Groebner Basis

Usage

get_groebner_basis(field, graph, sources, sinks)

Limitations

Does not support multi-edges

Inputs

Field is a type of field. Example:

GF(2)
The type of graph is a labelled digraph. The nodes need to be given names in order for the sinks and sources to be mapped. Example:

DiGraph({'s': ['t','u'], 't': ['x','y'], 'u': ['x','z'], 'x': ['y','z']})
The type of sources is a dictionary of lists, mapping a source node name to the list of processes inside it. Example:

sources = {'s': [0,1]}
The type of sinks is a dictionary of dictionary of lists, mapping a sink node name to to its demands. The demands are a dictionary mapping the sources from which the sink demands, to the processes from that source that are demanded. Example:

sinks = {'y': {'s':[0,1]} , 'z': {'s':[0,1]}}

Returns

It returns the groebner basis for that particular multicast network

----

Generates the sink matrix B to calculate the transfer matrix, defined as $(A(1-F)^{-1}B^T)$

```
def generate_matrix_sink(sinks, graph):
    vertices = graph.vertices()
    m = []
    for s in sinks:
        sources = sinks[s]
        for source in sources:
            sink_source_rows = sources[source]
            for i in sink_source_rows:
                row = []
                for v in vertices:
                    origin, destination, edge_name = v
                    if destination == s:

row.append(var('sink_'+str(origin)+str(destination)+str(source) +
str(i)))
                    else:
                        row.append(0)
                m.append(row)
    return matrix(SR, m, sparse=True)
```

Generates the source matrix A to calculate the transfer matrix, defined as $(A(1-F)^{-1}B^T)$

```
def generate_matrix_source(sources, graph):
    vertices = graph.vertices()
    m = []
    for s in sources:
        source_rows = sources[s]
        for i in source_rows:
            row = []
            for v in vertices:
                origin, destination, edge_name = v
                if origin == s:

row.append(var('source_'+str(origin)+str(destination)+str(i)))
                else:
                    row.append(0)
            m.append(row)
    return matrix(SR, m, sparse=True)
```

Generates the source matrix F to calculate the transfer matrix, defined as $(A(1-F)^{-1}B^T)$

```
def generate_labeled_line_graph_matrix(graph):
    line_graph = graph.line_graph()
    vertices = line_graph.vertices()
    adj_matrix = line_graph.adjacency_matrix()
    n_rows = adj_matrix.nrows()
    n_cols = adj_matrix.ncols()
    m = [[y for y in x] for x in adj_matrix]
    for i in range(n_rows):
        for j in range(n_cols):
             origin_1, destination_1, edge_name_1 = vertices[i]
             origin_2, destination_2, edge_name_2 = vertices[j]
             m[i][j] = m[i][j] *
var('e_'+str(origin_1)+str(destination_1)+'_'
                       +str(origin_2)+str(destination_2))

    return matrix(SR, m, sparse=True)
```

Genenerates the transfer matrix $(A(1-F)^{-1}B^T)$, that maps inputs to outputs

```
def generate_transfer_matrix(graph, sources, sinks):
    line_graph = graph.line_graph()
    F = generate_labeled_line_graph_matrix(graph)
    I = matrix.identity(F.nrows())
    A = generate_matrix_source(sources, line_graph)
    B = generate_matrix_sink(sinks, line_graph)
    return A*~(I-F)*B.T
```

Helper functions to find which rows in A and B are assigned to which source process or to which sink.

```
def get_source_process_rows(source_process_list, sources):
    source_rows ={}
    i=0
    for s in sources:
        source_rows[s] = {}
        dic = source_rows[s]
        for process in sources[s]:
            dic[process] = i
            i+=1
    rows = [source_rows[source][process]  for source in
source_process_list for process in source_process_list[source]]
    return rows

def get_sink_rows(sink_name, sinks):
    sink_rows ={}
    i=0
    for s in sinks:
        sink = sinks[s]
        len_sink = sum([len(sink[source]) for source in sink])
        sink_rows[s] = []
        for j in range(len_sink):
            sink_rows[s].append(i)
            i+=1
    rows = sink_rows[sink_name]
    return rows
```

Given the sink demands and the sources, these helper functions find which submatrices of the transfer function need to have determinant different from 0 and which entries need to be equal to 0.

```
def generate_nonnull_matrices(sources, sinks):
    nonnull_matrices = []
    for si in sinks:
        rows = get_source_process_rows(sinks[si], sources)
        columns = get_sink_rows(si, sinks)
        nonnull_matrices.append((rows, columns))
    return nonnull_matrices
```

```
def generate_null_elements(sources, sinks):
    n_processes = sum([len(sources[s]) for s in sources])
    n_demands = sum([len(sinks[sink][source]) for sink in sinks for
source in sinks[sink]])
    m = [[1 for i in range(n_demands)] for j in range(n_processes)]
    nonnull_matrices = generate_nonnull_matrices(sources, sinks)
    for rows,columns in nonnull_matrices:
        for r in rows:
            for c in columns:
                m[r][c] = 0
    return m
```

Calculates the ideal of the linear network problem, as stated on http://www.mit.edu/~medard/mpapers/aaatnetworkcoding.pdf.

It is the ideal of all values that should be equal to 0.

```
def get_ideal(graph, sources, sinks):
    transfer_matrix = generate_transfer_matrix(graph, sources, sinks)
    nonnull_matrices = generate_nonnull_matrices(sources, sinks)
    null_elements_indexes = generate_null_elements(sources, sinks)
    null_elements = [transfer_matrix[i][j] for i in
range(len(null_elements_indexes)) for j in
range(len(null_elements_indexes[0])) if null_elements_indexes[i][j]==1]
    nonnull_prod = reduce((lambda x,y: x*y),
(transfer_matrix.matrix_from_rows_and_columns(*row_column).det() for
row_column in nonnull_matrices))
    null_ideal = [1- var('normalizer')*nonnull_prod] + null_elements
    return null_ideal
```

