Mathematics and Statistics I
— Calculus and Linear Algebra

Lecture Notes

March 27, 2023

HOCHSCHULE FRESENIUS
UNIVERSITY OF APPLIED SCIENCES

This script aims to support my lecture at the HS Fresenius. It is incomplete and no substitute for taking actively part in class. Do not distribute without permission. I am thankful for comments and suggestions for improvement.

Contact

Prof. Dr. Stephan Huber
Hochschule Fresenius für Wirtschaft & Medien GmbH
Im MediaPark 4c
50670 Cologne

Office: 4b OG-1 Bü01
www.hs-fresenius.de

+49 221 973199-523
stephan.huber@hs-fresenius.de
www.t1p.de/stephanhuber

Contents

List of Solutions

Preface

Organizational Stuff

- Questions, comments, suggestions are welcome.
- You can contact me (at any time) in the lecture or via
  - phone (+49 221 973199-523),
  - e-mail (`Stephan.Huber@hs-fresenius.de`), or
  - in office (4b OG-1 Bü01).
- Material such as slides and lecture notes can be found on ILIAS.
- Workload: 62.5 h = 28 h (in-class) + 14 h (guided private study hours) - 27.5 h (private self-study).
- Credit Points: 5 (together with *Financial Mathematics*).
- Assessment: written exam of 45 minutes (we will see what COVID allows us to do...).
- Requirements for the Award of ECTS Points: A pass in this module is achieved when the overall grade is greater than or equal to 4.0.
- The students who have successfully completed the module are able to:
  - use methods of calculus and linear algebra to mathematical as well as economic problems,
  - describe the differentiation, integration and evaluation of common (low-dimensional) analytical functions and the solution of linear equation systems and eigenvalue problems employing matrix algebra techniques, and
  - execute analytical optimization techniques with and without constraints.
- Module Content
  - Numbers and Equations
  - Series, Power Series, Limits, Convergence
  - Functions
  - Differentiation and Integration
  - Vectors and Matrices
  - Linear Equation Systems
  - Optimization with Constraints: Lagrange Multipliers
- This lecture cannot cover all aspects of Calculus and Linear Algebra. However, it is a selection of important and fundamental concepts to understand how mathematics is applied to problems in the economics, in business world and in our private lifes'. For a deeper understanding and more

exercises, I recommend Stitz and Zeager (2013); Sydsæter et al. (2012); Chiang and Wainwright (2005); Openstax (2020); Simon and Blume (1994); Gonick (2011, 2015)

- Stitz, C. and Zeager, J. (2013). *Precalculus Version* $\pi = 3$*, Corrected Edition.* available at https://www.stitz-zeager.com/szprecalculus07042013.pdf,
- Sydsæter, K., Hammond, P. J., and Strøm, A. (2012). *Essential Mathematics for Economic Analysis.* Pearson Education, 4 edition,
- Chiang, A. C. and Wainwright, K. (2005). *Fundamental Methods of Mathematical Economics.* McGraw-Hill/Irwin,
- Openstax (2020). *Calculus: Volume 1.* Rice University. available on https://openstax.org/details/books/ca volume-1, and/or
- Simon, C. P. and Blume, L. (1994). *Mathematics for Economists*, volume 7. Norton New York.

I also recommend the comic introductions to algebra and calculus by

- Gonick, L. (2011). *The Cartoon Guide to Calculus.* William Morrow Paperbacks
- Gonick, L. (2015). *The Cartoon Guide to Algebra.* William Morrow Paperbacks.

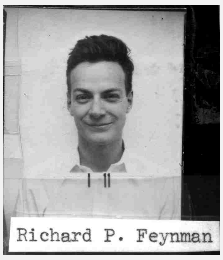
• My teaching principle is **KISS** which stands for **keep it simple and straightforward**.

The KISS principle states that most systems work best if they are kept simple rather than complicated; therefore, simplicity should be a key goal in design, and unnecessary complexity should be avoided.

> However, KISS does not mean that the course is easy! If you are not able to think logically or if you are not willing to work hard, you may have problems passing the course

Given your talent and mental capacities, I try to maximize your ability and self-confidence to solve future problems (in private and professional life).

How to prepare for the exam



Richard P. Feynman:

> "I don't know what's the matter with people: they don't learn by understanding; they learn by some other way by rote, or something. Their knowledge is so fragil!"

Stephan Huber:

> I agree with Feynman: The key to learning is understanding. However, I believe that there is no understanding without doing mathematics, that is, solving a problem yourself with a pencil and a blank sheet of paper without knowing the solution in advance.

- Study the lecture notes, i.e., try to understand the exercises and solve them yourself.
- Study the exercises, i.e., try to understand the rules of algebra and solve the problems yourself.
- Test yourself with past exams that you will find on ILIAS. The structure of the exam is more or less the same every semester.
- If you have the opportunity to form a group of students to study and prepare for the exam, make use of it. It is great to help each other, and it is very motivating to see that everyone has problems sometimes.
- If you have difficulties with some exercises and the solutions shown do not solve your problem, ask a classmate or contact me. I will do my best to help.

Chapter 1

Introduction

Why Mathematics and Statistics in IBM?

First semester: MAS-1

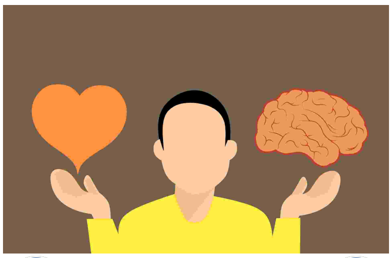
1. **Calculus and Linear Algebra** In many areas such as marketing, accounting and finance, you need a great deal of quantitative understanding. Knowledge of *Calculus (Analysis)* and *Linear Algebra* is essential for this.
2. **Descriptive Statistics** You will sometimes work empirically both in your studies and future work. Thus, you should be skilled in Descriptive Statistics to analyze and visualize one- and two-dimensional data sets and to know and understand the most common techniques for this purpose.

Second semester: MAS-2

1. Financial Mathematics and
2. Inferential Statistics

1.1 Motivation

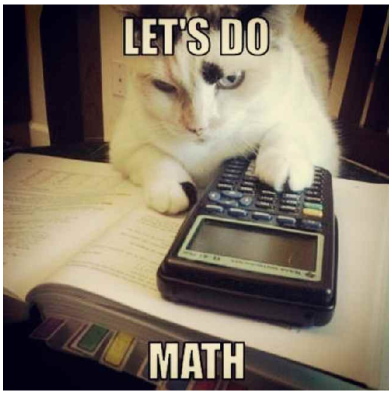
Math helps to distinguish logic from emotion



Emotions vs. Logic

For example, think of an investment case. Your gut feeling tells you that it is a brilliant idea to invest, but your analysis tells you that the probability of losing a lot of money is high. A good marketing strategy combines emotions with some logical aspects, such as 'This jacket is cheap and makes you feel good.' *fakenews* also usually try to convince you by appealing to your emotions based on a logical fallacy. Once feelings are involved, people tend to fall for logical fallacies quickly. They look for evidence that justifies and reinforces their feelings. Facts are only perceived selectively. This is sometimes the reason why it is very difficult to fight *fakenews*. The beauty of mathematics is that it is a science in which emotions play no role and therefore it is very unlikely to make mistakes or be blind in one eye.

Math is the language of logic



Unfortunately, logic and mathematics are less appealing to people than emotions. I understand that completely. Logic and math can be very troublesome and difficult. That is perhaps why many people tend to rely more on their gut feelings than on arguments based on logical thinking.

- Two necessary conditions for your success:
  1. Believe that you can learn math.
  2. Believe that being able to *do math* can make a difference for your future life.
- Maybe this helps to make the two conditions can hold for you: *Math is the hidden secret to understanding the world | Roger Antonsen* https://youtu.be/ZQElzjCsl9o

Carl Friedrich Gauss (1777-1855)

Maybe you have heard about story when that Gauss's teacher in primary school forced little Fritz and the whole class to calculate

$$1 + 2 + 3 + \cdots + 100 = ?$$

after the C.F. Gauss misbehaved. He produced the correct answer within seconds as he was able to find a pattern. He realized that pairwise addition of terms from opposite ends of the list yielded identical intermediate sums:

$$1 + 100 = 101,$$
$$2 + 99 = 101,$$
$$3 + 98 = 101,$$

and so on, for a total sum of

$$50 \cdot 101 = 5050.$$

Exercise 1.1 — Do it like Gauss (Solution → p. 12)

Calculate

$$2 - 4 + 6 - 8 + 10 - 12 + 14 - \ldots - 200 + 202 - 204 + 206 - 208 + 210 = ?$$

The solution is easy!

When I say *“it is easy”*, I don’t mean it is trivial

People who think math is hard and don’t believe they can master the logical steps in math often feel slightly offended when someone starts explaining math by saying, "This is easy" or something similar. Usually a teacher wants to encourage by saying, "This is easy." But students often just feel stupid.

I assure you, whenever I say *‘It’s easy”* I mean, "You can understand it, even though it may take some time. You can do it!" When I say it’s easy, I don’t mean it’s trivial and should be understood immediately. Understanding math can involve tedious algebra and a bunch of arithmetic rules that take time to understand. But once you know how to do math, it will become clear and hopefully easy.

Some paraphrased and funny examples on mathematical wording can be found here: Siegfried (1970)[1]. Also read Eldridge (2014)[2]

So please excuse me if you feel offended by my words from time to time. It is not my intention to make you feel stupid, the opposite is true. I want to keep it simple, stupid (**KISS**) and that sometimes makes me say that is simple or easy (instead of *you can understand...*). Please note that I don’t want you to feel stupid. So whenever you feel stupid, let me know! Ask questions to get rid of that feeling! Even though the question may sound silly to the other students in the class who have already figured out the trick. Asking these kinds of supposedly stupid questions is the smartest way to get better at math than those who are laughing at you! Anyway, if you are too embarrassed to ask in class: approach me, email me, call me, or just stop by my office (4c OG1 Room 1). I will try to explain everything to you face to face.

Solution to Exercise 1.1 — Do it like Gauss (Exercise → p. 11)

Solution 1 *We have 53 positive numbers:*

$$2 + 6 + 10 + 14 + 18 + ... + 202 + 206 + 210$$

we can build pairs

$$2 + 206 = 208$$
$$6 + 202 = 208$$
$$...$$

Thus, we have 26 pairs and we can easily sum up all positive numbers:

$$26 \cdot 208 + 210 = 5618$$

Analogously, we can do that for the 52 negative numbers:

$$4 + 8 + 12 + 16 + 20 + .. + 200 + 204 + 208$$

and the 26 pairs:

$$26 \cdot 212 = 5512.$$

Now, we just need to calculate

$$5618 - 5512 = 106$$

to come to a solution.

Solution 2 *There is a much nicer pattern:*

$$\underbrace{0+2}_{+2}\underbrace{-4+6}_{+2}\underbrace{-8+10}_{+2}-...\underbrace{-208+210}_{+2}$$

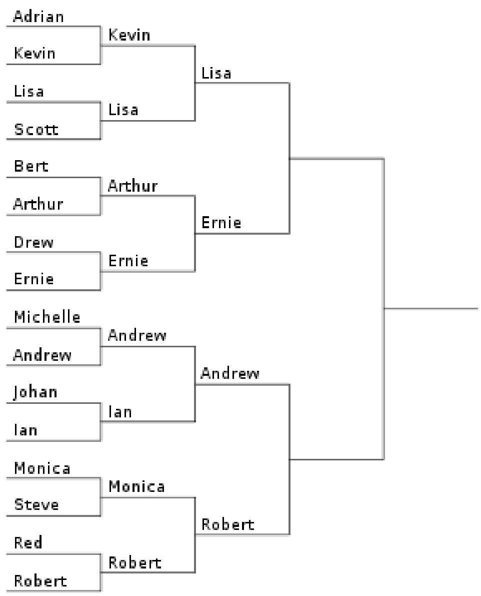
That means, we just need to calculate

$$53 \cdot 2 = 106$$

to come to a solution.

[1] Can be downloaded here: https://www.uibk.ac.at/econometrics/lit/siegfried_jpe_70.pdf
[2] Can be downloaded here: https://doi.org/10.1111/ecin.12041

Exercise 1.2 — Single-Elimination Tournament (Solution → p. 13)



In a single-elimination tournament 100 tennis player play for the cup. How many games must be played to know the winner of the tournament?

Solution to Exercise 1.2 — Single-Elimination Tournament (Exercise → p. 13)

In the tournament the loser of each match-up is immediately eliminated from the tournament. Thus, we will see 99 games as 99 players have to loose once. It is easy like that.

Exercise 1.3 — How Old is Grandpa? (Solution → p. 13)

Grandpa says: *"My son Alpha is 24 years younger than me. Alpha is 25 years older than my grandson Beta. Taken together, Beta and I have lived 73 years."*
How old is grandpa?

Solution to Exercise 1.3 — How Old is Grandpa? (Exercise → p. 13)

Let g, a, and b denote the age of grandpa, Adam, and Beta. Then, we can write the given information down like this:

$$g + b = 73 \tag{1.1}$$
$$a = b + 25 \tag{1.2}$$
$$a = g - 24 \tag{1.3}$$

Thus, we have a system of three equation with three unknowns. As we have not more unknowns than equations, the system of simultaneous equations can be solved. The strategy is to reduce this to two equations and two unknowns and then to one equation with one unknown.
For example, let us eliminate b. We will first eliminate it from equations (1) and (2) simply by adding them. We obtain:

$$(g + b) + (a - b) = 73 + 25 \tag{1.4}$$
$$\Leftrightarrow a = 98 - g \tag{1.5}$$

Plugging that into Eq. (3), we get

$$98 - g = g - 24 \quad (1.6)$$
$$\Leftrightarrow 2g = 122 \quad (1.7)$$
$$\Leftrightarrow g = 61 \quad (1.8)$$

Solution: Grandpa is 61 years old. (Alpha is 37 years, Beta is 12 years)

1.2 Helpful Stuff First

1.2.1 German — English

2 mal 4 ist 8	2 times 4 is 8
Bruch	fraction
gekürzter Bruch	reduced fraction
Zähler	numerator
Nenner	denominator
$\frac{123}{45}$	123 over 45
Dezimalzahl	decimal (number)
Klammer	bracket or paranthesis
Klammern	brackets or parentheses
eckige Klammer	square(d) bracket
geschweifte Klammer	curly bracket

quadrieren	to square
Wurzelziehen	to extract a root
n-te Wurzel	n-th root
Potenz	power
potenzieren	to raise to higher power
mit 2 potenzieren	to raise to the power of 2
logarithmieren	to take the logarithm of
Fakultät	factorial
Steigung	slope
Achsenabschnitt	intercept

Also see:
🔗 www.germanveryeasy.com/mathematics-in-german and
🔗 http://rkleiner.net/Vocab_Math_e-d.pdf and
🔗 http://www.zipcon.net/~swhite/docs/language/German/math.html

1.2.2 Greek Alphabet

α	A	alpha
β	B	beta
γ	Γ	gamma
δ	Δ	delta
ϵ	E	epsilon
ζ	Z	zeta
η	E	eta
θ	Θ	theta
ι	I	iota
κ	K	kappa
λ	Λ	lambda
μ	M	mu

ν	N	nu
ξ	Ξ	xi
o	O	omicron
π	Π	pi
ρ	R	rho
σ	Σ	sigma
τ	T	tau
υ	Υ	upsilon
ϕ	Φ	phi
χ	X	chi
ψ	Ψ	psi
ω	Ω	omega

1.2.3 Mathematical Symbols

Symbol	Meaning
+	Plus
−	Minus
±	Plus or Minus
×	Multiplied by
•	Multiplied by
÷	Divided by
/	Divided by
=	Equals
>	Is Greater Than
<	Is Less Than
≤	Is Less Than or Equal to
≥	Is Greater Than or Equal to
%	Percent
√	Root
π	Pi (3.1416)
°	Degree
∞	Infinity
≈	Is Approximately Equal to
∠	Angle
≡	Identical to, Congruent
≠	Does not equal
:	Ratio; Is to
::	As
{	Set (Left Side)
}	Set (Right Side)
′	First Derivative, Feet
″	Second Derivative, Inches
~	Difference
...	Ellipsis

Symbol	Meaning
!	Factorial
∀	For All
∃	Exists
Δ	Finite Difference, Increment
∴	Therefore
∵	Because
∗	Operation
⊥	Perpendicular
∝	Proportional To
∩	Intersection
∪	Union
⊂	Subset
⊄	Not a Subset
∈	Belongs To
∉	Does Not Belong To
∇	Vector Differential
⇔	Is Equivalent To
⇒	Implies
Σ	Sum of Terms (Sigma)
∫	Integral
Π	Product of Terms (Omega)
→	Approaches the Limit
∅	Empty Set
<	Mean (Left Side)
>	Mean (Right Side)
C	Complex Set
Z	Integer Set
N	Natural Set
R	Real Set

1.3 Numbers

In math you can express things very differently. It depends pretty much on how precise you want and/or need to be.

One way:

- Natural numbers $\mathbb{N}$ are also known as counting numbers: $1, 2, 3, 4, 5, \ldots$
- Whole numbers $\mathbb{N}_0$ are all natural numbers and the zero
- Integers $\mathbb{Z}$ are all whole numbers and the negative natural numbers
- Rational numbers (Fractions) $\mathbb{Q}$ are all numbers that can be expressed as the result of an integer divided by a natural number
- Irrational numbers are all real numbers that are not fractions
- Real numbers $\mathbb{R}$ are all rational and irrational numbers

Here is an alternative way to say basically the same:

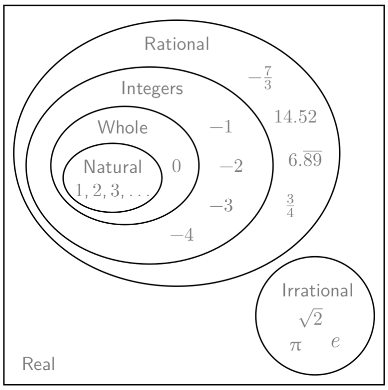
1. The **Empty Set**: $\emptyset = \{\} = \{x \mid x \neq x\}$. This is the set with no elements.[3] Like the number '0,' it plays a vital role in mathematics.
2. The **Natural Numbers**: $\mathbb{N} = \{1, 2, 3, \ldots\}$ The periods of ellipsis here indicate that the natural numbers contain 1, 2, 3, 'and so forth'.
3. The **Integers**: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
4. The **Rational Numbers**: $\mathbb{Q} = \left\{\frac{a}{b} \mid a \in \mathbb{Z} \,\text{and}\, b \in \mathbb{Z}\right\}$. Ratioal numbers are the ratios of integers (provided the denominator is not zero!) It turns out that another way to describe the rational

[3]Here we use the so-called *set-builder notation*. For example, $\{x \mid x$ is a letter in the word "pronghorns".$\}$ can be read like this: 'The set of elements x such that x is a letter in the word "pronghorns."' In each of the above cases, we may use the familiar equals sign '=' and write $S = \{p, r, o, n, g, h, s\}$ or $S = \{x \mid x$ is a letter in the word "pronghorns".$\}$. Clearly r is in S and q is not in S. We express these sentiments mathematically by writing $r \in S$ and $q \notin S$.

numbers is:

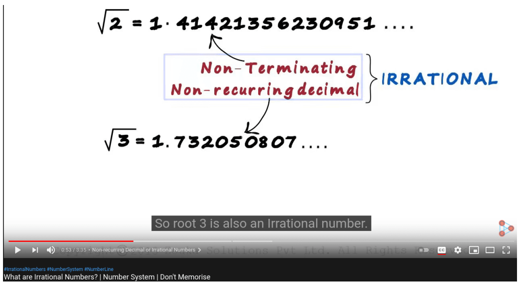
$$\mathbb{Q} = \{x \,|\, x \text{ possesses a repeating or terminating decimal representation.}\}$$

5. The **Real Numbers**: $\mathbb{R} = \{x \,|\, x \text{ possesses a decimal representation.}\}$
6. The **Irrational Numbers**: Real numbers that are not rational are called **irrational**. As a set, we have $\{x \in \mathbb{R} \,|\, x \notin \mathbb{Q}\}$. (There is no standard symbol for this set.) Every irrational number has a decimal expansion which neither repeats nor terminates.
7. The **Complex Numbers**: $\mathbb{C} = \{a + bi \,|\, a,b \in \mathbb{R} \text{ and } i = \sqrt{-1}\}$ (We will not deal with complex numbers in this course.)

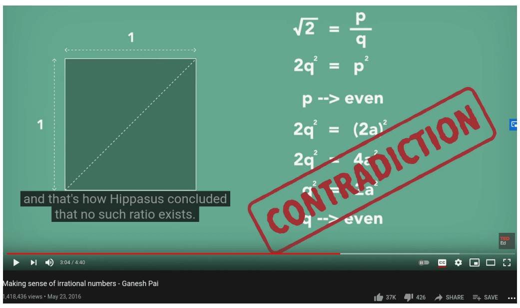


Homework: Download and check out Stitz and Zeager (2013)'s book. Especially if you feel like you need some math tutoring.

Irrational Numbers



Watch https://youtu.be/CtRtXoT_2Ps *What are Irrational Numbers?* by Don't Memorise and



https://youtu.be/sbGjr_awePE *Making sense of irrational numbers* by Ganesh Pai.

Chapter 2

Equations

2.1 Introduction to Equations

Most quantitative disciplines, such as science, economics, and areas of psychology and sociology, describe real-world situations in terms of equations.

An equation is a mathematical expression which states that one thing is equal to another. The meaning of a function is further explained in Openstax (2020, ch. 1.1).

Definition 1

A function f consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output. The set of inputs is called the domain of the function. The set of outputs is called the range of the function.

So,

$$2 + 3 = 5$$

is an equation, and so is

$$E = mc^2,$$

or

$$x - 7 = 25.$$

Each of these examples is subtly different:

- The first is an **identity**—it is always true.
- $E = mc^2$ is a **relationship**, defining E in terms of m and c, while
- $x - 7 = 25$ is an **equation** which is only true for certain values of x. In most algebraic contexts, at least one side of the equation will involve unknown elements, often denoted by x, y or z.

Please notice: while there are conventions in notation, there are no general rules how symbols should be assigned to denote specific elements or variables. For example, it is a standard naming convention in economics to denote the the profit of a firm with the Greek letter π because the word Profit begins with P. In macroeconomics, however, the rate of inflation is usually represented by π, too. This can be confusing, especially to students who are not familiar to the specific conventions. In school, the letter π is usually denoted to the number π (*dt.: Kreiszahl*) which is approximately equal to 3.14159. Actually, π—like all other letters and symbols—can be a placeholder for anything (unknown).

