

# Fractals Avoiding Fractal Configurations

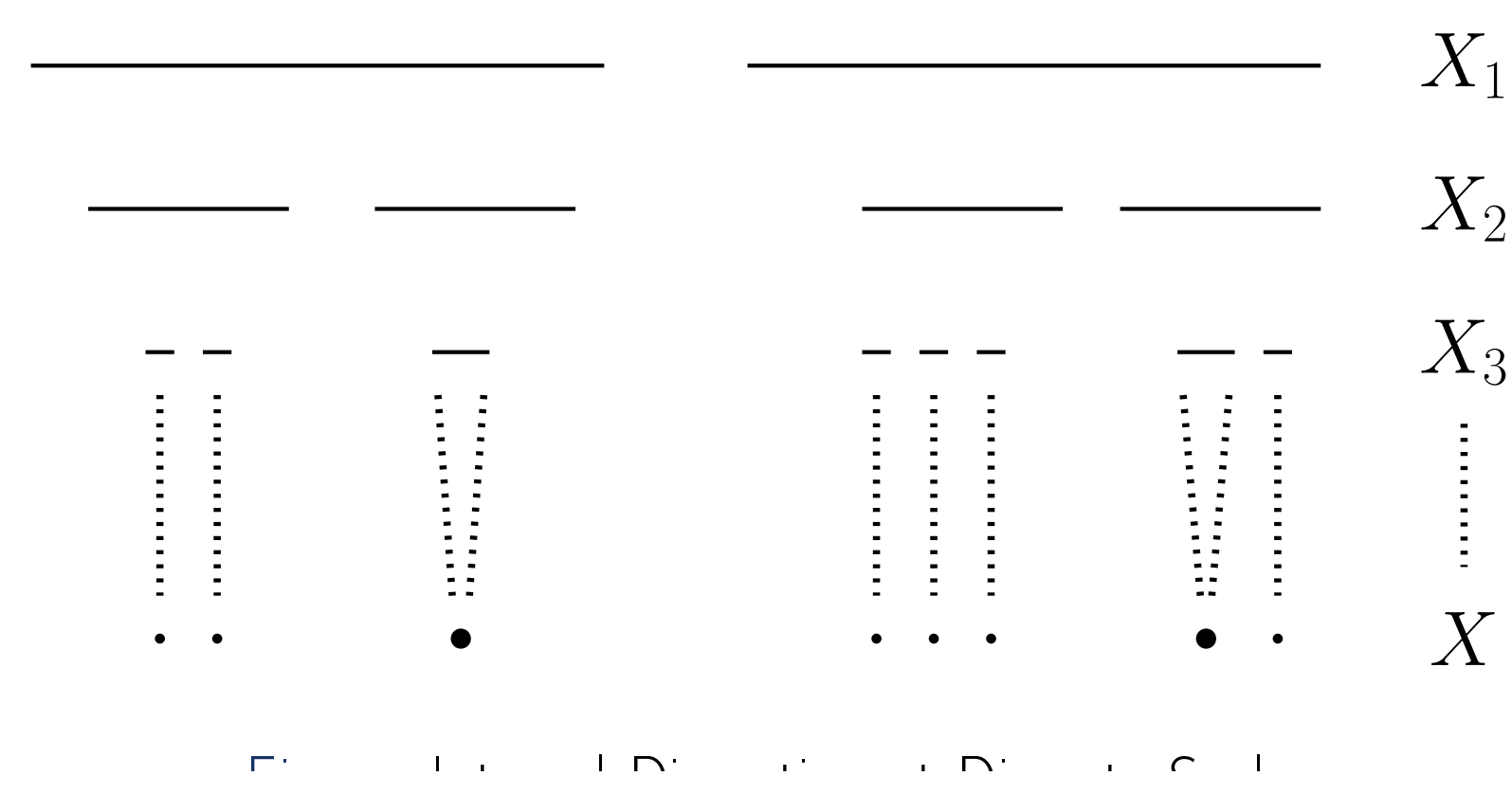
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## Method: Discretization of Scales

- Following Fraser, Pramanik, and Keleti, we construct solutions by repeatedly dissecting intervals, ala the construction of the Cantor set.
- If  $X$  is the decreasing limit of sets  $X_1, X_2, \dots$ , which are unions of intervals, we can discretize the problem so that we only have to avoid a discrete version of the configuration at each dissection.



- **Discrete configuration problem:** If  $x_1, \dots, x_d \in X_n$ , and  $f(x_1, \dots, x_d) = 0$ , then some  $x_i$  and  $x_j$  lie in a common interval in  $X_n$ .
- Provided that the discrete configuration problem is satisfied and the length of the intervals forming  $X_n$  tends to zero as  $n \rightarrow \infty$ ,  $X$  avoids all configurations.
- For technical reasons, we consider a slightly different discrete configuration problem where at each scale we consider a partition of  $X_n$  into unions of intervals, and the problem then becomes that if  $f(x_1, \dots, x_d) = 0$ , then some  $x_i$  and  $x_j$  lie in a common part of the partition of  $X_n$ .
- For fractal avoidance, the discrete criterion is obtained if whenever  $(x_1, \dots, x_d) \in X_n^d \cap Y$ , then  $|x_i - x_j| = o(1)$  for some indices  $i$  and  $j$ .

