Category Theory Course Notes

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Chapter 1

1.1 Definition of Category

A **category** (1-category) $\mathcal{C}$ consists of:

1 - A class $\mathrm{Ob}(\mathcal{C})$ of objects of $\mathcal{C}$

2 - $\forall X, Y \in \mathrm{Ob}(\mathcal{C})$.
a class $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of **morphisms** from X to Y

3 - $\forall X \in \mathrm{Ob}(\mathcal{C})$.
an **identity morphism** $id_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$

4 - $\forall X, Y, Z \in \mathrm{Ob}(\mathcal{C})$.
a **composition rule**:

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$
$$(g, f) \mapsto g \circ f$$

Such that it satisfies the following axioms:

1 - **Associativity of composition**:

$\forall X, Y, Z, W \in \mathrm{Ob}(\mathcal{C})$.
$\forall f \in \mathrm{Hom}_{\mathcal{C}}(X, Y), g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z), h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$.
$h \circ (g \circ f) = (h \circ g) \circ f$

2 - **Neutrality**:

$\forall X, Y \in \mathrm{Ob}(\mathcal{C})$.
$\forall f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.
$id_Y \circ f = f \wedge f \circ id_X = f$

1.2 Thin Categories

A category is **thin** if parallel morphisms are always the same, meaning that there is only one morphism between two objects.

In a thin category all morphisms are monic and epic.

1.3 Definition of Initial Object

An object I of a category $\mathcal{C}$ is **initial** (dual of terminal, special case of a colimit (of a functor from $\mathcal{C}$ to the empty category))
$\Updownarrow$
$\forall X \in \mathrm{Ob}(\mathcal{C})$.
$\exists! f \in \mathrm{Hom}_{\mathcal{C}}(I, X)$

1.4 Definition of Terminal Object

An object T of a category $\mathcal{C}$ is **terminal** (dual of initial, special case of limit (of a functor from the empty category to $\mathcal{C}$))
$\Updownarrow$
$\forall X \in \mathrm{Ob}(\mathcal{C})$.
$\exists! f \in \mathrm{Hom}_{\mathcal{C}}(X, T)$

1.5 Definition of Monomorphism

A morphism $f : X \to Y$ in a category $\mathcal{C}$ ($f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$) is a **monomorphism** (or monic in $\mathcal{C}$) (dual of epimorphism)
$\Updownarrow$
$\forall Z \in \mathrm{Ob}(\mathcal{C}). \forall p, q \in \mathrm{Hom}_{\mathcal{C}}(Z, X)$.
$f \circ p = f \circ q \implies p = q$

Example:
In **Set** monomorphisms are precisely the injective maps.

Monomorphisms “can be cancelled” from the left.

1.6 Definition of Split Monomorphism

A **split monomorphism** (dual of split epi) is a morphism $f : X \to Y$ such that there exists a morphism $g : Y \to X$ such that:

$$g \circ f = id_X$$

Proposition: every split mono is a mono.
Proposition: in **Set**, every mono $f : X \to Y$ where X is inhabited is a split mono, assuming LEM holds.

1.7 Definition of Epimorphism

A morphism $f : X \to Y$ in a category $\mathcal{C}$ ($f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$) is an **epimorphism** (or epic in $\mathcal{C}$) (dual of monomorphism)
$\Updownarrow$
$\forall Z \in \mathrm{Ob}(\mathcal{C}).\, \forall p, q \in \mathrm{Hom}_{\mathcal{C}}(Y, Z).$
$p \circ f = q \circ f \implies p = q$

Example:
In **Set** epimorphisms are precisely the surjective maps.

Epimorphisms "can be cancelled" from the right.

1.8 Definition of Split Epimorphism

A **split epimorphism** (dual of split mono) is a morphism $f : X \to Y$ such that there exists a morphism $g : Y \to X$ such that:

$$f \circ g = id_Y$$

Proposition: every split epi is an epi.
Proposition: in **Set**, every epi is a split epi $\iff$ assuming LEM holds.

