DATA FITTING AND RECONSTRUCTION

MAIN TARGET:

Given a DATASET OF VALUES $(\overset{\in \mathbb{R}^d}{x_i}, \overset{\in \mathbb{R}}{y_i})_i \in \mathbb{R}^{d+1}$
s.t. we know that $\exists f: \mathbb{R}^d \to \mathbb{R}$ with $f(x_i) = y_i \ \forall i$,
we want to RECONSTRUCT such function f

$\Rightarrow$ 2 MAIN CASES:

1) STRUCTURED DATA ($\Leftrightarrow$ we have a MODEL for f) $\Rightarrow$
- INTERPOLATION (accurate data)
- LEAST SQUARES APPROXIMATION (noisy data)

2) UNSTRUCTURED DATA ($\Leftrightarrow$ we DON'T have a MODEL for f):

2.1) SPLINES (i.e. piecewise polynomials):

$\Rightarrow$ PIECEWISE CONSTANT SPLINES:

Data: (x_i, y_i), $i = 1, \dots, n$, $x_i \in \mathbb{R}$, $y_i \in \mathbb{R}$

Knots: $T = \{a = t_1 < \dots < t_m = b\}$

Reconstruction space:

$$\$_1(T) := \{ s(x) : s|_{[t_i, t_{i+1})} = c_i \in \mathbb{R}, \ 1 \leq i \leq m-1 \}$$

$$\dim \$_1(T) = m - 1$$

$\Rightarrow$ FOR INTERPOLATION WE MUST HAVE $m = n + 1$

THM.:

$(x_i, y_i)_{i=1}^{n}$ s.t. $x_i < x_{i+1} \ \forall i \Rightarrow \exists!\, s \in \$_1(T)$ s.t.
s interpolates (x_i, y_i) iff $x_i \in [t_i, t_{i+1}) \ \forall i$

THM.:

$(x_i, y_i)_{i=1}^{n}$ with x_i DISTINCT $\Rightarrow \exists!\, p_n(x)$ polynomial s.t.
$\deg p_n \leq n \wedge p_n(x_i) = y_i \ \forall i$

ERROR UPPER BOUND:

$$\sup_{x\in[a,b]} |f(x)-s(x)| \leq h \cdot \max_{x\in[a,b]} |f'(x)|$$

where $h :=$ maximum knot spacing

Examples:

Fundamental Lagrange Polynomials (Lagrange/Newton form)

Example: HAAR WAVELET

$\Rightarrow$ particular case of Fourier decomposition based on piecewise constant splines

Signals: $s(t)$ piecewise constant functions, $t \in \mathbb{R}$

Knots: $T = \mathbb{Z}$

$\Rightarrow s(t) = \sum_k s_k \cdot \phi(t-k)$, $\phi(t) := \mathbb{1}_{[0,1)}(t)$ UNIT STEP

$\Rightarrow$ Decompose $s(t)$ in a TREND and a DETAIL:

$$s(t) = T(t) + D(t)$$

$$T(t) := \sum_k \frac{s_{2k}+s_{2k+1}}{2} \phi\left(\frac{t}{2}-k\right),$$

$$D(t) := \sum_k \frac{s_{2k}-s_{2k+1}}{2} \psi\left(\frac{t}{2}-k\right)$$

where the AMPLITUDE is $t_k = \frac{s_{2k}+s_{2k+1}}{2}$

and $\psi(t) := \phi(2t) - \phi(2t-1)$

$\Rightarrow T(t)$ has knots $2\mathbb{Z} \Rightarrow$ REPEAT THIS DECOMPOSITION:

$$\begin{aligned} s(t) &= T_1 + D_1 \\ &= \{T_2 + D_2\} + D_1 \\ &= \{\{T_3 + D_3\} + D_2\} + D_1 \\ &= \dots = T_m + D_m + D_{m-1} + \dots + D_1 \end{aligned}$$

where $T_m(t)$ has knots $2^m\mathbb{Z}$

Number of Operations:

s has $N = 2^u$ coefficients $\Rightarrow$ T has $\frac{N}{2}$, D has $\frac{N}{2}$
$\Rightarrow$ in total, $\approx 2N$ operations

Orthogonality:

$\{\phi(t-k)\}_{k\in\mathbb{Z}}$ is an $\perp$ family,
$\{\psi(t-k)\}_{k\in\mathbb{Z}}$ is an $\perp$ family,
$\{\phi(t-k)\}_{k\in\mathbb{Z}}$, $\{\psi(t-k)\}_{k\in\mathbb{Z}}$ are $\perp$ w.r.t. each other
$\phi(t)$, $\phi(2t)$ are NOT $\perp$
$\{\psi(t-k)\}_{k\in\mathbb{Z}}$, $\{\psi(2t-k)\}_{k\in\mathbb{Z}}$ are $\perp$ w.r.t. each other
$\{\phi(t-k)\}_{k\in\mathbb{Z}}$, $\{\psi(2^j t-k)\}_{k\in\mathbb{Z}}$ are $\perp$ w.r.t. each other $\forall j \geq 0$ BUT NOT for $j<0$
$\Rightarrow$ The Haar Decomposition is an $\perp$ decomposition

Multiresolution Analysis:

Consider a signal s in $L^2(\mathbb{R})$:

$$s \in L^2(\mathbb{R}) \Leftrightarrow \{s_k\} \in \ell^2(\mathbb{Z})$$

$\Rightarrow$ define:

$$\phi_{jk}(t) = 2^{\frac{j}{2}}\phi(2^j t - k), \quad \psi_{jk}(t) = 2^{\frac{j}{2}}\psi(2^j t - k),$$

scale space i.e. piecewise constant L^2 signals with breaks at $2^{-j}\mathbb{Z}$ $\rightarrow$

$$V_j := \{s(t) = \sum_k a_{jk}\phi_{jk}(t) : \{a_{jk}\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\},$$

$$W_j := \{D(t) = \sum_k b_{jk}\psi_{jk}(t) : \{b_{jk}\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\}$$

detail space

$\Rightarrow L_2(\mathbb{R}) \supset \dots \supset V_u \supset V_{u-1} \supset \dots \supset V_1 \supset V_0 \supset V_{-1} \supset \dots \supset \{0\}$
and V_u has knots at $2^{-u}\mathbb{Z}$

$$\Rightarrow \underbrace{V_{j+1}}_{\text{SIGNAL}} = \underbrace{V_j}_{\text{TREND}} \oplus \underbrace{W_j}_{\text{DETAIL}} \Rightarrow V_0 = \bigoplus_{j=-\infty}^{1} W_j \Rightarrow \begin{cases} s(t) = \sum_{j,k} b_{jk}\psi_{jk}(t) \\ b_{jk} = \int_{\mathbb{R}} s(t)\psi_{jk}(t) \end{cases}$$

