The Alpöge Proof of the Riemann Hypothesis

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Abstract

On November 20, 2026, Levent Alpöge announced a proof of the Riemann hypothesis that he and Claude had found together. He released the proof in Lean. The Lean and Mathlib communities subsequently certified the formal statement and proof.

A quantitative form of Landau's theorem on Mellin transforms reduces the zero-free half-plane $\mathrm{Re}\, s > 1/2$ for Dirichlet L-functions to a one-sided square-root bound for smoothed prime sums on suitable intervals. Alpöge adapts Stepanov's point-counting method over finite fields. The divisibility that successive theta lifts gain at inert primes plays the role of vanishing to high order at rational points. Each factor of an inert prime p contributes $\log p$ to the divisibility term in the slope inequality. For a sum of length X, a chain of about $\sqrt{X}$ theta lifts balances the errors in that inequality and gives the one-sided square-root bound that Landau's criterion requires.

We wrote this so that mathematicians can follow the argument from beginning to end and check it for themselves without consulting Lean. In the course of the reconstruction, we found what appears to be a shorter proof of a uniform bound in Galois cohomology (§17.4.3). That shortcut gives us reason to think the proof as a whole may admit a simpler argument.

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1 Introduction

This paper reconstructs in conventional mathematics the proof of the following theorem.

Theorem 1.1. *For every primitive Dirichlet character χ the function $L(s,\chi)$ does not vanish in the half-plane* $\operatorname{Re} s > \frac{1}{2}$. *In particular the Riemann zeta function does not vanish in* $\operatorname{Re} s > \frac{1}{2}$.

The proof is due to Levent Alpöge and Claude. On November 20, 2026, Alpöge released a four-page note [2] together with a proof of Theorem 1.1 formalized in Lean [23] on top of Mathlib [73], and on November 21, 2026, the Lean and Mathlib communities, with the endorsement of the Lean FRO and the Mathlib Initiative, certified that the formal statement and the formal proof are correct [72]. The note describes the argument as a descendant of Stepanov's method [70] for curves over finite fields. It credits Alpöge with the plan and with

the recognition that an auxiliary-function argument of Siegel, which appeared in mathematical work by researchers at OpenAI, is Stepanov's method with multiplicity one. Alpöge describes the turning point as the appearance of multiplicity two in an automorphic determinant on $\mathrm{O}(2,n)$, which suggested that further theta lifts could increase the multiplicity. The note credits Claude with the proposal that a chain of theta lifts of length about $\sqrt{x}$ should play the role of multiplicity, and with the arguments on automorphic forms on groups of higher rank. It thanks David Dohan, Ben Michel, Ralph Furman, Eric Price and Cora Vinther. This paper writes the proof as conventional mathematics that a reader can check without the formal development. The reconstruction is not complete. The paper proves Theorem 1.1 from the published theorems it cites and from a finite list of internal statements, each stated precisely where it is used and summarized in Subsection 18.7, whose conventional proofs it does not yet give. The formal development proves each of these statements. A few of the cited published theorems have not yet been compared line by line with their sources, and Subsection 18.7 lists them as well.

1.1 The architecture of the proof

For $x \geq 1$ let

$$G(x) = \sum_{n \leq x} \Lambda(n)\Big(\frac{x}{n} - 1\Big) - x \log x + (1+\gamma)x,$$

and let G_χ be the analogous function for a Dirichlet character χ, defined in (2.2). The Mellin transform of G has a simple pole at each zero of ζ in the critical strip, with residue $-1/(\rho(\rho-1))$ times the multiplicity. The proof has two parts.

Part I reduces Theorem 1.1 to one arithmetic statement. Section 2 proves a quantitative form of Landau's theorem on Dirichlet integrals of one sign. If a positive combination $G + \sum_i w_i G_{\chi_i}$ is at most Cx^θ for every x in windows $Y \leq x \leq Y^{3/2}$, for arbitrarily large Y, then ζ and the $L(s,\chi_i)$ do not vanish for $\operatorname{Re} s > \theta$. So it suffices to prove such a bound with $\theta = \frac{1}{2} + O(\varepsilon)$, on windows, for the powers of a fixed primitive character ψ_0. The bound comes from an abelian CM field E of degree $4n$ with n about $x^{2+8\varepsilon}$, cut out by ψ_0 and two auxiliary primes q and d. Section 3 proves two identities. The first, Proposition 3.1, shows that seven inequalities between slopes of hermitian lattices, heights of period maps and local terms bound a sum S over the primes inert in $\mathbb{Q}(\sqrt{-d})$ by an explicit main term. The second, Proposition 3.3, identifies S minus that main term with a combination of the functions G_ψ over the characters ψ of $\mathrm{Gal}(E/\mathbb{Q})$. Section 4 turns a bound for this combination, summed over many weights and many nodes, into a bound for a positive combination of the G_ψ at one point x. An average over translations removes the characters outside a small family, and an equidistributed choice of weights gives every remaining character a positive weight. The step leaves the values $|G_\psi(x)|$ for $3o$ auxiliary characters, where o is the order of ψ_0. Section 5 chooses q and d by a mean square over families of prime moduli so that these values are at most $x^{1/2+\varepsilon}$ on a whole window, without assuming any zero-free region beyond the classical one, and constructs the field E and the remaining data. Section 6 states the arithmetic theorem, Theorem A, and deduces Theorem 1.1 from it.

Part II concerns Theorem A. Bombieri's form of Stepanov's argument [10] bounds the number of rational points of a curve by comparing an auxiliary function of controlled pole order with its order of vanishing M at the rational points, and the two errors balance at $M = \sqrt{q}$. Over a number field the count of zeros against poles becomes a slope inequality of Bost's type [15] for a family of maps between hermitian lattices, and the order of vanishing becomes the number m of steps in a chain of theta lifts between unitary groups of growing rank. To each step the argument attaches a hermitian lattice over the ring of integers of E and a period map into the

dual of a lattice of automorphic forms. At a prime p inert in $\mathbb{Q}(\sqrt{-d})$, an elliptic curve with complex multiplication by $\mathbb{Q}(\sqrt{-d})$ has a formal group of height two, and the theta kernel of one lift is divisible by a power of p whose exponent at weight α has the main term $\sum_{i \text{ odd}} \alpha/p^i = \alpha p/(p^2-1)$. This main term matches the contribution of p to the sum S. Hecke operators at the inert primes cost a loss that a nondegeneracy theorem for integral representations of GL_k controls. The archimedean size of the period map is m times a quantity close to $\sum_t \log \Gamma(a_t)$, measured against the period of the CM curve, and arithmetic intersection theory on unitary Shimura varieties bounds the slope of the lattice of automorphic forms. Bessel Poincaré series on a unitary group realize the period maps at one complex place, and their near orthogonality gives the slope inequality for the family. At the places where the local type of the forms is twisted, lengths of Tate–Shafarevich groups of the CM field bound the congruence defects that enter the heights, through Galois representations attached to automorphic representations of unitary groups and p-adic Hodge theory. The two error terms balance at $m = \sqrt{x}$, as in Stepanov's argument. Section 1.4 describes the sections of Part II.

1.2 Contemporaneous accounts

Several groups began translating, mapping, explaining or distilling the formal development within days of its release. Zhang [82] wrote a preliminary mathematical account of the proof and of its principal ingredients, with an emphasis on their sources and history. Rubinstein [59, 60] maintains a guide to the proof together with a detailed exposition of the argument specialized to the Riemann zeta function, and his verification run compiled the formal development independently. Naskręcki [49] built a map of the formal proof, its structure and dependencies, and a record of its verification status, together with an atlas of the route. Epstein [25] wrote a human-readable blueprint that identifies the top-level statements, separates the principal mathematical crux from library material, and draws the dependency structure. Brox [17] leads a program to distill and simplify the formal proof into shorter human-readable proofs of the Riemann hypothesis and of the generalized Riemann hypothesis for Dirichlet characters, which is in progress. Slijepčević [67] wrote an outline of the proof in standard language, including the positivity argument, the theta-lift and lattice estimates, the arithmetic losses and the Galois-representation input. The present paper writes the argument as one continuous conventional proof with its open steps stated, so that a reader can judge its proofs and its cited external inputs without appeal to the formal development as mathematical authority.

1.3 How this account was obtained

The formal development has about 1.07 million declarations, of which about 767,000 lie on the dependency route of the final theorem, and a readable proof cannot arise by translating them one at a time. We began at the formal theorem `riemannHypothesis`, traced its dependencies backward, and identified the mathematical statements on which it depends. Whenever a region of the development implemented a recognizable argument, we stated and proved that argument at its natural mathematical level, and whenever the route reached an established theorem of the literature, we cited that theorem and did not use the formal development as its justification. This process does not repeat the formal verification of [72], and it did not involve reading every declaration. The author used Claude (Fable 5.2 and Opus 5.5), GPT (including GPT 5.6 Pro) and Kimi K3.1 Pro in the reconstruction. The formal certification shows that the Lean term proves the formal statement. The reconstruction writes the content of the internal dependencies as

mathematics, and a separate comparison of each cited theorem with its published statement checks hypotheses, normalizations and quantifiers.

Two findings of the reconstruction concern the strength of the internal statements. The bound for Tate–Shafarevich groups on windows of scales, Hypothesis 17.11, implies a bound in the direction of Vandiver's conjecture that is stronger than the published bounds of Soulé [68] and of de la Torre Durán [22] (Proposition 17.22). One branch of the formal proof uses an automorphy statement for residually reducible Galois representations of unitary type in three variables, (G3) of Subsection 17.4, which lies outside the hypotheses of the published automorphy lifting theorems for residually reducible representations. Second, the bound for the covolumes of the lattices of automorphic forms needs a constant independent of the level of a deep cover of a unitary Shimura variety, and the argument obtains it from the arithmetic Hilbert–Samuel theorem on each model of a tower of models and an exact identity at the good primes, without a uniform version of the Hilbert–Samuel theorem (Subsection 15.14.4).

1.4 Organization

Sections 2 to 6 form Part I. Part II begins in Section 7 with the theta kernels between hermitian lattices over the CM field and their covariance. Section 8 studies the theta function of a CM elliptic curve at an inert prime and proves the form of the kernels in the parameters of a formal group of height two. Section 9 classifies the semilinear pairings on that formal group and proves the valuation estimate for the kernels. Section 10 turns this estimate into divisibility of the composite of a chain of lifts. Section 11 constructs the lifts and shows that they form a chain datum. Section 12 proves the nondegeneracy of Hecke operators that controls the loss, and hypotheses (3.1) and (3.2). Section 13 proves the slope inequality (3.3) for the family of period maps, and Section 14 proves its automorphic inputs for Bessel Poincaré series. Section 15 computes the period of the CM curve and proves (3.4), (3.6) and (3.7). Section 16 proves (3.5). Section 17 treats the twisted places, the Tate–Shafarevich bound and the congruence bound. Section 18 constructs the data, the weights and the nodes, and proves Theorem A from the statements of the preceding sections. Subsection 18.7 collects the hypotheses used in the final deduction.

Part I
Reduction to an arithmetic theorem

2 One-sided bounds on windows

This section proves the analytic criterion of the paper. An upper bound $G_+(x) \leq Cx^\theta$ for a positive combination of smoothed prime sums, known for x in a sparse sequence of windows $Y \leq x \leq Y^\lambda$, excludes zeros of ζ and of the corresponding Dirichlet L-functions from the half-plane $\operatorname{Re} s > \theta$. The proof is a quantitative form of Landau's theorem on Dirichlet integrals of functions of one sign [38].

2.1 The smoothed prime sums

Let Λ denote the von Mangoldt function and γ Euler's constant. For $x \geq 1$ put

$$G(x) = \sum_{n \leq x} \Lambda(n)\Big(\frac{x}{n} - 1\Big) - x \log x + (1+\gamma)x, \tag{2.1}$$

and for a nontrivial Dirichlet character χ put

$$G_\chi(x) = \sum_{n \leq x} \Lambda(n) \operatorname{Re} \chi(n)\Big(\frac{x}{n} - 1\Big) + \operatorname{Re} \frac{L'(1,\chi)}{L(1,\chi)}\, x. \tag{2.2}$$

For $\operatorname{Re} s > 1$ and every integer $n \geq 1$,

$$\int_n^\infty \Big(\frac{x}{n} - 1\Big) x^{-s-1}\, dx = \frac{n^{-s}}{s(s-1)},$$
$$\int_1^\infty x \log x\; x^{-s-1}\, dx = \frac{1}{(s-1)^2}, \qquad \int_1^\infty x\; x^{-s-1}\, dx = \frac{1}{s-1}.$$

Summation over n gives, for $\operatorname{Re} s > 1$,

$$\int_1^\infty G(x) x^{-s-1}\, dx = F(s), \qquad \int_1^\infty G_\chi(x) x^{-s-1}\, dx = \tfrac{1}{2}\big(F_\chi(s) + F_{\bar\chi}(s)\big), \tag{2.3}$$

where

$$F(s) = -\frac{\zeta'(s)}{\zeta(s)}\,\frac{1}{s(s-1)} - \frac{1}{(s-1)^2} + \frac{1+\gamma}{s-1}, \qquad F_\chi(s) = -\frac{L'(s,\chi)}{L(s,\chi)}\,\frac{1}{s(s-1)} + \frac{L'(1,\chi)}{L(1,\chi)}\,\frac{1}{s-1}. \tag{2.4}$$

The expansions $-\zeta'(s)/\zeta(s) = (s-1)^{-1} - \gamma + O(s-1)$ and $1/(s(s-1)) = (s-1)^{-1} - 1 + O(s-1)$ show that F is analytic at $s = 1$. The same computation, together with $L(1,\chi) \neq 0$, shows that F_χ is analytic at $s = 1$. Both functions are meromorphic on $\mathbb{C}$. In the half-plane $\operatorname{Re} s > 0$ their poles are simple and lie at the zeros ρ of ζ and of $L(\cdot,\chi)$, with residues

$$\operatorname{Res}_{s=\rho} F = -\frac{m_\zeta(\rho)}{\rho(\rho-1)}, \qquad \operatorname{Res}_{s=\rho} F_\chi = -\frac{m_\chi(\rho)}{\rho(\rho-1)}, \tag{2.5}$$

where $m_\zeta(\rho)$ and $m_\chi(\rho)$ denote the orders of vanishing of ζ and of $L(\cdot,\chi)$ at ρ.

We use four classical facts [21]. The functions ζ and $L(\cdot,\chi)$ do not vanish on the half-plane $\operatorname{Re} s \geq 1$, and ζ does not vanish on the interval $0 < s < 1$. Mertens' estimate $\sum_{n \leq x} \Lambda(n)/n = \log x + O(1)$ and Chebyshev's estimate $\sum_{n \leq x} \Lambda(n) = O(x)$ hold, and for nontrivial χ the partial sums $\sum_{n \leq x} \Lambda(n)\chi(n)/n$ are bounded. These estimates give a constant B with

$$|G(x)| \leq Bx, \qquad |G_\chi(x)| \leq Bx \qquad (x \geq 1). \tag{2.6}$$

The function G is continuous, because the term of index n in (2.1) vanishes at $x = n$, and away from the integers $G'(x) = \sum_{n \leq x} \Lambda(n)/n - \log x + \gamma$ is bounded. Hence there is a constant K with

$$|G(x) - G(y)| \leq K|x - y| \qquad (x, y \geq 1), \tag{2.7}$$

and the same argument gives (2.7) for G_χ.

2.2 Three lemmas

Lemma 2.1. *Let $k \geq 0$ be an integer, let $U \geq 0$ and $a > 0$. If $b > a$, then $\int_0^U e^{-au} \frac{u^k}{k!}\, du \leq e^{(b-a)U} b^{-k-1}$. If $0 < b < a$, then $\int_U^\infty e^{-au} \frac{u^k}{k!}\, du \leq e^{-(a-b)U} b^{-k-1}$.*

Proof. For $b > a$ and $0 \leq u \leq U$ we have $e^{-au} = e^{-bu} e^{(b-a)u} \leq e^{(b-a)U} e^{-bu}$. For $0 < b < a$ and $u \geq U$ we have $e^{-au} = e^{-bu} e^{-(a-b)u} \leq e^{-(a-b)U} e^{-bu}$. In both cases the identity $\int_0^\infty e^{-bu} \frac{u^k}{k!}\, du = b^{-k-1}$ completes the proof. □

Lemma 2.2. *Let $0 < \varphi < \pi/2$ and let α be real. Every set of $\lceil 2\pi/\varphi \rceil$ consecutive integers contains an integer j with $\cos(\alpha + j\varphi) \geq \frac{1}{2}$.*

Proof. The set of angles with cosine at least $\frac{1}{2}$ is the union of the arcs $[-\pi/3, \pi/3] + 2\pi\mathbb{Z}$, of length $2\pi/3$, separated by gaps of length $4\pi/3$. The angles $\alpha + j\varphi$ advance in steps of length $\varphi < 2\pi/3$, so they cannot jump across an arc. A sequence of angles that starts in a gap leaves it after at most $\lceil 4\pi/(3\varphi) \rceil$ steps, and its first angle outside the gap lies in an arc. The count $\lceil 4\pi/(3\varphi) \rceil + 1$ is at most $\lceil 2\pi/\varphi \rceil$ because $2\pi/\varphi - 4\pi/(3\varphi) = 2\pi/(3\varphi) > 1$. □

Lemma 2.3. *Let $P \subset \mathbb{C}$ be a closed discrete set that is stable under complex conjugation. Let $\theta < 1$, let $\rho_0 = \beta_0 + i\gamma_0 \in P$ with $\theta < \beta_0 < 1$, and put $\theta' = (\theta + \beta_0)/2$. Let σ_1 be a real number with $\sigma_1 > 1 + (|\rho_0|^2 + \gamma_0^2)/(\beta_0 - \theta)$, and let r be the distance from σ_1 to P. Then $r < \sigma_1 - \theta'$, and every $p \in P$ with $|\sigma_1 - p| = r$ satisfies $\operatorname{Re} p > \theta'$. For all such σ_1 outside a countable set, the points of P at distance r from σ_1 form a single real point or a single pair of complex conjugate points.*

Proof. For $p \in P$ with $\operatorname{Re} p \leq \theta'$,

$$|\sigma_1 - p|^2 - |\sigma_1 - \rho_0|^2 = 2\sigma_1(\beta_0 - \operatorname{Re} p) + |p|^2 - |\rho_0|^2 \geq \sigma_1(\beta_0 - \theta) - |\rho_0|^2 > 0,$$

so p lies farther from σ_1 than ρ_0 does. Next,

$$(\sigma_1 - \theta')^2 - |\sigma_1 - \rho_0|^2 = (\beta_0 - \theta')(2\sigma_1 - \beta_0 - \theta') - \gamma_0^2 \geq (\beta_0 - \theta)(\sigma_1 - 1) - \gamma_0^2 > 0,$$

so $r \leq |\sigma_1 - \rho_0| < \sigma_1 - \theta'$. Finally, two points $p, p' \in P$ are equidistant from σ_1 exactly when $2\sigma_1(\operatorname{Re} p' - \operatorname{Re} p) = |p'|^2 - |p|^2$. If $\operatorname{Re} p \neq \operatorname{Re} p'$, this equation holds for one value of σ_1, and if $\operatorname{Re} p = \operatorname{Re} p'$, it forces $p' = p$ or $p' = \bar{p}$. The set P is countable, so the exceptional values of σ_1 form a countable set. □

2.3 The window theorem

Theorem 2.4. *Let $H\colon [1, \infty) \to \mathbb{R}$ be measurable with $|H(x)| \leq Bx$ for a constant B, and suppose that the function $A(s) = \int_1^\infty H(x) x^{-s-1}\, dx$, defined for $\operatorname{Re} s > 1$, extends to a meromorphic function on $\mathbb{C}$. Let $0 \leq \theta < 1$. Suppose that every pole p of A with $\operatorname{Re} p > \theta$ is simple and satisfies $\operatorname{Re} p < 1$, and that the residue of A at every real pole $p > \theta$ is positive. Suppose that there are constants $\lambda > 1$ and C such that, for arbitrarily large Y, the inequality $H(x) \leq Cx^\theta$ holds for every x with $Y \leq x \leq Y^\lambda$. Then A does not have a pole in the half-plane $\operatorname{Re} s > \theta$.*

The proof compares two evaluations of the Taylor coefficients $(-1)^k A^{(k)}(\sigma_1)/k!$ at a real point $\sigma_1 > 1$. The nearest pole of A forces these coefficients to be large for a positive proportion of k. The window bound forces them to be small for every k in a range proportional to $\log Y$, because the kernel $e^{-\sigma_1 u} u^k/k!$ concentrates near $u = k/(\sigma_1 - 1)$ with relative width of order $k^{-1/2}$.

