entire topological structure of a complex does *not* depend on the continuous embedding in the Euclidean space, but is completely governed by the discrete data of its vertex sets.

We will exploit this result to abstract the geometric hypotheses governing how simplices intersect using elementary set theory. By simply identifying each simplex with the finite set of its vertices, we can readily generalize the entire construction and safely discard the ambient Euclidean space.

Following this classical approach, we can now introduce a more "abstract" notion of simplicial complexes. Shifting to such a combinatorial perspective yields a framework that is more computational, more general, and strictly less geometric, so that we will be able to directly apply such a structure to posets in the next chapters, while still retaining the ability to switch between the combinatorial and geometric viewpoints.

Definition 2.2.1 (Abstract simplicial complex)**.** An *abstract simplicial complex* $\mathcal{K}$ on vertex set V is a non-empty collection of non-empty subsets of V satisfying the following conditions:

(i.) $\{v\} \in \mathcal{K}$ for each $v \in V$,

(ii.) if $G \in \mathcal{K}$ and $F \subseteq G$ with $F \neq \emptyset$, then $F \in \mathcal{K}$.

Elements of $\mathcal{K}$ are called *simplices* (or *faces*) of the complex, while the maximal faces with respect to inclusion are called *facets.* The dimension $\dim F$ of a face F is strictly one less than its cardinality, namely $\dim F = |F| - 1$. If all facets share the same dimension, we say that the complex is *pure.*

Note that this definition is purely set-theoretic. The condition (i.) corresponds to building the set of vertices, while (ii.) corresponds to the condition (i.) of Definition 2.1.2.

Given an abstract simplicial complex $\mathcal{K}$, a corresponding *geometric realization* can be formally constructed by gluing together geometric simplices along their shared faces. Conversely, every geometric complex naturally yields an *abstraction*.

Definition 2.2.2 (Abstraction of a geometric complex)**.** Given a geometric simplicial complex A, we define its *abstraction* $\mathcal{K}(A)$ as the abstract simplicial complex whose simplices are the vertex sets of the faces of A. Formally:

$$\mathcal{K}(A) = \big\{V(F) \bigm| F \text{ is a face of } A\big\},$$

where $V(F)$ denotes the set of vertices of the face F.

Definition 2.2.3 (Geometric realization)**.** Let $\mathcal{K}$ be an abstract simplicial complex. A *geometric realization* of $\mathcal{K}$ is a geometric simplicial complex whose abstraction is isomorphic to $\mathcal{K}$.

The notion of geometric realization acts as a partial inverse to abstraction, allowing us to map abstract data back into a topological space. To establish that every abstract complex admits at least one such realization, we now present a canonical construction.

Definition 2.2.4 (Standard geometric realization)**.** Given an abstract simplicial complex $\mathcal{K}$, we define its *standard geometric realization* $\|\mathcal{K}\|$ as follows. Let $V(\mathcal{K})$ be the set of vertices of $\mathcal{K}$, namely:

$$V(\mathcal{K}) = \{v \mid \{v\} \in \mathcal{K}\}.$$

Consider the vector space $\mathbb{R}^{(V(\mathcal{K}))}$ of functions from $V(\mathcal{K})$ to $\mathbb{R}$ with finite support. Then, for each simplex $F \in \mathcal{K}$, we define its realization $\|F\|$ to be the convex hull of the basis vectors corresponding to its vertices. Explicitly, we identify each $v \in V(\mathcal{K})$ with the basis vector $e_v \in \mathbb{R}^{(V(\mathcal{K}))}$, defined by:

$$e_v(w) = \begin{cases} 1 & \text{if } w = v, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the geometric realization $\|\mathcal{K}\|$ is defined as the union of all such geometric simplices:

$$\|\mathcal{K}\| = \bigcup_{F \in \mathcal{K}} \|F\|.$$

We endow $\|\mathcal{K}\|$ with the weak topology, under which a set $A \subseteq \|\mathcal{K}\|$ is closed if and only if $A \cap \|F\|$ is closed for each $F \in \mathcal{K}$.

Example 2.2.5. If we consider the abstract simplicial complex formed by all non-empty subsets of a finite set, its standard geometric realization yields precisely the standard geometric simplex. For instance, let:

$$A = \big\{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\big\}.$$

Then, its standard geometric realization is exactly the standard 2-simplex:

$$\|A\| = \Delta^2,$$

where we have naturally identified the function space $\mathbb{R}^{(\{a,b,c\})}$ with $\mathbb{R}^3$.

Remark 2.2.6. Note that, in general, the standard geometric realization $\|\mathcal{K}\|$ lives in a much higher-dimensional space than necessary, residing in an ambient space with as many coordinates as there are vertices in the complex. For instance, a hollow triangle in $\mathbb{R}^2$, such as the one in Example 2.1.4, once abstracted, would be formally realized in $\mathbb{R}^3$ rather than $\mathbb{R}^2$.

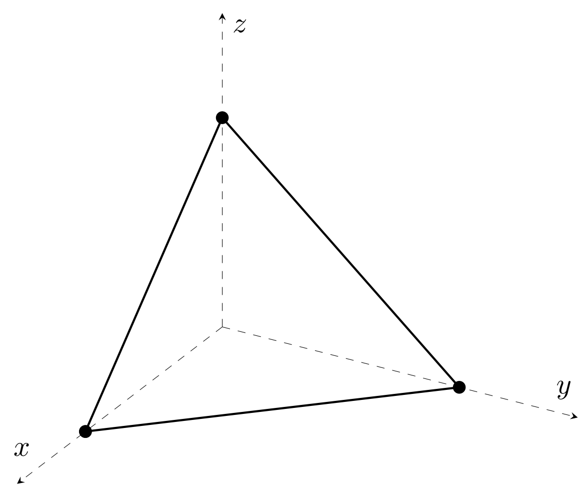


Figure 2.4: The standard geometric realization of the abstracted hollow triangle from Example 2.1.4, embedded in $\mathbb{R}^3$. This space coincides precisely with the boundary of the standard 2-simplex Δ^2.

However, this dimensional inflation is harmless. As the next proposition implies, if A is a geometric simplicial complex, the standard realization of its abstraction $\|\mathcal{K}(A)\|$ is naturally homeomorphic to the original space A.

Definition 2.2.7 (Simplicial isomorphism)**.** Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two abstract simplicial complexes. A bijection $\varphi : V(\mathcal{K}_1) \to V(\mathcal{K}_2)$ between their vertex sets is called a *simplicial isomorphism* if a subset $\{v_0, \ldots, v_k\}$ forms a k-simplex in $\mathcal{K}_1$ if and only if its image $\{\varphi(v_0), \ldots, \varphi(v_k)\}$ forms a k-simplex in $\mathcal{K}_2$.

Intuitively, a simplicial isomorphism corresponds to a relabeling of the vertices that preserves the combinatorial structure of the complex.

Two abstract simplicial complexes $\mathcal{K}_1$ and $\mathcal{K}_2$ are said to be *isomorphic*, denoted by $\mathcal{K}_1 \cong \mathcal{K}_2$, if there exists a simplicial isomorphism between their vertex sets.

Proposition 2.2.8. Let A and B be two geometric simplicial complexes such that their abstractions $\mathcal{K}(A)$ and $\mathcal{K}(B)$ are isomorphic. Then A and B are homeomorphic via a piecewise-linear mapping.

Proof. By hypothesis, there exists a simplicial isomorphism $\varphi\colon V(A) \to V(B)$. We construct a map $f\colon A \to B$ by linearly extending φ over the simplices. Let x be an arbitrary point in A. The point x belongs to some simplex $\sigma \in A$ with vertices $v_0, \ldots, v_k$, and can thus be uniquely expressed in barycentric coordinates as

$$x = \sum_{i=0}^{k} t_i v_i, \qquad \text{with } t_i \geq 0 \text{ and } \sum_{i=0}^{k} t_i = 1.$$

We define the image of x under f as:

$$f(x) = \sum_{i=0}^{k} t_i \varphi(v_i).$$

This mapping is well-defined: if x belongs to the intersection of two distinct simplices σ_1, $\sigma_2 \in A$, then x lies in their common face $\tau = \sigma_1 \cap \sigma_2$. When expressing x in barycentric coordinates with respect to the vertices of σ_1 (or respectively σ_2), the coefficients corresponding to vertices not in τ are identically zero. This ensures that $f(x)$ evaluates to the same result regardless of whether it is computed within σ_1 or σ_2.

By construction, f is affine on every simplex of A, making it a piecewise-linear map. Furthermore, since f is continuous on each closed simplex and these closed simplices form a fundamental cover of the space, f is globally continuous.

Finally, we can symmetrically apply the exact same linear extension to the inverse map $\varphi^{-1}\colon V(B) \to V(A)$ to construct a continuous piecewise-linear inverse $g\colon B \to A$. Therefore A and B are homeomorphic. □

Corollary 2.2.9. Let A be a geometric simplicial complex. Then $\|\mathcal{K}(A)\|$ is homeomorphic to A.

Proof. This follows immediately from Proposition 2.2.8, since A and $\|\mathcal{K}(A)\|$ have isomorphic abstractions. □

Remark 2.2.10. The minimal dimension of the embedding space for an arbitrary finite simplicial complex of dimension d is established by the Menger-Nöbeling theorem, which guarantees an embedding into $\mathbb{R}^{2d+1}$ (see Theorem 50.5 of [Mun00, Chapter 8, §50]).

This upper bound is also strictly tight, as the Van Kampen-Flores theorem demonstrates (see [Mat03, Chapter 5, §5.6]).

Remark 2.2.11. It is crucial to observe that while two abstract simplicial complexes that are isomorphic always yield homeomorphic geometric realizations, the converse is generally false. Two geometric simplicial complexes can be homeomorphic as topological spaces even if their underlying abstractions are fundamentally different.

As a fundamental counterexample, consider a standard 2-simplex $\mathcal{K}$ defined by the vertices $\{v_0, v_1, v_2\}$. Now, consider a second complex $\mathcal{L}$ obtained by placing a new vertex w in the interior of the triangle and connecting it to the original vertices, effectively dividing the space into three smaller 2-simplices: $\{v_0, v_1, w\}$, $\{v_1, v_2, w\}$, and $\{v_2, v_0, w\}$. Topologically, the geometric realizations $\|\mathcal{K}\|$ and $\|\mathcal{L}\|$ are identically homeomorphic; however, $\mathcal{K}$ has exactly 3 vertices while $\mathcal{L}$ has 4. Consequently, their underlying abstract simplicial complexes cannot possibly be the same. This is illustrated in Figure 2.5.

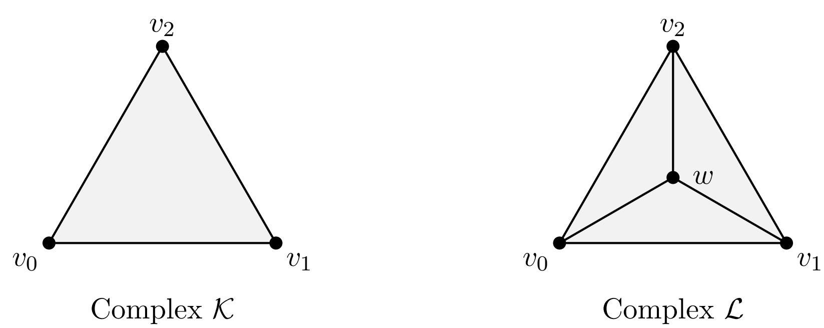


Figure 2.5: A standard 2-simplex (left) and the same geometric shape with an additional interior vertex (right).

As a result of Proposition 2.2.8, faces in the abstract complex perfectly correspond to geometric faces of the same dimension; in particular, abstract facets correspond precisely to geometric ones and dimensions are preserved.

Convention. In practice, we will often treat an abstract simplicial complex and its geometric realizations as interchangeable, without making a strict notational distinction. This abuse of notation is harmless, as they encode the exact same topological information, as formally justified by Proposition 2.2.8.

Convention. For the same reason, we will use the notation $\|\mathcal{K}\|$, previously reserved for the standard geometric realization, to denote any geometric realization of $\mathcal{K}$. Since all such realizations are homeomorphic, the specific choice of the embedding space does not matter.

Definition 2.2.12 (Generated subcomplex)**.** Let $\mathcal{K}$ be an abstract simplicial complex and let $F \in \mathcal{K}$ be a simplex. The *subcomplex generated by* F, denoted by $\langle F \rangle$, is the subcomplex consisting of F and all of its faces. Formally, it is defined as the set:

$$\langle F \rangle = \{\tau \in \mathcal{K} \mid \tau \subseteq F\}.$$

Definition 2.2.13 (Boundary of a simplex)**.** Let F be a simplex. The *boundary* of its generated subcomplex, denoted by $\partial\langle F \rangle$, is defined combinatorially as the subcomplex consisting of all the proper faces of F. Formally:

$$\partial\langle F \rangle = \{\tau \in \langle F \rangle \mid \tau \subsetneq F\}.$$

This discrete construction mirrors the topological boundary of the geometric realization of F.

Remark 2.2.14. From a computational and visual perspective, explicitly enumerating every single simplex of a simplicial complex is highly redundant. A simplicial complex $\mathcal{K}$ is completely determined by its facets. Letting $\mathcal{F}$ denote the set of all facets of $\mathcal{K}$, the entire complex is precisely the subcomplex generated by them; formally, $\mathcal{K} = \bigcup_{F \in \mathcal{F}} \langle F \rangle$. Consequently, encoding the complex in data structures or rendering its geometric realization only requires specifying this generating set of maximal simplices.