2.2 Terms Used for Equations

Many algebraic techniques are concerned with manipulating and solving equations to find these unknowns. The values of the unknown elements that satisfy the equality are called solutions of the equation.

An equation is written as two expressions connected by an equals sign (=). The expressions on the two sides of the equals sign are called the left-hand side and the right-hand side of the equation. Each side of an equation will contain one or more terms. For example, the equation

$$y = a \cdot x^2 + b \cdot x + c$$

has **left-hand side** (often abbreviated with LHS) y, consisting of just one term, and **right-hand side** (RHS) $a \cdot x^2 + b \cdot x + c$, which has three terms. The unknown elements are x and y, called **variables**, while a, b and c are **parameters**. If, additionally, one writes

$$y(x) = a \cdot x^2 + b \cdot x + c$$

then x is called the independent variable and y the dependent variable or the variable dependent on x.

2.3 Manipulation of Simple Equations

An equation can also be seen as a scale in which both sides are in balance. When a quantity is added or removed from one tray, the same quantity must be added or removed from the other tray to maintain the balance.

Likewise, a mathematical equation has to be preserved by performing the same operations (addition, subtraction, multiplication, division, power, logarithm, etc.) simultaneously on both sides of the equation.

Equations with one unknown Two equations that have exactly the same solution are **equivalent equations**. To get equivalent equations, do the following to both sides of the equality sign:

- add (or subtract) the same number,
- multiply (or divide) by the same number (different from 0!).

Exercise 2.1 — Equalize Me (Solution → p. 18)

Solve:

$$6p - \frac{1}{2}(2p - 3) = 3(1 - p) - \frac{7}{6}(p + 2)$$

Solution to Exercise 2.1 — Equalize Me (Exercise → p. 18)

$$p = -\frac{1}{11}$$

Chapter 3

Functions

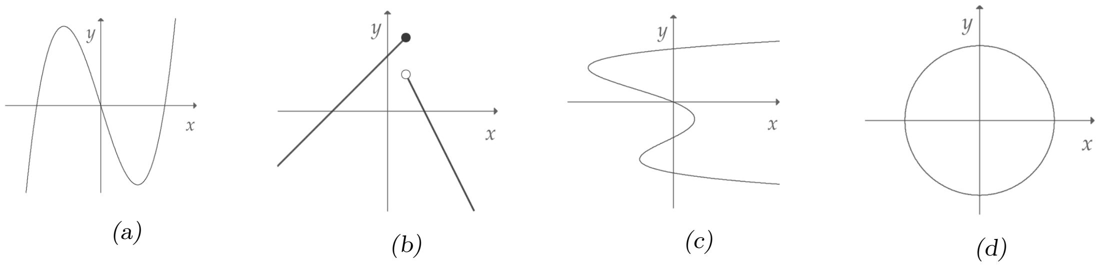


Figure 3.1: Function?

3.1 Introduction

Definition — Function: A relation in which each x-coordinate is matched with only one y-coordinate is said to describe y as a **function** of x.

Functions represent relationships between mathematical variables. They take an input, manipulate it in some way, and produce an output. For example, the function

$$f(x) = x + 2$$

takes an input of a real number x and produces an output $f(x)$ of two more than x.

In general, a real function f is a certain rule

$$f : D \to R, \quad x \mapsto y = f(x)$$

which maps any input x of a subset D of the real numbers called the function's domain onto an output $y = f(x)$ in another subset R of the real numbers called the range. If there is a value x with $f(x) = 0$, then x is called a zero of f.

In that form the function we mentioned above is

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto y = f(x) = x + 2$$

and has the zero $x = -2$.

Exercise 3.1 — Is it a function? (Solution → p. 68)

In Figure 3.1, four graphs are plotted. State which of these relations describes y as a function of x.

Note — (Inverse) Functions and Notation

Notation can confusing, there are various ways the disciplines of social sciences write things down. Very often they deny to be 100% correct in a strict mathematical sense. For example, instead of

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto y = f(x)$$

we just write in the following script

$$y = f(x).$$

The inverse function can be written like that

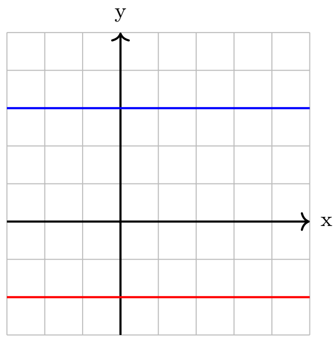
$$x = f^{-1}(y)$$

3.2 Constant Functions

The simplest real functions are the constant functions, which assign a certain real constant c to each real number x

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto y = f(x) = c$$

The graph of a constant function is a parallel to the x-axis. For example, with $c = 0$, we get the x-axis itself and with $c = 3$ and $c = -2$ the graphs

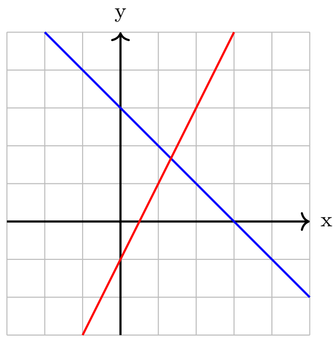


3.3 Linear Functions

A linear function is a function f of the form

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto y = f(x) = a \cdot x + b$$

where $a \neq 0$ and b are real constants. It has exactly one zero in $x = \frac{-b}{a}$. The graph of a linear function is a straight line with slope a and y-intercept b. For example, with $a = 2$, $b = -1$ or with $a = -1, b = 3$



3.4 Linear Functions in Economics

Relationship between price and demand If the price p increases, normally the demand x decreases. In economics, this relationship is often assumed to be linear. Thus, the **demand function** has the form

$$x = x(p) = -a \cdot p + b \text{ with positive } a \text{ and } b$$

and only makes sense for positive quantities x and prices p. For example

$$x = x(p) = -0.5 \cdot p + 50$$

This describes a function x depending on p. Since we seek for the reverse dependence $p = p(x)$ we can resolve this equation by p and get the **(inverse) demand function**

$$p = p(x) = -2 \cdot x + 100$$

Relationship between supply and demand If the price p of a good rises, a producer will increase the supply quantity in linear dependence. The **supply function** therefore has the form

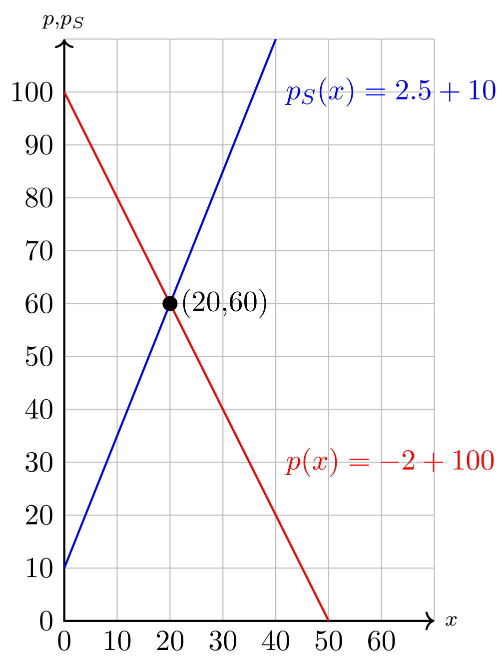
$$x = x_S(p) = a \cdot p - b \text{ with positive } a \text{ and } b$$

and only makes sense for positive quantities x and prices p. For example:

$$x_S(p) = 0.4 \cdot p - 4.$$

This describes a function x depending on p. Since we seek for the reverse dependence $p = p(x)$, we can resolve this equation by p and get the **(inverse) supply function**

$$p = p_S(x) = 2.5 \cdot x + 10$$



To calculate the equilibrium, i.e., the intersection of supply and demand, we need to set both functions equal and do some algebra:

$$2.5x + 10 = -2x + 100$$
$$\Leftrightarrow x = 20$$

Once we know that 20 units will be traded on the market in equilibrium, we can calculate the corresponding price:

$$p(20) = -2 \cdot 20 + 100 = 60 \quad \text{or}$$
$$p_S(20) = 2.5 \cdot 20 + 10 = 60$$

For the quantity $x^* = 20$ and the price $p^* = 60$ supply equals demand. At point (x^*, p^*) supply equals demand. That means the market is *balanced* or it is in *equilibrium* where no market participant wants to change (or can change).

Exercise 3.2 — Demand and Supply (Solution → p. 68)

Given the demand function

$$x(p) = -\frac{3}{4}p + 300$$

and the supply function

$$x_S(p) = \frac{5}{4}p - 100$$

Determine the equilibrium market price and quantity both graphically and algebraically.

3.5 Quadratic Functions

A quadratic function is a function of the form

$$f(x) = ax^2 + bx + c,$$

where a, b and c are real numbers with $a \neq 0$. The domain of a quadratic function is $(-\infty, \infty)$.

The most basic quadratic function is $f(x) = x^2$, whose graph is given in Figure 3.2. Its shape should look familiar from high school – it is called a **parabola**. The point $(0, 0)$ is called the **vertex** of the parabola.

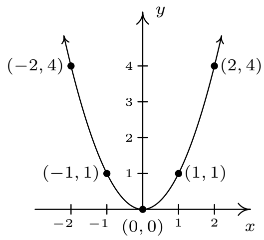


Figure 3.2: The graph of the basic quadratic function $f(x) = x^2$

3.5.1 Standard and general form of quadratic functions

Suppose f is a quadratic function.

- The **general form** of the quadratic function f is $f(x) = ax^2 + bx + c$, where a, b and c are real numbers with $a \neq 0$.
- The **standard form** of the quadratic function f is $f(x) = a(x - h)^2 + k$, where a, h and k are real numbers with $a \neq 0$.

One of the advantages of the standard form is that we can immediately read off the location of the **vertex**:

Note — Vertex Formula for Quadratics in Standard Form

For the quadratic function $f(x) = a(x - h)^2 + k$, where a, h and k are real numbers with $a \neq 0$, the vertex of the graph of $y = f(x)$ is (h, k).

Note — Completing the Square Method

To convert the general form into the standard form you can go on like this:
Step 1 Divide all terms by a, i.e., the coefficient of x^2.
Step 2 Complete the square (use $(\frac{b}{2})^2$)!) and balance it by adding the same value to the right side of the equation.
Here is a video entitled *How do you convert from standard form to vertex form of a quadratic*
https://youtu.be/pwITxyUghV0

Exercise 3.3 — Give Me The Standard Form (Solution → p. 68)

Convert the functions below from general form to standard form and assign the two graphs below to the two functions.

1. $f(x) = x^2 - 4x + 3$.
2. $g(x) = 6 - x - x^2$

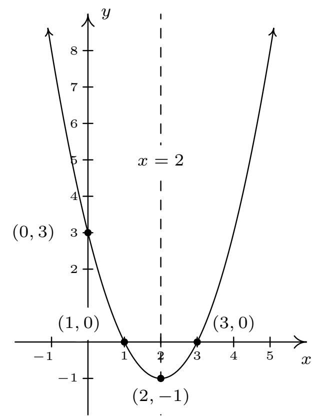


Figure 3.3

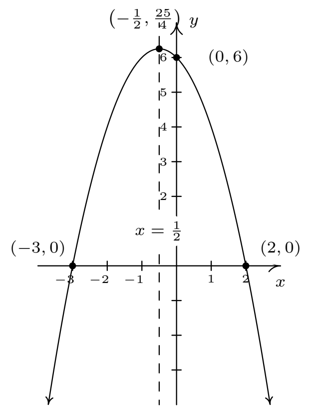


Figure 3.4

3.5.2 General Way to Come to the General Form

With Example 3.3 fresh in our minds, we are now in a position to show that every quadratic function can be written in standard form. We begin with $f(x) = ax^2 + bx + c$, assume $a \neq 0$, and complete the square in *complete* generality.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x\right) + c && \text{(Factor out coefficient of } x^2 \text{ from } x^2 \text{ and } x.) \\ &= a\left(x^2 + \frac{b}{a}x + \underline{\frac{b^2}{4a^2}} - \underline{\frac{b^2}{4a^2}}\right) + c && \left((\tfrac{b}{a}\cdot\tfrac{1}{2})^2 = \frac{b^2}{4a^2}\right) \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - a\left(\frac{b^2}{4a^2}\right) + c && \text{(Group the perfect square trinomial.)} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} && \text{(Factor and get a common denominator.)} \end{aligned}$$

Comparing this last expression with the standard form, we identify $(x - h)$ with $\left(x + \frac{b}{2a}\right)$ so that $h = -\frac{b}{2a}$. Instead of memorizing the value $k = \frac{4ac-b^2}{4a}$, we see that $f\left(-\frac{b}{2a}\right) = \frac{4ac-b^2}{4a}$.

Note — Vertex Formulas for Quadratic Functions

Suppose a, b, c, h and k are real numbers with $a \neq 0$.

- If $f(x) = a(x - h)^2 + k$, the vertex of the graph of $y = f(x)$ is the point (h, k).
- If $f(x) = ax^2 + bx + c$, the vertex of the graph of $y = f(x)$ is the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

There are two more results which can be gleaned from the completed-square form of the general form of a quadratic function,

$$f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

3.5.3 The Quadratic Formula

If a, b and c are real numbers with $a \neq 0$, then the solutions to $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Exercise 3.4 — Derive the Quadratic Formula (Solution → p. 69)

Show that the statement above is correct.

Discriminant Trichotomy

The square root of a negative number does not exist among the set of Real Numbers. Given that $\sqrt{b^2 - 4ac}$ is part of the Quadratic Formula, we will need to pay special attention to the radicand $b^2 - 4ac$. It turns out that the quantity $b^2 - 4ac$ plays a critical role in determining the nature of the solutions to a quadratic equation. It is given a special name.

Discriminant If a, b and c are real numbers with $a \neq 0$, then the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$ is the quantity $b^2 - 4ac$.

The discriminant *'discriminates'* between the kinds of solutions we get from a quadratic equation. Let a, b and c be real numbers with $a \neq 0$, these cases, and their relation to the discriminant, are as follows:

- If $b^2 - 4ac < 0$, the equation $ax^2 + bx + c = 0$ has no real solutions.
- If $b^2 - 4ac = 0$, the equation $ax^2 + bx + c = 0$ has exactly one real solution.
- If $b^2 - 4ac > 0$, the equation $ax^2 + bx + c = 0$ has exactly two real solutions.

Exercise 3.5 — Its All About Profits (1) (Solution → p. 70)

The profit of selling a product x is defined by Profit = Revenue − Cost, or $P(x) = R(x) - C(x)$. The inverse demand function is

$$p(x) = -1.5x + 250.$$

The cost, in dollars, to produce x goods is given as

$$C(x) = 80x + 150 \qquad \forall \quad x \geq 0.$$

1. Determine the functions for demand and revenue and discuss what these functions can tell you.
2. Determine the weekly profit function $P(x)$.
3. Graph $y = P(x)$. Include the x- and y-intercepts as well as the vertex and axis of symmetry.
4. Interpret the zeros of P.
5. Interpret the vertex of the graph of $y = P(x)$.
6. What should the price per unit be in order to maximize profit?

Exercise 3.6 — Its All About Profits (2) (Solution → p. 72)

Given the demand function

$$x(p) = -3 \cdot p + 600$$

and the cost function of a monopolist

$$C(x) = 180 \cdot x + 108$$

Determine

a) the fixed cost,

b) the variable cost,
c) the revenue function,
d) the profit function,
e) the quantity x for which the profit is maximal,
f) the maximal profit,
g) the factorization of the profit function, and
h) the quantities x for which the profit is positive.

3.6 System of Linear Equations with Two Unknowns

Find the values of x and y that satisfy both equations:

$$2x + 3y = 18 \qquad (*)$$
$$3x - 4y = -7 \qquad (**)$$

Use one of the following methods:

Method 1 Solve one of the equations for one of the variables in terms of the other; then substitute the result into the other equation.

Method 2 Eliminate one of the variables by adding or subtracting a multiple of one equation from the other.

Method 3 Solve both equations for the same variable and put this variable on the left hand side; then set the right hand sides of the two resulting equations equal.

Method 4 Use software.

3.6.1 Solutions — Method 1:

Solve one of the equations for one of the variables in terms of the other; then substitute the result into the other equation.

From (*)

$$\begin{aligned} 3y &= 18 - 2x \\ y &= 6 - \frac{2}{3}x \end{aligned}$$

inserting in (**) gives

$$\begin{aligned} 3x - 4\left(6 - \frac{2}{3}x\right) &= -7 \\ 3x - 24 + \frac{8}{3}x &= -7 \\ \frac{17}{3}x &= 17 \\ \frac{1}{3}x &= 1 \\ x &= 3 \end{aligned}$$

using $x = 3$ in (*) gives

$$\begin{aligned} 2 \cdot 3 + 3y &= 18 \\ 3y &= 12 \\ y &= 4 \end{aligned}$$

3.6.2 Solutions — Method 2:

Eliminate one of the variables by adding or subtracting a multiple of one equation from the other.

Multiply (*) by 4 and (**) by 3. This gives

$$
\begin{aligned}
8x + 12y &= 72 \\
9x - 12y &= -21
\end{aligned}
$$

Then add both equations. This gives

$$
\begin{aligned}
17x &= 51 \\
x &= 3
\end{aligned}
$$

Inserting this result in (*) gives

$$
\begin{aligned}
2 \cdot 3 + 3y &= 18 \\
3y &= 12 \\
y &= 4
\end{aligned}
$$

3.6.3 Solutions — Method 3:

Solve both equations for the same variable and put this variable on the left hand side; then set the right hand sides of the two resulting equations equal.

$$
\begin{aligned}
2x + 3y &= 18 \\
y &= 6 - \frac{2}{3}x \qquad (*') \\
3x - 4y &= -7 \\
y &= \frac{3}{4}x + \frac{7}{4} \qquad (**')
\end{aligned}
$$

Set (*') equal with (**'), we get

$$
\begin{aligned}
6 - \frac{2}{3}x &= \frac{1}{4}x + \frac{7}{4} \\
6 - \frac{7}{4} &= \frac{2}{3}x + \frac{3}{4}\times \\
4\frac{1}{4} &= 1\frac{5}{12}\times \\
\frac{17}{4} &= \frac{17}{12}\times \\
\frac{17}{4} \cdot \frac{12}{17} &= x \\
x &= 3
\end{aligned}
$$

and hence

$$
y = 4
$$

3.6.4 Solutions — Method 4:

There are some nice software packages available that allow to do analytical algebra with variables. The major players on the commercial market are Maple, Matlab, Mathematica. However, there are also open-source alternatives available such as SageMath. If you want learn one of these, I would go for SageMath. Not only because it is free of costs but also because it is based on the popular programming language Python. That means, if you use Sage in a mathematics, statistics, physics, or data-science class, you will learn Python along the way.

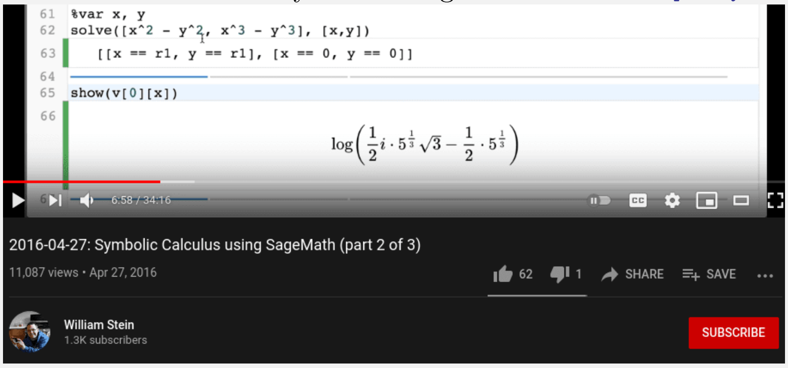
In the course, I will show you some free and easy to use tools that are available online. For a more complete list of software in general, I refer to
https://en.wikipedia.org/wiki/List_of_computer_algebra_systems

Go to https://www.wolframalpha.com/, type in `plot 2x+3y=18 plot 3x-4y=-7` and click Enter.

Also visit
https://www.wolframalpha.com/calculators/system-equation-calculator
and

https://www.mathpapa.com/algebra-calculator.html
and try to use this tool to solve the system of equations.

SageMath is a free open-source mathematics software system licensed under the GPL. It builds on top of many existing open-source packages: NumPy, SciPy, matplotlib, Sympy, Maxima, GAP, FLINT, R and many more. Access their combined power through a common, Python-based language or directly via interfaces or wrappers. A good way to start learning SageMath is the homepage of Gregory V. Bard who offers plenty of good ressources about SageMath, see: http://www.gregory-bard.com/Sage.html I also recommend the freely available book *Computational Mathematics with SageMath* from Zimmermann et al. (2018) which can be downloaded here: http://sagebook.gforge.inria.fr/english.html
Here is a video that demonstrates you how SageMath works: https://youtu.be/g8DI9eRgnBw



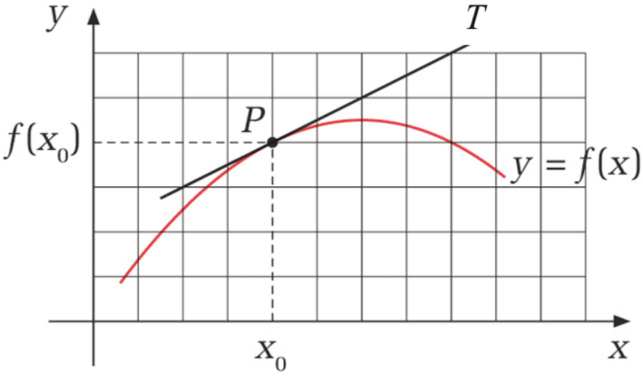
Maple is commercial PC program. In particular, it is a symbolic and numeric computing environment as well as a multi-paradigm programming language. It covers several areas of technical computing, such as symbolic mathematics, numerical analysis, data processing, visualization, and others. A toolbox, MapleSim, adds functionality for multidomain physical modeling and code generation. Maple's capacity for symbolic computing include those of a general-purpose computer algebra system. For instance, it can manipulate mathematical expressions and find symbolic solutions to certain problems, such as those arising from ordinary and partial differential equations.

MATLAB (an abbreviation of *matrix laboratory*) is a proprietary multi-paradigm programming language and numeric computing environment developed by MathWorks. MATLAB allows matrix manipulations, plotting of functions and data, implementation of algorithms, creation of user interfaces, and interfacing with programs written in other languages.