1.9 Definition of Isomorphism

A morphism $f : X \to Y$ in a category $\mathcal{C}$ ($f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$) is an **isomorphism**
$\Updownarrow$
$\exists g \in \mathrm{Hom}_{\mathcal{C}}(Y, X).$
$f \circ g = id_Y \wedge g \circ f = id_X$

$id_X \forall X \in \mathrm{Ob}(\mathcal{C})$ is always an isomorphisms for every category $\mathcal{C}$.
Objects X and Y in a category $\mathcal{C}$ are **isomorphic**
$\Updownarrow$
there exists an isomorphism between X and Y ($X \cong Y$)

In **Set**, if there exists an isomorphism between X and Y, X and Y are called eqinumerous.

1.10 Definition of Opposite Category

“The mother of all dualities”

Let $\mathcal{C}$ be a category. Then its opposite category $\mathcal{C}^{op}$ is the following category:

- $\mathrm{Ob}(\mathcal{C}^{op}) \coloneqq \mathrm{Ob}(\mathcal{C})$

- $\mathrm{Hom}_{\mathcal{C}^{op}}(X, Y) \coloneqq \mathrm{Hom}_{\mathcal{C}}(Y, X)$

- identities and composition inherited from $\mathcal{C}$
  $id_X \in \mathrm{Hom}_{\mathcal{C}}(X, X) = id_X^{op} \in \mathrm{Hom}_{\mathcal{C}^{op}}(X, X)$
  $f \circ g \coloneqq g^{op} \circ f^{op}$

Observations / Remarks:

- An object I of $\mathcal{C}$ is initial in $\mathcal{C}$
  $\Updownarrow$
  I is terminal when regarded as an object of $\mathcal{C}^{op}$

- A morphism in $\mathcal{C}$ is a monomorphism
  $\Updownarrow$
  it is an epimorphism in $\mathcal{C}^{op}$

1.11 Dualities

injective maps in **Set** (monomorphism in **Set**) $\leftrightarrow$ surjective maps in **Set** (epimorphism in **Set**)

$\leq \quad \leftrightarrow \quad \geq$

$\cap \quad \leftrightarrow \quad \cup$

$\{x\} \quad \leftrightarrow \quad \varnothing$

$\subset \quad \leftrightarrow$ quotient set

$\times$ (cartesian product) $\leftrightarrow$ disjoint union (tagged)

$f \circ g$ $\leftrightarrow$ $g \circ f$

1.12 Definition of Product

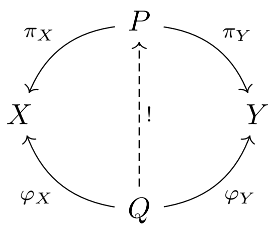
A **product** (dual of coproduct, special case of limit) of two objects X and Y in a category $\mathcal{C}$ consists of:

- an object P of $\mathcal{C}$

- a morphism $\pi_X : P \to X$ in $\mathcal{C}$

- a morphism $\pi_Y : P \to Y$ in $\mathcal{C}$

such that for every object Q of $\mathcal{C}$ together with morphisms $\varphi_X : Q \to X, \varphi_Y : Q \to Y$ there is exactly one morphism $Q \to P$ such that the following diagram commutes:



$\varphi_X = \pi_X \circ !$
$\varphi_Y = \pi_Y \circ !$

Remarks:
- π_X and π_Y are called projection morphisms (also in limits).
- Products are always associative and commutative up to isomorphism.
- There is also the notion of the (co) product of zero, one, three, four, ... objects.
- The zero case of a product is just a terminal object, an object with exactly one morphism from each object.

