Easy Parametricity [DRAFT]

Jem Lord

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This is a draft version of my forthcoming paper "Easy Parametricity". The draft is mathematically complete, but lacks a layer of polish and proofreading, and the introduction is only partially written.

1 Introduction [incomplete]

There is a beautiful class of metatheorems, known as 'parametricity results', which state that polymorphic functions – members of types like $\prod_{X:\mathcal{U}} . X \to X \to X$ which are defined over all types – behave uniformly with regards to the type provided, so, for example, $\prod_{X:\mathcal{U}} . X \to X \to X$ should have exactly two elements.

These are typically metatheorems. One does not obtain in the type theory that $\mathbf{2} \cong \prod_{X:\mathcal{U}} . X \to X \to X$, but rather that any closed term of type $\prod_{X:\mathcal{U}} . X \to X \to X$ is either $\lambda Xab.a$ or $\lambda Xab.b$.

The goal of this paper is to promote these metatheorems to theorems. This both enables parametricity theorems to be obtained in the same theory as they are formulated about (from only an additional axiom – no extra type formers, term formers, or equations), and greatly widens the scope of what they can be applied to.

For instance, compare the following two theorems.

Theorem (A). Suppose that in the theory $\mathcal{T}$ that $X : \mathcal{U} \vdash F[X] : \mathcal{U}$ and $X : \mathcal{U} \vdash G[X] : \mathcal{U}$, with X appearing only covariantly in F and G. If $\vdash \alpha : \prod_{X:\mathcal{U}} F[X] \to G[X]$, then the naturality equation $X : \mathcal{U}, Y : \mathcal{U}, h : X \to Y \vdash G[h] \circ \alpha_X \equiv \alpha_Y \circ F[h] : F[X] \to G[Y]$ holds.

Theorem (B). Assume the universe $\mathcal{U}$ satisfies the realignment parametricity axiom (RPA). For all functors $F, G : \mathbf{Set}_{\mathcal{U}} \to \mathbf{Set}_{\mathcal{U}}$, all families $\alpha : \prod_{X:\mathbf{Set}_{\mathcal{U}}} F(X) \to G(X)$ and all functions $h : X \to Y$, the naturality equation $G(h) \circ \alpha_X = \alpha_Y \circ F(h) : F(X) \to G(Y)$ holds.

Unlike theorem A, theorem B: is proved inside the type theory it is stated in; can be applied without any restrictions on the variables occurring free in the expressions F, G and α; can be applied to F,G which are functoral up to propositional equality but not judgemental equality; can be used in the presence of axioms (such as countable choice or from synthetic mathematics) which one may want to use when constructing such F, G or α; and is compatible with almost arbitrary extensions of the type theory under consideration.

Moreover, theorem B remains true if $F, G : \mathbf{Set}_{\mathcal{U}} \to \mathbf{Set}_{\mathcal{U}}$ are replaced by $F, G : \mathcal{C} \to \mathcal{D}$ for any category of algebras or grothendieck topos $\mathcal{C}$ and for any locally small category $\mathcal{D}$. Accepting RPA means you never again have to prove such transformations are natural. There are other significant consequences, particularly for impredicative or univalent reasoning, which are discussed later in the introduction.

Why accept the parametricity axiom RPA though?

Most common models $\mathbb{T}$ of dependent type theory can be extended to a new model $\mathbb{T}'$, adding universes and types but not adding new terms to existing types, where each universe of $\mathbb{T}$ is included in a universe of $\mathbb{T}'$ which satisfies RPA. If you care about type theory because you want to use it to reason inside such $\mathbb{T}$, then you may want to add RPA to your arsenal and work inside the larger model $\mathbb{T}'$. In particular, RPA is consistent and doesn't imply any 'taboos'. It seems plausible that such $\mathbb{T}$ could include all models of dependent type theory satisfying UIP (including the initial one) but this is not known.

Impredicatively, RPA implies that the impredicative encodings like $\prod_{X:\mathcal{U}} X^X \to X^X$ for $\mathbb{N}$, $\prod_{X:\mathcal{U}}(\prod_{i:I} A_i \to X) \to X$ for $\exists_{i:I} A_i$, and $\exists_{X:\mathcal{U}}(X \times A+1)^X \times X$ for $\mathsf{CoList}(A)$ all satisfy the desired universal properties and propositional η-rules.

Homotopically, RPA implies that any 'sufficiently-complete univalent wild category' has a *unique* way of composing k composable morphisms, so it's possible to state and obtain infinitely much coherence data.

1.1 How to read

The current (intended) dependency diagram is

$$\begin{tikzcd}
 & & 3 \arrow[d, dotted] \\
1 \arrow[r] & 2 \arrow[ur] \arrow[r] \arrow[dr] & 4 \arrow[d, dotted] \\
 & & 5
\end{tikzcd}$$

where a solid arrow indicates dependency, and a dotted arrow indicates that the latter section mostly uses only definitions of the former.

§1.4 describes how different users of type theories with different interests might use the results contained in this paper. §2 introduces some preliminary facts about orthogonality. §3 introduces the notion "$\mathcal{C}$ factors orthogonally to $\langle T_i \rangle_{i:I}$" for a (wild) category $\mathcal{C}$ and a type family $\langle T_i \rangle_{i:I}$, and uses this to derive many parametricity results. §4 introduces the parametricity axioms, and shows how they can be used to show that many (wild) categories $\mathcal{C}$ that occur in practice satisfy $\mathcal{C}$ factors orthogonally to $\langle A \rangle_{A:\mathcal{U}}$. §5 builds several models of the parametricity axioms.

1.2 Related work

Parametricity is usually formulated in a relational guise following Reynolds' relational parametricity [4], but there is precedent for a functor-based formulation of parametricity [3, 10].

This work takes a semantically-inspired approach to parametricity, axiomising the properties required to obtain parametricity results. This contrasts somewhat with the work of Cavallo and Harper [5] and Nuyts, Vezzosi, and Devriese [8], which each augment the syntax. Both enrich dependent type theory with additional judgemental structure (in [5], a type requiring substructural rules; in [8], comonadic modalities) and a notion of "bridge" in order to obtain polymorphism results.

Fully coherent impredicative encodings of higher inductive types have also previously been investigated—Awodey, Frey, and Speight [2] constructed coherent impredicative encodings for several HITs, albeit with elimination rules restricted to types of bounded h-level. Some work to remove these h-level restrictions was made in [15] for some HITs, although a complicated encoding was still required. The present work indicates that plausibly, in the presence of univalence, the naïve encodings for HITs obtain their full induction principles.

This work is closest in spirit to the recent [1], which obtains parametricity results in Cohesive HoTT with sufficient cohesion, also formulated through a bridge-like type. The current work differs from this in two key aspects: the parametricity we consider is functorial instead of relational in nature, and hence a priori more easily scalable to complex structures without having to apply parametricity individually at each step; and no cohesion structure is required (although a $\int$-like modality is helpful), which enables our results to automatically also hold in more general settings of impredicative universes and stably locally connected (∞-)toposes.

Because the current work can be carried out in a traditional type theory without any additional judgmental structure (unlike other works [5, 8, 1]), the standard semantics for MLTT can be immediately used and applied. Further, because this system applies to possibly-non-definable terms of the theory, parametricity results can be applied to functors whose well-definedness relies on additional axioms such as Countable Choice, as long as such are semantically justified[1].

1.3 Foundations and Notation

The ambient theory this work is done in is quite weak, roughly amounting to using a single universe, $\mathbf{1}$, Π, Σ, $=$ and function extensionality:

Definition 1.3.1 (Function extensionality). If $f, g : \prod_{a:A} B_a$ then $f = g$ iff $\prod_{a:A} f(a) = g(a)$.[2]

[1] For example, presheaf topoi over a topos satisfying countable choice (such as **Set**) also satisfy countable choice, so choosing a (sufficiently nontrivial) cohesive presheaf topos over **Set** will also satisfy CC, and hence both CC and Parametricity will be satisfied in the model.

[2] In absence of "all types are sets" the homotopy type theorist will be relieved to recall from

A "universe" is a type family $\mathcal{U}$ which is closed under $\mathbf{1}$, Σ, Π and $=$, and has a type $\mathbf{0}_{\mathcal{U}}$ for which for any $A : \mathcal{U}$, $(\mathbf{0}_{\mathcal{U}} \to A) \cong \mathbf{1}$ and $(\mathbf{0}_{\mathcal{U}} \to \mathcal{U}) \cong \mathbf{1}$. The weakness of this definition should be noted in Section 5 in particular, where various universes are constructed.

Some notation for type families:

- $\mathsf{el}_{\mathcal{U}}$ refers to the type family of all $\mathcal{U}$-small types.
- If A denotes some type family, $\{A\}$ is the one-element type family which takes value A
- If $\mathcal{C}$ is a (wild) category, $\mathrm{Hom}(\mathcal{C})$ is the $\mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C})$-indexed type family $\mathcal{C}(-,-)$.

1.4 A recipe for results

We have established various conditions on categories sufficient to get parametricity results, various axioms sufficient to get said conditions, and various models which model those axioms. Although this is all one *needs* to extract useful parametricity-type results, what we've done is more versatile than it may initially seem.

To that end several use cases are included below.

1.4.1 General constructive-compatible mathematics

Typical results arising from relational parametricity are of the form "if you explicitly write down a closed term F of type T, then F satisfies some equation/property obtained from the structure of the type T". However this tells you nothing about whether *internally* all F in T satisfy the property, and such results are obtained not within the type theory itself but by reasoning about its metatheory.

We do not have these shortcomings. The parametricity results obtained above are derived from an axiom (RPA) that one can add to the ambient theory of constructive mathematics $\mathcal{T}$ one is working within. Parametricity results can be obtained in the same manner as ordinary mathematical results with much less detour. The results obtained are also stronger on account of being internal – for example, whereas relational parametricity says that all closed terms of $\prod_{X:\mathcal{U}} X^X \to X^X$ are of the form $f \mapsto f^n$ for some natural n, RPA implies that this map $\mathbb{N} \to \prod_{X:\mathcal{U}} X^X \to X^X$ is an isomorphism (by Corollary 3.2.2 and Proposition 4.4.1).

As the statement is an axiom, it can be combined with reasoning obtained under other axioms. For example, suppose one wants to do some functional analysis under the assumption of countable choice (e.g. so that all sequentially-complete metric spaces are also filter-complete) or that $[0,1]$ is compact (as a topological space). This is not a problem, and parametricity results can be applied here too, because the parametricity axiom can be assumed in parallel with other axioms.

page 106 of the HoTT Book [17] that this implies the obvious function is an equivalence.

The results we get do seem to be of a different nature to those typical of relational parametricity, in that they're all[3] formulated in the language of category theory. For example, for nice enough categories we get that:

- transformations are natural e.g. $\mathsf{map}_f(\alpha_A(a)) = \alpha_B(\mathsf{map}_f(a))$ for $A \xrightarrow{f} B$ (Proposition 3.1.4)
- that preserving id is sufficient for functoriality e.g. $\mathsf{map}_g \circ \mathsf{map}_f = \mathsf{map}_{g\circ f}$ from just $\mathsf{map}_{\mathsf{id}} = \mathsf{id}$ (Proposition 3.1.6)
- that functors equal on objects are equal (Proposition 3.1.8)
- that the product of all objects of a category is initial e.g. $\prod_{X:\mathcal{U}, s:X\to X, z:X}(X, s, z)$ is $(\mathbb{N}, +1, 0)$ (Proposition 3.2.1)

wherein most complete categories are 'nice enough' by Proposition 4.4.1.

There are also lots of models of the parametricity axiom RPA. In particular one can obtain presheaf models as in Theorem 5.2.5 to get a universe $\mathcal{V}$ satisfying RPA. Moreover, $\mathcal{V}$ contains a copy $\Delta\mathcal{U} \subseteq \mathcal{V}$ of the base model's universe $\mathcal{U}$ as a subuniverse, and most axioms which held of $\mathcal{U}$ will also hold of $\Delta\mathcal{U}$, see Proposition 5.2.6. This lets one expand their internal logic (by moving to the presheaf category) and extend it with a universe satisfying RPA.

1.4.2 HoTT applications

There is not, as of writing, a definition of "$(\infty, 1)$-category" inside plain Homotopy Type Theory. So it is, of course, impossible to state theorems saying "$\mathcal{C}$ is an $(\infty, 1)$-category" or "F is an $(\infty, 1)$-functor". But, even if it were, one might imagine that the proofs of these things would be quite annoying, involving possibly an infinite tower of coherence conditions until all of the basic legwork and theory is set up.

Under the Univalent Parametricity Axiom, the situation is different, at least for a large class of $(\infty, 1)$-categories. For a locally $\mathcal{U}$-small wild category $\mathcal{C}$, the property "$\mathcal{C}$ factors orthogonally to $\mathcal{U}$" (a consequence of "$\mathcal{C}$ is univalent and complete" by Proposition 4.4.2) implies by Theorem 3.3.2 that there's a unique way to compose sequences of k composable morphisms, for any k, subject only to the condition that a sequence of id is sent to id; an infinite amount of coherence data follows from this assumption.

Perhaps one has, through some elaborate procedure, defined some potential functors $F, G : \mathbf{Ring} \times \mathbf{Ab\infty Grp} \to \mathbf{Ab\infty Grp}$ and transformation $\alpha : F \Rightarrow G$ in HoTT – or at least we suspect they're functors and a transformation, but we haven't yet managed to prove functoriality and naturality. Then under UPA, it would be sufficient to just prove that $F(\mathsf{id}, \mathsf{id}) = \mathsf{id}$, $G(\mathsf{id}, \mathsf{id}) = \mathsf{id}$, and some easy calculations about limits in $\mathbf{Ring}$ and $\mathbf{Ab\infty Grp}$: Theorem 3.3.2 gives that F, G are fully coherent functors, Proposition 3.1.4 gives that α is natural, and Proposition 4.4.2 implies the requisite hypotheses hold.

[3] It's unclear whether this is essential, or if a different technique using the same axiom could get results with statements more typical of relational parametricity.

The assumption that UPA holds for some universes $\mathcal{U}$ is, I believe, a reasonable axiom to assume. Corollary 5.2.2 gives an abundance of models of UPA, and Proposition 5.2.7 tells us that, if we wished to model in a grothendieck ∞-topos $\mathcal{E}$, there is a grothendieck topos $\mathcal{E}'$ which contains 'copies' of the standard universes of $\mathcal{E}$ (in a precise sense) which are each contained in superuniverses satisfying UPA. This indirectness is necessary: UPA for a universe $\mathcal{U}$ contradicts the (weak) law of excluded middle holding for propositions in $\mathcal{U}$ by Proposition 2.0.4 and contradicts $\mathsf{Prop}_{\mathcal{U}}$ being $\mathcal{U}$-small (as $A \perp A$ iff $A \cong 1$), but there's generally nothing to stop a universe $\mathcal{U}$ being contained in a larger universe $\mathcal{V}$ which satisfies UPA.

UPA is not so miraculous to be a cure-all to coherence-related troubles. Even in the complete univalent categories which it works well with, it seems to be unhelpful for showing that monoidal structures are coherent or that the associativity law for monads holds. However as it does let us define the concepts of "coherent functor" and "coherent transformation" between sufficiently-complete univalent categories, so I hold out hope for a tantalising conjecture:

Question. Is there a property of functors out of small 1-categories $\mathcal{C}$ which (i) under AC holds for the yoneda embedding in the quasi-category model of $(\infty, 1)$**Cat** and (ii) such that UPA implies there's a unique functor out of $\mathcal{C}$ for which it holds?

1.4.3 Impredicative uses

A universe $\mathcal{U}$ is **impredicative** when it is closed under *all* products, even when the indexing type is not $\mathcal{U}$-small. Although this approach has some drawbacks (for instance, it precludes the possibility that $\mathcal{U}$ contains its universe of propositions $\mathsf{Prop}_{\mathcal{U}}$), it is often quite useful. This is especially the case for semantics of programming languages, as it enables direct interpretation of the type-quantification seen in parametric polymorphism, like $\mathsf{concat} : \forall\alpha.\mathsf{list}(\alpha) \times \mathsf{list}(\alpha) \to \mathsf{list}(\alpha)$ in System F, encodings tricks like $\mathsf{list}(A) \coloneqq \forall\beta.(1 + A \times \beta) \to \beta$, avoids the need for a hierarchy of universes, and type operations of interest typically stay in $\mathcal{U}$[4].

Historically the main occurrence of parametricity results was in this impredicative setting; Reynolds' relational parametricity [**idk-yet**] allows one to show that for a wide variety of closed type-polymorphic expressions e that e satisfies a corresponding parametricity result. For example, consider the impredicative encoding $\mathsf{PolyNat} \coloneqq \forall\alpha.(\alpha \to \alpha) \to \alpha \to \alpha$ of the natural numbers. Each number n has a corresponding encoded $\underline{n} \coloneqq \lambda\alpha f x.f^n x$, and relational parametricity allows one to show that if $\vdash e : \mathsf{PolyNat}$ then for some natural number n that $\vdash e \equiv \underline{n}$. However, relational parametricity is insufficient too show the equation $e : \mathsf{PolyNat} \vdash e(\mathsf{PolyNat})(\underline{1+})(\underline{0}) = e$ (where $\underline{1+}(n) = \lambda\alpha f x.f(n\ \alpha f x)$)

[4]An exception to this is the powerset operation $\mathcal{P}(A) = A \to \mathsf{Prop}_{\mathcal{U}}$ used to model nondeterminism, as $\mathcal{P}(1)$ is never in $\mathcal{U}$. However the restricted powerset $\mathcal{P}_{\mathsf{ctb}}(A) \subseteq \mathcal{P}(A)$ of countable subsets of A does, and I believe this is also sufficient to model nondeterminism.

even though this holds for all $\underline{n}$ — and indeed, there are models [12] where this doesn't hold.

It is unfortunately not the case that impredicative universes necessarily satisfy a parametricity axiom, particularly as they may fail to have realignment types. The standard model of an impredicative universe, the category of assemblies, was shown to not necessarily satisfy the axiom IPA in Proposition 5.3.1. However, an internal presheaf model (Theorem 5.3.3) still gives a universe roughly equivalent to the universe of PERs which satisfies the parametricity axiom RPA (and Church's thesis), and the related cubical assemblies model has its universe of cubical PERs satisfy UPA.

2 Orthogonality

For types A, T, let $\mathbf{k}_{A,T}$ denote the function $(t \mapsto \lambda\_.t) : T \to (A \to T)$.

For types A and T, a function $f : A \to T$ is **constant** when there is some $t : T$ such that for all a in A, $f(a) = t$. Note that if T is not a hset or A is not inhabited then it may be possible that f is constant in multiple ways.

A is **orthogonal** to T (written $A \perp T$) when every function $f : A \to T$ is constant in a unique way i.e. there exists a unique $t : T$ such that $f = \mathsf{const}_t$. (Note that $A \perp T$ is *not* the same as $T \perp A$!)

Lemma 2.0.1. *A is orthogonal to T iff the function $\mathbf{k}_{A,T}$ is an equivalence.*

Proof. For any $\tau : A \to T$, "τ is constant" is equivalent to the fibre of f under $\mathbf{k}_{A,T}$, so "τ is constant" is always contractible iff $\mathbf{k}_{A,T}$ is an equivalence. □

Lemma 2.0.2. *If A is inhabited and every function $\tau : A \to T$ is constant then A is orthogonal to T.*

Proof. If every function $\tau : A \to T$ is constant then $\mathbf{k}_{A,T}$ has a section (with section given by witness of constancy). As there is some $a : A$, $t \mapsto t(a)$ is a retraction of $\mathbf{k}_{A,T}$. Hence $\mathbf{k}_{A,T}$ is an equivalence. □

We briefly note that our setting is necessarily not classical. The **weak law of excluded middle** (WLEM) states that for all propositions $\phi : \mathsf{Prop}$, $\neg\phi \vee \neg\neg\phi$.

Proposition 2.0.3. *Assume LEM. For sets A, B, $A \perp B$ iff either $A \cong 1$, $B \cong 1$ or A is nonempty and B is empty.*

Proof. Proceed by case analysis on the cardinality of $A \to B$. If $A \to B$ is empty, then B must be empty and A must be nonempty. If $A \to B$ has 1 element, then so does B. If $A \to B$ has multiple elements then B has multiple elements and A has 1 element (otherwise define a nonconstant function out of A by case analysis). □

Proposition 2.0.4. *WLEM is equivalent to the existence of a non-constant*[5] *function $f : \mathsf{Prop} \to \mathbf{2}$. $\mathsf{Prop} \perp \mathbf{2}$ is equivalent to $\neg$WLEM.*

[5] here we take a function f to be non-constant iff there are a, b with $f(a) \neq f(b)$

Proof. Equality in $\mathbf{2}$ is $\neg\neg$-stable and any $p : \mathsf{Prop}$ is not not in $\bot, \top$, so $f : \mathsf{Prop} \to \mathbf{2}$ is constant iff $f(\top) = f(\bot)$. Constancy of f is $\neg\neg$ stable, so $\mathsf{Prop} \perp \mathbf{2}$ iff no function $\mathsf{Prop} \to \mathbf{2}$ is non-constant.

Suppose WLEM holds. We can define a non-constant function $f : \mathsf{Prop} \to \mathbf{2}$ by $f(\phi) = 0$ if $\neg\phi$ and $f(\phi) = 1$ if $\neg\neg\phi$.

Let $f : \mathsf{Prop} \to \mathbf{2}$ be non-constant, so that $f(\top) \neq f(\bot)$; suppose wlog that $f(\top) = 1$ and $f(\bot) = 0$. Then for any proposition ϕ, $f(\phi) = 0$ or $f(\phi) = 1$, so $\phi \neq \top$ or $\phi \neq \bot$, so $\neg\phi \vee \neg\neg\phi$ and WLEM holds. □

Hence in order to use techniques, we are unable to globally assume the law of excluded middle and use classical mathematics. This is not a significant obstacle, even for the development of classical mathematics: we can have a classical universe $\mathcal{V}$ contained in a larger universe $\mathcal{U}$ in which the parametricity results are obtained. In particular, we may assert that $\mathcal{V}$ supports classical mathematics, as seen in the presheaf model Theorem 5.2.5 and Proposition 5.2.6.

Proposition 2.0.5. *For any type C, the types T with $C \perp T$ are closed under arbitrary products, dependent sum, and path types.*

Proof. Suppose $C \perp T_i$ for each $i : I$. Then the composite

$$\begin{aligned} C \to \prod_{i:I} T_i &\cong \prod_{i:I} C \to T_i \\ &\cong \prod_{i:I} T_i \end{aligned}$$

is inverse to $\mathbf{k}_{C,\prod_{i:I} T_i}$, so $\mathbf{k}_{C,\prod_{i:I} T_i}$ is an equivalence.

Suppose $C \perp T$ and for each $t : T$, $C \perp S_t$. Then the composite

$$\begin{aligned} C \to \sum_{t:T} S_t &\cong \sum_{f:C\to T} \prod_{c:C} S_{f(c)} \\ &\cong \sum_{t:T} \prod_{c:C} S_t \\ &\cong \sum_{t:T} S_t \end{aligned}$$

is inverse to $\mathbf{k}_{C,\sum_{t:T} S_t}$, so $\mathbf{k}_{C,\sum_{t:T} S_t}$ is an equivalence.

Suppose $C \perp T$. Note $C \perp \sum_{a,b:T} a = b$. Let h be the composite of the chain of equivalences

$$\begin{aligned} \sum_{a,b:T} (a = b) &\cong C \to \sum_{a,b:T} a = b \\ &\cong \sum_{f,g:C\to T} \prod_{c:C} f(c) = g(c) \\ &\cong \sum_{a,b:T} C \to (a = b). \end{aligned}$$

Then computation reveals that $h(a, b, p) = (a, b, \lambda c.p)$, so $C \perp (a = b)$ for all $a, b : T$. □

Lemma 2.0.6. *If $A \perp T$ and $f : A \to T$ and $a, b : A$, there is a path $\mathsf{cp}_f(a, b) : f(a) = f(b)$.*

Proof. There is a unique pair $(t : T, p : f = \mathsf{const}_t)$ by $A \perp T$. Let $\mathsf{cp}_f(a, b) = \mathsf{ap}_{\lambda g.ga}(p) \cdot \mathsf{ap}_{\lambda g.gb}(p)^{-1}$. □

Lemma 2.0.7. *Suppose that $A \perp T$, $t : A \to T$, $a, b : A$. If $p : a = b$, then $\mathsf{cp}_f(a, b) = \mathsf{ap}_f(p)$.*

Proof. Path induction on $p : a = b$ simplifies $\mathsf{cp}_f(a, b) = \mathsf{ap}_f(p)$ to $u \cdot u^{-1} = \mathsf{refl}$ for $u \equiv \mathsf{ap}_{\lambda g.ga}(\mathsf{refl})$. □

Lemma 2.0.8. *Suppose that $\sum_{a:A} B_a \perp T$, $\tau : A \to T$, $a : A$ and $b_0, b_1 : B_a$. If $b_0 = b_1$ then $\mathsf{cp}_{\tau \circ \pi_A}((a, b_0), (a, b_1)) = \mathsf{refl}_{\tau(a)}$.*

Proof. Immediate by induction on the provided path $b_0 = b_1$ and previous lemma. □

Proposition 2.0.9. *Suppose C has elements $0, 1 : C$, D is a type with $b : D$, and there's a function $f : C \times D \to D$ such that $f(0, d) = b$ and $f(1, d) = d$ always hold. Then if all functions $h : C \to T$ have $h(0) = h(1)$ (e.g. if $C \perp T$) then $D \perp T$.*

Proof. Given $\tau : D \to T$ and $d : D$, we show that $\tau(d) = \tau(b)$. As $C \perp T$, $\tau(f(-, d)) : C \to T$ satisfies $\tau(f(0, d)) = \tau(f(1, d))$, so

$$\tau(b) = \tau(f(0, d)) = \tau(f(1, d)) = \tau(d)$$

which is sufficient as D is inhabited. □

Corollary 2.0.10. *Suppose $\mathcal{U}$ satisfies RPE and that $a_0 \neq a_1 : A$ and A is locally $\mathcal{U}$-small. If $A \perp T$ then $\mathsf{Prop}_{\mathcal{U}} \perp T$*

Proof. Define $f : A \times \mathsf{Prop} \to \mathsf{Prop}$ by $f(a, \varphi) = (a = a_1) \to \varphi$. Then $f(a_0, \varphi) = \top$ and $f(a_1, \varphi) = \varphi$ and $A \perp T$ so $\mathsf{Prop}_{\mathcal{U}} \perp T$. □

2.1 Modalities

Fix a universe $\mathcal{V}$.