Wolfram Mathematica is a software system with built-in libraries for several areas of technical computing that allow symbolic computation, manipulating matrices, plotting functions and various types of data, implementation of algorithms, creation of user interfaces, and interfacing with programs written in other programming languages.

3.7 Derivatives (Slopes of Curves)

3.7.1 The first derivate



- For the graph representing the function $y = ax + b$ the slope was given by the number a.
- Consider some arbitrary function f. The slope of the corresponding graph at some point x_0 is the slope of the tangent to the graph at x_0.
- In the figure above, point P has the coordinates $(x_0, f(x_0))$.
- The straight line T is the tangent line to the graph at point P. It just *touches* the curve at point P.
- The slope of the graph at x_0 is the slope of T. This slope is $\frac{1}{2}$.

Definition: The slope of the tangent line at point $(x, f(x))$ is called the *derivative* of f at point x. This number is denoted by $f'(x)$.

- Read $f'(x)$ as "f prime x".
- We call $f'(x)$ the derivative of $f(x)$.
- In the figure the point $x = x_0$ was considered.
- The derivative of f at point x_0 was

$$f'(x_0) = \frac{1}{2}$$

  In the figure, P and Q are points on the curve (graph).
- The entire straight line through P and Q is called a secant.
- Keep P fixed, but move Q along the curve towards P. Then the secant rotates around P towards the limiting straight line T.
- T is the tangent (line) to the curve at P.
- In place of $f'(x)$ often y' or the differential notation of *Leibniz* is used:

$$\frac{\mathrm{d}y}{\mathrm{d}x}, \quad \mathrm{d}y/\mathrm{d}x, \quad \frac{\mathrm{d}f(x)}{\mathrm{d}x}, \quad \mathrm{d}f(x)/\mathrm{d}x, \quad \frac{\mathrm{d}}{\mathrm{d}x}f(x)$$

- The derivative $f'(x)$ can be used to define the notion of increasing and decreasing functions.
  - $f'(x) \geq 0$ for all x in $D_f \iff f$ is increasing in D_f
  - $f'(x) > 0$ for all x in $D_f \iff f$ is strictly increasing in D_f
  - $f'(x) \leq 0$ for all x in $D_f \iff f$ is decreasing in D_f
  - $f'(x) < 0$ for all x in $D_f \iff f$ is strictly decreasing in D_f

3.7.2 The second order derivative

- The derivate f' of a function $y = f(x)$ is called the *first derivate* of f.
- If f' is also differentiable, then we can differentiate f' in turn.
- The result is called the *second order derivative* and it is written as f'' or y''.

Definition: $f''(x)$ is the second order derivative of f evaluated at the particular point x.

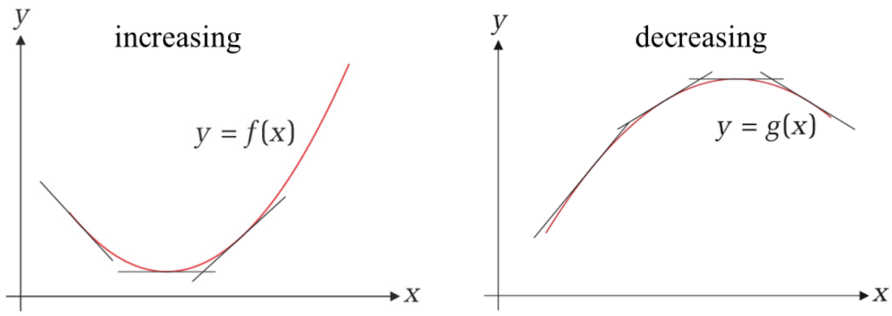
- f'' or y'' can be written in the differential notation as

$$\frac{d}{dx}\left[\frac{d}{dx}f(x)\right]$$

or more simply as

$$\frac{d^2f(x)}{dx^2} \quad \text{or} \quad \frac{d^2y}{dx^2}$$

- The second order derivative $f''(x)$ is the derivative of $f'(x)$. Therefore
  - $f''(x) \geq 0$ on I $\iff$ f' is increasing on I *(function is convex)*
  - $f''(x) \leq 0$ on I $\iff$ f' is decreasing on I *(function is concave)*
- The consequences are illustrated in the following figure.



Example: The first derivative of

$$f(x) = 2x^5 - 3x^3 + 2x$$

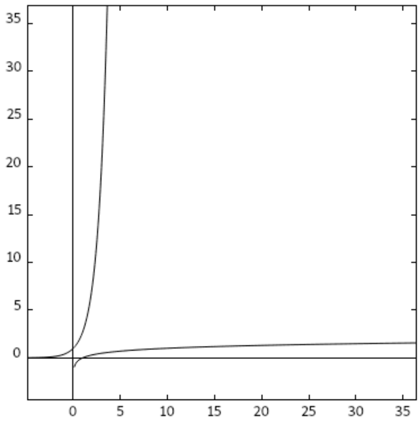
is

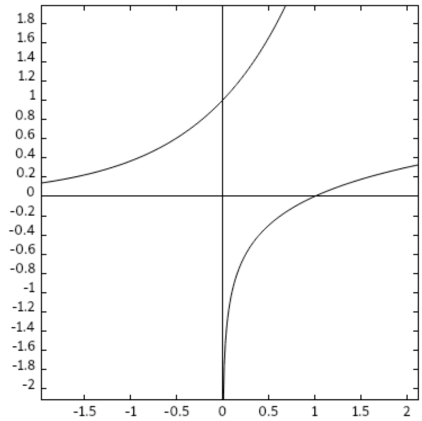
$$f'(x) = 10x^4 - 9x^2 + 2$$

Therefore, the second order derivative is

$$f''(x) = 40x^3 - 18x$$

3.8 Logarithmic and Exponential Function





- What is a logarithmic function?
- What is an exponential function?
- What is both good for?

3.8.1 Logarithmic Function

Maybe you have heard about the logarithm and I am quite sure you know the 'log' button on your calculator. However, do you really have an idea what it actually is?

Consider the following equations and then explain me what the *logarithm* is:

$$\begin{aligned} 2 \cdot 2 \cdot 2 \cdot 2 &= 16 \\ 2^4 &= 16 \\ x^n &= y \\ 2 &= 16^{\frac{1}{4}} \\ x &= y^{\frac{1}{n}} \\ \log_2 16 &= 4 \\ \log_x y &= n \end{aligned}$$

$$\begin{aligned} x^n &= y \\ \log x^n &= \log y \\ n \cdot \log x &= \log y \\ n &= \frac{\log y}{\log x} \\ 4 &= \frac{\log 16}{\log 2} \end{aligned}$$

$$\begin{aligned} \log_x n &= y \\ \log_{10} 16 &= 1.20411998265592 \\ \log_{10} 2 &= 0.301029995663981 \\ \log 16 &= 1.20411998265592 \\ \log 2 &= 0.301029995663981 \end{aligned}$$

Exercise 3.7 — Calculate a logarithmic function without a calculator (Solution → p. 72)

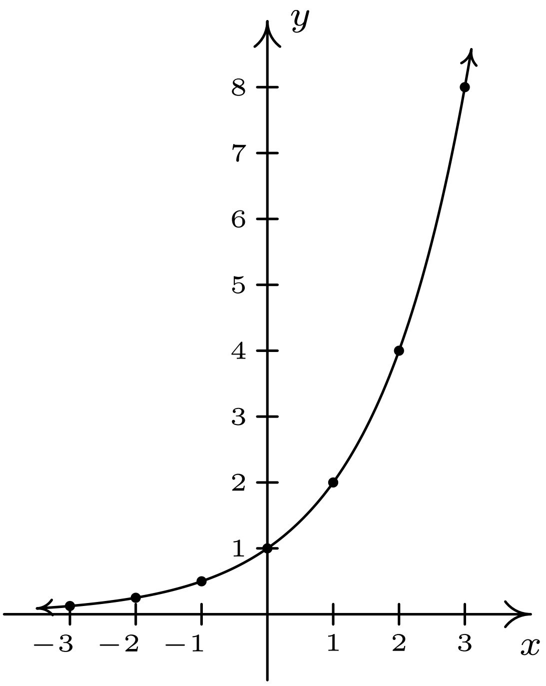
Hint: The result is an integer.

- $\log_2 16 =?$
- $\log_3 243 =?$
- $\log_5 125 =?$
- $\log_3 81 =?$
- $\log_2 \left(\frac{1}{8}\right)$

3.8.2 Exponential Function

Let us consider the function $f(x) = 2^x$.

x	$f(x)$	$(x, f(x))$
-3	$2^{-3} = \frac{1}{8}$	$\left(-3, \frac{1}{8}\right)$
-2	$2^{-2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$2^{-1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$2^0 = 1$	$(0, 1)$
1	$2^1 = 2$	$(1, 2)$
2	$2^2 = 4$	$(2, 4)$
3	$2^3 = 8$	$(3, 8)$



Definition A function of the form $f(x) = b^x$ where b is a fixed real number, $b > 0$, $b \neq 1$ is called **base b exponential function**

- Therefore, b is the factor by which $f(x)$ increases or decreases when x increases by one unit.
- For $b > 1$ the function $f(x)$ is strictly increasing.
- For $0 < b < 1$ the function $f(x)$ is strictly decreasing.

The Number e

The most important base for exponential functions is the irrational number

$$e \approx 2.71828182845904523536028747135266249775724709369995$$

It is sometimes called *Euler's number*, after the Swiss mathematician Leonhard Euler (1707-1783). It actually is this:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots$$

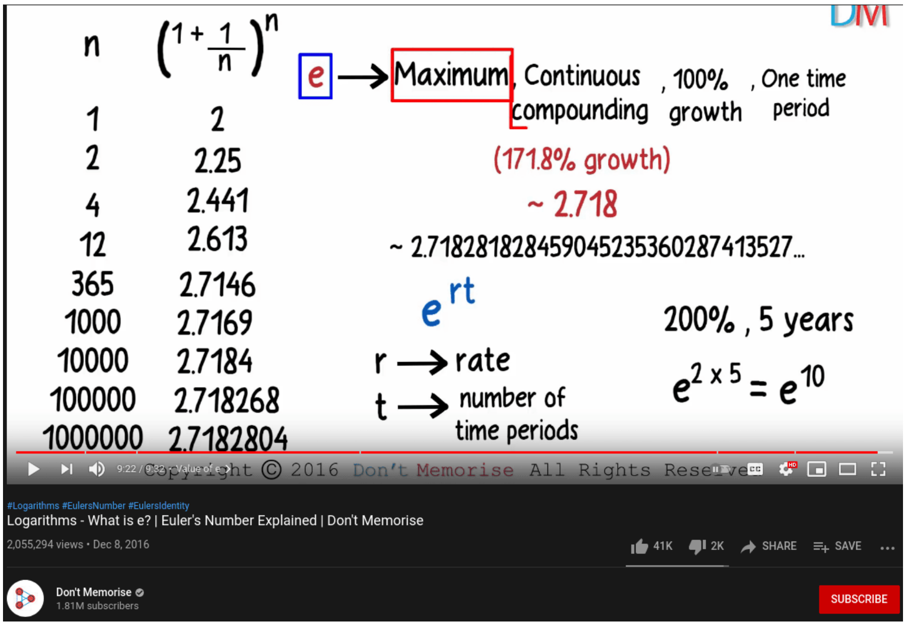
or as a Taylor series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

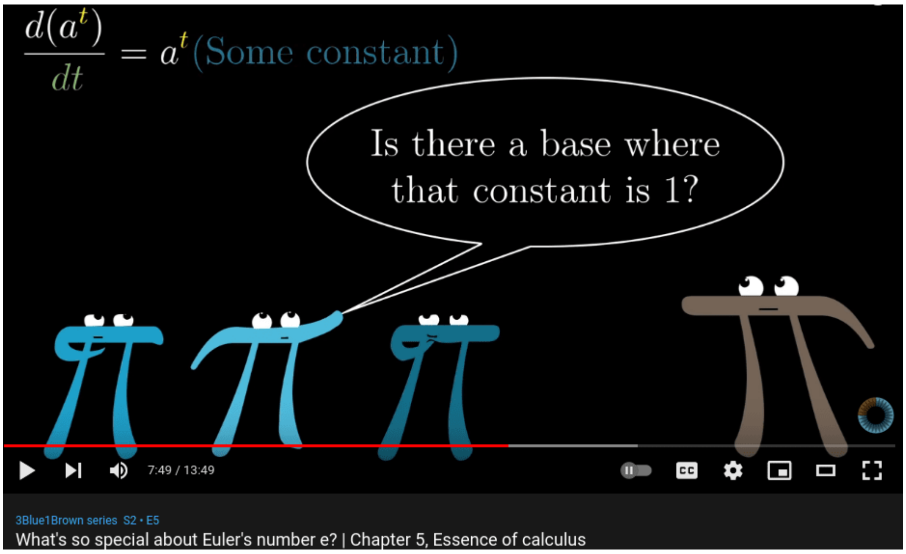
or as this

$$e^k = \lim_{n \to \infty} \left(1 + \frac{k}{n}\right)^n$$

Watch https://youtu.be/_-x90wGBD8U



Watch https://youtu.be/m2MIpDrF7Es



3.8.3 Logarithm and Growth Rates

Exercise 3.8 — Rule of 70 (Solution → p. 72)

The *Rule of 70* is often used to approximate the time required for a growing series to double. To understand this rule calculate how many periods it takes to double your money when it growth at a constant rate of 1% each period.

3.8.4 Difference of Logarithmic Functions and Growth Rates

Most data are recorded for discrete periods of time (e.g., quarters, years). Consequently, it is often useful to model economic dynamics in discrete periods of time. A good linear approximation to a growth rate from time $t = 0$ to $t = 1$ in x is $\ln x_0 - \ln x_1$:

$$\frac{x_1 - x_0}{x_0} \approx \ln x_1 - \ln x_0$$

Let us proof that with some numbers of per capita real GDP for the US and Japan in 1950 and 1989:

1950	1950	1989
US	8611	18317
Japan	1563	15101

What are the annual average growth rates over this period for the US and Japan? Here is one way to answer this question:

$$Y_{1989} = (1+g)^{39} \cdot Y_{1950}$$

Consequently,g can be calculated

$$(1+g) = \left(\frac{Y_{1989}}{Y_{1950}}\right)^{\frac{1}{39}}$$

Yielding $g = 0.0195$ for the US and $g = 0.0597$ for Japan. The US grew at an average growth rate of about 2% annually over the period while Japan grew at about 6% annually.

Log Growth Rates: The following method gives a close approximation to the answer above, and will be useful in other contexts. A useful approximation is that for any small number x:

$$\ln(1+x) \approx x$$

Now, we can take the natural log of both sides of

$$\frac{Y_{1989}}{Y_{1950}} = (1+g)^{39}$$

to get

$$\ln(Y_{1989}) - \ln(Y_{1950}) = 39 \cdot \ln(1+g)$$

which rearranges to

$$\ln(1+g) = \frac{\ln(Y_{1989}) - \ln(Y_{1950})}{39}$$

and using our approximation

$$g \approx \frac{\ln(Y_{1989}) - \ln(Y_{1950})}{39}$$

In other words, log growth rates are good approximations for percentage growth rates. Calculating log growth rates for the data above, we get $g \approx 0.0194$ for the U.S. and $g \approx 0.0582$ for Japan. The approximation is close for both.

3.8.5 Log Plots

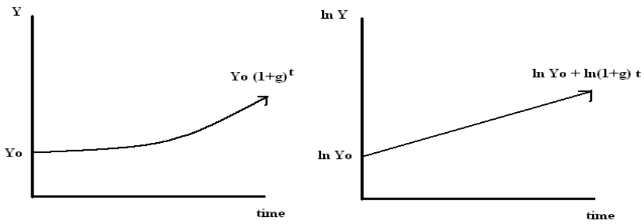
Recall that, with a constant growth rate g and starting from time 0, output in time t is

$$Y_t = (1+g)^t \cdot Y_0$$

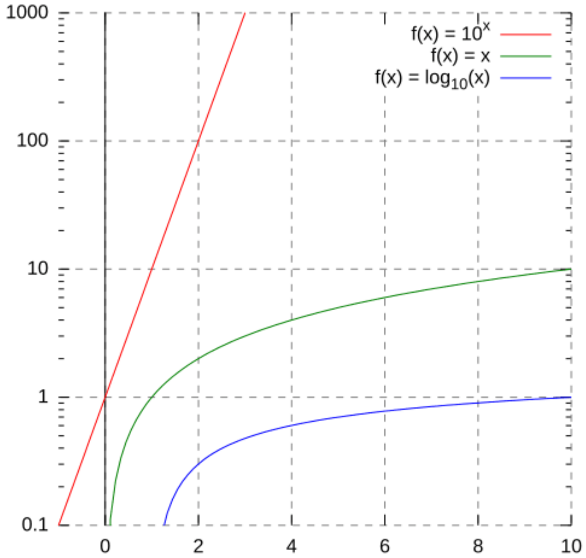
Taking natural logs of both sides,

$$\ln Y_t = \ln Y_0 + t \cdot \ln(1+g)$$

we see that log output is linear in time. Thus, if the growth rate is constant, a plot of log output against timewill yield a straight line. Consequently, plotting log output against time is a quick way to ***eyeball*** whether growth rates have changed over time

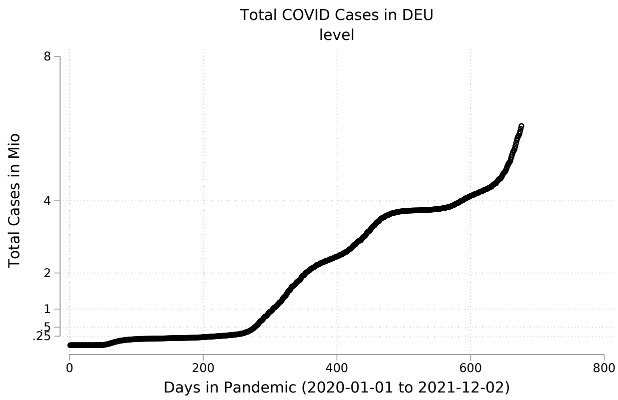


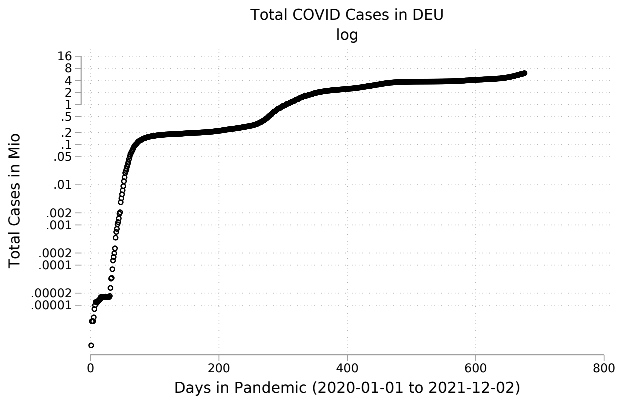
Here you see a semi-logarithmic plot that has one axis on a logarithmic scale, the other on a linear scale. It is useful for data with exponential relationships, where one variable covers a large range of values, or to zoom in and visualize that - what seems to be a straight line in the beginning - is in fact the slow start of a logarithmic curve that is about to spike and changes are much bigger than thought initially.

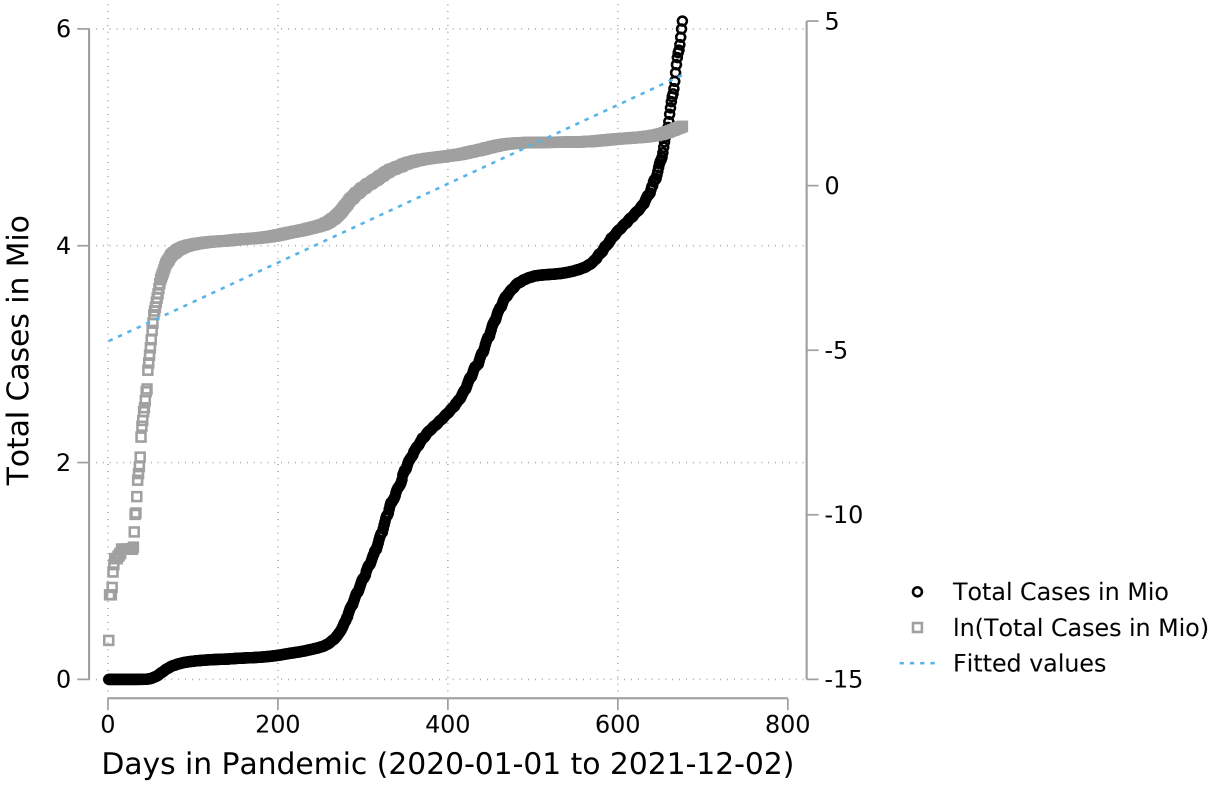


Exercise 3.9 — COVID and how to plot it (Solution → p. ??)