Example 2.2.15 (Graphs)**.** One must note that the set-theoretic foundation we are employing for defining abstract simplicial complexes imposes rigid constraints that may seem hidden at first in the geometric viewpoint. Since every simplex is defined as a unique subset of vertices, the structure imposes intrinsic topological restrictions: for instance, there can be at most one edge connecting any two distinct points as the set $\{A, B\}$ is uniquely determined by A and B. Moreover, emphasis must be put on the fact that an edge must connect two *distinct* points: loops are not allowed.

Thus, graphs constitute a class that is not entirely captured by simplicial complexes. The reason lies in the set-theoretic foundation we discussed earlier: loops are not allowed, and there can be at most one edge connecting two points. Graphs satisfying these two hypotheses are called *simple graphs*.

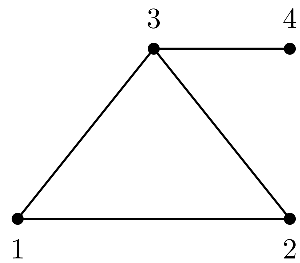


Figure 2.6: An example of a simple graph. As a simplicial complex, this is $\{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{2,3\},\{3,4\}\}$.

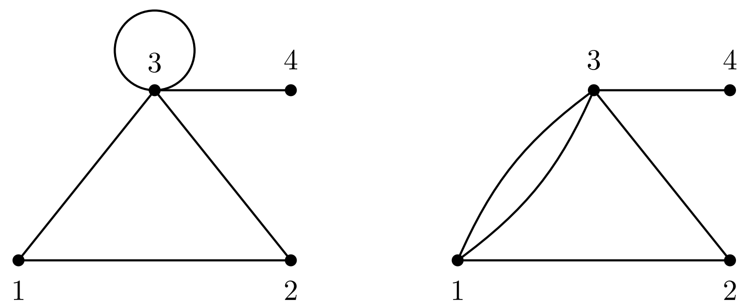


Figure 2.7: Two examples of graphs that *cannot* be described as simplicial complexes: a graph containing a loop on vertex 3 (left), and a multigraph with multiple edges connecting vertices 1 and 3 (right).

CHAPTER 3

Simplicial Homology

To rigorously investigate the topological properties of simplicial complexes, we equip them with an *orientation*—that is, a suitable ordering of the vertices of each simplex. This provides an underlying combinatorial structure that allows us to trace exactly how simplices interact and cancel out along their boundaries, a mechanism that is fundamental in defining *homology modules*.

Intuitively, homology acts as a mathematical scanner designed to detect and count the "holes" of a space across different dimensions. It formalizes a profound yet simple geometric observation: a hole exists whenever we find a closed shape (a *cycle*) that is not completely filled in by any higher-dimensional volume (a *boundary*). By algebraically defining homology modules as the quotient of cycles modulo boundaries, we will effectively translate the search for topological holes into straightforward linear algebra.

This approach offers a massive computational advantage over homotopy theory. While higher-dimensional homotopy groups π_n are notoriously difficult to compute even for simple spaces like spheres S^n, homology modules can be calculated by evaluating the ranks of matrices. Remarkably, despite this computational simplicity, homology modules remain robust homotopy invariants.

3.1 Orientation on a Simplicial Complex

Definition 3.1.1 (Orientation of a simplicial complex)**.** Let $\mathcal{K}$ be a simplicial complex with vertex set $V(\mathcal{K})$. An *orientation* on $\mathcal{K}$ is a partial order $\leq$ on $V(\mathcal{K})$ such that its restriction to the vertices of any simplex $F \in \mathcal{K}$ is a total order.

A simplicial complex endowed with an orientation is called an *oriented simplicial complex.*

Lemma 3.1.2. Each simplicial complex admits an orientation.

Proof. It suffices to endow the set of vertices with a total order. For instance, one can choose a well-order, whose existence is guaranteed by Theorem 1.3.8. □

The choice of an orientation induces a convenient notation for the simplices: a simplex $\{v_1, \ldots, v_n\}$ in $\mathcal{K}$ whose vertices satisfy $v_1 < \cdots < v_n$ will be denoted by $[v_1, \ldots, v_n]$.

Moreover, the standard n-simplex has a *natural* orientation, given by $e_1 < \ldots < e_{n+1}$.

Example 3.1.3. Consider the simplicial complex depicted in Figure 2.2. An orientation is induced on the entire complex by choosing the global vertex ordering:

$$v_1 < v_2 < v_3 < v_4 < v_5 < v_6.$$

Adopting the notation just established, we can explicitly represent the facets of the complex as follows: the edge is denoted by $[v_1, v_2]$, the triangle by $[v_2, v_3, v_4]$, and the tetrahedron by $[v_3, v_4, v_5, v_6]$.

3.2 Chain Spaces and the Boundary Map ∂

Definition 3.2.1 (Chain space)**.** Let $\mathcal{K}$ be an oriented simplicial complex and R be a commutative ring. We define the *n-th chain module* $C_n(\mathcal{K}; R)$ as the free R-module generated by the oriented n-simplices of $\mathcal{K}$. Each element of $C_n(\mathcal{K}; R)$ is referred to as an *n-chain.*

Convention. Whenever the coefficient ring R is clear from the context, we will drop it from the notation and simply write $C_n(\mathcal{K})$. In practice, we will primarily choose R to be either a field or the ring of integers $\mathbb{Z}$; in the fundamental case of integer coefficients, $C_n(\mathcal{K}; \mathbb{Z})$ naturally reduces to a free abelian group and will thus be called the *chain group.*

Definition 3.2.2 (Boundary map)**.** Let $\mathcal{K}$ be an oriented simplicial complex and R be a commutative ring. For each $n \geq 1$, we define the *n-th boundary map* $\partial_n \colon C_n(\mathcal{K}; R) \to C_{n-1}(\mathcal{K}; R)$ as the unique module homomorphism whose action on the generators (i.e., the oriented n-simplices) is given by:

$$\partial_n([w_0, w_1, \ldots, w_n]) = \sum_{i=0}^{n} (-1)^i \, [w_0, \ldots, \widehat{w}_i, \ldots, w_n],$$

where the symbol $\widehat{w}_i$ indicates that the i-th vertex is omitted from the sequence. For $n = 0$, we conventionally define $\partial_0 \colon C_0(\mathcal{K}; R) \to 0$ as the zero map.

Remark 3.2.3. In defining the boundary maps, we implicitly invoked the universal property of free modules, as described by Theorem 1.2.4.

Remark 3.2.4. The alternating signs in the boundary formula become incredibly intuitive if we interpret the oriented simplicial complex through the lens of a *directed* graph.

We can conceptually view an $(n+1)$-dimensional simplex as a directed "edge" that induces an oriented flow across its n-dimensional faces. For instance, in a standard 1-dimensional directed graph, the boundary of an oriented edge $[v_0, v_1]$ is simply $[v_1] - [v_0]$, where the signs denote the flow: the path exits the negative node and enters the positive one.

Generalizing this to dimension n, as we compute the boundary of an $(n+1)$-simplex, we conceptually trace a continuous flow across its boundary. In this context, the alternating sign $(-1)^i$ serves to reconcile the intrinsic orientations: it indicates whether the flow induced by the $(n+1)$-simplex aligns with the inherent orientation of its n-faces (yielding a positive sign), or if it opposes that orientation (yielding a negative sign).

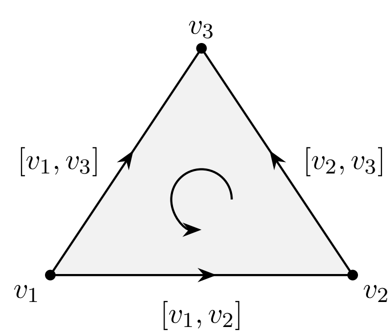


Figure 3.1: The standard 2-simplex with vertices ordered $v_1 < v_2 < v_3$. To trace the counter-clockwise boundary induced by the 2-simplex, one must traverse the left edge against its lexicographical orientation $[v_1, v_3]$, geometrically justifying the minus sign in ∂_2.

The core algebraic mechanism of homology theory rests upon a deeply intuitive geometric fact: the boundary of a boundary is always empty. For instance, the boundary of a solid 2-simplex (i.e., a triangle) is a closed loop of three edges; this loop itself has no endpoints, meaning its boundary is strictly zero.

Algebraically, this principle translates into one crucial identity: applying the boundary operator twice consecutively yields the zero map, written concisely as $\partial^2 = 0$. The following proposition formalizes this identity:

Proposition 3.2.5. Let $\mathcal{K}$ be an oriented simplicial complex and R be a commutative ring. Then, for each $n \geq 1$, $\partial_n \circ \partial_{n+1} = 0$.

Proof. By linearity, it suffices to prove the claim on the basis generators of $C_{n+1}(\mathcal{K}; R)$. Let $\sigma = [w_0, w_1, \ldots, w_{n+1}]$ be an oriented $(n+1)$-simplex. Applying the definition of the $(n+1)$-th boundary operator, we obtain:

$$\partial_{n+1}(\sigma) = \sum_{i=0}^{n+1} (-1)^i [w_0, \ldots, \widehat{w}_i, \ldots, w_{n+1}].$$

Next, we apply the n-th boundary operator to this result. Since ∂_n is a homomorphism, we can pass it inside the sum by linearity. When computing the boundary of $[w_0, \ldots, \widehat{w}_i, \ldots, w_{n+1}]$, we must remove one vertex w_j at a time. We split this operation into two cases depending on whether w_j appears before or after the already omitted vertex w_i:

- If $j < i$, the vertex w_j is still in the j-th position, so its removal contributes a sign of $(-1)^j$.
- If $j > i$, the vertex w_i has already been removed, meaning w_j has shifted one position to the left. Thus, it is now in the $(j-1)$-th position, contributing a sign of $(-1)^{j-1}$.

Expanding the double application of the boundary operator, we therefore get:

$$\begin{aligned}
\partial_n(\partial_{n+1}(\sigma)) &= \sum_{i=0}^{n+1} (-1)^i \, \partial_n\big([w_0, \ldots, \widehat{w}_i, \ldots, w_{n+1}]\big) \\
&= \sum_{0 \leq j < i \leq n+1} (-1)^i (-1)^j \, [w_0, \ldots, \widehat{w}_j, \ldots, \widehat{w}_i, \ldots, w_{n+1}] \\
&\quad + \sum_{0 \leq i < j \leq n+1} (-1)^i (-1)^{j-1} \, [w_0, \ldots, \widehat{w}_i, \ldots, \widehat{w}_j, \ldots, w_{n+1}].
\end{aligned}$$

We can simplify this expression by factoring out a negative sign from the second summation, since $(-1)^{j-1} = -(-1)^j$. Furthermore, observe that the indices of both summations describe exactly the same set of $(n-1)$-simplices. Consequently, the terms cancel out pairwise:

$$\begin{aligned}
\partial_n(\partial_{n+1}(\sigma)) &= \sum_{0 \leq j < i \leq n+1} (-1)^{i+j} \, [w_0, \ldots, \widehat{w}_j, \ldots, \widehat{w}_i, \ldots, w_{n+1}] \\
&\quad - \sum_{0 \leq i < j \leq n+1} (-1)^{i+j} \, [w_0, \ldots, \widehat{w}_i, \ldots, \widehat{w}_j, \ldots, w_{n+1}] \\
&= 0.
\end{aligned}$$

This concludes the proof. □

3.3 Cycles, Boundaries and Homology Modules

As a result of Proposition 3.2.5, each image of a ∂_i is naturally an element of the kernel of ∂_{i-1}, namely:

$$\operatorname{im}(\partial_i) \subseteq \ker(\partial_{i-1}).$$

This allows us to assemble the individual chain spaces and boundary maps into a single, cohesive *chain complex*, denoted by $(C_\bullet(\mathcal{K}), \partial_\bullet)$:

$$\cdots \longrightarrow C_{i+1}(\mathcal{K}) \xrightarrow{\partial_{i+1}} C_i(\mathcal{K}) \xrightarrow{\partial_i} C_{i-1}(\mathcal{K}) \longrightarrow \cdots \xrightarrow{\partial_1} C_0(\mathcal{K}) \xrightarrow{\partial_0} 0$$

Remark 3.3.1. If $\mathcal{K}$ is a finite-dimensional simplicial complex of dimension d, its corresponding chain complex is naturally *bounded*. Specifically, since the complex contains no simplices of dimension strictly greater than d, the chain spaces $C_n(\mathcal{K})$ are trivially zero for all $n > d$.

Thus, the non-trivial portion of the sequence is strictly finite:

$$C_d(\mathcal{K}) \xrightarrow{\partial_d} \cdots \xrightarrow{\partial_2} C_1(\mathcal{K}) \xrightarrow{\partial_1} C_0(\mathcal{K}) \xrightarrow{\partial_0} 0$$

Definition 3.3.2 (Cycles, boundaries and homology modules)**.** Let $\mathcal{K}$ be an oriented simplicial complex and R be a commutative ring. The *module of n-cycles* $Z_n(\mathcal{K}; R)$, the *module of n-boundaries* $B_n(\mathcal{K}; R)$, and the *n-th (simplicial) homology module* $H_n(\mathcal{K}; R)$ of the complex $\mathcal{K}$ are defined as those inherited directly from the chain complex $(C_\bullet(\mathcal{K}; R), \partial_\bullet)$.