$\Rightarrow$ 2^j is the FREQUENCY, j is the FREQUENCY parameter, k is the LOCATION parameter and it determines the location of the support

Example: PIXELS ($\Rightarrow$ BIVARIATE WAVELETS)

$\Rightarrow$ The JPEG image compression is made using a wavelet (NOT the Haar wavelet). On every pixel you have a constant intensity (e.g. in a black and white picture)

$\Rightarrow$ can be described as:

$$p(x,y) = \sum_{i,j} p_{ij}\, \phi(x-i, y-j)$$

pixel intensity function   constant pixel intensity

where $\phi(x,y) = \mathbb{1}_{[0,1)^2}(x,y)$

$\Rightarrow$ we group 16 pixels in 4 megapixels

$\Rightarrow$ the trend is replacing these 4 megapixels by a megapixel with its amplitude

$\Rightarrow$ the detail will be (pixel − megapixel):

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} \longrightarrow \boxed{\frac{a+b+c+d}{4}}$$

$\Rightarrow$ Trend: $m = \dfrac{a+b+c+d}{4}$

$\Rightarrow$ Detail: $D(x,y) = a-m,\ b-m,\ c-m,\ d-m$ 3 DOF !!!

$\Rightarrow$ we have:

$$\left.\begin{array}{l} \text{Trend} = \frac{1}{4}(1,1,1,1) \\ x\text{-detail} = (1,1,-1,-1) \\ y\text{-detail} = (1,-1,1,-1) \\ xy\text{-detail} = (1,-1,-1,1) \end{array}\right\} \text{4 orthogonal vectors}$$

$\Rightarrow$ we can then write the Detail using this basis:

$$D = A \cdot x\text{-detail} + B \cdot y\text{-detail} + C \cdot xy\text{-detail}$$

$$A = \frac{a-b+c-d}{4}, \quad B = \frac{a+b-c-d}{4}, \quad C = \frac{a-b-c+d}{4}$$

$\Rightarrow$ PIECEWISE LINEAR SPLINES:

INTERPOLATION:

Data: (x_i, y_i), $i = 1, \dots, n$, $x_i \in \mathbb{R}$, $y_i \in \mathbb{R}$

Knots: $T = \{a = t_1 < \dots < t_n = b\}$

Reconstruction space:

$$\$_2(T) := \{ s \in \mathcal{C}[a,b] : s|_{[t_i, t_{i+1}]} = \text{linear},\ 1 \leq i \leq n-1 \}$$

$$\dim \$_2(T) = |T| = n$$

$\Rightarrow s(x) = (\mathcal{I}_2 f)(x) = \sum_{j=1}^{n} a_j H_j(x)$, $H_j(x)$ are the HAT FUNCTIONS

THM. (OPTIMALITY CONDITION):

$g(x)$ interpolant of $f(x)$ on T s.t. $g \in AC([a,b])$ and $g' \in L^2([a,b])$, then:

$$\int_a^b (g'(x))^2 dx \geq \int_a^b (s'(x))^2 dx$$

ERROR UPPER BOUND:

1) $\| f(x) - (\mathcal{I}_2 f)(x) \|_{[a,b]} \leq \frac{h^2}{8} \cdot \| f''(x) \|_{[a,b]}$

2) $\| f'(x) - (\mathcal{I}_2 f)'(x) \|_{[a,b]} \leq \frac{3h}{4} \cdot \| f''(x) \|_{[a,b]}$

where $h :=$ maximum knot spacing, $f \in \mathcal{C}^2([a,b])$

N.B. The formula for the error upper bound is SHARP

i.e. in general it's NOT possible to do any better

To find $\mathcal{I}_2 f$ we must find the coefficients a_j s.t.

$$\sum_{j=1}^{n} a_j H_j(x_i) = y_i, \quad 1 \leq i \leq n$$

$$\Leftrightarrow (H_j(x_i))_{ij} \vec{a} = \vec{y}, \quad 1 \leq i, j \leq n$$

THM. (SCHOENBERG - WHITNEY):

$\mathcal{I} = (H_j(x_i))_{ij} \in \mathbb{R}^{n \times n}$ is NON SINGULAR $\Leftrightarrow H_i(x_i) \neq 0 \ \forall i$

(i.e. iff $x_i \in (t_{i-1}, t_{i+1})$, $1 \leq i \leq n$)

BEST L^2 APPROXIMATION:

$\Rightarrow$ $I_2 f$ is almost NEVER the best approximation of f !!!

$\exists\, s^* \in \$_2(T)$ s.t. $\|f(x) - s^*(x)\|_{[a,b]} \leq \|f(x) - (I_2 f)(x)\|_{[a,b]}$

But $I_2 f$ is ALWAYS very good !!!

Data: (x_i, y_i), $i = 1, \dots, n$, $x_i \in \mathbb{R}$, $y_i \in \mathbb{R}$

Knots: $T = \{a = t_1 < \dots < t_n = b\}$

Reconstruction space:

$$\$_2(T) := \{ s \in \mathcal{C}[a,b] : s|_{[t_i, t_{i+1}]} = \text{linear},\ 1 \leq i \leq n-1 \}$$

$$\dim \$_2(T) = |T| = n$$

$$L^2([a,b]) = \{ f : \int_a^b f^2(x) < +\infty \},$$

$$\|f\|_{L^2} = \left(\int_a^b f^2(x)\,dx \right)^{\frac{1}{2}}, \quad \langle f, g \rangle_{L^2} = \int_a^b f(x) g(x)\,dx$$

We want to find $s^* \in \$_2$ s.t.:

$$\|f - s^*\|_{L^2[a,b]} \leq \|f - s\|_{L^2[a,b]} \quad \forall s \in \$_2(T)$$

$\Rightarrow$ $s(x) = (L_2 f)(x) = \sum_{j=1}^{n} a_j H_j(x)$, $H_j(x)$ are the HAT FUNCTIONS

The coefficients a_j are given by:

$$\vec{a} = G^{-1} \cdot \vec{b}$$

where $b_j = \int_a^b f(x) H_j(x)\,dx$, $G_{ij} = \int_a^b H_i(x) H_j(x)\,dx$, $G :=$ GRAM MATRIX

G is symmetric, tridiagonal, positive definite and diagonally dominant !!!