The notion of modality has several equivalent presentations; for an introduction in the context of HoTT see "Modalities in homotopy type theory" [11].

Definition 2.1.1. A **modality** is an operation $\lozenge : \mathcal{V} \to \mathcal{V}$ which is equipped with:

- For each $A : \mathcal{V}$, a function $\eta_A : A \to \lozenge A$

- For every $A : \mathcal{V}$, $B : \lozenge A \to \mathcal{V}$, the map

$$- \circ \eta_A : \prod_{z:\lozenge A} \lozenge B(z) \to \prod_{a:A} \lozenge B(\eta_A a)$$

  is an equivalence.

The above definition of modality is equivalent to Definition 1.2 of *op. cit.*

For the remainder of the section, fix a modality $\lozenge$. A type T is **$\lozenge$-modal** if $\eta_T : T \to \lozenge T$ is an equivalence. A type C is **$\lozenge$-connected** if $\lozenge C$ is contractible.

Proposition 2.1.2. *If C is $\lozenge$-connected and T is $\lozenge$-modal then $C \perp T$.*

Proof. The function

$$- \circ \eta_C : \prod_{\_:\lozenge C} \lozenge T \to \prod_{\_:C} \lozenge T$$

is an equivalence. Via that $\eta_T : T \cong \lozenge T$ and $\lozenge C \cong \mathbf{1}$ we get that $\mathbf{k}_{C,T}$ is the composite of the equivalences

$$T \cong \lozenge T \cong (\lozenge T)^{\mathbf{1}} \cong (\lozenge T)^{\lozenge C} \cong (\lozenge T)^{C} \cong T^{C} \ .$$

□

Write $\mathcal{V}_\lozenge$ for the subuniverse of $\mathcal{V}$ on types which are $\lozenge$-modal.

Proposition 2.1.3. *$\mathcal{V}_\lozenge$ contains all types of the form $\lozenge A$ for $A : \mathcal{V}$, contains $\mathbf{1}$ and closed under dependent sums, path types, and A-indexed products for any A in $\mathcal{V}$.*

Proof. See Lemma 1.14 and Theorem 1.16 [11]. □

In the presence of higher inductive types, it is easy to construct many modalities.

Let B be a family of types in $\mathcal{V}$ indexed by a type in $\mathcal{V}$. A type T is **B-null** iff each B_i is orthogonal to T.

Theorem 2.1.4. *Suppose that $\mathcal{V}$ is a universe which admits HITs (or is a universe of sets and admits QITs).*

There is a modality $\lozenge_B$ on $\mathcal{V}$ such that $\mathcal{V}_{\lozenge_B}$ consists of precisely the B-null types.

Proof. See Theorem 2.16 and Theorem 2.17 [11]. □

3 Parametricity theorems

The parametricity results we shall obtain will be statements formulated with categories – for example, that under certain hypotheses that all unnatural transformations between $F, G : \mathcal{C} \to \mathcal{D}$ are natural. This generally follows from that, informally, the collection of all objects of the category $\mathcal{C}$ (and of its slices/coslices) is 'connected' or 'indiscrete', and the $\hom_{\mathcal{D}}(d, d')$ are 'discrete', so that all functions from the former to the latter are constant.

Instead of just focusing on when this happens for particularly nice categories, like **Set**, results are stated in (approximately) their full generality, which requires first defining some weakenings of 'category', 'natural transformation' etc. Our results are suitable both for ordinary 1-categories and for some potentially-incoherent wild categories. In the interests of generality, results are stated for wild categories.

Remark 3.0.1. The reader not interested in higher categories may safely refer to the equations

$$\begin{aligned} \text{wild category} &= \text{category} - \text{associativity} \\ \text{wild functor} &= \text{functor} - \text{preserving composition} \\ \text{wild transformation} &= \text{natural transformation} - \text{naturality} \end{aligned}$$

and then proceed to the next subsection.

Definition 3.0.2. A **magmoid** $\mathcal{C}$ consists of:

- a type $\mathrm{Ob}_{\mathcal{C}}$ of objects and for each X, Y in $\mathrm{Ob}_{\mathcal{C}}$, a type $\hom_{\mathcal{C}}(X, Y)$ of arrows from X to Y.
- for objects X, Y, Z and arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$ an arrow $X \xrightarrow{g \circ f} Z$.

A **wild category** $\mathcal{C}$ is a magmoid which further has:

- for all objects X an arrow $X \xrightarrow{\mathsf{id}_X} X$
- for all $X \xrightarrow{f} Y$ unitors $\mathsf{l}_f : \mathsf{id}_Y \circ f = f$ and $\mathsf{r}_f : f \circ \mathsf{id}_X = f$
- for all objects X a coherence law $\mathsf{c}_X : \mathsf{l}_{\mathsf{id}_X} = \mathsf{r}_{\mathsf{id}_X}$

And lastly, a 1**-category** is (as usual) a wild category where each $\hom_C(X, Y)$ is a h-set and which is associative in that for any $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, $h \circ (g \circ f) = (h \circ g) \circ f$.

Remark 3.0.3. The notion of wild category used here is unusual in that it lacks an associator and includes a unitor coherence homotopy. The associator is excluded as we usually don't need it, and as we would like to be able to derive associativity from appropriate coherence hypotheses anyway (see Proposition 3.1.7).

The unit coherence homotopy is useful to have, but also makes wild categories nicer in a precise sense: with it, the additional structure of a wild category atop a reflexive quiver actually exists uniquely in many cases of interest by Proposition 3.3.5.

There is a natural notion of wild category homomorphism — a map on objects and morphisms with paths for preservation of composition, identity arrows, left unitors, right unitors, and unitor coherences — which we won't use much.

Instead, a **wild functor** $F : C \to D$ consists of a function $F : Ob_C \to Ob_D$, for objects X, Y of C a function $F : \hom_C(X, Y) \to \hom_D(FX, FY)$ and for any object X of C a path $i^F_X : F(\mathsf{id}_X) = \mathsf{id}_{FX}$. For wild functors $F, G : C \to D$, a **wild transformation** $\alpha : F \to G$ consists of for each object X of C an arrow $FX \xrightarrow{\alpha_X} GX$.

Proposition 3.0.4. *For $\mathcal{C}$ a magmoid, the data of being a wild category is equivalent to giving every object X a left identity idl_X with left unitor and a right identity idr_X with right unitor.*[6]

Proof. Fix $\mathcal{C}$ a magmoid and suppose that each object X has a left identity idl_X with left unitors $\mathsf{l}_f : f = \mathsf{idl} \circ f$ and a right identity idr_X with right unitors $\mathsf{r}_f : f = f \circ \mathsf{idr}$.

Consider the following additional data extending this: paths $p_X : \mathsf{idl}_X = \mathsf{idr}_X$, (so that we can transfer over the unitors $\mathsf{r}'_f : f = f \circ \mathsf{id}$ as $\mathsf{r}_f \cdot (f \circ p_X)$) and paths $\mathsf{c}_X : l_{\mathsf{idl}_X} =_{\mathsf{idl}_X \circ \mathsf{idl}_X = \mathsf{idl}_X} (\mathsf{idl}_X \circ p_X) \cdot \mathsf{r}_{\mathsf{idl}_X}$. The total data of the left/right identity/unitors precisely furnishes $\mathcal{C}$ with the structure of a wild category with identities idl_X, unitors l_f and r'_f and coherences c_X.

On the other hand, the additional data is contractible: via the equivalences

$$\begin{aligned}
&(\mathsf{l}_{\mathsf{idl}_X} =_{\mathsf{idl}_X \circ \mathsf{idl}_X = \mathsf{idl}_X} (\mathsf{idl}_X \circ p_X) \cdot \mathsf{r}_{\mathsf{idl}_X}) \\
\cong\ &(\mathsf{l}_{\mathsf{idl}_X} \cdot \mathsf{r}^{-1}_{\mathsf{idl}_X} =_{\mathsf{idl}_X \circ \mathsf{idl}_X = \mathsf{idl}_X \circ \mathsf{idr}_X} (\mathsf{idl}_X \circ p_X)) \\
\cong\ &((\mathsf{idl}_X \circ)^{-1} (\mathsf{l}_{\mathsf{idl}_X} \cdot \mathsf{r}^{-1}_{\mathsf{idl}_X}) =_{\mathsf{idl}_X = \mathsf{idr}_X} p_X)
\end{aligned}$$

c_X (which is a member of the first type in the chain of equivalences above) says precisely that p_X is a specific path. □

Corollary 3.0.5. *Given the data of a magmoid $\mathcal{M}$ with identity morphisms $\mathsf{id}_\_$, left unitors $\mathsf{l}_\_$ and right unitors $\mathsf{r}_\_$, we can make $\mathcal{M}$ into a wild category with the same identity morphisms $\mathsf{id}_\_$ and same left unitors (or, symmetrically, same right unitors $\mathsf{r}_\_$).*

Proof. We can treat $\mathcal{M}$ as having separate left identity $\mathsf{idl}_\_ \equiv \mathsf{id}_\_$ and right identity $\mathsf{idr}_\_ \equiv \mathsf{id}_\_$, and then apply the previous proposition, noting the proof thereof doesn't change the identity morphisms $\mathsf{id}_\_$ or left unitors $\mathsf{l}_\_$. □

[6] This data unfortunately is not a property of a magmoid – consider the unique magmoid with two objects 0,1 and with $\hom(0,0) = \{\mathsf{id}\}$, $\hom(1,1) = \{\mathsf{id}\}$, $\hom(1,0) = \emptyset$, $\hom(0,1) = S^1$.

3.1 Obtaining parametricity results via factorisations

In general, we will obtain many theorems to the effect of that if $\mathcal{C}$ "factors orthogonally to" $\mathrm{Hom}(\mathcal{D})$ (defined below), then constructions which send arrows of $\mathcal{C}$ to arrows or arrow-identifications in $\mathcal{D}$ will be parametric. This condition will often hold; Section 4 will provide many instances (such as **??**) where $\mathcal{C}$ factors orthogonally to $\mathsf{el}_{\mathcal{U}}$, and hence to $\mathrm{Hom}(\mathcal{D})$ for any locally $\mathcal{U}$-small $\mathcal{D}$.

Definition 3.1.1. Let $\mathcal{C}$ be a wild category and $A \xrightarrow{f} B$ be a morphism in $\mathcal{C}$. Then a **factorisation** of f is a quadruple consisting of an object X, a morphism $A \xrightarrow{f_l} X$, a morphism $X \xrightarrow{f_r} B$ and an identification f_i of $f_r \circ f_l$ and f; this will be denoted $(f_i : f = A \xrightarrow{f_l} X \xrightarrow{f_r} B)$ or simply $(A \xrightarrow{f_l} X \xrightarrow{f_r} B)$ when f_i and f are clear from context (or when the choice of f_i isn't relevant).

$$\begin{tikzcd} & X \arrow[dr, "f_r"] \arrow[d, phantom, "\| \; f_i \; \|"] & \\ A \arrow[ur, "f_l"] \arrow[rr, "f"'] & {} & B \end{tikzcd}$$

Note that if the $\mathrm{hom}(X,Y)$ are all sets, it doesn't matter which f_i is used as they're all equal.

The collection of all factorisations of f is denoted $\mathrm{Fact}(f)$. For any morphism $A \xrightarrow{f} B$, we in particular have the **trivial left factorisation** $\mathrm{fact}_L(f) = (\mathsf{l}_f : f = A \xrightarrow{f} B \xrightarrow{\mathsf{id}_B} B)$ (see below left) and **trivial right factorisation** $\mathrm{fact}_R(f) = (\mathsf{r}_f : f = A \xrightarrow{\mathsf{id}_A} A \xrightarrow{f} B)$ (see below right).

$$\begin{tikzcd} & B \arrow[dr, "\mathsf{id}_B"] \arrow[d, phantom, "\| \; \mathsf{l}_f \; \|"] & \\ A \arrow[ur, "f"] \arrow[rr, "f"'] & {} & B \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dr, "f"] \arrow[d, phantom, "\| \; \mathsf{r}_f \; \|"] & \\ A \arrow[ur, "\mathsf{id}_A"] \arrow[rr, "f"'] & {} & B \end{tikzcd}$$

Note that the unit coherence $\mathsf{c}_A : \mathsf{l}_{\mathsf{id}_A} = \mathsf{r}_{\mathsf{id}_A}$ gives a path $\mathrm{fact}_=(A) : \mathrm{fact}_L(\mathsf{id}_A) = \mathrm{fact}_R(\mathsf{id}_A)$ (which is reflexivity on objects and morphisms).

When the wild category $\mathcal{C}$ is an actual 1-category and $A \xrightarrow{f} B$, $\mathrm{Fact}(f)$ can be identified with the type of objects of the slice-coslice category $({}_{A/}\mathcal{C})_{/f} \cong {}_{f/}(\mathcal{C}_{/B})$ and $\mathrm{fact}_L(f)$ and $\mathrm{fact}_R(f)$ with with the terminal and initial objects respectively.

We can now state the key concept needed for the parametricity theorems.

Definition 3.1.2. Let $\mathcal{C}$ be a wild category and $\langle T_i \rangle_{i:I}$ be a type family. $\mathcal{C}$ **factors orthogonally to** $\langle T_i \rangle_{i:I}$ iff for every morphism f of $\mathcal{C}$ and each $i : I$, $\mathrm{Fact}(f) \perp T_i$.

Remark 3.1.3. In practice, we will usually have that $\mathcal{C}$ factors orthogonally to $\mathsf{el}_{\mathcal{U}}$ (the family of all small types); we will also usually have that $\mathcal{C}$ is locally $\mathcal{U}$-small, so then automatically factors orthogonally to $\mathrm{Hom}(\mathcal{C})$. The reader can safely assume we work in this special case on a first read.

This is all that's needed to start getting parametricity results! Let's start with naturality.

Proposition 3.1.4. *Let $\mathcal{C}, \mathcal{D}$ be wild categories and $\mathcal{C}$ factor orthogonally to $\mathrm{Hom}(\mathcal{D})$. Let $F, G : \mathcal{C} \to \mathcal{D}$ be wild functors and $\alpha : F \to G$ be a wild transformation. Then α is natural: for any $A \xrightarrow{f} B$ in $\mathcal{C}$ the diagram*

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"] & GB \end{tikzcd}$$

commutes.

Proof. For a morphism $A \xrightarrow{f} B$ of $\mathcal{C}$, let

$$A \xrightarrow{f_l} X \xrightarrow{f_r} B$$

be a factorisation of f. We define a composite morphism $(Gf_r \circ \alpha_X) \circ Ff_l$:

$$\begin{tikzcd} FA \arrow[r, "Ff_l"] \arrow[d, dotted, "\alpha_A"] & FX \arrow[r, dotted, "Ff_r"] \arrow[d, "\alpha_X"] & FB \arrow[d, dotted, "\alpha_B"] \\ GA \arrow[r, dotted, "Gf_l"] & GX \arrow[r, "Gf_r"] & GB \end{tikzcd}$$

so as to get a function $N : \mathrm{Fact}(f) \to \hom_{\mathcal{D}}(FA, GB)$.

As $\mathcal{C}$ factors orthogonally to $\mathrm{Hom}(\mathcal{D})$, $\mathrm{Fact}(f) \perp \hom_{\mathcal{D}}(FA, GB)$ so N is constant.

In particular, $N(A \xrightarrow{f} B \xrightarrow{\mathsf{id}_B} B) = N(A \xrightarrow{\mathsf{id}_A} A \xrightarrow{f} B)$.

We can compute that

$$\begin{aligned} N(A \xrightarrow{f} B \xrightarrow{\mathsf{id}_B} B) &\equiv (G(\mathsf{id}_B) \circ \alpha_B) \circ Ff \\ &= (\mathsf{id}_{GB} \circ \alpha_B) \circ Ff \\ &= \alpha_B \circ Ff \end{aligned}$$

and similarly

$$\begin{aligned} N(A \xrightarrow{\mathsf{id}_A} A \xrightarrow{f} B) &\equiv (Gf \circ \alpha_B) \circ F(\mathsf{id}_A) \\ &= (Gf \circ \alpha_B) \circ \mathsf{id}_{FA} \\ &= Gf \circ \alpha_B \end{aligned}$$

so that in total

$$\begin{aligned} Gf \circ \alpha_B &= N(A \xrightarrow{\mathsf{id}_A} A \xrightarrow{f} B) \\ &= N(A \xrightarrow{f} B \xrightarrow{\mathsf{id}_B} B) \\ &= \alpha_B \circ Ff \end{aligned}$$

as required. $\square$

We can also obtain a coherence condition that says that the naturality square at id is the one arising from that functors preserve id:

Lemma 3.1.5. *In the above naturality square for $f = \mathsf{id}_A$, the obtained path $\alpha_A \circ F(\mathsf{id}_A) = G(\mathsf{id}_A) \circ \alpha_A$ is the composite $(\alpha_A \circ i^F_A) \cdot \mathsf{r}_{\alpha_A} \cdot \mathsf{l}_{\alpha_A} \cdot ((i^G_A)^{-1} \circ \alpha_A)$ in*

$$\alpha_A \circ F(\mathsf{id}_A) = \alpha_A \circ \mathsf{id}_{FA} = \alpha_A = \mathsf{id}_{GA} \circ \alpha_A = G(\mathsf{id}_A) \circ \alpha_A\ .$$

This is, of course, trivially true assuming UIP.

Proof. The path constructed in Proposition 3.1.4 is the composite

$$\begin{tikzcd}
(G(\mathsf{id}_A) \circ \alpha_A) \circ F(\mathsf{id}_A) \arrow[r, "p"'] & (G(\mathsf{id}_A) \circ \alpha_A) \circ F(\mathsf{id}_A) \arrow[d, "(G(\mathsf{id}_A) \circ \alpha_A) \circ i^F_A"] \\
(\mathsf{id}_{GA} \circ \alpha_A) \circ F(\mathsf{id}_A) \arrow[u, "(i^G_A \circ \alpha_A) \circ F(\mathsf{id}_A)"] & (G(\mathsf{id}_A) \circ \alpha_A) \circ \mathsf{id}_{FA} \arrow[d, "\mathsf{r}_{G(\mathsf{id}_A) \circ \alpha_A}"] \\
\alpha_A \circ F(\mathsf{id}_A) \arrow[u, "\mathsf{l}_{\alpha_A} \circ F(\mathsf{id}_A)"] & G(\mathsf{id}_A) \circ \alpha_A
\end{tikzcd}$$

wherein p is the path induced by constancy of N, and is equal to refl by Lemma 2.0.8 by using $\mathrm{fact}_=(A) : \mathrm{fact}_L(\mathsf{id}_A) = \mathrm{fact}_R(\mathsf{id}_A)$. The remainder is standard path manipulation (path induction on i^F_A and i^G_A and that the two ways to use l and r each once to go from $(\mathsf{id}_{GA} \circ \alpha_A) \circ \mathsf{id}_{FA}$ to α_A are equal). ☐

We get that 'wild' functors actually preserve composition.

Proposition 3.1.6. *Let $\mathcal{C}, \mathcal{D}$ be wild categories, $\mathcal{C}$ factor orthogonally to $\mathrm{Hom}(\mathcal{D})$ and $F : \mathcal{C} \to \mathcal{D}$ be a wild functor. Then F preserves composition: for any $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$ the diagram*

$$\begin{tikzcd}
& FB \arrow[dr, "Fg"] & \\
FA \arrow[ur, "Ff"] \arrow[rr, "F(g \circ f)"'] & & FC
\end{tikzcd}$$

always commutes.

Proof. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be morphisms in $\mathcal{C}$. For any factorisation $A \xrightarrow{f'} X \xrightarrow{g'} C$ of $g \circ f$, there is a morphism $N(A \xrightarrow{f'} X \xrightarrow{g'} C) :\equiv F(g) \circ F(f)$ defining a function $N : \mathrm{Fact}(g \circ f) \to \hom_{\mathcal{D}}(FA, FC)$.

N is constant, so in particular $N(\mathrm{fact}_R(g \circ f)) = N(A \xrightarrow{f} B \xrightarrow{g} C)$.

$$\begin{aligned}
F(g \circ f) &= F(g \circ f) \circ \mathsf{id}_{FA} \\
&= F(g \circ f) \circ F(\mathsf{id}_A) \\
&\equiv N(\mathrm{fact}_R(g \circ f)) \\
&= N(A \xrightarrow{f} B \xrightarrow{g} C) \\
&\equiv F(g) \circ F(f)
\end{aligned}$$

as required. ☐

We get that 'wild' categories actually have associative composition.

Proposition 3.1.7. *Let $\mathcal{C}$ be a wild category and $\mathcal{C}$ factor orthogonally to $\mathrm{Hom}(\mathcal{C})$. (Or, more weakly, that for any morphism f and $N : \mathrm{Fact}(f) \to \mathrm{hom}(A, D)$ that $N(\mathrm{fact}_L(f)) = N(\mathrm{fact}_R(f))$.) Then $\mathcal{C}$ is associative: for any $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, $h \circ (g \circ f) = (h \circ g) \circ f$.*

Proof. Fix morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ in $\mathcal{C}$. For each factorisation $B \xrightarrow{g_l} X \xrightarrow{g_r} C$ of g, define a morphism $N(B \xrightarrow{g_l} X \xrightarrow{g_r} C) : \mathrm{hom}_{\mathcal{C}}(A, D)$ as $(h \circ g_r) \circ (g_l \circ f)$.

This function $N : \mathrm{Fact}(g) \to \mathrm{hom}_{\mathcal{C}}(A, D)$ is constant, so $N(\mathrm{fact}_L(g)) = N(\mathrm{fact}_R(g))$. Hence:

$$\begin{aligned} h \circ (g \circ f) &= (h \circ \mathsf{id}_C) \circ (g \circ f) \\ &\equiv N(\mathrm{fact}_L(g)) \\ &= N(\mathrm{fact}_R(g)) \\ &\equiv (h \circ g) \circ (\mathsf{id}_B \circ f) \\ &= (h \circ g) \circ f \end{aligned}$$

as required. □

Surprisingly, the abuse of notation which results from talking about a functor just in terms of its action on objects has a justification.

Proposition 3.1.8. *Let $\mathcal{C}, \mathcal{D}$ be wild categories and $\mathcal{C}$ factor orthogonally to $\mathrm{Hom}(\mathcal{D})$. Let $F, G : \mathcal{C} \to \mathcal{D}$ be wild functors and suppose that for all objects A, $F(A) = G(A)$. Then $F = G$.*

Proof. We may freely assume by path induction that for $A \xrightarrow{f} B$, $Gf : \mathrm{hom}(FA, FB)$.

For a morphism $f : A \to B$ in $\mathcal{C}$, taking a factorisation

$$A \xrightarrow{f_l} X \xrightarrow{f_r} B$$

of f to the composite $FA \xrightarrow{Ff_l} FX \xrightarrow{Gf_r} FB$ defines a function $N : \mathrm{Fact}(f) \to \mathrm{hom}_{\mathcal{D}}(FA, FB)$. N is constant as $\mathrm{Fact}(f) \perp \mathrm{hom}_{\mathcal{D}}(FA, FB)$. Hence

$$\begin{aligned} F(f) &= \mathsf{id}_{FA} \circ F(f) \\ &= G(\mathsf{id}_A) \circ F(f) \\ &\equiv N(\mathrm{fact}_L(f)) \\ &= N(\mathrm{fact}_R(f)) \\ &\equiv G(f) \circ F(\mathsf{id}_B) \\ &= G(f) \circ \mathsf{id}_{FB} \\ &= G(f) \end{aligned}$$

This is sufficient in presence of UIP. Otherwise, we must show that the paths for "F/G preserves id" coincide.

