THE UNIVERSITY *of* EDINBURGH

The Maximum Codimension of a Salem Submanifold

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Research Problem

The *Fourier dimension* of a set $X \subset \mathbf{R}^n$ is the supremum of $s > 0$ for which there exists a finite measure μ satisfying a Fourier decay bound $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2}$ holds. An m-dimensional submanifold M of $\mathbf{R}^{m+k}$ is a *Salem set* if it has Fourier dimension m.

This research project determines for which $m, k \geq 0$ there exists a smooth, m dimensional submanifold of $\mathbb{R}^{m+k}$ which is a Salem set. It turns out this problem is related to the *Radon-Hurwitz numbers*. For an integer $m \geq 1$, if 2^{c+4d} is the largest power of two dividing m, where $0 \leq c \leq 3$ and $d \geq 0$, then $\rho(m) = 2^c + 8d$. For each m, $\rho(m)$ is the maximum number of everywhere linearly independent vector fields on S^{m-1}. Define $\rho(a) = 0$ if $a \in \mathbf{R} - \mathbf{Z}$.

Theorem. *There exists an m-dimensional smooth submanifold of $\mathbf{R}^{m+k}$ which is a Salem Set if and only if $k \leq \rho(m/2) + 1$. If $k > \rho(m/2) + 1$, then any smooth m dimensional submanifold of $\mathbf{R}^{m+k}$ has Fourier dimension at most $m - 1/3$.*

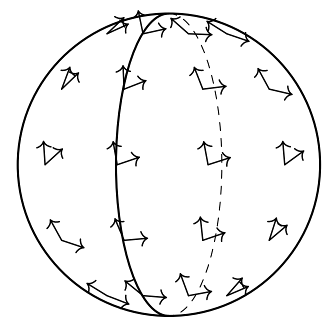
m	1	2	3	4	5	6	7	8
	$\mathbf{R}^2$	$\mathbf{R}^4$	$\mathbf{R}^4$	$\mathbf{R}^7$	$\mathbf{R}^6$	$\mathbf{R}^8$	$\mathbf{R}^8$	$\mathbf{R}^{13}$

Why The Radon-Hurwitz Numbers?

Let M be the graph of $\phi\colon \mathbf{R}^m \to \mathbf{R}^k$. For a smooth density μ on M,

$$\widehat{\mu}(\xi) = \int_{\mathbf{R}^m} a(x) e^{2\pi i(\alpha\cdot x + \eta\cdot\phi(x))}\, dx,$$

where $\xi = (\alpha, \eta)$ for $\alpha \in \mathbf{R}^m$ and $\eta \in \mathbf{R}^k$, and $a \in C_c^\infty(\mathbf{R}^m)$. Stationary phase tells us we only expect $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-m/2}$ if the Hessian of the phase is non-singular at each stationary point. If $H_i(x)$ is the Hessian of $\phi_i : \mathbf{R}^m \to \mathbf{R}$, then the Hessian of ϕ is a multiple of $\sum \eta_i H_i(x)$, which must be invertible for each $\eta \neq 0$ and each $x \in \text{supp}(a)$.



If $A_1, \dots, A_k$ are orthogonal $m \times m$ matrices s.t. $\sum \eta_i A_i$ is invertible for $\eta \neq 0$, and $A_1 = I$, then the vector fields

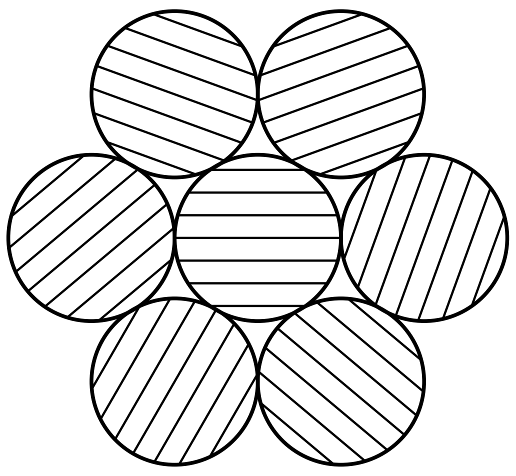
$$X_1, \dots, X_{k-1} : S^{m-1} \to \mathbf{R}^m$$

defined by letting $X_i(x)$ be the orthogonal projection of $\sum x_i A_i$ onto $T_x S^{m-1}$ are everywhere linearly independent. Using this, in 1965 Adams, Lax, and Phillips showed that $\rho(m/2) + 1$ is the largest dimension of a subspace of $m \times m$ symmetric matrices such that any non-zero element is non-singular.

Coverings by Parallel Tubes

Suppose M is an m-dimensional manifold in $\mathbf{R}^{m+k}$, where $k > \rho(m/2) + 1$, and μ is an arbitrary measure supported on M (not necessarily a smooth density) with $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2}$.

Then for each $x_0 \in M$ there exists $\xi_0 \in (T_{x_0} M)^\perp$ and $v_0 \in T_{x_0} M$ so that $x \mapsto \xi_0 \cdot x$ vanishes to first order at x_0 in all directions, and vanishes to second order in the direction v_0.