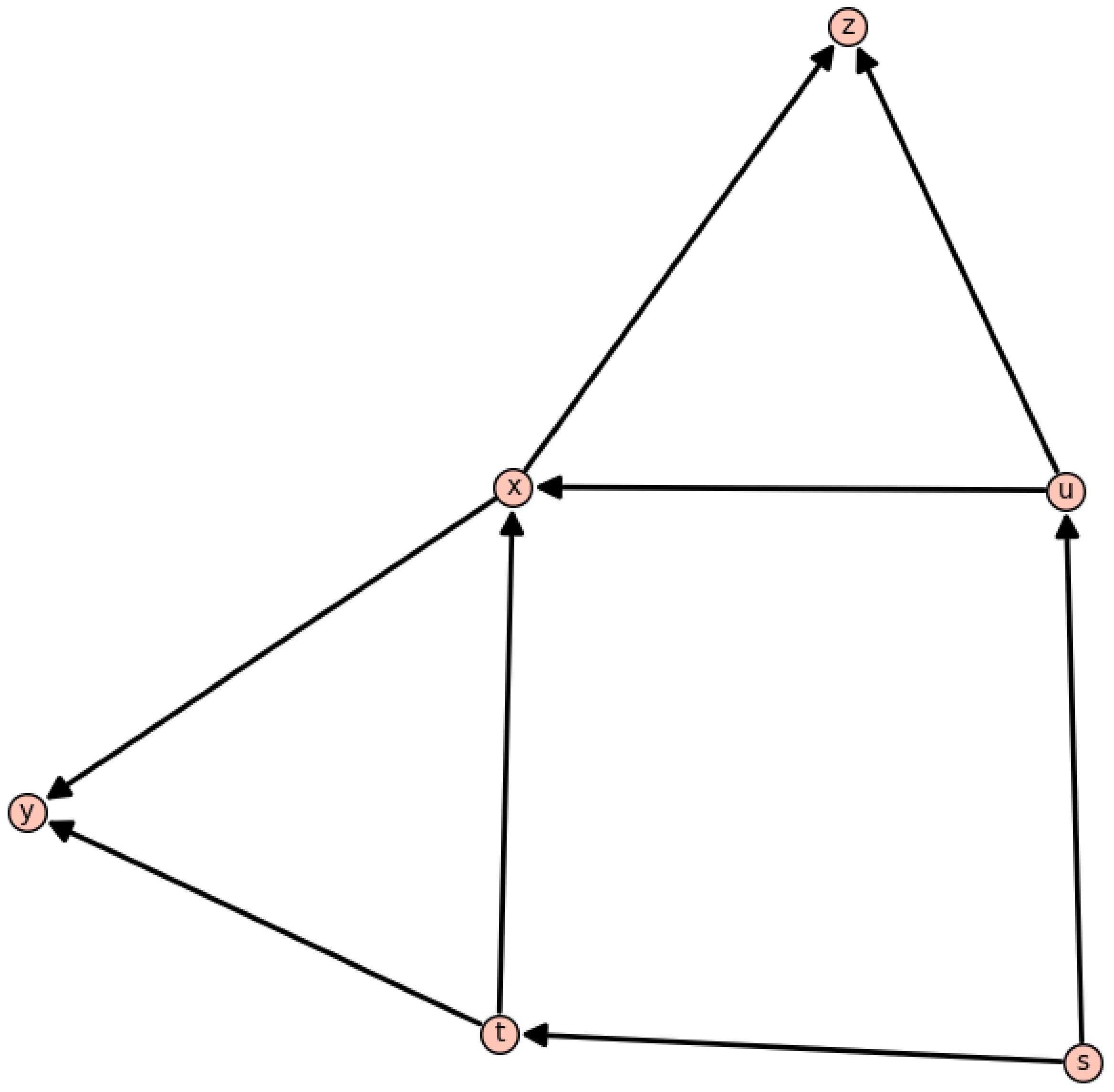
Calculates the groebner basis of the ideal of the linear network problem.

```
def get_groebner_basis(field, graph, sources, sinks):
    ideal = get_ideal(graph, sources, sinks)
    variables = [v for x in ideal for v in x.variables()]
    R = PolynomialRing(field, [x for x in set(variables)])
    I = R*ideal
    basis = I.groebner_basis()
    return basis
```

Example

```
%time
butterflyGraph =  DiGraph({'s': ['t','u'], 't': ['x','y'], 'u':
['x','z'], 'x': ['y','z']})

butterflyGraph.plot().show()

butterflysources = {'s': [0,1]}
butterflysinks = {'y': {'s':[0,1]} , 'z': {'s':[0,1]}}
g = get_groebner_basis(GF(2), butterflyGraph, butterflysources,
butterflysinks)
g2 = get_groebner_basis(GF(3), butterflyGraph, butterflysources,
butterflysinks)

print "has solution in GF2 => " + str(g != [1]) + "\n groebner_basis = "
+ str(g)
print ""
print "has solution in GF3 => " + str(g2 != [1]) + "\n groebner_basis =
" + str(g2)

print "6 nodes, 8 edges, 1 sources, 2 sinks"
```

```
has solution in GF2 => True
 groebner_basis =
[e_ux_xy*e_tx_xz*source_su0^2*e_st_tx*sink_uzs0*source_st1^2*sink_ty\
s0*e_su_uz*e_st_ty*sink_xzs1*sink_xys1*normalizer*e_su_ux +
e_ux_xy*e_tx_xz*e_st_tx*source_su1^2*sink_uzs0*sink_tys0*e_su_uz*sou\
rce_st0^2*e_st_ty*sink_xzs1*sink_xys1*normalizer*e_su_ux +
e_ux_xy*e_tx_xz*source_su0^2*e_st_tx*source_st1^2*sink_tys0*e_su_uz*\
e_st_ty*sink_xys1*normalizer*sink_xzs0*sink_uzs1*e_su_ux +
e_ux_xy*e_tx_xz*e_st_tx*source_su1^2*sink_tys0*e_su_uz*source_st0^2*\
e_st_ty*sink_xys1*normalizer*sink_xzs0*sink_uzs1*e_su_ux +
e_ux_xy*e_tx_xz*source_su0^2*e_st_tx*sink_tys1*sink_uzs0*source_st1^\
2*e_su_uz*e_st_ty*sink_xzs1*normalizer*e_su_ux*sink_xys0 +
e_ux_xy*e_tx_xz*e_st_tx*source_su1^2*sink_tys1*sink_uzs0*e_su_uz*sou\
rce_st0^2*e_st_ty*sink_xzs1*normalizer*e_su_ux*sink_xys0 +
e_ux_xy*e_tx_xz*source_su0^2*e_st_tx*sink_tys1*source_st1^2*e_su_uz*\
e_st_ty*normalizer*sink_xzs0*sink_uzs1*e_su_ux*sink_xys0 +
e_ux_xy*e_tx_xz*e_st_tx*source_su1^2*sink_tys1*e_su_uz*source_st0^2*\
e_st_ty*normalizer*sink_xzs0*sink_uzs1*e_su_ux*sink_xys0 + 1]

has solution in GF3 => True
 groebner_basis =
[e_ux_xy*e_tx_xz*source_su0^2*e_st_tx*sink_uzs0*source_st1^2*sink_ty\
s0*e_su_uz*e_st_ty*sink_xzs1*sink_xys1*normalizer*e_su_ux +
e_ux_xy*e_tx_xz*source_su0*e_st_tx*source_su1*sink_uzs0*source_st1*s\
ink_tys0*e_su_uz*source_st0*e_st_ty*sink_xzs1*sink_xys1*normalizer*e\
_su_ux +
e_ux_xy*e_tx_xz*e_st_tx*source_su1^2*sink_uzs0*sink_tys0*e_su_uz*sou\
rce_st0^2*e_st_ty*sink_xzs1*sink_xys1*normalizer*e_su_ux -
e_ux_xy*e_tx_xz*source_su0^2*e_st_tx*source_st1^2*sink_tys0*e_su_uz*\
e_st_ty*sink_xys1*normalizer*sink_xzs0*sink_uzs1*e_su_ux -
e_ux_xy*e_tx_xz*source_su0*e_st_tx*source_su1*source_st1*sink_tys0*e\
_su_uz*source_st0*e_st_ty*sink_xys1*normalizer*sink_xzs0*sink_uzs1*e\
_su_ux -
e_ux_xy*e_tx_xz*e_st_tx*source_su1^2*sink_tys0*e_su_uz*source_st0^2*\
e_st_ty*sink_xys1*normalizer*sink_xzs0*sink_uzs1*e_su_ux -
e_ux_xy*e_tx_xz*source_su0^2*e_st_tx*sink_tys1*sink_uzs0*source_st1^\
2*e_su_uz*e_st_ty*sink_xzs1*normalizer*e_su_ux*sink_xys0 -
e_ux_xy*e_tx_xz*source_su0*e_st_tx*source_su1*sink_tys1*sink_uzs0*so\
urce_st1*e_su_uz*source_st0*e_st_ty*sink_xzs1*normalizer*e_su_ux*sin\
k_xys0 -
e_ux_xy*e_tx_xz*e_st_tx*source_su1^2*sink_tys1*sink_uzs0*e_su_uz*sou\
rce_st0^2*e_st_ty*sink_xzs1*normalizer*e_su_ux*sink_xys0 +
e_ux_xy*e_tx_xz*source_su0^2*e_st_tx*sink_tys1*source_st1^2*e_su_uz*\
e_st_ty*normalizer*sink_xzs0*sink_uzs1*e_su_ux*sink_xys0 +
e_ux_xy*e_tx_xz*source_su0*e_st_tx*source_su1*sink_tys1*source_st1*e\
_su_uz*source_st0*e_st_ty*normalizer*sink_xzs0*sink_uzs1*e_su_ux*sin\
k_xys0 +
e_ux_xy*e_tx_xz*e_st_tx*source_su1^2*sink_tys1*e_su_uz*source_st0^2*\
e_st_ty*normalizer*sink_xzs0*sink_uzs1*e_su_ux*sink_xys0 + 1]
6 nodes, 8 edges, 1 sources, 2 sinks
CPU time: 0.23 s,  Wall time: 0.23 s
```