Proof. Suppose that A has a pole $\rho_0 = \beta_0 + i\gamma_0$ with $\beta_0 > \theta$, and let P be the set of poles of A. Since H is real, $A(\bar{s}) = \overline{A(s)}$, so P is stable under conjugation and the residues at conjugate poles are conjugate.

Step 1. The constants. Fix $\mu, \nu \in (0,1)$ with $\mu/\log(1+\mu) < \lambda^{1/3}$ and $-\log(1-\nu)/\nu < \lambda^{1/3}$. For real numbers $a > 0$ and $\eta \geq 0$ define

$$\xi_- = a\,\frac{(1+\mu)e^{\eta} - 1}{\log(1+\mu)}, \qquad \xi_+ = \frac{a\,\nu\lambda}{\eta - \log(1-\nu)}.$$

At $\eta = 0$ we have $\xi_-/a < \lambda^{1/3} < \lambda^{2/3} < \xi_+/a$. Both ratios are continuous in η and independent of a, so there is $\eta_0 > 0$ with $\xi_- < \xi_+$ whenever $0 \leq \eta \leq \eta_0$.

Step 2. The base point. Put $\theta' = (\theta + \beta_0)/2$. Lemma 2.3 gives a real number σ_1, as large as we please, such that $r = \operatorname{dist}(\sigma_1, P)$ satisfies $r < \sigma_1 - \theta'$, and such that the poles nearest to σ_1 form either one real pole p_1 or one conjugate pair $p_1, \bar{p}_1$ with $\operatorname{Im} p_1 > 0$. In both cases $\theta' < \operatorname{Re} p_1 < 1$. Put $a = \sigma_1 - 1$ and $\eta = \log(r/a)$. Then $r > a$, and $\eta \leq \log\big((\sigma_1 - \theta')/(\sigma_1 - 1)\big)$, which is at most η_0 once σ_1 is large. We fix such a σ_1, we fix $\xi_1 < \xi_2$ in the interval (ξ_-, ξ_+), and we put $\kappa = (\sigma_1 - \theta)/(\sigma_1 - \theta') > 1$.

Step 3. The Taylor coefficients from the poles. For an integer $k \geq 0$ put

$$T_k = \int_0^{\infty} H(e^u)\, e^{-\sigma_1 u}\, \frac{u^k}{k!}\, du = \frac{(-1)^k}{k!} A^{(k)}(\sigma_1).$$

Let R_1 be the residue of A at p_1. Subtracting the principal parts of A at the poles nearest to σ_1 leaves a function Q that is analytic on a disc $|s - \sigma_1| \leq r''$ with $r'' > r$, and Cauchy's estimate gives $|Q^{(k)}(\sigma_1)|/k! \leq M(r'')^{-k}$ with $M = \max_{|s-\sigma_1| = r''} |Q(s)|$. Since $\frac{(-1)^k}{k!}\frac{d^k}{ds^k}(s-p)^{-1} = (s-p)^{-k-1}$, a real nearest pole gives

$$T_k = R_1\, r^{-k-1} + \frac{(-1)^k}{k!} Q^{(k)}(\sigma_1), \qquad R_1 > 0.$$

For a conjugate pair write $\sigma_1 - p_1 = re^{-i\varphi}$ with $0 < \varphi < \pi/2$ and $R_1 = |R_1| e^{i\alpha}$. Then

$$T_k = 2|R_1|\, r^{-k-1} \cos\big(\alpha + (k+1)\varphi\big) + \frac{(-1)^k}{k!} Q^{(k)}(\sigma_1).$$

By Lemma 2.2 there is an integer K_0 such that every set of K_0 consecutive integers contains a k with $\cos(\alpha + (k+1)\varphi) \geq \frac{1}{2}$. Call such k good, and in the real case call every k good. Since $r'' > r$, there is k_0 such that every good $k \geq k_0$ satisfies

$$T_k \geq \tfrac{1}{2}|R_1|\, r^{-k-1}. \tag{2.8}$$

Step 4. The Taylor coefficients from the window. Let Y be one of the values in the hypothesis and put $L = \log Y$. We split the integral defining T_k at $u = L$ and at $u = \lambda L$. On the middle range $H(e^u) \leq Ce^{\theta u}$, hence

$$\int_L^{\lambda L} H(e^u) e^{-\sigma_1 u} \frac{u^k}{k!}\, du \leq C(\sigma_1 - \theta)^{-k-1} \leq C\kappa^{-k-1} r^{-k-1},$$

where the last inequality uses $r < \sigma_1 - \theta'$. On the outer ranges $|H(e^u)|e^{-\sigma_1 u} \le Be^{-au}$. Lemma 2.1 with $b = (1+\mu)r$ on $[0, L]$ and with $b = (1-\nu)a$ on $[\lambda L, \infty)$ gives

$$\int_0^L Be^{-au}\frac{u^k}{k!}\,du \le B\,r^{-k-1}\exp\Big(\log(1+\mu)\big(\xi_- L - (k+1)\big)\Big),$$
$$\int_{\lambda L}^{\infty} Be^{-au}\frac{u^k}{k!}\,du \le B\,r^{-k-1}\exp\Big(\big(\eta - \log(1-\nu)\big)\big((k+1) - \xi_+ L\big)\Big),$$

since $(1+\mu)r - a = \xi_- \log(1+\mu)$ and $r/((1-\nu)a) = e^\eta/(1-\nu)$. For every integer k with $\xi_1 L \le k+1 \le \xi_2 L$ both exponents are at most $-\delta L$, where

$$\delta = \min\big\{(\xi_1 - \xi_-)\log(1+\mu),\ (\xi_+ - \xi_2)(\eta - \log(1-\nu))\big\} > 0.$$

For every such k we therefore have

$$T_k \le r^{-k-1}\big(2Be^{-\delta L} + C\kappa^{-k-1}\big). \tag{2.9}$$

Step 5. The contradiction. The interval $[\xi_1 L, \xi_2 L]$ has length $(\xi_2 - \xi_1)L$, so for large L it contains K_0 consecutive values of $k+1$ with $k \ge k_0$, and one of these k is good. For that k, the inequalities (2.8) and (2.9) give $\frac{1}{2}|R_1| \le 2Be^{-\delta L} + C\kappa^{-\xi_1 L}$. The right side tends to zero as $L \to \infty$ along the admissible values of Y, while $|R_1| > 0$. Hence A does not have a pole with real part greater than θ. □

Corollary 2.5. *Let $\chi_1, \dots, \chi_N$ be nontrivial Dirichlet characters, let $w_1, \dots, w_N$ be positive real numbers, and put $G_+ = G + \sum_{i=1}^N w_i G_{\chi_i}$ with G and G_χ as in* (2.1) *and* (2.2). *Let $0 \le \theta < 1$, and suppose that there are constants $\lambda > 1$ and C such that, for arbitrarily large Y, the inequality $G_+(x) \le Cx^\theta$ holds for every x with $Y \le x \le Y^\lambda$. Then $\zeta(s) \ne 0$ and $L(s, \chi_i) \ne 0$ for every i and every s with $\operatorname{Re} s > \theta$.*

Proof. By (2.3), (2.4) and (2.6), the function $H = G_+$ satisfies $|H(x)| \le (1 + \sum_i w_i)Bx$, and its transform $A = F + \frac{1}{2}\sum_i w_i(F_{\chi_i} + F_{\bar\chi_i})$ is meromorphic on $\mathbb{C}$. By (2.5), the poles of A in $\operatorname{Re} s > \theta$ are simple, they lie at the zeros ρ of ζ, of the $L(\cdot, \chi_i)$ and of the $L(\cdot, \bar\chi_i)$, and

$$\operatorname{Res}_{s=\rho} A = -\frac{c_\rho}{\rho(\rho-1)}, \qquad c_\rho = m_\zeta(\rho) + \tfrac{1}{2}\sum_{i=1}^N w_i\big(m_{\chi_i}(\rho) + m_{\bar\chi_i}(\rho)\big) \ge 0.$$

These zeros satisfy $\operatorname{Re}\rho < 1$. At a real zero $\rho \in (\theta, 1)$ the residue is positive because $\rho(\rho-1) < 0$. Theorem 2.4 shows that A does not have a pole with $\operatorname{Re} s > \theta$, so $c_\rho = 0$ for every ρ in that half-plane. Each term of c_ρ is nonnegative and each w_i is positive, so each multiplicity vanishes. □

If E is an abelian number field with group X of primitive Dirichlet characters, then $\zeta_E(s) = \prod_{\chi \in X} L(s, \chi)$, and the weights $w_\chi = 1$ for the nontrivial $\chi \in X$ make G_+ the smoothed sum over the prime ideals of E. Part II proves a bound of this kind for a positive combination of characters attached to one CM field, and Corollary 2.5 then treats all of these characters at once.

3 The accounting and the excess identity

Part II attaches to a chain of m theta lifts a family of hermitian lattices, and it proves seven inequalities between their slopes, the heights of certain period maps and some local terms. This section proves that these seven inequalities, written as inequalities between real numbers, bound a sum S over primes by an explicit main term. It then proves that S minus this main term is a combination of the functions G and G_ψ of Section 2.1. The two results together reduce the arithmetic of Part II to an upper bound for that combination.

3.1 The accounting

For $y > 0$ put $\mathrm{ph}(y) = y \log y - y$.

Proposition 3.1. *Let $m, f, N_T \geq 1$ be integers, let $a_1, \ldots, a_{N_T}$ and $w_{j\rho}$ for $1 \leq j \leq m$ and $1 \leq \rho \leq f$ be positive real numbers, and put $A = \sum_t a_t$. Let $v > 0$ and let γ, λ_1, g be real numbers, and put $\log \Omega = \gamma + \log(\pi/v) - \lambda_1$. Let W be a finite nonempty set, and for $u \in W$ let $d_u, T_u > 0$ and $\mu_u^N, \mu_u^M, h_u, E_u$ be real numbers. Let $\mu^C, \mathcal{L}, \Gamma, S$ be real numbers and $e_1, \ldots, e_7 \geq 0$. Suppose that for every $u \in W$*

$$S \leq \Gamma + e_1, \tag{3.1}$$

$$\mu_u^M + \Gamma + \mathcal{L} - E_u - e_2 \leq \mu_u^N, \tag{3.2}$$

$$\frac{\sum_{u'} d_{u'}(\mu_{u'}^N - h_{u'})}{\sum_{u'} d_{u'}} - e_3 \leq \mu^C, \tag{3.3}$$

$$\mu^C + h_u - \mathcal{L} \leq \sum_{t=1}^{N_T} \left(m\,\mathrm{ph}(a_t) - m a_t \log \Omega - a_t \log T_u \right) + e_4, \tag{3.4}$$

$$E_u \leq mAg + e_5, \tag{3.5}$$

$$\log T_u \geq \sum_{j=1}^{m} \frac{1}{f} \sum_{\rho=1}^{f} \log \frac{w_{j\rho} v}{\pi} + mg - e_6, \tag{3.6}$$

$$\mu_u^M \geq -e_7. \tag{3.7}$$

Then

$$S \leq \sum_{j=1}^{m} \sum_{t=1}^{N_T} a_t \Big(\log a_t - 1 - \gamma + \lambda_1 - \frac{1}{f} \sum_{\rho=1}^{f} \log w_{j\rho} \Big) + e_1 + e_2 + e_3 + e_4 + e_5 + Ae_6 + e_7. \tag{3.8}$$

Proof. Write $\langle X \rangle = \sum_u d_u X_u / \sum_u d_u$ for the weighted mean over W. By (3.2) and (3.7), $\Gamma \leq \mu_u^N - \mathcal{L} + E_u + e_2 + e_7$ for every u. The weighted mean of this inequality and (3.3) give

$$\Gamma \leq \langle \mu^N - h \rangle + \langle h + E \rangle - \mathcal{L} + e_2 + e_7 \leq \mu^C - \mathcal{L} + \langle h + E \rangle + e_2 + e_3 + e_7.$$

Let R_u denote the right side of (3.4). The weighted mean of (3.4) is $\mu^C - \mathcal{L} \leq \langle R - h \rangle$, so the heights cancel and

$$\Gamma \leq \langle R + E \rangle + e_2 + e_3 + e_7.$$

By (3.5) and (3.6), and since $A \geq 0$,

$$\begin{aligned} R_u + E_u &\leq m \sum_t \left(a_t \log a_t - a_t - a_t \log \Omega \right) - A \log T_u + mAg + e_4 + e_5 \\ &\leq \sum_{j=1}^{m} \sum_t a_t \Big(\log a_t - 1 - \log \Omega - \frac{1}{f} \sum_{\rho} \log \frac{w_{j\rho} v}{\pi} \Big) + e_4 + e_5 + Ae_6. \end{aligned}$$

The terms mAg cancel, and $-\log \Omega - \log(v/\pi) = -\gamma + \lambda_1$, so the double sum equals the double sum in (3.8). Inequality (3.1) completes the proof. □

In Part II the index u runs over a set of auxiliary primes T_u, the number μ_u^N is the slope of a hermitian lattice N_u of rank d_u, μ_u^M is the slope of a base lattice, μ^C is the slope of the dual $R^\vee$

of a lattice R of automorphic forms, and h_u is the height of a period map $B_u\colon N_u \to R^\vee$. The number Γ is the total gain in divisibility of the m lifts at the finite places, and $\mathcal{L}$ and E_u are sums of local terms. Inequality (3.3) is the slope inequality for the maps B_u, and (3.4) bounds the archimedean size of B_u together with the slope of $R^\vee$. The factor m in front of $\mathrm{ph}(a_t)$ in (3.4), a quantity close to $\log\Gamma(a_t)$, appears because each of the m lifts contributes one copy of $\mathrm{ph}(a_t)$ to the height of B_u. In Part II the number v is $\frac{1}{2}\mathrm{rd}_E$, where $\mathrm{rd}_E = |d_E|^{1/[E:\mathbb{Q}]}$ is the root discriminant of the CM field E, so that $\log\Omega = \gamma + \log(2\pi/\mathrm{rd}_E) - \lambda_1$. Proposition 15.6 identifies this number with twice the normalized arithmetic degree of the line of periods of a CM elliptic curve, written in coordinates over E. The formula of Chowla and Selberg [20] with Dirichlet's class number formula gives $\gamma + \log(2\pi/\sqrt{d}) - \lambda_1$ for the period of the curve itself, and the two numbers differ by $\log(\mathrm{rd}_E/\sqrt{d})$, which comes from the different of $E/\mathbb{Q}(\sqrt{-d})$.

Remark 3.2. Proposition 3.1 has the structure of Stepanov's proof of the Riemann hypothesis for curves over finite fields [70], in the form of Bombieri [10]. There an auxiliary function vanishes to order M at every rational point, the count of zeros against poles bounds the number N of points by $N - 1 \le \ell + nq/M$, and the two error terms balance at $M = \sqrt{q}$. Over a number field the count of zeros against poles becomes a slope inequality of Bost's type [15] for a family of maps, which is (3.3), and the order of vanishing becomes the number m of lifts in (3.4). Inequality (3.3) bounds the mean slope of a target lattice by the slopes of the sources of a redundant family of maps, and Theorem 13.8 derives it from Bost's inequality for one map, the Harder–Narasimhan filtration and a count of frames at one complex place. Section 4 shows that the error terms balance at $m = \sqrt{x}$.

3.2 The excess identity

Proposition 3.3 below identifies the main term of (3.8) with the main term of a sum over primes indexed by a finite abelian group. Let H be a finite abelian group and $\mathcal{G} = (\mathbb{Z}/2)^2 \times H$, with elements $g = (e_1, e_2, h)$. Put

$$\Phi = \{g \in \mathcal{G} : e_1 = 0\}, \qquad f = |\Phi| = 2|H|, \qquad p_t(e_1, e_2, h) = (e_1 + 1,\, e_2 + 1,\, h + t) \quad (t \in H),$$

so that p_t maps $\mathcal{G} \setminus \Phi$ bijectively onto Φ. For each character ψ of $\mathcal{G}$ let $G_\psi\colon (0,\infty) \to \mathbb{C}$ be a function. For $a > 0$, $t \in H$ and a function $w\colon \Phi \to (0,\infty)$ define the excess

$$\mathrm{De}(a, t, w) = \frac{1}{f} \sum_{g \notin \Phi} \sum_{\psi} \overline{\psi(g)}\; w(p_t g)\; G_\psi\Big(\frac{a}{w(p_t g)}\Big), \tag{3.9}$$

where ψ runs over the $2f$ characters of $\mathcal{G}$.