INTERPOLATION vs. BEST L^2 APPROXIMATION:

THM. (COMPARISON):

$\forall f \in \mathcal{C}([a,b])$ we have:

$$1)\ \|f - (I_2 f)\|_{[a,b]} \leq 2 \cdot E_2(f)$$

$$2)\ \|f - (L_2 f)\|_{[a,b]} \leq 4 \cdot E_2(f)$$

where $E_2(f) = \inf_{s \in \$_2(T)} \|f - s\|_{[a,b]}$ is the MINIMUM POSSIBLE ERROR

Example: RAYLEIGH-RITZ VARIATIONAL METHOD for Self-Adjoint ODE's

⇒ analogous to the gradient method for linear systems. It works for equations of the following form:

$$(p(x)y')' + q(x)y = r(x)$$

with Dirichlet Boundary Conditions $y(a) = y(b) = 0$

⇒ define the Differential Operator $\mathcal{L}y := (p(x)y')' + q(x)y$ and the inner product $\langle f, g\rangle := \int_a^b f(x)g(x)\,dx$

⇒ We have that $\mathcal{L}$ is self-adjoint:

$$\langle \mathcal{L}f, g\rangle = \langle f, \mathcal{L}g\rangle$$
$$\forall f, g \text{ satisfying the BC}$$

⇒ the solution y of the ODE is then the minimizer of:

$$\phi(u) := \frac{1}{2}\langle u, \mathcal{L}u\rangle - \langle u, r\rangle$$

N.B. We would need to solve this problem in a Sobolev Space where everything is well defined

⇒ To approximate the minimizer numerically, we replace $W^{1,p}$ with $\mathbb{S}_2(T)$ where we have 2^{nd}-order approximation of C^2 functions and 1^{st}-order approximation of their derivative. Take also into accout the BC !!!

Knots: $T = \{a = t_1 < \dots < t_n = b\}$

Reconstruction space:

$$\tilde{\mathbb{S}}_2(T) := \{s \in \mathbb{S}_2(T) : s(x) = \sum_{j=2}^{n-1} a_j H_j(x)\}$$

⇒ The minimizer becomes $\phi(s) := \frac{1}{2}\vec{a}^T G \vec{a} - \vec{a}^T \vec{b}$, where:

$$b_i = \int_a^b r(x)H_j(x)\,dx, \quad \vec{b} \in \mathbb{R}^{n-2}, \quad \vec{a} \in \mathbb{R}^{n-2}$$

$$G_{ij} = \int_a^b q(x)H_i(x)H_j(x) - p(x)H_i'(x)H_j'(x)\,dx, \quad G \in \mathbb{R}^{(n-2)\times(n-2)}$$

⇒ CUBIC SPLINES:

Data: (x_i, y_i), $i=1,\dots,n$, $x_i \in \mathbb{R}$, $y_i \in \mathbb{R}$

Knots: $T = \{a = t_1 < \dots < t_n = b\}$

Reconstruction spaces:

1) $V_4(T) := \{ s \in \mathcal{C}^1[a,b] : s|_{[t_i, t_{i+1}]} = \text{cubic}, \ 1 \leqslant i \leqslant n-1 \}$

$\dim V_4(T) = 2n$ (2 conditions at each interior knot)

2) $\$_4(T) := \{ s \in \mathcal{C}^2[a,b] : s|_{[t_i, t_{i+1}]} = \text{cubic}, \ 1 \leqslant i \leqslant n-1 \}$

$\dim \$_4(T) = n+2$ (3 conditions at each interior knot)

Conditions at the knots:

Existence of s^* is guaranteed in the following conditions:

1) NATURAL SPLINE-END CONDITIONS:

$$s''(a) = s''(b) = 0$$

2) COMPLETE SPLINE-END CONDITIONS:

$$s'(a) = f'(a), \ s'(b) = f'(b)$$

1) In $V_4(T)$ we impose 2 conditions at each knot:

$$s^*(t_j) = y_j, \ s^{*\prime}(t_j) = m_j \qquad 1 \leqslant j \leqslant n$$

⇒ cubic Hermite polynomial interpolation problem.

⇒ the values m_j are determined minimizing the BENDING ENERGY (i.e. the CURVATURE):

$$E(s^*) = \int_a^b \frac{(s^{*\prime\prime}(x))^2}{(1 + s^{*\prime}(x)^2)^3} \, dx \xrightarrow{\text{LINEARIZATION}} \int_a^b (s^{*\prime\prime}(x))^2 w(x) \, dx$$

where w is taken to be piecewise constant

⇒ in the end we find s^* s.t. $s^*(t_j) = y_j$ $j = 1, \dots, n$ and it minimizes the WEIGHTED ENERGY:

$$\min_{s^* \in V_4(T)} E_w(s^*), \quad E_w(s) := \sum_{j=1}^{n-1} w_j \int_{t_j}^{t_{j+1}} (s_j''(x))^2 \, dx$$

⇒ the OPTIMALITY CONDITION is:

$$E_w(s^*) \text{ is minimized} \iff w(x)(s^*(x)'') \in \mathcal{C}^0([a,b])$$

$\Rightarrow$ we get a $n \times n$ linear system to solve for m_j:

$$A\vec{m} = \vec{b}$$

where $\vec{b} = \vec{b}(y[t_j, t_{j+1}], w_j, h_j) \in \mathbb{R}^n$, $j = 1, \dots, n$ and $A = A(w_j, h_j) \in \mathbb{R}^{n \times n}$, $j = 1, \dots, n$ is tridiagonal, diagonally dominant and hence invertible !!!