Denoting the coherences $i^F_X : F(\mathsf{id}_X) = \mathsf{id}_{FX}$ and $i^G_X : G(\mathsf{id}_X) = \mathsf{id}_{FX}$, we still need to show that $i^F_X \cdot (i^G_X)^{-1}$ coincides with the path constructed above when $f \equiv \mathsf{id}_A$.

In this case, the provided path $N(\mathrm{fact}_L(\mathsf{id}_{FX})) = N(\mathrm{fact}_R(\mathsf{id}_{FX}))$ equals $\mathsf{ap}_N(\mathrm{fact}_=(X))$ which (as N doesn't depend on the composition path) equals $\mathsf{refl}_{G(\mathsf{id}_X)\circ F(\mathsf{id}_X)}$.

Hence we just need to ensure commutativity of the diagram

$$
\begin{tikzcd}
 & G(\mathsf{id}_X) \arrow[dl, "i^G_X"'] & G(\mathsf{id}_X)\circ \mathsf{id}_{FX} \arrow[l, "\mathsf{r}_{G(\mathsf{id}_X)}"'] & G(\mathsf{id}_X)\circ F(\mathsf{id}_X) \arrow[l, "G(\mathsf{id}_X)\circ i^F_X"'] \arrow[dd, equal, "\mathsf{refl}"] \\
\mathsf{id}_{FX} & & & \\
 & F(\mathsf{id}_X) \arrow[ul, "i^F_X"] & id_{FX}\circ F(\mathsf{id}_X) \arrow[l, "\mathsf{l}_{F(\mathsf{id}_X)}"] & G(\mathsf{id}_X)\circ F(\mathsf{id}_X) \arrow[l, "i^G_X\circ F(\mathsf{id}_X)"]
\end{tikzcd}
$$

which, after using path induction on i^F_X and i^G_X (and simplifying any composition with refl), becomes $\mathsf{l}_{\mathsf{id}_X} = \mathsf{r}_{\mathsf{id}_X}$ which is witnessed by the wild category coherence c_X. ☐

Similarly to naturality squares commuting automatically, exrtanaturality squares commute automatically also. The wild category $\mathcal{C}^{\mathsf{op}}$ is defined in the usual way as having the same objects as $\mathcal{C}$, $\hom_{\mathcal{C}^{\mathsf{op}}}(C, C') = \hom_{\mathcal{C}}(C', C)$, reversed composition and with r and l swapped.

Proposition 3.1.9. *Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be wild categories and $\mathcal{C}$, $\mathcal{D}$ both factor orthogonally to $\mathrm{Hom}(\mathcal{E})$. Let $F : \mathcal{C} \times \mathcal{C}^{\mathsf{op}} \to \mathcal{E}$ and $G : \mathcal{D} \times \mathcal{D}^{\mathsf{op}} \to \mathcal{E}$ be wild functors. Suppose that for each object C of $\mathcal{C}$ and D of $\mathcal{D}$ there's a morphism $F(C, C) \xrightarrow{\alpha_{C,D}} G(D, D)$. Then the $\alpha_{C,D}$ are extranatural:*

For all $C \xrightarrow{g} C'$ in $\mathcal{C}$ and $D \xrightarrow{h} D'$ in $\mathcal{D}$, the squares below commute

$$
\begin{tikzcd}
F(C, C') \arrow[r, "F(\mathsf{id},g)"] \arrow[d, "F(g,\mathsf{id})"'] & F(C, C) \arrow[d, "\alpha_{C,D}"] \\
F(C', C') \arrow[r, "\alpha_{C',D}"'] & G(D, D)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(C, C) \arrow[r, "\alpha_{C,D}"] \arrow[d, "\alpha_{C,D'}"'] & G(D, D) \arrow[d, "G(h,\mathsf{id})"] \\
G(D', D') \arrow[r, "G(\mathsf{id},h)"'] & G(D', D)
\end{tikzcd}
$$

Proof. We prove this for the right extranaturality square; the left square is dual.

Let $D \xrightarrow{h_l} X \xrightarrow{h_r} D'$ be a factorisation of $D \xrightarrow{h} D'$, and consider the morphism $G(h_l, h_r) \circ \alpha_{C,X}$ from $F(C, C)$ to $G(D', D)$. When $(D \xrightarrow{h_l} X \xrightarrow{h_r} D') = \mathrm{fact}_L(h)$ this is the upper-right composite of the square, and when $(D \xrightarrow{h_l} X \xrightarrow{h_r} D') = \mathrm{fact}_L(h)$ this is the lower-left composite. Hence, as $\mathcal{D}$ factors orthogonally to $\hom_{\mathcal{E}}(F(C, C), G(D', D))$, these composites are equal. ☐

Corollary 3.1.10. *Let $\mathcal{D}$ be a category which factors orthogonally to $\mathsf{el}_{\mathcal{U}}$, and $F : \mathcal{D} \times \mathcal{D}^{\mathsf{op}} \to \mathcal{U}$ be a functor and for all $X : \mathcal{D}$, $\alpha_X : F(X, X)$. Then for any morphism $X \xrightarrow{g} Y$ of $\mathcal{C}$, $F(g, \mathsf{id}_X)(\alpha_X) =_{F(Y,X)} F(\mathsf{id}_Y, g)(\alpha_Y)$.*

Proof. In the above proposition, take $\mathcal{E}$ to be $\mathcal{U}$, $\mathcal{C}$ to be $\mathbf{1}$ and $F(c, c') = \mathbf{1}$ □

A weaker version of 'factoring orthogonally' which only cares about sending $\mathrm{fact}_L(f)$ and $\mathrm{fact}_R(f)$ to the same value is often sufficient for arguments, and is equivalent under mild hypotheses:

Lemma 3.1.11. *Let $\mathcal{C}$ be a wild category and $\langle T_i \rangle_{i:I}$ be a type family. Suppose that for any f,i and $M : \mathrm{Fact}(f) \to T_i$ that $M^f_= : M(\mathrm{fact}_L(f)) = M(\mathrm{fact}_R(f))$. If either*

1. *$\mathcal{C}$ is associative and the associator $\alpha_{f,g,h} : h \circ (g \circ f) = (h \circ g) \circ f$ satisfies coherence equations $\alpha_{f,\mathsf{id},h} = (h \circ \mathsf{l}_f) \cdot (\mathsf{r}_h \circ f)^{-1}$ and $\alpha_{\mathsf{id},g,h} = (h \circ \mathsf{r}_g) \cdot (\mathsf{r}_{h \circ g})^{-1}$*
2. *$\langle T_i \rangle_{i:I}$ includes both $\mathrm{Hom}(\mathcal{C})$ and $\langle f = g \rangle_{f,g:\mathrm{hom}(c,c'),c,c':\mathrm{Ob}_{\mathcal{C}}}$, and for any object X and $M : \mathrm{Fact}(\mathsf{id}_X) \to T_i$ as above (for T_i either $\mathrm{hom}(c, c')$ or $f =_{\mathrm{hom}(c,c')} g$), $M^{\mathsf{id}_X}_= = \mathsf{ap}_M(\mathrm{fact}_=(X))$*

then $\mathcal{C}$ factors orthogonally to $\langle T_i \rangle_{i:I}$.

Note that if the $\mathrm{hom}(c, c')$ are all sets then the extra hypotheses reduce to just "$\mathcal{C}$ is associative" and "$\langle T_i \rangle_{i:I}$ includes $\mathrm{Hom}(\mathcal{C})$" respectively.

Proof. Suppose that (1), and let $g : \mathrm{Fact}(f) \to T_i$. Let $(f_= : f = A \xrightarrow{f_l} X \xrightarrow{f_r} B)$ be a factorisation of f. It is now sufficient to establish constancy of g with $g(A \xrightarrow{f_l} X \xrightarrow{f_r} B) = g(\mathrm{fact}_R(f))$.

Define $M : \mathrm{Fact}(f_l) \to T_i$ as

$$\begin{aligned} &M(f_{l=} : f_l = A \xrightarrow{f_{ll}} Y \xrightarrow{f_{lr}} X) \\ &\coloneqq g(p : f = A \xrightarrow{f_{ll}} Y \xrightarrow{f_r \circ f_{lr}} B) \end{aligned}$$

wherein p is the composite path $\alpha^{-1}_{f_{ll},f_{lr},f_r} \cdot (f_r \circ f_{l=}) \cdot f_=$:

$$(f_r \circ f_{lr}) \circ f_{ll} \xrightarrow{\alpha^{-1}_{f_{ll},f_{lr},f_r}} f_r \circ (f_{lr} \circ f_{ll}) \xrightarrow{f_r \circ f_{l=}} f_r \circ f_l \xrightarrow{f_=} f$$

Then

$$\begin{aligned} g(f_= : f = A \xrightarrow{f_l} X \xrightarrow{f_r} B) &= g((\mathsf{r}_{f_r} \circ f_l) \cdot f_= : f = A \xrightarrow{f_l} X \xrightarrow{f_r \circ \mathsf{id}_X} B) \\ &= g(\alpha^{-1}_{f_l,\mathsf{id}_X,f_r} \cdot (f_r \circ \mathsf{l}_{f_l}) \cdot f_= : f = A \xrightarrow{f_l} X \xrightarrow{f_r \circ \mathsf{id}_X} B) \\ &\equiv M(\mathsf{l}_{f_l} : f_l = A \xrightarrow{f_l} X \xrightarrow{\mathsf{id}_X} X) \\ &= M(\mathsf{r}_{f_l} : f_l = A \xrightarrow{\mathsf{id}_A} A \xrightarrow{f_l} X) \\ &\equiv g(\alpha^{-1}_{\mathsf{id}_A,f_l,f_r} \cdot (f_r \circ \mathsf{r}_{f_l}) \cdot f_= : f = A \xrightarrow{\mathsf{id}_A} A \xrightarrow{f_r \circ f_l} B) \\ &= g(\mathsf{r}_{f_r \circ f_l} \cdot f_= : f = A \xrightarrow{\mathsf{id}_A} A \xrightarrow{f_r \circ f_l} B) \\ &= g(\mathrm{fact}_R(f)) \end{aligned}$$

where the second and second-to-last equalities follow directly from the additional coherences required of the associator.

Supposing (2), Proposition 3.1.7 provides an associator α; this is sufficient to obtain (1) in the presence of UIP. Otherwise the coherence equations in (1) must be proven still. Recall from that proposition that for $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ that $N : \mathrm{Fact}(g) \to \hom_{\mathcal{C}}(A, D)$ is defined as $N(B \xrightarrow{g_l} M \xrightarrow{g_r} C) = (h \circ g_r) \circ (g_l \circ f)$, and that $\alpha_{f,g,h} = (\mathsf{r}_h \circ (g \circ f))^{-1} \cdot N^{g}_{=} \cdot ((h \circ g) \circ \mathsf{l}_f)$.

The first coherence equation (for $\alpha_{f,\mathsf{id},h}$) corresponds to commutativity of the diagram

$$\begin{tikzcd}
(h \circ \mathsf{id}) \circ (\mathsf{id} \circ f) \arrow[rr, bend left, "N^{\mathsf{id}}_{=}"] \arrow[rr, equal, bend right, "\mathsf{refl}"'] \arrow[d, "\mathsf{r}_h \circ (\mathsf{id} \circ f)"'] & & (h \circ \mathsf{id}) \circ (\mathsf{id} \circ f) \arrow[d, "(h \circ \mathsf{id}) \circ \mathsf{l}_f"] \\
h \circ (\mathsf{id} \circ f) \arrow[r, "h \circ \mathsf{l}_f"'] & h \circ f & (h \circ \mathsf{id}) \circ f \arrow[l, "\mathsf{r}_h \circ f"]
\end{tikzcd},$$

the upper lozenge commutes as $\mathsf{ap}_N(\mathrm{fact}_=(X))$ is refl and the rectangle commutes by typical path algebra.

The second equation (for $\alpha_{\mathsf{id},g,h}$) corresponds to commutativity of the diagram

$$\begin{tikzcd}
(h \circ \mathsf{id}) \circ (g \circ \mathsf{id}) \arrow[rr, "N^{g}_{=}"] \arrow[d, "\mathsf{r}_h \circ (g \circ \mathsf{id})"'] & & (h \circ g) \circ (\mathsf{id} \circ \mathsf{id}) \arrow[d, "(h \circ g) \circ \mathsf{l}_{\mathsf{id}}"] \\
h \circ (g \circ \mathsf{id}) \arrow[r, "h \circ \mathsf{r}_g"'] & h \circ g & (h \circ g) \circ \mathsf{id} \arrow[l, "\mathsf{r}_{h \circ g}"]
\end{tikzcd}.$$

Abbreviate the composite path from $(h \circ \mathsf{id}) \circ (g \circ \mathsf{id})$ to $(h \circ g) \circ (\mathsf{id} \circ \mathsf{id})$ as $\beta_{g,h} : d_{g,h} = c_{g,h}$ so that the goal is that $N^{g}_{=} = \beta_{g,h}$.

Consider the function $M : \mathrm{Fact}(g) \to \sum_{m,n:\hom(B,D)} (m = n) \times (m = n)$ given by $M(B \xrightarrow{g_l} Y \xrightarrow{g_r} C) = (d_{g_r,h} \circ g_l, c_{g_r,h} \circ g_l, N^{g_r}_{=} \circ g_l, \beta_{g_r,h} \circ g_l)$. T may as well be closed under dependent sums; one obtains a $M^{g}_{=} : M(B \xrightarrow{g} C \xrightarrow{\mathsf{id}} C) = M(B \xrightarrow{\mathsf{id}} B \xrightarrow{g} C)$. This describes that the outer boundary of the cylinder below commutes

$$\begin{tikzcd}
d_{\mathsf{id},h} \circ g \arrow[r, bend left, "N^{\mathsf{id}}_{=} \circ g"] \arrow[r, bend right, "\beta_{\mathsf{id},h} \circ g"'] \arrow[d, "M^{g}_{=}"'] & c_{\mathsf{id},h} \circ g \arrow[d, "M^{g}_{=}"] \\
d_{g,h} \circ \mathsf{id} \arrow[r, bend left, "N^{g}_{=} \circ \mathsf{id}"] \arrow[r, bend right, "\beta_{g,h} \circ \mathsf{id}"'] & c_{g,h} \circ \mathsf{id}
\end{tikzcd}$$

and the top face of the cylinder commutes (for the same reason as the first coherence equation's rectangle commuting) so the bottom face of the cylinder must also commute, as required. □

Remark 3.1.12. The hypotheses cannot be weakened further. Let the set L contain points $0_L \neq 1_L$ and let $L \perp \mathbf{2}$. Consider the (set-truncated) wild category freely generated by arrows $a \xrightarrow{g} b \xrightarrow{h} c$ and an operation $n : \prod_f L \to \mathrm{Fact}(f)$ which has $n_f(0_L) = (\mathsf{id}, f)$ and $n_f(1_L) = (f, \mathsf{id})$. Then one can see that the factorisations of $h \circ g$ are only the $1 + L$ factorisations (g, h) and $n_{h\circ g}(t)$, and there is a nonconstant function $1 + L \to \mathbf{2}$.

3.2 Large (co)limits and impredicative encodings

An impredicative universe is a universe $\mathcal{U}$ where the product of any family of objects of $\mathcal{U}$ is still in $\mathcal{U}$, even if the indexing family isn't $\mathcal{U}$-small. Often in this case it is possible to do "impredicative encodings": for example, let the type $\mathsf{PolyNat} :\equiv \prod_{A:\mathcal{U}}(A \to A) \to (A \to A)$ of **church numerals** 'encode' the natural numbers via $\underline{n}_A(f) :\equiv f^n$ corresponding to n-times iterated composition. (The successor function $\underline{\mathtt{suc}}$ on church numerals is defined as $\underline{\mathtt{suc}}(n)_A(f) = f \circ n_A(f)$ which as expected obeys $\underline{\mathtt{suc}}(\underline{n}) = \underline{n+1}$.)

Although we can't generally prove that $(n \mapsto \underline{n}) : \mathbb{N} \to \mathsf{PolyNat}$ is an isomorphism without additional assumptions, relational parametricity lets one prove that the only close terms of $\mathsf{PolyNat}$ are of the form $\underline{n}$, which is often good enough.

However, we can use additional 'factors orthogonally' assumptions to conclude that $(n \mapsto \underline{n})$ *is* an isomorphism.

3.2.1 Large products

The notion of "product" P of objects $\langle A_i \rangle_i$ is well-defined in any wild category: an object P equipped with a family of morphisms $\rho_i : P \to A_i$ is a product iff for all objects B, the map $f \mapsto (\rho_- \circ f) : \hom(B, P) \to \prod_{i:I} \hom(B, A_i)$ is an equivalence.

Proposition 3.2.1. *Suppose that $\mathcal{C}$ is a wild category which factors orthogonally to $\hom(\mathcal{C})$, and I is an object of $\mathcal{C}$. Then I is a product of all objects of $\mathcal{C}$ iff I is initial.*

Proof. Let B be an object and let $\beta : \prod_{C:\mathcal{C}} \hom(B, C)$. Then β is an unnatural transformation from $\mathsf{const}_B : \mathcal{C} \to \mathcal{C}$ to $\mathsf{id}_{\mathcal{C}}$. Hence β is natural, and for all morphisms $C \xrightarrow{f} D$, $\beta_D = f \circ \beta_C$.

If (P, ρ_-) is a product of all objects of $\mathcal{C}$, then the function $\hom(P, P) \to \prod_{C:\mathcal{C}} \hom(P, C)$ defined by $f \mapsto (C \mapsto \rho_C \circ f)$ is an equivalence. For all objects C,

$$\rho_C \circ \mathsf{id}_P = \rho_C = \rho_C \circ \rho_P$$

by naturality of ρ at ρ_C, so $\mathsf{id}_P = \rho_P$. Hence, by naturality of ρ at $f : P \to C$,

$$\rho_C = f \circ \rho_P = f \circ \mathsf{id}_P = f \ ,$$

so P is initial.

Conversely, if P is initial we want to show the family of projections $!_C : P \to C$ make P a product. We show that $\pi_P : \prod_{C:\mathcal{C}} \hom(B, C) \to \hom(B, P)$ is inverse to $f \mapsto (!_{-} \circ f) : \hom(B, P) \to \prod_{C:\mathcal{C}} \hom(B, C)$. To show π_P is left inverse, for $f : \hom(B, P)$,

$$\pi_P(!_{-} \circ f) = !_P \circ f = \mathsf{id}_P \circ f = f \ .$$

To show π_P is right inverse, for $\beta : \prod_{C:\mathcal{C}} \hom(B, C)$,

$$(f \mapsto !_{-} \circ f)(\pi_P(\beta)) = (f \mapsto !_{-} \circ f)(\beta_P) = !_{-} \circ \beta_P = \beta_{-} \circ \mathsf{id}_B = \beta_{-}$$

with the second-to-last equality by naturality of $\beta : \mathsf{const}_B \Rightarrow \mathsf{id}_{\mathcal{C}}$. □

Now, form the wild category **DDS** with objects consisting of triples $|A| : \mathcal{U}$, $r_A : |A| \to |A|$ and $p_A : |A|$ ("discrete dynamical systems"), and morphisms $A \to B$ as functions $m : |A| \to |B|$ such that $m \circ r_A = r_B \circ m$ and $m(p_A) = p_B$.

Corollary 3.2.2. *If* **DDS** *factors orthogonally to* $\mathsf{el}_{\mathcal{U}}$ *and* $\mathsf{PolyNat}$ *is in* $\mathcal{U}$*, then:*

- $(\mathsf{PolyNat}, \underline{\mathsf{suc}}, \underline{0})$ *is initial in* **DDS**
- $\mathsf{PolyNat}$ *has the property of being a natural numbers object in* $\mathcal{U}$*,*
- *the "*η*-rule",* $n = n_{\mathsf{PolyNat}}\ \underline{\mathsf{suc}}\ \underline{0}$ *holds for all* $n : \mathsf{PolyNat}$*,*
- *If* $\mathbb{N}$ *is in* $\mathcal{U}$*, then* $(n \mapsto \underline{n})$ *is an isomorphism.*

Proof. $\prod_{X:\mathcal{U}}(X \to X) \to (X \to X) \cong \prod_{(|A|, r_A, p_A):\mathbf{DDS}} |A|$ and products in **DDS** are computed as in $\mathcal{U}$ so $\mathsf{PolyNat}$ is the initial DDS, which is always an NNO. id and $n \mapsto n_{\mathsf{PolyNat}}\ \underline{\mathsf{suc}}\ \underline{0}$ are both morphisms from $\mathsf{PolyNat}$ to itself so are equal by initiality. $(\mathbb{N}, 0, +1)$ is also the initial DDS (if in $\mathcal{U}$). □

Of course, this works just as well for obtaining the initial algebra of any endofunctor on $\mathcal{U}$.

Lemma 3.2.3. *Let* $\mathcal{C}$ *factor orthogonally to* $\mathcal{D}$ *and* 0 *be initial in* $\mathcal{C}$*. Then for any functor* $F : \mathcal{C} \to \mathcal{D}$*,* $\prod_{c:\mathcal{C}} F(c)$ *exists and is given by* $F(0)$ *and projection maps* $F(!_C) : F(0) \to F(C)$*.*

Proof. We want to show that for any object D of $\mathcal{D}$, that

$$f \mapsto F(!_{-}) \circ f : \hom(D, F(0)) \to \prod_{C:\mathcal{C}} \hom(D, F(C))$$

is an equivalence, by showing π_0 is its inverse. For one direction observe

$$\pi_0(F(!_{-}) \circ f) = F(!_0) \circ f = F(\mathsf{id}_{F(0)}) \circ f = \mathsf{id} \circ f = f \ .$$

Conversely, for $\beta : \prod_{C:\mathcal{C}} \hom(D, F(C))$,

$$(f \mapsto F(!_{-}) \circ f)(\pi_0(\beta)) = (f \mapsto F(!_{-}) \circ f)(\beta_0) = F(!_{-}) \circ \beta_0 = \beta_{-} \circ \mathsf{id}_D = \beta_{-}$$

with the second-to-last equality following from naturality of $\beta : \mathsf{const}_D \Rightarrow F$. □

3.2.2 Reflection

Assume for this subsubsection that $\mathcal{U}$ is an impredicative universe and $\mathcal{U}$ factors orthogonally to $\mathsf{el}_{\mathcal{U}}$.

For a type A (not necessarily in $\mathcal{U}$), let $\lozenge A$ be the type defined by $\prod_{X:\mathcal{U}}(A \to X) \to X$. Write $\eta_A : A \to \lozenge A$ for the evaluation function $\eta_A(a)_X(g) = g(a)$.

If $f : L \to A$ is a function for $A : \mathcal{U}$, define $\delta(f) : \lozenge L \to A$ as $\delta(f)(l) = l_A(f)$. f factors as the composite $L \xrightarrow{\eta_L} \lozenge L \xrightarrow{\delta(f)} A$.

Lemma 3.2.4. *If $\alpha : \lozenge L$, then α is natural: for any $f : X \to Y$ and $g : L \to X$, $f(\alpha_X(g)) = \alpha_Y(f \circ g)$.*

Proof. α defines a transformation between the functor $X \mapsto X^L$ and $X \mapsto X$ on $\mathcal{U} \to \mathcal{U}$. By Proposition 3.1.4, α is natural. □

Theorem 3.2.5. *$\delta : (L \to A) \to (\lozenge L \to A)$ is inverse to $\circ\eta_A : (\lozenge L \to A) \to (L \to A)$.*

Proof. First let $g : L \to A$. Then

$$((\circ\eta_L) \circ \delta)(g) \equiv \delta(g) \circ \eta_L \equiv l \mapsto \delta(g)(\eta_L(l)) \equiv l \mapsto \eta_L(l)(g) \equiv l \mapsto g(l) \equiv g$$

Conversely, let $f : \lozenge L \to A$. Then

$$\begin{aligned} \delta((\circ\eta_L)(f)) &\equiv \delta(f \circ \eta_L) \\ &\equiv \alpha \mapsto \alpha_A(f \circ \eta_L) \\ &= \alpha \mapsto f(\alpha_{\lozenge L}(\eta_L)) \end{aligned}$$

with the last equality by naturality of α, so it suffices to show that $\alpha = \alpha_{\lozenge L}(\eta_L)$ always.

Let $X : \mathcal{U}$ and $g : L \to X$. By naturality of α,

$$\alpha_X(\delta(g) \circ \eta_L) = \delta(g)(\alpha_{\lozenge L}(\eta_L)) \equiv \alpha_{\lozenge L}(\eta_L)_X(g)$$

but also

$$\delta(g) \circ \eta_L \equiv l \mapsto \delta(g)(\eta_L(l) \equiv l \mapsto \eta_L(l)_X(g) \equiv l \mapsto g(l) \equiv g$$

so $\alpha = \alpha_{\lozenge L}(\eta_L)$. □

Corollary 3.2.6. *For any $A : \mathcal{U}$, η_A has inverse $\delta(\mathsf{id}_A)$.*

In particular these can be seen to say that $\lozenge$ is a modality that witnesses that $\mathcal{U}$ is reflective in the collection of *all* types.