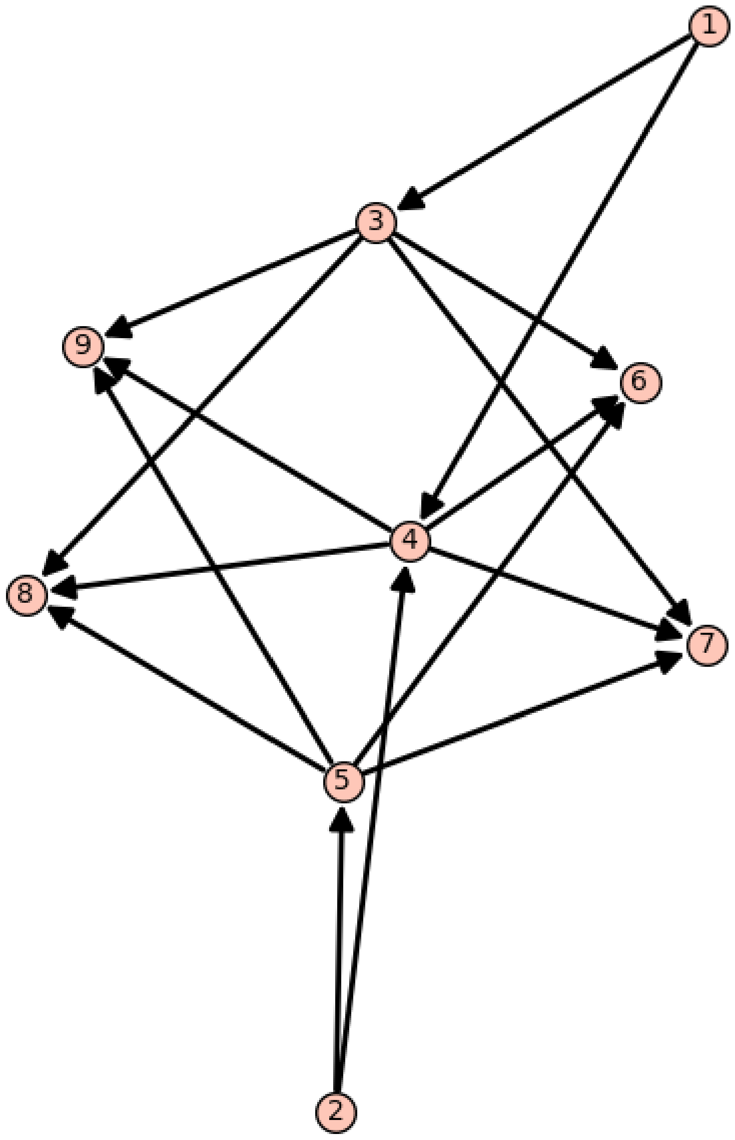
M network, from

On Coding for Non-Multicast Networks, by Medard, Muriel and Effros, Michelle and Karger, David and Ho

This network has no scalar solution

```
%time
mnetworkGraph = DiGraph({1: [3,4], 2: [4,5], 3: [6,7,8,9], 4: [6,7,8,9],
5: [6,7,8,9]})
mnetworkGraph.plot().show()
mnetworksources = {1: ['a','A'], 2:['b','B']}
mnetworksinks = {6: {1:['a'], 2:['b']}, 7: {1:['a'], 2:['B']}, 8: {1:
['A'], 2:['b']}, 9: {1:['A'], 2:['B']}}
g = get_groebner_basis(GF(2), mnetworkGraph, mnetworksources,
mnetworksinks)
g2 = get_groebner_basis(GF(3), mnetworkGraph, mnetworksources,
mnetworksinks)


print "has solution in GF2 => " + str(g != [1]) + "\n groebner_basis = "
+ str(g)
print ""
print "has solution in GF3 => " + str(g2 != [1]) + "\n groebner_basis =
" + str(g2)

print "9 nodes, 16 edges, 2 sources, 4 sinks"
```

```
has solution in GF2 => False
 groebner_basis = [1]

has solution in GF3 => False
 groebner_basis = [1]
9 nodes, 16 edges, 2 sources, 4 sinks
CPU time: 0.56 s,  Wall time: 0.56 s
```