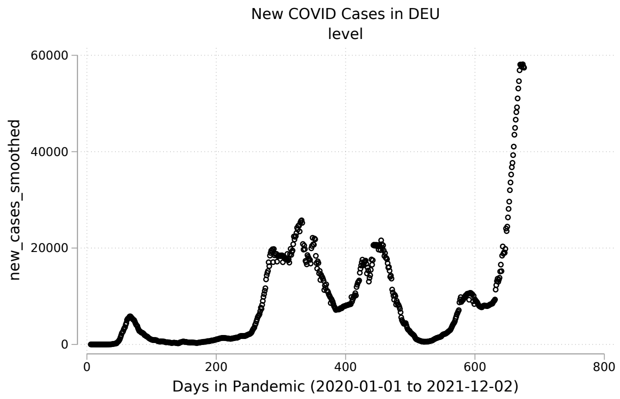
I downloaded the complete *Our World in Data COVID-19* dataset from `https://ourworldindata.org/coronavirus/country/germany`. I created some graphs which I show you below. Can you discuss the scalling and how to interpret it. What is your opinion on these graphs. Are some of them a bit misleading (at least if you don't look twice)?



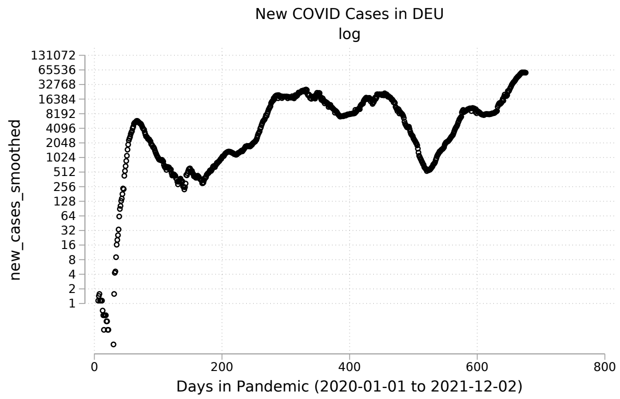




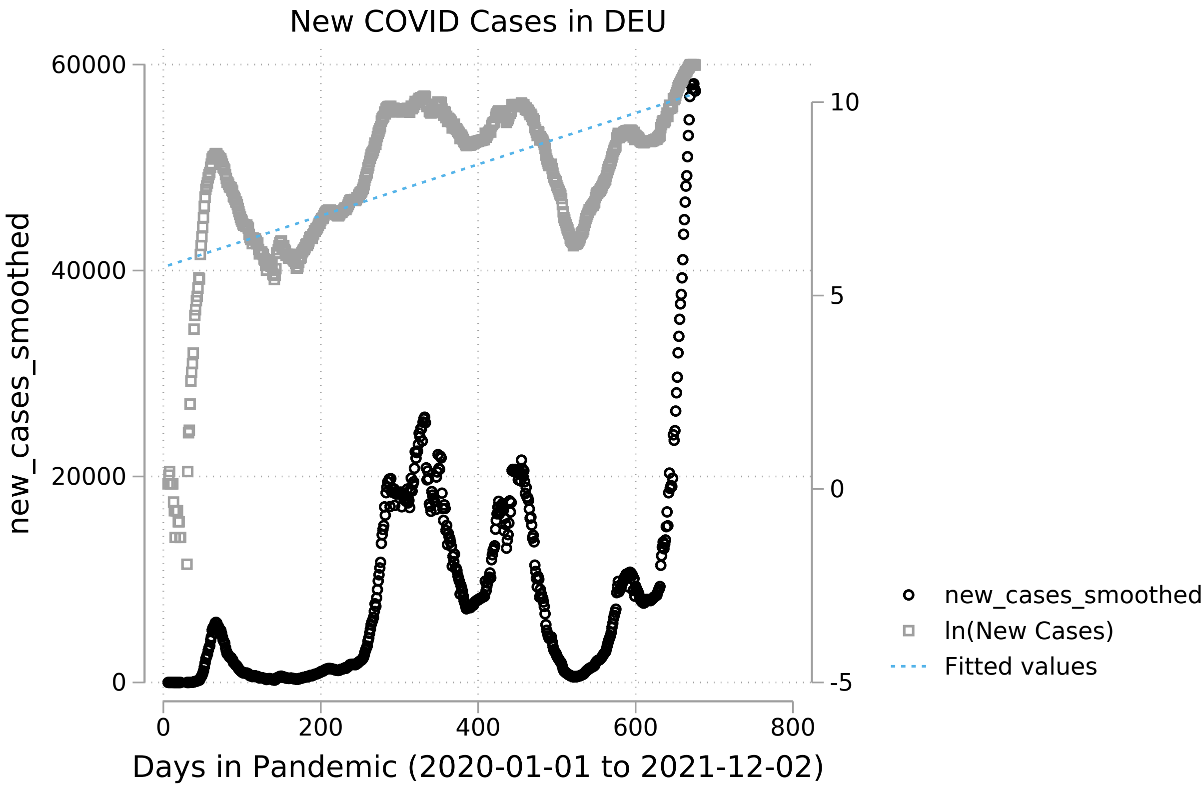
Total Cases in Mio
0
2
4
6
5
0
-5
-10
-15
0
200
400
600
800
Days in Pandemic (2020-01-01 to 2021-12-02)
Total Cases in Mio
ln(Total Cases in Mio)
Fitted values



New COVID Cases in DEU
level
new_cases_smoothed
0
20000
40000
60000
0
200
400
600
800
Days in Pandemic (2020-01-01 to 2021-12-02)



New COVID Cases in DEU
log
new_cases_smoothed
131072
65536
32768
16384
8192
4096
2048
1024
512
256
128
64
32
16
8
4
2
1
0
200
400
600
800
Days in Pandemic (2020-01-01 to 2021-12-02)



Exercise 3.10 — Investments over time (Solution → p. 73)

Describe the formulas to describe the growth process of an investment over time when time is discrete and when time is continuous.

Chapter 4

Series

4.1 Sequences

Sequences are very important mathematical objects, for example in financial mathematics as they can represent a stream of cash flows that occur at regular intervals.

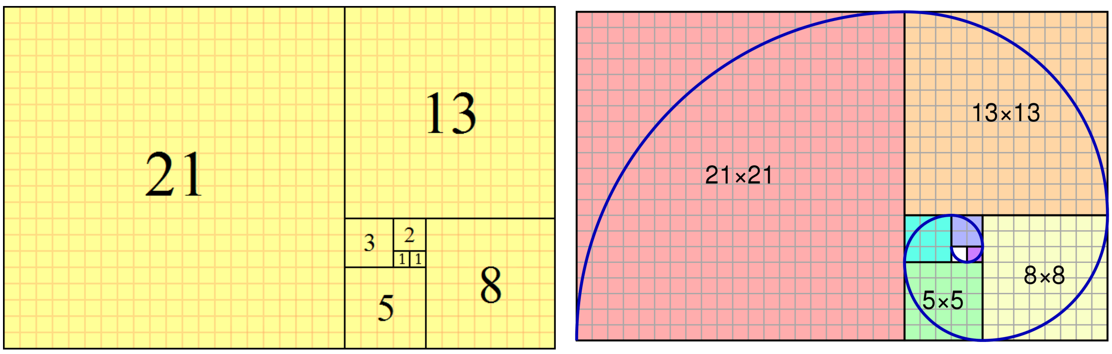
> A sequence is an ordered collection of (generally defined) objects with index in $\mathbb{N}$:
>
> $$a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots$$

Some sequences have useful characteristics that we will exploit here, in particular sequences that can be specified by *recursion*. A sequence is so specified when we can write

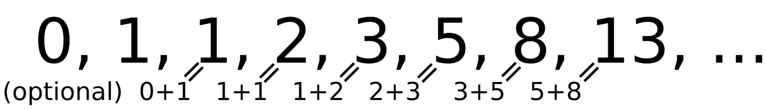
$$a_n = f(a_{n-1})$$

In other words, this recursive specification explains how one element in the sequence is obtained given the previous element(s). Specified in such a form, a sequence requires seeds, i.e., the first value(s). Notice that our goal will often be to find the pattern between the consecutive elements of the sequence. Two types of sequences are of special interest for us. They are described below, after the mention of a famous sequence.

The Fibonacci Sequence



As an famous illustration of a sequence defined recursively, we introduce the Fibonacci sequence (or Fibonacci numbers). These are the following.



The formal sequence is

$$F_0 = 1, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \forall \quad n \geq 2$$

When taking any two successive Fibonacci Numbers (2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...), their ratio tends to converge to a number that is known as the Golden Ratio which is symbolized with the Greek letter phi, ϕ, Φ.

A	B	$\frac{B}{A}$
2	3	1.5
3	5	1.666666666...
5	8	1.6
8	13	1.625
13	21	1.615384615...
...	...	...
144	233	1.618055556...
233	377	1.618025751...
...	...	...

The Golden Ratio

This particular number is an irrational number, is called the *Golden Ratio*, ϕ and can be defined by

$$\phi := \frac{a+b}{a} = \frac{a}{b}.$$

You can find the value of ϕ through simplifying the fraction and substituting in $\frac{b}{a} = \frac{1}{\phi}$:

$$\frac{a+b}{a} = \frac{a}{a} + \frac{b}{a} = 1 + \frac{b}{a} = 1 + \frac{1}{\phi}.$$

Therefore,

$$1 + \frac{1}{\phi} = \phi$$

Watch https://youtu.be/2tv6Ej6JVho *What is the Fibonacci Sequence & the Golden Ratio? Simple Explanation and Examples in Everyday Life*

> Solve
> $$\phi = 1 + \frac{1}{\phi}$$
> and calculate an approximation for ϕ.

$$x = 1 + \frac{1}{x}$$

$$x^2 = x + 1$$

$$x^2 - x - 1 = 0$$

$$\left(x - \frac{1}{2}\right)^2 - \frac{1}{4} - 1 = 0$$

$$\left(x - \frac{1}{2}\right)^2 = \frac{5}{4}$$

$$x - \frac{1}{2} = \left(\frac{5}{4}\right)^{\frac{1}{2}}$$
$$x - \frac{1}{2} = \pm\frac{\sqrt{5}}{2}$$
$$x = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$$
$$x_1 \approx 1.618033988749$$
$$x_2 \approx -0.618033989$$

Watch https://youtu.be/sj8Sg8qnj0g *The Golden Ratio (why it is so irrational) - Numberphile*

4.2 Finite Geometric Series

The sum of the first N elements of a sequence

$$s_N = a + ak + ak^2 + ak^3 + \ldots + ak^{N-2} + ak^{N-1}$$

is called a finite geometric series with quotient k To find the sum s_N of any series, multiply both sides of the above expression by k to obtain

$$ks_N = ak + ak^2 + ak^3 + \ldots + ak^{N-1} + ak^N$$

Then subtract this expression by the definition above

$$\begin{aligned} ks_N - s_N =& ak + ak^2 + ak^3 + \ldots + ak^{N-1} + ak^N \\ &-a - ak - ak^2 - ak^3 - \ldots - ak^{N-1} \\ &= ak^N - a \end{aligned}$$

For $k \neq 1$, this implies

$$s_N = a\frac{k^N - 1}{k - 1} = a\frac{1 - k^N}{1 - k}$$

Example: The geometric series for

$$2 + 4 + 8 + 16 + 32 + ... + 4096$$

is a sequence with $a = 2$ and $q = 2$:

$$\sum_{i=0}^{n} aq^{i-1} = \sum_{i=0}^{n} 2 \cdot 2^{i-1} = 2 \cdot 2^0 + 2 \cdot 2^1 + 2 \cdot 2^2 + ... + 2 \cdot 2^{n-1}$$

To find n algebraically, we can use the last element, a_n:

$$\begin{aligned} 2 \cdot 2^{n-1} &= 4096 \\ 2^{n-1} &= 2048 \\ (n-1)\ln 2 &= \ln 2048 \\ n &= \frac{\ln 2048}{\ln 2} + 1 = 12. \end{aligned}$$

Thus, the series has 12 elements:

$$\sum_{i=1}^{12} aq^{i-1} = \sum_{i=1}^{n} 2 \cdot 2^{i-1} = 2 \cdot 2^0 + 2 \cdot 2^1 + 2 \cdot 2^2 + ... + 2 \cdot 2^{11}$$

To find the sum of the series, s_N, we can apply the formula derived above:

$$s_N = a\frac{1 - k^N}{1 - k} = 2\frac{1 - 2^{12}}{1 - 2} = 2 \cdot (2^{12} - 1) = 8190$$

4.3 Infinite Geometric Series

A natural question arises, what happens when N becomes extremely large, i.e., when it tends to infinity? Notice that, as a first impression, it is not trivial to think of the sum of infinitely many elements. Using the formula above, the question here becomes

$$\lim_{N\to\infty} s_N = a\frac{1-k^N}{1-k}$$

This expression depends mainly on the behavior of k^N. If $k > 1$ or $k \leq -1$, then k^N does not tend to any limit and we say that the infinite series diverges. If $|k| < 1$, then k^N converges to 0. Thus, the sum also converges to the limit $\frac{a}{1-k}$ and we can note that

If $|k| < 1$, the infinite geometric series formula is,

$$\sum_{n=1}^{\infty} ak^{n-1} = a + ak + ak^2 + ak^3 + \ldots = \frac{a}{1-k}$$

When $a = 1$, this can be simplified to

$$1 + 1k + 1k^2 + 1k^3 + \ldots = \frac{1}{1-k}$$

Exercise 4.1 — Proof: 0.9999... is equal to 1 (Solution → p. 74)

Show that $0.\bar{9} = 1$ by applying an infinite geometric series.

Exercise 4.2 — Exponential Growth (Solution → p. 74)

Sketch a timeline for each of the following series:

- $a_t = a_{t-1} + g$
- $\ln(a_t)$
- $b_t = b_{t-1} \cdot (1 + g)$
- $\ln b_t$

Chapter 5

Constrained Optimization

5.1 Linear Programming

Linear Programming is a common technique for decision making under certainty. It allows to express a desired benefit (such as profit) as a mathematical function of several variables. The solution is the set of values for the independent variables (decision variables) that serves to maximize the benefit or to minimize the negative outcome under consideration of certain limits, a.k.a. constraints. The method usually follows a four step procedure:

1. state the problem;
2. state the decision variables;
3. set up an objective function;
4. clarify the constraints.

Exercise 5.1 — Production Under Constraints and Fixed Prices (Solution → p. 74)

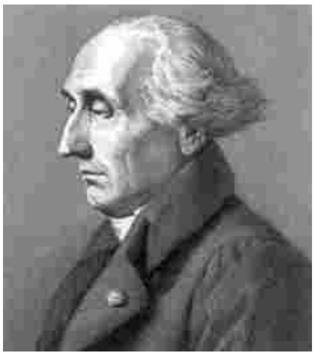
Consider a factory producing two products, product X and product Y. The problem is this: If you can realize $10.00 profit per unit of product X and $14.00 per unit of product Y, what is the production level of x units of product X and y units of product Y that maximizes the profit P each day? Your production, and therefore your profit, is subject to resource limitations, or constraints. Assume in this example that you employ five workers—three machinists and two assemblers—and that each works only 40 hours a week.

- Product X requires three hours of machining and one hour of assembly per unit.
- Product Y requires two hours of machining and two hours of assembly per unit.

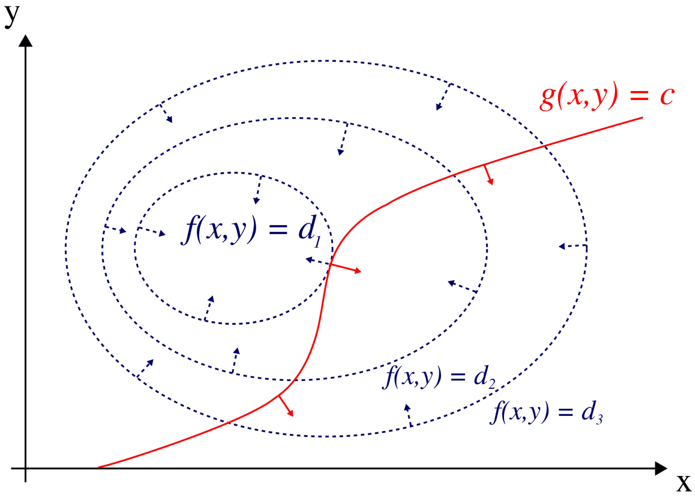
5.2 Lagrange Multiplier Method

The method is a strategy for finding the local maxima and minima of a function subject to constraints.[1]

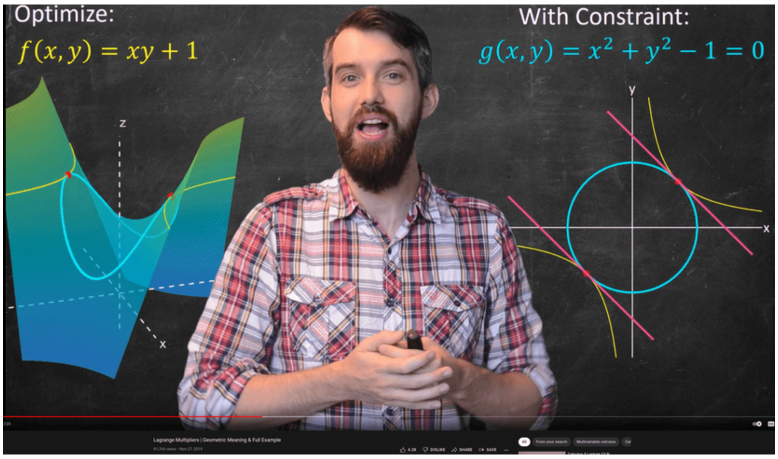
[1] For a detailed mathematical explanation, feel free to watch Aviv Censor's video https://youtu.be/AxEVJoxv-Z8.



Joseph-Louis Lagrange
(1736-1813)



The red curve shows the constraint $g(x, y) = c$. The blue curves are contours of $f(x, y)$. The point where the red constraint tangentially touches a blue contour is the maximum of $f(x, y)$ along the constraint, since $d1 > d2$.



https://youtu.be/8mjcnxGMwFo

Lagrange Multipliers | Geometric Meaning & Full Example

Figure 5.1: Lagrange Multiplier graphically explained

Step 1: The problem to be solved The problem that we want to solve can be written in the following way,

$$\begin{array}{ll} \max_{x,y} & F(x,y) \\ \text{s.t.} & g(x,y)=0 \end{array}$$

where $F(x,y)$ is the function to be maximized and $g(x,y)=0$ is the constraint to be respected. Notice that $\max_{x,y}$ means that we must solve (maximize) with respect to x and y.

Step 2: Define the Lagrangian Multiplier Define a new function, the *Lagrangian* $\mathcal{L}$, by combining the two functions of the problem and adding the new variable λ. The λ is called the *Lagrange Multiplier*.

$$\mathcal{L}(x,y,\lambda)=F(x,y)+\lambda g(x,y)$$

Step 3: Find the first order conditions Differentiate $\mathcal{L}$ w.r.t. x, y, and λ and equate the partial derivatives to 0:

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial x}=0 \Leftrightarrow \frac{\partial F(x,y)}{\partial x}+\lambda \frac{\partial g(x,y)}{\partial x}=0$$

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial y}=0 \Leftrightarrow \frac{\partial F(x,y)}{\partial y}+\lambda \frac{\partial g(x,y)}{\partial y}=0$$

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial \lambda}=0 \Leftrightarrow g(x,y)=0$$

Step 4: Solve the system of equations The solution to the system of three equations above gives the required optimal quantities.

Exercise 5.2 — Burgers and Drinks (Solution → p. 76)

Suppose you are in a fast food restaurant and you want to buy burgers and some drinks. You have €12 to spend, a burger costs €3 and a drink costs €2.

a) Assume that you want to spend all your money and that you can only buy complete units of each products. What are the possible choices of consumption?

b) Given your utility function $U(x,y)=B^{0.6}D^{0.4}$ calculate for each possible consumption point your overall utility. How will you decide?

c) Assume that you want to spend all your money and that both products can be bought on a metric scale where one burger weights 200 grams and a drink is 200 ml. How much of both goods would you consume now? *Hint: Use the Lagrangian multiplier method.*[a]

[a] Also see: http://www.sfu.ca/~wainwrig/5701/notes-lagrange.pdf

Exercise 5.3 — Consumption Choice (Solution → p. 77)

Suppose you want to spend your complete budget of €30,

$$I=30,$$

on the consumption of two goods, A and B. Further assume good A costs €6,

$$p_A=6,$$

and good B costs €4,

$$p_B=4$$

and that you want to maximize your utility that stems from consuming the two goods. Calculate how much of both goods to buy and consume, respectively, when your utility function is given as

$$U(A,B)=A^{0.8}B^{0.2}$$

Exercise 5.4 — Cost-Minimizing Combination of Factors (Solution → p. 78)

Using two input factors r_1 and r_2, a firm wants to produces a fixed quantity of a product, that is $x = 20$. Given the production function

$$x = \frac{5}{4} r_1^{\frac{1}{2}} r_2^{\frac{1}{2}}$$

and the factor prices

$$p_{r_1} = 1 \quad \text{and } p_{r_2} = 4.$$

calculate the cost-minimizing combination of factors (r_1, r_2).

Exercise 5.5 — Derivation of Demand Function (Solution → p. 78)

A representative consumer has on average the following utility function: $U = xy$, and faces a budget constraint of $B = P_x x + P_y y$, where B, P_x and P_y are the budget and prices, which are given. Solve the following choice problem:
Maximize $U = xy$ s.t. $B = P_x x + P_y y$.

