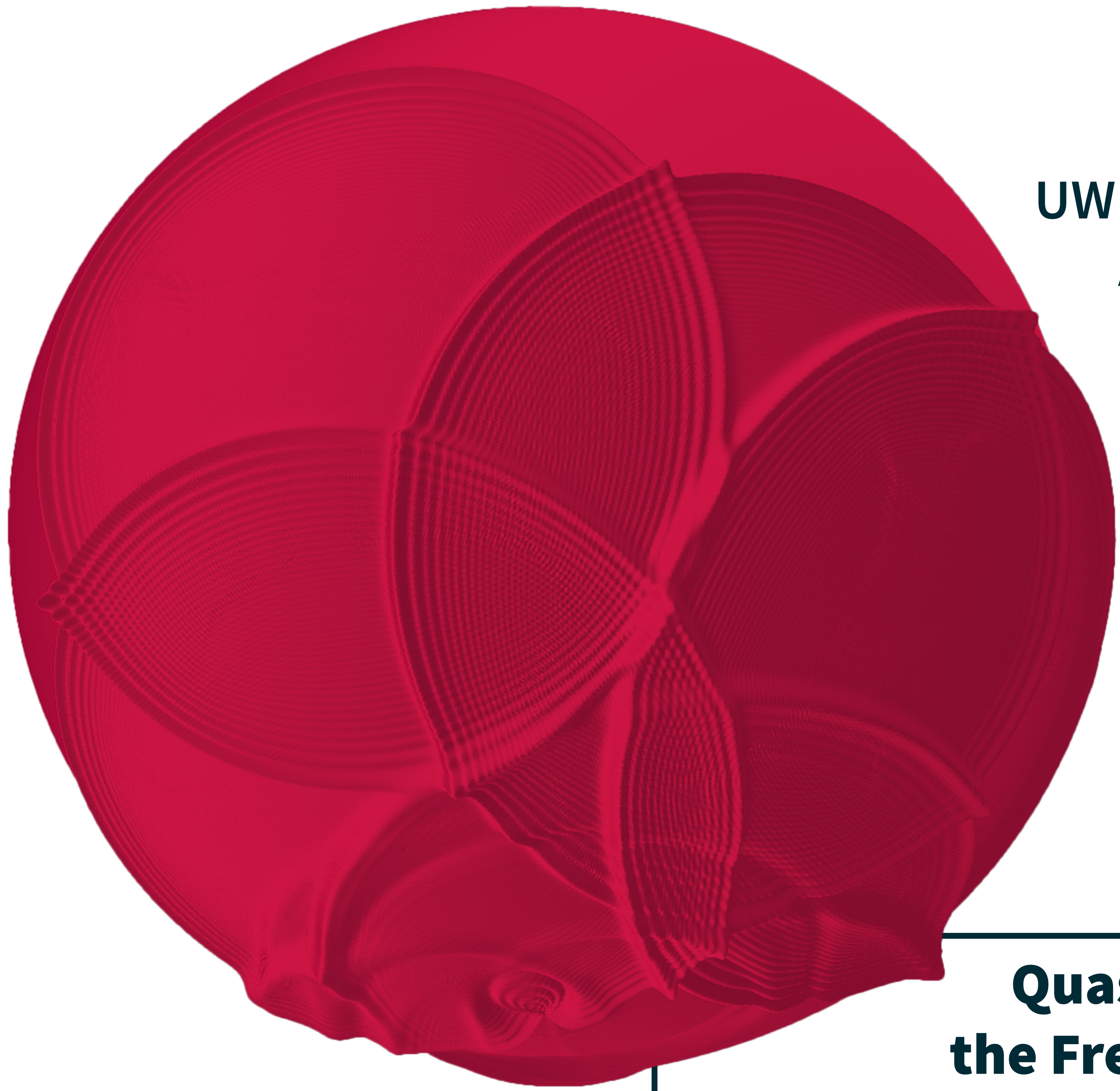
Characterizations of L^p Bounded Spectral Multipliers on the Sphere and Related Manifolds



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Main Research Question

What conditions are necessary and sufficient for dilates of a spectral multiplier operator on a manifold M to be bounded on $L^p(M)$?

Background

Let M be a compact Riemannian manifold, and let Δ be the Laplace-Beltrami operator on M. Then there is a discrete set $\Lambda \subset [0, \infty)$ such that every $f \in L^2(M)$ has an orthogonal decomposition $f = \sum_{\lambda \in \Lambda} f_\lambda$, where $\Delta f_\lambda = -\lambda^2 f_\lambda$.