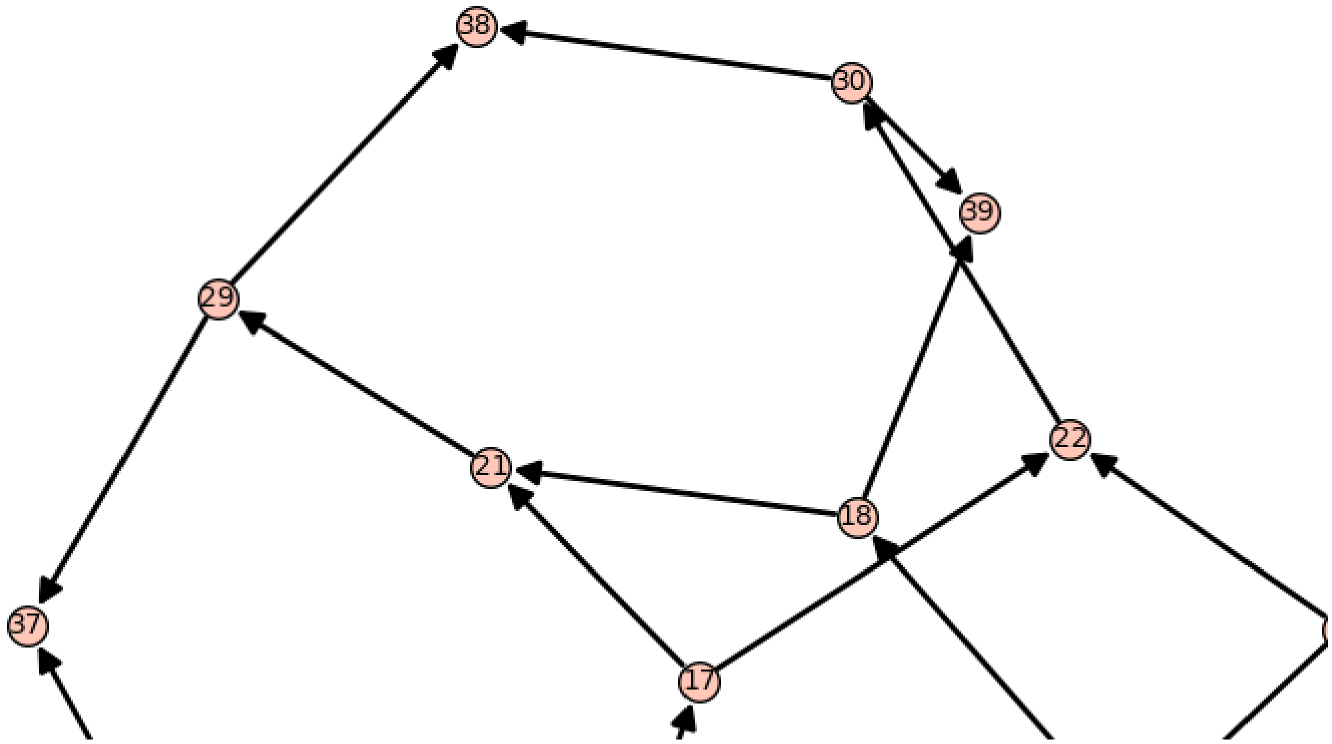


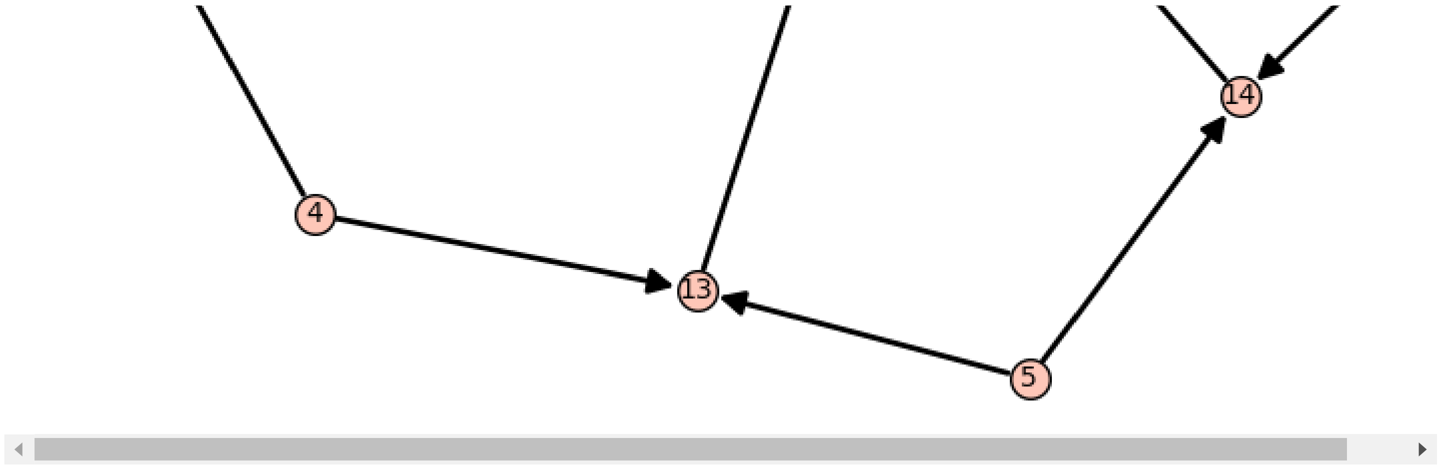
Network N1 from http://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=1468297

It has solution over even fields but no solution over odd fields.

```
%time
network1Graph = DiGraph({4: [13,37], 5: [13,14], 6: [14,22], 13: [17],
14: [18], 17: [21, 22], 18: [21, 39], 21: [29], 22: [30], 29: [37, 38],
30: [38, 39]})
network1Graph.plot().show()
network1sources = {4: ['a'], 5:['b'], 6:['c']}
network1sinks = {39: {4:['a']}, 38: {5:['b']}, 37: {6:['c']}}
g = get_groebner_basis(GF(2), network1Graph, network1sources,
network1sinks)
g2 = get_groebner_basis(GF(3), network1Graph, network1sources,
network1sinks)

print "has solution in GF2 => " + str(g != [1]) + "\n groebner_basis = "
+ str(g)
print ""
print "has solution in GF3 => " + str(g2 != [1]) + "\n groebner_basis =
" + str(g2)

print "14 nodes, 18 edges, 3 sources, 3 sinks"
```

```
has solution in GF2 => True
 groebner_basis = Polynomial Sequence with 25 Polynomials in 29
Variables

has solution in GF3 => False
 groebner_basis = [1]
14 nodes, 18 edges, 3 sources, 3 sinks
CPU time: 0.53 s,  Wall time: 0.53 s
```