Proposition 3.3. *Let H, $\mathcal{G}$, Φ, f and p_t be as above. For every prime p let I_p be a subgroup of $\mathcal{G}$ and σ_p an element of $\mathcal{G}$. Let d be a prime and S_0 a finite set of primes with $d \notin S_0$, and suppose that $I_p = \{0\}$ for $p \notin S_0 \cup \{d\}$, that $I_p \subset \Phi$ and $\sigma_p \in \Phi$ for $p \in S_0$, and that $I_d = \{0, (1,0,0)\}$. For a character ψ of $\mathcal{G}$ and a prime power p^i with $i \ge 1$ put*

$$\psi(p^i) = \frac{1}{|I_p|} \sum_{h \in I_p} \psi(i\sigma_p + h),$$

and put $\psi(n) = 0$ for the integers $n \ge 1$ that are not of this form. For every nontrivial ψ let ℓ_ψ be a complex number, put $G_\psi(y) = \sum_{n \le y} \Lambda(n)\psi(n)(y/n - 1) + \ell_\psi y$ for $y > 0$, and let G_ψ be the function

G of (2.1), *extended by the same formula to* $0 < y < 1$, *when* ψ *is trivial. Let* $\chi_1(e_1, e_2, h) = (-1)^{e_1}$, *and for* $i \geq 1$ *let* g_i *be the element of the coset* $i\sigma_d + I_d$ *that does not lie in* Φ. *Then for every* $a > 0$, *every* $t \in H$ *and every function* $w\colon \Phi \to (0, \infty)$,

$$2 \sum_{\substack{p \notin S_0 \cup \{d\} \\ \sigma_p \notin \Phi}} \log p \sum_{i \text{ odd}} \max\Big(\frac{a}{p^i} - w(p_t(i\sigma_p)), 0\Big) + \log d \sum_{i \geq 1} \max\Big(\frac{a}{d^i} - w(p_t g_i), 0\Big)$$

$$= a\Big(\log a - 1 - \gamma + \ell_{\chi_1} - \frac{1}{f} \sum_{\rho \in \Phi} \log w(\rho)\Big) + \mathrm{De}(a, t, w). \quad (3.10)$$

Proof. For $g \notin \Phi$ write $w_g = w(p_t g)$ and $y_g = a/w_g$, and put

$$A_g = \sum_{n \geq 1} \Lambda(n)\, c_g(n) \max\Big(\frac{a}{n} - w_g, 0\Big), \qquad c_g(n) = \frac{1}{|\mathcal{G}|} \sum_{\psi} \overline{\psi(g)}\, \psi(n).$$

We first show that the left side of (3.10) equals $2\sum_{g \notin \Phi} A_g$. By the orthogonality of characters, $c_g(p^i) = |I_p|^{-1}$ when $g \in i\sigma_p + I_p$, and $c_g(p^i) = 0$ otherwise. For $p \in S_0$ the coset $i\sigma_p + I_p$ lies in Φ, so these primes do not contribute. For $p \notin S_0 \cup \{d\}$ the coset is the point $i\sigma_p$, which lies outside Φ exactly when $\sigma_p \notin \Phi$ and i is odd, because Φ has index two. For $p = d$ the coset $i\sigma_d + I_d$ has two elements, one of which is $g_i \notin \Phi$, and $c_{g_i}(d^i) = \frac{1}{2}$.

Since $|\mathcal{G}| = 2f$ and the terms with $n > y_g$ vanish,

$$2A_g = \frac{1}{f} \sum_{\psi} \overline{\psi(g)}\, w_g \sum_{n \leq y_g} \Lambda(n)\psi(n)\Big(\frac{y_g}{n} - 1\Big).$$

For nontrivial ψ the inner sum equals $G_\psi(y_g) - \ell_\psi y_g$, and for the trivial character it equals $G(y_g) + y_g \log y_g - (1+\gamma)y_g$ by (2.1). With $w_g y_g = a$ we obtain

$$2A_g = \frac{1}{f} \sum_{\psi} \overline{\psi(g)}\, w_g\, G_\psi(y_g) + \frac{a}{f}\Big(\log \frac{a}{w_g} - 1 - \gamma\Big) - \frac{a}{f} \sum_{\psi \neq 1} \overline{\psi(g)}\, \ell_\psi.$$

The complement of Φ has f elements and p_t maps it bijectively onto Φ, so the first term sums over $g \notin \Phi$ to $\mathrm{De}(a, t, w)$, and the second sums to $a(\log a - 1 - \gamma) - \frac{a}{f}\sum_{\rho \in \Phi} \log w(\rho)$. The complement of Φ is a coset of Φ, so $\sum_{g \notin \Phi} \overline{\psi(g)}$ vanishes when ψ is nontrivial on Φ, and it equals $-f$ for $\psi = \chi_1$, the nontrivial character that is trivial on Φ. The third term therefore sums to $a\,\ell_{\chi_1}$. □

In the application, $\mathcal{G}$ is the Galois group of a CM field E that is abelian over $\mathbb{Q}$, the set Φ is a CM type, I_p is the inertia group and σ_p a Frobenius element at p, and $\ell_\psi = L'(1, \chi_\psi)/L(1, \chi_\psi)$ for the primitive Dirichlet character χ_ψ attached to ψ by class field theory. With this choice G_ψ is the function $G^*_{\chi_\psi}$ of (5.1) below, whose real part is the function G_{χ_ψ} of (2.2). The primes p with $\sigma_p \notin \Phi$ are the primes whose Frobenius element does not lie in the CM type, and for H trivial and $\mathcal{G}$ the Galois group of $\mathbb{Q}(\sqrt{-d})$ over $\mathbb{Q}$ they are the primes inert in $\mathbb{Q}(\sqrt{-d})$. Each such prime contributes $\log p \cdot \max(a/p^i - w, 0)$ for its odd powers, which has the size $a \log p \cdot p/(p^2 - 1)$ when w is small. Section 9 shows that a theta lift gains exactly this power of p in divisibility at an inert prime.

Corollary 3.4. *Let the notation be as in Proposition 3.3, and suppose that ℓ_{χ_1} is real. Let $T \subset H$ have N_T elements, let $a_t > 0$ for $t \in T$, let $m \geq 1$, and let $w_1, \dots, w_m \colon \Phi \to (0, \infty)$. Let S be the sum over $t \in T$ and $1 \leq j \leq m$ of the left side of* (3.10) *with $a = a_t$ and $w = w_j$. Suppose that S satisfies the conclusion* (3.8) *of Proposition 3.1 with these a_t, with $w_{j\rho} = w_j(\rho)$ for $\rho \in \Phi$, and with $\lambda_1 = \ell_{\chi_1}$. Then*

$$\sum_{t \in T} \sum_{j=1}^{m} \operatorname{Re} \operatorname{De}(a_t, t, w_j) \leq e_1 + e_2 + e_3 + e_4 + e_5 + A e_6 + e_7.$$

Proof. The left side of (3.10) is real, and so is the main term on its right side. Taking real parts in (3.10) and summing over t and j expresses S as the main term of (3.8) plus $\sum_{t,j} \operatorname{Re} \operatorname{De}(a_t, t, w_j)$. □

4 From the excess to a bound at a point

Corollary 3.4 bounds a sum of excesses over many weights a_t and many nodes $w_j(\rho)$. Corollary 2.5 needs an upper bound for a positive combination of the functions G_ψ at one point x. This section proves the step from the first to the second. Two averages make it work. An average over translations by a subgroup A_0 of H removes every character that is nontrivial on A_0, and the sum over a set T of weights that is equidistributed in a suitable sense gives each remaining character a positive weight. The cost of the step is the value $|G_\psi(x)|$ for finitely many auxiliary characters ψ, which Section 5 bounds.

4.1 Regularity of the smoothed sums

Lemma 4.1. *Let $\mathcal{G}$ and the functions G_ψ be as in Proposition 3.3, with $\ell_\psi = L'(1, \chi_\psi)/L(1, \chi_\psi)$ for the primitive Dirichlet characters χ_ψ attached to the nontrivial ψ. There is an absolute constant C_d such that for every character ψ of $\mathcal{G}$ the following hold.*

(i) For $A > 0$ and $0 < w, w' \leq A$, $|w\, G_\psi(A/w) - w' G_\psi(A/w')| \leq 3A|w - w'|/\min(w, w')$.

(ii) For $x \geq 2$ and $|y - x| \leq 1$, $|G_\psi(y) - G_\psi(x)| \leq \log(x+1) + C_d + |\ell_\psi|$, with $\ell_\psi = 0$ for trivial ψ.

Proof. Every coefficient satisfies $|\psi(n)| \leq 1$. Put $\Theta(y) = \sum_{n \leq y} \Lambda(n)$, which satisfies $0 \leq \Theta(y) \leq 1.04\, y$ for $y > 0$ by the theorem of Rosser and Schoenfeld [57, Theorem 12].

(i) For nontrivial ψ put $\varphi(w) = w G_\psi(A/w) = \sum_{n \leq A/w} \Lambda(n)\psi(n)(A/n - w) + \ell_\psi A$. The function φ is continuous, because the term of index n vanishes when $A/w = n$, and away from the finitely many points $w = A/n$ its derivative is $-\sum_{n \leq A/w} \Lambda(n)\psi(n)$, of absolute value at most $\Theta(A/w) \leq 1.04A/w$. For trivial ψ, the formula (2.1) gives $\varphi(w) = \sum_{n \leq A/w} \Lambda(n)(A/n - w) - A\log(A/w) + (1+\gamma)A$, whose derivative away from the points $w = A/n$ is $A/w - \Theta(A/w)$, of absolute value at most A/w. In both cases the mean value theorem on each interval between consecutive points A/n gives $|\varphi(w) - \varphi(w')| \leq 1.04A|w - w'|/\min(w, w')$.

(ii) The function G_ψ is continuous, and away from the integers its derivative is $\sum_{n \leq y} \Lambda(n)\psi(n)/n + \ell_\psi$ for nontrivial ψ and $\sum_{n \leq y} \Lambda(n)/n - \log y + \gamma$ for trivial ψ. Mertens' estimate gives $\sum_{n \leq y} \Lambda(n)/n \leq \log y + C$ for $y \geq 1$, so both derivatives are at most $\log(x+1) + C_d + |\ell_\psi|$ in absolute value for $x - 1 \leq y \leq x + 1$ and $x \geq 2$. □

4.2 The main step

Let H, $\mathcal{G}$, Φ, f and p_t be as in Section 3.2. Let A_0 be a subgroup of H and T_2 a nonempty subset of A_0, and put $\delta_2 = |T_2|/|H|$. For $u_0 \in A_0$ call an element $\rho = (0, e_2, h) \in \Phi$ thick with respect to u_0 when $e_2 = 1$ and $h \in T_2 + u_0$. Let Ψ_0 be the set of characters ψ of $\mathcal{G}$ with $\psi(0,0,s) = 1$ for every $s \in A_0$, a set with $4|H/A_0|$ elements.

Theorem 4.2. *Let the notation be as above, and for each character ψ of $\mathcal{G}$ let $G_\psi\colon (0,\infty) \to \mathbb{C}$ be a function. Let $K \subset \Psi_0$ be a set of characters that contains the trivial character and satisfies $\psi(1,0,0) = 1$ for every $\psi \in K$. Let $T \subset H$ have $N_T \geq 1$ elements, let $c, \nu \geq 0$, and suppose that every $\psi \in K$ that is nontrivial on $\{0\} \times \{0\} \times H$ satisfies*

$$\Big| \sum_{t \in T} \psi(0,0,t) - c\, N_T \Big| \leq \nu\, N_T. \tag{4.1}$$

Put $c_\psi = 1$ when $\psi \in K$ is trivial on $\{0\} \times \{0\} \times H$, and $c_\psi = c$ for the other $\psi \in K$. Let $x \geq 3$, let $b, b', X' > 0$ with $b'X' \leq xb$, and let $\omega, M \geq 0$ and $0 \leq \eta' \leq b$. Let $m \geq 1$ be an integer, let $a_t > 0$ satisfy $|a_t/b - x| \leq 1$ for $t \in T$, and for $u_0 \in A_0$ and $1 \leq j \leq m$ let $w_{u_0,j}\colon \Phi \to (0,\infty)$ satisfy $b \leq w_{u_0,j}(\rho) \leq b + \eta'$ when ρ is thick with respect to u_0 and $w_{u_0,j}(\rho) \leq 2b'$ otherwise. Suppose that for every character ψ of $\mathcal{G}$

$$\big|w\, G_\psi(A/w) - w' G_\psi(A/w')\big| \leq \frac{3A\,|w - w'|}{\min(w, w')} \qquad (A > 0,\ 0 < w, w' \leq A), \tag{4.2}$$

$$|G_\psi(y) - G_\psi(x)| \leq M \qquad (|y - x| \leq 1), \tag{4.3}$$

$$|G_\psi(y)| \leq \omega\, y \qquad (y \geq X'/3), \tag{4.4}$$

and that

$$\frac{1}{|A_0|} \sum_{u_0 \in A_0} \sum_{t \in T} \sum_{j=1}^{m} \operatorname{Re} \mathrm{De}(a_t, t, w_{u_0,j}) \leq \mathrm{Err}.$$

Then

$$N_T\, m\, b\, \frac{\delta_2}{2} \Big(\sum_{\psi \in K} c_\psi \operatorname{Re} G_\psi(x) - \nu \sum_{\psi \in K} |G_\psi(x)| - \sum_{\psi \in \Psi_0 \setminus K} |G_\psi(x)| - |\Psi_0|\, M \Big)$$
$$\leq \mathrm{Err} + N_T\, m \Big(2f\omega(x+1)b + 6f(x+1)\eta' \Big). \tag{4.5}$$

Proof. Fix $u_0 \in A_0$, $t \in T$ and j, and write $w = w_{u_0,j}$ and $a = a_t$, so that $(x-1)b \leq a \leq (x+1)b$. We split the sum over $g \notin \Phi$ in $\mathrm{De}(a, t, w)$ according to whether $p_t g$ is thick with respect to u_0.

The thin classes. If $p_t g$ is not thick, then $w(p_t g) \leq 2b'$, so $y = a/w(p_t g)$ satisfies

$$y \geq \frac{(x-1)b}{2b'} \geq \frac{(x-1)X'}{2x} \geq \frac{X'}{3},$$

and (4.4) gives $|w(p_t g)\, G_\psi(y)| \leq \omega a$. There are $2f$ characters and at most f such g, so the thin part of $\mathrm{De}(a, t, w)$ has absolute value at most $2f\omega a \leq 2f\omega(x+1)b$.

The nodes of the thick classes. If $p_t g$ is thick, then $b \leq w(p_t g) \leq b + \eta' \leq 2b \leq a$, and (4.2) with $A = a$ and $w' = b$ gives

$$\Big| w(p_t g)\, G_\psi\Big(\frac{a}{w(p_t g)}\Big) - b\, G_\psi\Big(\frac{a}{b}\Big) \Big| \leq \frac{3a\,\eta'}{b} \leq 3(x+1)\eta'.$$

Replacing every thick node by b therefore changes $\mathrm{De}(a,t,w)$ by at most $\frac{1}{f}\cdot f\cdot 2f\cdot 3(x+1)\eta' = 6f(x+1)\eta'$.

The mean over u_0. The class $g=(1,e_2,h)$ has $p_t g=(0,e_2+1,h+t)$, which is thick with respect to u_0 exactly when $e_2=0$ and $h=s+u_0-t$ for some $s\in T_2$. Hence for every character ψ of $\mathcal{G}$

$$\frac{1}{|A_0|}\sum_{u_0\in A_0}\sum_{\substack{g\notin\Phi\\ p_tg\ \text{thick}}}\overline{\psi(g)} = \overline{\psi(1,0,0)}\,\psi(0,0,t)\sum_{s\in T_2}\overline{\psi(0,0,s)}\,\frac{1}{|A_0|}\sum_{u_0\in A_0}\overline{\psi(0,0,u_0)}.$$

The mean over u_0 vanishes when $\psi\notin\Psi_0$ and equals 1 when $\psi\in\Psi_0$. For $\psi\in\Psi_0$ the sum over s equals $|T_2|$ because $T_2\subset A_0$, and $\epsilon_\psi=\psi(1,0,0)$ equals 1 or -1 because $(1,0,0)$ has order two. Since $|T_2|/f=\delta_2/2$, the mean over u_0 of the thick part of $\mathrm{De}(a_t,t,w_{u_0,j})$, with every thick node replaced by b, equals

$$\frac{\delta_2}{2}\,b\sum_{\psi\in\Psi_0}\epsilon_\psi\,\psi(0,0,t)\,G_\psi\Big(\frac{a_t}{b}\Big).$$

The sum over t. By (4.3), replacing $G_\psi(a_t/b)$ by $G_\psi(x)$ changes the last display by at most $\frac{\delta_2}{2}\,b\,|\Psi_0|\,M$. Put $S_\psi=\sum_{t\in T}\psi(0,0,t)$, so that $|S_\psi|\le N_T$. For $\psi\in K$ we have $\epsilon_\psi=1$ and, by (4.1), $|S_\psi-c_\psi N_T|\le \nu N_T$, where the left side vanishes when ψ is trivial on $\{0\}\times\{0\}\times H$. Therefore

$$\mathrm{Re}\sum_{\psi\in\Psi_0}\epsilon_\psi\,S_\psi\,G_\psi(x)\ge N_T\Big(\sum_{\psi\in K}c_\psi\,\mathrm{Re}\,G_\psi(x)-\nu\sum_{\psi\in K}|G_\psi(x)|-\sum_{\psi\in\Psi_0\setminus K}|G_\psi(x)|\Big).$$

Conclusion. For each j, the four estimates give

$$\begin{aligned}N_T\,b\,\frac{\delta_2}{2}\Big(\sum_{\psi\in K}c_\psi\,\mathrm{Re}\,G_\psi(x)-\nu\sum_{\psi\in K}|G_\psi(x)|-\sum_{\psi\in\Psi_0\setminus K}|G_\psi(x)|-|\Psi_0|\,M\Big)&\\ \le\frac{1}{|A_0|}\sum_{u_0\in A_0}\sum_{t\in T}\mathrm{Re}\,\mathrm{De}(a_t,t,w_{u_0,j})+N_T\Big(2f\omega(x+1)b+6f(x+1)\eta'\Big),&\end{aligned}$$

and the sum over j is (4.5). □

For $A_0=H$ and $K=\{1\}$ the set Ψ_0 consists of the trivial character and the three characters $(-1)^{e_1}$, $(-1)^{e_2}$ and $(-1)^{e_1+e_2}$, and (4.5) bounds $G(x)$ by the excess plus the three values $|G_\psi(x)|$. This is the case of one imaginary quadratic field, where the sum over inert primes controls $G-G_\chi$ and a bound for G needs a separate bound for $G_\chi(x)$.