1.13 Definition of Coproducts

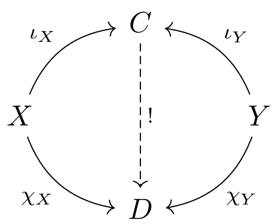
A **coproduct** (dual of product, special case of colimit) of two objects X and Y in a category $\mathcal{C}$ consists of:

- an object C of $\mathcal{C}$

- a morphism $\iota_X : X \to C$ in $\mathcal{C}$

- a morphism $\iota_Y : Y \to C$ in $\mathcal{C}$

such that for every object D of $\mathcal{C}$ together with morphisms $\chi_X : X \to D, \chi_Y : Y \to D$ there is exactly one morphism $C \to D$ which renders the following diagram commutative:



$\chi_X =! \circ \iota_X$
$\chi_Y =! \circ \iota_Y$

Remarks:
- Products in $\mathcal{C}^{op}$ are precisely coproducts in $\mathcal{C}$
- The zero case of a coproduct is the same as an initial object.

1.14 Definition of Functor

A (covariant) **functor** $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of:

- an object $F(X) \in \mathrm{Ob}(\mathcal{D})$ for each object $X \in \mathrm{Ob}(\mathcal{C})$

- a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$ for each morphism $f : X \to Y$ in $\mathcal{C}$

such that:

- $\forall X \in \mathrm{Ob}(\mathcal{C}).\, F(id_X) = id_{F(X)}$

- $\forall X, Y, Z \in \mathrm{Ob}(\mathcal{C}).\, \forall f : X \to Y \in \mathcal{C}, g : Y \to Z$ in $\mathcal{C}.\, F(g \circ f) = F(g) \circ F(f)$

Motto:
Functors $\mathcal{I} \to \mathcal{C}$ are $\mathcal{I}$-shaped **diagrams** in $\mathcal{C}$

Functors preserve commutative diagrams
Functors preserve isomorphisms

1.15 Definition of Contravariant Functor

A **contravariant functor** $\mathcal{C} \to \mathcal{D}$ is a covariant functor $\mathcal{C}^{op} \to \mathcal{D}$

1.16 Definition of Identity Functor

The **identity functor** $Id_{\mathcal{C}}$ on a category $\mathcal{C}$ is the following functor:

$Id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$
$X \mapsto X$
$f \mapsto f$

1.17 Definition of Constant Functor

Let X_0 be an object of a category $\mathcal{C}$.
The **constant functor** $Id_{\mathcal{C}}$ on X_0 is the following functor:

$Id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$
$X \mapsto X$
$f \mapsto f$

1.18 Forgetful Functors

A **forgetful functor** 'forgets' or drops some or all of the input's structure or properties 'before' mapping to the output.

Examples:

- From vector space category to group category

- From vector space category to set category

- From abelian group category to group category

1.19 Definition of Discrete Category

The **discrete category** associated with a set X, written $\mathcal{D}(X)$, is a category containing all the objects of X as objects, and no morphisms between different objects, just the identity morphisms.

1.20 Definition of Induced Functors

Claim:
Any map between sets can be turned into a functor.

Let $f : X \to Y$ be a map between sets.

Consider the discrete categories $\mathcal{D}(X), \mathcal{D}(Y)$.

Then f induces the following functor $\mathcal{D}(X) \to D(Y)$:
$x \mapsto f(x)$
$id_x \mapsto id_{f(x)}$

1.21 Definition of Essentially Surjective Functor

A functor $F : \mathcal{C} \to \mathcal{D}$ is **essentially surjective** iff:

$\forall Y \in \mathrm{Ob}(\mathcal{D}). \exists X \in \mathrm{Ob}(\mathcal{C}) | F(X) \cong Y$

1.22 Definition of Faithful Functor

A functor $F : \mathcal{C} \to \mathcal{D}$ is **faithful** iff:

$\forall X, Y \in \mathrm{Ob}(\mathcal{C})$.
$\forall f, g : X \to Y$ in $\mathcal{C}$
$F(f) = F(g) \implies f = g$

Reformulation: iff

$\forall X, Y \in \mathrm{Ob}(\mathcal{C})$.
$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
$f \mapsto F(f)$

is injective.