For a family of types $A : I \to \mathcal{U}$ (usually with I large), the existential type is defined via the impredicative encoding $\exists_{i:I} A_i \coloneqq \prod_{X:\mathcal{U}}(\prod_{i:I} A_i \to X) \to X$. The constructor $\langle i : I, a : A_i \rangle : \exists_{i:I} A_i$ is defined by $\langle i : I, a : A_i \rangle = \lambda Z.\lambda g.g_i(a)$. The eliminator $\exists\mathsf{rec}_C : (\prod_{i:I} A_i \to C) \to (\exists_{i:I} A_i) \to C$ is defined as $\exists\mathsf{rec}_C\ f\ u = u\ C\ f$. Note $\exists\mathsf{rec}_C\ f\ \langle i, a \rangle \equiv f_i(a)$.

Corollary 3.2.7. *For $B : \mathcal{U}$, $\exists\mathsf{rec}_B : \prod_{i:I} A_i \to B \cong (\exists_{i:I} .A_i) \to B$ forms a natural isomorphism in B. $(\exists_{i:I} A_i)$ is the I-indexed coproduct* in $\mathcal{U}$ *of the A_i.*

Proof. $\exists_{i:I} A_i \cong \lozenge(\sum_{i:I} A_i)$ □

Similarly to the treatment of the impredicative encoding of the natural numbers, we can also form impredicative encodings of terminal coalgebras.

Let $F : \mathcal{U} \to \mathcal{U}$ be a functor. There is a category F-CoAlg of F-coalgebras which has objects as pairs $(A : \mathcal{U}, a : A \to FA)$ and morphisms $(A, a) \to (B, b)$ as pairs $(f : A \to B, f_= : b \circ f = Ff \circ a)$; identity morphisms and composition are defined in the usual way (see Proposition 4.1.8 for a full definition).

Lemma 3.2.8. *For (A_i, a_i) a family of F-coalgebras, their coproduct has underlying object $\exists_{i:I} A_i$ and structure map $\exists\mathsf{rec}_{F(\exists_i A_i)}(\lambda i.F(\langle i, -\rangle) \circ a_i) : \exists_{i:I} A_i \to F(\exists_{i:I} A_i)$.*

This is a special case of that the forgetful functor F-$\mathsf{CoAlg} \to \mathcal{U}$ creates colimits.

Proof. For any F-coalgebra (B, b), the following sequence of isomorphisms holds naturally in (B, b):

$$
\begin{aligned}
&\hom_{F\text{-}\mathsf{CoAlg}}((\exists_{i:I} A_i, \exists\mathsf{rec}_{F(\exists_i A_i)}(\lambda i.F(\langle i, -\rangle) \circ a_i)), (B, b)) \\
\equiv &\sum_{f:(\exists_{i:I} A_i)\to B} b \circ f = F(f) \circ (\exists\mathsf{rec}_{F(\exists_i A_i)}(\lambda i.F(\langle i, -\rangle) \circ a_i)) \\
\cong &\sum_{f:(\exists_{i:I} A_i)\to B} \prod_{j:I} b \circ f \circ \langle j, -\rangle = F(f) \circ (\exists\mathsf{rec}_{F(\exists_i A_i)}(\lambda i.F(\langle i, -\rangle) \circ a_i)) \circ \langle j, -\rangle \\
\equiv &\sum_{f:(\exists_{i:I} A_i)\to B} \prod_{j:I} b \circ f \circ \langle j, -\rangle = F(f) \circ F(\langle j, -\rangle) \circ a_j \\
\cong &\sum_{f:(\exists_{i:I} A_i)\to B} \prod_{j:I} b \circ (f \circ \langle j, -\rangle) = F(f \circ \langle j, -\rangle) \circ a_j \\
\cong &\sum_{f:\prod_{j:I} A_j\to B} \prod_{j:I} b \circ f_j = F(f_j) \circ a_j \\
\cong &\prod_{j:I} \sum_{f:A_j\to B} b \circ f_j = F(f_j) \circ a_j \\
\equiv &\prod_{j:I} \hom_{F\text{-}\mathsf{CoAlg}}((A_i, a_i), (B, b))
\end{aligned}
$$

□

The impredicative encoding $(\nu F, \mathsf{next}_F)$ of the terminal coalgebra is defined as $\nu F :\equiv \exists_{X:\mathcal{U}}(X \to FX)\times X$ and $\mathsf{next}_F :\equiv \exists\mathsf{rec}_{F(\nu F)}\ \lambda X.\lambda(f, x).\ F(\langle X, (f, -)\rangle)(fx)$.

Corollary 3.2.9. *If F-CoAlg factors orthogonally to $\mathsf{el}_{\mathcal{U}}$ then $(\nu F, \mathsf{next}_F)$ is terminal in F-CoAlg.*

Proof. $(\nu F, \mathsf{next}_F)$ is (strictly) isomorphic to the coalgebra with object $\exists_{(X,f):\sum_{X:\mathcal{U}}(X\to FX)} X$ and structure map $\exists\mathsf{rec}_{F(\exists_{(X,f)} X)}\ \lambda(X,f).F(\langle(X,f),-\rangle)\circ f$ via sending $\langle X,(f,x)\rangle$ to $\langle(X,f),x\rangle$. This latter coalgebra is the coproduct of all objects of $F\text{-}\mathsf{CoAlg}$. By Proposition 3.2.1, it is terminal. ☐

We will later see in Proposition 4.4.4 that the parametricity axiom RPA implies that $F\text{-}\mathsf{CoAlg}$ factors orthogonally to $\mathsf{el}_{\mathcal{U}}$.

3.3 Parametricity and homotopical coherences

The above pattern (or even, the above specific results) are enough to derive most parametricity results about 1-categories one could ask for. Incredibly, it is also possible to prove an infinite amount of coherence for wild (∞-)categories.

Let $\mathcal{C}$ be a wild category. For $k \geq 0$, a **k-chain** is a sequence of morphisms

$$Z_0 \xrightarrow{z_0} Z_1 \xrightarrow{z_1} \cdots \xrightarrow{z_{k-1}} Z_k\ ;$$

also, the k-chain consisting only of Xs and id_X is denoted $X^{[k]}$.

For $\mathcal{C},\mathcal{D}$ wild categories and $|F| : |\mathcal{C}| \to |\mathcal{D}|$, a **$k$-ary composition map over** F is an operation which to each k-chain Z in $\mathcal{C}$ assigns a morphism $a(Z) : \hom_{\mathcal{D}}(|F|(Z_0), |F|(Z_k))$ such that for each object X of $\mathcal{C}$, there is a $a_=(X) : a(X^{[k]}) = \mathsf{id}_{|F|(X)}$. A **$k$-ary composition operator for** $\mathcal{C}$ is a k-ary composition map over $\mathsf{id}_{\mathcal{C}}$.

For $\mathcal{C} = \mathcal{D}$ and $|F| = \mathsf{id}$ this will be used to show that there's often a unique way of composing morphisms in $\mathcal{C}$. For general $|F|$, this will be used to show that, if $|F|$ can be extended to act on morphisms, then F is coherently a functor.

To facilitate the proof, two operations on composition maps are defined. For a morphism $f : \hom(FX, FY)$, abbreviate the path $(i^F_Y \circ f)\cdot \mathsf{l}_f : F(\mathsf{id}_Y)\circ f = f$ as L^F_f.

Given a k-ary composition map $(l, l_=)$ over $|F|$, define a $k+1$-ary composition map $(l^+, l^+_=)$ over $|F|$ by taking $l^+(X_0 \xrightarrow{x_0} \cdots X_k \xrightarrow{x_k} X_{k+1}) :\equiv F(x_k)\circ l(X_0 \xrightarrow{x_0} \cdots X_k)$ and $l^+_=(X)$ to be the composite of paths

$$F(\mathsf{id}_X)\circ l(X^{[k]}) \xrightarrow{\mathsf{L}^F_{l(X^{[k]})}} l(X^{[k]}) \xrightarrow{l_=(X)} \mathsf{id}_{FX}\ .$$

Given a $k+1$-ary composition map $(l, l_=)$ over $|F|$, define a k-ary composition map $(l^-, l^-_=)$ over $|F|$ by taking $l^-(X_0 \xrightarrow{x_0} \cdots X_k \xrightarrow{x_k} X_{k+1}) :\equiv l(X_0 \xrightarrow{x_0} \cdots X_k \xrightarrow{\mathsf{id}} X_k)$ and $l^-_=(X) :\equiv l_=(X)$.

Lemma 3.3.1. *Suppose $\mathcal{C},\mathcal{D}$ are wild categories, $F : \mathcal{C} \to \mathcal{D}$ is a wild functor, and $\mathcal{C}$ factors orthogonally to $\mathrm{Hom}(\mathcal{D})$. For any $k+1$-ary composition map u over $|F|$, $u = u^{-+}$.*

Theorem 3.3.2. *If $\mathcal{C},\mathcal{D}$ are wild categories, $F : \mathcal{C} \to \mathcal{D}$ is a wild functor, and $\mathcal{C}$ factors orthogonally to $\mathrm{Hom}(\mathcal{D})$, then for each k there's a unique k-ary composition map over $|F|$.*

Proof. Proceed by induction on k. For $k = 0$, there is a unique 0-ary composition map l^0 over $|F|$ given by $l^0(X) = \mathsf{id}_{FX}$ and $l^0_=(X) = \mathsf{refl}_{\mathsf{id}_{FX}}$.

Assume there is a unique k-ary composition map l. Then l^+ is a $k+1$-ary composition map. Moreover, for any $k+1$-ary composition map u, $u = u^{-+} = l^+$ by the previous lemma and $u^- = l$, so there's also a unique $k+1$-ary composition map. □

Proof of Lemma 3.3.1. Note the characterisation of the path type of k-ary composition maps over $|F|$: $(u, u_=) = (v, v_=)$ is equivalent to the type of pairs $p : \prod_{\vec{X}:k\text{-chain}} u(\vec{X}) = v(\vec{X})$ and $p_= : \prod_{X:\mathrm{Ob}\,\mathcal{C}} u_=(X) = p(X^{[k]}) \cdot v_=(X)$.

Let $X_0 \xrightarrow{x_0} \cdots \xrightarrow{x_{k-1}} X_k \xrightarrow{x_k} X_{k+1}$ be a $k+1$-chain. Define $u' : \mathrm{Fact}(x_k) \to \hom(FX_0, FX_{k+1})$ by

$$u'(y_p : x_k = X_k \xrightarrow{y_l} Y \xrightarrow{y_r} X_{k+1}) :\equiv F(y_r) \circ u(X_0 \xrightarrow{x_0} \cdots \xrightarrow{x_{k-1}} X_k \xrightarrow{y_l} Y)\ .$$

For the left factorisation $\mathrm{fact}_L(x_k)$, one obtains

$$\begin{aligned} u'(\mathrm{fact}_L(x_k)) &\equiv F(\mathsf{id}_{X_{k+1}}) \circ u(X_0 \xrightarrow{x_0} \cdots \xrightarrow{x_{k-1}} X_k \xrightarrow{x_k} X_{k+1}) \\ &= u(X_0 \xrightarrow{x_0} \cdots \xrightarrow{x_k} X_{k+1})\ . \end{aligned}$$

For the right factorisation $\mathrm{fact}_R(x_k)$, one obtains

$$\begin{aligned} u'(\mathrm{fact}_R(x_k)) &\equiv F(x_k) \circ u(X_0 \xrightarrow{x_0} \cdots \xrightarrow{x_{k-1}} X_k \xrightarrow{\mathsf{id}} X_k) \\ &\equiv F(x_k) \circ u^-(X_0 \xrightarrow{x_0} \cdots \xrightarrow{x_{k-1}} X_k) \\ &\equiv u^{-+}(X_0 \xrightarrow{x_0} \cdots \xrightarrow{x_k} X_{k+1})\ . \end{aligned}$$

As u' is a function from $\mathrm{Fact}(x_k)$ to $\hom_{\mathcal{D}}(FX_0, FX_{k+1})$ and $\mathcal{C}$ factors orthogonally to $\mathrm{Hom}(\mathcal{D})$, $\mathsf{cp}_{u'}(\mathrm{fact}_L(x_k), \mathrm{fact}_R(x_k)) : u'(\mathrm{fact}_L(x_k)) = u'(\mathrm{fact}_R(x_k))$, so we obtain a composite path

$$p(X_0 \xrightarrow{x_0} \cdots \xrightarrow{x_k} X_{k+1}) : u(X_0 \xrightarrow{x_0} \cdots \xrightarrow{x_k} X_{k+1}) = u^{-+}(X_0 \xrightarrow{x_0} \cdots \xrightarrow{x_k} X_{k+1})$$

defined as

$$p(X_0 \xrightarrow{x_0} \cdots \xrightarrow{x_k} X_{k+1}) :\equiv (\mathsf{L}^F_{u(X_0 \to \cdots \to X_{k+1})})^{-1} \cdot \mathsf{cp}_{u'}(\mathrm{fact}_L(x_k), \mathrm{fact}_R(x_k))\ .$$

It remains to show that for each object X that there is a path $p_=(X) : u_=(X) = p(X^{[k+1]}) \cdot u^{-+}(X)$.

Given already $p : u = u^{-+}$, we hence need to show that for each object X, the zig-zagged triangle of paths in the diagram below commutes:

$$\begin{tikzcd}
u(X^{[k+1]}) \arrow[r, squiggly, "p(X^{[k+1]})"'] \arrow[rr, squiggly, bend left, "u_{=}(X)"] \arrow[d, "(\mathsf{L}^F_{u(X^{[k+1]})})^{-1}"'] & u^{-+}(X^{[k+1]}) \arrow[r, squiggly, "u^{-+}_{\equiv}(X)"'] \arrow[d, equal, "\mathsf{refl}"] & \mathsf{id}_X \\
F(\mathsf{id}_X) \circ u(X^{[k+1]}) \arrow[d, Rightarrow, "\mathsf{refl}"'] & F(\mathsf{id}_X) \circ u^-(X^{[k]}) \arrow[r, "\mathsf{L}^F_{u^-(X^{[k+1]})}"'] & u^-(X^{[k]}) \arrow[u, "u^-_{\equiv}(X)"'] \\
u'(\mathrm{fact}_L(\mathsf{id}_X)) \arrow[r, bend left, "\mathsf{cp}_{u'}(\mathrm{fact}_L(\mathsf{id}_X), \mathrm{fact}_R(\mathsf{id}_X))"] \arrow[r, Rightarrow, bend right, "\mathsf{refl}"'] & u'(\mathrm{fact}_R(\mathsf{id}_X)) \arrow[u, Rightarrow, "\mathsf{refl}"'] &
\end{tikzcd}$$

To show that the zig-zagged triangle commutes, show all other 2-cells of the diagram and the exterior of the diagram commute. The left rectangle commutes by definition of $p(X^{[k+1]})$. The right square commutes by definition of $u^{-+}_{\equiv}(X)$. The bottom lozenge commutes by Lemma 2.0.8 applied to $\mathrm{fact}_{=}(X)$. The exterior of the diagram commutes as $u(X^{[k+1]}) \equiv u^-(X^{[k]})$ and $u_{=}(X) \equiv u^-_{\equiv}(X)$, and $(\mathsf{L}^F_{u(X^{[k+1]})})^{-1} \cdot \mathsf{L}^F_{u(X^{[k+1]})} = \mathsf{refl}$. Pasting together these four cells gives a filler $p_{=}(X)$ for the zig-zagged triangle, as required. ☐

Corollary 3.3.3. *If $\mathcal{C}$ is a wild category and $\mathcal{C}$ factors orthogonally to $\mathrm{Hom}(\mathcal{C})$, then for each k there is a* unique *k-ary composition operation. It takes a sequence of k composable morphisms*

$$C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{k-1}} C_k$$

to a morphism in $\hom_{\mathcal{C}}(C_0, C_k)$ subject to sending $(\mathsf{id}_X, \cdots \mathsf{id}_X)$ to id_X.

Below gf is written as shorthand for $g \circ f$ for composable morphisms g, f.

Corollary 3.3.4. *If $\mathcal{C}$ is a wild category and $\mathcal{C}$ factors orthogonally to $\mathrm{Hom}(\mathcal{C})$, then there is an associator $\alpha_{f,g,h} : (hg)f = h(gf)$ for $\mathcal{C}$ which both has a filler for the pentagon equation*

$$\begin{tikzcd}
& (ih)(gf) \arrow[dr, "\alpha_{gf,h,i}"] & \\
((ih)g)f \arrow[ur, "\alpha_{f,g,ih}"] \arrow[d, "\alpha_{g,h,i}\ f"'] & & i(h(gf)) \\
(i(hg))f \arrow[rr, "\alpha_{f,hg,i}"'] & & i((hg)f) \arrow[u, "i\ \alpha_{f,g,h}"']
\end{tikzcd}$$

and a filler for the K_5-associahedron

$$\begin{tikzcd}
 & & ((ji)(hg))f \arrow[rrr] & & & (((ji)h)g)f \\
 & (ji)((hg)f) \arrow[ur] & (j(i(hg)))f \arrow[u] \arrow[r] & (j((ih)g))f \arrow[r] & ((j(ih))g)f \arrow[ur] & \\
(ji)(h(gf)) \arrow[ur] \arrow[rrrrr] & & & & & ((ji)h)(gf) \arrow[uu] \\
 & j(i((hg)f)) \arrow[uu] \arrow[r] & j((i(hg))f) \arrow[uu] \arrow[r] & j(((ih)g)f) \arrow[uu] & (j(ih))(gf) \arrow[uu] \arrow[ur] & \\
j(i(h(gf))) \arrow[uu] \arrow[ur] \arrow[rrr] & & & j((ih)(gf)) \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

Proof. The associator is the unique 3-ary composition operation.

The two 4-ary operations which take f, g, h, i to $((ih)g)f$ and $i(h(gf))$ are equal by paths $\alpha_{f,g,ih} \cdot \alpha_{gf,h,i}$ and $\alpha_{g,h,i}f \cdot \alpha_{f,hg,i} \cdot i\alpha_{f,g,h}$; these paths are equal as they are between members of a contractible type.

Similarly, the associator and pentagon define S^2-many 5-ary operations as depicted above, and these are all equal, which corresponds to a filler for K_5 existing. □

3.3.1 Detecting some coherent $(\infty, 1)$-categories

Defining the notion of fully coherent $(\infty, 1)$-category is open in HoTT, but it's still possible to identify examples. Theorem 3.3.2 gives a large swath of examples of such identifiably fully coherent $(\infty, 1)$-categories (although the hypotheses of the theorem typically need a parametricity axiom to be satisfiable). This can be turned into a definition of a class of $(\infty, 1)$-categories.

A **reflexive quiver** $\mathcal{C}$ consists of a type $\mathrm{Ob}(\mathcal{C})$ of objects, for each $X, Y : \mathrm{Ob}(\mathcal{C})$ a type $\hom_{\mathcal{C}}(X, Y)$ of arrows from X to Y, and for each $X : \mathrm{Ob}(\mathcal{C})$ an arrow $\mathsf{id}_X : \hom_{\mathcal{C}}(X, X)$.

Proposition 3.3.5. *Let $\mathcal{C}$ be a reflexive quiver, and suppose that for $k = 1, 2$ there is a unique k-ary composition operator for $\mathcal{C}$. Then there is a unique way to extend the reflexive quiver structure of $\mathcal{C}$ to a wild category structure.*

Proof. The type which needs be shown contractible is the type of quadruples

$$\begin{aligned}
c &: \prod_{X \xrightarrow{f} Y \xrightarrow{g} Z : \mathcal{C}} \hom(X, Z) \\
l &: \prod_{X \xrightarrow{f} Y : \mathcal{C}} c(X \xrightarrow{f} Y \xrightarrow{\mathsf{id}} Y) = f \\
r &: \prod_{X \xrightarrow{f} Y : \mathcal{C}} c(X \xrightarrow{\mathsf{id}} X \xrightarrow{f} Y) = f \\
p &: \prod_{X : \mathcal{C}} l(X \xrightarrow{\mathsf{id}} X) = r(X \xrightarrow{\mathsf{id}} X)
\end{aligned}$$

and this is evidently equivalent to the type of sextuples

$$
\begin{aligned}
c &: \prod_{X \xrightarrow{f} Y \xrightarrow{g} Z : \mathcal{C}} \hom(X, Z) & c_= &: \prod_{X:\mathcal{C}} c(X \xrightarrow{\mathsf{id}} X \xrightarrow{\mathsf{id}} X) = \mathsf{id}_X \\
l &: \prod_{X \xrightarrow{f} Y : \mathcal{C}} c(X \xrightarrow{f} Y \xrightarrow{\mathsf{id}} Y) = f & l_= &: \prod_{X:\mathcal{C}} c_=(X) = l(X \xrightarrow{\mathsf{id}} X) \\
r &: \prod_{X \xrightarrow{f} Y : \mathcal{C}} c(X \xrightarrow{\mathsf{id}} X \xrightarrow{f} Y) = f & r_= &: \prod_{X:\mathcal{C}} c_=(X) = r(X \xrightarrow{\mathsf{id}} X)
\end{aligned}
$$

via

$$l(X \xrightarrow{\mathsf{id}} X) = r(X \xrightarrow{\mathsf{id}} X) \cong \sum_{c_= : c(X \xrightarrow{\mathsf{id}} X \xrightarrow{\mathsf{id}} X) = \mathsf{id}_X} l(X \xrightarrow{\mathsf{id}} X) = c_= \times c_= = r(X \xrightarrow{\mathsf{id}} X)\,.$$

The type of pairs $(c, c_=)$ as in the first row of this sextuple is precisely the type of a 2-ary composition operator, which there is a unique instance of.

Note the characterisation of the path type of 1-ary composition operators $(u, u_=) = (v, v_=)$ as equivalent to the type of pairs $q : \prod_{X_0 \xrightarrow{x} X_1 : \mathcal{C}} u(X_0 \xrightarrow{x} X_1) = v(X_0 \xrightarrow{x} X_1)$ and $q_= : \prod_{X : \mathrm{Ob}\,\mathcal{C}} u_=(X) = q(X \xrightarrow{\mathsf{id}} X) \cdot v_=(X)$.

The type of pairs $(l, l_=)$ as in the second row is, by this characterisation, equivalent to the path type $((X \xrightarrow{f} Y) \mapsto c(X \xrightarrow{f} Y \xrightarrow{\mathsf{id}} Y), X \mapsto c_=(X)) = (f \mapsto f, X \mapsto \mathsf{refl}_{\mathsf{id}_X})$ of 1-ary composition operators. As there is a unique 1-ary composition operation, this type is contractible.

The type of pairs $(r, r_=)$ as in the third row is equivalent to the path type $((X \xrightarrow{f} Y) \mapsto c(X \xrightarrow{\mathsf{id}} X \xrightarrow{f} Y), X \mapsto c_=(X)) = (f \mapsto f, X \mapsto \mathsf{refl}_{\mathsf{id}_X})$ so this type is contractible.

Hence the type of sextuples $(c, c_=, l, l_=, r, r_=)$ is equivalently a dependent sum of contractible types, so is contractible. □

Corollary 3.3.6. *For a reflexive quiver $\mathcal{C}$, "$\mathcal{C}$ has a wild category structure for which $\mathcal{C}$ factors orthogonally to $\hom(\mathcal{C})$" is a (homotopy-)proposition.*

This implies that many $(\infty, 1)$-categories can be identified from their underlying reflexive quivers.

Definition 3.3.7. A **parametrically coherent locally $\mathcal{U}$-small ∞-category** is a reflexive quiver $\mathcal{C}$ with $\hom_{\mathcal{C}}$ landing in $\mathcal{U}$-small types for which there exists a (necessarily unique) wild category structure on $\mathcal{C}$ for which $\mathcal{C}$ factors orthogonally to $\mathsf{el}_{\mathcal{U}}$. A **parametrically coherent functor** $\mathcal{C} \to \mathcal{D}$ is a reflexive quiver homomorphism (i.e. wild functor).

3.3.2 Constructing arrow wild categories

Let $\mathcal{C}$ be a wild category which factors orthogonally to $\mathrm{Hom}(\mathcal{C})$. We want to construct the arrow category $\mathcal{C}^{\to}$ of $\mathcal{C}$.

- Objects of $\mathcal{C}^{\to}$ are triples $(A_0, A_1, A_\to)$ for A_0, A_1 objects of $\mathcal{C}$ and $A_\to : \hom_{\mathcal{C}}(A_0, A_1)$.