4.3 The sizes

Part II bounds the excess by $x^{C\varepsilon}$ times the quantity $\mathcal{B}$ of (4.8). Proposition 4.3 divides $\mathcal{B}$ by the factor $N_T m b\,\delta_2/2$ on the left of (4.5) and obtains a square-root bound.

Proposition 4.3. *Let $x\ge 3$, $\varepsilon>0$ and $C\ge 0$, let $n,N_T,N_2,m\ge 1$ be integers and $a,b,k>0$ real numbers, and suppose that*

$$a\le 2xb,\qquad N_T\ge\tfrac{1}{2}x^{-\varepsilon/8}\,n,\qquad x^{-\varepsilon}n\le N_2\le n, \tag{4.6}$$

$$x^{1/2-\varepsilon}\le m\le x^{1/2+\varepsilon/2},\qquad xm\le k\le n^2x^{1/2-\varepsilon},\qquad x^{3\varepsilon/8}\ge 26. \tag{4.7}$$

Put $\delta_2 = N_2/n$ and

$$\mathcal{B} = N_T\, a\Big(1+\frac{m^2}{k}\Big) + m\Big(2\,n\,m\,b\,x^{\varepsilon} + \frac{N_2\,k\,b\,x^{2\varepsilon}}{n^2}\Big). \tag{4.8}$$

Then

$$x^{C\varepsilon}\,\mathcal{B} \le \frac{10}{13}\,x^{1/2+(C+3)\varepsilon}\cdot N_T\,m\,b\,\frac{\delta_2}{2}.$$

Proof. We divide $\mathcal{B}$ by $N_T m b\,\delta_2/2$ and bound the three terms with (4.6) and (4.7). In the first term $m/k \le 1/x$ and $1/\delta_2 \le x^{\varepsilon}$, so

$$\frac{2a}{m\,b\,\delta_2}\Big(1+\frac{m^2}{k}\Big) = \frac{2a}{b\,\delta_2}\Big(\frac{1}{m}+\frac{m}{k}\Big) \le 4x^{1+\varepsilon}\Big(x^{-1/2+\varepsilon}+\frac{1}{x}\Big) = 4x^{1/2+2\varepsilon}+4x^{\varepsilon}.$$

Since $n/N_T \le 2x^{\varepsilon/8}$,

$$\frac{2m\cdot 2\,n\,m\,b\,x^{\varepsilon}}{N_T\,m\,b\,\delta_2} = \frac{4\,n\,m\,x^{\varepsilon}}{N_T\,\delta_2} \le 8\,x^{1/2+21\varepsilon/8}.$$

Since $N_2/\delta_2 = n$ and $k \le n^2 x^{1/2-\varepsilon}$,

$$\frac{2m\,N_2\,k\,b\,x^{2\varepsilon}}{n^2\,N_T\,m\,b\,\delta_2} = \frac{2\,k\,x^{2\varepsilon}}{n\,N_T} \le \frac{2\,n\,x^{1/2+\varepsilon}}{N_T} \le 4\,x^{1/2+9\varepsilon/8}.$$

Each of the four bounds is at most its coefficient times $x^{1/2+21\varepsilon/8}$, so

$$\frac{\mathcal{B}}{N_T\,m\,b\,\delta_2/2} \le 20\,x^{1/2+21\varepsilon/8} = 20\,x^{-3\varepsilon/8}\,x^{1/2+3\varepsilon} \le \frac{20}{26}\,x^{1/2+3\varepsilon}.$$ □

After division by $N_T m b\,\delta_2/2$ the bracket $\mathcal{B}$ has the four shapes

$$\frac{x}{m},\qquad \frac{xm}{k},\qquad m\,x^{c\varepsilon},\qquad \frac{k\,x^{c\varepsilon}}{n\,N_T}.$$

The first comes from slopes that do not grow with the number of lifts, and the third from a cost that each lift adds. They balance at $m=\sqrt{x}$, which is the balance $M=\sqrt{q}$ of Remark 3.2 with the number of lifts in the role of the order of vanishing. The second shape is at most 1 because $xm \le k$, and the fourth is at most $x^{1/2+c\varepsilon}$ because $k \le n^2 x^{1/2-\varepsilon}$. The two conditions on k are compatible when $n^2 \ge x^{1/2+\varepsilon}m$, so the group H must have at least about $\sqrt{x}$ elements.

Corollary 4.4. *Suppose that the hypotheses of Theorem 4.2 hold, and that the hypotheses of Proposition 4.3 hold with $n=|H|$ and $N_2=|T_2|$. Suppose further that*

$$\mathrm{Err} \le x^{C\varepsilon}\,\mathcal{B},\qquad 2f(x+1)\omega \le \frac{1}{x},\qquad 6f(x+1)\eta' \le \frac{b}{x},$$
$$4x^{\varepsilon-1} + \nu\sum_{\psi\in K}|G_\psi(x)| + |\Psi_0|\,M \le \frac{3}{13}\,x^{1/2+(C+3)\varepsilon}.$$

Then

$$\sum_{\psi\in K} c_\psi\,\mathrm{Re}\,G_\psi(x) \le x^{1/2+(C+3)\varepsilon} + \sum_{\psi\in\Psi_0\setminus K}|G_\psi(x)|.$$

Proof. We divide (4.5) by $N_T m b\,\delta_2/2$. Proposition 4.3 bounds the term with Err by $\frac{10}{13}x^{1/2+(C+3)\varepsilon}$. The conditions on ω and η' bound the last bracket of (4.5) by $2b/x$, and $2b/x$ divided by $b\,\delta_2/2$ equals $4/(x\delta_2) \le 4x^{\varepsilon-1}$. The last hypothesis bounds this term, together with the terms of (4.5) that contain ν and M, by $\frac{3}{13}x^{1/2+(C+3)\varepsilon}$. □

5 The auxiliary characters and the host field

Corollary 4.4 leaves two kinds of terms. The first is the bound (4.4) at very large scales, and the second is the value $|G_\psi(x)|$ for the characters $\psi \in \Psi_0 \setminus K$. In the application the characters of $\Psi_0 \setminus K$ are products of characters of a fixed modulus with the quadratic characters of two auxiliary primes q and d. The conductors of these characters exceed every power of x, so the required bound at the single point x does not follow from any zero-free region for a fixed L-function. This section chooses q and d so that the bound holds on a whole window of values of x, by a mean square over a family of prime moduli. It then constructs the CM field E cut out by q, d and the target character, together with the remaining data that the arithmetic part of the argument uses.

5.1 Notation and classical statements

For a Dirichlet character χ put $\Theta(u,\chi) = \sum_{n\le u} \Lambda(n)\chi(n)$ and $\Theta(u) = \sum_{n\le u} \Lambda(n)$. For a primitive character χ modulo $N \ge 3$ and $y \ge 1$ put

$$G_\chi^*(y) = \sum_{n\le y} \Lambda(n)\chi(n)\Big(\frac{y}{n} - 1\Big) + \frac{L'(1,\chi)}{L(1,\chi)}\, y, \tag{5.1}$$

so that the function (2.2) is $G_\chi = \operatorname{Re} G_\chi^*$ and $|G_\chi(y)| \le |G_\chi^*(y)|$. A zero of $L(s,\chi)$ means a zero in the strip $0 < \operatorname{Re} s < 1$. Write $N(\sigma, T, \chi)$ for the number of zeros ρ with $\operatorname{Re}\rho \ge \sigma$ and $|\operatorname{Im}\rho| \le T$, and $\operatorname{li}(X) = \int_2^X dt/\log t$. We use the following eight statements, in which every constant is absolute.

(C1) Chebyshev [19]. There is $c_0 \ge 1$ with $\Theta(u) \le c_0 u$ and $\pi(u)\log u \le c_0 u$ for every $u \ge 1$.

(C2) The prime number theorem for characters [32, Theorem 5.27], [21, Ch. 20]. There are $0 < c_1 \le \frac{1}{2}$ and $C_1 \ge 1$ such that for every primitive character χ modulo $N \ge 3$ and every $u \ge 2$

$$\Big|\Theta(u,\chi) + \frac{u^\beta}{\beta}\Big| \le C_1\, u\, (\log N)^4 \exp\Big(-\frac{c_1 \log u}{\sqrt{\log u} + \log N}\Big),$$

where the term with β is present when χ is real and $L(s,\chi)$ has a real zero $\beta > 1 - c_1/\log N$, and is absent otherwise. Moreover $|\Theta(u) - u| \le C_1 u \exp(-c_1\sqrt{\log u})$ for $u \ge 2$, and the series $\sum_n \Lambda(n)\chi(n)/n$ converges to $-L'(1,\chi)/L(1,\chi)$.

(C3) Page [55], [21, Ch. 14]. There is $c_2 > 0$ such that for every $R \ge 3$ at most one primitive real character of conductor at most R has a real zero $\beta > 1 - c_2/\log R$.

(C4) Zeros and the explicit formula [21, Ch. 16 and 19]. Let χ be a primitive character modulo $N \ge 3$. The map $\rho \mapsto 1 - \bar\rho$ permutes the zeros of $L(s,\chi)$, and for every real t the number of zeros with $|\operatorname{Im}\rho - t| \le 1$ is at most $C_2 \log(N(|t|+2))$. For $2 \le T \le u$,

$$\Big|\Theta(u,\chi) + \frac{u^{\beta_1}}{\beta_1} + \sum_{|\operatorname{Im}\rho|<T} \frac{u^\rho}{\rho}\Big| \le C_2\Big(\frac{u\log^2(Nu)}{T} + u^{1/4}\log u\Big),$$

where the term with β_1 is present when the zero β of (C2) exists, in which case $\beta_1 = \beta$ and the sum omits β_1 and $1 - \beta_1$.

(C5) Montgomery's density theorem [47, Theorem 12.2]. There is C_3 such that for $Q, T \geq 2$ and $\frac{4}{5} \leq \sigma \leq 1$

$$\sum_{N \leq Q} \sum_{\chi} N(\sigma, T, \chi) \leq C_3 \, (Q^2 T)^{5(1-\sigma)/2} \log^{14}(QT),$$

where the inner sum runs over the primitive characters modulo N.

(C6) Primes in the class 3 modulo 8 [21, Ch. 20]. There is U_0 such that for $U \geq U_0$ the number of primes $q \equiv 3 \pmod 8$ with $\sqrt{U} < q \leq U$ is at least $U/(5 \log U)$.

(C7) Primes in progressions of large modulus. There are $c_6 > 0$, A_0' and C_0 with the following property. For every integer M divisible by 8, consider the pairs (χ_1, β_1) in which χ_1 is a real nonprincipal character modulo M and β_1 is a real zero of $L(s, \chi_1)$ with $1 - 1/(8 \log M) < \beta_1 < 1$. If such pairs exist, one of them can be chosen so that the following estimate holds with the chosen pair, and if no such pair exists the estimate holds with the second form of the main term. For every $X \geq 2$ with $C_0 \log M \cdot \log(\log M + 2) \leq \log X$ and every b coprime to M, the number $\pi(X; M, b)$ of primes $p \leq X$ with $p \equiv b \pmod M$ satisfies

$$\big|\pi(X; M, b) - \mathfrak{m}(X; M, b)\big| \leq A_0' \Big(e^{-c_6 \log X / \log M} + e^{-\sqrt{c_6 \log X}} \Big) \, \mathfrak{m}(X; M, b),$$

where $\mathfrak{m}(X; M, b) = \big(\operatorname{li}(X) - \chi_1(b) \operatorname{li}(X^{\beta_1}) \big) / \varphi(M)$ for the chosen pair, and $\mathfrak{m}(X; M, b) = \operatorname{li}(X)/\varphi(M)$ when no such pair exists.

(C8) The distance of a real zero from 1 [21, Ch. 14]. There is $0 < c_7 \leq 1$ such that every real zero β of $L(s, \chi)$, for a real nonprincipal character χ modulo $M \geq 3$, satisfies $\beta \leq 1 - c_7/(M^{1/2} \log^2 M)$.

Statement (C7) is the prime number theorem in arithmetic progressions in the range of Linnik, with the exceptional zero kept in the main term. We derive it from Gallagher's theorem on primes in short intervals [27, Theorem 7], together with the uniqueness of the exceptional zero recorded in [27, Section 4, display (26)] and the bound (C8). In Gallagher's notation the starred sum $\sum_{\chi}^{*}$ runs over the primitive characters χ modulo q, and for $q = 1$ it consists of the trivial character. We state the two inputs with Gallagher's constants renamed, so that they do not clash with the letters of this section.

(G1) Gallagher [27, Section 4, display (26)], [21, Ch. 14]. There is $c_* > 0$ such that for every $T \geq 3$ at most one L-function of a primitive character of modulus at most T has a zero s with $\operatorname{Re} s > 1 - c_*/\log T$ and $|\operatorname{Im} s| \leq T$. If such a zero exists, it is real and simple, and no other zero of that L-function lies in this region. It is called the exceptional zero.

(G2) Gallagher [27, Theorem 7]. There are $a_* > 0$, $0 < b_* \leq 1$, $d_* > 0$ and C_* with the following property. Let $x \geq 2$, R and h be real numbers with $\exp(\sqrt{\log x}) \leq R \leq x^{b_*}$ and $x/R \leq h \leq x$. For a primitive character χ modulo $q > 1$ put $\Delta(\chi) = \sum_{x < p \leq x+h} \chi(p) \log p$, and for the trivial character put $\Delta(\chi) = \sum_{x < p \leq x+h} \log p - h$. Then

$$\sum_{q \leq R} \sum_{\chi}^{*} |\Delta(\chi)| \leq C_* \, h \, \exp\Big(- a_* \frac{\log x}{\log R} \Big),$$

with the following modification. If for a primitive character χ' of modulus $q' \leq R$ the function $L(s, \chi')$ has an exceptional zero $1 - \delta$ with $\delta \log R \leq d_*$, then there is $\xi \in [x, x+h]$ such that the estimate holds after $\Delta(\chi')$ is replaced by $\Delta(\chi') + h\xi^{-\delta}$, and in this case its right side may be multiplied by $(\delta \log R)^2$.

Gallagher's proof of (G2) takes $T = R^5$ in (G1), and the factor $(\delta \log R)^2$ comes from the Deuring–Heilbronn widening of the zero-free region together with Siegel's bound $\delta \gg_\varepsilon T^{-\varepsilon}$, so the constant C_* is not effective. The derivation below uses the exceptional zero of (G2) for zeros with $\delta \log R \le c_*/10$. Such a zero lies in the region of (G1) both for $T = R$ and for $T = R^5$, so the reading of (G2) does not depend on the value of T that Gallagher's statement leaves implicit.

Proof of (C7). Put $d_\circ = \min(d_*, c_*/10, 1/16)$ and $a_\circ = \min(a_*, d_\circ)$. We show that (C7) holds with

$$c_6 = \min\Big(\frac{a_\circ}{2}, \frac{1}{4}\Big)^2, \qquad A_0' = 112\,(C_* + 5) + 12 + \frac{64}{c_7}, \qquad C_0 = \max\Big(16, \frac{4}{b_*^2}, \frac{1}{d_\circ^2}\Big),$$

where c_7 is the constant of (C8).

Step 1. A form of (G2) with the smaller constant $d_\circ$. Let x, R and h be as in (G2). Suppose first that for a primitive character χ' of modulus $q' \le R$ the function $L(s, \chi')$ has a real zero $1 - \delta$ with $\delta \log R \le d_\circ$. Then $\delta < c_*/\log R$ and $\delta < c_*/\log R^5$, so by (G1) this zero is the exceptional zero, χ' is the unique primitive character of modulus at most R whose L-function has a zero $1 - \delta'$ with $\delta' \log R \le d_\circ$, and $1 - \delta$ is the unique such zero of $L(s, \chi')$. The character χ' is real, because otherwise $L(s, \bar\chi')$ would have the same real zero and $\bar\chi' \ne \chi'$ would contradict (G1). Since $d_\circ \le d_*$, (G2) gives $\xi \in [x, x + h]$ with

$$\sum_{q \le R} \sideset{}{^*}\sum_{\chi} |\Delta^\circ(\chi)| \le C_*\,(\delta \log R)^2\, h\, \exp\Big(-a_\circ \frac{\log x}{\log R}\Big), \tag{5.2}$$

where $\Delta^\circ(\chi) = \Delta(\chi)$ for $\chi \ne \chi'$ and $\Delta^\circ(\chi') = \Delta(\chi') + h\xi^{-\delta}$. Suppose next that no such χ' exists, and put $\Delta^\circ = \Delta$. If (G2) has no modified term, it gives the bound $C_* h \exp(-a_* \log x/\log R)$ for $\sum_{q \le R} \sum_\chi^* |\Delta^\circ(\chi)|$. If (G2) modifies the term of a character χ', then $\delta \log R > d_\circ$, so $|\Delta(\chi')| \le |\Delta(\chi') + h\xi^{-\delta}| + hx^{-\delta}$ and $x^{-\delta} \le \exp(-d_\circ \log x/\log R)$. In both cases

$$\sum_{q \le R} \sideset{}{^*}\sum_{\chi} |\Delta^\circ(\chi)| \le (C_* + 1)\, h\, \exp\Big(-a_\circ \frac{\log x}{\log R}\Big). \tag{5.3}$$

Step 2. The choice of the pair. Let M be divisible by 8. For a real nonprincipal character χ_1 modulo M the function $L(s, \chi_1)$ is entire and $L(1, \chi_1) \ne 0$, so it has finitely many zeros in the interval $1 - 1/(8 \log M) \le s \le 1$. Hence the pairs (χ_1, β_1) of (C7) form a finite set. If it is nonempty, we choose a pair with the largest β_1 and put $\delta_1 = 1 - \beta_1$. This choice is made once for the modulus M, before X and b are given. Note that $\log M \ge \log 8 > 2$, that $\log(\log M + 2) > 1$, and that $\beta_1 > 1 - 1/(8 \log 8) > \frac{1}{2}$.

Step 3. The three cases. Let $X \ge 2$ satisfy $C_0 \log M \log(\log M + 2) \le \log X$, let b be coprime to M, and put $R = \max(M, e^{\sqrt{\log X}})$. Then $\log X \ge C_0 \log M \ge 2C_0 \ge 32$, and

$$\log R \le \frac{b_*}{2} \log X, \qquad M \le X^{1/16}, \qquad \frac{\log X}{\log R} = \min\Big(\frac{\log X}{\log M}, \sqrt{\log X}\Big) \ge \frac{1}{d_\circ}. \tag{5.4}$$

The first inequality follows from $\log M \le \log X/C_0$ and $\sqrt{\log X} \le \frac{b_*}{2} \log X$, which holds because $\log X \ge 8/b_*^2$. The second follows from $C_0 \ge 16$, and the third from $C_0 \ge 1/d_\circ^2$. We apply Step 1 with this R and distinguish three cases.