Explicitly, they are defined as:

$$\begin{aligned} Z_n(\mathcal{K}; R) &= \ker(\partial_n), \\ B_n(\mathcal{K}; R) &= \operatorname{im}(\partial_{n+1}), \\ H_n(\mathcal{K}; R) &= Z_n(\mathcal{K}; R)/B_n(\mathcal{K}; R). \end{aligned}$$

Convention. Whenever the coefficient ring R is clear from the context, or when a particular result is entirely independent of its choice, we will omit R from the notation and simply write $Z_n(\mathcal{K})$, $B_n(\mathcal{K})$, and $H_n(\mathcal{K})$, as previously established for the chain modules. Furthermore, when $R = \mathbb{Z}$, these modules naturally reduce to abelian groups and will be referred to as the *cycle group*, *boundary group*, and *homology group* (or integral homology), respectively.

Example 3.3.3. We now rigorously compute the first homology group H_1 of the hollow triangle X from Figure 3.2.

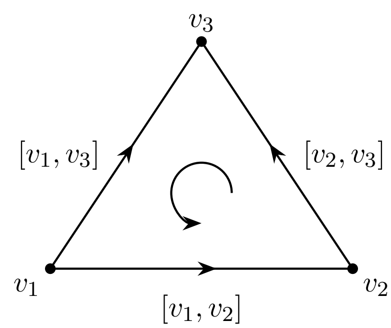


Figure 3.2: A hollow triangle with vertices ordered $v_1 < v_2 < v_3$.

The chain group $C_1(X)$ is freely generated by the edges. By fixing the lexicographically ordered bases, we can represent the boundary operator ∂_1 as a labeled matrix:

$$\partial_1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} [v_1, v_2] & [v_1, v_3] & [v_2, v_3] \end{array} \\ \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

The module of 1-cycles, $Z_1(X) = \ker \partial_1$, corresponds to the null space of this matrix, which is generated by the vector $(1, -1, 1)^\top$. Translating this vector back into our basis yields the fundamental cycle of the complex:

$$z = [v_1, v_2] + [v_2, v_3] - [v_1, v_3].$$

Since X contains no 2-dimensional simplices, the chain group $C_2(X)$ is trivial. Consequently, the module of 1-boundaries, $B_1(X) = \operatorname{im} \partial_2$, is also trivial. Therefore, the first homology group is isomorphic to the module of 1-cycles:

$$H_1(X) = Z_1(X)/B_1(X) \cong Z_1(X).$$

Because $Z_1(X)$ is freely generated by the single non-trivial cycle z, we conclude that the hollow triangle possesses exactly one 1-dimensional hole. In other words:

$$H_1(X) \cong \mathbb{Z}.$$

Example 3.3.4. To illustrate what homology computes, we can compare the solid 2-simplex in Figure 3.1 with the hollow triangle analyzed in Example 3.3.3.

Since both complexes share the exact same vertices and edges, the module of 1-cycles $Z_1(X)$ is exactly the same and is generated by the fundamental cycle:

$$z = [v_1, v_2] + [v_2, v_3] - [v_1, v_3].$$

However, the crucial distinction lies in whether this cycle constitutes the boundary of a 2-dimensional region. In the hollow triangle, the interior is absent, meaning z is a cycle, but *not* a boundary.

Conversely, in the solid triangle, the chain group $C_2(X)$ is freely generated by the 2-simplex $[v_1, v_2, v_3]$. When we apply the boundary operator ∂_2 to this generator, we obtain:

$$\partial_2([v_1, v_2, v_3]) = [v_2, v_3] - [v_1, v_3] + [v_1, v_2] = z.$$

This tells us that the module of 1-boundaries, $B_1(X) = \operatorname{im} \partial_2$, is generated by the exact same cycle z. Because the cycle acts precisely as the boundary of the 2-dimensional interior, there is no 1-dimensional hole.

This specific algebraic condition of being a cycle without being a boundary is exactly what homology modules capture to formalize the concept of a topological hole. Computing the first homology group of the solid 2-simplex makes this explicit:

$$H_1(X) = Z_1(X)/B_1(X) = \langle z \rangle / \langle z \rangle \cong 0.$$

Thus, unlike the hollow triangle, the first homology group of the solid 2-simplex is trivial.

3.4 Reduced Simplicial Homology and the Augmentation Map

We first show that $H_0(\mathcal{K})$ is a free module of rank equal to the number of connected components of the complex $\mathcal{K}$.

Proposition 3.4.1. Let $\mathcal{K}$ be an oriented simplicial complex and R be a commutative ring. If $\mathcal{K}$ has exactly d connected components, then its 0-th homology module is isomorphic to the free R-module of rank d, namely:

$$H_0(\mathcal{K}; R) \cong R^d.$$

Proof. By definition, the 0-th homology module is the quotient $H_0(\mathcal{K}; R) = \ker \partial_0 / \operatorname{im} \partial_1$. Since ∂_0 is the zero map by convention, we have $\ker \partial_0 = C_0(\mathcal{K}; R)$.

The submodule $\operatorname{im} \partial_1$ is generated by the boundaries of the 1-simplices. For any oriented edge $e = [u, v]$ in $\mathcal{K}$, its boundary is $\partial_1(e) = v - u$. Consequently, in the quotient module $H_0(\mathcal{K}; R)$, the homology classes of the two vertices coincide: $[u] = [v]$.

Extending this relation transitively, two vertices represent the same generator in $H_0(\mathcal{K}; R)$ if and only if they can be joined by a sequence of edges—that is, if and only if they belong to the same connected component of $\mathcal{K}$.

By choosing exactly one arbitrary vertex from each of the d connected components, we obtain a set of d linearly independent generators for the quotient. Therefore, $H_0(\mathcal{K}; R)$ is freely generated by these d elements, yielding the isomorphism $H_0(\mathcal{K}; R) \cong R^d$. □

As a result of Proposition 3.4.1, the homology modules of a point are all trivial, *except for* $n = 0$, for which the homology module is isomorphic to R, since a point has only one connected component.

We are then motivated to introduce a variation of our usual chain complex $(C_\bullet, \partial_\bullet)$ to make it such that all the resulting homology modules are trivial for a point.

Definition 3.4.2 (Augmentation map)**.** Let $\mathcal{K}$ be an oriented simplicial complex and R be a commutative ring. We define the *augmentation map* ϵ from $C_0(\mathcal{K}; R)$ to R as the unique R-module homomorphism whose action on each generator (vertex) v is given by $\epsilon(v) = 1$.

Explicitly, when evaluated on an arbitrary 0-chain, the map simply returns the sum of its coefficients:

$$\epsilon\left(\sum_{i=1}^{k} t_i v_i\right) = \sum_{i=1}^{k} t_i,$$

where $t_i \in R$ and the elements v_1, ..., v_k are distinct vertices of $\mathcal{K}$.

Proposition 3.4.3. Let $\mathcal{K}$ be an oriented simplicial complex and R be a commutative ring. If ϵ is the augmentation map, then $\epsilon \circ \partial_1 = 0$.

Proof. By the linearity of module homomorphisms, it suffices to prove the claim on the basis generators of $C_1(\mathcal{K}; R)$. Let $e = [v_i, v_j]$ be an arbitrary oriented 1-simplex in $\mathcal{K}$, then:

$$\begin{aligned}
(\epsilon \circ \partial_1)([v_i, v_j]) &= \epsilon\big(\partial_1([v_i, v_j])\big) \\
&= \epsilon(v_j - v_i) \\
&= \epsilon(v_j) - \epsilon(v_i) \\
&= 1 - 1 \\
&= 0.
\end{aligned}$$

This concludes the proof. □

Therefore, we can *augment* the original chain complex $(C_\bullet, \partial_\bullet)$ we built upon the boundary maps to include ϵ instead of ∂_0:

$$\cdots \longrightarrow C_1(\mathcal{K}) \xrightarrow{\partial_1} C_0(\mathcal{K}) \xrightarrow{\epsilon} R \longrightarrow 0 \ .$$

This is called the *augmented chain complex* $\widetilde{C}_\bullet(\mathcal{K}; R)$ of the oriented simplicial complex $\mathcal{K}$ on the ring R. We now define its algebraic tools:

Definition 3.4.4 (Reduced homology)**.** Let $\mathcal{K}$ be an oriented simplicial complex and R be a commutative ring. The *module of reduced n-cycles* $\widetilde{Z}_n(\mathcal{K};R)$, the *module of reduced n-boundaries* $\widetilde{B}_n(\mathcal{K};R)$, and the *$n$-th reduced (simplicial) homology module* $\widetilde{H}_n(\mathcal{K};R)$ are defined as those inherited directly from the augmented chain complex $\widetilde{C}_\bullet(\mathcal{K};R)$.

Explicitly, since the augmented complex differs from the standard chain complex only in dimension zero, the reduced modules perfectly coincide with the usual homology modules for all $n \geq 1$:

$$\begin{aligned}
\widetilde{Z}_n(\mathcal{K};R) &= \ker(\partial_n) = Z_n(\mathcal{K};R),\\
\widetilde{B}_n(\mathcal{K};R) &= \operatorname{im}(\partial_{n+1}) = B_n(\mathcal{K};R),\\
\widetilde{H}_n(\mathcal{K};R) &= \widetilde{Z}_n(\mathcal{K};R)/\widetilde{B}_n(\mathcal{K};R) = H_n(\mathcal{K};R).
\end{aligned}$$

For the exceptional case $n = 0$, the 0-th reduced modules are explicitly defined as:

$$\begin{aligned}
\widetilde{Z}_0(\mathcal{K};R) &= \ker(\epsilon),\\
\widetilde{B}_0(\mathcal{K};R) &= \operatorname{im}(\partial_1) = B_0(\mathcal{K};R),\\
\widetilde{H}_0(\mathcal{K};R) &= \widetilde{Z}_0(\mathcal{K};R)/\widetilde{B}_0(\mathcal{K};R).
\end{aligned}$$

The definition of the augmentation map ϵ is motivated by the following proposition, which allows us to recover the standard homology module $H_0(\mathcal{K};R)$ by means of $\widetilde{H}_0(\mathcal{K};R)$:

Proposition 3.4.5. Let $\mathcal{K}$ be an oriented non-empty simplicial complex and R be a commutative ring. Then:

$$H_0(\mathcal{K};R) \cong \widetilde{H}_0(\mathcal{K};R) \oplus R.$$

Proof. Let us fix an arbitrary vertex $v_0 \in \mathcal{K}$. Using the augmentation map $\epsilon\colon C_0(\mathcal{K};R) \to R$, we can identically rewrite any 0-chain $c \in C_0(\mathcal{K};R)$ as follows:

$$c = (c - \epsilon(c)v_0) + \epsilon(c)v_0.$$

Applying ϵ to the first term yields:

$$\begin{aligned}
\epsilon(c - \epsilon(c)v_0) &= \epsilon(c) - \epsilon(c)\epsilon(v_0)\\
&= \epsilon(c) - \epsilon(c)\\
&= 0.
\end{aligned}$$

This implies that the term $c - \epsilon(c)v_0$ belongs to $\ker\epsilon$. The second term naturally belongs to the free submodule generated by v_0, which is isomorphic to R. This yields a direct sum decomposition at the chain level:

$$C_0(\mathcal{K};R) = \ker\epsilon \oplus Rv_0.$$

We know by Proposition 3.4.3 that $\operatorname{im}\partial_1 \subseteq \ker \epsilon$, thus:

$$H_0(\mathcal{K};R) = \frac{C_0(\mathcal{K};R)}{\operatorname{im}\partial_1} = \frac{\ker\epsilon \oplus Rv_0}{\operatorname{im}\partial_1 \oplus \{0\}} \cong \frac{\ker\epsilon}{\operatorname{im}\partial_1} \oplus \frac{Rv_0}{\{0\}} \cong \widetilde{H}_0(\mathcal{K};R) \oplus R. \quad \square$$

Remark 3.4.6. If the complex $\mathcal{K}$ consists of a single point, we know that $H_0(\mathcal{K};R) \cong R$. Thus, by Proposition 3.4.5:

$$R \cong \widetilde{H}_0(\mathcal{K};R) \oplus R.$$

This forces the reduced module to be trivial, i.e., $\widetilde{H}_0(\mathcal{K};R) = 0$. Therefore, the reduced homology of a point vanishes in every dimension, fulfilling our initial motivation.

3.5 Betti Numbers, Euler–Poincaré Characteristic and Homotopy Invariance

The homology modules of a simplicial complex encode its topological features in a computable algebraic format. Over a field, the ranks of these modules are invariant topological properties of the space.

Definition 3.5.1 (Betti numbers)**.** Let $\mathcal{K}$ be a simplicial complex and F be a field. The n-th *Betti number* $\beta_n(\mathcal{K};F)$ is the dimension of the n-th homology vector space $H_n(\mathcal{K};F)$ over F:

$$\beta_n(\mathcal{K};F) = \dim_F H_n(\mathcal{K};F).$$

Remark 3.5.2. By virtue of the *Universal Coefficient Theorem for Homology* (see [Hat02, Section 3.A]), if the field F has characteristic zero, the Betti numbers $\beta_n(\mathcal{K};F)$ are entirely independent of the specific choice of the field.

In such cases, they invariably coincide with the Betti numbers computed over the rational numbers, which in turn equal the rank of the free abelian part of the homology groups $H_n(\mathcal{K};\mathbb{Z})$.