- Morphisms $A \to B$ of $\mathcal{C}^{\to}$ are triples $(f_0, f_1, f_\to)$ for $A_0 \xrightarrow{f_0} B_0$, $A_1 \xrightarrow{f_1} B_1$ and $f_\to : f_1 \circ A_\to = B_\to \circ f_0$:

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "A_\to"'] & B_0 \arrow[d, "B_\to"] \\ A_1 \arrow[r, "f_1"'] \arrow[ur, dashed, Rightarrow, "f_\to"] & B_1 \end{tikzcd}$$

- The identity morphism $A \xrightarrow{\mathsf{id}_A} A$ of $\mathcal{C}$ is given by $(\mathsf{id}_{A_0}, \mathsf{id}_{A_1}, \mathsf{l}_{A_\to} \cdot \mathsf{r}^{-1}_{A_\to})$.

- The composite of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ as in

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "A_\to"'] & B_0 \arrow[r, "g_0"] \arrow[d, "B_\to"] & C_0 \arrow[d, "C_\to"] \\ A_1 \arrow[r, "f_1"'] \arrow[ur, dashed, Rightarrow, "f_\to"] & B_1 \arrow[r, "g_1"'] \arrow[ur, dashed, Rightarrow, "g_\to"] & C_1 \end{tikzcd}$$

  is given by $(g_0 \circ f_0, g_1 \circ f_1, p)$ where p is the path composite of

$$\begin{aligned} & (g_1 \circ f_1) \circ A_\to \overset{\alpha^{-1}}{=} g_1 \circ (f_1 \circ A_\to) \\ \overset{g_1 \circ f_\to}{=} \; & g_1 \circ (B_\to \circ f_0) \overset{\alpha}{=} (g_1 \circ B_\to) \circ f_0 \\ \overset{g_\to \circ f_0}{=} \; & (C_\to \circ g_0) \circ f_0 \overset{\alpha^{-1}}{=} C_\to \circ (g_0 \circ f_0) \end{aligned}$$

- The left unitor requires showing that the composite of the diagram below-left is the diagram below-right

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "A_\to"'] & B_0 \arrow[r, "\mathsf{id}"] \arrow[d, "B_\to"] & B_0 \arrow[d, "B_\to"] \\ A_1 \arrow[r, "f_1"'] \arrow[ur, dashed, Rightarrow, "f_\to"] & B_1 \arrow[r, "\mathsf{id}"'] \arrow[ur, dashed, Rightarrow, "\mathsf{l}_{A_\to} \cdot \mathsf{r}^{-1}_{A_\to}"] & B_1 \end{tikzcd} \qquad \begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "A_\to"'] & B_0 \arrow[d, "B_\to"] \\ A_1 \arrow[r, "f_1"'] \arrow[ur, dashed, Rightarrow, "f_\to"] & B_1 \end{tikzcd}$$

  Using the obvious paths $\mathsf{l}_{f_0} : \mathsf{id}_{B_0} \circ f_0 = f_0$ and $\mathsf{l}_{f_1} : \mathsf{id}_{B_1} \circ f_1 = f_1$, this reduces to showing that the five squares of paths below commute:

$$
\begin{tikzcd}
(\mathsf{id}_{B_1} \circ f_1) \circ A_{\to} \arrow[r, Rightarrow, "\mathsf{l}_{f_1 \circ A_{\to}}"'] \arrow[d, Rightarrow, "\alpha^{-1}"'] & f_1 \circ A_{\to} \arrow[d, equal] \\
\mathsf{id}_{B_1} \circ (f_1 \circ A_{\to}) \arrow[r, Rightarrow, "\mathsf{l}_{f_1 \circ A_{\to}}"'] \arrow[d, Rightarrow, "\mathsf{id}_{B_1} \circ f_{\to}"'] & f_1 \circ A_{\to} \arrow[d, Rightarrow, "f_{\to}"'] \\
\mathsf{id}_{B_1} \circ (B_{\to} \circ f_0) \arrow[r, Rightarrow, "\mathsf{l}_{B_{\to} \circ f_0}"'] \arrow[d, Rightarrow, "\alpha"'] & B_{\to} \circ f_0 \arrow[d, equal] \\
(\mathsf{id}_{B_1} \circ B_{\to}) \circ f_0 \arrow[r, Rightarrow, "\mathsf{l}_{B_{\to}} \circ f_0"'] \arrow[d, Rightarrow, "(\mathsf{l}_{B_{\to}} \cdot \mathsf{r}_{B_{\to}}^{-1}) \circ f_0"'] & B_{\to} \circ f_0 \arrow[d, equal] \\
(B_{\to} \circ \mathsf{id}_{B_0}) \circ f_0 \arrow[r, Rightarrow, "\mathsf{r}_{B_{\to}} \circ f_0"'] \arrow[d, Rightarrow, "\alpha^{-1}"'] & B_{\to} \circ f_0 \arrow[d, equal] \\
B_{\to} \circ (\mathsf{id}_{B_0} \circ f_0) \arrow[r, Rightarrow, "B_{\to} \circ \mathsf{l}_{f_0}"'] & B_{\to} \circ f_0
\end{tikzcd}
$$

The second and fourth of these squares commute for general path-algebra reasons. The first, third and fifth are all triangle equations for associativity, which are immediate consequences of that there's a unique 3-ary composition operator.

- The right unitor is constructed symmetrically to the left unitor.
- The unit coherence, instead of being constructed directly, is constructed by appeal to Proposition 3.0.4; this may result in replacing the right unitor (analogously to adjusting the data of an equivalence of categories to be an adjoint equivalence).

This is enough to construct the wild category $\mathcal{C}^{\to}$, but it remains to show that $\mathcal{C}^{\to}$ factors orthogonally to $\mathrm{Hom}(\mathcal{C}^{\to})$. We show that when $\mathcal{C}$ factors orthogonally to T (and $\mathrm{Hom}\,\mathcal{C}$), then $\mathcal{C}^{\to}$ does also (which is sufficient as the types T that $\mathcal{C}^{\to}$ factors orthogonally to are closed under dependent sum and path types by Proposition 2.0.5, and the hom-types of $\mathcal{C}^{\to}$ are constructed from those of $\mathcal{C}$ in this way).

Given a morphism $f : A \to B$ in $\mathcal{C}^{\to}$, a factorisation of f consists of both a factorisation of f_1

$$
\begin{tikzcd}
A_0 \arrow[rr, "f_0"] \arrow[d, "A_{\to}"] & & B_0 \arrow[d, "B_{\to}"'] \\
A_1 \arrow[urr, Rightarrow, dashed, "f_{\to}"] \arrow[r, "f_{1l}"] \arrow[rr, bend right, "f_1"'] & X_1 \arrow[r, "f_{1r}"] & B_1
\end{tikzcd}
$$

and an 8-tuple consisting of X_0, g_0, g_1, g_2, three commutative squares/triangles, and a path witnessing the composite of those squares/triangles is the path $f_\to$:

$$\begin{tikzcd}
A_0 \arrow[rr, "f_0"] \arrow[dr, dashed, "g_0"] \arrow[ddr, "f_{1l}\circ A_\to"'] & & B_0 \arrow[dd, "B_\to"'] \\
& X_0 \arrow[ur, dashed, "g_2"] \arrow[d, dashed, "g_1"] & \\
& X_1 \arrow[r, "f_{1r}"] & B_1
\end{tikzcd}$$

The type of factorisations of f_1 is orthogonal to T, and types orthogonal to T are closed under dependent sums, so it only remains to show that the type of 8-tuples above is orthogonal to T.

Although in principle it's possible to show this directly, we instead prove it in the special case that $\mathcal{C}$ has pullbacks:

Lemma 3.3.8. *If $\mathcal{C}$ is a wild category which factors orthogonally to $\mathrm{Hom}(\mathcal{C})$ and T, and $\mathcal{C}$ has pullbacks, then $\mathcal{C}^\to$ factors orthogonally to T.*

A pullback of $A \to C \leftarrow B$ is defined as in the usual way as a representation of $\mathrm{hom}(-,A) \times_{\mathrm{hom}(-,C)} \mathrm{hom}(-,B)$.

Proof. The type of 8-tuples as described above is equivalent to an object X_0, a morphism from X_0 to $X_1 \times_{B_1} B_0$, a morphism from A_0 to X_0, and a path witnessing this composite $A_0 \to X_1 \times_{B_1} B_0$ corresponds to the commutative square $f_\to$. This is just the data of a factorisation of the morphism $u : A_0 \to X_1 \times_{B_1} B_0$ corresponding to $(f_{1l\circ A_\to}, f_0, f_\to)$, so this type of 8-tuples is equivalent to the type of factorisations of u, which is known to factor orthogonally to T. □

Remark 3.3.9. For deriving many parametricity results the notion '$\mathcal{C}$ factors orthogonally to $\langle T_i \rangle_{i:I}$' can be replaced by the weaker'for any objects A, C of $\mathcal{C}$ and any $i : I$, the composition map $c : (\sum_B \mathrm{hom}(A,B) \times \mathrm{hom}(B,C)) \to \mathrm{hom}(A,C)$ has $(T_i)^c$ be an equivalence'. We cannot do this weakening here: although $\mathcal{C}^\to$ will still be a wild category (and we can lift associativity etc. from that of $\mathcal{C}$) there is in general no way to conclude that $\mathcal{C}^\to$ factors orthogonally in this weak sense to T from that $\mathcal{C}$ does.

4 Obtaining orthogonality hypotheses from parametricity axioms

[References only sections 1, 2 and subsections 3.0.0 – 3.2.0.]

In the previous section it was established that many parametricity results follow from various hypotheses that '$\mathcal{C}$ factors orthogonally to T'. We now move our attention to how to obtain such hypotheses, and how they follow from some parametricity axioms.

Most categories of interest $\mathcal{C}$ are locally small, so for locally $\mathcal{U}$-small $\mathcal{C}$, for $\mathcal{C}$ to factor orthogonally to $\mathrm{Hom}(\mathcal{C})$ it is sufficient that $\mathrm{Fact}(f) \perp A$ for any $A : \mathcal{U}$ and any morphism f of $\mathcal{C}$.

Definition 4.0.1. A (typically large) type C is ($\mathcal{U}$-)**superconnected** iff for every $T : \mathcal{U}$, $C \perp T$. ("Maps into small types are constant.")

Example 4.0.2. If $\lozenge$ is a modality, every $\lozenge$-connected type is $\mathcal{U}_\lozenge$-superconnected.

In general, to get that a type D is $\mathcal{U}$-superconnected it often suffices to find a single $\mathcal{U}$-superconnected type I and show that there are lots of functions from I to D.

Lemma 4.0.3. *Suppose $\mathbb{I}$ is superconnected with elements $0, 1 : \mathbb{I}$, D is a type with $b : D$, and there's a function $f : \mathbb{I} \times D \to D$ such that $f(0, d) = b$ and $f(1, d) = d$ always hold. Then D is superconnected also.*

Proof. Immediate from Proposition 2.0.9. $\square$

Following on from this idea, the parametricity axioms will involve saying that some type I is $\mathcal{U}$-superconnected, and that there are 'enough' functions from I to $\mathcal{U}$.

4.1 Isomorphs and Realignment

In order to state the axioms, the following notions are needed for the universe of types $\mathcal{U}$. However as they are more generally useful, they are stated in the general case with any wild category $\mathcal{C}$ in place of $\mathcal{U}$.

Definition 4.1.1. For a wild category $\mathcal{C}$, an **isomorphism** is a morphism $A \xrightarrow{e} B$ such that for all objects X, the induced functions $e\circ : \mathrm{hom}(X, A) \to \mathrm{hom}(X, B)$ and $\circ e : \mathrm{hom}(B, X) \to \mathrm{hom}(A, X)$ are equivalences. $A \cong B$ denotes the type of morphisms which are isomorphisms.

For an object A of $\mathcal{C}$, an **isomorph** of A is an object B equipped with an isomorphism $e : A \cong B$; the type of isomorphs of A is denoted $\mathsf{Isos}_{\mathcal{C}}(A)$.

For a type φ, a wild category $\mathcal{C}$ has φ**-realignment** iff for every object A of $\mathcal{C}$, there is a specified section s^{φ}_A of the diagonal map $\mathsf{Isos}_{\mathcal{C}}(A) \to (\varphi \to \mathsf{Isos}_{\mathcal{C}}(A))$.

Lastly, a **realigning $\mathcal{U}$-family** for $\mathcal{C}$ is a family $(\varphi_p)_{p:\mathbb{P}}$ of $\mathcal{U}$-small types for which $\mathcal{C}$ has specified φ_p-realignment for all p and there are specified $0_{\mathbb{P}}, 1_{\mathbb{P}} : \mathbb{P}$ with $\varphi_{1_{\mathbb{P}}} \cong \mathbf{1}$, $\varphi_{0_{\mathbb{P}}} \cong \mathbf{0}_{\mathcal{U}}$ and $(\mathbf{0}_{\mathcal{U}} \to \mathrm{Ob}(\mathcal{C})) \cong \mathbf{1}$[7].

If $\mathcal{C}$ is univalent (for any objects A, B the function $(A = B) \to (A \cong B)$ sending refl_A to id_A is an equivalence) then the $\mathsf{Isos}_{\mathcal{C}}(A)$ all have a unique element, so $\mathcal{C}$ has φ-realignment for all φ and the entire notion trivialises.

Note that all wild categories $\mathcal{C}$ (with $(\mathbf{0}_{\mathcal{U}} \to \mathrm{Ob}(\mathcal{C})) \cong \mathbf{1}$) have $\mathbf{0}_{\mathcal{U}}$-realignment, via the constant function $\_ \mapsto (A, \mathsf{id}) : (\mathbf{0}_{\mathcal{U}} \to \mathsf{Isos}_{\mathcal{C}}(A)) \to \mathsf{Isos}_{\mathcal{C}}(A)$; all wild categories $\mathcal{C}$ uniquely have $\mathbf{1}$-realignment via evaluation at $() : \mathbf{1}$. A wild category

[7]This final condition is of course automatic if $\mathbf{0}_{\mathcal{U}} \cong \mathbf{0}$, which is almost always the case

$\mathcal{C}$ has $\mathbf{2}$-realignment iff $\mathcal{C}$ is univalent, so usually we care about φ-realignment only for propositions φ.

It is *not* the case that the type, $\mathsf{Isos}_{\mathcal{C}}(A)$ or whether $\mathcal{C}$ has φ-realignment, is invariant under equivalence of categories with $\mathcal{C}$. For example, for any set A with $a : A$, the indiscrete category with A objects and a unique morphism between any two objects has $\mathsf{Isos}_{\mathsf{Indisc}(A)}(a) \cong A$, and $\mathsf{Indisc}(2)$ has φ-realignment iff $\neg\varphi \vee \neg\neg\varphi$. In **??** it's noted that if $\mathbf{Set}$ has φ-realignment and $\mathcal{C}$ is a locally small category then the essential image of the yoneda embedding $\mathbf{y} : \mathcal{C} \to [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$ also has φ-realignment.

Lemma 4.1.2. *Suppose that $(\varphi_p)_{p:\mathbb{P}}$ is a realigning $\mathcal{U}$-family for $\mathcal{C}$. Then for all types T and all objects A of $\mathcal{C}$, $\mathbb{P} \perp T$ implies $\mathsf{Isos}_{\mathcal{C}}(A) \perp T$.*

Proof. Assume $\mathbb{P} \perp T$, let A be an object of $\mathcal{C}$, and let $\tau : \mathsf{Isos}_{\mathcal{C}}(A) \to T$. Let $s_A^{\varphi_p} : (\varphi \to \mathsf{Isos}_{\mathcal{C}}(A)) \to \mathsf{Isos}_{\mathcal{C}}(A)$ denote the section of diagonal map provided by φ_p-realignment. Consider any $B : \mathsf{Isos}_{\mathcal{C}}(A)$. Then

$$\tau(B) = \tau(s_A^{1_{\mathbb{P}}}(\lambda\_.B)) = \tau(s_A^{0_{\mathbb{P}}}(\lambda\_.B)) = \tau(s_A^{0_{\mathbb{P}}}(!_{\mathbf{0}_{\mathcal{U}} \to \mathsf{Isos}_{\mathcal{C}}(A)}))$$

where $!_{\mathbf{0}_{\mathcal{U}} \to \mathsf{Isos}_{\mathcal{U}}(A_i)}$ is the unique function of that type, the second equality is by $s_A^{1_{\mathbb{P}}}$ necessarily being evaluation, and the third equality is by $\varphi \mapsto \tau(s_A^{\varphi}(\lambda\_.B)) : \mathbb{P} \to T$ being constant. ☐

In particular if $\mathbb{P}$ is $\mathcal{U}$-superconnected then so are the $\mathsf{Isos}_{\mathcal{C}}(A)$.

We will see in §4.3 that $\mathsf{Isos}_{\mathcal{C}}(-)$ landing in $\mathcal{U}$-superconnected types is one of the main ingredients for showing that $\mathcal{C}$ factors orthogonally to $\mathsf{el}_{\mathcal{U}}$. This leads us to compute the types $\mathsf{Isos}_{\mathcal{C}}(-)$ for various categories $\mathcal{C}$.

Lemma 4.1.3. *If $\mathcal{C}$ is a wild category and has an associator $\alpha_{f,g,h} : (h \circ g) \circ f = h \circ (g \circ f)$ for all composable triples of morphisms f, g, h, then the following are equivalent for a morphism $A \xrightarrow{e} B$:*

1. *e is an isomorphism*
2. *For all objects X, the function $\circ e : \hom(B, X) \to \hom(A, X)$ is an equivalence*
3. *For all objects Y, the function $e\circ : \hom(Y, A) \to \hom(Y, B)$ is an equivalence*
4. *There are morphisms $B \xrightarrow{l} A$ and $B \xrightarrow{r} A$ such that $l \circ e = \mathsf{id}_A$ and $e \circ r = \mathsf{id}_B$*

Proof. By definition, (1) holds iff both (2) and (3) hold. We show that (2) and (4) are equivalent; that (3) and (4) are equivalent then follows from applying the same argument in the opposite category.

Assume (2). Define $B \xrightarrow{e'} A$ as $e' :\equiv (\circ e)^{-1}(\mathsf{id}_A)$. Then by construction $e' \circ e = (\circ e)^{-1}(\mathsf{id}_A) \circ e = \mathsf{id}_A$, so e' is left-inverse to e. Hence for any morphism

$A \xrightarrow{x} X$, $(x \circ e') \circ e = x \circ (e' \circ e) = x \circ \mathsf{id}_A = x$ so $\circ e'$ is inverse to $\circ e$. In particular, $e \circ e' = (\mathsf{id}_B \circ e) \circ e' = \mathsf{id}_B$, so e' is also a right-inverse to e. Hence (4).

Assume (4), with left-inverse l and right-inverse r to e. For any morphism $A \xrightarrow{x} X$, $x = x \circ \mathsf{id}_A = x \circ (e \circ r) = (x \circ e) \circ r$, so $\circ r$ is left-inverse to $\circ e$. Similarly for any morphism $B \xrightarrow{x} X$, $x = x \circ \mathsf{id}_B = x \circ (l \circ e) = (x \circ l) \circ e$, so $\circ l$ is right-inverse to $\circ e$. Hence $\circ e$ is an equivalence, so (2).

Lastly, (1), (2) and (3) are all clearly hprops; we show (4) is also. If e is an isomorphism, the data of a left inverse $(l : \mathrm{hom}(B, A), l_= : l \circ e = \mathsf{id}_A)$ is equivalent to the data $(l : \mathrm{hom}(B, A), l_= : l = (\circ e)^{-1}\mathsf{id}_A)$, which is contractible. Similarly for right inverses. □

Corollary 4.1.4. *If $\mathcal{C}$ is a wild category and has an associator $\alpha_{f,g,h} : (h \circ g) \circ f = h \circ (g \circ f)$ for all composable triples of morphisms f, g, h, then for any object X, $\mathsf{Isos}_{\mathcal{C}}(X) \cong \mathsf{Isos}_{\mathcal{C}^{\mathsf{op}}}(X)$.*

Proof. Isomorphisms $X \xrightarrow{e} Y$ out of X are in bijection with isomorphisms $X \xleftarrow{e^{-1}} Y$ into X. □

Let $p : \mathcal{E} \to \mathcal{C}$ be a wild functor. p is an **amnestic isofibration** iff p preserves isomorphisms and for every object E of $\mathcal{E}$ and isomorphism $f : pE \to C$ in $\mathcal{C}$, there is a unique isomorphism $\hat{f} : E \to \hat{C}$ which is sent to f by p.

Lemma 4.1.5. *Let $p : \mathcal{E} \to \mathcal{C}$ be an amnestic isofibration. For any object E of $\mathcal{E}$, $\mathsf{Isos}_{\mathcal{E}}(E) \cong \mathsf{Isos}_{\mathcal{C}}(pE)$.*

Proof. The equivalence is given by sending $E \xrightarrow{f} C$ to $pE \xrightarrow{pf} pC$; this is a function as p preserves isomorphisms. As p is an amnestic isofibration, the fibres of $p : \mathsf{Isos}_{\mathcal{E}}(E) \cong \mathsf{Isos}_{\mathcal{C}}(pE)$ are contractible, so this is an equivalence. □

Corollary 4.1.6. *Let $p : \mathcal{E} \to \mathcal{C}$ be an amnestic isofibration. If $(\varphi_p)_{p:\mathbb{P}}$ is a realigning $\mathcal{U}$-family for $\mathcal{C}$ (and $(\mathbf{0}_{\mathcal{U}} \to \mathrm{Ob}(\mathcal{E})) \cong \mathbf{1}$) then $(\varphi_p)_{p:\mathbb{P}}$ is also a realigning $\mathcal{U}$-family for $\mathcal{E}$.*

Typically, when $\mathcal{C}$ is some category, $\mathcal{E}$ is some category whose objects are objects of $\mathcal{C}$ with some extra structure and morphisms are morphisms of $\mathcal{C}$ preserving this extra structure, then the forgetful functor $p : \mathcal{E} \to \mathcal{C}$ will typically be an amnestic isofibration. For example:

Proposition 4.1.7. *The following functors between 1-categories are all amnestic isofibrations:*

- *The forgetful functor $T\mathsf{Alg} \to \mathcal{C}$ associated to any endofunctor or monad T on a 1-category $\mathcal{C}$*
- *The functor $\mathcal{C}^{\mathcal{D}} \to \mathcal{C}^{\mathrm{Ob}\,\mathcal{D}}$ given by composition with the inclusion $\mathrm{Ob}\,\mathcal{D} \hookrightarrow \mathcal{D}$ for any 1-categories $\mathcal{C}$, $\mathcal{D}$*
- *The forgetful functors $\mathbf{Pos} \to \mathbf{Set}$, $\mathbf{Top} \to \mathbf{Set}$ and $\mathbf{Met} \to \mathbf{Set}$*

- *For any functor $F : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$, the projection $\mathrm{elts}(F) \to \mathcal{C}$ from the category of elements of F is an amnestic isofibration*

As are the following wild functors between wild categories:

- *Any replete subcategory inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$ ($\mathcal{C}$ is a replete subcategory of $\mathcal{D}$ iff whenever $f : c \cong c'$ in $\mathcal{D}$ and $c \in \mathcal{C}$ then $c', f \in \mathcal{C}$ also)*

- *If G and F are both amnestic isofibrations, then so is the composite $\mathcal{E} \xrightarrow{G} \mathcal{D} \xrightarrow{F} \mathcal{C}$*

- *If the wild functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ is an amnestic isofibration and the wild categories $\mathcal{D}$ and $\mathcal{C}$ have associators then $\mathcal{D}^{\mathsf{op}} \xrightarrow{F^{\mathsf{op}}} \mathcal{C}^{\mathsf{op}}$ is an amnestic isofibration*

Proof. Each of these are immediate from direct calculation, using Corollary 4.1.4 for the final bullet. □

Lastly, let's provide an example of a nontrivial amnestic isofibration between wild categories.

Proposition 4.1.8. *Let $\mathcal{C}$ be a wild category which factors orthogonally to $\mathrm{hom}(\mathcal{C})$, and let $F : \mathcal{C} \to \mathcal{C}$ be a wild functor. Then the wild category $F\mathsf{Alg}$ of F-algebras exists, and the forgetful functor $F\mathsf{Alg} \to \mathcal{C}$ is an amnestic isofibration.*

Proof. The construction of $F\mathsf{Alg}$ is similar to the construction of $\mathcal{C}^{\to}$ from Section 3.3.2.

- Objects of $F\mathsf{Alg}$ are pairs of an object and a morphism $(A : \mathrm{Ob}\,\mathcal{C}, a : \mathrm{hom}(FA, A))$.

- Morphisms $(A, a) \to (B, b)$ are quadruples $(f : \mathrm{hom}(A, B), f' : \mathrm{hom}(FA, FB), f_= : f \circ a = b \circ f', f'_= : f' = Ff)$. The data of the second and fourth components is contractible, but stating in this way allows easier definitions of composition and unitors.

- The identity morphism on (A, a) is given by $(\mathsf{id}_A, \mathsf{id}_{FA}, \mathsf{l}_a \cdot \mathsf{r}_a^{-1}, (i^F_A)^{-1})$.