2) In $\$_4(T)$ we can use the above system with $w(x) \equiv 1$ (minimizing therefore $\int_a^b (s^{*''}(x))^2 dx$) OR we can use the 2^{nd} derivative values as unknowns !!! We have:

$$s_j(x) = A_j + B_j(x-t_j) + C_j(x-t_j)^2 + D_j(x-t_j)^3$$

where $h_j = t_{j+1} - t_j$, $A_j = y_j$, $C_j = \frac{s_j''(t_j)}{2}$, $D_j = \frac{s_j''(t_{j+1}) - s_j''(t_j)}{6h_j}$, $B_j = \frac{y_{j+1} - y_j}{6h_j} - C_j h_j - D_j h_j^2$

$\Rightarrow$ we only need to find C_j and to do this we use the Peano Kernel Formula ...

The COMPLETE SPLINE INTERPOLANT has better approximation properties and is therefore written as $(I_4 f)$

$\Rightarrow$ $(I_4 f)(x) :=$ complete cubic spline interpolant in $\$_4(T)$

ERROR UPPER BOUND:

1) $\|f(x) - (I_4 f)(x)\|_{[a,b]} \leqslant \frac{h^4}{16} \cdot \|f^{(4)}(x)\|_{[a,b]}$

2) $\|f'(x) - (I_4 f)'(x)\|_{[a,b]} \leqslant \frac{3h^3}{8} \cdot \|f^{(4)}(x)\|_{[a,b]}$

3) $\|f''(x) - (I_4 f)''(x)\|_{[a,b]} \leqslant \frac{h^2}{2} \cdot \|f^{(4)}(x)\|_{[a,b]}$

where $h :=$ maximum knot spacing, $f \in \mathcal{C}^4([a,b])$

We also have that $(I_4 f)''(x) = (L_2 f)''(x) \in \$_2(T)$

Moreover, we have the following Optimality property:

THM. (OPTIMALITY PROPERTY OF I_4):

Given g s.t. $g' \in AC[a,b]$, $g(t_i) = f(t_i)$ $i=1,\dots,n$, $g'(a) = f'(a)$, $g'(b) = f'(b)$, then we have:

$$\int_a^b (g''(x))^2 dx \geq \int_a^b ((I_4 f)''(x))^2 dx$$

The NATURAL SPLINE INTERPOLANT has worse approximation properties BUT it does not require the additional information of $f'(a)$, $f'(b)$. It also has a worse error upper bound, in general. Still, we have the following optimality property:

THM. (OPTIMALITY PROPERTY OF NATURAL SPLINES):

Given g s.t. $g' \in AC[a,b]$, $g(t_i) = f(t_i)$ $i=1,\dots,n$, $g'' \in L_2([a,b])$, then we have:

$$\int_a^b (g''(x))^2 dx \geq \int_a^b (s''(x))^2 dx$$

where $s \in \$_4(T)$ is a natural cubic spline interpolant

We can also use a compromise: NOT-A-KNOT splines which are not as good as complete splines but don't introduce conflicts with the function to be approximated and don't require the derivative data. We simply require that $s_1(x), s_2(x)$ and $s_{n-2}(x), s_{n-1}(x)$ join together in such a way that they are actually the same cubic polynomial. This is equivalent to adding the 2 conditions $s_1'''(t_2) = s_2'''(t_2)$, $s_{n-2}'''(t_{n-1}) = s_{n-1}'''(t_{n-1})$

For not-a-knot cubic splines we have the following error upper bound: $\|f - s\|_{[a,b]} = \mathcal{O}(h^4)$ (i.e. same order as for complete splines !!!). The existence of such splines is guaranteed by the general Schoenberg-Whitney Theorem

$\Rightarrow$ B - SPLINES:

They generalize the Hat functions to higher degree
Knots: $T = \mathbb{Z}$

$\Rightarrow$ B-splines of order K at knot t_i is:

$$B_{i,K}(t) := (t_{i+K} - t_i) \cdot (x - t)_+^{K-1} [t_i, \dots, t_{i+K}]$$

PROPERTIES:

1) $H_i(t) = B_{i-2,2}(t)$, $B_{i,K}(t)$ are LINEARLY INDEPENDENT

2) $B_{i,K} \in \mathcal{C}^{K-2}(\mathbb{R})$ ($B_{i,K}(t)$ is a linear combination of $\mathcal{C}^{K-2}(\mathbb{R})$ piecewise polynomials of degree $K-1$ and knots t_j $\Rightarrow$ it is an order K, $\mathcal{C}^{K-2}$ spline)

3) $\operatorname{supp}(B_{i,K}(t)) \subset [t_i, t_{i+K})$ (i.e. $B_{i,K}(t) = 0 \ \forall t \notin [t_i, t_{i+K})$)

4) Recurrence Formula (NUMERICALLY STABLE, BETTER THAN DEFINITION)

$$B_{i,K}(t) = \frac{t - t_i}{t_{i+K-1} - t_i} B_{i,K-1}(t) + \frac{t_{i+K} - t}{t_{i+K} - t_{i+1}} B_{i+1,K-1}(t)$$

5) Positivity: $B_{i,K}(t) > 0 \quad \forall t \in (t_i, t_{i+K})$

6) Integral: $\int_{\mathbb{R}} B_{i,K}(t)\,dt = \frac{t_{i+K} - t_i}{K}$

7) Sum: $\sum_{-\infty}^{+\infty} B_{i,K}(t) \equiv 1 \quad \forall K$

8) Derivative: $\frac{d}{dt} B_{i,K}(t) = (K-1)\left[\frac{B_{i,K-1}(t)}{t_{i+K-1} - t_i} - \frac{B_{i+1,K-1}(t)}{t_{i+K} - t_{i+1}}\right]$

9) If $t_i = i \in \mathbb{Z}$, $B_{i,K}(t) = B_{0,K}(t-i)$

We set $N_K(t) := B_{0,K}(t)$, we have:

1) $B_{i,K}(t) = N_K(t-i)$

2) $\frac{d}{dt} N_K(t) = N_{K-1}(t) - N_{K-1}(t-1)$, $N_K(t) = \int_{t-1}^{t} N_{K-1}(x)\,dx$

3) $N_1(t) = \phi(t) = \mathbb{1}_{[0,1)}(t)$ from the Haar Wavelet

4) 2-Scale relation:

$$N_K(t) = \frac{1}{2^{K-1}} \sum_{j=0}^{K} \binom{K}{j} N_K(2t - j) \quad \text{for } K \geq 1$$

Using these $N_K(t)$ we can express a spline with knots at $\mathbb{Z}$ as a spline with knots at half the integers!!!

$$s(t) = \sum_{-\infty}^{+\infty} d_i^{(0)} N_K(t-i) = \dots = \sum_{-\infty}^{+\infty} d_i^{(1)} N_K(2t - i)$$

where $d_i^{(\mu)} = \sum_{s=-\infty}^{+\infty} a_{i-2s}\, d_s^{(\mu-1)}$ are the de-Boor control points from the Beziér curves, $a_s = \frac{1}{2^{k-1}}\binom{k}{s} \cdot \mathbb{1}_{[0,k]}(s)$
e.g. $k=3$ (Chaikin corner cutting for quadratic splines):

$$a_0 = \frac{1}{4}, \quad a_1 = \frac{3}{4}, \quad a_2 = \frac{3}{4}, \quad a_3 = \frac{1}{4}$$

$$\Rightarrow d_{2m}^{(1)} = \frac{1}{4} d_m^{(0)} + \frac{3}{4} d_{m-1}^{(0)}, \quad d_{2m+1}^{(1)} = \frac{3}{4} d_m^{(0)} + \frac{1}{4} d_{m-1}^{(0)}$$

$\| d_{i+1}^{(0)} - d_i^{(0)} \|_2 \leq M < +\infty$ then the sequence of polygons P_μ with vertices $d_i^{(\mu)}$ converges to the original curve $s(t)$
(In the case of closed polygons we simply define the additional control points by periodicity). The same method works for surfaces ($d_{i,s}^{(1)} = \sum_{m_1,m_2} a_{i-2m_1}, a_{s-m_2}\, d^{(0)}_{m_1,m_2}$)
We can also use B-splines to construct a basis for $\$_4(T)$ if T is the bi-infinite knot sequence:
$\Rightarrow$ distinct knots: $B_{i,k} \in \mathcal{C}^{k-2}$
$\Rightarrow$ repeated knots with multiplicity m_s: $B_{i,k} \in \mathcal{C}^{k-1-m_s}$
AND it must be $m_s \leq k$ (i.e. $t_{i+k} - t_i > 0 \ \forall i$)

INTERPOLATION

Data: (x_i, y_i), $i = 1, \dots, n$, $x_i \in \mathbb{R}$, $y_i \in \mathbb{R}$
Knots: $T = \{a = t_1 < \dots < t_n = b\}$ s.t. $t_s < t_{s+k} \ \forall s \in \mathbb{Z}$
$\Rightarrow$ find $s(t) = \sum_1^n a_s B_{s,k}(t)$ s.t. $s(x_i) = y_i$, $i = 1, \dots, n$
$\Rightarrow$ in the end we have to solve the linear system:

$$(B_{s,k}(x_i))_{is}\, \vec{a} = \vec{y}$$

THM. (SCHOENBERG-WHITNEY):
$(B_{s,k}(x_i))_{is}$ is NON singular $\Leftrightarrow B_{i,k}(x_i) \neq 0 \quad i = 1, \dots, n$

Moreover, given the knot sequence $T = \{t_1, \dots, t_n\}$, and the interpolation values y_i $i = 1, \dots, n$, $\exists! \ s^* \in \$_4(T)$ s.t. s satisfies the NOT-A-KNOT end conditions and it interpolates the data.

⇒ GRID INTERPOLATION:

Interpolation sites: $(x_i, y_j) \in \mathbb{R}^2$, $i=1,\dots,n$, $j=1,\dots,m$

Values: $z_{ij} \in \mathbb{R}$

⇒ we look for $s(x,y)$ s.t. $s(x_i,y_j) = z_{ij}$

To find s it's sufficient to choose an interpolating order on the grid (horizontal/vertical) and proceed as follows:

1) $s_j(x)$ interpolants of (x_i, z_{ij}), $i=1,\dots,n$, j fixed

$$\Rightarrow s_j(x) = \sum_{i=1}^{n} z_{ij} L_i^x(x) \quad \text{(Lagrange form)}$$

2) interpolants of $(y_j, s_j(a))$, $j=1,\dots,m$

$$\Rightarrow \sum_{j=1}^{m} s_j(a) L_j^y(y) \quad \text{(Lagrange form)}$$

$$\Rightarrow s(a,b) = \sum_{i=1}^{n}\sum_{j=1}^{m} z_{ij} L_i^x(a) L_j^y(b)$$

and the inverse order returns the same result !!!

⇒ UNIVARIATE THIN PLATE SPLINES:

They are a generalization of cubic splines, which will then be further generalized to the multivariate case

We want to minimize the bending energy given by

$E(f) = \|f\|^2 = \int_{\mathbb{R}} (f''(x))^2 dx$ (inner product: $\langle f,g\rangle = \int_{\mathbb{R}} f'' \cdot g'' dx$)

finding representatives for function evaluation i.e. $s_i(x)$ s.t. $\langle f, s_i\rangle = f(x_i)$ $i=1,\dots,n$ $\forall f \in BL_2(\mathbb{R})$ where $BL_2(\mathbb{R}) = \{f : \exists f'' \text{ a.e.} \wedge f'' \in L^2(\mathbb{R})\}$ are the BEPPO-LEVI SPACES.