Convention. Whenever the field F is omitted from the notation, the symbol $\beta_n(\mathcal{K})$ implicitly denotes the n-th Betti number computed over a field of characteristic zero.

Intuitively, β_0 counts the connected components, β_1 the 1-dimensional “holes” or cycles, β_2 the 2-dimensional voids, and so on.

Definition 3.5.3 (Euler–Poincaré characteristic)**.** The *Euler characteristic* (or Euler–Poincaré characteristic) $\chi(\mathcal{K})$ of a finite simplicial complex $\mathcal{K}$ of dimension d is defined as the alternating sum of the number of its faces:

$$\chi(\mathcal{K}) = \sum_{i=0}^{d} (-1)^i f_i,$$

where f_i is the number of i-dimensional simplices in $\mathcal{K}$.

The following theorem connects this purely combinatorial count to the topological invariants given by the Betti numbers:

Theorem 3.5.4. For any finite d-dimensional simplicial complex $\mathcal{K}$ and any field F, the Euler characteristic satisfies:

$$\chi(\mathcal{K}) = \sum_{i=0}^{d} (-1)^i \beta_i(\mathcal{K}; F).$$

Proof. By the rank-nullity theorem applied to the boundary map ∂_i, we have:

$$\dim C_i(\mathcal{K}; F) = \dim(\ker \partial_i) + \dim(\operatorname{im} \partial_i) = \dim Z_i(\mathcal{K}; F) + \dim B_{i-1}(\mathcal{K}; F),$$

where we use B_{-1} to denote the trivial module. Taking the alternating sum over all i's, we obtain:

$$\sum_{i=0}^{d} (-1)^i \dim C_i(\mathcal{K}; F) = \sum_{i=0}^{d} (-1)^i \dim Z_i(\mathcal{K}; F) + \sum_{i=0}^{d} (-1)^i \dim B_{i-1}(\mathcal{K}; F).$$

By shifting the index in the second sum, we get:

$$\sum_{i=0}^{d} (-1)^i \dim B_{i-1}(\mathcal{K}; F) = -\sum_{i=-1}^{d-1} (-1)^i \dim B_i(\mathcal{K}; F).$$

Substituting this back into the original equation yields:

$$\sum_{i=0}^{d} (-1)^i \dim C_i(\mathcal{K}; F) = \sum_{i=0}^{d} (-1)^i (\dim Z_i(\mathcal{K}; F) - \dim B_i(\mathcal{K}; F)).$$

Finally, since $H_i(\mathcal{K}; F) = Z_i(\mathcal{K}; F)/B_i(\mathcal{K}; F)$, we know that $\beta_i(\mathcal{K}; F)$ is equal to $\dim Z_i(\mathcal{K}; F) - \dim B_i(\mathcal{K}; F)$, which concludes the proof:

$$\sum_{i=0}^{d} (-1)^i \dim C_i(\mathcal{K}; F) = \sum_{i=0}^{d} (-1)^i \beta_i(\mathcal{K}; F).$$

□

Similarly, one can define the *reduced Euler characteristic* $\widetilde{\chi}(\mathcal{K})$ using the reduced Betti numbers $\widetilde{\beta}_i(\mathcal{K}; F) = \dim_F \widetilde{H}_i(\mathcal{K}; F)$, leading to the straightforward relation:

$$\widetilde{\chi}(\mathcal{K}) = \chi(\mathcal{K}) - 1.$$

Now we cite the central theorem that allows us to use all these tools as topological invariants.

Theorem 3.5.5 (Homotopy invariance of simplicial homology)**.** If two simplicial complexes $\mathcal{K}_1$ and $\mathcal{K}_2$ are homotopy equivalent, then both their standard and reduced homology modules are isomorphic:

$$H_n(\mathcal{K}_1) \cong H_n(\mathcal{K}_2) \quad \text{and} \quad \widetilde{H}_n(\mathcal{K}_1) \cong \widetilde{H}_n(\mathcal{K}_2) \quad \text{for all } n.$$

A complete proof of Theorem 3.5.5 is provided as Theorem 19.2 in [Mun84, Chapter 2, §18], which relies fundamentally on the preliminary results established in [Mun84, Chapter 2, §16-17], particularly the *Simplicial Approximation Theorem*.

The following classical theorem establishes the structure of the integral homology groups for a wedge sum of spheres:

Theorem 3.5.6. Let $\mathcal{K}$ be a finite d-dimensional simplicial complex such that its geometric realization $\|\mathcal{K}\|$ has the homotopy type of a wedge of spheres. Let r_i denote the number of i-dimensional spheres in the wedge. Then, the reduced integral homology of $\mathcal{K}$ is given by:

$$\widetilde{H}_i(\mathcal{K};\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{r_i} & \text{if } 1 \leq i \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

The proof of Theorem 3.5.6 is a direct consequence of Corollary 2.25 in [Hat02].

Remark 3.5.7. In light of Remark 3.5.2, if $\mathcal{K}$ satisfies the hypotheses of Theorem 3.5.6, its reduced homology over any field F of characteristic zero explicitly counts the spheres in the wedge. Specifically, if the wedge contains r_i spheres of dimension i, we have:

$$\widetilde{H}_i(\mathcal{K};F) \cong F^{r_i} \quad \text{for } 1 \leq i \leq d.$$

CHAPTER 4

Order Complexes and Posets

We now introduce the central concept of an *order complex*, a construction that associates an abstract simplicial complex with any given partially ordered set. Likewise, any simplicial complex yields a poset through its *face poset*. This elegant duality mirrors in some way the interplay between geometric and abstract simplicial complexes we explored in the previous chapters, thus establishing a rigorous bridge between order theory and algebraic topology.

By formalizing this correspondence, we essentially endow posets with a *topology*. This framework then allows us to directly translate topological invariants—such as connectedness and homology modules—into intrinsic combinatorial properties of the underlying poset, and vice versa.

4.1 From Posets to Simplicial Complexes and Vice Versa

Definition 4.1.1 (Order complex)**.** Let $(P, \leq)$ be a partially ordered set. The *order complex* of P, denoted by $\Delta(P)$, is the abstract simplicial complex whose vertex set is P and whose simplices are exactly the chains of P.

Remark 4.1.2. In light of Remark 2.2.14, encoding or drawing the order complex of a poset is conceptually straightforward. Since the simplices of an order complex correspond directly to the chains of the underlying poset, the facets of the complex correspond exactly to the *maximal chains* of the poset. Therefore, for computational purposes or to visualize the space, it is entirely sufficient to determine the maximal chains; they act as the maximal building blocks that fully generate the topological structure.

Example 4.1.3. Consider the poset $P = \{a, b, c, d\}$ endowed with the strict order relations $a < c$, $a < d$, $b < c$, and $b < d$, as shown in the Hasse diagram of Figure 4.1.

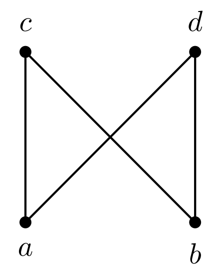


Figure 4.1: The Hasse diagram of the poset $P = \{a, b, c, d\}$.

We can now determine the topological structure of the order complex $\Delta(P)$. By definition, the vertices of $\Delta(P)$ coincide precisely with the elements of P. To construct the simplices of the complex, we examine the chains of the poset.

In this specific case, the maximal chains are exactly four:

$$a < c, \;\; a < d, \;\; b < c, \;\; \text{and } b < d.$$

These chains correspond directly to the edges:

$$\{a, c\}, \; \{a, d\}, \; \{b, c\}, \text{ and } \{b, d\}.$$

Geometrically, gluing these four edges at their shared vertices yields precisely a hollow square, as illustrated in Figure 4.2.

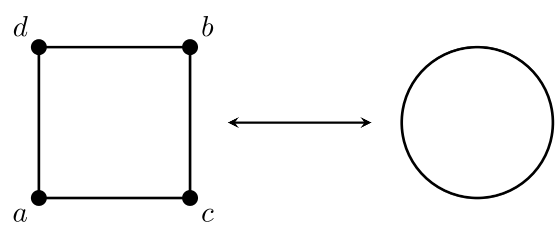


Figure 4.2: The geometric realization of the order complex $\Delta(P)$ for the poset $P = \{a, b, c, d\}$, constructed by gluing four edges at their vertices.

Remark 4.1.4. As we saw in Example 4.1.3, there are specific cases where the order complex of a poset is homeomorphic to its Hasse diagram. It is worth noting that, in this topological context, we rigorously treat the Hasse diagram as an abstract 1-dimensional simplicial complex. Consequently, any accidental edge crossings that may appear in planar drawings, such as the one from Figure 4.1, are purely "graphical artifacts"; topologically, the diagram is assumed to be embedded in a space of sufficiently high dimension to avoid any self-intersections.

Furthermore, since the Hasse diagram represents the covering relations of P, it *naturally* embeds as a 1-dimensional geometric subcomplex within the realization $\|\Delta(P)\|$.

However, in general, the two structures are topologically distinct. To see why, consider a totally ordered chain of three elements, such as $P = \{a, b, c\}$ endowed with the strict relations $a < b < c$. Its order complex $\Delta(P)$ contains the single maximal chain $a < b < c$, which translates exactly into a 2-simplex.

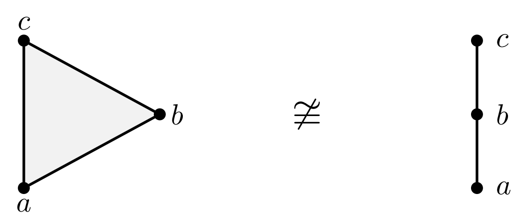


Figure 4.3: A visual comparison between the geometric realization of the order complex $\|\Delta(P)\|$ (left) and the Hasse diagram (right) for the totally ordered chain $P = \{a, b, c\}$.

Conversely, the Hasse diagram exclusively traces the covering relations $a < b$ and $b < c$. By explicitly omitting the transitive relation $a < c$, it fails to close the loop, forming merely a 1-dimensional path (i.e., a line segment), as illustrated in Figure 4.3. Consequently, the Hasse diagram does not generally coincide *even* with the underlying graph of the order complex, since the latter includes an explicit edge for every transitive relation.

In the same spirit as defining the geometric realization of an abstract simplicial complex, we now introduce a construction that acts as a "right inverse" to the operation of taking the order complex of a poset. Conceptually, this operation can be viewed as a "poset realization" of a simplicial complex.

Definition 4.1.5 (Face poset)**.** Let $\mathcal{K}$ be an abstract simplicial complex. The *face poset* of $\mathcal{K}$, denoted by $(FP(\mathcal{K}), \subseteq)$, is the partially ordered set whose elements are the simplices of $\mathcal{K}$, ordered by standard set inclusion.

Proposition 4.1.6. Let $\mathcal{K}$ be an abstract simplicial complex. Then $\|\mathcal{K}\|$ is homeomorphic to $\|\Delta(FP(\mathcal{K}))\|$ via a piecewise-linear map.

Proof. The order complex $\Delta(FP(\mathcal{K}))$ has as its vertices the elements of $FP(\mathcal{K})$, which are precisely the simplices of $\mathcal{K}$. By definition, a simplex in $\Delta(FP(\mathcal{K}))$

corresponds to a finite chain of strict inclusions

$$\sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_m$$

among the simplices of $\mathcal{K}$.

To establish the homeomorphism, we construct a map $f\colon \|\Delta(FP(\mathcal{K}))\| \to \|\mathcal{K}\|$. We begin by defining f on the vertices of $\Delta(FP(\mathcal{K}))$. For each vertex $\sigma \in FP(\mathcal{K})$, we map it to the barycenter b_σ of the corresponding geometric simplex in $\|\mathcal{K}\|$. If a simplex is generated by the vertices $\{v_0, \ldots, v_k\}$, its barycenter is given by $b_\sigma = \frac{1}{k+1}\sum_{i=0}^{k} v_i$.

Next, we extend f linearly over the simplices. Given any simplex in $\Delta(FP(\mathcal{K}))$ represented by a chain $C = \{\sigma_0, \sigma_1, \ldots, \sigma_m\}$, the corresponding geometric barycenters $b_{\sigma_0}, b_{\sigma_1}, \ldots, b_{\sigma_m}$ are affinely independent in $\|\mathcal{K}\|$. Therefore, we can linearly extend f to map the abstract simplex C bijectively onto the geometric simplex spanned by $\{b_{\sigma_0}, \ldots, b_{\sigma_m}\}$.

Since f is constructed as a piecewise-linear affine map on each closed simplex, it is continuous everywhere. Consequently, f is a global homeomorphism. $\square$

Remark 4.1.7. The complex $\Delta(FP(\mathcal{K}))$ is known as the *barycentric subdivision* of $\mathcal{K}$, which is homeomorphic to $\mathcal{K}$ as a result of Proposition 4.1.6. Geometrically, each abstract face $\{v_0, v_1, \ldots, v_k\}$ (for short, $v_0 v_1 \cdots v_k$) in the poset $FP(\mathcal{K})$ corresponds to a newly inserted vertex, positioned exactly at the barycenter of the geometric simplex spanned by $v_0, \ldots, v_k$, as illustrated in Figure 4.4.