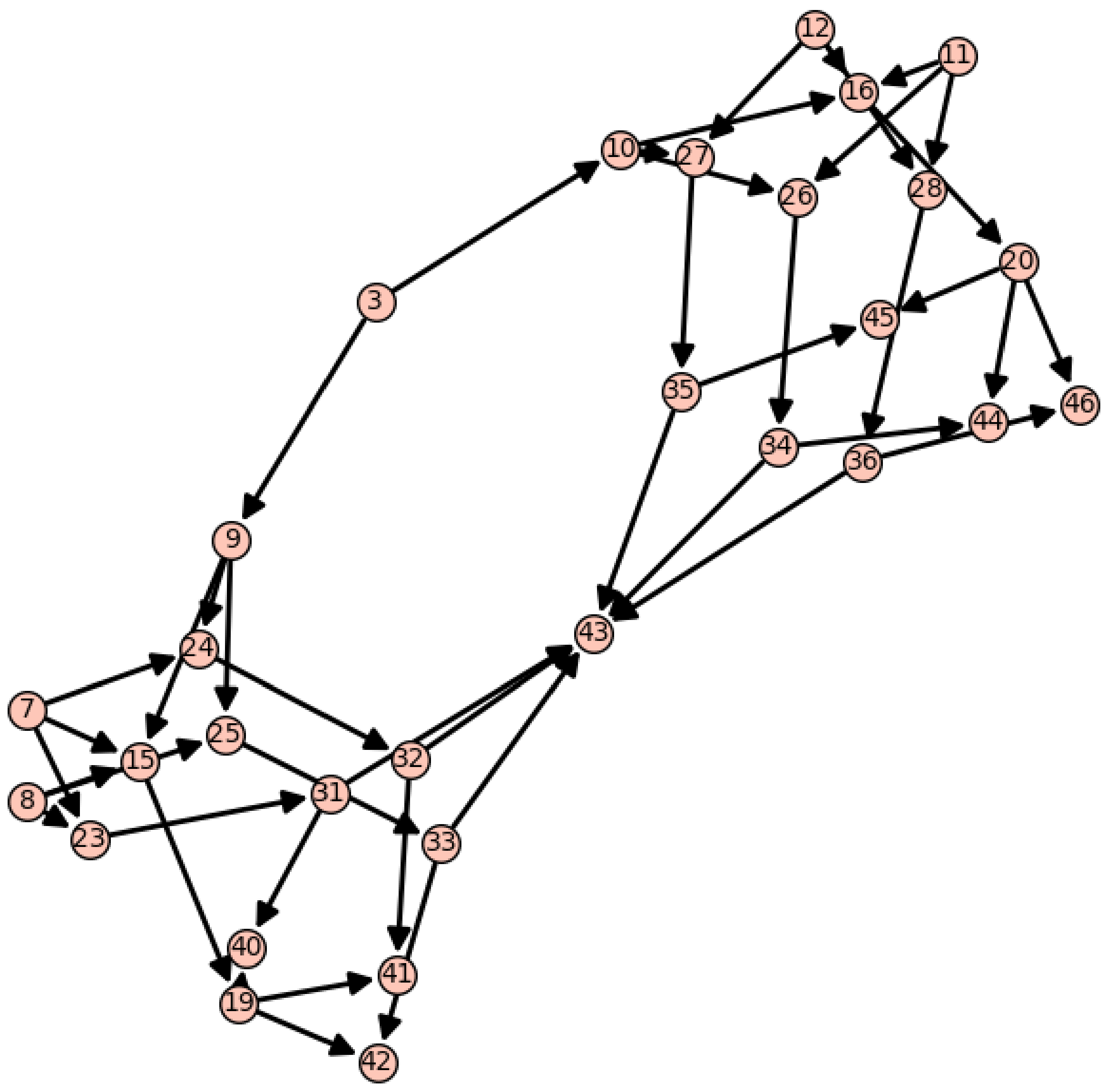


Network N2 from http://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=1468297

It has a scalar linear solution over any ring where is a unit, but has no linear solution for any vector dimension over a finite field with characteristic two. Also, the coding capacity of is .

```
%time
network2Graph = DiGraph({3: [9,10], 7: [23,15,24], 8: [23,15,25], 9:
[15, 24, 25], 10: [26, 16, 27], 11: [26, 16, 28], 12: [16, 27, 28], 15:
[19], 16: [20], 19: [40, 41, 42], 20: [44, 45, 46], 23: [31], 24: [32],
25: [33], 26: [34], 27: [35], 28: [36], 31: [40,43], 32: [41,43], 33:
[42,43], 34: [44,43], 35: [45,43], 36: [46,43]})
network2Graph.plot().show()
network2sources = {3: ['c'], 7:['a'], 8:['b'], 11:['d'], 12:['e']}
network2sinks = {40: {3:['c']}, 41: {8:['b']}, 42: {7:['a']}, 43: {3:
['c']}, 44: {12:['e']}, 45: {11:['d']}, 46: {3:['c']}}
g = get_groebner_basis(GF(2), network2Graph, network2sources,
network2sinks)
g2 = get_groebner_basis(GF(3), network2Graph, network2sources,
network2sinks)

print "has solution in GF2 => " + str(g != [1]) + "\n groebner_basis = "
+ str(g)
print ""
print "has solution in GF3 => " + str(g2 != [1]) + "\n groebner_basis =
" + str(g2)

print "30 nodes, 46 edges, 5 sources, 7 sinks"
```

```
    has solution in GF2 => False
     groebner_basis = [1]

    has solution in GF3 => True
     groebner_basis = Polynomial Sequence with 2025 Polynomials in 75
    Variables
    30 nodes, 46 edges, 5 sources, 7 sinks
    CPU time: 4.97 s,  Wall time: 4.98 s
```