1.23 Definition of Full Functor

A functor $F : \mathcal{C} \to \mathcal{D}$ is **full** iff:

$\forall X, Y \in \mathrm{Ob}(\mathcal{C})$.
$\forall g : F(X) \to F(Y)$ in $\mathcal{D}$
$\exists f : X \to Y$ in $\mathcal{C} | F(f) = g$

Reformulation: iff

$\forall X, Y \in \mathrm{Ob}(\mathcal{C})$.
$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
$f \mapsto F(f)$

is surjective.

1.24 Definition of Fully Faithful Functor

A functor is **fully faithful** iff it is full and faithful.

Reformulation: iff

$\forall X, Y \in \mathrm{Ob}(\mathcal{C})$.
$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
$f \mapsto F(f)$

is bijective.

1.25 Definition of Elementary Equivalence

An **elementary equivalence** is a fully faithful, essentially surjective functor.

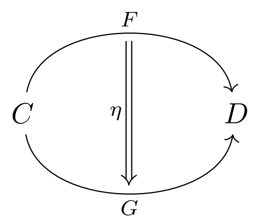
1.26 Definition of Equivalence of Categories

Categories are called **equivalent** iff there is an elementary equivalence between them.

Remark:
Equivalent categories have exactly the same categorical properties.

1.27 Definition of Natural Transformation

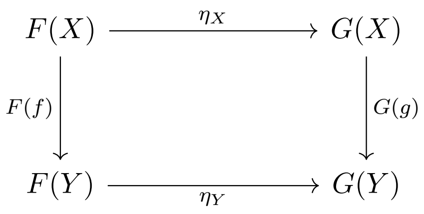
A **natural transformation** $\eta : F \Rightarrow G$ between two functors $F, G : C \to D$



consists of:

- for each object $X \in \mathrm{Ob}(\mathcal{C})$ a morphism $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$

such that for all morphisms $f : X \to Y$ in $\mathcal{C}$, the **naturality square** commutes:



$G(f) \circ \eta_X = \eta_Y \circ F(f)$

Motto:
Natural transformations are **uniform** families of morphisms.

1.28 Definition of Functor Category

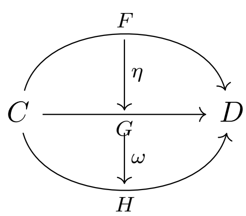
Let $\mathcal{C}, \mathcal{D}$ be categories.
The **functor category** $[\mathcal{C}, \mathcal{D}]$ has:

- as objects: all functors $\mathcal{C} \to \mathcal{D}$

- as morphisms: $\mathrm{Hom}_{[\mathcal{C},\mathcal{D}]}(F, G) := \{h : F \Rightarrow G | h \text{ is a natural transformation}\}$

- as identity: for the object F, the identity $id_F : F \Rightarrow F$
  $(id_F)_X : F(X) \to F(X)$
  given by $id_{F(X)}$

- as composition rule:
  $(\omega \circ \eta)_X := \omega_X \circ \eta_X$



$\eta_X : F(X) \to G(X)$
$\omega_X : G(X) \to H(X)$

$(\omega \circ \eta)_X : F(X) \to H(X)$

and $\omega \circ \eta$ should be natural.

1.29 Definition of Small Category

A category $\mathcal{C}$ is small when $\mathrm{Ob}(\mathcal{C})$ is just a set and not a proper class.

1.30 Definition of Category of Categories

The **1-category of 1-categories**, **Cat** has:

- as objects: all categories

- as morphisms: $\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D}) \coloneqq \{F : \mathcal{C} \to \mathcal{D} | F \text{ is a functor}\}$

- as identities $Id_{\mathcal{C}}$ (the identity functor)

- as composition rule:
  $F : \mathcal{C} \to \mathcal{D}$
  $G : \mathcal{D} \to \mathcal{E}$

  $G \circ F : \mathcal{C} \to \mathcal{E}$
  $X \mapsto G(F(X))$
  $f \mapsto G(F(f))$

There are two issues with this definition:

- Size issue (in ZFC). (it's too big, the objects don't fit in a proper class?)
  Remedies:

  - just consider the category of small categories

  - switch foundations

- It ignores natural transformations
  Remedy:
  Consider the 2-category of 1-categories