## Our Research Problem: How Large can Sets with a Fixed Irregularity Be?

- The irregularities commonly manifest as avoiding the zero set of a function.
- Largeness is quantified by the Hausdorff dimension of the irregular set.
- Examples of such problems including finding a large set  $X \subset \mathbf{R}^3$  such that the angles formed by any three distinct points in  $X$  are distinct.
- **Configuration Avoidance:** Find  $X$  such that for distinct  $x_1, \dots, x_d \in X$ ,  $f(x_1, \dots, x_d) \neq 0$ .
- Our new method of finding  $X$  more naturally considers a generalization of configuration avoidance.
- **Fractal Avoidance:** Given  $Y \subset \mathbf{R}^d$ , find  $X$  such that  $X^d \cap Y \subset \Delta$ , where  $\Delta = \{x : x_i = x_j \text{ for some } i, j\}$ . Generalize configuration avoidance by setting  $Y = f^{-1}(0)$ .

## Main Result:

### Theorem.

If the zero set of a function  $f : \mathbf{R}^{nd} \rightarrow \mathbf{R}$  is  $\alpha$  dimensional, then we can find  $X \subset \mathbf{R}^d$  with Hausdorff dimension  $(nd - \alpha)/(n - 1)$  such that  $f(x_1, \dots, x_d) \neq 0$  for distinct  $x_1, \dots, x_d \in X$ .

- Extends results of Pramanik and Fraser (2018) which give the result when  $f$  is smooth and nonsingular.
- If  $Y \subset \mathbf{R}$  has dimension  $\alpha$ , we can find a set  $X \subset \mathbf{R}$  of dimension  $1 - \alpha$  such that  $X + X$ ,  $X - X$ , and  $X \cdot X$  avoids elements of  $Y$ . We hope to extend this result to finding  $X$  as a vector space over  $\mathbf{Q}$ .
- Given a 1 dimensional set  $Y$  such that  $Y \cap L$  is zero dimensional for each straight line  $L$ , and a projection  $\pi$  such that  $\pi(Y)$  has non-empty interior, we can find a  $1/2$  dimensional subset  $X$  not containing the vertices of any isocles triangles.