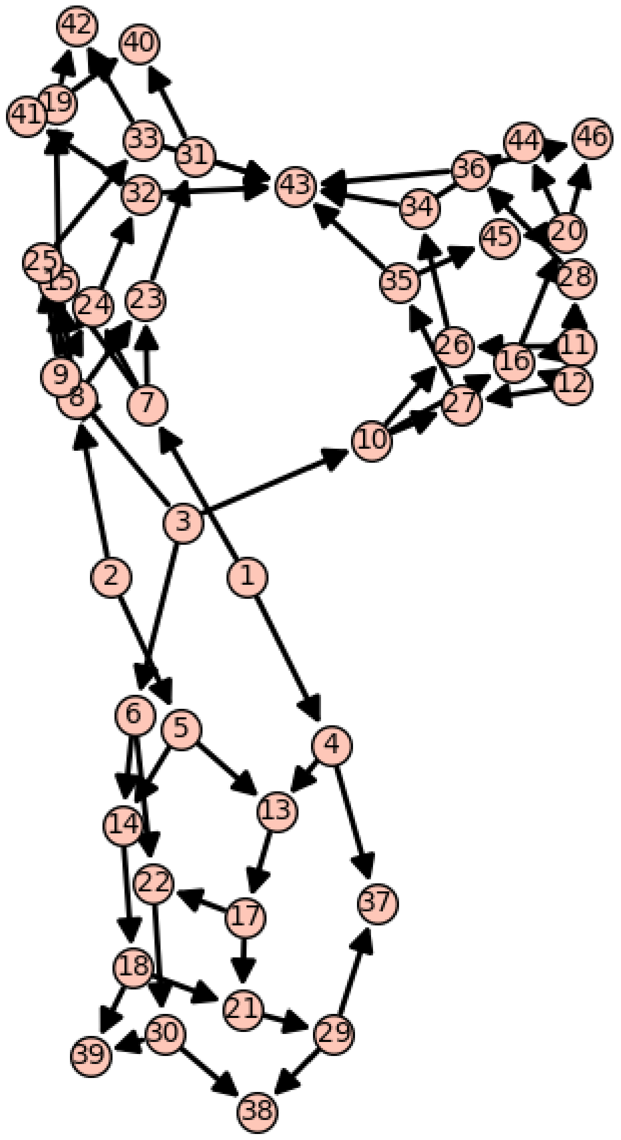


Network N3 from http://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=1468297

It has no scalar nor vector solutions.

```
%time
network3Graph = DiGraph({4: [13,37], 5: [13,14], 6: [14,22], 13: [17],
14: [18], 17: [21, 22], 18: [21, 39], 21: [29], 22: [30], 29: [37, 38],
30: [38, 39], 3: [6,9,10], 7: [23,15,24], 8: [23,15,25], 9: [15, 24,
25], 10: [26, 16, 27], 11: [26, 16, 28], 12: [16, 27, 28], 15: [19], 16:
[20], 19: [40, 41, 42], 20: [44, 45, 46], 23: [31], 24: [32], 25: [33],
26: [34], 27: [35], 28: [36], 31: [40,43], 32: [41,43], 33: [42,43], 34:
[44,43], 35: [45,43], 36: [46,43], 1: [4,7], 2: [5,8]})

network3Graph.plot().show()


network3sources = {1: ['a'], 2:['b'], 3: ['c'], 11:['d'], 12:['e']}
network3sinks = {40: {3:['c']}, 41: {2:['b']}, 42: {1:['a']}, 43: {3:
['c']}, 44: {12:['e']}, 45: {11:['d']}, 46: {3:['c']}, 39: {1:['a']},
38: {2:['b']}, 37: {3:['c']}}

g = get_groebner_basis(GF(2), network3Graph, network3sources,
network3sinks)
g2 = get_groebner_basis(GF(3), network3Graph, network3sources,
network3sinks)

print "has solution in GF2 => " + str(g != [1]) + "\n groebner_basis = "
+ str(g)
print ""
print "has solution in GF3 => " + str(g2 != [1]) + "\n groebner_basis =
" + str(g2)

print "46 nodes, 69 edges, 5 sources, 10 sinks"
```

```
has solution in GF2 => False
 groebner_basis = [1]

has solution in GF3 => False
 groebner_basis = [1]
46 nodes, 69 edges, 5 sources, 10 sinks
CPU time: 2.52 s,  Wall time: 2.52 s
```



Network N1 From:

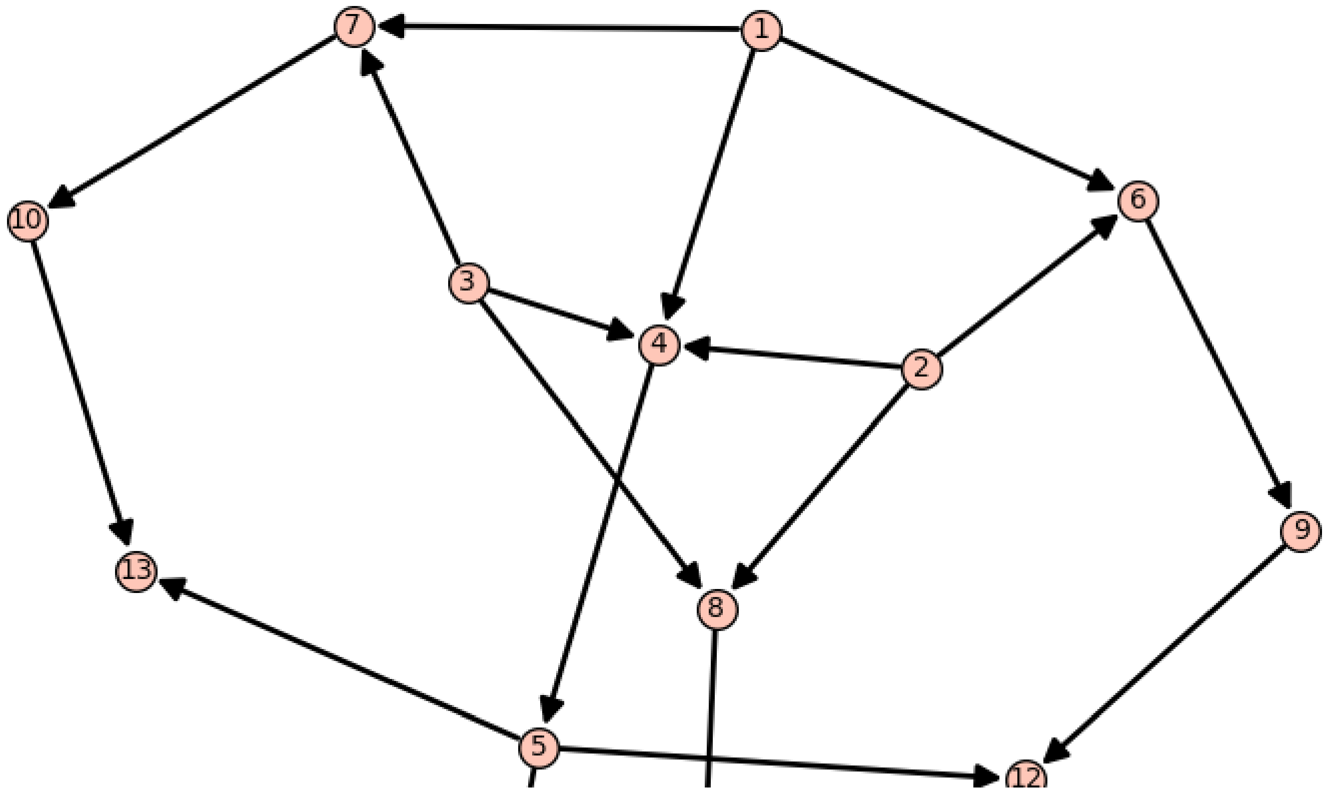
Unachievability of Network Coding Capacity
Randall Dougherty, Chris Freiling, and Kenneth Zeger, Fellow, IEEE