Exercise 5.6 — Cobb-Douglas and Demand (Solution → p. 78)

A consumer who has a Cobb-Douglas utility function $u(x, y) = Ax^\alpha y^\beta$ faces the budget constraint $px + qy = I$, where A, α, β, p, and q are positive constants. Solve the consumer's problem:

$$\max Ax^\alpha y^\beta \quad \text{subject to} \quad px + qy = I$$

Exercise 5.7 — Lagrange with n-constraints (Solution → p. 79)

Write down the Lagrangian multiplier for the following minimization problem:
Minimize $f(\mathbf{x})$ subject to:

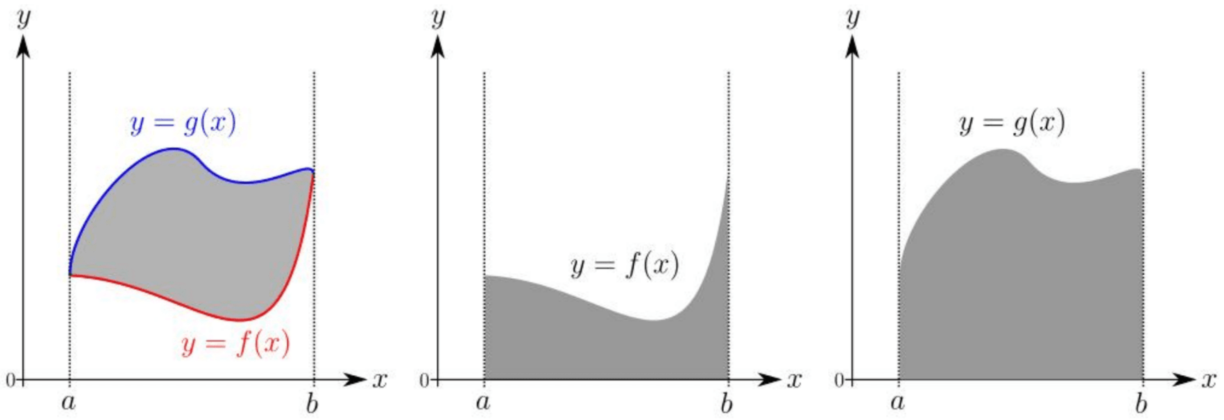
$$\begin{aligned} g_1(\mathbf{x}) &= 0 \\ g_2(\mathbf{x}) &= 0 \\ &\vdots \\ g_n(\mathbf{x}) &= 0, \end{aligned}$$

where n denotes the number of constraints.

Chapter 6

Integration

- ***Differentiation*** is used to describe the run of a curve of a function.
- ***Integration*** is used to calculate areas that are enclosed by curves of functions.
- The **area** A of a region which lies under a function $y = g(x)$ and above the function $y = f(x)$ (so for $f(x) = 0$ above the x-axis) enclosed between the limits $x = a$ and $x = b$



The Antiderivative (Primitive)

The area A can be calculated using the abbreviation $h(x) = g(x) - f(x)$ by the integral

$$A = \int_a^b h(x)\mathrm{d}x = H(b) - H(a)$$

where $H(x)$ is a primitive of the function $h(x)$ that is

$$H'(x) = h(x)$$

Examples

Primitive $H(x)$	Function $h(x)$	Derivative $h'(x)$
$\frac{1}{a+1} \cdot x^{a+1}$	$x^a (a \neq -1 \text{ real! })$	$a \cdot x^{a-1}$
$\ln(x)$	$x^{-1} = \frac{1}{x}$	$-1/x^2$
e^x	e^x	e^x
$x \cdot \ln(x) - x$	$\ln(x)$	$1/x$

Calculation Steps

The area A can be calculated by the steps

1. Sketch the area
2. Determine the boundaries a and b if necessary
3. Set up the integral
4. Find the primitive $F(x)$ for $f(x)$
5. Evaluate $F(b) - F(a)$

Exercise 6.1 — Integral I (Solution → p. 79)

Calculate the area in the first quadrant bounded by

$$y = 4x - x^2$$

and the x-axis.

Exercise 6.2 — Integral II (Solution → p. 80)

Calculate the area in the first quadrant bounded by the following three curves

$$y = x^2 - 4$$
$$y = 0$$
$$x = 4.$$

Exercise 6.3 — Integral III (Solution → p. 80)

Calculate the area in the first quadrant of the region enclosed by

$$y = e^x,$$
$$y = x - 1,$$
$$x = 2$$

Exercise 6.4 — Consumer and Producer Surplus (Solution → p. 81)

Suppose that for $0 \leqslant x \leqslant 80$ the demand for a good is given by function

$$p(x) = -0.01x^2 - 0.2x + 80$$

and the supply by the function

$$p_S(x) = 0.03x^2 + 0.6x + 20.$$

Determine the consumer and producer surplus. Sketch both functions and mark the consumer and producer surplus.

Chapter 7

Matrices and Vectors

7.1 Introduction

- A matrix allows the representation of information that is usually displayed in a table in equations.
- Hence, operations like addition or multiplication can be applied to a set of variables at once.
- This can simplify the solution methods in case of simultaneous linear equations.
- Moreover, it can help a lot to ease the notation and *keep the overview* and simplify data management.
- Suppose a firm produces three types of goods that are sold to two customers. The monthly sales are given in the following table:

	Monthly sales		
Sold to costumer	$G1$	$G2$	$G3$
$C1$	7	3	4
$C2$	1	5	6

- If we know from context what the numbers represent we may ignore table headings and write the content in matrix form:

$$\mathbf{A} = \begin{bmatrix} 7 & 3 & 4 \\ 1 & 5 & 6 \end{bmatrix}$$

- This is a matrix of order 2×3 (2 rows, 3 columns). The element in the first row, second column is denoted by a_{12}.
- A (general) matrix $\mathbf{D}$ of order 3×2 would be written:

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \end{bmatrix}$$

- A matrix consisting of one row is a row vector.
- A matrix consisting of one column is a column vector.
- A matrix consisting of one row and one column is a scalar.

7.2 Basic Operations

7.2.1 Transposition

- In the first example, the different goods were represented in columns, whereas the different customers were listed in the two rows. A transposed matrix contains the same information but in a rearranged form. We put then the goods in the rows and the customers in the columns.
- Let's take the original matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 7 & 3 & 4 \\ 1 & 5 & 6 \end{bmatrix}$$

- We can now get the transposed matrix $\mathbf{A}^T$ and denote it by $\mathbf{B}$

$$\mathbf{B} = \begin{bmatrix} 7 & 1 \\ 3 & 5 \\ 4 & 6 \end{bmatrix} = \mathbf{A}^T$$

- Obviously, $\mathbf{A}^T = \mathbf{B}$ is equivalent to $\mathbf{B}^T = \mathbf{A}$
- Moreover, if the order of $\mathbf{A}$ is $m \times n$, then the order of the transposed $\mathbf{A}^T$ is $n \times m$

7.2.2 Matrix Addition

If we add matrices, we have to calculate the sum of elements in the same position. Consequently, addition is only possible if the matrices are of equal order.

Two matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

are added together component by component

$$A + B = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

From the definition we can see that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

Economic Example

- Assume the monthly sales in January given by $\mathbf{A}$ and in February given by $\mathbf{B}$.

$$\mathbf{B} = \begin{bmatrix} 6 & 2 & 1 \\ 0 & 4 & 4 \end{bmatrix}$$

- Find the sales in the first two months $\mathbf{C}$.

$$\mathbf{C} = \begin{bmatrix} 7+6 & 3+2 & 4+1 \\ 1+0 & 5+4 & 6+4 \end{bmatrix} = \begin{bmatrix} 13 & 5 & 5 \\ 1 & 9 & 10 \end{bmatrix}$$

Example

$$\begin{pmatrix} 3 & -5 \\ 0 & 12 \end{pmatrix} + \begin{pmatrix} -2 & 7 \\ 8 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 8 & 6 \end{pmatrix}$$

7.2.3 Scalar Multiplication

A matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is multiplied by a scalar, that is a real number, say s, we have to multiply each element of the matrix with the scalar:

$$s \cdot A = s \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} s \cdot a & s \cdot b \\ s \cdot c & s \cdot d \end{pmatrix}$$

Example

$$(-2) \cdot \begin{pmatrix} 3 & -5 \\ 0 & 12 \end{pmatrix} = \begin{pmatrix} -6 & 10 \\ 0 & -24 \end{pmatrix}$$

Economic Example Assume the monthly sales are always the same. Then, the sales per year can be calculated by multiplying $\mathbf{A}$ with 12.

$$\mathbf{B} = 12 \cdot \mathbf{A} = \begin{bmatrix} 12 \cdot 7 & 12 \cdot 3 & 12 \cdot 4 \\ 12 \cdot 1 & 12 \cdot 5 & 12 \cdot 6 \end{bmatrix} = \begin{bmatrix} 84 & 36 & 48 \\ 12 & 60 & 72 \end{bmatrix}$$

The usual properties of multiplication operations are applicable, that implies:

$$k \cdot (\mathbf{A} + \mathbf{B}) = k \cdot \mathbf{A} + k \cdot \mathbf{B}$$
$$k \cdot (l \cdot \mathbf{A}) = (k \cdot l) \cdot \mathbf{A}$$

7.2.4 Matrix Multiplication

Multiply two matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

using the rule "every row with every column"

$$A \cdot B = \begin{pmatrix} a \cdot e + b \cdot g & a \cdot f + b \cdot h \\ c \cdot e + d \cdot g & c \cdot f + d \cdot h \end{pmatrix}$$

Note A matrix multiplication $A \cdot B$ is only possible if the number of columns of A is equal to the number of rows of B, that is,
A is an $(m \times n)$ -matrix and
B is an $(n \times l)$ -matrix.
The matrix product $A \cdot B$ then yields a $(m \times l)$ -matrix.

Example 1:

$$A \cdot B = \begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix} \cdot \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix} \quad \neq \quad B \cdot A = \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ 3 & -2 \end{pmatrix}$$

Example 2: The (3×2) -matrix A times the (2×4) -matrix B results in

$$B = \underbrace{\begin{pmatrix} 4 & -1 & 2 & 0 \\ -2 & 3 & 1 & -3 \end{pmatrix}}_{(2x4)}, \quad A = \underbrace{\begin{pmatrix} 0 & 3 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}}_{(3x2)}$$

$$A \cdot B = \underbrace{\begin{pmatrix} 0 & 3 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}}_{(3x2)} \cdot \underbrace{\begin{pmatrix} 4 & -1 & 2 & 0 \\ -2 & 3 & 1 & -3 \end{pmatrix}}_{(2x4)} = \underbrace{\begin{pmatrix} -6 & 9 & 3 & -9 \\ 6 & 1 & 5 & -3 \\ 4 & -1 & 2 & 0 \end{pmatrix}}_{(3x4)}$$

So $A \cdot B$ is a (3×4) -matrix, **but $B \cdot A$ is not possible!**

7.2.5 Matrix-Vector Product

A matrix A with n columns can always be multiplied by a vector $\vec{x}$, which has n rows.

Example
With the matrix

$$A = \begin{pmatrix} 7 & 1 & -3 \\ 2 & -1 & 5 \end{pmatrix}$$

and the vector

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

one gets the matrix-vector product

$$A \cdot \vec{x} = \begin{pmatrix} 7x_1 + x_2 - 3x_3 \\ 2x_1 - x_2 + 5x_3 \end{pmatrix}$$

Exercise 7.1 — Matrices (Solution → p. 82)

Calculate with the matrices

$$A = \begin{pmatrix} 1 & -2 & 4 \\ -3 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 0 & 3 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 5 & -1 & 0 \\ -4 & 6 & 7 \\ -3 & 2 & -2 \end{pmatrix}$$

if possible the following products:

1. $A \cdot B$
2. $B \cdot A$
3. $A \cdot C$
4. $C \cdot A$
5. A^2
6. $B \cdot C$
7. C^2
8. $(A \cdot B)^2$
9. $A \cdot C \cdot B$

Check out https://www.symbolab.com/solver/matrix-vector-calculator

Chapter 8

Rules of Algebra

Let a, b, c, and d be any real numbers.

8.1 Basic Arithmetic

a) $a + b = b + a$

b) $(a + b) + c = a + (b + c)$

c) $a + 0 = a$

d) $a + (-a) = 0$

e) $ab = ba$

f) $(ab)c = a(bc)$

g) $1 \cdot a = a$

h) $aa^{-1} = 1$, for $a \neq 0$

i) $(-a)b = a(-b) = -ab$

j) $(-a)(-b) = ab$

k) $a(b + c) = ab + ac$

l) $(a + b)c = ac + bc$

m) $(a + b)^2 = a^2 + 2ab + b^2$

n) $(a - b)^2 = a^2 - 2ab + b^2$

o) $(a + b)(a - b) = a^2 - b^2$

8.2 Rules for Fractions

a) $\frac{a}{b} + \frac{a}{d} = \frac{ad+bc}{bd}$

b) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

c) $\frac{a}{b} : \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$

d) $\frac{a}{b} \cdot \frac{c}{c} = \frac{ac}{bc}$

8.3 Rules for Powers

a) $x^0 = 1$

b) $x^a \cdot x^b = x^{a+b}$

c) $\frac{x^a}{x^b} = x^{a-b}$

d) $x^a \cdot y^a = (x \cdot y)^a$

e) $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$

f) $(x^a)^b = x^{a \cdot b}$

g) $x^a \cdot y^b = z^c \Leftrightarrow x^{\frac{a}{c}} \cdot y^{\frac{b}{c}} = z$

Exercise 8.1 — Apply Basic Rules (Solution → p. 52)

Rearrange/simplify:

a) $$\frac{x}{y}-\frac{y}{x}=?$$

b) $$\frac{x^2+1}{x-1}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}=?$$

c) $$\frac{1}{\left(\frac{1}{x}\right)}=?$$

d) $$\frac{x}{1-x}\cdot\frac{1}{x}=?$$

e) $$\lambda^2 y+(\lambda x)\cdot(\lambda y)=?$$

f) $$x\cdot x^2\cdot x^5=?$$

g) $$\frac{(1-x)^2}{(1-x)^4}=?$$

h) $$\frac{x^{a+1}}{x^{b+1}}=?$$

i) $$\frac{1}{x^3}+\frac{1}{x^2}=?$$

j) $$\left(\frac{x-y}{x}\right)^2\cdot\left(\frac{y}{x-y}\right)^2=?$$

k) $$(-x^3)^2=?$$

l) $$(x^{-1})^{-1}=?$$

m) $$(-x^2)^3=?$$

Solution to Exercise 8.1 — Apply Basic Rules (Exercise → p. 51)

a) $\frac{x}{y}-\frac{y}{x}=\frac{x^2-y^2}{x\cdot y}$

b) $\frac{x^2+1}{x-1}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}=\frac{x+\frac{1}{x}}{1-\frac{1}{x}}$

c) $\frac{1}{\left(\frac{1}{x}\right)}=1\cdot\frac{x}{1}=x$

d) $\frac{x}{1-x}\cdot\frac{1}{x}=\frac{x\cdot 1}{x\cdot(1-x)}=\frac{1}{1-x}$

e) $\lambda^2 y+(\lambda x)\cdot(\lambda y)=\lambda^2 y+\lambda^2 xy=\lambda^2(y+xy)$

f) $x\cdot x^2\cdot x^5=x^{1+2+5}=x^8$

g) $\frac{(1-x)^2}{(1-x)^4}=(1-x)^{2-4}=(1-x)^{-2}=\frac{1}{(1-x)^2}$

h) $\frac{x^{a+1}}{x^{b+1}}=x^{a+1-b-1}=x^{a-b}=\frac{x^a}{x^b}$

i) $\frac{1}{x^3}+\frac{1}{x^2}=\frac{x^2+x^3}{x^3\cdot x^2}=\frac{x^2(x)}{x^5}=\frac{1+x}{x^3}$

j) $\left(\frac{x-y}{x}\right)^2\cdot\left(\frac{y}{x-y}\right)^2=\left[\frac{(x-y)\cdot y}{x\cdot(x-y)}\right]^2=\left(\frac{y}{x}\right)^2=\frac{y^2}{x^2}$

k) $(-x^3)^2=[(-1)\cdot x^3]^2=(-1)^2\cdot x^6=x^6$

l) $(x^{-1})^{-1}=x^{(-1)\cdot(-1)}=x^1=x$

m) $(-x^2)^3=[(-1)\cdot x^2]^{-3}=(-1)^3\cdot x^{-6}=\frac{1}{(-1)^3\cdot x^6}=-\frac{1}{x^6}$

8.4 Rules for Roots

a) $a^{\frac{1}{2}}=\sqrt{a}$

b) $\sqrt[n]{a}=a^{\frac{1}{n}}$ (valid if $a\geq 0$)

c) $a^{\frac{m}{n}}=\sqrt[n]{a^m}$

d) $\frac{1}{\sqrt[n]{a^m}}=a^{-\frac{m}{n}}$

e) $\sqrt{ab} = \sqrt{a}\sqrt{b}$

f) $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

g) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Exercise 8.2 — Apply Rules for Roots (Solution → p. 53)

a) $x\sqrt{x} = ?$

b) $\frac{x}{\sqrt{x}} = ?$

c) $\sqrt{\sqrt{x}} = ?$

d) $\sqrt{x\sqrt{x}} = ?$

e) $\sqrt{\frac{1}{x^4}} = ?$

f) $\frac{1}{x} \cdot \sqrt{y} \cdot \sqrt{\frac{x}{y}} = ?$

Solution to Exercise 8.2 — Apply Rules for Roots (Exercise → p. 53)

a) $x\sqrt{x} = x^1 \cdot x^{\frac{1}{2}} = x^{\frac{3}{2}} = \sqrt{x^3}$

b) $\frac{x}{\sqrt{x}} = x^1 \cdot x^{-\frac{1}{2}} = x^{\frac{1}{2}} = \sqrt{x}$

c) $\sqrt{\sqrt{x}} = \left(x^{\frac{1}{2}}\right)^{\frac{1}{2}} = x^{\frac{1}{4}} = \sqrt[4]{x}$

d) $\sqrt{x\sqrt{x}} = \sqrt{x\sqrt{x}} = \left(x^1 \cdot x^{\frac{1}{2}}\right)^{\frac{1}{2}} = \left(x^{\frac{3}{2}}\right)^{\frac{1}{2}} = x^{\frac{3}{4}} = \sqrt[4]{x^3}$

e) $\sqrt{\frac{1}{x^4}} = x^{-\frac{4}{2}} = x^{-2} = \frac{1}{x^2}$

f) $\frac{1}{x} \cdot \sqrt{y} \cdot \sqrt{\frac{x}{y}} = x^{-1} \cdot y^{\frac{1}{2}} \cdot x^{\frac{1}{2}} \cdot y^{-\frac{1}{2}} = x^{-\frac{1}{2}} \cdot y^0 = \frac{1}{\sqrt{x}}$

8.5 Rules for Quadratic Function

- The **roots** of a quadratic function, i.e. $ax^2 + bx + c = 0$, are defined by the formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  if $b^2 - 4ac \geq 0$.

- Factorization of a quadratic function also gives us the roots:

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

Example:

$$G(x) = -3x^2 + 90x - 375 = -3(x-5)(x-25)$$

Exercise 8.3 — Quadratic Functions (Solution → p. 54)

Caculate:

a) $(a+b)^2$

b) $(a-b)^2$

c) $(a+b)\cdot(a-b)$

d) $(x-\sqrt{2})(x+\sqrt{2})$

e) $(a-\frac{b}{2})^2$

f) $(a+b)^2\cdot(a-b)^2$

Solution to Exercise 8.3 — Quadratic Functions (Exercise → p. 54)

a) $(a+b)^2 = a^2 + 2ab + b^2$

b) $(a-b)^2 = a^2 - 2a + b^2$

c) $(a+b)\cdot(a-b) = a^2 - b^2$

d) $(x-\sqrt{2})(x+\sqrt{2}) = x^2 - \left(\sqrt{2}\right)^2 = x^2 - 2$

e) $(a-\frac{b}{2})^2 = a^2 - 2a\frac{b}{2} + \frac{b^2}{4} = a^2 - ab + \frac{b^2}{4}$

f) $(a+b)^2\cdot(a-b)^2 = [(a+b)\cdot(a-b)]^2 = \left[a^2-b^2\right]^2 = a^4 - 2a^2b^2 + b^4$

8.6 Rules of Differentiation

$$\begin{aligned}
\text{Rule 1:}\quad & f(x) = A \\
& \Rightarrow f'(x) = 0 \\
\text{Rule 2:}\quad & f(x) = A + g(x) \\
& \Rightarrow f'(x) = g'(x) \\
\text{Rule 3:}\quad & f(x) = Ag(x) \\
& \Rightarrow f'(x) = Ag'(x) \\
\text{Rule 4 (power):}\quad & f(x) = x^a \\
& \Rightarrow f'(x) = ax^{a-1}
\end{aligned}$$

with a being an arbitrary constant.

If both f and g are differentiable at x, then the sum $f+g$ and the difference $f-g$ are both differentiable at x, and

$$\text{Rule 5 (sums):} \quad h(x) = f(x) \pm g(x)$$
$$\Rightarrow \quad h'(x) = f'(x) \pm g'(x)$$

If both f and g are differentiable at x, then so is $h = f \cdot g$, and

$$\text{Rule 6 (products):} \quad h(x) = f(x) \cdot g(x)$$
$$\Rightarrow \quad h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Example: The function

$$h(x) = \left(x^3 - x\right)\left(5x^4 + x^2\right)$$

can be written as

$$h(x) = f(x) \cdot g(x)$$

with

$$f(x) = (x^3 - x)$$
$$g(x) = (5x^4 + x^2)$$

Therefore

$$\begin{aligned} h'(x) &= f'(x) \cdot g(x) + f(x) \cdot g'(x) \\ &= \left(3x^2 - 1\right)\left(5x^4 + x^2\right) + \left(x^3 - x\right)\left(20x^3 + 2x\right) \\ &= 35x^6 - 20x^4 - 3x^2 \end{aligned}$$

If both f and g are differentiable at x and $g(x) \neq 0$, then $h = f/g$ is differentiable at x, and

$$\text{Rule 7 (quotient):} \quad h(x) = \frac{f(x)}{g(x)}$$
$$\Rightarrow \quad h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

Example: The derivative of the function

$$h(x) = \frac{3x - 5}{x - 2} = \frac{f(x)}{g(x)}$$

is

$$\begin{aligned} h'(x) &= \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2} \\ &= \frac{3 \cdot (x - 2) - (3x - 5) \cdot 1}{(x - 2)^2} \\ &= \frac{-1}{(x - 2)^2} \end{aligned}$$

Note that $h(x)$ is strictly decreasing at all $x \neq 2$.

If g is differentiable at x and f is differentiable at $u = g(x)$, then the composite function $h(x) = f(g(x))$ is differentiable at x, and

$$\text{Rule 8 (sums):} \quad h'(x) = f'(u) \cdot g'(x) = f'(g(x)) \cdot g'(x)$$

In words: First differentiate the exterior function with respect to the interior function (kernel), then multiply by the derivative of the interior function.