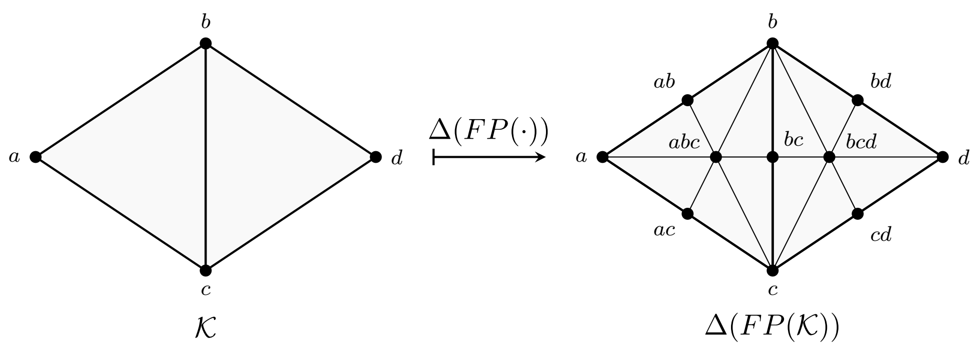


Figure 4.4: A simplicial complex $\mathcal{K}$ formed by two glued 2-simplices (left) and its barycentric subdivision $\Delta(FP(\mathcal{K}))$ (right).

4.2 Poset Topology and Reduction to the Proper Part

Associating a poset with its order complex *naturally* yields a topology, called the *poset topology*:

Definition 4.2.1 (Poset topology)**.** Let $(P, \leq)$ be a partially ordered set. The *poset topology* of P is defined as the weak topology of the geometric realization of its order complex $\|\Delta(P)\|$.

Convention. Throughout the remainder of this work, by a standard abuse of notation also seen in [Wac06], any topological property or algebraic invariant attributed directly to a poset P is understood to refer to its poset topology, namely to its geometric realization $\|\Delta(P)\|$.

For instance, writing $H_2(P; \mathbb{C}) \cong \mathbb{C}^2$ implicitly means $H_2(\|\Delta(P)\|; \mathbb{C}) \cong \mathbb{C}^2$.

Remark 4.2.2. Note that a poset P and its dual P^* share the exact same poset topology. Indeed, the chains of P and P^* consist of the exact same elements, traversed in opposite orders.

Since a simplex in the order complex is determined solely by the *unordered* set of elements in a chain, we have an equality of simplicial complexes:

$$\Delta(P) = \Delta(P^*).$$

Consequently, the geometric realizations of P and P^* are *identical*, meaning $\|\Delta(P)\| = \|\Delta(P^*)\|$.

Remark 4.2.3. Observe that if a poset P possesses a global maximum $\hat{1}$ or a global minimum $\hat{0}$, its poset topology is quite trivial. Indeed, in such cases, every simplex in the order complex $\Delta(P)$ is joined to the vertex corresponding to this global extremum.

Geometrically, this means that the realization $\|\Delta(P)\|$ is a cone with one extremum acting as its apex. Consequently, P is contractible via a deformation retraction onto this apex, yielding trivial reduced homology modules.

This topological behavior motivates the following definition:

Definition 4.2.4 (Proper part of a poset)**.** Let $(P, \leq)$ be a partially ordered set. We define its *proper part*, denoted by $\overline{P}$, as the subposet obtained by removing its global maximum $\hat{1}$ and global minimum $\hat{0}$, provided they exist. Formally:

$$\overline{P} = P \setminus \{\hat{0}, \hat{1}\}.$$

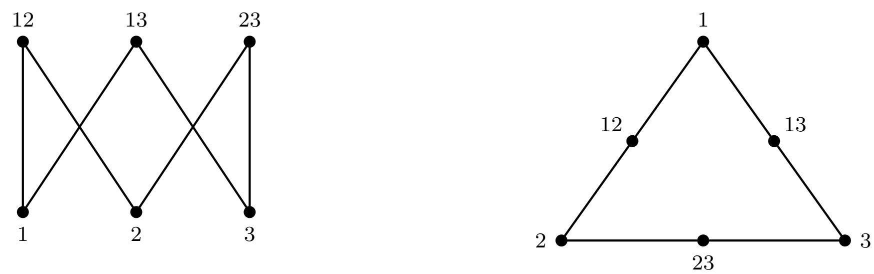
Remark 4.2.5. Naturally, if the resulting proper part $\overline{P}$ happens to contain a new global maximum or minimum, this stripping process can be applied iteratively. One can repeatedly extract the "proper part" until the remaining subposet lacks any unique top or bottom element, isolating the topologically non-trivial core of the original poset.

Example 4.2.6 (Boolean algebras and spheres)**.** Let B_n be the Boolean algebra of all subsets of $\{1, \ldots, n\}$ ordered by inclusion. We denote by $\overline{B}_n = B_n \setminus \{\emptyset, \{1, \ldots, n\}\}$ its proper part.

The order complex $\Delta(\overline{B}_n)$ turns out to be homeomorphic to an $(n-2)$-dimensional sphere:

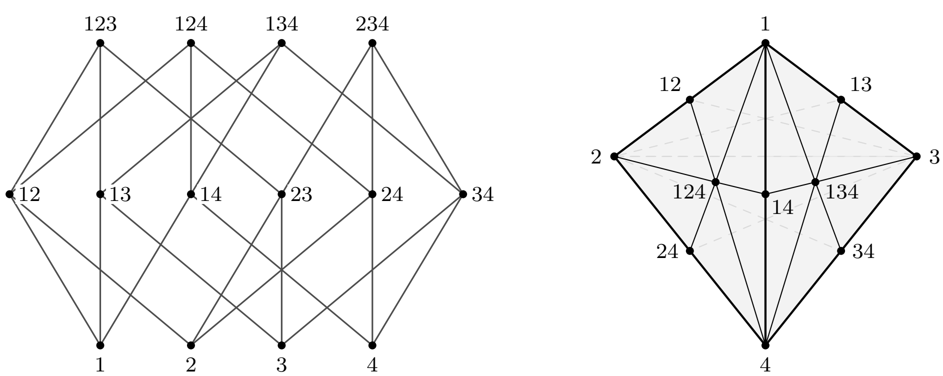
$$\Delta(\overline{B}_n) \cong S^{n-2}.$$

To see why, observe that $B_n \setminus \{\emptyset\}$ can be *naturally* thought of as the face poset of the $(n-1)$-simplex. By additionally removing the unique maximal element $\{1, \ldots, n\}$, we are topologically removing the solid interior volume of the simplex, thus leaving only its outer boundary, $\partial\Delta^{n-1}$, which is homeomorphic to S^{n-2}, as illustrated for $n = 3$ and $n = 4$ in Figure 4.5 and Figure 4.6.



$\overline{B}_3$ $\Delta(\overline{B}_3) \cong S^1$

Figure 4.5: The reduced Boolean algebra $\overline{B}_3$ (left) and its order complex (right).



$\overline{B}_4$ $\Delta(\overline{B}_4) \cong S^2$

Figure 4.6: The reduced Boolean algebra $\overline{B}_4$ (left) and its order complex (right).

4.3 Canonical Orientation and Poset Homology

As established in the foundations of simplicial homology, defining the boundary maps of a generic abstract simplicial complex requires an orientation, which typically involves making an arbitrary choice to order the global vertex set.

However, the order complex $\Delta(P)$ of a poset stands out precisely because it avoids this arbitrariness: it is endowed with an intrinsic orientation.

Since the vertices of $\Delta(P)$ are the elements of P, and every simplex is by definition a totally ordered chain, the poset's structure already dictates an internal ordering of every single face. This remarkable fact leads directly to the following definition:

Definition 4.3.1 (Canonical orientation of the order complex)**.** Let $(P, \leq)$ be a partially ordered set and $\Delta(P)$ its order complex. The *canonical orientation* of $\Delta(P)$ is the *natural* orientation induced directly by the partial order of P.

Specifically, any k-simplex $\{x_0, x_1, \dots, x_k\}$ in $\Delta(P)$ naturally inherits the total order of its vertices from the unique strict chain $x_{i_0} < x_{i_1} < \cdots < x_{i_k}$ that generates it in the underlying poset P.

Convention. We will always implicitly assume the canonical orientation in the computation of homology modules of posets.

Example 4.3.2. Building upon Example 4.1.3, consider again the poset $P = \{a, b, c, d\}$ defined by the strict covering relations $a < c$, $a < d$, $b < c$, and $b < d$.

These relations canonically dictate the orientation of the edges in its order complex $\Delta(P)$. Since the edges correspond exactly to these strict inclusions, the canonical orientation naturally directs them from the minimal elements (a, b) toward the maximal elements (c, d), without the need for any arbitrary vertex labeling, as illustrated in Figure 4.7.

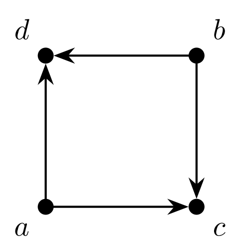


Figure 4.7: The order complex $\Delta(P)$ endowed with its canonical orientation. The edges are naturally directed from the sources (a, b) to the sinks (c, d), mirroring the strict inequalities of the underlying poset.

Since the geometric realization $\|\Delta(P)\|$ is homeomorphic to the circle S^1, the simplicial homology groups of P with integer coefficients vanish in all degrees except for $n = 0$ and $n = 1$. Specifically, we have:

$$H_0(P; \mathbb{Z}) \cong H_1(P; \mathbb{Z}) \cong \mathbb{Z}.$$

The canonical orientation established above provides the precise algebraic requirements needed to explicitly compute these homology groups through the straightforward use of linear algebra.

CHAPTER 5

Shellable Complexes and Edge-Lexicographic Shellability

In the previous chapters, we established the foundational combinatorial structures of simplicial and order complexes. We now introduce the concept of *shellability*, a purely combinatorial property describing how a simplicial complex can be "assembled" piece by piece.

Remarkably, this discrete assembling process dictates the homotopy type of the space, as illustrated by Theorem 5.1.6 and Remark 5.1.7.

The final part of this chapter is then devoted to the theory of *EL-shellability*. Rather than manually constructing a valid shelling sequence for the facets of the complex, this powerful technique builds such a sequence by simply assigning ordered labels to its covering relations.

5.1 Shellable Simplicial Complexes and Homology Facets

Originally, the concept of shellability was studied strictly in the context of *pure* simplicial complexes. However, we will consider its extension to the non-pure case. This framework was established in the 1990s, notably in the foundational works of Björner and Wachs [BW96, Wac06].

Definition 5.1.1 (Shellable simplicial complex)**.** Let $\mathcal{K}$ be a simplicial complex. A *shelling* of $\mathcal{K}$ is a linear ordering F_1, F_2, ..., F_n of its facets such that for every $1 < k \leq n$, the intersection of the subcomplex generated by F_k with the union of the subcomplexes generated by all preceding facets is generated by a non-empty set of maximal proper faces of F_k. Formally, the subcomplex

$$\left(\bigcup_{i=1}^{k-1} \langle F_i \rangle\right) \cap \langle F_k \rangle$$

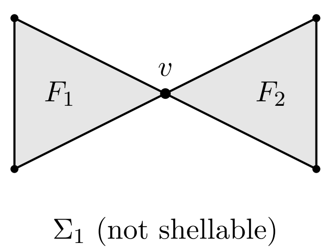
is required to be a pure simplicial complex of dimension $\dim F_k - 1$.

If such an ordering exists, the complex $\mathcal{K}$ is said to be *shellable*.

Example 5.1.2. Let us consider two different complexes featuring a "pinch point" at a single vertex v, as illustrated by Figure 5.1.

Firstly, let Σ_1 be a pure simplicial complex consisting of two 2-dimensional facets, F_1 and F_2, which intersect exclusively at the vertex v. Any shelling attempt must start with one facet, say F_1, and add F_2. According to the definition, the intersection $\langle F_1 \rangle \cap \langle F_2 \rangle = \{v\}$ must be pure of dimension $\dim(F_2) - 1 = 1$. However, a single vertex has dimension 0; thus, the condition fails, and Σ_1 is not shellable.

Let now Σ_2 be a non-pure complex consisting of a 2-simplex F and a hollow triangle, formed by three edges E_1, E_2, E_3. Suppose F and E_1 share the vertex v. We can then construct a valid shelling order: F, E_1, E_2, E_3.



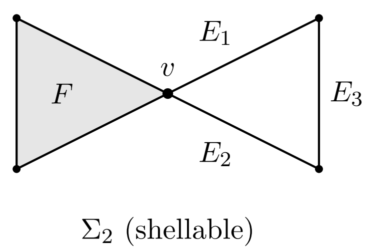


Figure 5.1: Two complexes with a "pinch point" at v. In Σ_1, attaching the 2-simplex F_2 requires a 1-dimensional intersection, causing the shelling to fail. In Σ_2, attaching the 1-simplex E_1 only requires a 0-dimensional intersection (i.e., the vertex v), yielding a valid shelling.

In Example 5.1.2 the shelling sequence began with the facet of maximal dimension. The next proposition proves that the initial element of any valid shelling is *always* forced to have a dimension equal to the overall dimension of the complex:

Proposition 5.1.3. Let $\mathcal{K}$ be a shellable simplicial complex of dimension d. If F_1, F_2, ..., F_n is a valid shelling of $\mathcal{K}$, then the initial facet F_1 must have maximal dimension, that is, $\dim F_1 = d$.

Proof. We proceed by contradiction. Suppose that $\dim F_1 < d$. Since the overall dimension of the complex $\mathcal{K}$ is d, there must exist at least one facet in the shelling sequence with dimension d.

Let F_k be the first facet in the ordering such that $\dim F_k = d$. Because $\dim F_1 < d$, we are guaranteed that $k > 1$.