Given any bounded function $m : \Lambda \to \mathbb{C}$, we define a *spectral multiplier operator*

$$m\left(\sqrt{-\Delta}\right) f = \sum_{\lambda \in \Lambda} m(\lambda) f_\lambda.$$

Our research question more precisely asks to find necessary and sufficient conditions on a function m so that

$$\sup_{R>0} \|m(R\sqrt{-\Delta})\|_{L^p(M) \to L^p(M)} < \infty.$$

Theorem If $M \in \{S^d, \mathbb{RP}^d, \mathbb{CP}^{d/2}, \mathbb{HP}^{d/4}, \mathbb{OP}^2\}$ and $1 < p < 2\left(\frac{d-1}{d+1}\right)$, and $m : (0, \infty) \to \mathbb{C}$ has compact support, then

$$\sup_{R>0} \|m(R\sqrt{-\Delta})\|_{L^p(M) \to L^p(M)} \sim \left(\int_0^\infty \langle t \rangle^{(d-1)(1-p/2)} |\widehat{m}(t)|^p \, dt \right)^{1/p}$$

Corollary If M is as above, then for $1 < p < 2\left(\frac{d-1}{d+1}\right)$, if $m : (0, \infty) \to \mathbb{C}$ has compact support, then

$$\sup_{R>0} \|m(R\sqrt{-\Delta})\|_{L^p(M) \to L^p(M)} \sim \|F_m\|_{L^p(M) \to L^p(M)}$$

where F_m is the radial Fourier multiplier on $\mathbf{R}^d$ given by

$$F_m f(x) = \int m(|\xi|) \widehat{f}(\xi) e^{2\pi i \xi \cdot x} \, d\xi.$$

This **Corollary** is the first general transference principle in the literature which gives bounds from $\mathbb{R}^d$ to M for any compact manifold M and any exponent p.

Reduction to the Wave Equation

Applying the Fourier inversion formula, we conclude that

$$m(R\sqrt{-\Delta}) = \int_{-\infty}^{\infty} R^{-1} \widehat{m}(R^{-1} t) e^{2\pi i t \sqrt{-\Delta}} \, dt$$

Thus we see that the main **Theorem** is closely related to local smoothing phenomena for the wave equation

Quasi-Orthogonality for '1D Averages' of the Frequency Localized Wave Equation

Fix $R \geqslant 1$. For each $x \in M$, consider an L^1 normalized function $f_x \in C^\infty(M)$ localized in a $1/R$ neighborhood of x, and frequency localized at scale R. For each k, consider a set $X_k \subset M \times [2^k, 2^{k+1}]$ such that $\#(X_k \cap B_r) \leqslant (Ru)r$ for all balls B_r of radius $1/R \leqslant r \leqslant 2^k/R$. Then we prove that

$$\Big\| \sum_k \sum_{(x,t) \in X_k} e^{2\pi i t \sqrt{-\Delta}} f_x \Big\|_{L^2(M)} \lesssim R^d (\log u) u^{\frac{2}{d-1}} \left(\sum_k 2^{k(d-1)} \#(X_k) \right)^{1/2}.$$

To prove this we use the Lax parametrix to write the wave equation as a Fourier integral operator, which reduces the problem to a geometric argument.

Density Decompositions

We then employ a density decomposition argument. Let $X = \bigcup_{k=0}^\infty$, where $X_k \subset M \times [2^k, 2^{k+1}]$. Then we can write $X_k = \bigcup_{l=0}^\infty X_k(2^l)$, such that $X_k(2^l)$ satisfies the assumptions required to obtain quasi-orthogonality bounds, with $u = 2^l$, and $X_k(2^l)$ is clustered when $l \gg 1$, i.e. we can find balls $B_1, \dots, B_N \in M \times [2^k, 2^{k+1}]$ such that $\sum_\alpha |B_\alpha| \leqslant 2^{-l}/R \# X_k$. Interpolating between the L^2 bounds above and simple L^1 bounds yields L^p estimates that are sufficient to establish the main **Theorem**.

What's Next?

- Using atomic decompositions and Littlewood-Paley theory to remove the compactness assumption in the **Theorem**.
- Extending the **Theorem** to other general manifolds, on, such as the class of Zoll Manifolds (manifolds with periodic geodesic flow).
- Dualizing Argument to obtain further results.
- Improving range of p in the **Theorem** using more sophisticated incidence geometry.