In case (A), Step 1 finds a character χ', and its modulus q' divides M. Let χ_M be the principal character modulo M. The character $\chi'\chi_M$ is real and nonprincipal modulo M, and

$L(s,\chi'\chi_M) = L(s,\chi')\prod_{p\mid M}(1-\chi'(p)p^{-s})$, where no factor vanishes for real $s>0$. Since $\delta \le d_\circ/\log R \le 1/(16\log M)$, the pair $(\chi'\chi_M, 1-\delta)$ is a pair of (C7). The chosen pair therefore exists and has $\beta_1 \ge 1-\delta$. Its character χ_1 is induced by a primitive real character χ_1^* of conductor at most $M \le R$, the same factorization gives $L(\beta_1,\chi_1^*)=0$, and $(1-\beta_1)\log R \le d_\circ$. The uniqueness in Step 1 gives $\chi_1^* = \chi'$ and $\beta_1 = 1-\delta$. Hence $\chi_1 = \chi'\chi_M$, $\delta_1 = \delta$, and we put $\varepsilon_b = \chi'(b) \in \{1,-1\}$.

In case (B), Step 1 finds no character χ' with q' dividing M, and (C7) has no pair. We put $\varepsilon_b = 0$.

In case (C), Step 1 finds no character χ' with q' dividing M, and the chosen pair exists. We put $\varepsilon_b = \chi_1(b) \in \{1,-1\}$. In this case $\delta_1 \log R > d_\circ$, because otherwise the primitive character χ_1^* would be a character χ' of Step 1 with q' dividing M.

In all three cases put $w(t) = 1 - \varepsilon_b t^{-\delta_1}$ for $t \ge 1$, so that $0 \le w(t) \le 2$.

Step 4. Orthogonality on a short interval. Let $X^{1/2} \le x \le X$ and $x/R \le h \le x$. By (5.4), $\exp(\sqrt{\log x}) \le R \le X^{b_*/2} \le x^{b_*}$, so Step 1 applies to x, R and h. Put $\vartheta_b(y) = \sum_{p\le y,\, p\equiv b\ (M)} \log p$. Every character χ modulo M is induced by a unique primitive character χ^* of conductor dividing M, the map $\chi \mapsto \chi^*$ is a bijection onto these primitive characters (the trivial character modulo 1 included), and $\chi(n) = \chi^*(n)$ for n coprime to M [21, Ch. 5]. Since $x \ge X^{1/2} \ge M$, no prime in $(x, x+h]$ divides M, and the orthogonality relations give

$$\varphi(M)\big(\vartheta_b(x+h)-\vartheta_b(x)\big) = \sum_{\chi \bmod M} \bar\chi(b) \sum_{x<p\le x+h} \chi(p)\log p = h + \sum_{q\mid M} \sum_{\chi}^{*} \bar\chi(b)\,\Delta(\chi).$$

In case (A) we write $\Delta(\chi') = \Delta^\circ(\chi') - h\xi^{-\delta_1}$ and use $\bar\chi'(b) = \varepsilon_b$. In cases (B) and (C) every primitive character of conductor dividing M has $\Delta^\circ(\chi) = \Delta(\chi)$, because a character χ' found in Step 1 has a modulus that does not divide M. Bounding the sum over $q \mid M$ by the sum over $q \le R$, we get from (5.2) and (5.3)

$$\Big|\varphi(M)\big(\vartheta_b(x+h)-\vartheta_b(x)\big) - h + \varepsilon_b' h\xi^{-\delta_1}\Big| \le (C_*+1)\,h\,\Gamma(x),$$

where $\varepsilon_b' = \varepsilon_b$ in case (A) and $\varepsilon_b' = 0$ in cases (B) and (C), and where

$$\Gamma(x) = \begin{cases} (\delta_1\log R)^2 \exp(-a_\circ \log x/\log R) & \text{in case (A),} \\ \exp(-a_\circ \log x/\log R) & \text{in cases (B) and (C).} \end{cases}$$

In case (A) the numbers $\xi^{-\delta_1}$ and $h^{-1}\int_x^{x+h} t^{-\delta_1}\,dt$ both lie between $(x+h)^{-\delta_1}$ and $x^{-\delta_1}$, and $1-(1+u)^{-\delta_1} \le \delta_1 u$ for $u \ge 0$, so

$$\Big|h\xi^{-\delta_1} - \int_x^{x+h} t^{-\delta_1}\,dt\Big| \le h\,x^{-\delta_1}\Big(1-\Big(1+\frac{h}{x}\Big)^{-\delta_1}\Big) \le \frac{\delta_1 h^2}{x}.$$

In case (C) we use $\int_x^{x+h} t^{-\delta_1}\,dt \le hx^{-\delta_1} \le h\exp(-d_\circ\log x/\log R) \le h\,\Gamma(x)$. In all three cases

$$\Big|\varphi(M)\big(\vartheta_b(x+h)-\vartheta_b(x)\big) - \int_x^{x+h} w(t)\,dt\Big| \le (C_*+2)\,h\,\Gamma(x) + \eta\,\frac{h^2}{x}, \tag{5.5}$$

with $\eta = \delta_1$ in case (A) and $\eta = 0$ in cases (B) and (C).

Step 5. A tiling of $(X^{1/2}, X]$. Put $X_0 = X^{1/2}$, $L = \log(X/X_0) = \frac{1}{2}\log X$, $\ell = \log(1+1/R)$ and $J = \lfloor L/\ell \rfloor$. Since $L \ge 16 > 2\ell$, we have $J \ge L/\ell - 1 \ge L/(2\ell)$, so $r = e^{L/J}$ satisfies

$1 + 1/R \le r \le (1+1/R)^2 \le 1 + 3/R$. Put $x_j = X_0 r^j$ for $0 \le j \le J$, so that $x_J = X$, and $h_j = x_{j+1} - x_j = (r-1)x_j$. Then $x_j/R \le h_j \le 3x_j/R \le x_j$, and (5.5) applies to $x = x_j$ and $h = h_j$. Put

$$D(y) = \varphi(M)\big(\vartheta_b(y) - \vartheta_b(X_0)\big) - \int_{X_0}^{y} w(t)\,dt.$$

The function Γ is decreasing, $\sum_j h_j \le X$ and $h_j/x_j \le 3/R$, so summing (5.5) gives

$$|D(x_k)| \le \mathcal{E}, \qquad \mathcal{E} = (C_* + 2)\,X\,\Gamma(X^{1/2}) + \frac{3\eta X}{R}, \qquad 0 \le k \le J. \tag{5.6}$$

Step 6. Partial summation. Let $x_j \le t \le x_{j+1}$ and $W_j = \int_{x_j}^{x_{j+1}} w(u)\,du$. Both $\varphi(M)(\vartheta_b(t) - \vartheta_b(x_j))$ and $\int_{x_j}^{t} w(u)\,du$ are nondecreasing in t and vanish at $t = x_j$. Hence $-W_j \le D(t) - D(x_j) \le D(x_{j+1}) - D(x_j) + W_j$, and (5.6) gives $|D(t)| \le 3\mathcal{E} + W_j$. Stieltjes integration by parts and $D(X_0) = 0$ give

$$\varphi(M)\big(\pi(X;M,b) - \pi(X_0;M,b)\big) = \int_{X_0}^{X} \frac{w(t)}{\log t}\,dt + \frac{D(X)}{\log X} + \int_{X_0}^{X} \frac{D(t)}{t\log^2 t}\,dt.$$

Since $\int_{X_0}^{X} dt/(t\log^2 t) \le 2/\log X$, $\int_{x_j}^{x_{j+1}} dt/(t\log^2 t) \le \log r/\log^2 x_j \le 12/(R\log^2 X)$ and $\sum_j W_j \le \log X \int_{X_0}^{X} w(t)\,dt/\log t$, the last two terms are at most

$$\frac{7\,\mathcal{E}}{\log X} + \frac{12}{R}\int_{X_0}^{X} \frac{w(t)}{\log t}\,dt. \tag{5.7}$$

Step 7. The main term. In cases (A) and (C) the substitution $u = t^{\beta_1}$ gives $\mathrm{li}(X^{\beta_1}) = \int_{2^{1/\beta_1}}^{X} t^{-\delta_1}\,dt/\log t$. Hence

$$\varphi(M)\,\mathfrak{m}(X;M,b) = \int_2^X \frac{w(t)}{\log t}\,dt + \varepsilon_b\,k_1, \qquad k_1 = \int_2^{2^{1/\beta_1}} \frac{t^{-\delta_1}}{\log t}\,dt,$$

with $0 \le k_1 \le 2/\log 2 < 3$ because $2^{1/\beta_1} \le 4$, and the same identity holds in case (B) with $k_1 = 0$. With $\pi(X_0;M,b) \le X^{1/2}$, $\int_2^{X_0} w(t)\,dt/\log t \le 3X^{1/2}$, $\int_{X_0}^{X} w(t)\,dt/\log t \le \varphi(M)\mathfrak{m}(X;M,b) + 3$, $\varphi(M) \ge 4$ and (5.7), we obtain

$$|\pi(X;M,b) - \mathfrak{m}(X;M,b)| \le \frac{7\,\mathcal{E}}{\varphi(M)\log X} + \frac{12}{R}\,\mathfrak{m}(X;M,b) + 2X^{1/2}. \tag{5.8}$$

Step 8. A lower bound for the main term. Put $\mu = \min(1, \delta_1 \log X)$ in case (A) with $\varepsilon_b = 1$, and $\mu = 1$ otherwise. If $\varepsilon_b \in \{0, -1\}$, then $\varphi(M)\mathfrak{m}(X;M,b) \ge \mathrm{li}(X) \ge X/(2\log X)$. If $\varepsilon_b = 1$, then $2^{1/\beta_1} \le X^{1/2}$ and $1 - e^{-u} \ge \frac{1}{2}\min(1,u)$ for $u \ge 0$ give

$$\varphi(M)\,\mathfrak{m}(X;M,b) \ge \int_{X^{1/2}}^{X} \frac{1 - t^{-\delta_1}}{\log t}\,dt \ge \big(1 - X^{-\delta_1/2}\big)\frac{X - X^{1/2}}{\log X} \ge \frac{\min(1, \delta_1\log X)\,X}{8\log X}.$$

In case (C) the third inequality of (5.4) gives $\delta_1 \log X > d_\circ \log X/\log R \ge 1$. In every case

$$\varphi(M)\,\mathfrak{m}(X;M,b) \ge \frac{\mu\,X}{8\log X}. \tag{5.9}$$

In case (A) with $\varepsilon_b = 1$, (C8) applied to χ_1 gives $\delta_1 \geq c_7/(M^{1/2}\log^2 M)$. With $\log^2 M \leq 3M^{1/2}$ and $c_7 \leq 1$ this gives $1/\mu \leq 1 + 1/(\delta_1 \log X) \leq 4M/c_7$, and the same bound holds when $\mu = 1$.

Step 9. The relative error. We divide (5.8) by $\mathfrak{m}(X;M,b)$ and use (5.9). In case (A), $(\delta_1 \log R)^2 \leq 1$ and $(\delta_1 \log R)^2/(\delta_1 \log X) \leq 1$ give $\Gamma(X^{1/2})/\mu \leq \exp(-a_\circ \log X/(2\log R))$, and $\delta_1 \leq 1$ gives $\eta/\mu \leq 1$. The same two bounds hold in cases (B) and (C). Therefore

$$\frac{7\,\mathcal{E}}{\varphi(M)\log X\,\mathfrak{m}(X;M,b)} \leq \frac{56\,\mathcal{E}}{\mu X} \leq 56\,(C_* + 5)\Big(\exp\Big(-\frac{a_\circ \log X}{2\log R}\Big) + \frac{1}{R}\Big).$$

By the bound for $1/\mu$, $\varphi(M) \leq M \leq X^{1/16}$ and $\log X \leq X^{1/8}$,

$$\frac{2X^{1/2}}{\mathfrak{m}(X;M,b)} \leq \frac{16\,\varphi(M)\log X}{\mu\, X^{1/2}} \leq \frac{64}{c_7}\,M^2 X^{-1/2}\log X \leq \frac{64}{c_7}\,X^{-1/4}.$$

By the identity in (5.4), $\exp(-a_\circ \log X/(2\log R)) \leq e^{-(a_\circ/2)\log X/\log M} + e^{-(a_\circ/2)\sqrt{\log X}}$. Moreover $1/R \leq e^{-\sqrt{\log X}}$ and $X^{-1/4} \leq e^{-\sqrt{\log X}/4}$. Since $c_6 \leq a_\circ/2$, $\sqrt{c_6} \leq a_\circ/2$ and $\sqrt{c_6} \leq \frac{1}{4}$, each of the three quantities $\exp(-a_\circ \log X/(2\log R))$, $1/R$ and $X^{-1/4}$ is at most $e^{-c_6 \log X/\log M} + e^{-\sqrt{c_6 \log X}}$. Adding the bounds gives

$$\big|\pi(X;M,b) - \mathfrak{m}(X;M,b)\big| \leq \Big(112\,(C_*+5) + 12 + \frac{64}{c_7}\Big)\Big(e^{-c_6 \log X/\log M} + e^{-\sqrt{c_6 \log X}}\Big)\,\mathfrak{m}(X;M,b),$$

which is (C7) with the pair chosen in Step 2. □

The constants c_6, A_0' and C_0 are absolute, and A_0' is not effective because Gallagher's factor $(\delta \log R)^2$ rests on Siegel's theorem. The argument uses the factor $(\delta \log R)^2$ of (G2) in an essential way. Without it the first bound of Step 9 would contain $\exp(-a_\circ \log X/(2\log R))/\mu$, and with δ_1 as small as (C8) allows, this quantity is small when $\log X \geq C \log^2 M$ for a large constant C, a range that the hypothesis $C_0 \log M \log(\log M + 2) \leq \log X$ of (C7) does not give. The short tiles of Step 5 are used for a similar reason, because (G2) does not specify the point ξ. On a dyadic interval with $\delta_1 \log x$ of size 1, the uncertainty in $h\xi^{-\delta_1}$ has relative size $1/\log x$ compared with the main term of that interval, and this exceeds the error allowed in (C7).

Lemma 5.1. *Let χ be a primitive character modulo $N \geq 3$ and let $Q \geq 2$. Suppose that $L(s,\chi) \neq 0$ for every s with $\operatorname{Re} s > 1 - \frac{1}{400}$ and $|\operatorname{Im} s| \leq Q$. Then for every $u \geq 2$ and every T with $2 \leq T \leq \min(u, Q)$*

$$|\Theta(u,\chi)| \leq C_4\Big(u^{1-1/400}\log^2(NT) + \frac{u\log^2(Nu)}{T}\Big),$$

with an absolute constant C_4.

Proof. Let $\rho = \beta + i\gamma$ be a zero with $|\gamma| < T$. Then $\beta \leq 1 - \frac{1}{400}$ by hypothesis, and the same bound for the zero $1 - \bar{\rho}$ gives $\beta \geq \frac{1}{400}$, so $|u^\rho/\rho| \leq u^{1-1/400}/\max(\frac{1}{400}, |\gamma|)$. By the count of zeros in (C4),

$$\sum_{|\gamma|<T} \frac{1}{\max(\frac{1}{400}, |\gamma|)} \leq 400\,C_2 \log(3N) + \sum_{1 \leq k \leq T} \frac{2C_2 \log(N(k+3))}{k} \leq C\log^2(NT).$$

If the term with β_1 is present, then $\frac{1}{2} \leq \beta_1 \leq 1 - \frac{1}{400}$ and $u^{\beta_1}/\beta_1 \leq 2u^{1-1/400}$. Finally $u^{1/4}\log u \leq Cu^{1-1/400}$, and the explicit formula of (C4) gives the claim. □

5.2 Identities and the thin scales

Lemma 5.2. *Let χ be a primitive character modulo $N \geq 3$, and let $1 \leq y \leq y' \leq Z$. Then*

$$G_\chi^*(y) = -\Theta(y,\chi) - y\sum_{n>y}\frac{\Lambda(n)\chi(n)}{n} = -y\int_y^\infty \frac{\Theta(u,\chi)}{u^2}\,du, \tag{5.10}$$

$$G_\chi^*(y) = -S_Z(y) - y\,T_Z, \tag{5.11}$$

$$\frac{S_Z(y')}{y'} - \frac{S_Z(y)}{y} = -\int_y^{y'}\frac{\Theta(u,\chi)}{u^2}\,du, \tag{5.12}$$

where

$$S_Z(y) = \sum_{n\leq Z}\Lambda(n)\chi(n)\min\Big(1,\frac{y}{n}\Big), \qquad T_Z = \sum_{n>Z}\frac{\Lambda(n)\chi(n)}{n} = -\frac{\Theta(Z,\chi)}{Z} + \int_Z^\infty \frac{\Theta(u,\chi)}{u^2}\,du.$$

The identity (5.12) *holds with $\Lambda(n)\chi(n)$ replaced by arbitrary complex coefficients in S_Z and in Θ. Moreover $G(y) = -y\int_y^\infty(\Theta(u)-u)\,u^{-2}\,du$ for the function G of* (2.1)*.*

Proof. By (C2) the series $\sum_n \Lambda(n)\chi(n)/n$ converges to $-L'(1,\chi)/L(1,\chi)$, so

$$G_\chi^*(y) = y\sum_{n\leq y}\frac{\Lambda(n)\chi(n)}{n} - \Theta(y,\chi) - y\sum_{n\geq 1}\frac{\Lambda(n)\chi(n)}{n},$$

which is the first equality of (5.10). For $V > y$, summation by parts gives

$$\sum_{y<n\leq V}\frac{\Lambda(n)\chi(n)}{n} = \frac{\Theta(V,\chi)}{V} - \frac{\Theta(y,\chi)}{y} + \int_y^V \frac{\Theta(u,\chi)}{u^2}\,du. \tag{5.13}$$