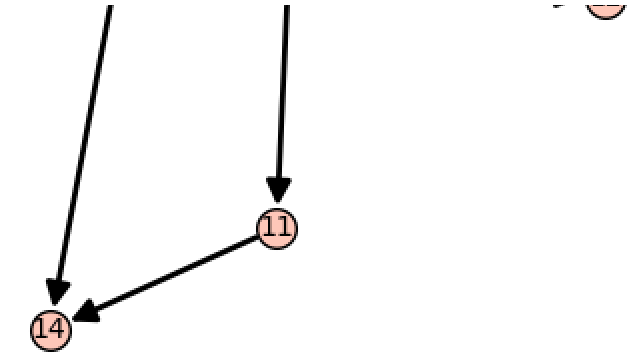
Only has solution over alphabets that have property P, as defined in the paper

```
%time
networkN1Graph = DiGraph({1: [4,6,7], 2: [6,4,8], 3: [4,7,8], 4: [5], 5:
[12,13,14], 6: [9], 7: [10], 8: [11], 9: [12], 10: [13], 11: [14]})
networkN1Graph.plot().show()
networkN1Sources = {1: ['a'], 2:['b'], 3:['c']}
networkN1Sinks = {12: {3:['c']}, 13: {2:['b']}, 14: {1:['a']}}
g = get_groebner_basis(GF(2), networkN1Graph, networkN1Sources,
networkN1Sinks)
g2 = get_groebner_basis(GF(3), networkN1Graph, networkN1Sources,
networkN1Sinks)
```

```
print "has solution in GF2 => " + str(g != [1]) + "\n groebner_basis = "
+ str(g)
print ""
print "has solution in GF3 => " + str(g2 != [1]) + "\n groebner_basis =
" + str(g2)

print str(networkN1Graph.order()) + " nodes, " +
str(networkN1Graph.size()) +" edges, 3 sources, 3 sinks"
```

```
has solution in GF2 => True
 groebner_basis = Polynomial Sequence with 22 Polynomials in 31
Variables

has solution in GF3 => True
 groebner_basis = Polynomial Sequence with 22 Polynomials in 31
Variables
14 nodes, 19 edges, 3 sources, 4 sinks
CPU time: 0.38 s,  Wall time: 0.39 s
```





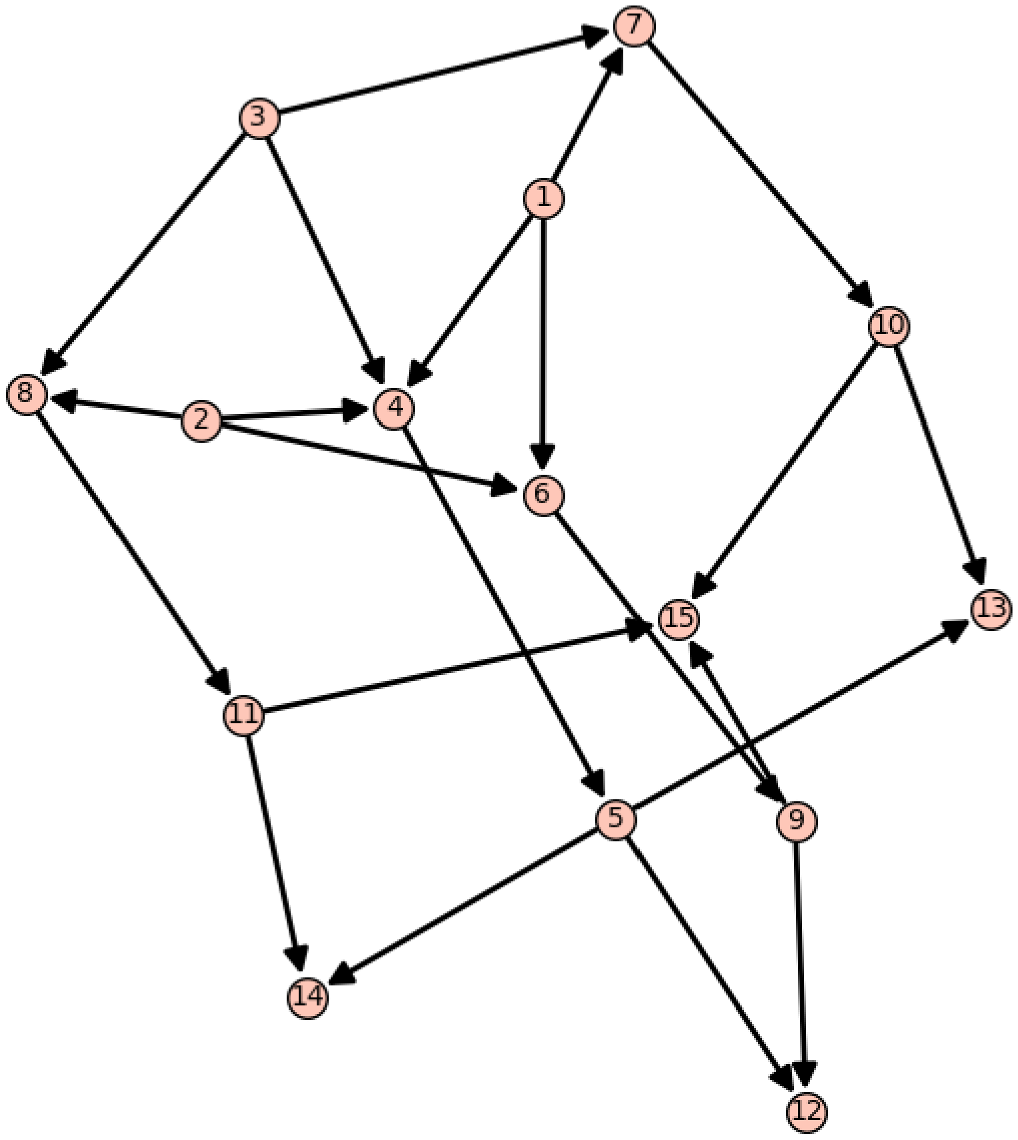
Network N2 From:

Unachievability of Network Coding Capacity
Randall Dougherty, Chris Freiling, and Kenneth Zeger, Fellow, IEEE