By our choice of k, every preceding facet F_i, for $1 \leq i < k$, has dimension strictly less than d, meaning $\dim F_i \leq \dim F_k - 1$.

Now, consider the intersection of F_k with any previously attached facet F_i. Since F_i and F_k are both facets of $\mathcal{K}$, F_i cannot be fully contained within F_k. Consequently, their intersection $\langle F_i \rangle \cap \langle F_k \rangle$ must be a proper face of F_i. Thus we have:

$$\dim\big(\langle F_i \rangle \cap \langle F_k \rangle\big) \leq \dim F_i - 1 \leq (\dim F_k - 1) - 1 = \dim F_k - 2.$$

This inequality holds for all $i < k$. Therefore, the total intersection of F_k with the entire past subcomplex $\mathcal{K}_k = \bigcup_{i=1}^{k-1} \langle F_i \rangle$ is a union of faces, each of dimension at most $\dim F_k - 2$. Thus:

$$\dim\big(\mathcal{K}_k \cap \langle F_k \rangle\big) \leq \dim F_k - 2.$$

However the intersection $\mathcal{K}_k \cap \langle F_k \rangle$ must be a pure complex of dimension $\dim F_k - 1$, leading to a contradiction. □

In order to fully state the topological implications of shellability, we first need to identify which specific steps in a shelling process actually alter the topology of the space by creating "holes".

Definition 5.1.4 (Homology facet)**.** Let $\mathcal{K}$ be a shellable simplicial complex and let F_1, F_2, ..., F_n be a shelling of its facets. A facet F_k is called a *homology facet* if its entire boundary is contained within the union of the subcomplexes generated by all preceding facets. Formally, this occurs when:

$$\partial \langle F_k \rangle \subseteq \bigcup_{i=1}^{k-1} \langle F_i \rangle.$$

Geometrically, when a non-homology facet is added during a shelling, it is attached along a contractible piece of its boundary, thus preserving the current homotopy type. Conversely, adding a homology facet effectively "closes a bubble", increasing the homology of the space.

Example 5.1.5. Let $\mathcal{K}$ be the boundary of a tetrahedron, which is a pure simplicial complex of dimension 2 consisting of four triangular facets F_1, F_2, F_3, F_4. Any linear ordering of these facets forms a shelling.

- Adding F_1 gives a contractible space (a single triangle).

- Adding F_2 attaches it to F_1 along a single edge. The space remains contractible.
- Adding F_3 attaches it to $F_1 \cup F_2$ along two edges. The space is still contractible (homeomorphic to a 2-disk).
- Finally, adding F_4 attaches it along its *entire* boundary (three edges) to $F_1 \cup F_2 \cup F_3$.

Thus, F_4 is the only homology facet. As expected, attaching F_4 closes the void, yielding a space homeomorphic to the 2-sphere S^2.

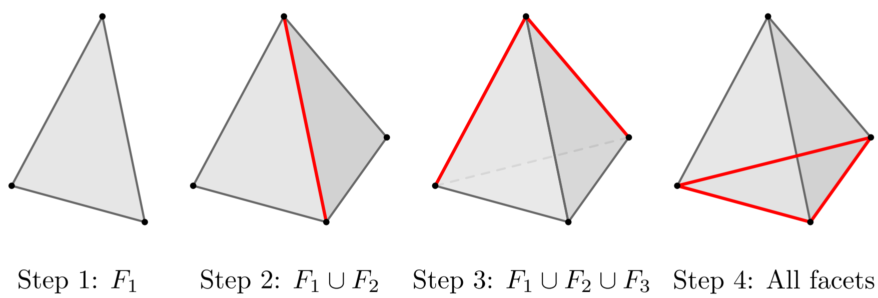


Figure 5.2: The shelling process of a tetrahedron boundary. At each step k, the red edges denote the intersection of the new facet F_k with the previous subcomplex $\mathcal{K}_k$. The addition of the final facet F_4 closes the internal void, acting as a homology facet and granting the space the homotopy type of S^2.

With this terminology in place, we can state the Björner–Wachs theorem, which completely characterizes the homotopy type of any shellable complex. We will prove this theorem in Section 5.2.

Theorem 5.1.6 (Björner and Wachs [BW96, Theorem 4.1])**.** Let $\mathcal{K}$ be a shellable simplicial complex of dimension d. Then its geometric realization $\|\mathcal{K}\|$ has the homotopy type of a wedge of spheres, where, for each integer i, the number of i-dimensional spheres S^i in the wedge is exactly the number of i-dimensional homology facets in the shelling.

Remark 5.1.7. Combining Theorem 3.5.6 and Theorem 5.1.6, the reduced integral homology of a finite shellable simplicial complex $\mathcal{K}$ of dimension d is completely and explicitly determined:

$$\widetilde{H}_i(\mathcal{K};\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{r_i} & \text{if } 0 \leq i \leq d, \\ 0 & \text{otherwise,} \end{cases}$$

where r_i is the number of i-dimensional homology facets in any valid shelling of $\mathcal{K}$. Consequently, the reduced i-th Betti number of the complex is precisely equal to r_i.

Example 5.1.8. We can readily illustrate Theorem 5.1.6 by revisiting the shellable complex from Example 5.1.2. In that construction, the shelling process contains exactly one 1-dimensional homology facet, which occurs when the final edge closes the 1-dimensional loop.

According to Theorem 5.1.6, this single homology facet dictates that the complex must have the homotopy type of a circle S^1. Indeed, by collapsing the contractible 2-simplex F, it is evident that the remaining space is homotopy equivalent to S^1.

5.2 The Björner–Wachs Theorem via the Rearrangement Lemma

Before diving into the technical details, we outline the general strategy of the proof of Theorem 5.1.6.

Let $\mathcal{K}$ be a shellable simplicial complex. Suppose $\mathcal{K}$ admits a shelling F_1, F_2, ..., F_n. Our goal is to rearrange this sequence into a new shelling where the first j facets $(F_1, \ldots, F_j)$ are *not* homology facets, and the remaining ones $(F_{j+1}, \ldots, F_n)$ are all homology facets, for some $1 \leq j \leq n$. By doing so, as we attach the first j facets, we incrementally construct a contractible simplicial complex at each step, as suggested by Example 5.1.5. Subsequently, each homology facet attached at the end effectively creates a sphere, since the entirety of its boundary is contracted to a point within the already assembled contractible subcomplex.

The approach we adopt to formalize this result follows the original work of Björner and Wachs in [BW96].

Building upon this, we can associate to each facet in a shelling a specific set of vertices that completely encodes how it attaches to the previous ones:

Definition 5.2.1 (Restriction map)**.** Given a shelling F_1, ..., F_n of $\mathcal{K}$ with successively generated subcomplexes $\mathcal{K}_j = \bigcup_{i=1}^{j-1} \langle F_i \rangle$, we define the *restriction* of the facet F_k as the set of vertices:

$$R(F_k) = \{x \in F_k \mid F_k \setminus \{x\} \in \mathcal{K}_k\}.$$

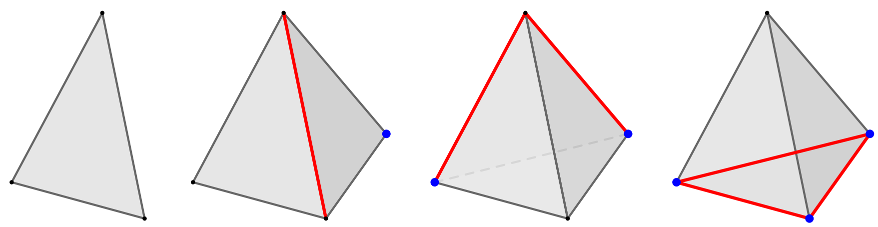
Example 5.2.2. Let us revisit the shelling of the tetrahedron boundary from Example 5.1.5, consisting of four 2-dimensional facets F_1, F_2, F_3, F_4. We compute the restriction $R(F_k)$ at each step by checking which codimension-1 faces (i.e., the edges of the current triangle) are already present in the previous subcomplex $\mathcal{K}_k$.

(i.) The subcomplex $\mathcal{K}_1$ is empty. Thus, $R(F_1) = \emptyset$.

(ii.) The facet F_2 attaches to F_1 along a single edge. Let $x \in F_2$ be the unique vertex opposite to this shared edge. By definition, the edge $F_2 \setminus \{x\}$ is precisely the one contained in $\mathcal{K}_2 = \langle F_1 \rangle$. Thus, $R(F_2) = \{x\}$.

(iii.) The facet F_3 attaches to $F_1 \cup F_2$ along two edges. Therefore, there are two distinct vertices, say $y, z \in F_3$, whose opposite edges lie in the past complex $\mathcal{K}_3$. Thus, $R(F_3) = \{y, z\}$.

(iv.) The final facet F_4 attaches along its *entire* boundary (three edges) to the past complex $\mathcal{K}_4$. For every vertex $v \in F_4$, the opposite edge $F_4 \setminus \{v\}$ is already present. Thus, $R(F_4) = F_4$.



Step 1: F_1 Step 2: $F_1 \cup F_2$ Step 3: $F_1 \cup F_2 \cup F_3$ Step 4: All facets

Figure 5.3: The shelling process of a tetrahedron boundary. At each step k, the red edges denote the intersection of the new facet F_k with the previous subcomplex $\mathcal{K}_k$. The vertices in F_k opposite to these edges form the restriction map $R(F_k)$, highlighted with solid blue dots.

Remark 5.2.3. Let $\mathcal{K}_k = \bigcup_{i=1}^{k-1} \langle F_i \rangle$ be the subcomplex built before the k-th step of the shelling. The restriction set $R(F_k)$ cannot be a face of any facet of $\mathcal{K}_k$; namely, it is not contained in any F_i for $i < k$.

Indeed, suppose by contradiction that $R(F_k) \subseteq F_i$ for some $i < k$. Then we would have $R(F_k) \subseteq \mathcal{K}_k \cap F_k$. By the shelling condition, the intersection $\mathcal{K}_k \cap F_k$ is pure of codimension 1, meaning it is the union of maximal proper faces of the form $F_k \setminus \{x\}$ for certain vertices $x \in F_k$. Thus, $R(F_k)$ must be entirely contained in at least one such face:

$$R(F_k) \subseteq F_k \setminus \{x\} \in \mathcal{K}_k.$$

This inclusion logically forces $x \notin R(F_k)$. However, by the exact definition of the restriction set, $R(F_k)$ consists precisely of all such vertices x, yielding $x \in R(F_k)$, leading to a contradiction.

Moreover, every proper subset $A \subsetneq R(F_k)$ is already a face of $\mathcal{K}_k$: since A is a proper subset, there exists some $x \in R(F_k)$ such that $A \subseteq R(F_k) \setminus \{x\}$. Consequently, $A \subseteq F_k \setminus \{x\}$, which is contained in $\mathcal{K}_k$. Thus, $R(F_k)$ is the minimal face of F_k not contained in $\mathcal{K}_k$.

As an immediate consequence of Remark 5.2.3, we obtain the following lemma.

Lemma 5.2.4. Let $F_1, F_2, \dots, F_n$ be a shelling of $\mathcal{K}$ with restriction map R. Let $\mathcal{K}_k = \bigcup_{i=1}^{k-1} \langle F_i \rangle$ be the subcomplex constructed prior to the k-th step of the shelling. Then,

$$\mathcal{K}_{k+1} = \mathcal{K}_k \sqcup [R(F_k), F_k],$$

where $[R(F_k), F_k]$ denotes the interval of faces σ satisfying $R(F_k) \subseteq \sigma \subseteq F_k$.

Proof. By definition, the complex generated up to the $(k+1)$-th step is given by the union:

$$\mathcal{K}_{k+1} = \mathcal{K}_k \cup \langle F_k \rangle.$$

To express this as a disjoint union, we must characterize the faces of F_k that are not already contained in $\mathcal{K}_k$. Let $\sigma \subseteq F_k$ be a face. By Remark 5.2.3, $R(F_k)$ is the unique minimal face of F_k not contained in $\mathcal{K}_k$. Therefore, $\sigma \notin \mathcal{K}_k$ if and only if it contains this minimal face, which is strictly equivalent to the condition $R(F_k) \subseteq \sigma \subseteq F_k$.

This implies that the set of new faces added at the k-th step is exactly the interval $[R(F_k), F_k]$. Consequently, the union can be naturally rewritten as a disjoint union:

$$\mathcal{K}_{k+1} = \mathcal{K}_k \sqcup [R(F_k), F_k],$$

which concludes the proof. □

Remark 5.2.5. By recursively applying Lemma 5.2.4 and noting that $\mathcal{K}_1 = \emptyset$, it is straightforward to show by induction that the subcomplex $\mathcal{K}_k$ globally decomposes into a disjoint union of intervals:

$$\mathcal{K}_k = \bigsqcup_{i=1}^{k-1} [R(F_i), F_i].$$

Remark 5.2.6. A facet F_k is fixed by the restriction map if $R(F_k) = F_k$. By Definition 5.2.1, this equality holds if and only if for every vertex $x \in F_k$, the opposite codimension-1 face $F_k \setminus \{x\}$ is already contained in the previous subcomplex $\mathcal{K}_k$. Geometrically, this means that the *entire boundary* of F_k is glued to the past complex. Therefore, the facets fixed by the map R are exactly the homology facets of the shelling, as per Definition 5.1.4.