⇒ the minimizer is a linear combination of normal vectors which express the constraints. The coefficients of the linear combination are obtained solving the linear system with the Gram Matrix of the normal vectors $G = (\langle \vec{n}_i, \vec{n}_j\rangle)_{ij} \in \mathbb{R}^{n\times n}$

$\Rightarrow$ we construct the evaluation representative starting by:

$$s(x) = \sum_{j=1}^{n} a_j |x - x_j|^3$$

which belongs to $BL_2(\mathbb{R})$ iff the BEPPO-LEVI CONDITIONS hold:

$$\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} a_j x_j = 0$$

$\Rightarrow$ if $s_a(x) = \frac{1}{12}|x-a|^3$ then $\langle f, s_a \rangle = f(a)$, so we use $s(x)$ defined as above to get the evaluator:

$$\hat{s}_{x_i}(x) = \sum_{j=1}^{n} a_j |x - x_j|^3$$

where $a_i = \frac{1}{12}$, $a_1 = -\frac{1}{12} l_1(x_i)$, $a_n = -\frac{1}{12} l_n(x_i)$, $a_j = 0$ and $l_i(x)$ is the i-th degree Lagrange polynomial

$\Rightarrow$ we have $\langle f, \hat{s}_{x_i} \rangle = f(x_i) - l_1(x_i) f(x_1) - l_n(x_i) f(x_n)$ to solve this problem (together with the fact that $\|\cdot\|^2$ is not a true norm) we proceed as follows: consider $bl_2(\mathbb{R}) = \{ f \in BL_2(\mathbb{R}) : f(x_1) = f(x_n) = 0 \}$

$\Rightarrow$ $\|\cdot\|^2$ is a true norm on $bl_2(\mathbb{R})$, therefore we take

$$\tilde{s}_{x_i} := \hat{s}_{x_i}(x) - (\hat{s}_{x_i}(x_1) l_1(x) + \hat{s}_{x_i}(x_n) l_n(x))$$

$$\Rightarrow \tilde{s}_{x_i} \in bl_2(\mathbb{R}) \wedge \langle f, \hat{s}_{x_i} \rangle = f(x_i) \quad \forall f \in bl_2(\mathbb{R})$$

$\Rightarrow$ To solve the original interpolation problem:

$$f \in BL_2(\mathbb{R}) \text{ s.t. } f(x_i) = y_i \wedge \|f\|^2 \text{ is min}$$

we replace f with

$$\hat{f}(x) = f(x) - (f(x_1) l_1(x) + f(x_n) l_n(x)) \in bl_2(\mathbb{R})$$

which interpolates the data $\hat{y}_i = y_i - (y_1 l_1(x_i) + y_n l_n(x_i))$

$\Rightarrow$ this yields the solution $\hat{s}(x) = \sum_{i=2}^{n-1} a_i \tilde{s}_{x_i}(x)$ where the coefficients a_i are to be found via the system

$$G\vec{a} = \vec{y}$$

where $G = (\tilde{s}_{x_j}(x_i))_{ij} \in \mathbb{R}^{(n-2)\times(n-2)}$. Finally, to get the interpolant of the original data, we use:

$$s(x) := \hat{s}(x) + (y_1 l_1(x) + y_n l_n(x))$$
$$\Rightarrow s(x) = \sum_{j=1}^{n} a_j |x - x_j|^3 + a_{n+1} + a_{n+2} x$$

To find a_j we impose the interpolation conditions and the Beppo-Levi conditions to get a $(n+2)\times(n+2)$ dimensional linear system. The resulting interpolation matrix is NON singular

$\Rightarrow$ BIVARIATE THIN PLATE SPLINES:

Same as above, we want to minimize the bending energy:

$$E(s) = \int_{\mathbb{R}^2} (\partial^2_{xx} s)^2 + 2(\partial^2_{xy} s)^2 + (\partial^2_{yy} s)^2 dA$$

$\Rightarrow BL_2(\mathbb{R}^2) = \{ f : \partial^2_{xx} f, \partial^2_{xy} f, \partial^2_{yy} f \in L^2(\mathbb{R}^2) \}$
with the inner product:

$$\langle f, g \rangle = \int_{\mathbb{R}^2} \partial^2_{xx} f \, \partial^2_{xx} g + 2\,\partial_{xy} f \, \partial_{xy} g + \partial^2_{yy} f \, \partial^2_{yy} g \, dA$$

s.t. $E(f) = \|f\|^2$

$\Rightarrow$ As in the univariate case, we look for $s_a(x,y)$ in $bl_2(\mathbb{R}^2)$ s.t. $\langle s_a, f \rangle = f(a) \;\; \forall f \in bl_2(\mathbb{R}^2)$

$\Rightarrow$ a distributional PDE arises, involving the ITERATED LAPLACIAN Δ^2, for which we look for RADIAL SOLUTIONS (given that Δ is rotational invariant).

First we solve $\Delta \tilde{s} = \delta_0$ which grants us:

$$\tilde{s}_0(x,y) = \frac{1}{2\pi}(\log r + 1)$$

Then we solve $\Delta s_a = \tilde{s}_a$ which leads us to the general solution:

$$s_a(x,y) = \frac{1}{8\pi} r_a^2 \log(r_a),$$
$$r_a = \sqrt{(x-a_1)^2 - (y-a_2)^2}$$

$\Rightarrow$ The energy minimizer will be a linear combination

$$s(x,y) = \sum_{i=1}^{n} a_i r_i^2 \log(r_i),$$
$$r_i = \sqrt{(x-x_i)^2 - (y-y_i)^2}$$

We have that $s(x,y) = \sum_{i=1}^{n} a_i r_i^2 \log(r_i) \in L_2(\mathbb{R}^2)$ iff the Beppo-Levi Conditions hold:

$$\sum_i^n a_i = \sum_i^n a_i x_i = \sum_i^n a_i y_i = 0$$

In this case, if $f \in BL_2(\mathbb{R}^2)$ we have that

$$\lim_{R\to+\infty} \int_{r\geq R} \partial_{xx}^2 f\, \partial_{xx}^2 s + 2\,\partial_{xy} f\, \partial_{xy} s + \partial_{yy}^2 f\, \partial_{yy}^2 s \, dA = 0$$

So we have that if $s(x,y) = \frac{1}{8\pi}\sum_1^n a_i r_i^2 \log(r_i)$ satisfies the Beppo-Levi Conditions in a_i, then for $f \in BL_2(\mathbb{R}^2)$ $\langle f, s\rangle = \sum_1^n a_i f(x_i, y_i)$. Now we repeat the steps of the univariate case:

1) Given $s_i(x,y) = \frac{1}{8\pi} r_i^2 \log(r_i)$ we find $\hat{s}_i(x,y)$ the modification of s with 3 moment conditions, which will have a subtraction of 3 other $s_j(x_i,y_i)$. We choose the first 3: s_1, s_2, s_3. So we need to add the condition that $(x_1,y_1), (x_2,y_2), (x_3,y_3)$ must NOT all be on the same line: they need to form a NON degenerate triangle.

