# A graph with four card types and five vertex orbits: the $k = 4$ case of Kourovka Notebook Problem 20.71

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**Abstract**

A *card* of a finite simple graph $G$ is a vertex-deleted induced subgraph $G - v$. Problem 20.71 of the Kourovka Notebook (J. Lauri, communicated by V. Pannone) asks, for $k = 2, 3, 4$, whether a connected graph with exactly $k$ isomorphism types of cards can have more than $k$ orbits of $\mathrm{Aut}(G)$ on vertices. We answer the case $k = 4$ affirmatively with a graph on 9 vertices, which an exhaustive census shows to be of minimum order and unique up to complementation; curiously, no example of order 10 exists for any $k \in \{2, 3, 4\}$. The verification is elementary, and is additionally certified in Lean 4. The same census shows that any example for $k = 2$ or $k = 3$ must have at least 13 vertices.

## 1 Introduction

Let $G$ be a finite simple undirected graph with vertex set $V(G)$, $|V(G)| = n$. The *cards* of $G$ are the $n$ induced subgraphs $G - v$, $v \in V(G)$. Vertices in one orbit of $\mathrm{Aut}(G)$ yield isomorphic cards, so the partition of $V(G)$ by card type is coarser than the orbit partition; in particular a graph with exactly $k$ card types has at least $k$ orbits, and vertices with isomorphic cards lying in distinct orbits are the classical *pseudosimilar* vertices [1, 3, 4]. If all cards are mutually isomorphic ($k = 1$), it is known that $G$ must be vertex-transitive, i.e. the orbit count is exactly 1 [2, Problem 20.71, comment].

Problem 20.71 of the Kourovka Notebook [2], due to J. Lauri and communicated by V. Pannone, asks:

**Problem 1** ([2, 20.71])**.** *For $k = 2, 3, 4$: can a connected graph with exactly $k$ isomorphism types of cards have more than $k$ orbits of* $\mathrm{Aut}(G)$ *on $V(G)$?*

Examples are recorded there with 5 card types and 6 orbits, and with 6 card types and 7 orbits. We settle the case $k = 4$:

**Theorem 2.** *There is a connected graph $W$ on 9 vertices with exactly four isomorphism types of cards and exactly five vertex orbits.*

*IIT Madras. Verification scripts and a Lean 4 certificate: `https://github.com/infinityscroll/kourovka-20-71`.

**Proposition 3** (computational)**.** *By exhaustive search over connected unlabeled graphs:*

1. *no graph of order at most* 8 *has* $k \in \{2,3,4\}$ *card types and more than* $k$ *orbits, so* $W$ *has minimum order;*
2. *at order* 9 *there are exactly two such graphs for* $k = 4$*, namely* $W$ *and its complement* $\overline{W}$ *(card types and orbits are invariant under complementation, and* $\overline{W}$ *is again connected), so the example is unique up to complementation; at order* 10 *there is none for any* $k \in \{2,3,4\}$*;*
3. *no graph of order at most* 12 *has* $k \in \{2,3\}$ *card types and more than* $k$ *orbits. Consequently any affirmative example for the two remaining cases of Problem 1 has at least* 13 *vertices.*

## 2 The witness

Let $W$ be the graph on $\{0,1,\ldots,8\}$ with the 16 edges

$$03,\ 04,\ 07,\ 14,\ 15,\ 16,\ 25,\ 26,\ 27,\ 28,\ 36,\ 37,\ 38,\ 48,\ 57,\ 68$$

(graph6 code `HCpbdgy`). $W$ is connected: $03, 04, 07, 14, 15, 16, 25, 28$ is a spanning tree.

### 2.1 Exactly five orbits

Colour refinement starting from degrees stabilises at the partition

$$\pi \ = \ \{0,5\} \mid \{1,4\} \mid \{2,3\} \mid \{6,8\} \mid \{7\},$$

and every automorphism preserves the cells of $\pi$. The involution

$$\sigma = (0\,5)(1\,4)(2\,3)(6\,8)$$

is an automorphism of $W$ (a direct check on the edge list), so each cell of $\pi$ is a single orbit: there are exactly five orbits, and $\mathrm{Aut}(W) = \{\mathrm{id}, \sigma\}$ (the exhaustive check over all 9! permutations confirms no further automorphisms, though for Theorem 2 the count of orbits needs only $\pi$ and $\sigma$).