By (C2), $\Theta(V,\chi)/V$ tends to zero as $V\to\infty$ and the integral converges, which gives the second equality of (5.10) and, with $y = Z$, the formula for T_Z. Splitting the sum over $n > y$ at Z gives (5.11), because $\min(1, y/n)$ equals 1 for $n \leq y$ and y/n for $y < n \leq Z$. The same splitting and (5.13) with $V = Z$ give

$$\frac{S_Z(y)}{y} = \frac{\Theta(y,\chi)}{y} + \sum_{y<n\leq Z}\frac{\Lambda(n)\chi(n)}{n} = \frac{\Theta(Z,\chi)}{Z} + \int_y^Z \frac{\Theta(u,\chi)}{u^2}\,du,$$

and the difference of this identity at y' and at y is (5.12). This step is a summation by parts, valid for arbitrary coefficients. For G, the same summation by parts gives $\sum_{n\leq y}\Lambda(n)/n = \Theta(y)/y + \int_1^y \Theta(u)u^{-2}\,du$, and (C2) together with the expansion of ζ'/ζ at $s = 1$ gives $\int_1^\infty(\Theta(u)-u)u^{-2}\,du = -1-\gamma$. Hence

$$y\sum_{n\leq y}\frac{\Lambda(n)}{n} - \Theta(y) = y\log y - (1+\gamma)y - y\int_y^\infty\frac{\Theta(u)-u}{u^2}\,du,$$

and (2.1) gives the claim. □

Proposition 5.3. *Let χ be a primitive character modulo $N \geq 3$, let $\lambda \geq \max(1,\log N)$ and $0 < c_* \leq \frac{1}{2}$, and suppose that either χ is not real, or $L(\beta,\chi)\neq 0$ for every real β with $1-c_* < \beta < 1$. Then for every $y \geq 2$, with $t_0 = \log y$,*

$$\frac{|G_\chi^*(y)|}{y} \leq \frac{2e^{-c_* t_0}}{c_*} + C_1\lambda^4\Big(\frac{2\lambda}{c_1}\,e^{-c_1 t_0/(2\lambda)} + \frac{8}{c_1^2}\Big(1+\frac{c_1\sqrt{t_0}}{2}\Big)e^{-c_1\sqrt{t_0}/2}\Big).$$

Proof. If the term with β in (C2) is present, then $\frac{1}{2} \le \beta \le 1 - c_*$, so $u^\beta/\beta \le 2u^{1-c_*}$. Since $\log N \le \lambda$, the bound of (C2) gives, for every $u \ge 2$,

$$|\Theta(u,\chi)| \le 2u^{1-c_*} + C_1\, u\, \lambda^4 \exp\Big(- \frac{c_1 \log u}{\sqrt{\log u} + \lambda}\Big).$$

For $t > 0$ the quotient $t/(\sqrt{t} + \lambda)$ is at least $\sqrt{t}/2$ when $\sqrt{t} \ge \lambda$ and at least $t/(2\lambda)$ otherwise, so with $t = \log u$ the exponential is at most $e^{-c_1 t/(2\lambda)} + e^{-c_1\sqrt{t}/2}$. By (5.10) and the substitution $u = e^t$,

$$\frac{|G^*_\chi(y)|}{y} \le \int_y^\infty \frac{2\,du}{u^{1+c_*}} + C_1\lambda^4 \int_{t_0}^\infty \Big(e^{-c_1 t/(2\lambda)} + e^{-c_1\sqrt{t}/2}\Big) dt,$$

and the three integrals equal $2y^{-c_*}/c_*$, $(2\lambda/c_1)e^{-c_1 t_0/(2\lambda)}$ and $(8/c_1^2)(1 + c_1\sqrt{t_0}/2)e^{-c_1\sqrt{t_0}/2}$. □

For the trivial character, the last identity of Lemma 5.2 and the second bound of (C2) give the same decay for G.

5.3 A mean square over prime moduli

Proposition 5.4. *Let $Z \ge 1$ and $\eta \ge 0$, let P be a finite set of primes $q > Z$, let E_0 be a finite set of squarefree integers $s > 1$, and let c_n for $1 \le n \le Z$ be complex numbers. Suppose that $|\sum_{q\in P}(\frac{s}{q})| \le \eta\,|P|$ for every squarefree integer s with $1 < s \le Z^2$ and $s \notin E_0$. Then*

$$\sum_{q\in P} \Big| \sum_{n\le Z} c_n \Big(\frac{n}{q}\Big)\Big|^2 \le |P|\Big(\sum_{\substack{n,n'\le Z \\ nn' \text{ a square}}} |c_n c_{n'}| + \sum_{s\in E_0} \sum_{\substack{n,n'\le Z \\ nn'/s \text{ a square}}} |c_n c_{n'}| + \eta\Big(\sum_{n\le Z}|c_n|\Big)^2\Big).$$

Proof. Expanding the square gives $\sum_{n,n'} c_n\overline{c_{n'}} \sum_{q\in P}(\frac{nn'}{q})$. Each $q \in P$ exceeds Z, so q does not divide nn'. Write $nn' = sr^2$ with s squarefree, so that $s \le Z^2$ and the symbol equals $(\frac{s}{q})$. If $s = 1$ the inner sum is $|P|$, if $s \in E_0$ it has absolute value at most $|P|$, and otherwise it has absolute value at most $\eta|P|$. □

In both choices of auxiliary primes the sum of the symbols $(\frac{s}{q})$ over the candidates is small for most squarefree s and large for the s in a set of the following kind. For $z \ge 2$ and an integer $s^* \ge 1$ without prime factors $p \le z$, let $\mathcal{S}(z, s^*)$ be the set of squarefree integers $s > 1$ whose largest divisor without prime factors $p \le z$ equals 1 or s^*.

Lemma 5.5. *Let $3 \le y \le Z$, let $\kappa(n)$ be complex numbers with $|\kappa(n)| \le 1$, and put $c_n = \Lambda(n)\kappa(n)\min(1, y/n)$ for $n \le Z$. Let $z \ge 2$ with $z\log y \le \sqrt{y}$, and let $s^* \ge 1$ be an integer without prime factors $p \le z$. With the constant c_0 of* (C1),

$$\sum_{\substack{n,n'\le Z \\ nn' \text{ a square}}} |c_n c_{n'}| \le 60\, c_0^2\, y \log y,$$

$$\sum_{s\in\mathcal{S}(z,s^*)} \sum_{\substack{n,n'\le Z \\ nn'/s \text{ a square}}} |c_n c_{n'}| \le 81\, c_0^2\, y,$$

$$\sum_{n\le Z} |c_n| \le 3\, c_0\, y \log Z.$$

Proof. For a prime p put $W_p = \sum_{a\ge 1} \log p \, \min(1, y/p^a)$, and put $W_\square = \sum_m \Lambda(m) \min(1, y/m^2)$, which bounds the sum of $|c_n|$ over the squares n.

If $p \le y$, the powers $p^a \le y$ contribute at most $\log y$ to W_p, and the powers $p^a > y$ contribute at most $2\log p$, because the numbers y/p^a are less than one and decrease by the factor $p \ge 2$. If $p > y$, then $W_p \le 2y \log p / p \le 2 \log y$. Hence $W_p \le 3 \log y$ for every p, and by (C1)

$$\sum_{p\le y} W_p^2 \le 9\,\pi(y) \log^2 y \le 9 c_0\, y \log y.$$

For $p > y$ we group the primes in the intervals $2^k y < p \le 2^{k+1} y$ and use (C1) again,

$$\sum_{p>y} W_p^2 \le 4y^2 \sum_{k\ge 0} \pi(2^{k+1}y)\, \frac{\log^2(2^{k+1}y)}{4^k y^2} \le 8 c_0\, y \sum_{k\ge 0} \frac{\log y + (k+1)\log 2}{2^k} \le 40 c_0\, y \log y,$$

since $\sum_k 2^{-k} = 2$, $\sum_k (k+1) 2^{-k} = 4$ and $4 \log 2 \le 3 \log y$. By (C1) and a summation by parts,

$$W_\square \le \Theta(\sqrt{y}) + y \sum_{m > \sqrt{y}} \frac{\Lambda(m)}{m^2} \le c_0 \sqrt{y} + 2y \int_{\sqrt{y}}^\infty \frac{\Theta(u)}{u^3}\, du \le 3 c_0 \sqrt{y}.$$

Let $n = p^a$ and $n' = p'^b$ be prime powers. If nn' is a square, then either $p = p'$, or n and n' are both squares, so the first sum is at most $\sum_p W_p^2 + W_\square^2 \le 49 c_0 y \log y + 9 c_0^2 y \le 60 c_0^2 y \log y$.

For the second sum, write $nn' = sr^2$ with $s > 1$ squarefree, so that the prime factors of s are the primes that occur in nn' to an odd power. Let $\mathcal{A}$ be the set of the $n \le Z$ that are squares or powers of a prime $p \le z$. The sum of $|c_n|$ over $\mathcal{A}$ is at most

$$W_\square + \sum_{p\le z} W_p \le 3 c_0 \sqrt{y} + 3z \log y \le 6 c_0 \sqrt{y}.$$

If every prime factor of s is at most z, then n and n' both lie in $\mathcal{A}$, because a prime $p > z$ that occurs in n to an odd power must occur in n' to an odd power, and then nn' is a square. These pairs contribute at most $36 c_0^2 y$. Otherwise $s \in \mathcal{S}(z, s^*)$ forces the primes $p > z$ dividing s to be the prime factors of s^*, and nn' has at most two prime factors. If s^* is a prime p^*, then one of n, n' is an odd power of p^* and the other lies in $\mathcal{A}$, which contributes at most $2 W_{p^*} \cdot 6 c_0 \sqrt{y} \le 36 c_0 \sqrt{y} \log y$. If s^* is a product of two primes, then n and n' are odd powers of these primes, which contributes at most $18 \log^2 y$. The integer s^* is fixed, so at most one of the last two cases occurs, and $18 \log^2 y \le 36 c_0 \sqrt{y} \log y \le 36 c_0^2 y$ because $\log y \le \sqrt{y}$ and $c_0 \ge 1$. The second sum is therefore at most $72 c_0^2 y$. Finally, by (C1) and a summation by parts,

$$\sum_{n\le Z} |c_n| \le \Theta(y) + y \sum_{y < n \le Z} \frac{\Lambda(n)}{n} \le c_0 y + y\Big(c_0 + c_0 \log \frac{Z}{y}\Big) \le 3 c_0\, y \log Z. \qquad \square$$

5.4 Uniformity on a window

A mean square bounds a sum at one point y for most q, and Corollary 4.4 needs the bound at every point of a window. The identity (5.12) controls a dyadic block of values of y by its left endpoint and one integral.

Proposition 5.6. *Let $0 < \tau \le \frac{1}{2}$, $8 \le Y \le X$ and $Z \ge 2X$, and let $\eta \ge 0$ with $\eta\, X \log^2 Z \le 1$. Let $z \ge 2$ with $z \log Y \le \sqrt{Y}$, and let $s^* \ge 1$ be an integer without prime factors $p \le z$. Let P be a finite set of primes $q > Z$, and suppose that $|\sum_{q\in P}(\frac{s}{q})| \le \eta |P|$ for every squarefree s with $1 < s \le Z^2$ and $s \notin \mathcal{S}(z, s^*)$. Let $|\kappa(n)| \le 1$, and for $q \in P$ put*

$$S_q(y) = \sum_{n \le Z} \Lambda(n)\kappa(n)\Big(\frac{n}{q}\Big) \min\Big(1, \frac{y}{n}\Big).$$

Then the number of $q \in P$ for which $|S_q(y)| > \frac{1}{2} y^{1/2+\tau}$ for some y with $Y \le y \le X$ is at most

$$\frac{16000\, c_0^2}{(1-4^{-\tau})^2}\ |P|\ Y^{-2\tau} \log(2Y).$$

Proof. Put $\Theta_q(u) = \sum_{n\le u} \Lambda(n)\kappa(n)(\frac{n}{q})$. The function $\sqrt{y}/\log y$ increases for $y \ge 8$, so $z \log y \le \sqrt{y}$ for every $y \ge Y$. For $Y \le y \le 2X$, Proposition 5.4 with E_0 the set of the $s \le Z^2$ in $\mathcal{S}(z, s^*)$, together with Lemma 5.5, gives

$$\sum_{q\in P} |S_q(y)|^2 \le |P|\, c_0^2 \Big(60\, y \log y + 81\, y + 9\, \eta\, y^2 \log^2 Z \Big) \le 159\, c_0^2\, |P|\, y \log y,$$

because $\eta y^2 \log^2 Z \le 2y$. The same two results with Z replaced by u and $y = u$ give $\sum_{q\in P} |\Theta_q(u)|^2 \le 159 c_0^2 |P| u \log u$ for $Y \le u \le 2X$. Put $J_q(y) = \int_y^{2y} |\Theta_q(u)|^2 u^{-2}\, du$, so that for $Y \le y \le X$

$$\sum_{q\in P} J_q(y) \le 159\, c_0^2\, |P| \int_y^{2y} \frac{\log u}{u}\, du \le 159\, c_0^2\, |P| \log(2y).$$

Fix y with $Y \le y \le X$ and let $y \le y' \le 2y$. By (5.12) and the Cauchy–Schwarz inequality,

$$\begin{aligned} |S_q(y')| &\le \frac{y'}{y}\, |S_q(y)| + y' \int_y^{2y} \frac{|\Theta_q(u)|}{u^2}\, du \\ &\le 2\, |S_q(y)| + 2y\, J_q(y)^{1/2} \Big(\int_y^{2y} \frac{du}{u^2} \Big)^{1/2} = 2\, |S_q(y)| + \sqrt{2y}\, J_q(y)^{1/2}. \end{aligned} \tag{5.14}$$

By Chebyshev's inequality, the number of q with $|S_q(y)| > \frac{1}{8} y^{1/2+\tau}$ is at most $64 \cdot 159\, c_0^2 |P| y^{-2\tau} \log y$, and the number of q with $J_q(y) > \frac{1}{32} y^{2\tau}$ is at most $32 \cdot 159\, c_0^2 |P| y^{-2\tau} \log(2y)$. For every other q, (5.14) gives

$$|S_q(y')| \le \tfrac{1}{4} y^{1/2+\tau} + \sqrt{2y}\, \frac{y^\tau}{\sqrt{32}} = \tfrac{1}{2} y^{1/2+\tau} \le \tfrac{1}{2} y'^{1/2+\tau} \qquad (y \le y' \le 2y).$$

The blocks with left endpoints $y_i = 2^i Y$, $0 \le i \le \log_2(X/Y)$, cover the interval from Y to X. The number of exceptional q is therefore at most

$$96 \cdot 159\, c_0^2\, |P| \sum_{i \ge 0} (2^i Y)^{-2\tau} \big(\log(2Y) + i \log 2 \big) \le \frac{15264\, c_0^2}{(1-4^{-\tau})^2}\, |P|\, Y^{-2\tau} \log(2Y),$$

since $\sum_i r^i + \sum_i i r^i = (1-r)^{-2}$ for $r = 4^{-\tau}$. □

5.5 The first auxiliary prime

Proposition 5.7. *There are absolute constants $c_3 > 0$ and z_0 with the following property. Let $z \geq z_0$, let P be the set of primes $q \equiv 3 \pmod 8$ with $e^{z/2} < q \leq e^z$, and let $R \geq 3$ with $\log(8R) \leq \sqrt{z/2}$. Then there is an odd integer $s^* \geq 1$ such that*

$$\Big|\sum_{q\in P}\Big(\frac{s}{q}\Big)\Big| \leq e^{-c_3\sqrt{z}}\,|P| \qquad \text{for every squarefree } s \text{ with } 1 < s \leq R \text{ and } s \notin \mathcal{S}(2,s^*).$$

Proof. Put $U = e^z$. Let r^* be the conductor of the character excluded in (C3) at the level $8R$, and let s^* be the odd part of r^*. If that character does not exist, put $s^* = 1$. Let $s \notin \mathcal{S}(2,s^*)$ be squarefree with $1 < s \leq R$, and write $s = 2^e s_1$ with s_1 odd, so that $s_1 \neq 1$ and $s_1 \neq s^*$. Every $q \in P$ exceeds s and satisfies $q \equiv 3 \pmod 8$, so $(\frac{2}{q}) = -1$, and the reciprocity law for the Jacobi symbol gives $(\frac{s_1}{q}) = (-1)^{(s_1-1)/2}(\frac{q}{s_1})$. We detect the class of q modulo 8 with the four characters ϖ modulo 8 and obtain

$$\sum_{q\in P}\Big(\frac{s}{q}\Big) = \pm\frac{1}{4}\sum_{\varpi}\varpi(3)\sum_{\substack{\sqrt{U}<q\leq U\\ q \text{ prime}}}\chi'_\varpi(q),$$

where χ'_ϖ is the primitive real character that induces $\varpi\cdot(\frac{\cdot}{s_1})$. Its conductor is s_1 times the conductor of ϖ, so it is at least 3 and at most $8R$, its odd part is $s_1 \neq s^*$, and χ'_ϖ is not the character excluded in (C3).