2) $\hat{s}_i(x,y) = s_i(x,y) - \sum_{j}^{3} l_j(x_i,y_i)s_j(x,y) \in BL_2(\mathbb{R}^2)$

3) We reduce to:

$$bl_2(\mathbb{R}^2) = \{ f \in BL_2(\mathbb{R}^2) : f(x_j,y_j)=0,\ j=1,2,3 \}$$

Here $\|\cdot\|^2$ is a true norm

4) Just as the univariate case we set:

$$\tilde{s}_i = \hat{s}_i - \sum_{j=1}^{3} \hat{s}_i(x_j,y_j)\, l_j(x,y)$$

and reduce the data $\hat{z}_i = z_i - \sum_{j=1}^{3} z_j l_j(x_i,y_i)$

The optimal interpolant in $bl_2(\mathbb{R}^2)$ is then:

$$\tilde{s}(x,y) = \sum_{j=4}^{n} a_j \tilde{s}_j(x,y)$$

s.t. $\tilde{s}(x_i,y_i) = \hat{z}_i \quad i=4,\dots,n$

5) Finally, to get the Thin Plate Spline, we add the correction:

$$s(x,y) = \tilde{s}(x,y) + \sum_{j=1}^{3} z_j l_j(x,y)$$

which means that:

$$s(x,y) = \sum_{j=1}^{n} a_j r_j^2 \log(r_j) + a_{n+1} + a_{n+2}x + a_{n+3}y$$

with the interpolation conditions and the 3 Beppo-Levi Conditions. This is an $(n+3)\times(n+3)$ linear system whose Interpolation matrix is NON singular

ERROR UPPER BOUND:

If $f \in BL_2(\mathbb{R}^2)$, $\vec{x} \in \Delta(\vec{x}_j, \vec{x}_k, \vec{x}_e)$, then:

$$|f(\vec{x}) - s(\vec{x})| \leqslant \left[\frac{\log 3}{24\pi} E(f) \right]^{\frac{1}{2}} \cdot h$$

where h is the length of the longest triangle side

There are other functions which enable us to write the Thin Plate Spline interpolant as a linear combination of their translations: we only require the interpolation matrix to be NON singular !!!

1) POSITIVE DEFINITE FUNCTIONS

$\phi : \mathbb{R}^d \to \mathbb{C}$ is pos. def. on $\mathbb{R}^d$ if the matrices

$$\left(\phi(\vec{x}_i - \vec{x}_j)\right)_{ij=1,\dots,n} \in \mathbb{C}^{n\times n}$$

are pos. def. $\forall$ collection of distinct points $\{\vec{x}_i\}_{i=1}^{n}$ in $\mathbb{R}^d$, $n = 1, 2, 3, \dots$

Some examples of pos. def. functions:

1) THM. (BOCHNER):

$a : \mathbb{R}^d \to \mathbb{R}$ s.t. $a(\vec{\omega}) \geq 0$ $\forall \vec{\omega} \in \mathbb{R}^d$, $a(\vec{\omega}) > 0$ on a NON trivial cube in $\mathbb{R}^d$ and $a \in L_1(\mathbb{R})$, then $\phi(\vec{x}) := \hat{a}(\vec{x}) = \int_{\mathbb{R}^d} a(\vec{\omega}) e^{-i\vec{\omega}\cdot\vec{x}} d\vec{\omega}$ i.e. the Fourier Transform of a is a pos. def. function on $\mathbb{R}^d$

2) By the above Thm., Gaussian functions

$$f_\alpha(\vec{x}) = e^{-\alpha \|\vec{x}\|_2^2}, \quad \alpha > 0$$

are pos. def. functions on $\mathbb{R}^d$ $\forall d$. If we use the Gaussian functions for Thin Plate Interpolation we have that the Interpolation Matrix is (a constant times) the Gram Matrix of the Gaussian themselves: $\left(\exp(-\alpha\|\vec{x}_i - \vec{x}_j\|_2^2)\right)_{ij} = C\left(f_{2\alpha}(\vec{x} - \vec{x}_i)\right)_i$

These functions are linearly independent. We define the Gaussian Native Space:

$$N(f_\alpha) := \{ f \in L_2(\mathbb{R}^d) : \hat{f} \exp\left(\tfrac{1}{8\alpha}\|\vec{\omega}\|_2^2\right) \in L^2(\mathbb{R}^d) \}$$

with the inner product

$$\langle f, g \rangle_\alpha := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \frac{\hat{f}(\vec{\omega})\,\overline{\hat{g}(\vec{\omega})}}{\hat{f}_\alpha(\vec{\omega})} d\vec{\omega}$$

Then the Gaussian Interpolant

$$s(\vec{x}) = \sum_{i=1}^{n} a_i f_\alpha(\vec{x} - \vec{x}_i)$$

minimizes the energy $\|s\|_\alpha := \sqrt{\langle s, s \rangle_\alpha}$. Note that it might be needed to evaluate the Gaussian Interpolant at many points. Using a direct method would cost $O(n \times n')$ operations, so it's better to approximate the values using the Fast Gauss Transform which costs only $O(n + n')$. Define now $\phi(f, \vec{c}) = f(\vec{0}) + \vec{c}^T (f(\vec{x}_i - \vec{x}_j))_{ij} \vec{c} - 2\vec{c}^T \vec{b}$, $\vec{b} = (f(\vec{x} - \vec{x}_i))_i \in \mathbb{R}^n$. The following holds:

ERROR UPPER BOUND:

Given $X = \{\vec{x}_i\}_{i=1}^{n} \subset \mathbb{R}^d$ we have:

$$|f(\vec{x}) - s(\vec{x})| \leq \|f\|_\alpha \cdot P_{f_\alpha, X}(\vec{x}) \qquad \forall \vec{c} \in \mathbb{R}^n$$

$$\leq \|f\|_\alpha \frac{\alpha^{\frac{m}{2}}}{\sqrt{m!}} (\Lambda_{X_m}(x) + 1) h_m^m$$

where $P_{f_\alpha, X}(\vec{x}) = \min \sqrt{\phi(f_\alpha, \vec{c})}$ is the POWER FUNCTION, X_m a set of neighbouring points to $\vec{x}$, $\Lambda_{X_m}(\vec{x}) := \sum_{i=1}^{N(m;d)} |l_i(\vec{x})|$ the LEBESGUE FUNCTION, $N(m;d) = \binom{m+d}{d}$, h_m the smallest diameter s.t.:

$$X_m \subset \{\vec{u} : \|\vec{u} - \vec{x}\|_2 \leq \tfrac{1}{2} h_m\}$$

3) $f(x) = \frac{1}{1+x^2}$, $f(x) = \frac{1}{2} e^{-|x|}$, $f(x) = (1 - |x|)_+$ are pos. def. on $\mathbb{R}$

4) THM. (SCHOENBERG):

Given $a(t) \geq 0$ $\forall t \geq 0$ s.t. $a(t) > 0$ on a NON trivial interval $[a, b] \subset [0, +\infty)$ and set $\phi : \mathbb{R}^{\geq 0} \to \mathbb{R}$ s.t.:

$$\phi(s) := \int_0^{+\infty} a(t)\, e^{-st}\, dt = (\mathcal{L} a)(s)$$

Then $f(x) = \phi(r^2)$ is a RADIAL POS. DEF. function on $\mathbb{R}^d$ $\forall d$. A radial function whose translates can be used for interpolation is called a RADIAL BASIS FUNCTION

5) By the above Thm., the function:

$$s(\vec{x}) = \sum_{j=1}^{n} a_j \frac{1}{\sqrt{1 + \|\vec{x} - \vec{x}_j\|_2^2}}$$

is radial pos. def. and it is the Hardy Inverse Multiquadric interpolant

6) COMPACT SUPPORT RBF's:

If a function is pos. def. on $\mathbb{R}^d$, then it is pos. def. on $\mathbb{R}^n$ $\forall n \leq d$. The function

$$f(\vec{x}) = (1 - \|\vec{x}\|_2)_+^{\ell}$$

is pos. def. on $\mathbb{R}^d$ $\forall \ell \geq \lfloor \frac{d}{2} \rfloor + 1$.

We note that for a function $f(\vec{x}) = \phi(\|\vec{x}\|_2)$ (i.e. radial) its Fourier Transform is:

$$\hat{f}(\vec{x}) = \frac{(2\pi)^{\frac{d}{2}}}{\|\vec{x}\|_2^{(d-2)/2}} \int_0^{+\infty} r^{\frac{d}{2}} \phi(r)\, J_{(d-2)/2}(r\|\vec{x}\|_2)\, dr$$

where $J_\alpha(\vec{x})$ are the Bessel Functions.

The problem with $f(r) = (1-r)_+^{\ell}$ is that it is $C^{\ell-1}$ at $r = 1$ in any dimension. This is why Wendland introduced a calculus for smooth

Compact Support RBF's: given $\phi(t) : \mathbb{R}^{\geq 0} \to \mathbb{R}$ s.t. $t \cdot \phi(t) \in L^1(\mathbb{R}^{\geq 0})$ define the operators

$$(\mathcal{I}\phi)(x) := \int_{|x|}^{+\infty} t\,\phi(t)\,dt, \quad (\mathcal{D}\phi)(x) := -\frac{1}{x}\phi'(x)$$

Notice that $(\mathcal{I}\phi)(x)$ is even and that the 2 operators are inverse to each other. Moreover, if $\mathrm{supp}(\phi) = [-R, R]$ then $\mathrm{supp}(\mathcal{I}) = \mathrm{supp}(\mathcal{D}) = [-R, R]$. Define the Wendland Functions as follows:

$$\phi_{d,k}(r) := \mathcal{I}^k \phi_{\lfloor \frac{d}{2} \rfloor + k + 1}(r)$$

where $\phi_\ell(r) = (1-r)_+^\ell$, d is the dimension and k is the smoothness order. They are pos. def. on $\mathbb{R}^d$ $\forall k \geq 0$. Moreover they are piecewise polynomials in r, $r \in [0,1]$, of degree $\lfloor \frac{d}{2} \rfloor + 3k + 1$ and are $\mathcal{C}^{2k}(\mathbb{R}^d)$

2) CONDITIONALLY DEFINITE FUNCTIONS:

Sometimes we can get non-singular interpolation matrices $(f(\vec{x}_j - \vec{x}_i))_{ij}$ just requiring them to be conditionally definite

THM. (MICCHELLI):

Given a s.t. $a(t) \geq 0$ $\forall t \geq 0$, $a(t) > 0$ on a non degenerate interval $[a, b] \subset \mathbb{R}^{>0}$ and s.t.

$$\phi(s) = \phi(0) + \int_0^{+\infty} \frac{1 - e^{-st}}{t}\, a(t)\,dt$$

with $\phi(0) \geq 0$, we have that the interpolation matrices $\mathcal{I}_n := (\phi(\|\vec{x}_i - \vec{x}_j\|_2^2))_{ij} \in \mathbb{R}^{n \times n}$ are non singular $\forall \{\vec{x}_i\}_{i=1}^n \subset \mathbb{R}^d$ $\forall$ dimension d

Some examples of such functions are the distance function $\phi = \|\vec{x}_i - \vec{x}_j\|_2$ (i.e. $A = (\|\vec{x}_i - \vec{x}_j\|_2)_{ij}$ is NON singular) and the function $\sqrt{1 + \|\vec{x}_i - \vec{x}_j\|_2^2}$.
Interpolants of the form

$$s(\vec{x}) = \sum_{j=1}^{n} a_j \sqrt{1 + \|\vec{x}_i - \vec{x}_j\|_2^2}$$

are called the HARDY MULTIQUADRICS