### 2.2 Exactly four card types

The degree multisets of the nine cards take exactly four values:

| | |
|---|---|
| $W-0,\ W-5:$ | $(2,3,3,3,3,4,4,4)$ |
| $W-1,\ W-4:$ | $(2,2,3,3,4,4,4,4)$ |
| $W-2,\ W-3,\ W-6,\ W-8:$ | $(2,3,3,3,3,3,3,4)$ |
| $W-7:$ | $(2,2,3,3,3,3,4,4)$ |

so there are at least four card types. Conversely $\sigma$ identifies the cards within the pairs $\{0,5\}$, $\{1,4\}$, $\{2,3\}$, $\{6,8\}$, and the explicit bijections

$$\begin{aligned} W-2 \to W-6:&\ \ 0\mapsto 2,\ 1\mapsto 4,\ 3\mapsto 7,\ 4\mapsto 8,\ 5\mapsto 1,\ 6\mapsto 0,\ 7\mapsto 5,\ 8\mapsto 3,\\ W-2 \to W-8:&\ \ 0\mapsto 3,\ 1\mapsto 1,\ 3\mapsto 7,\ 4\mapsto 6,\ 5\mapsto 4,\ 6\mapsto 5,\ 7\mapsto 0,\ 8\mapsto 2, \end{aligned}$$

are isomorphisms (checks on the edge lists), so the four vertices $2, 3, 6, 8$ give one card type. Hence there are exactly four card types, with deletion classes $\{0, 5\}, \{1, 4\}, \{2, 3, 6, 8\}, \{7\}$.

The point of the example is that the card class $\{2, 3, 6, 8\}$ is the union of the *two* orbits $\{2, 3\}$ and $\{6, 8\}$: these four vertices are pairwise pseudosimilar across the two orbits. This proves Theorem 2. □

*Remark* 4. The verification above is deliberately software-independent: one involution, two explicit card isomorphisms, four degree multisets, and colour refinement, all checkable by hand. A machine-checked Lean 4 certificate of the statement (connectivity, card-type count 4, orbit count 5), which compiles against Mathlib without `sorry` or added axioms, is provided in the repository, together with scripts reproducing every claim. Its finite decisions use `native_decide`, so this certificate also trusts Lean's native compiler.

## 3 The census

Proposition 3 rests on exhaustive generation of connected unlabeled graphs with `nauty`'s `geng` [5], in the counts recorded at OEIS A001349 (11,716,571 graphs of order 10; 1,006,700,565 of order 11), filtered in two sound pruning stages: a graph with at most 3 card types has at most 3 distinct degrees (isomorphic cards force equal deleted-vertex degrees), and the partition of cards by the invariant pair (degree multiset, triangle count) is coarser than the partition into isomorphism types. Cards of the graphs surviving both filters were classified exactly by canonical labelling (`pynauty`), and orbit counts of the resulting candidates computed exactly. At orders 10 and 11 respectively, 741,839 and 27,219,751 graphs passed the degree filter, 8,059 and 60,536 required canonical labelling, and none was an example for $k \in \{2, 3\}$. The census was completed at order 12: all 164,059,830,476 connected graphs (again matching OEIS A001349 exactly) were generated, of which 1,905,839,762 passed the degree filter and 32,500 required canonical labelling; none is an example for $k \in \{2, 3\}$. Beyond order 12, structured sweeps also found no example: all regular graphs of orders 13–14 and all graphs of order 13 whose degrees lie in a window $\{a, a + 1\}$, $2 \le a \le 5$ (836,922,152 graphs). Here degrees $d \le (n - 1)/2$ (resp. the low windows) suffice: card types and orbit counts are complement-invariant, and although complementation need not preserve connectivity, the complement of a disconnected graph is always connected, so the high slices are covered by additionally testing the 2,844 disjoint unions of low-slice connected components — none of which is an example either. Full data are in the repository.

## 4 Remaining cases

The cases $k = 2$ and $k = 3$ of Problem 1 remain open; by Proposition 3 any example has order at least 13. It seems plausible that the difficulty is genuine: fewer card types force the graph closer to vertex-transitivity, while extra orbits require asymmetry, and the tension is sharpest at $k = 2$ where the $k = 1$ rigidity theorem sits immediately below.

## Disclosure

The search, the write-up and the Lean formalization were prepared with substantial assistance from Claude (Anthropic); the witness was additionally re-verified by independent brute-force computation, and the central claim is machine-checked.

## References


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