- The composite of morphisms $(A, a) \xrightarrow{(f, f', f_=, f'_=)} (B, b) \xrightarrow{(g, g', g_=, g'_=)} (C, c)$ has its first, second and third components given by the second, first and third components of the $\mathcal{C}^{\to}$-composition of the morphisms $(FA, A, a) \xrightarrow{(f', f, f_=)} (FB, B, b) \xrightarrow{(g', g, g_=)} (FC, C, c)$, and fourth component, lying in $g' \circ f' = F(g \circ f)$, given after path induction on $f'_=$ and $g'_=$ as the path in $Fg \circ Ff = F(g \circ f)$ from both being the unique 2-ary composition map over F from Theorem 3.3.2.

- Similarly, the left unitor, right unitor, and unit coherence for $F\mathsf{Alg}$ each have the first/second/third components of the path defined as the second/first/third components of the corresponding path in $\mathcal{C}^{\to}$, and have their fourth components given by (if necessary) path induction on the $f'_=$ and then uniqueness of the 1-ary or 0-ary composition map over F.

It isn't necessary to have explicit descriptions for the unitors or unit coherence, although one could in principle construct them explicitly by using Theorem 3.3.2 to get many coherence laws for composition in $\mathcal{C}$ and for F, and working it out by hand.

It is easy to see that the forgetful functor U from $F\mathsf{Alg}$ to $\mathcal{C}$ which selects the first components of objects and morphisms is a wild functor. Also if morphism e in $F\mathsf{Alg}$ has a left and right inverse, then this is preserved by U, so by Lemma 4.1.3 U preserves isomorphisms.

Now let $A \xrightarrow{f} B$ be an isomorphism in $\mathcal{C}$, and (A, a) be a F-algebra. We would like to show that this data has a unique lift to a morphism $\hat{f}$ with $U\hat{f} = f$ i.e. that the type of quadruples

$$(b : \hom(FB, B), f' : \hom(FA, FB), f_= : f \circ a = b \circ f', f'_= : f' = Ff)$$

is contractible. Path induction on $f'_= : f' = Ff$ shows this is equivalent to the type of pairs

$$(b : \hom(FB, B), f_= : f \circ a = b \circ Ff)\ .$$

As F preserves composition, F preserves having a left/right inverse, so preserves isomorphisms. Hence $\circ Ff$ is an equivalence, and this type is equivalent in turn to the type of pairs

$$(b : \hom(FB, B), f_= : (\circ Ff)^{-1}(f \circ a) = b)$$

which is evidently contractible. Hence U is an amnestic isofibration. □

Having described the $\mathsf{Isos}_{\mathcal{C}}(-)$ of many categories $\mathcal{C}$, let's briefly turn to φ-realignment. For a category $\mathcal{C}$ and type A, and $X : A \to \mathrm{Ob}(\mathcal{C})$ the type $\mathsf{Isos}_{\mathcal{C}^A}(X)$ is easily seen to be equivalent to $\prod_{a:A} \mathsf{Isos}_{\mathcal{C}}(X(a))$, but we don't know much about this type. In particular, having $\mathsf{Isos}_{\mathcal{C}}(X(a)) \perp T$ for each $a : A$ is in general insufficient to ensure that $\prod_{a:A} \mathsf{Isos}_{\mathcal{C}}(X(a)) \perp T$.

Fortunately, this is not the case with φ-realignment.

Proposition 4.1.9. *Let $\varphi : \mathcal{U}$, and suppose that for each $i : I$ that $\mathcal{C}_i$ is a wild category with (a specified) φ-realignment. Then $\prod_{i:I} \mathcal{C}_i$ has a φ-realignment.*

Proof. For any object X of $\prod_i \mathcal{C}_i$, note that $\mathsf{Isos}_{\prod_{i:I} \mathcal{C}_i}(X) \cong \prod_{i:I} \mathsf{Isos}_{\mathcal{C}_i}(X_i)$. Then the sections $(\varphi \to \mathsf{Isos}_{\mathcal{C}_i}(X_i)) \to \mathsf{Isos}_{\mathcal{C}_i}(X_i)$ combine to give a section $(\varphi \to \prod_{i:I} \mathsf{Isos}_{\mathcal{C}_i}(X_i)) \to \prod_{i:I} \mathsf{Isos}_{\mathcal{C}_i}(X_i)$. □

4.2 The Parametricity Axioms

There are three parametricity axioms we're interested in, which we will discuss in decreasing order of strength (and can each be stated independently):

Definition 4.2.1 (Univalent Parametricity Axiom)**.** The universe $\mathcal{U}$ satisfies the univalent parametricity axiom (UPA) iff $\mathcal{U}$ is $\mathcal{U}$-superconnected and univalent.

Definition 4.2.2 (Realignment Parametricity Axiom)**.** The universe $\mathcal{U}$ satisfies the realignment parametricity axiom (RPA) iff there is a specified realigning $\mathcal{U}$-family $(\varphi_p)_{p:\mathbb{P}}$ for $\mathcal{U}$ for which $\mathbb{P}$ is $\mathcal{U}$-superconnected.

Definition 4.2.3 (Isomorph Parametricity Axiom)**.** The universe $\mathcal{U}$ satisfies the isomorph parametricity axiom (IPA) iff the following hold:

1. $\mathcal{U}$ is $\mathcal{U}$-superconnected
2. For any type $I : \mathcal{U}$ and family $A : I \to \mathcal{U}$, $\prod_{i:I} \mathsf{Isos}_{\mathcal{U}}(A_i)$ is $\mathcal{U}$-superconnected

4.2.1 Implication between the Axioms

As mentioned above, these axioms are in descending order of strength.

Proposition 4.2.4. *The universe $\mathcal{U}$ satisfies UPA iff it is univalent and satisfies RPA.*

Proof. If $\mathcal{U}$ satisfies UPA, taking $\mathbb{P} = \mathcal{U}$ shows $\mathcal{U}$ satisfies RPA also: as the $\mathsf{Isos}_{\mathcal{U}}(A)$ are contractible, P-realignment holds for all types P.

Conversely suppose $\mathcal{U}$ is univalent and satisfies RPA via the type family $|-| : \mathbb{P} \to \mathcal{U}$. Then for any $T : \mathcal{U}$, $\tau : \mathcal{U} \to T$ and $B : \mathcal{U}$,

$$\tau(B) = \tau(B^{\mathbf{1}}) = \tau(B^{|1_{\mathbb{P}}|}) = \tau(B^{|0_{\mathbb{P}}|}) = \tau(B^{\mathbf{0}_{\mathcal{U}}}) = \tau(\mathbf{1})$$

where the middle equality holds via constancy of $\tau(B^{|-|}) : \mathbb{P} \to T$, so τ is constant. □

Proposition 4.2.5. *If the universe $\mathcal{U}$ satisfies RPA, then it satisfies IPA.*

Proof. For each $\varphi : \mathbb{P}$, let $s^{\varphi}_{A} : (|\varphi| \to \mathsf{Isos}_{\mathcal{U}}(A)) \to \mathsf{Isos}_{\mathcal{U}}(A)$ be the section provided by $|\varphi|$-realignment.

Firstly, let $I : \mathcal{U}$ and $A : I \to \mathcal{U}$. To show $\prod_{i:I} \mathsf{Isos}_{\mathcal{U}}(A_i)$ is $\mathcal{U}$-superconnected, consider any $B : \prod_{i:I} \mathsf{Isos}_{\mathcal{U}}(A_i)$, $T : \mathcal{U}$ and $\tau : \prod_{i:I} \mathsf{Isos}_{\mathcal{U}}(A_i) \to X$. Then

$$\tau(B) = \tau(\lambda i.B_i) = \tau(\lambda i.s^{1_{\mathbb{P}}}_{A_i}(\lambda \_.B_i)) = \tau(\lambda i.s^{0_{\mathbb{P}}}_{A_i}(\lambda \_.B_i)) = \tau(\lambda i.s^{0_{\mathbb{P}}}_{A_i}(!_{\mathbf{0}_{\mathcal{U}} \to \mathsf{Isos}_{\mathcal{U}}(A_i)}))$$

where $!_{\mathbf{0}_{\mathcal{U}} \to \mathsf{Isos}_{\mathcal{U}}(A_i)}$ is the unique function of that type, the second equality is by $s^{1_{\mathbb{P}}}_{A_i}$ necessarily being evaluation, and the third equality is by $\varphi \mapsto \tau(\lambda i.s^{\varphi}_{A_i}(\lambda \_.B_i)) : \mathbb{P} \to T$ being constant.

Now, to show $\mathcal{U}$ is $\mathcal{U}$-superconnected, let $T : \mathcal{U}$ and $\tau : \mathcal{U} \to T$. For any $B : \mathcal{U}$,

$$\tau(B) = \tau(B^{\mathbf{1}}) = \tau(B^{|1_{\mathbb{P}}|}) = \tau(B^{|0_{\mathbb{P}}|}) = \tau(B^{\mathbf{0}_{\mathcal{U}}}) = \tau(\mathbf{1})$$

where the first equality holds by constancy of $(B', e : B \cong B') \mapsto \tau(B') : \mathsf{Isos}_{\mathcal{U}}(B) \to T$, the middle equality holds by constancy of $\tau(B^{|-|}) : \mathbb{P} \to T$, and the last equality holds by constancy of $(B', e : \mathbf{1} \cong B') \mapsto \tau(B') : \mathsf{Isos}_{\mathcal{U}}(\mathbf{1}) \to T$, so τ is constant. □

4.2.2 A road not taken

There are several different possible formulations of the parametricity axiom, and it's not obvious how to find one with the "right" strength.

In a different direction, most models that we will consider arise from a stronger version of SRPA, where $\mathcal{U}$ is the subuniverse of some larger universe $\mathcal{V}$ consisting of the $\lozenge$-modal types. This could be stated as an axiom e.g. "the universe $\mathcal{V}$ has Prop-realignment types and a sufficiently well-behaved and nontrivial modality $\lozenge$" and state all of the results about $\mathcal{V}_\lozenge$ instead of for $\mathcal{U}$. Although mathematically sound and sufficient, this excludes some models of interest (such as realisability models that come with one impredicative universe only, or if the type theory lacks the HITs/QITs to construct $\lozenge$) and shifts some of the focus from $\mathcal{U}$, the universe we care about, to $\mathcal{V}$, which is introduced to be a technical tool.

4.3 Bridges

We now move on to a notion of a "bridge" which will be used to show that many categories $\mathcal{C}$ factor orthogonally to $\mathsf{el}_\mathcal{U}$. The idea is that for a $\mathcal{U}$-superconnected type $\mathbb{I}$, if there are enough functions from $\mathbb{I}$ to factorisations of a morphism f, then $\mathrm{Fact}(f)$ should be $\mathcal{U}$-superconnected too.

Definition 4.3.1. Let $(\mathbb{I}, 0_\mathbb{I}, 1_\mathbb{I})$ be a bipointed type. A wild category $\mathcal{C}$ is **$\mathbb{I}$-bridgeable** iff every $i : \mathbb{I}$ and morphism $C \xrightarrow{f} D$ is equipped with a factorisation $B_i(f) : \mathrm{Fact}(f)$ of f, subject to the requirements that the left factor of $B_i(f)$ is an isomorphism if $i = 0$ or $f = \mathsf{id}$, and the right factor of $B_i(f)$ is an isomorphism if $i = 1$ or $f = \mathsf{id}$.

Each B_i will provide a section of the object component of the composition functor $\mathcal{C}^{\bullet\to\bullet\to\bullet} \to \mathcal{C}^{\bullet\to\bullet}$. No requirement that this section acts on morphisms or is functorial is imposed (although this can be useful and occurs in most examples). Of these functorial factorisations of $\mathcal{C}$, B_0 is initial and B_1 is terminal.

Diagramatically, this situation looks like a method of interpolating between the right and left factorisations of f, as below:

$$\begin{tikzcd}
 & & C \arrow[d, dash, "\cong"] \arrow[dr, "f"] & \\
B_0(f) = & C \arrow[ur, "\mathsf{id}"] \arrow[r, "l_0"] & X_0 \arrow[r, "r_0"] & D \\
B_S(f) = & C \arrow[r, "l_S"] & X_S \arrow[r, "r_S"] & D \\
B_1(f) = & C \arrow[r, "l_1"] \arrow[dr, "f"'] & X_1 \arrow[r, "r_1"] \arrow[d, dash, "\cong"] & D \\
 & & D \arrow[ur, "\mathsf{id}"'] &
\end{tikzcd}$$

Theorem 4.3.2. *Let $\mathcal{C}$ be a wild category which is $(\mathbb{I}, 0_{\mathbb{I}}, 1_{\mathbb{I}})$-bridgeable, and $\langle T_j \rangle_{j:J}$ a type family that includes $\hom(\mathcal{C})$. If for all $j : J$, $\mathbb{I} \perp T_j$ and for all objects A of $\mathcal{C}$, $\mathsf{Isos}_{\mathcal{C}}(A) \perp T_j$ and $\mathsf{Isos}_{\mathcal{C}^{\mathsf{op}}}(A) \perp T_j$, then $\mathcal{C}$ factors orthogonally to $\langle T_j \rangle_{j:J}$.*

Proof. The theorem will follow by verifying the hypotheses for Lemma 3.1.11. Note that by Proposition 2.0.5, we may assume that $\langle T_j \rangle_{j:J}$ includes $\langle f = g \rangle_{f,g:\hom(c,c'),c,c':\mathrm{Ob}_{\mathcal{C}}}$. First, for any

Let $C \xrightarrow{f} E$ be a morphism of $\mathcal{C}$ and $B : \mathbb{I} \to \mathrm{Fact}(f)$ witness $\mathbb{I}$-bridgeability. Write $B_i(f) \equiv (p_i : f = C \xrightarrow{l_i} X_i \xrightarrow{r_i} E)$. Let $\tau : \mathrm{Fact}(f) \to T_j$.

Firstly, the following three equalities shall be established in turn

$$\tau(\mathrm{fact}_L(f)) = \tau B_1(f) = \tau B_0(f) = \tau(\mathrm{fact}_R(f)) \ .$$

To show $\tau(\mathrm{fact}_L(f)) = \tau B_1(f)$, define $M_L : \mathsf{Isos}_{\mathcal{C}^{\mathsf{op}}}(E) \to \mathrm{Fact}(f)$ by

$$M_L(A, e : A \cong E) = (\mathsf{invpath}_{((\circ e)^{-1} f) \circ e = f} : f = C \xrightarrow{(\circ e)^{-1} f} A \xrightarrow{e} E) : \mathsf{Isos}_{\mathcal{C}^{\mathsf{op}}}(E) \to \mathrm{Fact}(f)$$

and note that $\tau \circ M_L : \mathsf{Isos}_{\mathcal{C}^{\mathsf{op}}}(E) \to T_j$ is constant and that there are paths $p_1 : M_L(E, \mathsf{id}_E) = \mathrm{fact}_L(f)$ and $p_2 : M_L(B_1(f)_{middle,right}) = B_1(f)$, hence $\tau(\mathrm{fact}_L(f)) = \tau(B_1(f))$.

To show $\tau B_1(f) = \tau B_0(f)$, note that $i \mapsto \tau B_i(f) : \mathbb{I} \to T_j$ is constant, and apply this to $1_{\mathbb{I}}, 0_{\mathbb{I}}$.

To show $\tau B_0(f) = \tau(\mathrm{fact}_R(f))$, define $M_R : \mathsf{Isos}_{\mathcal{C}}(C) \to \mathrm{Fact}(f)$ by

$$M_R(A, e : C \cong A) = (\mathsf{invpath}_{e \circ ((e \circ)^{-1} f) = f} : f = C \xrightarrow{e} A \xrightarrow{(e \circ)^{-1} f} E) : \mathsf{Isos}_{\mathcal{C}}(C) \to \mathrm{Fact}(f)$$

and note that $\tau \circ M_R : \mathsf{Isos}_{\mathcal{C}}(C) \to T_j$ is constant and that $p_3 : M_R(B_0(f)_{middle,left}) = B_0(f)$ and $p_4 : M_R(C, \mathsf{id}_C) = \mathrm{fact}_R(f)$, hence $\tau(B_0(f)) = \tau(\mathrm{fact}_R(f))$.

This creates a path $\tau^{=} : \tau(\mathrm{fact}_L(f)) = \tau(\mathrm{fact}_R(f))$ as this composite:

$$\begin{tikzcd}
\tau(\mathrm{fact}_L(f)) \arrow[d, "\mathsf{ap}_\tau(p_1^{-1})"] & \tau(B_1(f)) \arrow[r, "\mathsf{cp}_{\tau B_-(f)}(-,-)"] & \tau(B_0(f)) \arrow[d, "\mathsf{ap}_\tau(p_3^{-1})"] & \tau(\mathrm{fact}_R(f)) \\
\tau M_L(E, \mathsf{id}_E) \arrow[r, "\mathsf{cp}_{\tau M_L}(-,-)"'] & \tau M_L(B_1(f)_{middle,right}) \arrow[u, "\mathsf{ap}_\tau(p_2)"'] & \tau M_R(B_0(f)_{middle,left}) \arrow[r, "\mathsf{cp}_{\tau M_R}(-,-)"'] & \tau M_R(C, \mathsf{id}_C) \arrow[u, "\mathsf{ap}_\tau(p_4)"']
\end{tikzcd} .$$

In order to apply Lemma 3.1.11 and deduce the theorem, it remains to prove that when f is id_C that $\tau^{=}$ is $\mathsf{ap}_\tau(\mathrm{fact}_{=}(C))$. Note that this is automatic when $\langle T_j \rangle_j$ only contains sets.

Let $\mathrm{isofact}(C)$ be the type of factorisations of id_C for which both the left morphism and right morphism are isomorphisms. Define $\tau' : \mathrm{isofact}(C) \to T_j$ as the restriction of τ. As being an isomorphism is a property, this is a subtype of $\mathrm{Fact}(\mathsf{id}_C)$. By Proposition 3.1.7, composition is associative, so when s, r are morphisms with $r \circ s = \mathsf{id}$, s is an isomorphism iff r is (by the usual proof), and s, r uniquely determine each other.

In particular, this means that M_L, $B_-(\mathsf{id}_C)$ and M_R all factor through the inclusion of $\mathrm{isofact}(C)$ into $\mathrm{Fact}(\mathsf{id}_C)$. Denote these restricted functions by M_L^r, $B_-(\mathsf{id}_C)^r$ and M_R^r. Additionally $M_R^r : \mathsf{Isos}_{\mathcal{C}}(C) \to \mathrm{isofact}(C)$ is an equivalence, so $\mathrm{isofact}(C) \perp T_j$.

This lets us show that the previous diagram is equal to the one below, wherein $\langle q, \_\rangle$ is the path sent to q by the inclusion $\mathrm{isofact}(C) \hookrightarrow \mathrm{Fact}(\mathsf{id}_C)$.

$$\begin{tikzcd}
\tau'(\mathrm{fact}_L(\mathsf{id}_C), _) \arrow[d, "\mathsf{ap}_{\tau'}\langle p_1, _\rangle^{-1}"] & & \tau'(B_1(\mathsf{id}_C)^r) \arrow[r, "\mathsf{cp}_{\tau' B_-(\mathsf{id}_C)^r}(_,_)"] & \tau'(B_0(\mathsf{id}_C)^r) \arrow[d, "\mathsf{ap}_{\tau'}\langle p_3, _\rangle^{-1}"'] & & \tau'(\mathrm{fact}_R(\mathsf{id}_C), _) \\
\tau'(M_L^r(E, \mathsf{id}_E)) \arrow[rr, "\mathsf{cp}_{\tau' M_L}(_,_)"'] & & \tau' M_L^r(B_1(\mathsf{id}_C)_{m,r}) \arrow[u, "\mathsf{ap}_{\tau'}\langle p_2, _\rangle"'] & \tau' M_R^r(B_0(\mathsf{id}_C)_{m,l}) \arrow[rr, "\mathsf{cp}_{\tau' M_R^r}(_,_)"'] & & \tau' M_R^r(C, \mathsf{id}_C) \arrow[u, "\mathsf{ap}_{\tau'}\langle p_4, _\rangle"]
\end{tikzcd}$$

Additionally, $\mathsf{ap}_{\tau}(\mathrm{fact}_=(C)) = \mathsf{ap}_{\tau'}\langle \mathrm{fact}_=(C), \_\rangle$, so it's sufficient to show the composite path above equals $\mathsf{ap}_{\tau'}\langle \mathrm{fact}_=(C), \_\rangle$. As $\mathrm{isofact}(C) \perp T_j$, τ' is constant, so there is a path $q : \tau' = \lambda\_.t$ for some $t : T_j$. Induction on q reduces this to proving $\mathsf{refl}_t = \mathsf{refl}_t$. □

Corollary 4.3.3. *Assume $\mathcal{U}$ satisfies RPA via $(\varphi_p)_{p:\mathbb{P}}$, and let $\mathcal{C}$ be a locally $\mathcal{U}$-small wild category which is $(\mathbb{I}, 0_{\mathbb{I}}, 1_{\mathbb{I}})$-bridgeable for $\mathcal{U}$-superconnected $\mathbb{I}$. If $(\varphi_p)_{p:\mathbb{P}}$ is a realigning $\mathcal{U}$-family for $\mathcal{C}$, then $\mathcal{C}$ factors orthogonally to $\mathsf{el}_{\mathcal{U}}$.*

Proof. Immediate from Theorem 4.3.2 and Lemma 4.1.2. □

Typically, showing that a wild category $\mathcal{C}$ factors orthogonally to T has been reduced into two parts: showing that $\mathsf{Isos}_{\mathcal{C}}(A) \perp T$, which has been covered previously for many wild categories $\mathcal{C}$; and showing that T is $\mathbb{I}$-bridgeable for some appropriate $\mathbb{I}$.

4.3.1 Obtaining bridgeable 1-categories

Let A be a set. An **A-ary pullback** is a limit of a diagram whose shape $\Delta A * 1$ has A objects together with a terminal object $\top$ and only identity morphisms and morphisms into $\top$. For example, here are the diagram shapes for 0, 1, 2 and 3-ary pullbacks:

$$\top \qquad \begin{tikzcd} \top \\ \bullet \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} & \top & \\ \bullet \arrow[ur] & & \bullet \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & \top & \\ \bullet \arrow[ur] & \bullet \arrow[u] & \bullet \arrow[ul] \end{tikzcd}$$

Lemma 4.3.4. *Let $\mathbb{I}$ be a strictly bipointed set and $\mathcal{C}$ be a 1-category which for all $i : \mathbb{I}$ has all $(i = 1_{\mathbb{I}})$-ary pullbacks. Then $\mathcal{C}$ is $\mathbb{I}$-bridgeable.*

Proof. Let $C \xrightarrow{f} D$ be a morphism of $\mathcal{C}$. For each $i : \mathbb{I}$, let φ be the proposition $i = 1_{\mathbb{I}}$. Let $J_\varphi : (\varphi * \mathbf{1}) \to \mathcal{C}$ be the diagram for a φ-ary pullback where we take the φ nonterminal objects to C, the terminal object to D, and the nonidentity morphisms to f. Note there is a cone from C to J_p which takes the morphisms $C \to C$ to be id_C and the morphisms $C \to D$ to be f.

J_φ has a limit $\lim J_\varphi$. The cone from C to J_φ gives a morphism $C \xrightarrow{c_\varphi} \lim J_\varphi$ by the universal property of limits. The projection map at the terminal object of the diagram J_φ gives a morphism $\lim J_\varphi \xrightarrow{\pi_\varphi} D$. The composite of these two morphisms is f, so this defines a factorisation of f for each φ. In particular, this defines a function $i \mapsto (C \xrightarrow{l_i} X_i \xrightarrow{r_i} D) : \mathbb{I} \to \mathrm{Fact}(f)$ by $(C \xrightarrow{l_i} X_i \xrightarrow{r_i} D) = (C \xrightarrow{c_{i=1_\mathbb{I}}} \lim J_{i=1_\mathbb{I}} \xrightarrow{\pi_{i=1_\mathbb{I}}} D)$.

In case i is $1_\mathbb{I}$, the factorisation $C \xrightarrow{l_1} X_1 \xrightarrow{r_1} D$ has that l_1 is an isomorphism (as 1-ary pullbacks select the source). In case i is $0_\mathbb{I}$, the factorisation $C \xrightarrow{l_0} X_0 \xrightarrow{r_0} D$ has that r_0 is an isomorphism (as 0-ary pullbacks select the target). □

4.3.2 Obtaining bridgeable wild categories

The basic idea for obtaining bridgeable wild categories $\mathcal{C}$ is the same as for 1-categories: if enough subfinite wide pullbacks exist then $\mathcal{C}$ is bridgeable. To do this, the wild category under consideration is taken to be associative, and extra care is taken when defining limits.

For this subsubsection, fix a wild category $\mathcal{C}$ and an associator $\alpha_{f,g,h} : (h \circ g) \circ f = h \circ (g \circ f)$ for all composable triples of morphisms $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$ in $\mathcal{C}$.