Notice how $R(F_4) = F_4$ in Example 5.2.2 perfectly aligns with our statement that F_4 is the only homology facet of this shelling, as per Remark 5.2.6.

Next, we state a lemma concerning the rearrangement of the homology facets introduced in the proof outline of Theorem 5.1.6.

Lemma 5.2.7 (Rearrangement lemma, [BW96, Lemma 2.7])**.** Let F_1, F_2, ..., F_n be a shelling of $\mathcal{K}$ with restriction map R. Let $F_{i_1}, F_{i_2}, \ldots, F_{i_n}$ be the rearrangement obtained by first taking all facets F such that $R(F) \neq F$ in their initially induced order, followed by all remaining facets in an arbitrary order. Then, this rearrangement is also a valid shelling of $\mathcal{K}$ with the exact same restriction map R.

Proof. By Remark 5.2.6, a facet F_k is a homology facet if and only if $R(F_k) = F_k$. According to Lemma 5.2.4, the set of new faces added to the complex at the k-th step is precisely the interval $[R(F_k), F_k]$. Thus, if F_k is a homology facet, this interval collapses to the single element $\{F_k\}$. This implies that a homology facet contributes *no proper faces* to the evolving complex; it solely adds the maximal simplex itself.

Now, consider the rearranged sequence G_1, G_2, ..., G_n, where the first m facets are the non-homology facets (maintaining their original relative order), and the remaining facets are the homology ones (in any arbitrary order).

For any non-homology facet $G_b = F_q$ (with $b \leq m$), its intersection with the subcomplex generated by the newly formed prefix G_1, ..., G_{b-1} must consist entirely of proper faces (since no two distinct facets can contain each other). Because the homology facets that originally preceded F_q contributed no proper faces to the complex, their removal from the prefix does not alter the pool of available lower-dimensional faces.

Consequently, the intersection of G_b with its new prefix is absolutely identical to its intersection with its original prefix. Therefore, the shelling condition is seamlessly satisfied, and the restriction set $R(G_b)$ remains strictly unchanged.

Finally, consider any homology facet G_j placed at the end of the sequence (with $j > m$). In the original shelling, the entire boundary of each homology facet was already generated by previous steps. Since the union of all non-homology facets $\bigcup_{a=1}^{m} \langle G_a \rangle$ contains every proper face ever generated in the entire shelling process, the complete boundary ∂G_j is guaranteed to be contained in this new prefix. Therefore, attaching G_j at the end trivially satisfies the shelling condition (the intersection is the entire boundary, which is inherently pure of codimension 1), and it enforces $R(G_j) = G_j$, thus preserving the restriction map. □

With these tools, we are now ready to prove the Björner–Wachs theorem.

Proof of Theorem 5.1.6. Let $\mathcal{H} = \{F \in \mathcal{K} \mid R(F) = F\}$ be the set of all homology facets as per Remark 5.2.6, and let $\mathcal{K}^*$ be the subcomplex generated by the remaining non-homology facets. By Lemma 5.2.7, the induced order of the facets in $\mathcal{K}^*$ forms a valid shelling of $\mathcal{K}^*$, possessing the same restriction map R.

Let F_k be the k-th facet of $\mathcal{K}^*$ in this new shelling, and set $\mathcal{K}^*_k = \bigcup_{i=1}^{k-1} \langle F_i \rangle$. Because F_k is not a homology facet, we have $R(F_k) \subsetneq F_k$. Therefore, the face $R(F_k)$ is a proper face of F_k. Moreover $R(F_k)$ is not contained in any other facet by Remark 5.2.3.

Removing $R(F_k)$ and all faces containing it corresponds exactly to removing the interval $[R(F_k), F_k]$. By Lemma 5.2.4, this operation collapses the complex, yielding $\mathcal{K}^*_{k+1} \setminus [R(F_k), F_k] = \mathcal{K}^*_k$. Geometrically, because $R(F_k) \subsetneq F_k$, $R(F_k)$ acts as a "free face"; this allows us to continuously retract F_k by "pushing" its volume through the open face $R(F_k)$ and flattening it onto the remaining boundary glued to $\mathcal{K}^*_k$. Topologically, this makes $\mathcal{K}^*_k$ a deformation retract of $\mathcal{K}^*_{k+1}$, meaning they are homotopy equivalent. By induction, since the base case is a single simplex (which is contractible), we conclude that the entire subcomplex $\mathcal{K}^*$ is contractible.

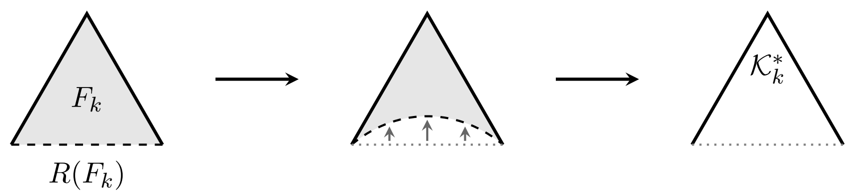


Figure 5.4: Geometric representation of an elementary collapse on a 2-simplex. The free face $R(F_k)$ (dashed) allows for the continuous retraction of the volume of F_k without altering its homotopy type, flattening it completely onto the remaining boundary glued to $\mathcal{K}^*_k$.

Finally, the original space $\mathcal{K}$ is obtained from the contractible space $\mathcal{K}^*$ by attaching the geometric simplices corresponding to the homology facets in $\mathcal{H}$, each along its entire boundary.

We invoke a standard result in algebraic topology: passing to a quotient space modulo a contractible subspace does not alter the homotopy type in these hypotheses (see [Hat02, Proposition 0.17]). Thus, when the subcomplex $\|\mathcal{K}^*\|$ is contracted to a single distinguished point, each attached i-dimensional cell from $\mathcal{H}$ is deformed into an i-sphere S^i. Outside this common basepoint, these spheres are topologically independent. Therefore, the resulting space is a wedge of spheres, with one sphere for each homology facet of the corresponding dimension. $\square$

5.3 Edge-Lexicographic Labelings and EL-Shellability

To overcome the difficulty of explicitly constructing a shelling order for $\Delta(P)$, we introduce a powerful labeling technique on the Hasse diagram of the poset. This method, formalized by Björner and Wachs [Bjö80], allows us to deduce the shellability of the entire complex simply by looking at the labels on the edges of the poset.

Let $E(P)$ denote the set of covering relations in a poset P, geometrically representing the edges of its Hasse diagram.

Definition 5.3.1 (Edge-labeling)**.** An *edge-labeling* of a bounded poset P is a map $\lambda\colon E(P) \to \mathbb{Z}$.

Any chain $C = \{x_0 < \cdots < x_k\}$ naturally reads a sequence of labels:

$$\lambda(C) = \lambda(x_0, x_1)\,\lambda(x_1, x_2)\,\cdots\,\lambda(x_{k-1}, x_k).$$

We say that C is *strictly increasing* if

$$\lambda(x_0, x_1) < \lambda(x_1, x_2) < \cdots < \lambda(x_{k-1}, x_k).$$

Furthermore, we can compare two chains lexicographically by comparing their label sequences, as per Definition 1.3.14.

Definition 5.3.2 (EL-labeling)**.** An edge-labeling $\lambda\colon E(P) \to \mathbb{Z}$ is called an *Edge-Lexicographic labeling* (or *EL-labeling*) if for every interval $[x, y]$ in P, the following two conditions hold:

(i.) There exists a unique strictly increasing maximal chain C in $[x, y]$.

(ii.) The sequence $\lambda(C)$ is strictly lexicographically smaller than the label sequence of any other maximal chain in $[x, y]$.

If a poset admits an EL-labeling, it is called *EL-shellable*.

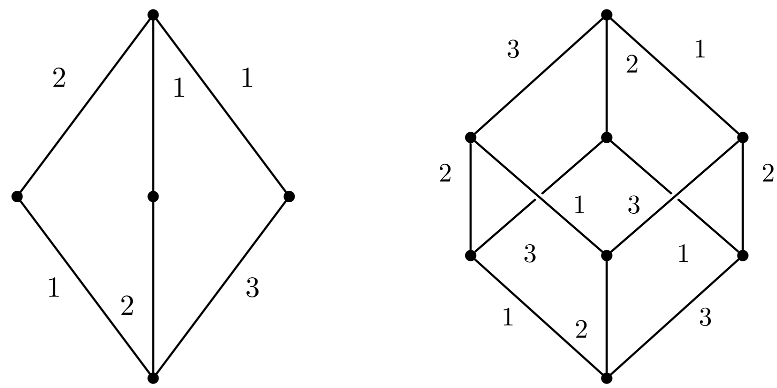


Figure 5.5: Two examples of posets admitting valid EL-labelings.

Remark 5.3.3. Note that the map assigning each maximal chain its sequence of edge labels via an EL-labeling λ is not necessarily injective. That is, distinct maximal chains C_1 and C_2 may yield identical label sequences, meaning $\lambda(C_1) = \lambda(C_2)$. Thus, to establish a strict total order on the set of all maximal chains, we must arbitrarily break any such ties. We formalize this concept in the following definition.

Definition 5.3.4 (λ-compatible linear order)**.** Let P be a poset equipped with an edge-labeling $\lambda\colon E(P) \to \mathbb{Z}$. A linear order $\ll$ on the set of maximal chains of P is said to be λ-*compatible* if, for any two maximal chains m and m' of P, the following implication holds:

$$\lambda(m) <_L \lambda(m') \implies m \ll m',$$

where $<_L$ denotes the strict lexicographic order on the associated label sequences.

This theory culminates in the use of EL-labelings as a simple yet powerful tool for investigating shellability, as we will show by proving Theorem 5.3.7. However, to rigorously manipulate the shelling order without violating shellability, we first introduce a more operative, but equivalent, definition of a shelling:

Lemma 5.3.5 ([BW96, Lemma 2.3])**.** Let $\mathcal{K}$ be an abstract simplicial complex. An ordering F_1, F_2, ..., F_n of the facets of $\mathcal{K}$ is a shelling if and only if for every i and j with $1 \le i < j \le n$, there exists an index k with $1 \le k < j$ and a vertex $x \in F_j$ such that the following combinatorial condition holds:

$$\langle F_i \rangle \cap \langle F_j \rangle \subseteq \langle F_k \rangle \cap \langle F_j \rangle = \langle F_j \setminus \{x\} \rangle.$$

Proof. Let $\mathcal{K}_j = \bigcup_{r=1}^{j-1} \langle F_r \rangle$ denote the subcomplex generated by the first $j-1$ facets. Recall that the definition of a shelling requires the intersection $\mathcal{K}_j \cap \langle F_j \rangle$

to be pure of dimension $\dim F_j - 1$ for all j. We prove the two implications separately.

$\boxed{\Longrightarrow}$ Assume the ordering is a shelling. Let $1 \leq i < j \leq n$. The intersection $\langle F_i \rangle \cap \langle F_j \rangle$ is a subcomplex of $\mathcal{K}_j \cap \langle F_j \rangle$. Since the ordering is a shelling, $\mathcal{K}_j \cap \langle F_j \rangle$ is pure of dimension $\dim F_j - 1$. This implies that $\langle F_i \rangle \cap \langle F_j \rangle$ is contained within some maximal face F of dimension $\dim F_j - 1$.

Moreover, any face of $\langle F_j \rangle$ of dimension $\dim F_j - 1$ is obtained by removing a single vertex, and thus F can be written as $F_j \setminus \{x\}$ for some $x \in F_j$. Since F belongs to the intersection $\mathcal{K}_j \cap \langle F_j \rangle$, it must be fully contained in at least one previously attached facet F_k for some $k < j$. Therefore we get:

$$\langle F_i \rangle \cap \langle F_j \rangle \subseteq \langle F_k \rangle \cap \langle F_j \rangle = F = \langle F_j \setminus \{x\} \rangle.$$

$\boxed{\Longleftarrow}$ Assume the combinatorial condition holds. We must show that for any j, the intersection $\mathcal{K}_j \cap \langle F_j \rangle$ is pure of dimension $\dim F_j - 1$.

Let σ be an arbitrary face in $\mathcal{K}_j \cap \langle F_j \rangle$. By the definition of the generated subcomplex, there exists some $i < j$ such that $\sigma \in \langle F_i \rangle \cap \langle F_j \rangle$. By our assumed condition, there exists an index $k < j$ and a vertex $x \in F_j$ such that:

$$\sigma \in \langle F_i \rangle \cap \langle F_j \rangle \subseteq \langle F_k \rangle \cap \langle F_j \rangle = \langle F_j \setminus \{x\} \rangle.$$

This tells us two critical things. First, the face $\langle F_j \setminus \{x\} \rangle$ is contained in $\langle F_k \rangle$, meaning it is entirely part of the past complex $\mathcal{K}_j$. Second, our arbitrary face σ is contained within $\langle F_j \setminus \{x\} \rangle$.

Since $\langle F_j \setminus \{x\} \rangle$ has dimension $\dim F_j - 1$, we have shown that every face in the intersection $\mathcal{K}_j \cap \langle F_j \rangle$ is contained in a face of dimension $\dim F_j - 1$ that also belongs to the intersection. Thus, the intersection is pure of dimension $\dim F_j - 1$, fulfilling the definition of a shelling.