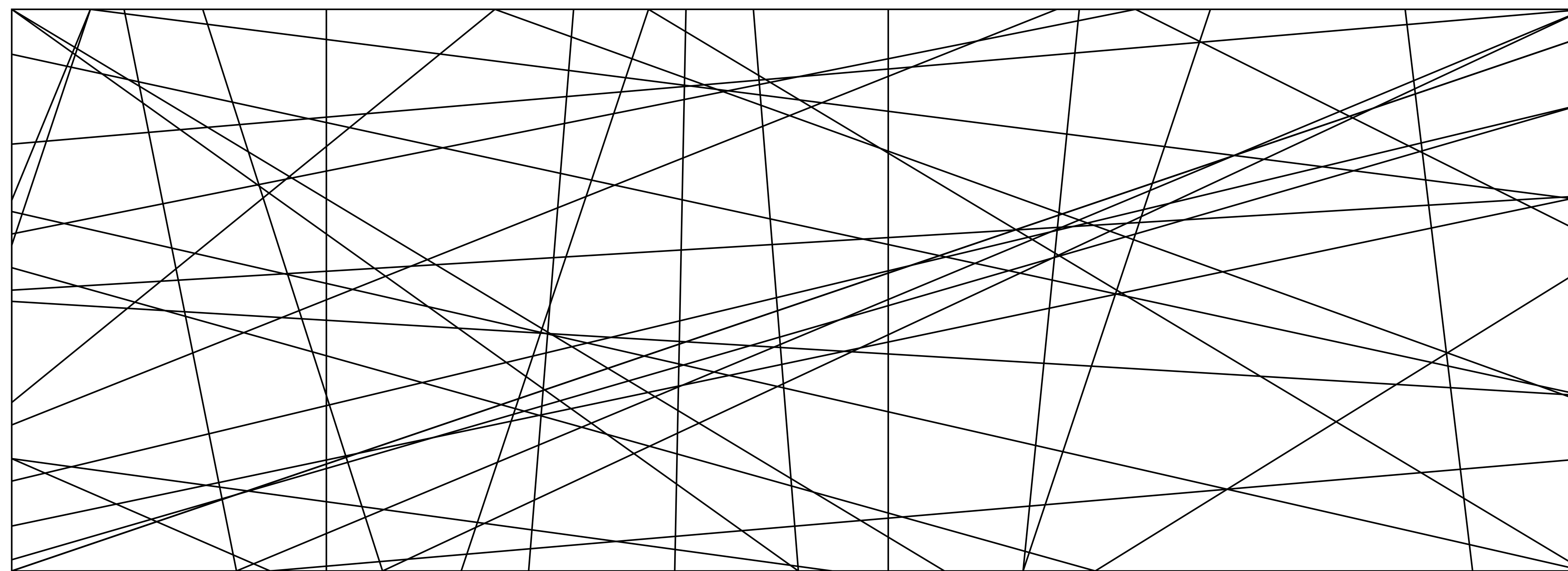
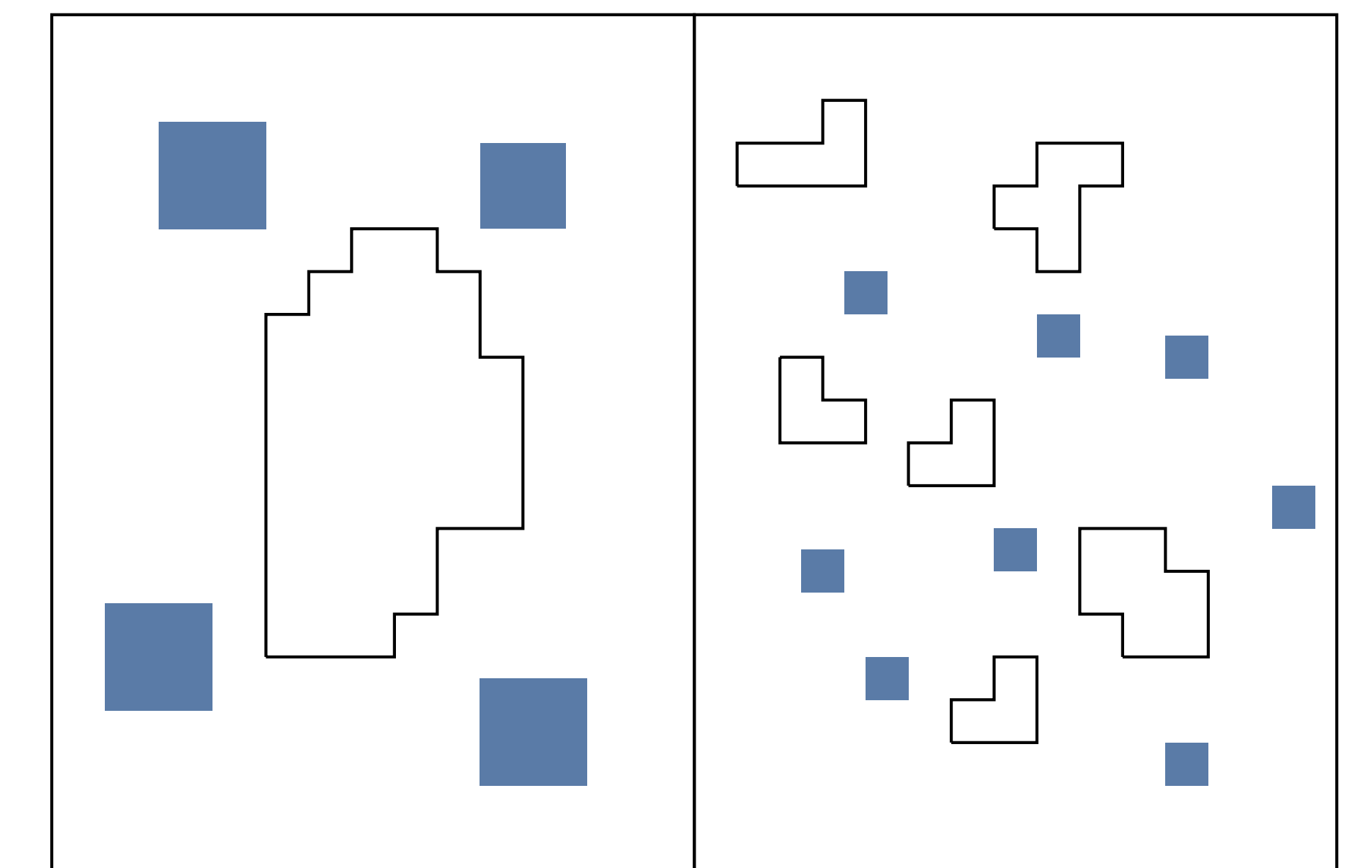


Figure: Our method can now find a full dimensional  $X$  avoiding  $Y$  formed by uncountably many lines that aren't too 'bushy'.

## Method: Random Selection

- The dimension of  $Y$  gives us very little structural information about  $Y$ , so it behaves like a random distribution of mass.
- To combat this, we choose random interval dissections to form  $X$  at each discrete scale, pruning intersections with  $Y$ .
- If  $Y$  concentrates at a particular location, the random choice of  $X$  can stay away from this location. On the other hand, if  $Y$  is spread out rather uniformly, we can spread out  $X$  uniformly while still avoiding the elements of  $Y$ .



- Our method currently works when the dimension of  $Y$  is quantified as the box counting dimension.
- We are currently working on using hyperdyadic scaling to extend the result where  $Y$  is quantified by its Hausdorff dimension.
- The construction parallels a random construction of an independent set in a hypergraph, similar to Turan's theorem.
- We are also currently looking at using other techniques on hypergraphs to improve the dimension of  $X$  when  $Y$  has more structure.