Let I be a set, T be an object of $\mathcal{C}$, and $D_i \xrightarrow{d_i} T$ be a morphism for each $i : I$. For an object X of $\mathcal{C}$, a **cone** with tip X over this data consists of a morphism $\pi_\top : \hom(X, T)$ and for each $i : I$ both a morphism $\pi_i : \hom(X, D_i)$ and a path $\pi_i^= : d_i \circ \pi_i = \pi_\top$. Denote the collection of cones with tip X as $\mathsf{Cone}_{T,D,d}(X)$, or $\mathsf{Cone}(X)$ when unambiguous. For a morphism $f : Y \to X$ and a cone $(\pi_\top, \pi_-, \pi_-^=)$ with tip X, there is a cone $(f^*\pi_\top, f^*\pi_-, f^*\pi_-^=)$ with tip Y which is defined by $f^*\pi_\top :\equiv \pi_\top \circ f$, $f^*\pi_i :\equiv \pi_i \circ f$ and $f\pi_i^= :\equiv \alpha^{-1}_{f,\pi_i,d_i} \cdot (\pi_-^= \circ f) : d_i \circ (\pi_i \circ f) = \pi_\top \circ f$. A **$I$-ary pullback** of this data consists of an object P and a cone π with tip P such that for all objects X the function $f \mapsto f^*\pi : \hom(X, P) \to \mathsf{Cone}(X)$ is an equivalence.

Proposition 4.3.5. *Let $\mathbb{I}$ be a strictly bipointed set. If for all $i : \mathbb{I}$, $\mathcal{C}$ has all $(i = 1_\mathbb{I})$-ary pullbacks, then $\mathcal{C}$ is $\mathbb{I}$-bridgeable.*

Proof. Let $C \xrightarrow{f} D$ be a morphism of $\mathcal{C}$. For any $i : \mathbb{I}$, let φ be the proposition $i = 1_\mathbb{I}$.

Consider cones over the diagram with φ morphisms into D, each given by $C \xrightarrow{f} D$. This has a limit (P, p). Note there is a cone with tip C given by $c :\equiv (f, (\mathsf{id}_C)_{\_:\varphi}, (\mathsf{r}_f)_{\_:\varphi})$.

As (P, p) is a limit, there is a unique morphism $l : \hom(C, P)$ with $w : l^*(p) = c$. Write the components of w as $w_\top : p_\top \circ l = f$, and for $u : \varphi$, $w_u : p_u \circ l = \mathsf{id}_C$ and $w_u^= : \alpha^{-1}_{l,p_u,f} \cdot (p_u^= \circ l) = \mathsf{r}_f$. Then $(w_\top : f = C \xrightarrow{l} P \xrightarrow{p_\top} D)$ is a factorisation of f, parametrised by $i : \mathbb{I}$.

To deduce $\mathcal{C}$ is $\mathbb{I}$-bridgeable, it remains to show that if $i = 0_{\mathbb{I}}$ then $p_\top$ is an isomorphism and if $i = 1_{\mathbb{I}}$ then l is an isomorphism.

Suppose that $i = 0_{\mathbb{I}}$. Then the composite of the equivalences

$$\hom(X, P) \xrightarrow{f \mapsto f^*(p)} \mathsf{Cone}(X) \xrightarrow{-_\top} \hom(X, D)$$

(using that $(i = 0_{\mathbb{I}})$ is empty in the second case) is $\circ p_\top$, so the functions $\circ p_\top$ are all equivalences, and by Lemma 4.1.3 $p_\top$ is an isomorphism.

Suppose that $i = 1_{\mathbb{I}}$. Then there is a unique $u : \varphi.\ w_u$ witnesses that l has a left inverse p_u, so (by associativity) $\circ p_u$ is one-sided inverse to $\circ l$, and l is an isomorphism if p is. The function $(c_\top, c_u, c_u^{=}) \mapsto c_u : \mathsf{Cone}(X) \to \hom(X, C)$ is an equivalence, by forgetting $(c_\top, c_u^{=})$ in the singleton type of $\mathsf{id}_C \circ c_u$. Then the composite of equivalences

$$\hom(X, P) \xrightarrow{f \mapsto f^*(p)} \mathsf{Cone}(X) \xrightarrow{-_u} \hom(X, D)$$

is $\circ p_u$, so the functions $\circ p_u$ are all equivalences, and by Lemma 4.1.3 p_u and hence l are isomorphisms. □

4.4 Various results we can now deduce

Proposition 4.4.1. *Suppose $\mathcal{U}$ satisfies IPA. If $\mathcal{C}$ is a complete category which admits an amnestic isofibration to $\mathbf{Set}^X$ for some $X : \mathcal{U}$ then $\mathcal{C}$ factors orthogonally to $\mathsf{el}_{\mathcal{U}}$. This includes the following categories:*

- **Set**, **Pos**, **Top**, **Met**[8]
- *$\mathcal{C}^{\mathcal{D}}$ for any $\mathcal{C}$ in this list and any category $\mathcal{D}$*
- *Any Grothendieck topos (presented as a reflective subcategory of a presheaf category)*
- *Any category of algebras (models of an algebraic theory)*
- *Any reflective subcategory of any category in this list*

Proof. Follows straightforwardly from Theorem 4.3.2 using Lemma 4.3.4 and completeness to obtain bridges and using Lemma 4.1.5 and IPA to obtain the $\mathsf{Isos}_{\mathcal{C}}(-) \perp T$.

Showing that each item in the list has an amnestic isofibration to some $\mathbf{Set}^X$ follows from Proposition 4.1.7. □

Proposition 4.4.2. *Suppose $\mathcal{U}$ satisfies UPA. If $\mathcal{C}$ is a complete univalent category then $\mathcal{C}$ factors orthogonally to $\mathsf{el}_{\mathcal{U}}$. This includes the following categories:*

[8] where **Met** uses short i.e. distance-nonincreasing morphisms and distances are valued in lower reals

Proof. Follows straightforwardly from Theorem 4.3.2 using Lemma 4.3.4 and completeness to obtain bridges and using univalence to obtain $\mathsf{Isos}_{\mathcal{C}}(A) \cong \mathbf{1}$ hence the $\mathsf{Isos}_{\mathcal{C}}(-) \perp T$. □

Proposition 4.4.3. *Suppose $\mathcal{U}$ satisfies UPA. Any univalent category with all filtered colimits factors orthogonally to $\mathcal{U}$. In particular, this includes any finitely*[9] *accessible category.*

Proof. This follows from Theorem 4.3.2 using that univalent categories $\mathcal{C}$ have $\mathsf{Isos}_{\mathcal{C}}(A) \perp T$ automatically by virtue of $\mathsf{Isos}_{\mathcal{C}}(A) \cong 1$, and bridges given by Lemma 4.3.4 using that φ-ary pushouts are directed colimits when φ is a proposition. □

Proposition 4.4.4. *If $\mathcal{U}$ satisfies RPA, then the following wild categories factor orthogonally to $\mathsf{el}_{\mathcal{U}}$:*

- *$\mathcal{U}^A$ and $(\mathcal{U}^A)^{\mathsf{op}}$ for any type A*
- *The wild category of (co)algebras of any wild functor $F : \mathcal{U}^A \to \mathcal{U}^A$ (assuming, in the coalgebra case, that $\mathcal{U}$ has pushouts)*

Proof. Let $(\varphi_p)_{p:\mathbb{P}}$ be the realigning $\mathcal{U}$-family provided by RPA. $\mathcal{U}^A$ and $(\mathcal{U}^A)^{\mathsf{op}}$ also have this be a realigning $\mathcal{U}$-family by Proposition 4.1.9 and Proposition 4.1.7.

$\mathcal{U}^A$ has wide pushouts and wide pullbacks, constructed in the usual way, hence are in particular $\mathbb{P}$-bridgeable. (Note that in $\mathcal{U}$ the I-ary pushout of $C \xrightarrow{f_i} B_i$ can be given by coequaliser of two morphisms $C \to C + \sum_i B_i$, and that coequaliser and $+$ can be cosntructed from pushout and $\mathbf{0}_{\mathcal{U}}$.) Hence by Theorem 4.3.2 $\mathcal{U}^A$ and $(\mathcal{U}^A)^{\mathsf{op}}$ factor orthogonally to $\mathsf{el}_{\mathcal{U}}$.

By Proposition 4.1.8, the category of algebras of a wild functor $F : \mathcal{U}^A \to \mathcal{U}^A$ admits an amnestic isofibration to $\mathcal{U}^A$. Similarly for the category of coalgebras (after applying opposites twice, using Proposition 4.1.7).

The wild category of F-algebras has its limits computed as in $\mathcal{U}^A$ and the opposite of the wild category of F-coalgebras has its limits computed as in $(\mathcal{U}^A)^{\mathsf{op}}$. The proofs that their limits (in particular, wide pullbacks) can be constructed in this way is exactly as in the 1-categorical case, appealing to Theorem 3.3.2 in order to get all of the required coherence. Then Theorem 4.3.2 gives that these categories factor orthogonally to $\mathsf{el}_{\mathcal{U}}$. □

Proposition 4.4.5. *Suppose that $\mathcal{U}$ is a univalent universe and is closed under $+$. Then $\mathcal{U}$ satisfies UPA iff for all $A : \mathcal{U}$, the map $(a \mapsto \lambda X.\lambda f.f(a)) : A \to \prod_{X:\mathcal{U}}(A \to X) \to X$ is an equivalence.*

[9]Removing "finitely" seems to be nontrivial, both because of the lack of a constructive theory of sound limit doctrines as of writing and because φ-ary pushouts aren't necessarily κ-directed when κ includes infinite sets

Proof. Firstly suppose $\mathcal{U}$ satisfies UPA. Note that by the above, the wild category $\mathcal{U}$ factors orthogonally to $\mathsf{el}_{\mathcal{U}}$. Then $\prod_{X:\mathcal{U}}(A \to X) \to X$ is the type of transformations from the functor $-^A$ to $\mathsf{id}_{\mathcal{U}}$. All such transformations are natural by Proposition 3.1.4, so any $\alpha : \prod_{X:\mathcal{U}}(A \to X) \to X$ satisfies $\alpha_X(f) = \alpha_X(f \circ \mathsf{id}_A) = f(\alpha_A(\mathsf{id}_A))$ for all $f : A \to X$. This shows $\alpha \mapsto \alpha_A(\mathsf{id}_A)$ is inverse to $(a \mapsto \lambda X.\lambda f.f(a))$, so this is an equivalence.

Conversely, suppose that for all $A : \mathcal{U}$, the map $(a \mapsto \lambda X.\lambda f.f(a)) : A \to \prod_{X:\mathcal{U}}(A \to X) \to X$ is an equivalence. Let $c : \mathcal{U} \to A$ and $u : \mathcal{U}$ be arbitrary. Then define $\alpha : \prod_{X:\mathcal{U}}(A \to X) \to X$ as $\alpha_X(f) = f(c(\mathsf{isEquiv}(f) \times u))$. Note $\alpha_A(\mathsf{id}_A) = c(u)$. By assumption $\alpha_X(f) = f(\alpha_A(\mathsf{id}_A)) = f((u))$ always holds. In particular, for $f = \mathsf{inl} : A \to (A + 1)$,

$$\mathsf{inl}(c(0)) = \alpha_X(f) = f(c(u)) = \mathsf{inl}(c(u))$$

so $c(0) = c(u)$. As u was arbitrary, c is constant, so $\mathcal{U} \perp A$ for all $A : \mathcal{U}$. $\square$

4.5 Impredicative universes are almost parametric

For this subsection, suppose $\mathcal{U}$ is a universe closed under arbitrary (possibly-not-$\mathcal{U}$-small) products.

Recall the definitions $\Diamond A :\equiv \prod_{X:\mathcal{U}} X^A \to X$ and $\eta_A :\equiv \lambda a X f.f(a) : A \to \Diamond A$ from Section 3.2.2.

Lemma 4.5.1. *For an inhabited large type P, the following are equivalent:*

1. *P is $\mathcal{U}$-superconnected*
2. *η_P is constant*
3. *$\Diamond P \cong \mathbf{1}$*

Proof. (1) implies (3): let $p : P$ and $q_0, q_1 : \Diamond P$, then $q_0 = \mathsf{const}_{q_0}(p) = \mathsf{const}_{q_1}(p) = q_1$.

(3) implies (2) as functions into $\mathbf{1}$ are (uniquely) constant. In particular, if (3) then (2) is a hprop also.

(2) implies (1): constancy of η_P implies there is $c : \Diamond P$ for which for any $p, q : P$, $A : \mathcal{U}$ and function $f : P \to A$, $f(p) = \eta_P(p)(A)(f) = c(A)(f)$, so f is constant with value $c(A)(f)$. As P is inhabited, Lemma 2.0.2 implies $P \perp A$.

(1) and (3) are hprops by construction, and (2) is a hprop if true (via (3)) so is a hprop, hence these bi-implications are equivalences. $\square$

Suppose that there is a poset $(\mathsf{Prop}_\neg, \leq)$ with the following properties:

- $\leq$ is valued in $\mathcal{U}$-small propositions
- $\mathsf{Prop}_\neg$ is a complete lattice
- The only function $f : \mathsf{Prop}_\neg \to \mathsf{Prop}_\neg$ with $f(\bot) = \bot$ and $f(\top) = \top$ is id
- $\bot \neq \top$

One might try to define $\mathsf{Prop}_\neg$ to be $\{A : \mathcal{U} \mid A \cong (A \to \mathbf{0}) \to \mathbf{0}\} / \leftrightarrow$, but this set might not form a complete lattice. If there is a type $\mathsf{Prop}_\mathcal{U}$ of $\mathcal{U}$-small propositions which satisfies propositional extensionality (such as if $\mathcal{U}$ is univalent), we can take $\mathsf{Prop}_\neg = \{\varphi : \mathsf{Prop}_\mathcal{U} \mid \varphi = \neg\neg\varphi\}$ and $a \leq b$ iff $a \to b$.

Theorem 4.5.2. $\mathsf{Prop}_\neg$ *is not not* $\mathcal{U}$*-superconnected.*

Proof. Suppose towards contradiction that $\neg(\eta_{\mathsf{Prop}_\neg}(\bot) = \eta_{\mathsf{Prop}_\neg}(\top))$. Let f be the composite

$$\mathsf{Prop}_\neg \xrightarrow{\eta_{\mathsf{Prop}_\neg}} \Diamond\mathsf{Prop}_\neg \xrightarrow{=\eta_{\mathsf{Prop}_\neg}(\top)} \mathsf{Prop}_\neg$$

and note that $f(\bot) = \bot$ so $f = \mathsf{id}$ by the third bullet above.

Hence $\mathsf{Prop}_\neg$ is isomorphic to the set of fixed points of $\eta_{\mathsf{Prop}_\neg} \circ (= \eta_{\mathsf{Prop}_\neg}(\top)) : \Diamond\mathsf{Prop}_\neg \to \Diamond\mathsf{Prop}_\neg$ so the set $\mathsf{Prop}_\neg$ is essentially $\mathcal{U}$-small. For notational convenience we assume it is.

This lets us construct the covariant endofunctor $P : \mathbf{Set} \to \mathbf{Set}$ with $P(X) = {\mathsf{Prop}_\neg}^X$ and for $f : X \to Y$, $P(f)(\varphi)(y) = \bigwedge_{x \in f^{-1}\{y\}} \varphi(x)$. The category of P-algebras is large-complete because $\mathbf{Set}$ is and the forgetful functor to $\mathbf{Set}$ creates all limits; the limit of the identity functor on P-$\mathbf{Alg}$ is an initial algebra $I \cong {\mathsf{Prop}_\neg}^I$. Then by Lawvere's fixed point theorem or Cantor's diagonal argument, $\neg : \mathsf{Prop}_\neg \to \mathsf{Prop}_\neg$ defined by $\neg p = \bigwedge_{p=\top} \bot$ has a fixed point, which is absurd (for p a fixed point of $\neg$, $p = \top$ iff $p \neq \top$).

Hence $\neg\neg(\eta_{\mathsf{Prop}_\neg}(\bot) = \eta_{\mathsf{Prop}_\neg}(\top))$. If $\eta_{\mathsf{Prop}_\neg}(\bot) = \eta_{\mathsf{Prop}_\neg}(\top)$ then for any $\tau : \mathsf{Prop}_\neg \to T$, define the function f as $f(b) = \tau(\varphi \wedge b)$ so that

$$\tau(\bot) = \tau(\varphi \wedge \bot) = f(\bot) = \eta_{\mathsf{Prop}_\neg}(\bot)_T(f) = \eta_{\mathsf{Prop}_\neg}(\top)_T(f) = f(\top) = \tau(\varphi \wedge \top) = \tau(\varphi)\,,$$

so τ is constant, so $\mathsf{Prop}_\neg$ is $\mathcal{U}$-superconencted. As actually it is only the case that $\neg\neg(\eta_{\mathsf{Prop}_\neg}(\bot) = \eta_{\mathsf{Prop}_\neg}(\top))$, we unconditionally get that $\mathsf{Prop}_\neg$ is not not $\mathcal{U}$-superconencted. $\square$

Question. Is $\mathsf{Prop}_\neg$ always $\mathcal{U}$-superconnected?

Corollary 4.5.3. *If* $\mathcal{U}$ *is a univalent universe then* $\neg\neg$*UPA(*$\mathcal{U}$*).*

Corollary 4.5.4. *Let* $T : \mathcal{U}$ *and suppose that for any* $a, b : T$ *that* $\neg\neg(a = b)$ *implies* $a = b$*. Then* $\mathsf{Prop}_\mathcal{U} \perp T$*.*

Proof. If $\tau : \mathsf{Prop}_\mathcal{U} \to T$ then as $\neg\neg(\eta_{\mathsf{Prop}}(\bot) = \eta_{\mathsf{Prop}}(\top))$, then $\neg\neg(\tau(\varphi) = \tau(\psi))$. Hence $\tau(\varphi) = \tau(\psi)$ and $\mathsf{Prop}_\mathcal{U} \perp T$. $\square$

In particular, this means that impredicative universes which are univalent or with realignment types automatically satisfy "enough" parametricity to get naturality in some restricted cases, such as the following:

Let $\mathbf{Set}^{sep}_{\neg\neg}$ denote the full subcategory of $\mathbf{Set}$ on '$\neg\neg$-separated' sets A satisfying $\forall a, b : A. \neg\neg(a = b) \to a = b$.

Corollary 4.5.5. *Assume realignment types exist, and let* $F, G : \mathbf{Set}^{sep}_{\neg\neg} \to \mathbf{Set}^{sep}_{\neg\neg}$ *be functors and* $\alpha_A : F(A) \to G(A)$ *for each* $\neg\neg$*-separated set* A*. Then* α *is natural.*

Proof. As realignment types exist, $\mathbf{Set}$ is $\mathsf{Prop}_{\mathcal{U}}$-gluable so its replete subcategory $\mathbf{Set}_{\neg\neg}^{sep}$ is also. As $\mathbf{Set}_{\neg\neg}^{sep}$ has all small limits and $\mathsf{Prop}_{\mathcal{U}} \perp T$ when T is $\neg\neg$-separated, Theorem 4.3.2 and Lemma 4.3.4 tell us $\mathbf{Set}_{\neg\neg}^{sep}$ factors orthogonally to each $\neg\neg$-separated set T. In particular, for $A, B : \mathbf{Set}_{\neg\neg}^{sep}$, $\mathrm{hom}(A, B)$ is $\neg\neg$-separated also, so Proposition 3.1.4 lets us conclude the result. □

5 Models of the axioms

The previous sections have established that parametricity axioms, in particular the realignment parametricity axiom (see Definition 4.2.2), are sufficient to derive powerful and useful parametricity results. However, and axiom is useless without models. As IPA is a consequence of RPA by Proposition 4.2.5 and UPA is equivalent to RPA plus univalence by Proposition 4.2.4, focus will primarily be on models of RPA.

Most models considered are constructed in three steps. Firstly, the base model $\mathcal{S}$ is extended into a presheaf model $[\mathcal{C}^{\mathsf{op}}, \mathcal{S}]$. Secondly, a universe $\mathcal{U}$ in $\mathcal{S}$ is lifted to a universe $\widetilde{\mathcal{U}}$ of $[\mathcal{C}^{\mathsf{op}}, \mathcal{S}]$; this is then shown to have realignment or univalence as appropriate. Thirdly, a (usually reflective) subuniverse $\widetilde{\mathcal{U}}_{\lozenge}$ of $\widetilde{\mathcal{U}}$ is produced, and $\widetilde{\mathcal{U}}_{\lozenge}$ is shown to satisfy the realignment/univalence and orthogonality conditions needed for it to model the parametricity axiom.

There are also two previously-studied models of type theory which are shown to model the Univalent Parametricity Axiom. In Cohesive HoTT [14], the internal universe of discrete ($\int$-modal) objects often satisfies UPA. Also, the cubical assemblies model [16] of HoTT validates UPA for the universe of modest assemblies.

5.1 Realignment Types in Presheaf Categories

To construct models of RPA in $\mathbf{PSh}(\mathcal{C})$, it is necessary that $\mathbf{PSh}(\mathcal{C})$ has φ-realignment types for enough propositions φ.

Let $\mathcal{U}$ be a universe in $\mathbf{Set}$ and $\mathcal{C}$ be a category in $\mathbf{Set}$. A universe $\widetilde{\mathcal{U}}$ in $\mathbf{PSh}(\mathcal{C})$ is defined following the Hofmann-Streicher lifting [7]. Recall that for a presheaf $F : \mathbf{PSh}(\mathcal{C})$, that the slice category $\mathbf{PSh}(\mathcal{C})/F$ is equivalent to the presheaf category $\mathbf{PSh}(\mathrm{elts}(F))$ for $\mathrm{elts}(F)$ the category of elements of F, whose objects are pairs $(c : \mathcal{C}, x : F(c))$ and morphisms $(c, x) \to (c', x')$ are morphisms $f : c \to c'$ such that $F(f)(x') = x$.

The presheaf $\widetilde{\mathcal{U}}$ is defined by

$$\begin{aligned}
\widetilde{\mathcal{U}}(c) &\coloneqq \mathrm{hom}_{\mathbf{Cat}}((\mathcal{C}/c)^{\mathsf{op}}, \mathcal{U}) \\
\widetilde{\mathcal{U}}(f : c \to c') &: \mathrm{hom}_{\mathbf{Cat}}((\mathcal{C}/c')^{\mathsf{op}}, \mathcal{U}) \to \mathrm{hom}_{\mathbf{Cat}}((\mathcal{C}/c)^{\mathsf{op}}, \mathcal{U}) \\
\widetilde{\mathcal{U}}(f : c \to c') &\coloneqq x \mapsto x \circ (\mathcal{C}/f)^{\mathsf{op}}
\end{aligned}$$

The type family $\mathsf{el}_{\widetilde{\mathcal{U}}}$, seen as a presheaf on $\mathrm{elts}(\widetilde{\mathcal{U}})$, is given by

$$\begin{aligned}\mathsf{el}_{\widetilde{\mathcal{U}}}(c : \mathcal{C}, x : (\mathcal{C}/c)^{\mathsf{op}} \to \mathcal{U}) &:= x(\mathsf{id}_c)\\ \mathsf{el}_{\widetilde{\mathcal{U}}}(f : (c,x) \to (c',x')) &: x'(\mathsf{id}_{c'}) \to x'(f)\\ \mathsf{el}_{\widetilde{\mathcal{U}}}(f : (c,x) \to (c',x')) &:= x'(f : \hom_{(\mathcal{C}/c')^{\mathsf{op}}}(f, \mathsf{id}_{c'}))\end{aligned}$$

noting in the bottom two lines that $x = x' \circ (\mathcal{C}/f)^{\mathsf{op}}$ and hence $x(\mathsf{id}_c) = x'(f)$.

This construction can be done for an arbitrary type family.

Proposition 5.1.1. *Let $\mathcal{U}$ be a universe in* **Set**, *$\mathbb{P}$ be a family of $\mathcal{U}$-small sets, and $\mathcal{C}$ be a category in* **Set**. *If in* **Set**, *$\mathcal{U}$ has φ-realignment for all $\varphi : \mathbb{P}$, then in* **PSh**$(\mathcal{C})$, *$\widetilde{\mathcal{U}}$ has φ-realignment for all $\varphi : \widetilde{\mathbb{P}}$.*

Proof. This is proved by Proposition 10 of Uemura [16], taking $\mathsf{Cof} = \mathbb{P}$. Uemura's axiom 10 holding for Cof and $\mathcal{U}$ is equivalent to the category $\mathcal{U}$ having φ-realignment for all $\varphi : \mathsf{Cof}$. The proof doesn't use axioms 6, 7 and 9, or that $\mathcal{U}$ is impredicative. □

This proof is valid constructively, which will be important for application to the category of assemblies later.