Only has solution over alphabets that have property P, as defined in the paper, and no elements of order 2.

```
%time
networkN2Graph = DiGraph({1: [4,6,7], 2: [6,4,8], 3: [4,7,8], 4: [5], 5:
[12,13,14], 6: [9], 7: [10], 8: [11], 9: [12, 15], 10: [13, 15], 11:
[14, 15]})
networkN2Graph.plot().show()
networkN2Sources = {1: ['a'], 2:['b'], 3:['c']}
networkN2Sinks = {12: {3:['c']}, 13: {2:['b']}, 14: {1:['a']}, 15: {3:
['c']}}
g = get_groebner_basis(GF(2), networkN2Graph, networkN2Sources,
networkN2Sinks)
g2 = get_groebner_basis(GF(3), networkN2Graph, networkN2Sources,
networkN2Sinks)


print "has solution in GF2 => " + str(g != [1]) + "\n groebner_basis = "
+ str(g)
print ""
print "has solution in GF3 => " + str(g2 != [1]) + "\n groebner_basis =
" + str(g2)

print str(networkN2Graph.order()) + " nodes, " +
str(networkN2Graph.size()) +" edges, 3 sources, 4 sinks"
```

```
    has solution in GF2 => False
     groebner_basis = [1]

    has solution in GF3 => True
     groebner_basis = Polynomial Sequence with 57 Polynomials in 37
    Variables
    15 nodes, 22 edges, 3 sources, 4 sinks
    CPU time: 0.47 s,  Wall time: 0.47 s
```



Network N3 From:

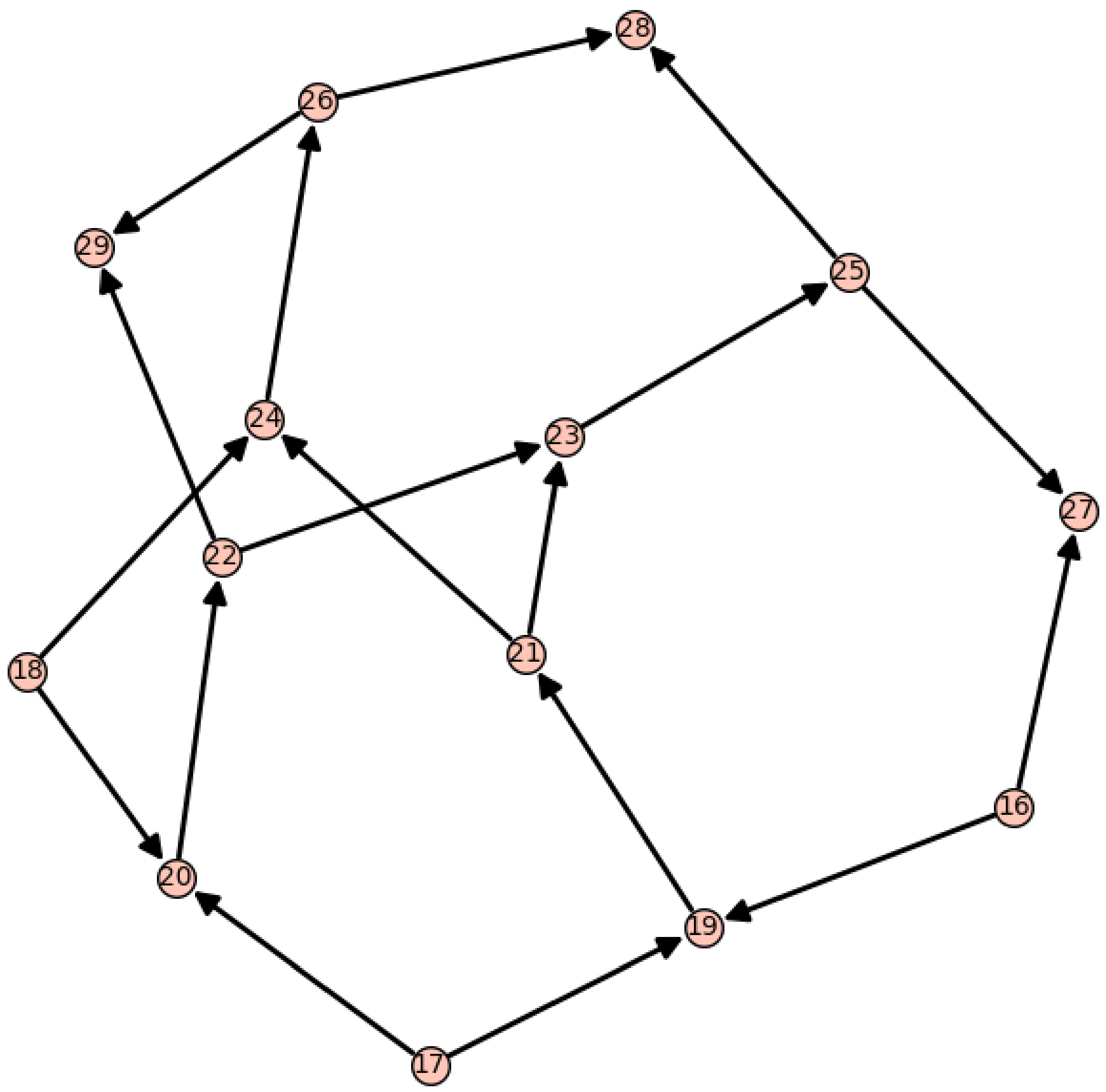
Unachievability of Network Coding Capacity
Randall Dougherty, Chris Freiling, and Kenneth Zeger, Fellow, IEEE

Only has solution over alphabets that have property P, as defined in the paper, and all non zero elements have order 2.

```
%time
networkN3Graph = DiGraph({16: [19, 27], 17: [19, 20], 18: [20, 24], 19:
[21], 20: [22], 21: [23, 24], 22: [23, 29], 23: [25], 24: [26], 25: [27,
28], 26: [28, 29]})
networkN3Graph.plot().show()
networkN3Sources = {16: ['a'], 17:['b'], 18:['c']}
networkN3Sinks = {27: {18:['c']}, 28: {17:['b']}, 29: {16:['a']} }
g = get_groebner_basis(GF(2), networkN3Graph, networkN3Sources,
networkN3Sinks)
g2 = get_groebner_basis(GF(3), networkN3Graph, networkN3Sources,
networkN3Sinks)
```

```
g3 = get_groebner_basis(GF(4), networkN3Graph, networkN3Sources,
networkN3Sinks)


print "has solution in GF2 => " + str(g != [1]) + "\n groebner_basis = "
+ str(g)
print ""
print "has solution in GF3 => " + str(g2 != [1]) + "\n groebner_basis =
" + str(g2)
print ""
print "has solution in GF4 => " + str(g3 != [1]) + "\n groebner_basis =
" + str(g3)

print str(networkN3Graph.order()) + " nodes, " +
str(networkN3Graph.size()) +" edges, 3 sources, 3 sinks"
```

```
    has solution in GF2 => True
     groebner_basis = Polynomial Sequence with 25 Polynomials in 29
    Variables

    has solution in GF3 => False
     groebner_basis = [1]

    has solution in GF4 => True
     groebner_basis = Polynomial Sequence with 25 Polynomials in 29
    Variables
    14 nodes, 18 edges, 3 sources, 3 sinks
    CPU time: 0.57 s,  Wall time: 0.60 s
```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