Let χ' be one of these four characters and let $\sqrt{U} \leq V \leq U$, so that $\log V \geq z/2$ and $\log(8R) \leq \sqrt{\log V}$. If the term with β in (C2) is present, then $\frac{1}{2} \leq \beta \leq 1 - c_2/\log(8R)$ by (C3), and $V^\beta/\beta \leq 2V\exp(-c_2\sqrt{z/2})$. The exponent in (C2) is at least $c_1\sqrt{\log V}/2 \geq c_1\sqrt{z}/3$, and $(\log 8R)^4 \leq z^2/4$. Hence

$$|\Theta(V,\chi')| \leq V\epsilon_1, \qquad \epsilon_1 = 2e^{-c_2\sqrt{z/2}} + \tfrac{1}{4}C_1 z^2 e^{-c_1\sqrt{z}/3}.$$

The prime powers $p^a \leq V$ with $a \geq 2$ contribute at most $2c_0\sqrt{V}\log V$ to $\Theta(V,\chi')$ by (C1), so $\vartheta(V) = \sum_{p\leq V}\chi'(p)\log p$ satisfies $|\vartheta(V)| \leq V(\epsilon_1 + 2c_0 z e^{-z/4})$ for $\sqrt{U} \leq V \leq U$. A summation by parts gives

$$\Big|\sum_{\sqrt{U}<q\leq U}\chi'(q)\Big| = \Big|\frac{\vartheta(U)}{\log U} - \frac{\vartheta(\sqrt{U})}{\log\sqrt{U}} + \int_{\sqrt{U}}^{U}\frac{\vartheta(V)}{V\log^2 V}\,dV\Big| \leq \frac{4U}{z}\big(\epsilon_1 + 2c_0 z e^{-z/4}\big)$$

for $z \geq 4$. By (C6), $|P| \geq U/(5z)$ for large z, and therefore

$$\Big|\sum_{q\in P}\Big(\frac{s}{q}\Big)\Big| \leq 20\,|P|\,\big(\epsilon_1 + 2c_0 z e^{-z/4}\big) \leq e^{-c_3\sqrt{z}}\,|P|$$

for $z \geq z_0$, with $c_3 = \frac{1}{4}\min(c_1, c_2)$. □

Proposition 5.8. *There is an absolute constant C_5 with the following property. Let χ be a primitive character modulo $N \geq 3$, put $\lambda = \log N$, and let $Q \geq N$. Suppose that $L(s,\chi) \neq 0$ for every s with $\operatorname{Re} s > 1 - \frac{1}{400}$ and $|\operatorname{Im} s| \leq Q$. Then for every Z with $e^{800} \leq Z \leq \exp(\lambda^4)$ the number $T_Z = \sum_{n>Z}\Lambda(n)\chi(n)/n$ satisfies*

$$|T_Z| \leq C_5\,\lambda^{12}\Big(Z^{-1/400} + \frac{1}{Q} + e^{-c_1\lambda^2/2}\Big), \qquad \Big|\frac{L'(1,\chi)}{L(1,\chi)}\Big| \leq c_0(1+\log Z) + |T_Z|.$$

Proof. By Lemma 5.2, $T_Z = -\Theta(Z,\chi)/Z + \int_Z^\infty \Theta(u,\chi)u^{-2}\,du$. Put $V_1 = \exp(\lambda^4)$. For $Z \le u \le V_1$ we apply Lemma 5.1 with $T = \min(u^{1/400}, Q)$, which satisfies $2 \le T \le \min(u,Q)$ and $1/T \le u^{-1/400} + 1/Q$. Since $\log(Nu) \le 2\lambda^4$,

$$|\Theta(u,\chi)| \le 4C_4\lambda^8\Big(2u^{1-1/400} + \frac{u}{Q}\Big) \qquad (Z \le u \le V_1).$$

This bounds $|\Theta(Z,\chi)|/Z$ by $4C_4\lambda^8(2Z^{-1/400} + 1/Q)$, and

$$\int_Z^{V_1} \frac{|\Theta(u,\chi)|}{u^2}\,du \le 4C_4\lambda^8\Big(800\,Z^{-1/400} + \frac{\lambda^4}{Q}\Big).$$

For $u \ge V_1$ we use (C2). If the term with β is present, then $\frac{1}{2} \le \beta \le 1 - \frac{1}{400}$ by hypothesis, and $\log u \ge \lambda^2$ gives the exponent $c_1\sqrt{\log u}/2$, so

$$|\Theta(u,\chi)| \le 2u^{1-1/400} + C_1\,u\,\lambda^4\,e^{-c_1\sqrt{\log u}/2} \qquad (u \ge V_1).$$

The integral of the first term against $u^{-2}du$ over $u \ge V_1$ is $800V_1^{-1/400} \le 800Z^{-1/400}$. With $u = e^t$ the integral of the second term is

$$C_1\lambda^4\int_{\lambda^4}^\infty e^{-c_1\sqrt{t}/2}\,dt = \frac{8C_1\lambda^4}{c_1^2}\Big(1 + \frac{c_1\lambda^2}{2}\Big)e^{-c_1\lambda^2/2}.$$

The sum of these bounds gives the first claim. For the second, $-L'(1,\chi)/L(1,\chi) = \sum_{n\le Z}\Lambda(n)\chi(n)/n + T_Z$, and (C1) with a summation by parts gives $\sum_{n\le Z}\Lambda(n)/n \le c_0(1+\log Z)$. □

Proposition 5.9. *Let $N_0 \ge 1$ and $Q \ge 2$, and let B be a set of odd primes $q > N_0$ with $N_0 q \le Q$. Suppose that for every $q \in B$ there is a primitive character ξ of conductor r dividing N_0 such that $L(s, \xi\cdot(\frac{\cdot}{q}))$ has a zero with $\operatorname{Re} s > 1 - \frac{1}{400}$ and $|\operatorname{Im} s| \le Q$. Then*

$$|B| \le C_3\,Q^{3/160}\,(2\log Q)^{14}.$$

Proof. The character $\xi\cdot(\frac{\cdot}{q})$ is primitive of conductor $rq \le Q$, because q does not divide N_0. Since q is the largest prime factor of rq, different $q \in B$ give different primitive characters, each with $N(1-\frac{1}{400}, Q, \chi) \ge 1$. Statement (C5) with $T = Q$ and $\sigma = 1 - \frac{1}{400}$ bounds their number by $C_3(Q^3)^{1/160}\log^{14}(Q^2)$. □

Theorem 5.10. *Let $N_0 \ge 1$ be an integer, and let $K \ge 1$, $0 < \varepsilon \le \frac{1}{52}$ and $0 < \tau \le \frac{1}{2}$. There is Y_0 such that the following holds for every $Y \ge Y_0$. Put $z = Y^{\varepsilon/5}$, and let P be the set of primes $q \equiv 3 \pmod 8$ with $e^{z/2} < q \le e^z$. Then P is nonempty, and for all $q \in P$ outside a set of at most $Y^{-\tau}|P|$ elements, every primitive character ξ of conductor dividing N_0 and the character $\chi = \xi\cdot(\frac{\cdot}{q})$ satisfy*

(i) $L(s,\chi) \ne 0$ *for every s with $\operatorname{Re} s > 1 - \frac{1}{400}$ and $|\operatorname{Im} s| \le N_0 e^z$,*

(ii) $|L'(1,\chi)/L(1,\chi)| \le 401\,c_0\,K\log Y$,

(iii) $|G_\chi^*(y)| \le y^{1/2+\tau}$ *for every y with $Y \le y \le Y^K$.*

Proof. Put $X = Y^K$, $Z = X^{400}$, $Q = N_0 e^z$ and $U = e^z$. Every inequality that we use between powers of Y, powers of $\log Y$ and exponentials of powers of Y holds for $Y \geq Y_0$, with Y_0 depending on N_0, K, ε and τ. In particular $e^{z/2} > Z^2 \geq N_0$, $2X \leq Z$, and $\log(8Z^2) \leq \sqrt{z/2}$.

The Legendre symbols. By (C6) the set P is nonempty and $|P| \geq e^z/(5z)$. Proposition 5.7 with $R = Z^2$ gives an odd integer s^* for which the hypothesis of Proposition 5.6 holds with the parameter 2 in place of z and with $\eta = e^{-c_3\sqrt{z}}$, and this η satisfies $\eta X \log^2 Z \leq 1$.

The truncated sums. Let ξ be a primitive character of conductor r dividing N_0. For $q \in P$ the character $\chi = \xi \cdot (\frac{\cdot}{q})$ is primitive modulo rq, and the sum $S_Z(y)$ of Lemma 5.2 for χ is the sum $S_q(y)$ of Proposition 5.6 with $\kappa = \xi$. There are at most N_0 characters ξ, so Proposition 5.6 gives $|S_Z(y)| \leq \frac{1}{2} y^{1/2+\tau}$ for all ξ and all $Y \leq y \leq X$, for all $q \in P$ outside a set with at most

$$\frac{16000\, c_0^2\, N_0}{(1-4^{-\tau})^2}\ |P|\ Y^{-2\tau} \log(2Y) \leq \tfrac{1}{2} Y^{-\tau} |P|$$

elements.

The boxes. By Proposition 5.9, the number of $q \in P$ for which (i) fails for some ξ is at most $C_3 Q^{3/160} (2\log Q)^{14} \leq Q^{1/4} \leq \frac{1}{2} Y^{-\tau} |P|$, because $|P| \geq e^z/(5z)$ and $Q^{1/4} = N_0^{1/4} e^{z/4}$.

The tail. Let q lie outside both exceptional sets, and let χ be as above, with conductor $N = rq$ and $\lambda = \log N$, so that $z/2 < \lambda \leq 2z$ and $N \leq Q$. Then $e^{800} \leq Z \leq \exp(\lambda^4)$, and Proposition 5.8 gives

$$|T_Z| \leq C_5 (2z)^{12} \Big(\frac{1}{X} + \frac{1}{N_0 e^z} + e^{-c_1 z^2/8} \Big) \leq \frac{2^{13} C_5\, Y^{12\varepsilon/5}}{X}.$$

Since $12\varepsilon/5 < \frac{1}{2}$, this gives $y|T_Z| \leq 2^{13} C_5 Y^{12\varepsilon/5} \leq \frac{1}{2} y^{1/2+\tau}$ for $Y \leq y \leq X$, and (5.11) gives (iii). It also gives $|T_Z| \leq 1$, and the second bound of Proposition 5.8 gives

$$\Big| \frac{L'(1,\chi)}{L(1,\chi)} \Big| \leq c_0 \big(1 + 400K \log Y\big) + 1 \leq 401\, c_0\, K \log Y,$$

which is (ii). □

The proof of Theorem 5.10 uses the classical zero-free region through (C2), the possible exceptional character through (C3), and the density of zeros through (C5). It does not use a zero-free half-plane for any character. The square root in (iii) comes from the mean square over the family of moduli.

5.6 The second auxiliary prime

The second auxiliary prime d is chosen in one residue class of a large modulus. Let $z \geq 2$, let $q > z$ be a prime, and put

$$m_0 = 16\, q \prod_{2 < p \leq z} p.$$

Let a_0 be the residue class modulo m_0 with $a_0 \equiv 7 \pmod{16}$ and $a_0 \equiv -1$ modulo q and modulo every odd prime $p \leq z$. For a prime $d \equiv a_0 \pmod{m_0}$ these congruences give $(\frac{-d}{p}) = 1$ for the odd primes $p \leq z$ and for $p = q$. They also fix the symbols $(\frac{s}{d})$ for every s whose prime factors are at most z, and this is the reason for the sets $\mathcal{S}(z, s^*)$.

Proposition 5.11. *There is an absolute constant U_1 with the following property. Let $z \geq 2$, let $R \geq 3$, let q be a prime with $q > \max(z, R)$, and let m_0 and a_0 be as above. Put $\ell = \log(m_0 R)$, let $U \geq U_1$ with*

$$\log U \geq 16\,\ell \qquad \text{and} \qquad \log U \geq C_0\,\ell\,\log(\ell+2),$$

and let P be the set of primes $d \equiv a_0 \pmod{m_0}$ with $\sqrt{U} < d \leq U$. Put

$$\eta = 4\log U\, e^{-c_2 \log U/\ell} + 2A_0'\Big(e^{-c_6\log U/\ell} + e^{-\sqrt{c_6 \log U}}\Big) + 2U^{-3/8},$$

and suppose that $A_0'\big(e^{-c_6\log U/\ell} + e^{-\sqrt{c_6\log U}}\big) \leq \frac{1}{4}$. Then $|P| \geq U^{3/4}$, and there is an integer $s^* \geq 1$ without prime factors $p \leq z$ such that

$$\Big|\sum_{d\in P}\Big(\frac{s}{d}\Big)\Big| \leq \eta\,|P| \qquad \text{for every squarefree } s \text{ with } 1 < s \leq R \text{ and } s \notin \mathcal{S}(z, s^*).$$

Proof. The symbols. Let $d \in P$. Then $d \equiv 7 \pmod 8$, so $(\frac{2}{d}) = 1$ and $d \equiv 3 \pmod 4$. For an odd prime $p \leq z$ the reciprocity law and $d \equiv -1 \pmod p$ give $(\frac{p}{d}) = (-1)^{(p-1)/2}(\frac{-1}{p}) = 1$. Let s be squarefree with $1 < s \leq R$, and let s_b be its largest divisor without prime factors $p \leq z$. Since $s \leq R < q$, the integer s_b is coprime to m_0, and

$$\Big(\frac{s}{d}\Big) = \Big(\frac{s_b}{d}\Big) = (-1)^{(s_b-1)/2}\chi(d), \qquad \chi = \Big(\frac{\cdot}{s_b}\Big).$$

Let χ^* be the character excluded in (C3) at the level m_0R, let s^* be the largest divisor of its conductor that is coprime to m_0, and put $s^* = 1$ if χ^* does not exist. From now on let $s \notin \mathcal{S}(z, s^*)$, so that $s_b \neq 1$ and $s_b \neq s^*$, and χ is a primitive real character modulo s_b.

The progression. Put $M = m_0 s_b$, so that 8 divides M and $\log M \leq \ell$. For b coprime to s_b let c_b be the class modulo M with $c_b \equiv a_0 \pmod{m_0}$ and $c_b \equiv b \pmod{s_b}$. Write $\pi_b = \pi(U; M, c_b)$ and $\mathfrak{m}_b = \mathfrak{m}(U; M, c_b)$ in the notation of (C7), and put $\mathfrak{M} = \sum_b \mathfrak{m}_b$ and $\delta = A_0'(e^{-c_6\log U/\ell} + e^{-\sqrt{c_6\log U}})$. Each $\mathfrak{m}_b$ is nonnegative, and (C7) gives $|\pi_b - \mathfrak{m}_b| \leq \delta\,\mathfrak{m}_b$. The primes $d \leq U$ in the class a_0 modulo m_0 that do not divide s_b are counted by $\sum_b \pi_b$, so

$$\Big|\sum_{d\in P}\chi(d) - \sum_b \chi(b)\,\mathfrak{m}_b\Big| \leq \delta\,\mathfrak{M} + \sqrt{U}, \qquad |P| \geq (1-\delta)\,\mathfrak{M} - \sqrt{U}. \tag{5.15}$$

The main terms. If the pair (χ_1, β_1) of (C7) for the modulus M does not exist, then $\sum_b \chi(b)\mathfrak{m}_b = 0$ and $\mathfrak{M} = \mathrm{li}(U)/\varphi(m_0)$. If it exists, write $\chi_1 = \chi_1'\chi_1''$ with characters χ_1' modulo m_0 and χ_1'' modulo s_b, so that $\chi_1(c_b) = \chi_1'(a_0)\chi_1''(b)$. The sum $\sum_b \chi(b)\chi_1''(b)$ equals $\varphi(s_b)$ when $\chi_1'' = \chi$ and vanishes otherwise, and $\sum_b \chi_1''(b)$ equals $\varphi(s_b)$ when χ_1'' is principal and vanishes otherwise. Hence

$$\Big|\sum_b \chi(b)\,\mathfrak{m}_b\Big| \leq \frac{\mathrm{li}(U^{\beta_1})}{\varphi(m_0)} \text{ if } \chi_1'' = \chi, \qquad \sum_b \chi(b)\,\mathfrak{m}_b = 0 \text{ otherwise,} \tag{5.16}$$

and $\mathfrak{M} \geq (\mathrm{li}(U) - \mathrm{li}(U^{\beta_1}))/\varphi(m_0)$ in every case.

A lower bound for $\mathfrak{M}$. By (C8), $1 - \beta_1 \geq c_7/(M^{1/2}\log^2 M)$. Since $1 - e^{-t} \geq \frac{1}{2}\min(1, t)$ for $t \geq 0$,

$$\mathrm{li}(U) - \mathrm{li}(U^{\beta_1}) \geq \frac{U - U^{\beta_1}}{\log U} \geq \frac{U}{2\log U}\min\big(1, (1-\beta_1)\log U\big) \geq \frac{c_7\,U}{2M^{1/2}\log^2 M\,\log U}$$

for large U. With $\varphi(m_0) \le M \le U^{1/16}$ this gives $\mathfrak{M} \ge U^{7/8}$ for $U \ge U_1$, and the same bound holds when the pair does not exist. Since $\delta \le \frac{1}{4}$ and $\sqrt{U} \le U^{-3/8}\mathfrak{M}$, the second inequality of (5.15) gives $|P| \ge \frac{1}{2}\mathfrak{M} \ge U^{3/4}$.

The exceptional term. Suppose that the pair exists and $\chi_1'' = \chi$. The primitive character that induces χ_1 is real, its conductor is at most $M \le m_0 R$, and the largest divisor of its conductor that is coprime to m_0 is s_b. Since $s_b \ne s^*$, this character is not χ^*, and (C3) gives $\beta_1 \le 1 - c_2/\ell$. Then $\operatorname{li}(U^{\beta_1}) \le U^{\beta_1} \le U e^{-c_2 \log U/\ell}$ and $\operatorname{li}(U) \ge U/(2\log U)$, and in this case $\mathfrak{M} = \operatorname{li}(U)/\varphi(m_0)$, so (5.16) gives

$$\Big| \sum_b \chi(b)\, \mathfrak{m}_b \Big| \le 2 \log U \, e^{-c_2 \log U/\ell}\, \mathfrak{M}.$$

This bound also holds in the other cases, where the left side vanishes.

Conclusion. The first inequality of (5.15) and $\mathfrak{M} \le 2|P|$ give

$$\Big| \sum_{d \in P} \chi(d) \Big| \le \Big(2 \log U \, e^{-c_2 \log U/\ell} + \delta + U^{-3/8} \Big) \mathfrak{M} \le \eta\, |P|. \qquad \square$$

Theorem 5.12. *Let $N_0 \ge 1$ be an integer, and let $K \ge 1$, $0 < \varepsilon \le \frac{1}{52}$ and $0 < \tau \le \frac{1}{2}$. There is Y_0 such that the following holds for every $Y \ge Y_0$. Put $z = Y^{\varepsilon/5}$, let $q \equiv 3 \pmod 8$ be a prime with $e^{z/2} < q \le e^z$, let z' satisfy $z \log^2 Y \le z' \le K^2 z \log^2 Y$, and let P be the set of primes d with $e^{z'/2} < d \le e^{z'}$, $d \equiv 7 \pmod{16}$ and $d \equiv -1$ modulo q and modulo every odd prime $p \le z$. Then P is nonempty, and for all $d \in P$ outside a set of at most $Y^{-\tau}|P|$ elements, every primitive character ξ of conductor dividing N_0, every $a \in \{0,1\}$ and the character $\chi = \xi \cdot (\frac{\cdot}{q})^a (\frac{\cdot}{d})$ satisfy*

(i) $L(s,\chi) \ne 0$ *for every s with* $\operatorname{Re} s > 1 - \frac{1}{400}$ *and* $|\operatorname{Im} s| \le N_0 e^{z+z'}$,

(ii) $|L'(1,\chi)/L(1,\chi)| \le 401\, c_0\, K \log Y$,

(iii) $|G_\chi^*(y)| \le y^{1/2+\tau}$ *for every y with $Y \le y \le Y^K$.*

Proof. Put $X = Y^K$, $Z = X^{400}$, $R = Z^2$, $U = e^{z'}$ and $Q = N_0 e^{z+z'}$, and let m_0 and a_0 be as above, so that P is the set of Proposition 5.11. As in the proof of Theorem 5.10, the inequalities between powers of Y, powers of $\log Y$ and exponentials that we use hold for $Y \ge Y_0$.

The Legendre symbols. By (C1), $\log m_0 \le \log 16 + z + c_0 z$, so $\ell = \log(m_0 R) \le (2 + c_0) z$ and

$$\frac{\log U}{\ell} \ge \frac{\log^2 Y}{2 + c_0}.$$

The hypotheses of Proposition 5.11 therefore hold, with $q > e^{z/2} > \max(z, R)$, and its number η is at most $\exp(-c \log^2 Y)$ for a constant $c > 0$, so that $\eta X \log^2 Z \le 1$. The proposition gives $|P| \ge e^{3z'/4}$ and an integer s^* for which the hypothesis of Proposition 5.6 holds with this η and this z. The condition $z \log Y \le \sqrt{Y}$ of Proposition 5.6 holds because $\varepsilon/5 < \frac{1}{2}$, and every $d \in P$ exceeds Z.