Example: Let $f(u) = u^3$ and $g(x) = 2 - x^2$. The derivative of

$$h(x) = f(g(x)) = \left(2 - x^2\right)^3$$

is

$$\begin{aligned} h'(x) &= f'(g(x)) \cdot g'(x) \\ &= 3\left(2 - x^2\right)^2 \cdot (-2x) \\ &= -6x\left(4 - 4x^2 + x^4\right) \\ &= -6x^5 + 24x^3 - 24x \end{aligned}$$

Exercise 8.4 — First Derivates (Solution → p. 56)

$$\begin{aligned} f(x) = 5 \Rightarrow &\quad f'(x) = ? \\ f(x) = 5 + 2x \Rightarrow &\quad f'(x) = ? \\ f(x) = 5 \cdot 2x \Rightarrow &\quad f'(x) = ? \\ f(x) = x^3 \Rightarrow &\quad f'(x) = ? \\ f(x) = 3x^8 \Rightarrow &\quad f'(x) = ? \\ h(x) = x^3 - 5x^{-2} \Rightarrow &\quad h'(x) = ? \end{aligned}$$

Solution to Exercise 8.4 — First Derivates (Exercise → p. 56)

$$\begin{aligned} f(x) = 5 \Rightarrow &\quad f'(x) = 0 \\ f(x) = 5 + 2x \Rightarrow &\quad f'(x) = 2 \\ f(x) = 5 \cdot 2x \Rightarrow &\quad f'(x) = 5 \cdot 2 = 10 \\ f(x) = x^3 \Rightarrow &\quad f'(x) = 3x^2 \\ f(x) = 3x^8 \Rightarrow &\quad f'(x) = 3 \cdot 8x^7 = 24x^7 \\ h(x) = x^3 - 5x^{-2} \Rightarrow &\quad h'(x) = 3x^2 - \left(-2 \cdot 5x^{-3}\right) = 3x^2 + 10x^{-3} \end{aligned}$$

8.7 Rules for Summation

The **summation sign**, $\sum$, allows for a compact formulation of lengthy expressions.

Examples

$$N_1 + N_2 + N_3 + N_4 + N_5 + N_6 = N_1 + N_2 + \ldots + N_6 = \sum_{i=1}^{6} N_i$$

$$a_1(1-a_1) + a_2(1-a_2) + a_3(1-a_3) + a_4(1-a_4) + a_5(1-a_5) = \sum_{i=1}^{5} a_i(1-a_i)$$

$$(b)^3 + (2b)^4 + (3b)^5 + (4b)^6 + (5b)^7 + (6b)^8 = \sum_{i=1}^{6} (ib)^{2+i}$$

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \sum_{i=1}^{5} i^2$$

$$(5 \cdot 3 - 3) + (5 \cdot 4 - 3) + (5 \cdot 5 - 3) = \sum_{k=3}^{5} (5k-3)$$

$$(x_{3j} - \bar{x}_j)^2 + (x_{4j} - \bar{x}_j)^2 + \ldots + (x_{nj} - \bar{x}_j)^2 = \sum_{i=3}^{n} (x_{ij} - \bar{x}_j)^2$$

Rule (Additivity Property)

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

Rule (Homogeneity Property)

$$\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$$

and if $a_i = 1$ for all i then

$$\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i = c(n \cdot 1) = cn$$

Rule (Rule for Sums)

$$\sum_{i=1}^{n} i = 1 + 2 + \ldots + n = \frac{1}{2} n(n+1)$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{1}{2} n(n+1)\right)^2 = \left(\sum_{i=1}^{n} i\right)^2$$

$$\sum_{i=0}^{n} a^i = \frac{1 - a^{n+1}}{1 - a}$$

The double sum notation allows us to write lengthy expressions in a compact way.

$$\sum_{i=1}^{Z} b_i \sum_{j=1}^{S} a_{ij} b_j = \sum_{i=1}^{Z} \sum_{j=1}^{S} a_{ij} b_i b_j = \sum_{j=1}^{S} \sum_{i=1}^{Z} a_{ij} b_i b_j = \sum_{j=1}^{S} b_j \sum_{i=1}^{Z} a_{ij} b_i$$

Consider some summation sign $\sum_{i=1}^{Z}$. All variables with index i must be to the right of that summation sign.

Examples

$$\begin{aligned} & b_1\left(a_{11}b_1 + a_{12}b_2 + \ldots + a_{1S}b_S\right) \\ + & b_2\left(a_{21}b_1 + a_{22}b_2 + \ldots + a_{2S}b_S\right) \\ & \vdots \\ + & b_S\left(a_{S1}b_1 + a_{S2}b_2 + \ldots + a_{SS}b_S\right) \end{aligned}$$

This sum can be written in the form

$$\sum_{i=1}^{S} b_i\left(a_{i1}b_1 + a_{i2}b_2 + \ldots + a_{iS}b_S\right)$$

Writing the brackets in a more compact form like

$$\sum_{i=1}^{S} b_i \sum_{j=1}^{S} a_{ij}b_j = \sum_{i=1}^{S}\sum_{j=1}^{S} a_{ij}b_ib_j$$

Writing it in the following forms is not admissable:

$$\sum_{i=1}^{Z} b_j \sum_{j=1}^{S} a_{ij}b_i, \quad b_i \sum_{i=1}^{Z}\sum_{j=1}^{S} a_{ij}b_j, \text{ or } \quad \sum_{i=1}^{Z} a_{ij} \sum_{j=1}^{S} b_ib_j,$$

8.8 Rules for Logarithms and Exponentials

8.8.1 Exponential Function

$$\begin{aligned} y = f(x) = e^x = exp(x) & \quad \text{is strictly increasing } \forall x \in \mathbb{R} \\ D_f = \mathbb{R} =]-\infty, \infty[& \\ e^x > 0 \forall x \in \mathbb{R} & \\ e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{24} + \ldots & \\ e^0 = 1 & \\ e^1 = e = 2.718\ldots \text{Euler number} & \\ \lim_{x\to\infty} e^x \to \infty & \quad \text{and} \\ \lim_{x\to-\infty} e^x \to 0 & \end{aligned}$$

8.8.2 Logarithmic Function

$$\begin{aligned} y = f(x) = \ln(x) & \quad \text{is strictly increasing} \\ D_f = \mathbb{R}^+ =]0, \infty[& \\ \ln(x) = e^{\ln x} & \\ \ln 1 = 0 & \\ \ln e = 1 & \end{aligned}$$

8.8.3 Rules to Transform exp

$$e^{x+y} = e^x \cdot e^y$$
$$e^{x-y} = \frac{e^x}{e^y}$$
$$e^{ax} = (e^x)^a$$
$$e^{\ln x} = x$$

Exercise 8.5 — Play with exp (Solution → p. 59)

a) $e^{\frac{1}{2}\ln 16} =?$ wegen $e^{ax} =?$

b) $e^{-x} =?$

c) $\frac{e^x}{e^{-x}} =?$

d) $e^{1-x} \cdot e^{1+x} =?$

e) $e^{1-x^2} = 1$ Write as an exponential function.

Solution to Exercise 8.5 — Play with exp (Exercise → p. 59)

a) $e^{\frac{1}{2}\ln 16} = \left(e^{\ln 16}\right)^{\frac{1}{2}} = 16^{\frac{1}{2}} = \sqrt{16} = 4$ *because of* $e^{ax} = (e^x)^a$ *and* $e^{\ln x} = x$

b) $e^{-x} = (e^x)^{-1} = \frac{1}{e^x}$ *with* $\lim_{x\to\infty} e^{-x} = 0$ *und* $\lim_{x\to-\infty} e^{-x} = \infty$

c) $\frac{e^x}{e^{-x}} = e^{x-(-x)} = e^{2x}$ *because of* $\frac{e^x}{e^y} = e^{x-y}$

d) $e^{1-x} \cdot e^{1+x} = e^{1-x+1+x} = e^2$ *because of* $e^x \cdot e^y = e^{x+y} \Rightarrow x_1 = 1, x_2 = -1$ *because of* $e^{\ln x} = x$ *and* $e^0 = 1$

e) $e^{1-x^2} = 1 \Rightarrow \ln e^{1-x^2} = \ln 1 \Rightarrow 1 - x^2 = 0 \Rightarrow x^2 = 1 \Rightarrow x_1 = 1, x_2 = -1$ *because of* $\ln e^x = x$ *and* $\ln 1 = 0$

8.8.4 Rules to Transform log

$$\ln(x \cdot y) = \ln x + \ln y$$
$$\ln(\frac{x}{y}) = \ln x - \ln y$$
$$\ln x^a = a \cdot \ln x$$
$$\ln e^x = x$$

Exercise 8.6 — Play with log (Solution → p. 60)

a) $\ln \frac{1}{\sqrt{x}} =?$

b) $\ln x - \ln 2x =?$ wegen $\ln x - \ln y =?$

c) $\ln(x-1) + \ln(x+1) =?$ wegen $\ln x + \ln y =?$

d) $\ln\left(2 - x^2\right) = 0$

e) $f(x) = \ln(x+1)$ Solve for x.

Solution to Exercise 8.6 — Play with log (Exercise → p. 59)

a) $\ln \frac{1}{\sqrt{x}} = \ln x^{-\frac{1}{2}} = -\frac{1}{2} \ln x$ *because of* $\ln x^a = a \cdot \ln x$

b) $\ln x - \ln 2x = \ln \frac{x}{2x} = \ln \frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$ *because of* $\ln x - \ln y = \ln \frac{x}{y}$ *and* $\ln 1 = 0$

c) $\ln(x-1) + \ln(x+1) = \ln(x-1)(x+1) = \ln\left(x^2 - 1\right)$ *because of* $\ln x + \ln y = \ln(x \cdot y)$

d) $\ln\left(2 - x^2\right) = 0 \Rightarrow e^{\ln\left(2-x^2\right)} = e^0 \Rightarrow 2 - x^2 = 1 \Rightarrow x^2 = 1 \Rightarrow \Rightarrow x_1 = 1, x_2 = -1$ *because of* $e^{\ln x} = x$ *and* $e^0 = 1$

e)

$$f(x) = \ln(x+1)$$

You get the inverse function f^{-1} *by solving* x *like this:*

$$y = \ln(x+1) \Rightarrow e^y = e^{\ln(x+1)} = x + 1 \Rightarrow x = e^y - 1$$

by changing x *and* y *we get*

$$y = f^{-1}(x) = e^x - 1$$

8.8.5 Rules for Derivatives of Both Functions

$$f(x) = e^x \Rightarrow f'(x) = e^x$$

$$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$$

$$\left(e^{f(x)}\right)' = f'(x) \cdot e^{f(x)}$$

$$(\ln f(x))' = f'(x) \cdot \frac{1}{f(x)}$$

Exercise 8.7 — Derive log and exp (Solution → p. 60)

a) $f(x) = x \cdot e^{1-x}$

b) $f(x) = \frac{e^x}{e^x + 1}$

c) $f(x) = \ln(2x - 4)$

d) $f(x) = \ln\sqrt{x^2 + 1} = \ln\left(x^2 + 1\right)^{\frac{1}{2}} = \frac{1}{2} \ln\left(x^2 + 1\right)$

e) $f(x) = (\ln x)^2$

f) $f(x) = \frac{x}{\ln x}$

Solution to Exercise 8.7 — Derive log and exp (Exercise → p. 60)

a) $f(x) = x \cdot e^{1-x}$

$$f'(x) = 1 \cdot e^{1-x} + x \cdot e^{1-x} \cdot (-1) = e^{1-x} \cdot (1 - x)$$

b) $f(x) = \frac{e^x}{e^x+1}$

$$f'(x) = \frac{e^x\left(e^x+1\right) - e^x \cdot e^x}{\left(e^x+1\right)^2} = \frac{e^x}{\left(e^x+1\right)^2} \cdot \left(e^x + 1 - e^x\right) =$$

c) $f(x) = \ln(2x-4)$

$$f'(x) = \frac{1}{2x-4} \cdot 2 = \frac{1}{x-2}$$

d) $f(x) = \ln\sqrt{x^2+1} = \ln\left(x^2+1\right)^{\frac{1}{2}} = \frac{1}{2}\ln\left(x^2+1\right)$

$$f(x) = \frac{1}{2} \cdot \frac{2x}{x^2+1} = \frac{x}{x^2+1}$$

e) $f(x) = (\ln x)^2$

$$f'(x) = 2 \cdot \ln x \cdot \frac{1}{x} = \frac{2}{x}\ln x$$

f) $f(x) = \frac{x}{\ln x}$

$$f'(x) = \frac{1 \cdot \ln x - x \cdot \frac{1}{x}}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$$

Exam appendix: Rules of algebra

Basic Arithmetic

a) $a + b = b + a$

b) $(a + b) + c = a + (b + c)$

c) $a + 0 = a$

d) $a + (-a) = 0$

e) $ab = ba$

f) $(ab)c = a(bc)$

g) $1 \cdot a = a$

h) $aa^{-1} = 1$, for $a \neq 0$

i) $(-a)b = a(-b) = -ab$

j) $(-a)(-b) = ab$

k) $a(b + c) = ab + ac$

l) $(a + b)c = ac + bc$

m) $(a + b)^2 = a^2 + 2ab + b^2$

n) $(a - b)^2 = a^2 - 2ab + b^2$

o) $(a + b)(a - b) = a^2 - b^2$

Rules for Fractions

a) $\frac{a}{b} + \frac{a}{d} = \frac{ad+bc}{bd}$

b) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

c) $\frac{a}{b} : \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$

d) $\frac{a}{b} \cdot \frac{c}{c} = \frac{ac}{bc}$

Rules for Powers

a) $x^0 = 1$

b) $x^a \cdot x^b = x^{a+b}$

c) $\frac{x^a}{x^b} = x^{a-b}$

d) $x^a \cdot y^a = (x \cdot y)^a$

e) $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$

f) $(x^a)^b = x^{a \cdot b}$

g) $x^a \cdot y^b = z^c \Leftrightarrow x^{\frac{a}{c}} \cdot y^{\frac{b}{c}} = z$

Rules for Roots

a) $a^{\frac{1}{2}} = \sqrt{a}$

b) $\sqrt[n]{a} = a^{\frac{1}{n}}$ (valid if $a \geq 0$)

c) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

d) $\frac{1}{\sqrt[n]{a^m}} = a^{-\frac{m}{n}}$

e) $\sqrt{ab} = \sqrt{a}\sqrt{b}$

f) $\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$

g) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Rules for Quadratic Function

- The **roots** of a quadratic function, i.e. $ax^2 + bx + c = 0$, are defined by the formula
$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
if $b^2 - 4ac \geq 0$.
- Factorization of a quadratic function also gives us the roots:
$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

Rules of Differentiation

$$\begin{aligned}
\text{Rule 1:} \quad & f(x) = A \\
& \Rightarrow f'(x) = 0 \\
\text{Rule 2:} \quad & f(x) = A + g(x) \\
& \Rightarrow f'(x) = g'(x) \\
\text{Rule 3:} \quad & f(x) = Ag(x) \\
& \Rightarrow f'(x) = Ag'(x) \\
\text{Rule 4 (power):} \quad & f(x) = x^a \\
& \Rightarrow f'(x) = ax^{a-1}
\end{aligned}$$

with a being an arbitrary constant.

If both f and g are differentiable at x, then the sum $f+g$ and the difference $f-g$ are both differentiable at x, and

Rule 5 (sums): $h(x) = f(x) \pm g(x)$

$$\Rightarrow \quad h'(x) = f'(x) \pm g'(x)$$

If both f and g are differentiable at x, then so is $h = f \cdot g$, and

Rule 6 (products): $h(x) = f(x) \cdot g(x)$

$$\Rightarrow \quad h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

If both f and g are differentiable at x and $g(x) \neq 0$, then $h = f/g$ is differentiable at x, and

Rule 7 (quotient): $h(x) = \dfrac{f(x)}{g(x)}$

$$\Rightarrow \quad h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

If g is differentiable at x and f is differentiable at $u = g(x)$, then the composite function $h(x) = f(g(x))$ is differentiable at x, and

Rule 8 (sums): $h'(x) = f'(u) \cdot g'(x) = f'(g(x)) \cdot g'(x)$

In words: First differentiate the exterior function with respect to the interior function (kernel), then multiply by the derivative of the interior function.

Rules for Summation

The **summation sign**, $\sum$, allows for a compact formulation of lengthy expressions.

Rule (Additivity Property)

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

Rule (Homogeneity Property)

$$\sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i$$

and if $a_i = 1$ for all i then

$$\sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i = c(n \cdot 1) = cn$$

Rule (Rule for Sums)

$$\sum_{i=1}^{n} i = 1 + 2 + \ldots + n = \frac{1}{2} n(n+1)$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{1}{2} n(n+1)\right)^2 = \left(\sum_{i=1}^{n} i\right)^2$$

$$\sum_{i=0}^{n} a^i = \frac{1 - a^{n+1}}{1 - a}$$

The double sum notation allows us to write lengthy expressions in a compact way.

$$\sum_{i=1}^{Z} b_i \sum_{j=1}^{S} a_{ij} b_j = \sum_{i=1}^{Z} \sum_{j=1}^{S} a_{ij} b_i b_j = \sum_{j=1}^{S} \sum_{i=1}^{Z} a_{ij} b_i b_j = \sum_{j=1}^{S} b_j \sum_{i=1}^{Z} a_{ij} b_i$$

Consider some summation sign $\sum_{i=1}^{Z}$. All variables with index i must be to the right of that summation sign.

Rules for Logarithms and Exponentials

Exponential Function

$$y = f(x) = e^x = exp(x) \quad \text{is strictly increasing } \forall x \in \mathbb{R}$$
$$D_f = \mathbb{R} =]-\infty, \infty[$$
$$e^x > 0 \forall x \in \mathbb{R}$$
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{24} + \dots$$
$$e^0 = 1$$
$$e^1 = e = 2.718\dots \text{Euler number}$$
$$\lim_{x \to \infty} e^x \to \infty \quad \text{and}$$
$$\lim_{x \to -\infty} e^x \to 0$$

Logarithmic Function

$$y = f(x) = \ln(x) \quad \text{is strictly increasing}$$
$$D_f = \mathbb{R}^+ =]0, \infty[$$
$$\ln(x) = e^{\ln x}$$
$$\ln 1 = 0$$
$$\ln e = 1$$

Rules to Transform log

$$\ln(x \cdot y) = \ln x + \ln y$$
$$\ln(\frac{x}{y}) = \ln x - \ln y$$
$$\ln x^a = a \cdot \ln x$$
$$\ln e^x = x$$

Rules to Transform exp

$$e^{x+y} = e^x \cdot e^y$$
$$e^{x-y} = \frac{e^x}{e^y}$$
$$e^{ax} = (e^x)^a$$
$$e^{\ln x} = x$$

Rules for Derivatives of Both Functions

$$f(x) = e^x \Rightarrow f'(x) = e^x$$
$$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$$
$$\left(e^{f(x)}\right)' = f'(x) \cdot e^{f(x)}$$
$$(\ln f(x))' = f'(x) \cdot \frac{1}{f(x)}$$

Chapter 9

Additional Exercises

Exercise 9.1 — Equations with Parameters (Solution → p. 82)

Equations can be used to describe a relationship between two variables (e.g., x and y). Given the following equations:

$$y_1 = 10x_1$$

$$y_2 = 3x_2 + 4$$

$$y_3 = -\frac{8}{3}x_3 - \frac{7}{2}.$$

a) Solve for the variable x.

b) Is there a solution for x if

- $y^a_{\{1;2;3\}} = 1$
- $y^b_{\{1;2;3\}} = 10$
- $y^c_{\{1;2;3\}} = 0$

Exercise 9.2 — Inverse Functions (Solution → p. 83)

Find the inverse of each of the following functions:

a) $y = -2x + 6$

b) $y = \sqrt{3x - 4}$

c) $y = (x + 1)^2$

d) $y = \frac{x-1}{x-2}$

Exercise 9.3 — Equivalent Equations (Solution → p. 84)

Solve:

a) $\frac{x+2}{x-2} - \frac{8}{x^2-2x} = \frac{2}{x}$

b) $\frac{z}{z-5} + \frac{1}{3} = \frac{-5}{5-z}$

c) $x^2 + 8x - 9 = 0$

Visit

https://www.wolframalpha.com/examples/mathematics/plotting-and-graphics/

and draw each of the functions above.

Visit

https://www.mathpapa.com/algebra-calculator.html

or

https://www.wolframalpha.com/calculators/equation-solver-calculator

or

https://quickmath.com/

and use it to solve the equations above.

Exercise 9.4 — Two Nonlinear Equations (Solution → p. 85)

Assume market demand is given by $P + Q^2 + Q = 11$ and market supply by $2P - 2Q^2 + Q - 4 = 0$. Calculate equilibrium price and quantity.

Exercise 9.5 — Production Possibility Frontier Curve (Solution → p. 86)

A firm can produce different amount of two goods, X and Y. The production possibility frontier (PPF) curve shows the maximum output of either goods attainable for any given level of output of the other, is

$$X^2 + 2X = 10 - Y.$$

What are the largest amounts of X and Y that can be produced? Can you sketch the PPF curve using an online tool?

Exercise 9.6 — Nonlinear Average Costs (Solution → p. 86)

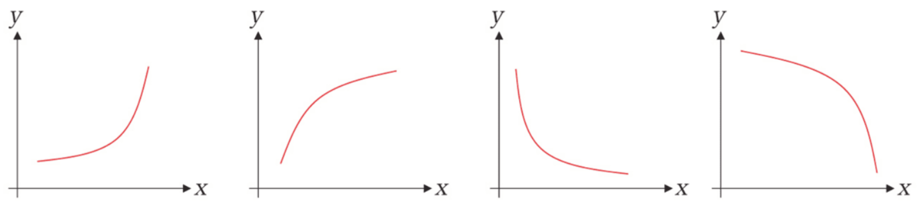
Total costs of a firm are 550 when output is 100 and fixed costs are 50. Assume that the total cost function is linear

- Graph the average cost function.
- What are the asymptotes of the average cost function?
- Find the inverse form of the average cost function and give the values for which it is valid.
- If total costs decrease to 450 , assuming fixed costs and output remain unchanged, calculate the average cost function and say for what values the inverse function is valid.

Exercise 9.7 — To Be Convex or Not to Be (Solution → p. ??)

a) Fill in the blanks:
   Der Bauch vom T-Rex ist________(konvex/konkav).
   In konKAVE Dinge kann man________(?) reinschütten.

b) Characterize the following functions with respect to $f(x)'$ and $f(x)''$.