□

Remark 5.3.6. The combinatorial condition $\langle F_i \rangle \cap \langle F_j \rangle \subseteq \langle F_k \rangle \cap \langle F_j \rangle = \langle F_j \setminus \{x\} \rangle$ from Lemma 5.3.5 translates the topological requirement of pure intersections into a precise geometric constraint. The equality $\langle F_k \rangle \cap \langle F_j \rangle = \langle F_j \setminus \{x\} \rangle$ guarantees that the newly introduced facet F_j shares an entire codimension-1 face with at least one previously attached facet F_k.

However, the mechanism of the lemma lies in the inclusion relation: it dictates that any other incidental contact with an earlier facet F_i must be absorbed within this primary attachment wall. Geometrically, this prevents the new facet from touching the existing complex along isolated, lower-dimensional intersections. In essence, the condition ensures that every "weak" intersection is strictly a subface of a "strong", codimension-1 intersection.

Theorem 5.3.7 (Björner and Wachs [BW96, Theorem 5.8])**.** Let P be a bounded poset equipped with an EL-labeling λ, and let $\ll$ be a λ-compatible linear ordering of the maximal chains of P. Then the following statements hold:

(i.) The ordering $\ll$ defines a shelling of the order complex $\Delta(P)$.

(ii.) The restriction of $\ll$ to the maximal chains of the proper part $\overline{P}$ induces a shelling of $\Delta(\overline{P})$.

(iii.) The homology facets of $\Delta(\overline{P})$ are precisely those corresponding to weakly decreasing maximal chains in $\overline{P}$ with respect to $\ll$.

While Björner originally proved this result in [Bjö80] for pure posets, we present the proof for non-pure posets.

Proof. Let M be the set of all maximal chains, which constitute the facets of $\Delta(P)$. Let λ be an EL-labeling on P. Let $\ll$ be any λ-compatible linear order on M. We prove (i.) by proving that $\ll$ is a valid shelling order.

By Lemma 5.3.5, we must show that for any two maximal chains $k, m \in M$ such that $k \ll m$, there exists a third maximal chain $h \in M$ satisfying three conditions:

(i.) $h \ll m$,

(ii.) $k \cap m \subseteq h \cap m$,

(iii.) $h \cap m = m \setminus \{m_e\}$ for some vertex $m_e \in m$.

Let us write the two maximal chains explicitly using covering relations:

$$k = \left\{\hat{0} = k_0 \lessdot k_1 \lessdot \cdots \lessdot k_r = \hat{1}\right\},$$
$$m = \left\{\hat{0} = m_0 \lessdot m_1 \lessdot \cdots \lessdot m_t = \hat{1}\right\}.$$

Suppose $k \ll m$. Let d be the greatest integer such that the two chains share the exact same bottom segment: $m_i = k_i$ for all $0 \leq i \leq d$. Let m_g be the lowest vertex in m strictly above m_d that also belongs to k, so that $m_g = k_f$ for some integer $f > d$. Since both chains must end at $\hat{1}$, such a vertex is guaranteed to exist. By this construction, the chains split immediately after m_d and reunite at m_g. Thus, $g - d \geq 2$, and for all intermediate indices $d < i < g$, the vertices of m are strictly disjoint from k. This configuration is illustrated in Figure 5.6.

Let c_{bottom} be the shared lower chain $m_0 \lessdot \cdots \lessdot m_d$ and consider now the interval $[m_d, m_g]$. Inside this interval, the chain m follows the path $c_m = m_d \lessdot \cdots \lessdot m_g$, while k follows $c_k = k_d \lessdot \cdots \lessdot k_f$.

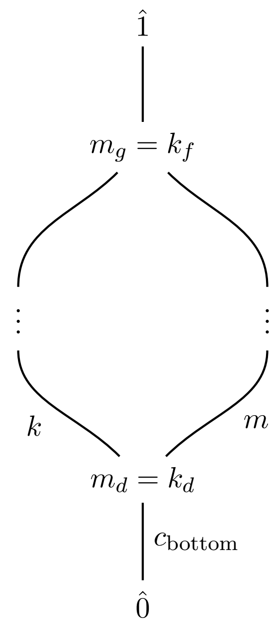


Figure 5.6: The behavior of chains k and m, illustrating their shared bottom segment c_{bottom}, their divergence after m_d, and their subsequent reunion at m_g.

By Definition 5.3.2, the interval $[m_d, m_g]$ must contain a *unique* maximal chain with strictly increasing labels, and this unique chain is lexicographically strictly smaller than any other chain in the interval. If c_m were this unique increasing chain, we would have $\lambda(c_m) <_L \lambda(c_k)$. Since k and m share the identical root c_{bottom} and diverge at the very next step, their lexicographic relationship is completely determined by the first steps of c_m and c_k. This implies that the full label sequence of m is lexicographically smaller than k, meaning $\lambda(m) <_L \lambda(k)$. This forces $m \ll k$, which contradicts our initial assumption that $k \ll m$.

Because c_m cannot be an increasing chain, there exists an index e, with $d < e < g$, such that the sequence fails to strictly increase across the vertex m_e:

$$\lambda(m_{e-1}, m_e) \geq \lambda(m_e, m_{e+1}).$$

Now, we focus on the smaller sub-interval $[m_{e-1}, m_{e+1}]$. The subchain $m_{e-1} \lessdot m_e \lessdot m_{e+1}$ is not an increasing label sequence, so it is not the lexicographically first path in this small interval. Therefore, there exists an alternative increasing chain

$$c_{\text{alt}} = m_{e-1} \lessdot x_1 \lessdot \cdots \lessdot x_s \lessdot m_{e+1},$$

whose label sequence comes strictly earlier in the lexicographical order. The replacement of m_e by this alternative path is schematically shown in Figure 5.7.

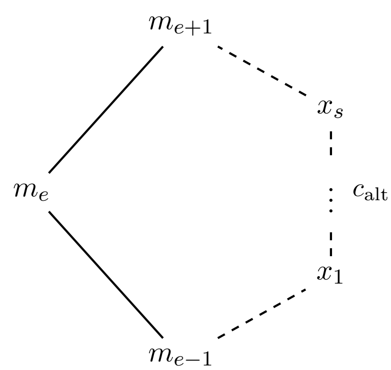


Figure 5.7: The local sub-interval $[m_{e-1}, m_{e+1}]$ containing the descent via m_e and the alternative lexicographically smaller path c_{alt}.

We construct the new maximal chain h by taking the original chain m and replacing the single vertex m_e with the alternative path c_{alt}:

$$h = \{\hat{0} = m_0 \lessdot \cdots \lessdot m_{e-1} \lessdot x_1 \lessdot \cdots \lessdot x_s \lessdot m_{e+1} \lessdot \cdots \lessdot \hat{1}\}.$$

Because h matches m exactly up to m_{e-1} but takes a lexicographically smaller path immediately after, we have $\lambda(h) <_L \lambda(m)$, which satisfies our first condition $h \ll m$.

Next, we determine the intersection $h \cap m$. The chain h contains every vertex of m except m_e. Thus, exactly one vertex of m is missing in h:

$$h \cap m = m \setminus \{m_e\}.$$

Finally, since $d < e < g$, the vertex m_e resides purely in the "split" section and therefore $m_e \notin k$. This means that removing m_e from m does not remove any vertex that m shares with k. Consequently:

$$k \cap m \subseteq m \setminus \{m_e\} = h \cap m.$$

We have successfully found a chain $h \ll m$ that perfectly satisfies all the criteria for a valid shelling. A similar argument can be applied to $\Delta(\overline{P})$ to establish (ii.).

Finally, to prove (iii.), recall that in a shelled simplicial complex, a facet m is a homology facet if and only if its entire boundary is contained in the union of the preceding facets. For the order complex $\Delta(\overline{P})$, this means that for every vertex $m_i \in m$, there must exist an earlier maximal chain $h \ll m$ such that $h \cap m = m \setminus \{m_i\}$.

As established in our local construction (see Figure 5.7), such an earlier chain h exists if and only if the subchain $m_{i-1} \lessdot m_i \lessdot m_{i+1}$ fails to be strictly increasing, allowing us to replace m_i with a lexicographically smaller path. By Definition 5.3.2, the lexicographically first chain in any interval is strictly increasing and unique. Consequently, an earlier chain replacing m_i exists for *every* vertex m_i if and only if the label sequence of m has no strict ascents—that is, if and only if the labels along m are weakly decreasing. This concludes the proof. $\square$

Example 5.3.8. Consider the poset P whose Hasse diagram is illustrated in Figure 5.8 (left), equipped with an EL-labeling λ. The proper part $\overline{P} = \{a, b, c, d, e, f, g\}$ consists of seven elements partitioned into two ranks. Since the maximal chains in $\overline{P}$ have length 1, the order complex $\Delta(\overline{P})$ is a 1-dimensional simplicial complex—that is, a simple graph—with seven vertices and eight edges.

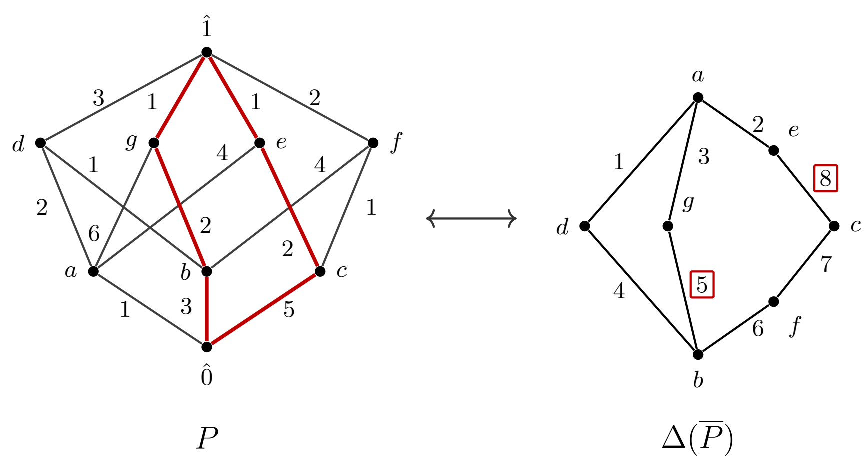


Figure 5.8: A poset P equipped with an EL-labeling (left) and its order complex $\Delta(\overline{P})$ (right). The facets F_5 and F_8 (highlighted by boxed labels) are the only homology facets in the induced shelling.

By part (ii.) of Theorem 5.3.7, the shelling order of $\Delta(\overline{P})$ is obtained by reading the label sequences of the eight maximal chains from $\hat{0}$ to $\hat{1}$ in lexicographical order and associating them with the corresponding facets:

- $c_1 = (1, 2, 3) \implies F_1 = \{a, d\}$,
- $c_2 = (1, 4, 1) \implies F_2 = \{a, e\}$,
- $c_3 = (1, 6, 1) \implies F_3 = \{a, g\}$,

- $c_4 = (3, 1, 3) \implies F_4 = \{b, d\}$,
- $c_5 = (3, 2, 1) \implies F_5 = \{b, g\}$,
- $c_6 = (3, 4, 2) \implies F_6 = \{b, f\}$,
- $c_7 = (5, 1, 2) \implies F_7 = \{c, f\}$,
- $c_8 = (5, 2, 1) \implies F_8 = \{c, e\}$.

Notice that exactly two chains, c_5 and c_8, have weakly decreasing label sequences along their paths in P. By part (iii.) of the theorem, the facets F_5 and F_8 are precisely the homology facets of $\Delta(\overline{P})$, which shows that the order complex has the homotopy type of a wedge of two spheres of dimension 1, namely $S^1 \vee S^1$.

Note. While EL-labelings offer a straightforward approach, they are not sufficient to characterize all shellable posets. A powerful generalization is provided by *Chain-Lexicographic shellability* (CL-shellability), also introduced by Björner and Wachs [BW96].

Unlike an EL-labeling, where the label of a covering relation $x \lessdot y$ is fixed globally, a CL-labeling assigns labels to the edges of a maximal chain based on the specific sequence used to reach them.

Bibliography

[BGS24] Andrea Bandini, Patrizia Gianni, and Enrico Sbarra. *Commutative algebra through exercises*. Springer, 2024.

[Bjö80] Anders Björner. Shellable and Cohen-Macaulay partially ordered sets. *Transactions of the american mathematical society*, 260(1):159–183, 1980.

[Bjö95] Anders Björner. Topological methods. *Handbook of combinatorics*, 2:1819–1872, 1995.

[BW96] Anders Björner and Michelle Wachs. Shellable nonpure complexes and posets. I. *Transactions of the American mathematical society*, 348(4):1299–1327, 1996.

[BW97] Anders Björner and Michelle Wachs. Shellable nonpure complexes and posets. II. *Transactions of the American Mathematical Society*, 349(10):3945–3975, 1997.

[DP02] Brian A Davey and Hilary A Priestley. *Introduction to lattices and order*. Cambridge university press, 2002.

[Hal98] Paul R. Halmos. *Naive Set Theory*. Undergraduate Texts in Mathematics. Springer, 1998.

[Hat02] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[Mat03] Jiří Matoušek. *Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry*. Springer, Berlin, Heidelberg, 2003.

[Mun84] James R. Munkres. *Elements of Algebraic Topology*. Addison-Wesley, Reading, MA, 1984.

[Mun00] James R. Munkres. *Topology*. Prentice Hall, Upper Saddle River, NJ, 2nd edition, 2000.

[Sta11] Richard P. Stanley. *Enumerative Combinatorics*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2 edition, 2011.

[Wac06] Michelle L. Wachs. Poset topology: Tools and applications, 2006. URL: `https://arxiv.org/abs/math/0602226`, `arXiv:math/0602226`.