  The 2-category of 1-categories has:

- as objects: all 1-categories

- as morphisms: functors

- as -2-morphisms / 2-cells: natural transformations

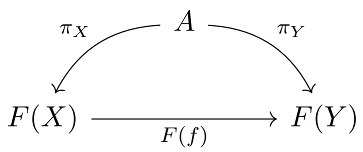
1.31 Definition of Cone

A **cone** (dual of cocone) of a diagram (functor) $F : \mathcal{I} \to \mathcal{C}$ in a category $\mathcal{C}$ consists of:

- an object A of $\mathcal{C}$ (the "tip" of the cone)

- for each object $X \in \mathrm{Ob}(\mathcal{C})$, a morphism $\pi_X : A \to F(X)$

such that for all morphisms $f : X \to Y$ in $\mathcal{I}$, the triangle:



$\pi_Y = \pi_X \circ F(f)$

commutes.

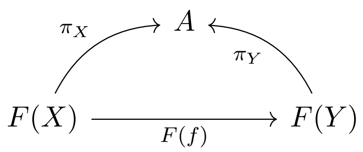
1.32 Definition of Cocone

A **cocone** (dual of cone) of a diagram (functor) $F : \mathcal{I} \to \mathcal{C}$ in a category $\mathcal{C}$ consists of:

- an object A of $\mathcal{C}$ (the "tip" of the cocone)

- for each object $X \in \mathrm{Ob}(\mathcal{C})$, a morphism $\pi_X : F(X) \to A$

such that for all morphisms $f : X \to Y$ in $\mathcal{I}$, the triangle:

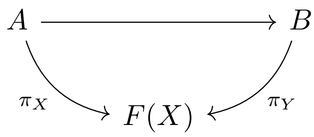


$$\pi_X = \pi_Y \circ F(f)$$

commutes.

1.33 Definition of Morphism Between Cones

A **morphism** between a cone $(A, (\pi_X)_X)$ and a further cone $(B, (\phi_X)_X)$ of a diagram $F : \mathcal{I} \to \mathcal{C}$ consists of a morphism $f : A \to B$ in $\mathcal{C}$ such that:



$$\pi_X = \pi_Y \circ f$$

commutes.

1.34 Definition of Limit

A **limit** (dual of colimit) of a diagram $F : \mathcal{I} \to \mathcal{C}$ is a **terminal cone** of F, that is, a terminal object in the category of of cones of cones of F.

Remark:
A terminal object of $\mathcal{C}$ is the limit of the unique functor from the empty category to $\mathcal{C}$.

1.35 Definition of Colimit

A **colimit** (dual of limit) of a diagram $F : \mathcal{I} \to \mathcal{C}$ is an **initial cocone** of F.

Remark:
An initial object of $\mathcal{C}$ is the colimit of the unique functor from the empty category to $\mathcal{C}$.

1.36 Definition of Equalizer of Two Set-Theoretic Maps

Let $f, g : X \to Y$. Then the **equalizer** of f and g is the following function:

$$Eq(f, g) = x \in X | f(x) = g(x)$$

1.37 Definition of Pullback

A **pullback** P (also called fiber product of the domains over the codomain) (dual of pushout) is the limit of a diagram consisting of two morphisms $f : X \to Z$ and $g : Y \to Z$ with a common codomain.

It comes equipped with two natural morphisms $P \to X$ and $P \to Y$.

1.38 Definition of Pushout

A **pushout** P (also called fibered coproduct) (dual of pullback) is the colimit of a diagram consisting of two morphisms $f : Z \to X$ and $g : Z \to Y$ with a common domain.

It comes equipped with two morphisms $X \to P$ and $Y \to P$.

1.39 Definition of Small Diagram

A **small diagram** in $\mathcal{C}$ is a diagram $\mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is a small category.

1.40 Definition of Complete Cateogory

A category $\mathcal{C}$ is **complete** (dual of cocomplete) iff every small diagram in $\mathcal{C}$ has a limit (it has all small limits).