Corollary 5.1.2. *Let $\mathcal{U}$ be a universe in* **Set** *and $\mathcal{C}$ be a category in* **Set**. *Then $\widetilde{\mathbf{2}}$ is a realigning $\widetilde{\mathcal{U}}$-family for the universe $\widetilde{\mathcal{U}}$.*

Proof. Immediate from the preceding theorem, that $\mathcal{U}$ has both $\mathbf{0}_{\mathcal{U}}$- and $\mathbf{1}$-realignment, and that the elements $\mathsf{const}_0, \mathsf{const}_1$ of $\widetilde{\mathbf{2}}$ have $\mathsf{el}_{\widetilde{\mathbf{2}}}(\mathsf{const}_0) \cong \Delta\mathbf{0}_{\mathcal{U}} \cong \mathbf{0}_{\widetilde{\mathcal{U}}}$ and $\mathsf{el}_{\widetilde{\mathbf{2}}}(\mathsf{const}_1) \cong \Delta\mathbf{1} \cong \mathbf{1}$. □

Note that $\widetilde{\mathbf{2}}(c)$ can be identified with the set of sieves S on $c : \mathcal{C}$ for which a morphism f into c is either in S or not in S. In particular, if LEM holds in **Set**, $\widetilde{\mathbf{2}}$ is the subobject classifier Ω of **PSh**$(\mathcal{C})$.

It is also known that, if **Set** satisfies the axiom of choice, then any grothendieck topos over **Set** has φ-realignment for all propositions $\varphi : \Omega$ as shown in Corollary 4.3.3 of "Strict universes for Grothendieck topoi" [6], with the equivalence to the notion of realignment used herein given by Lemmas 5.1.5 and 5.1.6 of that paper.

5.2 Parametricity axioms in reflective subuniverses

5.2.1 Generalities regarding reflective universes

Proposition 5.2.1. *Let $\mathcal{U}$ be a universe and $(\varphi_p)_{p:\mathbb{P}}$ be a realigning $\mathcal{U}$-family of propositions for $\mathcal{U}$. Let $(A_i)_{i:I}$ be a $\mathcal{U}$-small family of inhabited $\mathcal{U}$-small types, and $\lozenge$ be the modality generated by nullifying the $(A_i)_{i:I}$. If $\mathbb{P}$ is $\lozenge$-connected then $\mathcal{U}_\lozenge$ satisifes RPA.*

Proof. As the subuniverse $\mathcal{U}_\lozenge \subseteq \mathcal{U}$ is closed under isomorphism, $\mathsf{Isos}_{\mathcal{U}_\lozenge}(A) \cong \mathsf{Isos}_{\mathcal{U}}(A)$, so if $\mathcal{U}$ has φ-realignment then so does $\mathcal{U}_\lozenge$.

Let φ be a proposition in $\mathcal{U}$. For each $i : I$, as A_i is inhabited, $(A_i \to \varphi) \leftrightarrow \varphi$, so $A_i \perp \varphi$. Hence φ is $\lozenge$-modal, so in $\mathcal{U}_\lozenge$, and $(\varphi_p)_{p:\mathbb{P}}$ is a realigning $\mathcal{U}$-family for $\mathcal{U}$.

Lastly, if $\lozenge\mathbb{P}$ is $\lozenge$-connected, then $\mathbb{P} \perp A$ for any $A : \mathcal{U}_\lozenge$. □

Corollary 5.2.2. *Let $\mathcal{U}$ be a univalent universe and $\lozenge$ be the modality generated by requiring that the $\mathcal{U}$-small types $(A_i)_{i:I}$ are all null. If all A_i are inhabited and there is some A_i which is a set containing unequal points $a_0 \neq a_1$ then UPA holds for $\mathcal{U}_\lozenge$.*

Proof. If $\mathcal{U}$ is univalent, $\mathcal{U}$ has φ-realignment for all types φ, so by the above theorem if $\mathsf{hProp}_{\mathcal{U}}$ is $\lozenge$-connected then $\mathcal{U}_\lozenge$ satisfies RPA. Corollary 2.0.10 implies that $\mathsf{hProp}_{\mathcal{U}}$ is $\lozenge$-connected and Proposition 4.2.4 gives that as $\mathcal{U}$ is univalent and satisfies RPA, it satisfies UPA. □

Also, note that $\mathcal{U}_\lozenge$ has HITs which eliminate into $\lozenge$-modal types:

Proposition 5.2.3. *Let $\mathcal{U}$ be a universe which has HITs (resp. QITs) and modality $\lozenge$ generated by requiring that the $\mathcal{U}$-small types $(A_i)_{i:I}$ are all null. Then the reflective subuniverse $\mathcal{U}_\lozenge$ has HITs (resp. QITs) restricted to have* small elimination *wherein the induction principle is restricted to land in type families in $\mathcal{U}_\lozenge$.*

Proof. To any HIT or QIT $\alpha(x : X)$, define a new HIT/QIT $\alpha'(x : X)$ which has the constructors of $\alpha(x : X)$ and the constructors of $\lozenge$. As the constructors include those of $\lozenge$, the produced type must be $\lozenge$-modal. Eliminating from α' requires eliminating the constructors of α and also proving that the output types are all $\lozenge$-modal, which (as a type can be $\lozenge$-modal in at most one way) is equivalent to eliminating the constructors of α, restricted to eliminating into types that are $\lozenge$-modal. □

Remark 5.2.4. It is not generally the case that the $\mathcal{U}_\lozenge$-HITs agree with the HITs: for example, if some A_i is $\mathbf{2}$ then the $\mathcal{U}_\lozenge$-inductive type for $\mathbf{1}+\mathbf{1}$ is $\lozenge\mathbf{2} \cong \mathbf{1}$, not $\mathbf{2}$. Parameters in $\mathcal{U}$ can also pose a problem: given $B : \mathcal{U}$, the inductive type α that has a single unary constructor $B \to \alpha$ will become $\lozenge B$ as a $\mathcal{U}_\lozenge$-type.

This naturally leads to the question of what circumstances are sufficient for $\mathcal{U}_\lozenge$-HITs to agree with $\mathcal{U}$-HITs.

Question. Suppose that $\lozenge$ preserves finite colimits. Which HITs/QITs is $\mathcal{U}_\lozenge$ closed under?

5.2.2 Models in presheaf categories

Let $\mathbf{FinSet}_{\neq\emptyset}$ be the category of finite nonempty sets. In what follows $\mathbf{FinSet}_{\neq\emptyset}$ could be replaced by any small category $\mathcal{C}$ such that $\mathcal{C}$ has finite products, all objects of $\mathcal{C}$ have a morphism from $\mathbf{1}$, some object has multiple distinct morphisms from $\mathbf{1}$, and for which "$f : 1 \to b$ factors through $f : a \to b$" is a decidable property of morphisms.

Theorem 5.2.5. *Let $\mathcal{U}$ be a universe of sets in $\mathbf{Set}$. Let $\lozenge$ be the modality in $\mathbf{PSh}(\mathbf{FinSet}_{\neq\emptyset})$ which nullifies $\mathbf{y}n$ for each $n \in \mathbf{FinSet}_{\neq\emptyset}$. Then the universe $\widetilde{\mathcal{U}}_\lozenge$ in $\mathbf{PSh}(\mathbf{FinSet}_{\neq\emptyset})$ satisfies RPA. Moreover, the universe $\Delta\mathcal{U}$ is included as a subuniverse of $\widetilde{\mathcal{U}}_\lozenge$.*

Proof. Firstly, we wish to apply Proposition 5.2.1 in $\mathbf{PSh}(\mathbf{FinSet}_{\neq\emptyset})$. $(\mathbf{y}n)_{n:\Delta\mathbf{FinSet}_{\neq\emptyset}}$ is a $\widetilde{\mathcal{U}}$-small family of inhabited types. Also, $(\mathsf{el}_{\widetilde{\mathbf{2}}}\varphi)_{\varphi:\widetilde{\mathbf{2}}}$ is a realigning $\widetilde{\mathcal{U}}$-family of propositions for $\widetilde{\mathcal{U}}$ by Corollary 5.1.2.

Instead of applying Proposition 5.2.1 directly to the family $(\mathsf{el}_{\widetilde{\mathbf{2}}}\varphi)_{\varphi:\widetilde{\mathbf{2}}}$, it will be applied to the subfamily $(\mathsf{el}_{\widetilde{\mathbf{2}}}(b = \mathbf{y}1))_{b:\mathbf{y2}}$ formed by restricting along the morphism $(- = \mathbf{y}1) : \mathbf{y2} \to \widetilde{\mathbf{2}}$ (which concretely can be defined as an element of $\hom_{\mathbf{Cat}}((\mathbf{FinSet}_{\neq\emptyset}/\mathbf{2})^{\mathsf{op}}, \mathbf{2})$ by sending morphisms $f : \mathbf{n} \to \mathbf{2}$ whose image includes 1 to 1 and other morphisms to 0).

As this family includes the propositions $\mathbf{0}_{\widetilde{\mathcal{U}}}$ and $\mathbf{1}$, and $\mathbf{y2}$ is $\lozenge$-connected (on account of being in the generating family of nullified types), $\widetilde{\mathcal{U}}_\lozenge$ satisfies RPA.

To see that $\Delta\mathcal{U}$ is included in $\widetilde{\mathcal{U}}_\lozenge$, it suffices that $\prod_{A:\Delta\mathcal{U}}\prod_{C:\Delta\mathbf{FinSet}_{\neq\emptyset}} \mathbf{y}C \perp A$. Using that dependent products over ΔX are equivalent to products over X, this can be checked for each A and C individually.

For any set A in $\mathcal{U}$ and any object C of $\mathbf{FinSet}_{\neq\emptyset}$, we see in $\mathbf{PSh}(\mathbf{FinSet}_{\neq\emptyset})$ that naturally in $D : \mathbf{FinSet}_{\neq\emptyset}$:

$$(\Delta A)(D) \equiv (\Delta A)(C\times D) \cong \hom(\mathbf{y}(C\times D), \Delta A) \cong \hom(\mathbf{y}C\times\mathbf{y}D, \Delta A) \cong (\Delta A)^{\mathbf{y}C}(D)$$

with these isomorphisms composing to the D-component of the diagonal map $\Delta A \to (\Delta A)^{\mathbf{y}C}$, so $\mathbf{y}C \perp \Delta A$. □

It was noted in Proposition 2.0.3 that orthogonality properties that LEM implies that $\perp$ is trivial, and so a nontrivial universe $\mathcal{U}$ cannot satisfy both RPA and LEM. However, it is still possible for two universes to coexist with a smaller one satisfying LEM or AC contained in a larger one satisfying RPA:

Proposition 5.2.6. *Let $\mathcal{U}$ be a universe in $\mathbf{Set}$. If $\mathcal{U}$ contains a classifier for $\mathcal{U}$-small subobjects, satisfies LEM, satisfies RPA, or satisfies AC, then so does $\Delta\mathcal{U}$ in $\mathbf{PSh}(\mathbf{FinSet}_{\neq\emptyset})$.*

Proof. Note that the diagonal functor $\Delta : \mathbf{Set} \to \mathbf{PSh}(\mathcal{C})$ preserves indexed (co)products, 2, $\exists$, $\forall$, $\to$, and so on. As the properties "$\mathcal{U}$ contains a classifier for $\mathcal{U}$-small subobjects", "$\mathcal{U}$ satisfies LEM", "$\mathcal{U}$ satisfies RPA" and "$\mathcal{U}$ satisfies AC" of a family of sets $\mathcal{U}$ can all be phrased in terms of these connectives preserved by Δ, if these properties held for $\mathcal{U}$ in $\mathbf{Set}$ then they also hold for $\Delta\mathcal{U}$ in $\mathbf{PSh}(\mathbf{FinSet}_{\neq\emptyset})$. $\square$

Lastly, for illustration purposes, we give a description of the $\widetilde{\mathcal{U}}_\lozenge$-small types. These are precisely the types which are both $\widetilde{\mathcal{U}}$-small and which are $\lozenge$-modal. A presheaf $F : \mathbf{FinSet}_{\neq\emptyset}^{\mathsf{op}} \to \mathbf{Set}$ is $\widetilde{\mathcal{U}}$-small iff F factors through the inclusion $\mathcal{U} \hookrightarrow \mathbf{Set}$. A presheaf $F : \mathbf{FinSet}_{\neq\emptyset}^{\mathsf{op}} \to \mathbf{Set}$ is $\lozenge$-modal iff it is constant, as nullifying at $\mathbf{y}n$ implies that the the function $\hom(\mathbf{y}k, F) \cong \hom(\mathbf{y}k \times \mathbf{y}n, F)$ induced by the projection $\mathbf{y}k \times \mathbf{y}n \to \mathbf{y}k$ is an isomorphism, which in particular implies $F(!_n) : F(\mathbf{1}) \cong F(n)$.

This in particular implies that the global points functor $\hom_{\mathbf{PSh}(\mathbf{FinSet}_{\neq\emptyset})}(\mathbf{1}, -) : \mathbf{PSh}(\mathbf{FinSet}_{\neq\emptyset}) \to \mathbf{Set}$ sends the internal category $\widetilde{\mathcal{U}}_\lozenge$ to a category equivalent to $\mathcal{U}$.

More generally, let I be a type and $(X_i)_{i:I}$ be a type family over I. For the type family X to be $\widetilde{\mathcal{U}}$-small (i.e. a pullback of $\mathsf{el}_{\widetilde{\mathcal{U}}}$), it is necessary and sufficient that for each $c : \hom(\mathbf{y}n, I)$ that $c^*X : \mathbf{PSh}(\mathbf{FinSet}_{\neq\emptyset})/\mathbf{y}n \cong \mathbf{PSh}(\mathbf{FinSet}_{\neq\emptyset}/n)$ is valued in $\mathcal{U}$-small sets (up to a family of isomorphisms parametrised in (n, c)). $(X_i)_{i:I}$ is moreover $\widetilde{\mathcal{U}}_\lozenge$-small iff for any k, n in $\mathbf{FinSet}_{\neq\emptyset}$ the function $X(n) \to X(n \times k) \times_{I(n\times k)} I(n)$ induced by the total space projection $X \to I$ and the product projection $n \times k \to n$ is an isomorphism.

5.2.3 Models in ∞-topoi

Recall that classically any $(\infty, 1)$-topos $\mathcal{E}$ has a presentation as a model category for which it models HoTT with strict univalent universes [13].

Let $\square$ be some 1-category with finite products for which $\hom(1, -)$ is faithful and conservative, and with an object $\mathbb{I}$ with two distinct points, such as the category $\mathbf{FinSet}_{\neq\emptyset}$ of nonempty finite sets.

Proposition 5.2.7. *Assume the axiom of choice. Let κ be an inaccessible cardinal and $\mathcal{E}$ be a locally λ-presentable $(\infty, 1)$-topos for $\lambda < \kappa$ regular. Then $\mathcal{E}$ has a morphism $U_\kappa^{\mathcal{E}}$ satisfying univalence and for which the relatively κ-presentable morphisms of $\mathcal{E}$ are the pullbacks of $U_\kappa^{\mathcal{E}}$.*

There is a presentation of $[\square^{\mathsf{op}}, \mathcal{E}]$ as a type theoretic model topos for which there are morphisms $\Delta U_\kappa^{\mathcal{E}}, U_\kappa^p, U_\kappa^{[\square^{\mathsf{op}}, \mathcal{E}]}$ with:

1. *$\Delta U_\kappa^{\mathcal{E}}$ is internally a pullback of U_κ^p along an embedding*
2. *U_κ^p is internally a pullback of $U_\kappa^{[\square^{\mathsf{op}}, \mathcal{E}]}$ along an embedding*
3. *$U_\kappa^{[\square^{\mathsf{op}}, \mathcal{E}]}$ is a univalent universe in the sense of [13, Definition 5.1] and is closed under Σ types, Π types, binary sum types and path types, and contains 0,1,2,$\mathbb{N}$ and S^n*
4. *U_κ^p satisfies UPA*
5. *$\Delta U_\kappa^{\mathcal{E}}$ is equivalent to $U_\kappa^{\mathcal{E}}$ as a $\mathcal{E}$-indexed family*

Proof. The main statement of this theorem and (3) are both contained in [13, Theorem 11.2].

(2) and (4) follow by taking U^p_κ to internally be the subuniverse on objects A for which $\mathbf{y}\mathbb{I} \perp A$ holds internally for $\mathbb{I}$ the interval object of $\square$. (It is not given that this subuniverse is reflective as $U_\kappa^{[\square^{\mathsf{op}},\mathcal{E}]}$ isn't stated to be closed under HITs.)

(5) follows from that the diagonal functor $\mathcal{E} \to [\square^{\mathsf{op}}, \mathcal{E}]$ is fully faithful and preserves finite limits. (1) follows from that $\Delta U^{\mathcal{E}}_\kappa$ is internally a pullback of $U_\kappa^{[\square^{\mathsf{op}},\mathcal{E}]}$ as Δ preserves relatively κ-presentable morphisms (and is pullback via an *embedding* as $\Delta U^{\mathcal{E}}_\kappa$ is univalent as Δ is a locally cartesian closed $\mathcal{E}$-indexed functor (hence preserves univalence of type families)), and that this embedding factors through the embedding from (2) using the same proof as the last paragraph of Theorem 5.2.5. ☐

Remark 5.2.8. The notion of universe used by Shulman [13] is relative to a chosen model category presenting type theory and corresponds to classifying a specified notion of fibred structure. It is stronger than the notion of universes used herein, although it seems plausible that U^p_κ and $\Delta U^{\mathcal{E}}_\kappa$ are universes in that stronger sense also.

5.2.4 Models via cohesive HoTT

Recall Axiom C2 of Cohesive HoTT [14]:

Definition 5.2.9 (Axiom C2 of Cohesive HoTT)**.** In the context of cohesive HoTT, axiom C2 is said to hold (for a universe $\mathcal{U}$) when there is a type family $R :: I \to \mathcal{U}$ such that a crisp type $A :: \mathcal{U}$ is discrete (qua $(-)_\flat : \flat A \to A$ being an equivalence) iff $R_i \perp A$ for all $i : I$ and, additionally, there is some $r : \prod_{i:I} R_i$ and some $i_0 : I$ and $r_0, r_1 : R_{i_0}$ such that R_{i_0} is a set and $r_0 \neq r_1$.

Theorem 5.2.10. *Let $\mathcal{U}$ be a univalent universe satisfying axiom C2. The subuniverse $\mathcal{U}_{\mathsf{discrete}}$ of types which are discrete satisfies UPA.*

Proof. Note that by definition (specifically, Definition 8.2 [14]), a type $A : \mathcal{U}$ is discrete iff $R_i \perp A$ for all $i : I$. Hence Proposition 5.2.1 immediately gives SRPA, and Proposition 4.2.4 gives UPA. ☐

The corresponding modality $\lozenge$ is referred to as “shape” $\int$ in cohesive HoTT.

Note that in particular for any $A :: \mathcal{U}$, $\flat A$ is in $\mathcal{U}_{\mathsf{discrete}}$, so the universe Disc of crisp discrete types is a subuniverse of $\mathcal{U}_{\mathsf{discrete}}$. The axiom C2 is also consistent with the axiom $\flat$LEM stating that LEM holds in Disc.

5.3 Realisability Models

It is unfortunately the case that the parametricity axioms fail in the category of assmeblies with the standard choice of universe.

Proposition 5.3.1. *For any nontrivial PCA $\mathbb{A}$, IPA is false in the impredicative universe* $\mathbf{PER}$ *in* $\mathbf{Asm}(\mathbb{A})$.

Proof. It is sufficient to find PERs A, B and a function $\mathsf{Isos}_{\mathbf{PER}}(A) \to B$ which isn't constant.

For a PER A, $\mathsf{Isos}_{\mathbf{PER}}(A)$ is the assembly described as follows: the underlying set consists of pairs (S, e) for S a PER and $e : |S| \cong |A|$ an isomorphism of PERs, and $a \in \mathbb{A}$ realises (S, e) iff $\mathsf{fst}\ a$ tracks $e : |S| \to |A|$ and $\mathsf{snd}\ a$ tracks $e^{-1} : |A| \to |S|$.

Let $A = \mathbb{A}$. Then a realises $(A, \mathsf{id}) \in \mathsf{Isos}_{\mathbf{PER}}(\mathbb{A})$ iff for all $x : \mathbb{A}$, $\mathsf{fst}\ a\ x = x$ and $\mathsf{snd}\ a\ x = x$. If a in addition realises $(B, e) \in \mathsf{Isos}_{\mathbf{PER}}(\mathbb{A})$, then as $\mathsf{fst}\ a$ tracks e, $e([x]) = [x]$, and as e is invertible, $(B, e) = (\mathbb{A}, \mathsf{id})$.

Now define R as the PER $a\ R\ b$ iff (a realises $(A, \mathsf{id}) \in \mathsf{Isos}_{\mathbf{PER}}(\mathbb{A})$ iff b realises $(A, \mathsf{id}) \in \mathsf{Isos}_{\mathbf{PER}}(\mathbb{A})$), and define $f : \mathsf{Isos}_{\mathbf{PER}}(\mathbb{A}) \to R$ by taking (B, e) to the equivalence class of any realiser of (B, e). This function is tracked by id, and is nonconstant as there are nonidentity isomorphisms out of $\mathbb{A}$ (such as $\mathbb{A} \cong (\mathbf{1} \times \mathbb{A})$ with the standard encoding). ☐

Remark 5.3.2. For some PCAs, there is stronger failure of parametricity results. Rummelhoff has shown [12] that for a well-chosen combinatory algebra $\mathbb{A}$, there are nonstandard numbers in $\prod_X X^X \to X^X$ in the PER semantics over $\mathbb{A}$, and direct computation shows that this PER is isomorphic to the assembly internally computed by $\prod_{X:\mathcal{U}} X^X \to X^X$ for $\mathcal{U}$ the universe of PERs over $\mathbb{A}$. In contrast, it is noted there that $\prod_{X:\mathcal{U}} X^X \to X^X$ is standard for PCAs with strong equality, such as $(\mathbb{N}, \varphi_{-}(-))$.

There are two potential solutions to this problem: either try to select another impredicative universe in $\mathbf{Asm}(\mathbb{A})$ which does satisfy the parametricity axiom, or modify the category of assemblies to get models where RPA does hold. We take the latter route.

Fortunately, we can still use assemblies to get a model of the parametricity axiom, and this model will still embed $\mathbf{PER}$.

Theorem 5.3.3. *The category of internal functors* $[\mathbf{FinSet}^{\mathsf{op}}_{\neq\emptyset}, \mathbf{Asm}(\mathbb{A})]$ *contains an impredicative universe $\mathcal{U}$ which satisfies RPA. Moreover, the externalisation of this internal category $\mathcal{U}$ is equivalent to the category* $\mathbf{PER}_{\mathbb{A}}$ *of PERs over $\mathbb{A}$.*

Proof. Immediate from Theorem 5.2.5 and the discussion immediately following, applied in the internal logic of $\mathbf{Asm}(\mathbb{A})$. ☐

Theorem 5.3.4. *In Uemura's cubical assemblies model [16] of realisability, UPA holds for the universe of modest assemblies $\mathcal{U}$ (and any reflective subuniverse thereof).*

Proof. Work internal to the category of assemblies. Let $\square$ be the category of cubes used in [16], $I = \nabla 2$ be the assembly with two points $0, 1$ with are both realised by all members of the PCA, and ΔI be the constant preasheaf valued in

I. As this is a constant presheaf, is discrete and has a fibration structure i.e. is interpreted as a type. For any $a, b : I$, $a = b$ is a ≤ 1-element PER so I is locally $\mathbf{PER}$-small; ΔI is similarly locally $\mathcal{U}$-small. Hence, by **??** and univalence of $\mathcal{U}$, to conclude UPA it suffices to show that ΔI is $\mathcal{U}$-superconnected.

To show this, we need that $[\![A : \mathcal{U} \vdash \Delta I \perp A]\!]$. As $\mathbf{PSh}(\square)/\mathcal{U} \cong \mathbf{PSh}(\int_{\square}\mathcal{U})$ (the category of elements of $\mathcal{U}$), this is equivalent to asking that, internal to $\mathbf{PSh}(\int_{\square}\mathcal{U})$, that $u^*\Delta I \perp \mathcal{E}$ where u^* is the pullback functor (and in particular $u^*\Delta I$ is the constant I-valued presheaf on $\int_{\square}\mathcal{U}$) and $\mathcal{E}$ is the family of $\mathcal{U}$-small types, indexed over $\mathcal{U}$. $\mathcal{E}$ in particular is valued in $\mathbf{PER}$ and $u^*\Delta I$ is well-supported, so by [16, Proposition 21] the diagonal function $\mathcal{E} \to (u^*\Delta I \to \mathcal{E})$ is an isomorphism internal to $\mathbf{PSh}(\int_{\square}\mathcal{U})$, hence $u^*\Delta I \perp \mathcal{E}$ as desired.

Lastly let $\mathcal{V}$ be a reflective subuniverse of $\mathcal{U}$. $\mathcal{U}$ is univalent so $\mathcal{V}$ is. Given $A : \mathcal{V}$ and $\mathcal{U} \perp A$ then, as $\mathcal{V}$ is a retract of $\mathcal{U}$, $\mathcal{V} \perp A$: for any $f : \mathcal{V} \to A$, $f = f \circ \lozenge \circ \iota = \mathsf{const}_a \circ \iota = \mathsf{const}_a$ for a $a : A$. $\square$

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