Exercise 9.8 — Summation Notation (Solution → p. 87)

1. Find the following sums.
   (a) $\displaystyle\sum_{k=1}^{4} \frac{13}{100^k}$

   (b) $\displaystyle\sum_{n=0}^{4} \frac{n!}{2}$

   (c) $\displaystyle\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n}(x-1)^n$

2. Write the following sums using summation notation.
   (a) $1+3+5+\ldots+117$

   (b) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+-\ldots+\frac{1}{117}$

   (c) $0.9+0.09+0.009+\ldots 0.\underbrace{0\cdots0}_{n-1\text{ zeros}}9$

Chapter 10

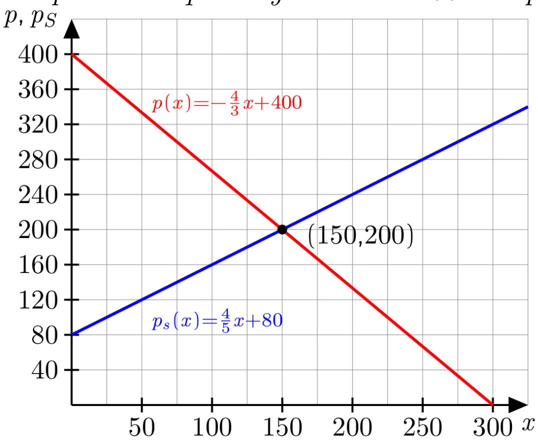
Solutions

Solution to Exercise 3.1 — Is it a function? (Exercise → p. 19)

Relations (a) and (b) are functions.

Solution to Exercise 3.2 — Demand and Supply (Exercise → p. 21)

- $x = x(p) = -\frac{3}{4} + 300 \Leftrightarrow p(x) = -\frac{4}{3}x + 400$
- $p_S(x) = -\frac{4}{5}x + 80$
- *The equilibrium market price and quantity is in* $x = 150$ *and* $p = 200$.



Solution to Exercise 3.3 — Give Me The Standard Form (Exercise → p. 23)

1.

$$\begin{aligned} f(x) &= x^2 - 4x + 3 && \text{(take } (\tfrac{b}{2})^2\text{)} \\ &= \left(x^2 - 4x + \underline{4} - \underline{4}\right) + 3 \\ &= \left(x^2 - 4x + 4\right) - 4 + 3 && \text{(Group the perfect square trinomial.)} \\ &= (x-2)^2 - 1 \end{aligned}$$

From the standard form we can immediately (if desired) produce a sketch of the graph of f, *as shown in Figure 3.3.*

2. *Here, our first step is to factor out the* (-1) *from both the* x^2 *and* x *terms. We then follow the completing the square recipe.*

$$\begin{aligned} g(x) &= -x^2 - x + 6 \\ &= (-1)\left(x^2 + x\right) + 6 && \text{(take } (\tfrac{b}{2})^2\text{)} \\ &= (-1)\left(x^2 + x + \underline{\frac{1}{4}} - \underline{\frac{1}{4}}\right) + 6 \\ &= (-1)\left(x^2 + x + \frac{1}{4}\right) + (-1)\left(-\frac{1}{4}\right) + 6 \\ & && \text{(Group the perfect square trinomial.)} \\ &= -\left(x + \frac{1}{2}\right)^2 + \frac{25}{4} \end{aligned}$$

Using the standard form, we can again obtain the graph of g*, as shown in Figure 3.4.*

Solution to Exercise 3.4 — Derive the Quadratic Formula (Exercise → p. 24)

$$\begin{aligned} ax^2 + bx + c &= 0 \\ a\left(x^2 + \frac{b}{a}x\right) &= -c \\ a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) &= -c + \frac{b^2}{4a} \\ a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a} \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\ x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

Alternatively, you can also do this:

$$\begin{aligned}
a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}&=0\\
a\left(x+\frac{b}{2a}\right)^2&=-\frac{4ac-b^2}{4a}\\
\frac{1}{a}\left[a\left(x+\frac{b}{2a}\right)^2\right]&=\frac{1}{a}\left(\frac{b^2-4ac}{4a}\right)\\
\left(x+\frac{b}{2a}\right)^2&=\frac{b^2-4ac}{4a^2}\\
x+\frac{b}{2a}&=\pm\sqrt{\frac{b^2-4ac}{4a^2}} && \text{extract square roots}\\
x+\frac{b}{2a}&=\pm\frac{\sqrt{b^2-4ac}}{2a}\\
x&=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}\\
&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}
\end{aligned}$$

Solution to Exercise 3.5 — Its All About Profits (1) (Exercise → p. 24)

1. *Since* $R(x,p)=x(p)p(x)=xp$, *we substitute* $p(x)=-1.5x+250$ *to get*

$$R(x)=x(-1.5x+250)=-1.5x^2+250x.$$

From the price-demand function $p(x)$ *(or inverse demand function), we get the demand function is*

$$x=166\frac{2}{3}-\frac{2}{3}p.$$

Thus, demand is restricted to $0\le x\le 166$, $R(x)$ *is restricted to these values of* x *as well. Summing up, the weekly revenue made by selling* x *goods was found to be*

$$R(x)=-1.5x^2+250x$$

with the restriction that $0\le x\le 166$.

2. *To find the profit function* $P(x)$, *we subtract*

$$P(x)=R(x)-C(x)=\left(-1.5x^2+250x\right)-(80x+150)=-1.5x^2+170x-150.$$

Since the revenue function is valid when $0\le x\le 166$, P *is also restricted to these values.*

3. *To find the* x*-intercepts, we set* $P(x)=0$ *and solve* $-1.5x^2+170x-150=0$. *The mere thought of trying to factor the left hand side of this equation could do serious psychological*

damage, so we resort to the quadratic formula. Identifying $a = -1.5$, $b = 170$, *and* $c = -150$, *we obtain*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-170 \pm \sqrt{170^2 - 4(-1.5)(-150)}}{2(-1.5)} = \frac{-170 \pm \sqrt{28000}}{-3} = \frac{170 \pm 20\sqrt{70}}{3}.$$

We get two x*-intercepts:* $\left(\frac{170-20\sqrt{70}}{3}, 0\right)$ *and* $\left(\frac{170+20\sqrt{70}}{3}, 0\right)$.

To find the y*-intercept, we set* $x = 0$ *and find* $y = P(0) = -150$ *for a* y*-intercept of* $(0, -150)$.

To find the vertex, we use the fact that $P(x) = -1.5x^2 + 170x - 150$ *is in the general form of a quadratic function. Substituting* $a = -1.5$ *and* $b = 170$, *we get* $x = -\frac{170}{2(-1.5)} = \frac{170}{3}$.

To find the y*-coordinate of the vertex, we compute* $P\left(\frac{170}{3}\right) = \frac{14000}{3}$ *and find that our vertex is* $\left(\frac{170}{3}, \frac{14000}{3}\right)$. *The axis of symmetry is the vertical line passing through the vertex so it is the line* $x = \frac{170}{3}$.

To sketch a reasonable graph in Figure 10.1, we approximate the x*-intercepts,* $(0.89, 0)$ *and* $(112.44, 0)$, *and the vertex,* $(56.67, 4666.67)$.[a]

4. *The zeros of* P *are the solutions to* $P(x) = 0$, *which we have found to be approximately* 0.89 *and* 112.44. *These are the 'break-even' points of the profit function, where enough product is sold to recover the cost spent to make the product. More importantly, we see from the graph that as long as* x *is between* 0.89 *and* 112.44, *the graph* $y = P(x)$ *is above the* x*-axis, meaning* $y = P(x) > 0$ *there. This means that for these values of* x, *a profit is being made. Since* x *represents the weekly sales of the respective good, we round the zeros to positive integers and have that as long as* 1, *but no more than* 112 *goods are sold weekly, the retailer will make a profit.*

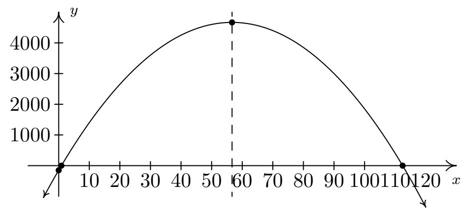


Figure 10.1: The graph of the profit function $P(x)$

5. *From the graph, we see that the maximum value of* P *occurs at the vertex, which is approximately* $(56.67, 4666.67)$. *As above,* x *represents the weekly sales, so we can't sell* 56.67 *units of the good. Comparing* $P(56) = 4666$ *and* $P(57) = 4666.5$, *we conclude that we will make a maximum profit of* \$4666.50 *if we sell* 57 *units of* x.
6. *In the previous part, we found that we need to sell* 57 *goods per week to maximize profit. To find the price per unit, we substitute* $x = 57$ *into the price-demand function to get* $p(57) = -1.5(57) + 250 = 164.5$. *The price should be set at* \$164.50.

[a] Note that in order to get the x-intercepts and the vertex to show up in the same picture, we had to scale the x-axis differently than the y-axis in Figure 10.1. This results in the left-hand x-intercept and the y-intercept being uncomfortably close to each other and to the origin in the picture.

Solution to Exercise 3.6 — Its All About Profits (2) (Exercise → p. 24)

a) *fixed cost is* $C(0) = 108$

b) *variable cost is* $C(x) - C(0) = 180x$

c) $R(x) = xp(x)$ *where* $p(x)$ *results from* $x(p)$ *by resolving to* p, *i.e.,*

$$x = x(p) = -3p + 600$$

$$3p = -x + 600$$

$$p = -\frac{1}{3}x + 200$$

$$\Rightarrow R(x) = xp(x) = x\left(-\frac{1}{3}x + 200\right)$$

$$= -\frac{1}{3}x^2 + 200x$$

d) $G(x) = R(x) - C(x) = -\frac{1}{3}x^2 + 20x - 108$

e) *Since* $G(x)$ *is a parabola with* $a = -\frac{1}{3}$ *its graph is opened down and thus* $G(x)$ *has its maximum in the position* $x = -\frac{b}{2a} = -\frac{20}{2(-\frac{1}{3})} = 30$

f) $G(30) = -\frac{1}{3} \cdot 30^2 + 20 \cdot 30 - 108 = 190$

g) $G(x) = 0 \Leftrightarrow x^2 - 60x + 324 = 0 \Rightarrow x_1, x_2 = 30 \pm \sqrt{30^2 - 324} = 30 \pm 24 = 6.54 \Rightarrow G(x) = -\frac{1}{3} \cdot (x-6) \cdot (x-54) = \frac{1}{3} \cdot (x-6) \cdot (54-x)$

h) *For* $6 < x < 54$, *because then both factors are negative.*

Solution to Exercise 3.7 — Calculate a logarithmic function without a calculator (Exercise → p. 30)

- $\log_2 16 = 4$ *because* $2^4 = 16$
- $\log_3 243 = 5$ *because* $3^5 = 243$
- $\log_5 125 = 3$ *because* $5^3 = 125$
- $\log_3 81 = 4$ *because* $3^4 = 81$
- $\log_2\left(\frac{1}{8}\right) = -3$ *because* $2^{-3} = \frac{1}{8}$

Solution to Exercise 3.8 — Rule of 70 (Exercise → p. 32)

Let X *be the initial value of a growing variable, and* Y *denote the terminal value at time* $t + n$. *The relationship between the two is given by*

$$Y = X(1+g)^n$$

where g *is the annual growth rate. As we are interested in the time span required for* X *to double,* $Y = 2$ *and*

$$2 = (1+g)^n$$

Taking natural logs which is the logarithm to the base of e, we get

$$\ln 2 = n \ln(1+g)$$

and hence

$$n = \frac{\ln 2}{\ln(1+g)} \qquad (*).$$

This is the exact time periods required for a growing variable to double its size. One can approximate n using the definition of e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + R_n$$

where the remainder term $R_n \to 0$ *as* $n \to \infty$*. Ignoring high order terms, for small* x*, it may be approximated by*

$$e^x \approx 1 + x$$

Taking logarithms of both sides, we get

$$x \approx \ln(1+x) \qquad (**)$$

Using (**), equation (*) may be approximated as

$$n \approx \frac{\ln 2}{g} = \frac{0.693147}{g} \cong \frac{70}{g\%}$$

This is the origin of the Rule of 70.
Using the number e right away is simpler:

$$\begin{aligned}
(e^r)^t &= 2 \\
\ln e^{rt} &= \ln 2 \\
rt &= \ln 2 \\
t &= \frac{\ln 2}{r} \\
t &\approx \frac{0.693147}{r}
\end{aligned}$$

Solution to Exercise 3.10 — Investments over time (Exercise → p. 36)

The formula under discrete time is:

$$Y_t = Y_0 \cdot (1+g)^t$$

The formula under continuous time is:

$$Y_t = Y_0 \cdot e^{gt}$$

Solution to Exercise 4.1 — Proof: 0.9999... is equal to 1 (Exercise → p. 40)

$$0.999\cdots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = \frac{9}{10} \cdot \underbrace{\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right)}_{\text{infinite geometric series}}$$

plugging $a = 1$ and $k = \frac{1}{10}$ in the infinite geometric series formula, we get

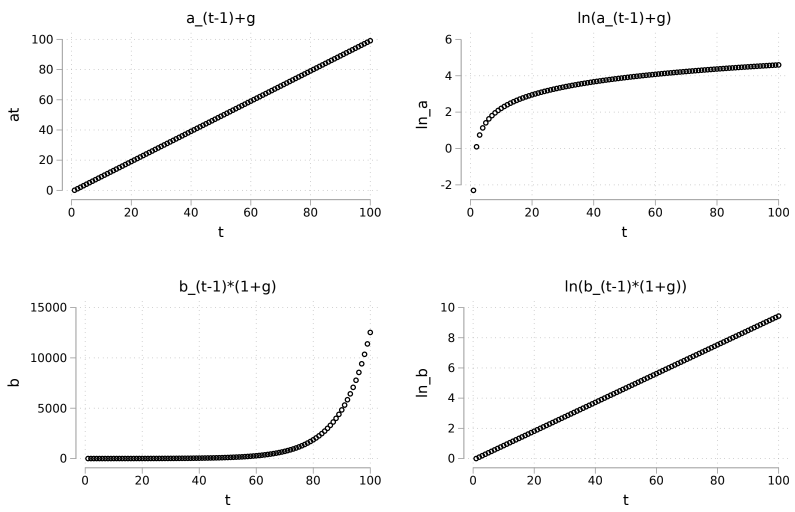
$$\sum_{n=1}^{\infty} ak^{n-1} = \frac{1}{1 - \frac{1}{10}} = \frac{1}{\frac{9}{10}} = \frac{10}{9}$$

and hence, we can plug that back into the equation above to get:

$$0.999\cdots = \frac{9}{10} \cdot \frac{10}{9} = 1$$

q.e.d.

Solution to Exercise 4.2 — Exponential Growth (Exercise → p. 40)



Solution to Exercise 5.1 — Production Under Constraints and Fixed Prices (Exercise → p. 41)

1. State the problem: *How many of product X and product Y to produce to maximize profit?*

2. Decision variables: *Let x = number of product X to produce per day*
Let y = number of product Y to produce per day

3. Objective function: *Maximize*

$$P = 10x + 14y$$

4. Constraints:

- *machine time=120h*

- *assembling time=80h*
- *hours needed for production of one good:*
  machine time: $x \rightarrow 3h$ *and* $y \rightarrow 2h$
  assembling time: $x \rightarrow 1h$ *and* $y \rightarrow 2h$

Thus, we get:

$$3x + 2y \leq 120 \quad \Leftrightarrow y \leq 60 - \frac{3}{2}x \quad \textit{(hours of machining time)}$$

$$x + 2y \leq 80 \quad \Leftrightarrow y \leq 40 - \frac{1}{2}x \quad \textit{(hours of assembly time)}$$

Since there are only two products, these limitations can be shown on a two-dimensional graph (Figure 10.2). Since all relationships are linear, the solution to our problem will fall at one of the corners.

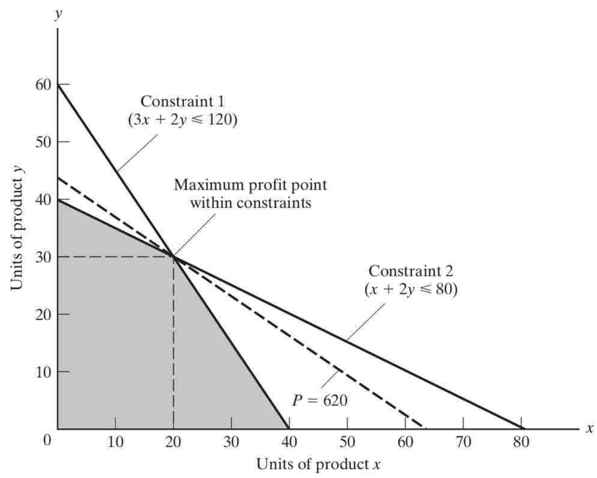


Figure 10.2: Linear program example: constraints and solution

To draw the isoprofit function in a plot with the good y on the y-axis and good x on the x-axis, we can re-arrange the objective function to get

$$y = \frac{1}{14}P - \frac{10}{14}x$$

To illustrate the function let us consider some arbitrarily chosen levels of profit in Figure 10.3:

- *\$350 by selling 35 units of X or 25 units of Y*
- *\$700 by selling 70 units of X or 50 units of Y*
- *\$620 by selling 62 units of X or 44.3 units of Y.*

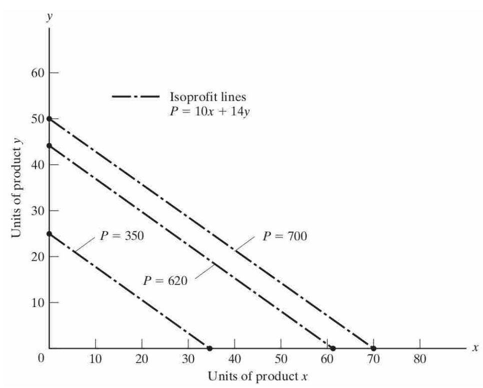


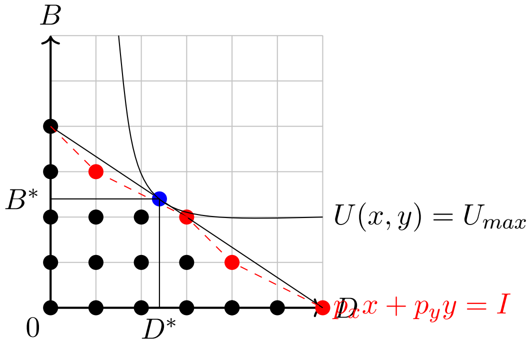
Figure 10.3: Linear program example: isoprofit lines

To find the solution, begin at some feasible solution (satisfying the given constraints) such as $(x, y) = (0, 0)$, and proceed in the direction of steepest ascent of the profit function (in this case, by increasing production of Y at \$14.00 profit per unit) until some constraint is reached. Since assembly hours are limited to 80, no more than 80/2, or 40, units of Y can be made, earning

*40 · $14.00, or $560 profit. Then proceed along the steepest allowable ascent from there (along the assembly constraint line) until another constraint (machining hours) is reached. At that point, (x,y) = (20,30) and profit P = (20 * +10.00) + (30 * +14.00), or $620. Since there is no remaining edge along which profit increases, this is the optimum solution.*

Solution to Exercise 5.2 — Burgers and Drinks (Exercise → p. 43)

a) *In the plot, I marked all 19 possible bundles of burger and drinks of consumption. The budget constraint is shown by the solid line.*



b) *We now should calculate the utility of all 19 points but only the red dots denote choices that may yield an optimal utility. The best utility is at when we buy 2 burger and 3 drinks:*

$$U = 2^{.6}3^{.4} = 2.35$$

c) *Solve:*

$$\mathcal{L} = B^{0.6}D^{0.4} + \lambda(3B + 2D - 12)$$

FOC:

$$3B + 2D - 12 = 0$$

$$.6B^{-0.4}D^{0.4} + 3\lambda = 0$$

$$.4B^{0.6}D^{-0.6} + 2\lambda = 0$$

Solving the second and third FOC for λ, substituting λ gives:

$$B = D$$

which we can plug in the first FOC to get

$$B^* = 2.4 \quad \text{and} \quad D^* = 2.4$$

Solution to Exercise 5.3 — Consumption Choice (Exercise → p. 43)

$$\mathcal{L} = A^{0.8}B^{0.2} + \lambda(6A + 4B - 30)$$

$$FOC: \frac{\partial \mathcal{L}}{\partial \lambda} = 6A + 4B - 30 = 0 \qquad (*)$$

$$\frac{\partial \mathcal{L}}{\partial A} = 0.8A^{-0.2}B^{0.2} + 6\lambda = 0 \qquad (**)$$

$$\frac{\partial \mathcal{L}}{\partial B} = 0.2A^{0.8}B^{-0.8} + 4\lambda = 0 \qquad (***)$$

System of 3 equation with 3 unknowns can be solved in various ways. The easiest way is to solve (**) and (***) for λ and substitute it out:

1. *solve for* λ

$$-\frac{2}{15}A^{-0.2}B^{0.2} = \lambda \qquad (**')$$

$$-\frac{1}{20}A^{0.8}B^{-0.8} = \lambda \qquad (***')$$

2. *set both equations equal by substituting* λ *and solve for* B

$$\frac{2}{15}A^{-0.2}B^{0.2} = \frac{1}{20}A^{0.8}B^{-0.8}$$

$$B = 0.375A \qquad (****)$$

3. *Now, plug in* $(****)$ *into* $(*)$ *to get a number for* A

$$30 = 6A + 4 \cdot 0.375A$$

$$\Leftrightarrow 30 = 7.5A$$

$$\Leftrightarrow A = 4$$

4. *Use* $A = 4$ *in* $(*)$ *to get a number for* B

$$30 = 6 \cdot 4 + 4B$$

$$\Leftrightarrow 6 = 4B$$

$$B = \frac{6}{4} = 1.5$$

Thus, we'd consume 4 units of good A and 1.5 of good B.