Assuming LEM, the only categories which have **all** limits or **all** colimits are (some) thin categories.

1.41 Definition of Cocomplete Category

A category $\mathcal{C}$ is **cocomplete** (dual of complete) iff every small diagram in $\mathcal{C}$ has a colimit (it has all small colimits).

$\mathcal{C}$ complete $\iff$ $\mathcal{C}^{op}$ cocomplete.

1.42 Definition of Presheaf

A **presheaf** (plural presheaves) on a category $\mathcal{C}$ is a functor $\mathcal{C}^{op} \to \mathbf{Set}$

Motto:
we picture a presheaf F on $\mathcal{C}$ as an "ideal, fictional, object of $\mathcal{C}$" in that we know its relation to actual objects of $\mathcal{C}$

1.43 Definition of $\hat{X}$

$\hat{X}$ (**X hat**) is a presheaf:

$\mathcal{C}^{op} \to \mathbf{Set}$
$T \mapsto \mathrm{Hom}_{\mathcal{C}}(T, X)$

1.44 Definition of Representable Presheaf

A presheaf $F : \mathcal{C}^{op} \to \mathbf{Set}$ is representable iff:

$\exists X \in \mathrm{Ob}(\mathcal{C}) : F \cong \hat{X}$

1.45 Definition of Adjoint Functors

Let $F : C \to D$, $G : D \to C$

Then, $F \dashv G$ "F is **left adjoint** to G"
(or $G \vdash F$ ("G is **right adjoint** to F"))

iff for every object $X \in \mathrm{Ob}(\mathcal{C}), Y \in \mathrm{Ob}(\mathcal{D})$ there is an isomorphism:

$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$

naturally in X and Y.

Every adjunction $L \dashv R$ gives rise to a monad:

The monad functor will be: $M \coloneqq R \circ L$

The natural transformation:
$\eta : Id \Rightarrow M$

will be given by:
$\eta_X : X \to R(L(X))$

which is in 1:1 correspondence with:
$id_{RL(X)} : RL(X) \to RL(X)$

since:
$\mathrm{Hom}(LA, B) \cong \mathrm{Hom}(A, RB)$

which means that:

$LA \to B$

is in 1:1 correspondence with:
$A \to RB$

The natural transformation:
$\mu : M \circ M \Rightarrow M$

will be given by:
$\mu_X : RLRL(X) \to RL(X)$

induced from:
$LRL(X) \to L(X)$

which is in 1:1 correspondence with:
$id_{RL(X)} : RL(X) \to RL(X)$

Remark:
The monad axioms should also be checked.

1.46 Currying Adjunction

The "product-Hom adjunction" or **currying adjunction** is the following:

$\_ \times S \dashv \mathrm{Hom}_{\mathbf{Set}}(S, \_)$

$\mathrm{Hom}_{\mathbf{Set}}(X \times S, Y) \cong \mathrm{Hom}_{\mathbf{Set}}(X, Hom_{\mathbf{Set}}(S, Y))$

1.47 Adjunction of Logical Connectives

"$\exists$" $\dashv$ "extending the context" $\dashv$ "$\forall$"

The left adjunctions means that it is possible to freely convert between proofs of the following kind:

"Assume $\exists x \in X : A(x)$ ... Hence B." ($\exists x \in X : A(x) \vdash B$)

and

"Let $x \in X$ be arbitrary. Assume $A(x)$ ... Hence B." ($A(x) \vdash_{x \in X} B$)

The right adjunction means that it is possible to freely convert between proofs of the following kind:

“Let $x \in X$ be arbitrary. Assume A ... Hence $B(x)$.” $(A \vdash_{x \in X} B(x))$

and

“Assume A. ... Hence $\forall x \in X : B(x)$.” $(A \vdash (\forall x \in X.\, B(x)))$

1.48 Monoids

A **monoid** consists of:

- a set M

- an element $e \in M$

- an operation $\circ : M \times M \to M$

such that:

- $\forall x \in M.\, x \circ e = x = e \circ x$

- $\forall x, y, z \in M.\, (x \circ y) \circ z = x \circ (y \circ z)$

1.49 Monoids Categorically

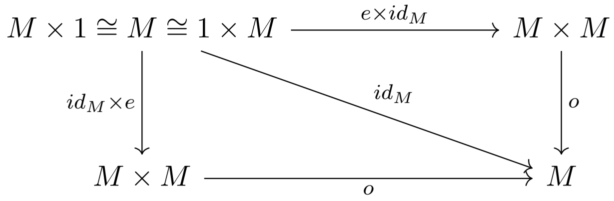
Equivalently, a **monoid** consists of:

- an object M

- a morphism 1 from a terminal object to every other object.

- a map $M \times M \to M$

such that certain diagrams commute.



$$\begin{array}{ccc} (M \times M) \times M \cong M \times M \times M \cong M \times (M \times M) & \xrightarrow{id_M \times o} & M \times M \\ \downarrow{\scriptstyle o \times id_M} & & \downarrow{\scriptstyle o} \\ M \times M & \xrightarrow[o]{} & M \end{array}$$

1.50 Definition of Monoidal Category

A **monoidal category** (sometimes called tensor category) consists of:

- a category $\mathcal{C}$

- a functor $* : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

- an object $1 \in \mathrm{Ob}(\mathcal{C})$

- natural isomorphisms:

  - $1 * X \cong X$

  - $X * 1 \cong X$

  - $X * (Y * Z) \cong (X * Y) * Z$

such that certain coherence conditions are satisfied.

Remark:
In any monoidal category one can speak of **monoid objects**.

1.51 Definition of Monad

A **monad** over a category $\mathcal{C}$ consists of:

- a functor $M : \mathcal{C} \to \mathcal{C}$

- a natural transformation $\eta : Id_{\mathcal{C}} \Rightarrow M$

- a natural transformation $\mu : M \circ M \Rightarrow M$

such that certain diagrams commute.

$$\begin{array}{ccc} M \circ \mathrm{Id}_C = M = \mathrm{Id}_C \circ M & \xrightarrow{\eta * id_M} & M \times M \\ \downarrow{\scriptstyle id_M * \eta} & \searrow{\scriptstyle id_M} & \downarrow{\scriptstyle \mu} \\ M \times M & \xrightarrow[\mu]{} & M \end{array}$$

$$\begin{array}{ccc} M \circ M \circ M & \xrightarrow{id_M * \mu} & M \times M \\ \downarrow{\scriptstyle \mu * id_M} & & \downarrow{\scriptstyle \mu} \\ M \times M & \xrightarrow[\mu]{} & M \end{array}$$

Every monad is given rise to by an adjunction (always of a free and forgetful functor pair).
There are two ways of factorizing a monad into adjoint functors, one is the Kleisli category.

1.52 Definition of Kleisli Category

The **Kleisli category** $\mathcal{C}_M$ of a monad M in a category $\mathcal{C}$ is the following category:

- objects: objects of $\mathcal{C}$

- morphisms: $\mathrm{Hom}_{\mathcal{C}_M}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(X, M(Y))$

1.53 Definition of Cobordism Category

The category **nCob** ("the **cobordism category**") has:

- as objects $(n-1)$-dimensional oriented manifolds

- as morphisms: n-dimesional cobordisms between those

1.54 Definition of Category of Hilbert Spaces

Hilb is the **category of Hilbert spaces** (vector spaces with additional structure).

Hilbert spaces are important in quantum physics, because they can be used to model "slices" of spacetime.

1.55 Definition of Topological Quantum Field Theory

A **topological quantum field theory** (in spacetime dimension n) is a monoidal functor between the monoidal categories **nCob** and **Hilb**:

$$Z : \mathbf{nCob} \to \mathbf{Hilb}$$

Z maps each $(n-1)$-dimensional slice of n-dimensional spacetime to the Hilbert space modelling that slice, and Z maps a morphism $X \to Y$ in **nCob** to the "propagator" $Z(X) \to Z(Y)$.