Then we can divide the radius

$$R^{-1} \times R^{-1/3} \times \cdots \times R^{-1/3}$$

slab centered at x_0 with smallest length in the direction ξ_0 into a union of parallel

$$R^{-1} \times R^{-1/2} \times \cdots \times R^{-1/2} \times R^{-1/3}$$

tubes T pointing in the direction v_0, upon which $x \mapsto \xi_0 \cdot x$ varies by $O(R^{-1})$.

A variant of the Esseen concentration bound implies $\mu(T) \lesssim R^{-s/2}$ for each T. Summing over T, i.e. over the $O(R^{m/3})$ balls of radius $R^{-1/3}$ and the $O(R^{(m-1)(1/2-1/3)})$ tubes in each ball, gives

$$\mu(\mathbf{R}^{m+k}) \lesssim R^{m/3+(m-1)(1/2-1/3)-s/2}.$$

If $s > m - 1/3$, taking $R \to \infty$ gives $\mu = 0$.

Variant of Esseen Concentration Inequality

Let T be a tube considered in the previous section. Suppose

$$\chi(x) = \chi_0(\pi_{\xi_0}(x))\chi_1(\pi_{\xi_0}^\perp(x))$$

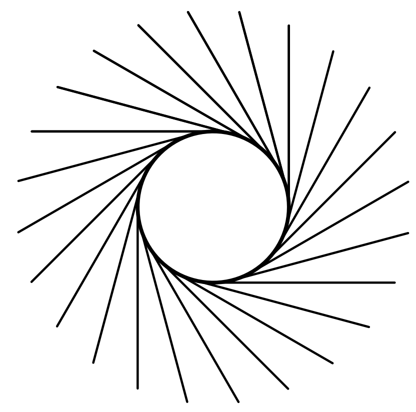
is L^∞ normalized, where $\chi_0 \geq 0$ is smooth, compactly supported, and adapted to an interval of length R^{-1}, and $\chi_1 \geq 0$ is Schwartz and adapted to the $R^{-1/3} \times R^{-1/2} \times \cdots \times R^{-1/2}$ tube pointing in the direction v_0, with compactly supported Fourier transform. Then

$$\mu(T) \lesssim \left| \int \chi(x) e^{Riv_0 \cdot x} d\mu(x) \right| = \left| \int \widehat{\chi}(\xi)\widehat{\mu}(\xi - Rv_0)\, d\xi \right| \lesssim R^{-s/2}.$$

More Applications of Proof Technique

A *hypersurface of constant nullity k* is a hypersurface such that at each point, exactly k principal curvatures vanish. Examples include cones, cylinders, and tangent developables of curves in $\mathbf{R}^3$.

Theorem. *If M is a hypersurface of constant nullity k in $\mathbf{R}^n$, then M has Fourier dimension $n - 1 - k$.*



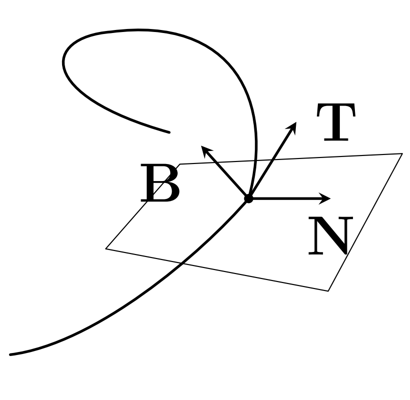
The vanishing of k principal curvatures implies M is foliated by k-dimensional planes, and upon each of these planes the normal vectors of M are constant. If μ is supported on M with $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2}$, and T is an

$$R^{-1} \times R^{-1/2} \times \cdots \times R^{-1/2} \times 1 \times \cdots \times 1$$

plank with sidelength R^{-1} normal to M, sidelength 1 in the directions of the foliation, and sidelengths $R^{-1/2}$ in directions perpendicular to the foliation, then $x \mapsto \xi_0 \cdot x$ varies by $O(R^{-1})$ on T, where ξ_0 is normal to the plank, and the Esseen concentration variant implies $\mu(T) \lesssim R^{-s/2}$. Since M is covered by $O(R^{\frac{n-1-k}{2}})$ such planks, we conclude that $\mu(\mathbf{R}^n) \lesssim R^{\frac{n-1-k-s}{2}}$. If $s > n - 1 - k$, we take $R \to \infty$ and conclude $\mu = 0$, which completes the proof.

A *non-degenerate curve* in $\mathbf{R}^n$ is the image of $\gamma : I \to \mathbf{R}^n$ such that at each $t \in I$, $\gamma^{(1)}, \dots, \gamma^{(n)}$ are linearly independent.

Theorem. *Such curves have Fourier dimension $2/n$.*



We cover a length $R^{-1/n}$ curve segment by a thickness R^{-1} neighborhood Θ of a hyperplane normal to the binormal vector to the curve. Then if μ is supported on the curve and $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2}$, then $\mu(\Theta) \lesssim R^{-s/2}$, and summing over $O(R^{1/n})$ segments gives $\mu(\mathbf{R}^n) \lesssim R^{1/n - s/2}$. So if $s > 2/n$, we conclude $\mu = 0$.