Solution to Exercise 5.4 — Cost-Minimizing Combination of Factors (Exercise → p. 44)

$$\mathcal{L} = r_1 + 4r_2 + \lambda\left(\frac{5}{4}r_1^{\frac{1}{2}}r_2^{\frac{1}{2}} - 20\right)$$

Taking the FOC we get

$$r_1 = 4r_2$$

using that in the constraint, we get $r_1 = 32$ *and* $r_2 = 8$.
Also see `https://t1p.de/huber-lagr1` *which uses this tool:* `https://www.emathhelp.net/calculators/calculus-3/lagrange-multipliers-calculator/`

Solution to Exercise 5.5 — Derivation of Demand Function (Exercise → p. 44)

The Lagrangian for this problem is

$$Z = xy + \lambda\,(P_x x + P_y y - B)$$

The first order conditions are

$$\begin{aligned} Z_x &= y + \lambda P_x = 0 \\ Z_y &= x + \lambda P_y = 0 \\ Z_\lambda &= -B + P_x x + P_y y = 0 \end{aligned}$$

Solving the first order conditions yield the following solutions

$$x^M = \frac{B}{2P_x} \qquad y^M = \frac{B}{2P_y} \qquad \lambda = \frac{B}{2P_x P_y}$$

where x^M *and* y^M *are the consumer's demand functions.*

Solution to Exercise 5.6 — Cobb-Douglas and Demand (Exercise → p. 44)

The Lagrangian is

$$\mathcal{L}(x, y) = Ax^\alpha y^\beta + \lambda(px + qy - I)$$

Therefore, the first-order conditions are

$$\mathcal{L}'_x(x, y) = A\alpha x^{\alpha-1}y^\beta - \lambda p = 0 \qquad (*)$$

$$\mathcal{L}'_y(x, y) = Ax^\alpha \beta y^{\beta-1} - \lambda q = 0 \qquad (**)$$

$$px + qy - I = 0 \qquad (***)$$

Solving $(*)$ *and* $(**)$ *for* λ *yields*

$$\lambda = \frac{A\alpha x^{\alpha-1}y^{\beta-1}y}{p} = \frac{Ax^{\alpha-1}x\beta y^{\beta-1}}{q}$$

Canceling the common factor $Ax^{\alpha-1}y^{\beta-1}$ *from the last two fractions gives*

$$\frac{\alpha y}{p} = \frac{x\beta}{q}$$

and therefore

$$qy = px\frac{\beta}{\alpha}$$

Inserting this result in $(***)$ *yields*

$$px + px\frac{\beta}{\alpha} = I$$

Rearranging gives

$$px\left(\frac{\alpha+\beta}{\alpha}\right) = I$$

Solving for x *yields the following demand function*

$$x = \frac{\alpha}{\alpha+\beta}\frac{I}{p}$$

Inserting

$$px = qy\frac{\alpha}{\beta}$$

in $(***)$ *gives*

$$qy\frac{\partial}{\beta} + qy = I$$

and therefore the demand function

$$y = \frac{\beta}{\alpha+\beta}\frac{I}{q}$$

Solution to Exercise 5.7 — Lagrange with n-constraints (Exercise → p. 44)

$$\mathcal{L}(x_1,\ldots,x_m,\lambda_i,\ldots,\lambda_n) = f(\mathbf{x}) + \lambda_1 g_1(\mathbf{x}) + \lambda_2 g_2(\mathbf{x}) + \ldots + \lambda_n g_n(\mathbf{x})$$

The points of local minimum would be the solution of the following equations:

$$\frac{\partial\mathcal{L}}{\partial x_j} = 0 \quad \forall j = 1\ldots m$$

$$g_i(\mathbf{x}) = 0 \quad \forall i = 1\ldots n$$

Solution to Exercise 6.1 — Integral I (Exercise → p. 46)

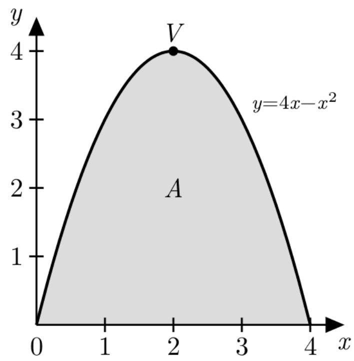
Since

$$y = 4x - x^2 = x\cdot(4-x) = 0 \quad\Leftrightarrow\quad x_1 = 0 \text{ or } x_2 = 4,$$

the position of the vertex of this parabola is in

$$x = \frac{0+4}{2} = 2 \quad \text{and so} \quad V = (2,4)$$

$$
\begin{aligned}
A &= \int_0^4 \left(4x - x^2\right) \mathrm{d}x \\
&= \left[2x^2 - \frac{1}{3}x^3\right]_{x=0}^{x=4} \\
&= \left(2 \cdot 4^2 - \frac{1}{3}4^3\right) - \left(2 \cdot 0^2 - \frac{1}{3}0^3\right) \\
&= 32 - \frac{64}{3} \\
&= \frac{32}{3}
\end{aligned}
$$



Solution to Exercise 6.2 — Integral II

(Exercise → p. 46)

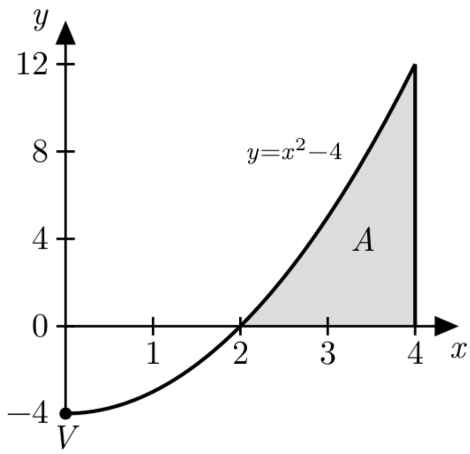
Since

$$y = x^2 - 4 = 0 \Leftrightarrow x = \pm 2,$$

the position of the vertex of this parabola is in

$$x = \frac{-2+2}{2} = 0 \quad \textit{and thus} \quad V = (0, -4).$$

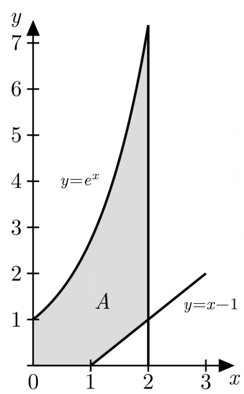
$$
\begin{aligned}
A &= \int_2^4 \left(x^2 - 4\right) \mathrm{d}x \\
&= \left[\frac{1}{3}x^3 - 4x\right]_{x=2}^{x=4} \\
&= \frac{64}{3} - 16 - \left(\frac{8}{3} - 8\right) \\
&= \frac{32}{3}
\end{aligned}
$$



Solution to Exercise 6.3 — Integral III

(Exercise → p. 46)

$$
\begin{aligned}
A &= \int_0^1 (e^x - 0)\, \mathrm{d}x + \int_1^2 (e^x - (x-1))\, \mathrm{d}x \\
&= e^x\big|_{x=0}^{2} + \left(e^x - \frac{1}{2}x^2 + x\right)\Big|_{x=1}^{x=2} \\
&= e - e^0 + e^2 - 2 + 2 - \left(e - \frac{1}{2} + 1\right) \\
&= e^2 - \frac{3}{2}
\end{aligned}
$$



Solution to Exercise 6.4 — Consumer and Producer Surplus (Exercise → p. 46)

Because for $0 \leqslant x \leqslant 80$ *the demand is given by function*

$$p(x) = -0.01x^2 - 0.2x + 80$$

and the supply by the function

$$p_S(x) = 0.03x^2 + 0.6x + 20$$

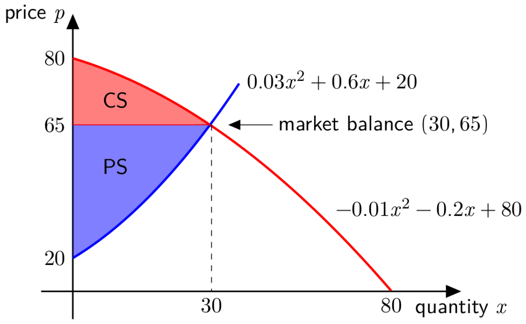
one has in the market balance

$$p_S(x) = p(x) \iff -0.01x^2 - 0.2x + 80 = 0.03x^2 + 0.6x + 20$$

that reduces to

$$0.04x^2 + 0.8x - 60 = 0.04 \cdot (x - 30)(x + 50) = 0$$

Thus the market balance is $(x_0, p_0) = (30, p(30)) = (30, 65)$.



To calculate the consumer and producer surplus is straightforward:

$$\begin{aligned} CS &= \int_0^{x_0} p(x)\mathrm{d}x - x_0 p_0 \\ &= \int_0^{30} \left(-0.01x^2 - 0.2x + 80\right) \mathrm{d}x - 30 \cdot 65 \\ &= \left[-\frac{0.01}{3}x^3 - 0.1x^2 + 80x\right]_{x=0}^{30} - 1950 \\ &= 270 \end{aligned}$$

$$\begin{aligned} PS &= x_0 p_0 - \int_0^{x_0} p_S(x)\mathrm{d}x \\ &= 30 \cdot 65 - \int_0^{30} \left(0.03x^2 + 0.6x + 20\right) \mathrm{d}x \\ &= 1950 - \left[0.01x^3 + 0.3x^2 + 20x\right]_{x=0}^{x=30} \\ &= 810 \end{aligned}$$

Solution to Exercise 7.1 — Matrices (Exercise → p. 50)

$$A = \begin{pmatrix} 1 & -2 & 4 \\ -3 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 3 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 5 & -1 & 0 \\ -4 & 6 & 7 \\ -3 & 2 & -2 \end{pmatrix}$$

1. $A \cdot B = \begin{pmatrix} 0 & 1 \\ -1 & -9 \end{pmatrix}$
2. $B \cdot A = \begin{pmatrix} -9 & 0 & -3 \\ -1 & -4 & 7 \\ 1 & -2 & 4 \end{pmatrix}$
3. $A \cdot C = \begin{pmatrix} 1 & -5 & -22 \\ -12 & 1 & 2 \end{pmatrix}$
4. *$C \cdot A$ is impossible*
5. *A^2 is impossssible*
6. *$B \cdot C$ is impossible*
7. $C^2 = \begin{pmatrix} 29 & -11 & -7 \\ -65 & 54 & 28 \\ -17 & 11 & 18 \end{pmatrix}$
8. $(A \cdot B)^2 = \begin{pmatrix} -1 & -9 \\ 9 & 80 \end{pmatrix}$
9. $A \cdot C \cdot B = \begin{pmatrix} -32 & -2 \\ 4 & -35 \end{pmatrix}$

Solution to Exercise 9.1 — Equations with Parameters (Exercise → p. 65)

a)

$$y_1 = 10x_1 \quad \Rightarrow \quad x_1 = \frac{y_1}{10}$$

$$y_2 = 3x_2 + 4 \quad \Rightarrow \quad x_2 = \frac{y_2 - 4}{3} = \frac{1}{3}y_2 - \frac{4}{3}$$

$$y_3 = -\frac{8}{3}x_3 - \frac{7}{2} \quad \Rightarrow \quad x = -\frac{y_3 + \frac{7}{2}}{\frac{8}{3}} = -\frac{3}{8}y_3 - \frac{21}{16}.$$

b)

- $y^a_{\{1;2;3\}} = 1$

$$x^a_1 = \frac{1}{10}$$

$$x^a_2 = \frac{1-4}{3} = 1$$

$$x^a_3 = -\frac{3}{8} \cdot 1 - \frac{21}{16} = -\frac{27}{16}.$$

- $y^b_{\{1;2;3\}} = 10$

$$x^a_1 = \frac{10}{10} = 1$$

$$x^a_2 = \frac{10-4}{3} = 2$$

$$x^a_3 = -\frac{3}{8} \cdot 10 - \frac{21}{16} = -\frac{81}{16}.$$

- $y^c_{\{1;2;3\}} = 0$

$$x^a_1 = \frac{0}{10} = 0$$

$$x^a_2 = \frac{0-4}{3} = -\frac{4}{3}$$

$$x^a_3 = -\frac{3}{8} \cdot 0 - \frac{21}{16} = -\frac{21}{16}.$$

Solution to Exercise 9.2 — Inverse Functions (Exercise → p. 65)

a) $x = \frac{6-y}{2}$

b)

$$y = \sqrt{3x-4}$$

$$y = (3x-4)^{\frac{1}{2}}$$

$$y^2 = 3x-4$$

$$x = \frac{y^2+4}{3}$$

c)

$$y = (x+1)^2$$

$$y^{\frac{1}{2}} = x+1$$

$$x = y^{\frac{1}{2}} - 1$$

$$x = \sqrt{y} - 1$$

d)

$$y = \frac{x-1}{x-2}$$

$$(x-2)y = x-1$$

$$yx - 2y = x - 1$$

$$yx - x = 2y - 1$$

$$x(y-1) = 2y - 1$$

$$\frac{x(y-1)}{y-1} = \frac{2y-1}{y-1}$$

$$x = \frac{2y-1}{y-1}$$

Solution to Exercise 9.3 — Equivalent Equations (Exercise → p. 65)

Solve:

a) The equation is not defined for $x = 2$, $x = 0$.

$$\frac{x+2}{x-2} - \frac{8}{x^2-2x} = \frac{2}{x}$$

$$\frac{x(x+2)}{x(x-2)} - \frac{8}{x(x-2)} = \frac{2(x-2)}{x(x-2)} \quad (\textit{ for } x \neq 2 \textit{ and } x \neq 0)$$

$$x(x+2) - 8 = 2(x-2)$$

$$x^2 + 2x - 8 = 2x - 4$$

$$x^2 = 4$$

$$x = -2$$

This is the only solution, since for $x = 2$ the equation is not defined.

b)

$$\frac{z}{z-5} + \frac{1}{3} = \frac{-5}{5-z}$$

$$z + \frac{z-5}{3} = 5$$

$$3z + z - 5 = 15$$

$$4z = 20$$

$$z = 5$$

For $z = 5$, the equation is not defined. Thus, no solution exists.

c)

$$\begin{aligned} x^2 + 8x &= 9 \\ x^2 + 2 \cdot 4 \cdot x &= 9 \\ x^2 + 2 \cdot 4 \cdot x + 4^2 &= 9 + 4^2 \\ (x+4)^2 &= 25 \end{aligned}$$

Therefore, the solutions are $x_1 = 1$ and $x_2 = -9$.

Alternative 2: you can use the quadratic formula:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{8^2 - 4 \cdot 1 \cdot (-9)}}{2 \cdot 1} = \frac{-8 \pm 10}{2}$$

Alternative 3: If x_1 and x_2 are the solutions of $ax^2 + bx + c = 0$, then

$$ax^2 + bx + c = 0 \quad \Leftrightarrow \quad a\,(x - x_1)\,(x - x_2) = 0.$$

Thus, you can re-write the equation like this:

$$(x-1)(x+9) = 0$$

Solution to Exercise 9.4 — Two Nonlinear Equations (Exercise → p. 66)

We rewrite the first equation to

$$P = 11 - Q^2 - Q$$

and the second equation to

$$\begin{aligned} 2P &= 2Q^2 - Q + 4 \\ P &= Q^2 - 0.5Q + 2. \end{aligned}$$

Equalizing both expressions results in

$$11 - Q^2 - Q = Q^2 - 0.5Q + 2.$$

Sorting gives

$$2Q^2 + 0.5Q - 9 = 0$$

or

$$4Q^2 + Q - 18 = 0.$$

Using the quadratic formula $\left(\frac{-b \pm \sqrt{b^2-4ac}}{2a}\right)$ with $a = 4, b = 1$ and $c = -18$ gives

$$Q_1 = -2.25 \qquad \text{and} \qquad Q_2 = 2,$$

but only the second value makes sense in this question. We may use that in one of the original equations to get

$$P = 5.$$

Solution to Exercise 9.5 — Production Possibility Frontier Curve (Exercise → p. 66)

X is as large as possible if $Y = 0$. *Therefore*

$$X^2 + 2X = 10.$$

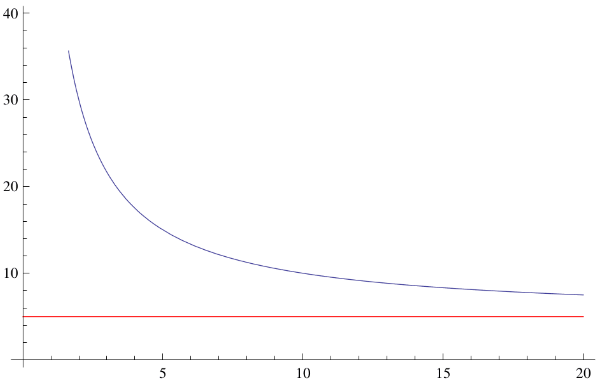
Applying the quadratic formula results in $X_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ *with* $a = 1, b = 2$ *and* $c = -10$ *and*

$$X = -1 \pm \sqrt{11}.$$

Since X should be positive we have $X = -1 + \sqrt{11} \approx 2.3166$. *Y is as large as possible if* $X = 0$. *Therefore* $0 = 10 - Y$, *resulting in* $Y = 10$.

Solution to Exercise 9.6 — Nonlinear Average Costs (Exercise → p. 66)

Total costs are variable costs plus fixed costs. If fixed costs are 50, *variable costs are 500 or 5 per unit. The resulting cost function is* $C = 5Q + 50$
Average costs are given by $AC = \frac{50}{Q} + 5$.



We can rewrite the function to

$$AC \cdot Q = 5Q + 50$$

or

$$(AC - 5)(Q - 0) = 50,$$

thus, asymptotic average costs are $AC = 5$
The inverse is

$$Q = \frac{50}{AC - 5}.$$

For positive quantities we should have $Z > 5$.
If fixed costs are 50, *variable costs are 400 or 4 per unit. The resulting cost function is*

$$C = 4Q + 50.$$

Average costs are given by

$$AC = \frac{50}{Q} + 4.$$

The inverse is

$$Q = \frac{50}{AC - 4}$$

and valid for $AC > 4$

Solution to Exercise 9.8 — Summation Notation (Exercise → p. 67)

1. (a) *We substitute* $k = 1$ *into the formula* $\frac{13}{100^k}$ *and add successive terms until we reach* $k = 4$.

$$\begin{aligned}\sum_{k=1}^{4} \frac{13}{100^k} &= \frac{13}{100^1} + \frac{13}{100^2} + \frac{13}{100^3} + \frac{13}{100^4} \\ &= 0.13 + 0.0013 + 0.000013 + 0.00000013 \\ &= 0.13131313\end{aligned}$$

(b) *Proceeding as in (a), we replace every occurrence of* n *with the values* 0 *through* 4 *to get*

$$\begin{aligned}\sum_{n=0}^{4} \frac{n!}{2} &= \frac{0!}{2} + \frac{1!}{2} + \frac{2!}{2} + \frac{3!}{2} = \frac{4!}{2} \\ &= \frac{1}{2} + \frac{1}{2} + \frac{2\cdot 1}{2} + \frac{3\cdot 2\cdot 1}{2} + \frac{4\cdot 3\cdot 2\cdot 1}{2} \\ &= \frac{1}{2} + \frac{1}{2} + 1 + 3 + 12 \\ &= 17\end{aligned}$$

(c) *We proceed as before, replacing the index* n, *but* not *the variable* x, *with the values* 1 *through* 5 *and adding the resulting terms.*

$$\begin{aligned}\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n}(x-1)^n &= \frac{(-1)^{1+1}}{1}(x-1)^1 + \frac{(-1)^{2+1}}{2}(x-1)^2 + \frac{(-1)^{3+1}}{3}(x-1)^3 \\ &\quad + \frac{(-1)^{1+4}}{4}(x-1)^4 + \frac{(-1)^{1+5}}{5}(x-1)^5 \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}\end{aligned}$$

2. *The key to writing these sums with summation notation is to find the pattern of the terms.*

(a)

$$1 + 3 + 5 + \ldots + 117 = \sum_{n=1}^{59}(2n-1)$$

(b) *We rewrite all of the terms as fractions, the subtraction as addition, and associate the negatives '−' with the numerators to get*

$$\frac{1}{1} + \frac{-1}{2} + \frac{1}{3} + \frac{-1}{4} + \ldots + \frac{1}{117}$$

The numerators, 1, −1, etc. can be described by the geometric sequence $c_n = (-1)^{n-1} \forall n \geq 1$.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \ldots + \frac{1}{117} \quad = \quad \sum_{n=1}^{117} \frac{(-1)^{n-1}}{n}$$

(c)

$$0.9 + 0.09 + 0.009 + \ldots 0.\underbrace{0 \cdots 0}_{n-1 \text{ zeros}} 9 \quad = \quad \sum_{k=1}^{n} \frac{9}{10^k}$$

Acknowledgments

Some parts of these notes are build on other sources including the following:

- Stitz and Zeager (2013) which is licensed under the is licensed under a Creative Commons Attribution Non-Commercial ShareAlike 4.0 International License (CC BY-NC-SA 4.0);
- Openstax (2020) which is licensed under a Creative Commons Attribution Non-Commercial ShareAlike 4.0 International License (CC BY-NC-SA 4.0);
- icons stem from `https://uxwing.com` and are public domain.
- I am also thankful to Prof. Dr. Dirk Daemen for sharing his slides and exercises.

Bibliography

Chiang, A. C. and Wainwright, K. (2005). *Fundamental Methods of Mathematical Economics*. McGraw-Hill/Irwin. [Cited on page 8.]

Eldridge, D. S. (2014). A comment on Siegfried's first lesson in econometrics. *Economic Inquiry*, 52(1):503–504. [Cited on page 12.]

Gonick, L. (2011). *The Cartoon Guide to Calculus*. William Morrow Paperbacks. [Cited on page 8.]

Gonick, L. (2015). *The Cartoon Guide to Algebra*. William Morrow Paperbacks. [Cited on page 8.]

Openstax (2020). *Calculus: Volume 1*. Rice University. available on https://openstax.org/details/books/calculus-volume-1. [Cited on pages 8, 17, and 88.]

Siegfried, J. J. (1970). A first lesson in econometrics. *Journal of Political Economy*, 78(6):1378–1379. [Cited on page 12.]

Simon, C. P. and Blume, L. (1994). *Mathematics for Economists*, volume 7. Norton New York. [Cited on page 8.]

Stitz, C. and Zeager, J. (2013). *Precalculus Version $\pi = 3$, Corrected Edition*. available at https://www.stitz-zeager.com/szprecalculus07042013.pdf. [Cited on pages 8, 16, and 88.]

Sydsæter, K., Hammond, P. J., and Strøm, A. (2012). *Essential Mathematics for Economic Analysis*. Pearson Education, 4 edition. [Cited on page 8.]

Zimmermann, P., Casamayou, A., Cohen, N., Connan, G., Dumont, T., Fousse, L., Maltey, F., Meulien, M., Mezzarobba, M., and Pernet, C. (2018). *Computational Mathematics with SageMath*. SIAM. [Cited on page 27.]