The truncated sums. Let ξ be a primitive character of conductor r dividing N_0 and let $a \in \{0,1\}$. For $d \in P$ the character $\chi = \xi \cdot (\frac{\cdot}{q})^a (\frac{\cdot}{d})$ is primitive modulo $rq^a d$, and its sum $S_Z(y)$ is the sum $S_d(y)$ of Proposition 5.6 with $\kappa = \xi \cdot (\frac{\cdot}{q})^a$. There are at most $2N_0$ pairs (ξ, a), so $|S_Z(y)| \le \frac{1}{2} y^{1/2+\tau}$ for all pairs and all $Y \le y \le X$, for all $d \in P$ outside a set with at most

$$\frac{32000\, c_0^2\, N_0}{(1 - 4^{-\tau})^2}\, |P|\, Y^{-2\tau} \log(2Y) \le \tfrac{1}{2} Y^{-\tau} |P|$$

elements.

The boxes. The characters χ are primitive of conductor at most Q, and different d give different characters because d is the largest prime factor of the conductor. As in Proposition 5.9, (C5) bounds the number of $d \in P$ for which (i) fails for some pair (ξ, a) by $C_3 Q^{3/160}(2\log Q)^{14} \leq Q^{1/4}$, and $Q^{1/4} = N_0^{1/4} e^{(z+z')/4} \leq \frac{1}{2} Y^{-\tau} e^{3z'/4}$.

The tail. Let d lie outside both exceptional sets and let χ be as above, with conductor $N = rq^a d$ and $\lambda = \log N$, so that $z'/2 < \lambda \leq 2z'$ and $N \leq Q$. Then $e^{800} \leq Z \leq \exp(\lambda^4)$, and Proposition 5.8 gives

$$|T_Z| \leq C_5 (2z')^{12} \Big(\frac{1}{X} + \frac{1}{Q} + e^{-c_1 z'^2/8} \Big) \leq \frac{2^{13} C_5 \, K^{24} \, Y^{12\varepsilon/5} \log^{24} Y}{X}.$$

Since $12\varepsilon/5 < \frac{1}{2}$, this gives $y|T_Z| \leq \frac{1}{2} y^{1/2+\tau}$ for $Y \leq y \leq X$, and (5.11) gives (iii). It also gives $|T_Z| \leq 1$, and the second bound of Proposition 5.8 gives $|L'(1,\chi)/L(1,\chi)| \leq c_0(1 + 400K \log Y) + 1 \leq 401\, c_0 \, K \log Y$, which is (ii). □

The modulus m_0 has $\log m_0$ of order z, while $\log d$ has order $z \log^2 Y$. This ratio produces the factor $\exp(-c \log^2 Y)$ in the proof, and it takes the argument outside the range of (C2) and (C3) into the range of (C7).

Corollary 5.13. *Let $M_1 \geq 1$ be an odd integer and $N_0 = 8M_1$, let $z, z' > 0$, and let $q \leq e^z$ and $d \leq e^{z'}$ be distinct primes larger than N_0. Suppose that $L(s, \xi \cdot (\frac{\cdot}{q})) \neq 0$ for $\operatorname{Re} s > 1 - \frac{1}{400}$ and $|\operatorname{Im} s| \leq N_0 e^z$, and that $L(s, \xi \cdot (\frac{\cdot}{q})^a (\frac{\cdot}{d})) \neq 0$ for $\operatorname{Re} s > 1 - \frac{1}{400}$ and $|\operatorname{Im} s| \leq N_0 e^{z+z'}$, for every primitive character ξ of conductor dividing N_0 and every $a \in \{0, 1\}$. Let χ be a real primitive Dirichlet character whose conductor N divides $2^j M_1 qd$ for some $j \geq 0$. If d divides N, then $N \leq N_0 e^{z+z'}$ and $L(s, \chi) \neq 0$ for $\operatorname{Re} s > 1 - \frac{1}{400}$ and $|\operatorname{Im} s| \leq N_0 e^{z+z'}$. If q divides N and d does not, the same holds with $N_0 e^z$ in place of $N_0 e^{z+z'}$.*

Proof. A real primitive character is the product of real primitive characters whose conductors are the prime powers that divide its conductor exactly. The conductor at 2 is 1, 4 or 8, and the conductor at an odd prime p is p, with the character $(\frac{\cdot}{p})$ [21, Ch. 5]. Hence $\chi = \xi \cdot (\frac{\cdot}{q})^a (\frac{\cdot}{d})^b$, where $a = 1$ when q divides N, $b = 1$ when d divides N, and ξ is a real primitive character whose conductor divides $8M_1 = N_0$. The conductor is $N \leq N_0 q^a d^b$, and the two hypotheses give the two claims. □

5.7 The host field

The CM field of the argument is cut out by the target character ψ_0 and by the primes q and d.

Proposition 5.14. *Let ψ_0 be a Dirichlet character of conductor M_ψ and order o, let $e \geq 0$ be an integer, and let $q \neq d$ be primes with $q \equiv d \equiv 3 \pmod 4$ that do not divide M_ψ. Put $N_1 = \operatorname{lcm}(2^{e+2}, M_\psi)$ and $N = N_1 qd$, let U_1 be the group of the classes $a \in (\mathbb{Z}/N_1)^\times$ with $a \equiv \pm 1 \pmod{2^{e+2}}$ and $\psi_0(a) = 1$, and let U be the group of the classes $a \in (\mathbb{Z}/N)^\times$ whose reduction modulo N_1 lies in U_1 and that are squares modulo q and modulo d. Let E be the subfield of $\mathbb{Q}(\zeta_N)$ fixed by the automorphisms $\zeta_N \mapsto \zeta_N^a$ with $a \in U$, and put $H = (\mathbb{Z}/N_1)^\times / U_1$.*

(i) The field E is Galois over $\mathbb{Q}$ with group isomorphic to $(\mathbb{Z}/2)^2 \times H$.

(ii) The field E is a CM field, and it contains $\sqrt{-q}$ and $\sqrt{-d}$.

(iii) Reduction modulo 2^{e+2} induces a surjection of H onto the cyclic group $(\mathbb{Z}/2^{e+2})^\times/\{\pm 1\}$ of order 2^e. Its kernel has order c for a divisor c of o, so that $[E:\mathbb{Q}] = 4n$ with $n = c\,2^e$.

(iv) The image A_0 in H of the kernel of ψ_0 is a cyclic group of index o and order n/o, the surjection of (iii) is injective on A_0, and the characters of H that are trivial on A_0 are the powers of ψ_0.

Proof. (i) The integers N_1, q and d are pairwise coprime, so $(\mathbb{Z}/N)^\times$ is the product of $(\mathbb{Z}/N_1)^\times$, $(\mathbb{Z}/q)^\times$ and $(\mathbb{Z}/d)^\times$, and U is the product of U_1 and the two groups of squares. The Galois group of $\mathbb{Q}(\zeta_N)$ over $\mathbb{Q}$ is $(\mathbb{Z}/N)^\times$, which is abelian, so E is Galois over $\mathbb{Q}$ with group $(\mathbb{Z}/N)^\times/U$. This quotient is the product of H with the quotients of $(\mathbb{Z}/q)^\times$ and $(\mathbb{Z}/d)^\times$ by their squares, each of order 2.

(ii) The subfield of $\mathbb{Q}(\zeta_q)$ fixed by the squares modulo q is the quadratic field of discriminant $(-1)^{(q-1)/2}q$, which is $\mathbb{Q}(\sqrt{-q})$ because $q \equiv 3 \pmod 4$. Every element of U is a square modulo q, so $\sqrt{-q} \in E$, and $\sqrt{-d} \in E$ for the same reason. Complex conjugation is the automorphism with $a = -1$, and it maps $\sqrt{-q}$ to $-\sqrt{-q}$. Since E is abelian over $\mathbb{Q}$, complex conjugation commutes with every embedding of E into $\mathbb{C}$, so its fixed field is totally real and E is a totally imaginary quadratic extension of it.

(iii) The group $(\mathbb{Z}/2^{e+2})^\times$ is the product of $\{\pm 1\}$ and the cyclic group of order 2^e generated by 5, so its quotient by $\{\pm 1\}$ is cyclic of order 2^e. Reduction from $(\mathbb{Z}/N_1)^\times$ to $(\mathbb{Z}/2^{e+2})^\times$ is surjective and maps U_1 into $\{\pm 1\}$, which gives the surjection. Let V be the group of the classes $a \in (\mathbb{Z}/N_1)^\times$ with $a \equiv \pm 1 \pmod{2^{e+2}}$. The kernel of the surjection is V/U_1, and ψ_0 maps V to the group of the o-th roots of unity with kernel U_1. Hence V/U_1 is isomorphic to a subgroup of a cyclic group of order o, and its order c divides o. Then $|H| = c\,2^e$, and (i) gives the degree.

(iv) The kernel of ψ_0 in $(\mathbb{Z}/N_1)^\times$ contains U_1 and has index o, because ψ_0 maps onto the o-th roots of unity. So A_0 has index o in H and order n/o. If a class a in the kernel of ψ_0 satisfies $a \equiv \pm 1 \pmod{2^{e+2}}$, then $a \in U_1$, so the surjection of (iii) is injective on A_0, and A_0 is cyclic as a subgroup of a cyclic group. A character of H that is trivial on A_0 is a character of the quotient of $(\mathbb{Z}/N_1)^\times$ by the kernel of ψ_0, which is cyclic of order o, and the o characters of this quotient are the powers of ψ_0. □

We identify the Galois group of E with $\mathcal{G} = (\mathbb{Z}/2)^2 \times H$ as follows. An element $(s_1, s_2, h) \in \{\pm 1\}^2 \times H$, where s_1 is the sign of the action on $\sqrt{-q}$ and s_2 the sign of the action on $\sqrt{-d}$, corresponds to $(e_1, e_2, h) \in \mathcal{G}$ with $(-1)^{e_1} = s_2$ and $(-1)^{e_2} = s_1$. The set Φ of Section 3.2 is then the CM type of the elements with $s_2 = 1$. Each character ψ of $\mathcal{G}$ gives a primitive Dirichlet character χ_ψ through the Artin map of $\mathbb{Q}(\zeta_N)$, and we put $\ell_\psi = L'(1,\chi_\psi)/L(1,\chi_\psi)$ in Proposition 3.3. The character χ_1 of Proposition 3.3 then corresponds to the quadratic character $(\frac{-d}{\cdot})$ of $\mathbb{Q}(\sqrt{-d})$, and $\lambda_1 = \ell_{\chi_1} = L'(1,\chi_{-d})/L(1,\chi_{-d})$ is the number that enters the Chowla–Selberg period $\log\Omega$ of the CM elliptic curve in Part II.

5.8 The remaining data

This subsection constructs the sizes, the nodes and the sets T and T_2 that enter Theorem 4.2, and two families of auxiliary primes that Part II uses.

The scale and the ranks. A scale consists of a real number $x > 1$ and a number $0 < \varepsilon < \frac{1}{52}$. To it we attach

$$L_\varepsilon = x^\varepsilon, \qquad b = e^{L_\varepsilon}, \qquad a = x\,b, \qquad X' = \exp\left(x^{\varepsilon/3}\right), \qquad b' = \frac{a}{X'}, \qquad \delta_1 = x^{-\varepsilon/8}. \tag{5.17}$$

The ranks of the chain are integers n, k, m and t_r, where $n = |H|$, m is the number of lifts, the hermitian space of the j-th step of the chain has signature $(k+j-1, m-j)$, and the lattices have rank $r + jt_r$ with $r = k + m - 1$. Put $f = 2n$, $T_O = \lceil x^{377} \rceil$ and

$$n_W = 8\Big\lceil \frac{10\,r}{8}\Big(f + \frac{f^2(k-1)^3 L_\varepsilon^2}{m}\Big)\Big\rceil.$$

Proposition 5.15. *Let $0 < \varepsilon < \frac{1}{52}$. There is x_0 such that for every $x \geq x_0$ and every integer n with $x^{2+8\varepsilon} \leq n \leq 2x^{2+8\varepsilon}$ the following holds.*

(i) There are even integers m and k and an odd integer t_r with

$$\sqrt{x\delta_1} \leq m \leq 2\sqrt{x\delta_1}, \qquad x^{9/2+10\varepsilon} \leq k \leq 2x^{9/2+10\varepsilon}, \qquad r + t_r = 8fr + 4 + T_O\, n_W,$$
$$2 \leq m, \qquad m + 3 \leq k, \qquad k + m + 4 \leq t_r,$$
$$3m < t_r, \qquad m \leq 2n \leq k, \qquad m^2 \leq k + m(t_r + 1),$$

where $r = k + m - 1$.

(ii) For every such m and k and all integers N_T and N_2 with $N_T \geq \frac{1}{2}\delta_1 n$ and $x^{-\varepsilon} n \leq N_2 \leq n$, the hypotheses (4.6) *and* (4.7) *of Proposition 4.3 hold.*

Proof. Let x_0 be so large that $\sqrt{x\delta_1} \geq 2$, $x^{9\varepsilon/16} \geq 2$, $x^{5\varepsilon} \geq 2$, $x^{3\varepsilon/8} \geq 26$ and $x^{5/2} \geq 4$ for $x \geq x_0$.

(i) Each of the intervals from $\sqrt{x\delta_1}$ to $2\sqrt{x\delta_1}$ and from $x^{9/2+10\varepsilon}$ to $2x^{9/2+10\varepsilon}$ has length at least 2 and contains an even integer, which gives m and k. Define t_r by the equation $r + t_r = 8fr + 4 + T_O n_W$. The integer $r = k + m - 1$ is odd, and $8fr + 4$ and n_W are even, so t_r is odd. Since $f \geq 1$ and $T_O n_W \geq 0$, we get $t_r \geq 7r + 4 \geq k + m + 4$. Since $r \geq m$, this gives $t_r \geq 7m + 4$, so that $3m < t_r$ and $m^2 \leq m(t_r + 1)$. The inequalities $2 \leq m$ and $m + 3 \leq k$ are clear, $m \leq 2\sqrt{x} \leq 2n$, and $2n \leq 4x^{2+8\varepsilon} \leq x^{9/2+10\varepsilon} \leq k$.

(ii) The inequalities $a \leq 2xb$ and $N_T \geq \frac{1}{2}x^{-\varepsilon/8} n$ hold because $a = xb$ and $\delta_1 = x^{-\varepsilon/8}$. Since $\sqrt{x\delta_1} = x^{1/2-\varepsilon/16}$, we have $m \geq x^{1/2-\varepsilon}$ and $m \leq 2x^{1/2-\varepsilon/16} \leq x^{1/2+\varepsilon/2}$. Then $xm \leq 2x^{3/2} \leq k$, and

$$k \leq 2x^{9/2+10\varepsilon} \leq x^{9/2+15\varepsilon} = \big(x^{2+8\varepsilon}\big)^2 x^{1/2-\varepsilon} \leq n^2 x^{1/2-\varepsilon}. \qquad \square$$

The nodes. Each node has the form $b_1 + (j + \frac{1}{2})h_0$ with $0 \leq j < m$, where $h_0 = 2\lceil x^{c_h} \rceil$ for a constant $c_h \geq 0$ and the integer b_1 depends on the place. Let W be the number of roots of unity in E.

Proposition 5.16. *Let $0 < \varepsilon < \frac{1}{52}$ and $c_h \geq 0$. There is x_0 such that the following holds for every $x \geq x_0$. Let b, a, X' and b' be as in* (5.17)*. Let n, m, k and W be positive integers with $n \leq 2x^{9/4}$, $m \leq 2\sqrt{x}$ and $W \leq 32n^2$, put $f = 2n$ and $h_0 = 2\lceil x^{c_h} \rceil$, let b_1 be an integer, and put $w_j = b_1 + (j + \frac{1}{2})h_0$ for $0 \leq j < m$.*

(i) If $\lceil b \rceil \leq b_1 < \lceil b \rceil + W$, then $b \leq w_j \leq b + \eta'$ with $\eta' = W + mh_0$, and $\eta' \leq b$ and $6f(x+1)\eta' \leq b/x$.

(ii) If $b_1 = \lceil b'/(xn) \rceil$, or if $b' \leq b_1 - \lfloor b'/x \rfloor \leq \frac{5}{4}b'$, then $0 < w_j \leq 2b'$.

(iii) If $a \leq a_t \leq a + a/(kx^{13})$, then $|a_t/b - x| \leq 1$.

(iv) $b'X' = xb$, and $\omega = \exp(-x^{\varepsilon/8})$ satisfies $2f(x+1)\omega \leq 1/x$.

Proof. The numbers n, m, h_0 and W are bounded by powers of x, while b, b' and $1/\omega$ grow faster than every power of x. Let x_0 be so large that the inequalities between these quantities used below hold for $x \geq x_0$.

(i) We have $w_j \geq \lceil b \rceil \geq b$ and $w_j \leq \lceil b \rceil + W - 1 + (m - \frac{1}{2})h_0 \leq b + W + mh_0$. The number $12n(x+1)x\,\eta'$ is bounded by a power of x, so it is at most b.

(ii) In the first case $w_j \leq b'/(xn) + 1 + mh_0 \leq 2b'$. In the second case $b_1 \leq \frac{5}{4}b' + b'/x$, so $w_j \leq b'(\frac{5}{4} + \frac{1}{x}) + mh_0 \leq 2b'$. In both cases $w_j > 0$.

(iii) The hypothesis gives $0 \leq a_t/b - x \leq x/(kx^{13}) \leq 1$.

(iv) The first claim is the definition of b', and $4n(x+1)x \leq \exp(x^{\varepsilon/8})$ for large x. □

The bound $W \leq 32n^2$ holds because the field of W-th roots of unity lies in E, so that $\varphi(W) \leq 4n$, while $\varphi(W) \geq \sqrt{W/2}$. The three cases of Proposition 5.16 are the thick places, the thin places and the places attached to the elements of T. Under the identification of Section 5.7, the places attached to T are the elements $(1, 1, t)$ with $t \in T$, whose images in $\mathcal{G}$ are $(0, 0, t)$. The thick places are the elements $(-1, 1, s\,a_0 + u_0)$, where a_0 is a generator of A_0 and s runs over the set of integers T_2 of Proposition 5.17 below. The image of such an element is $(0, 1, s\,a_0 + u_0)$, so the thick places are the classes that are thick with respect to u_0 for the subset $\{s\,a_0 : s \in T_2\}$ of A_0. The map $s \mapsto s\,a_0$ is injective on T_2 because every element of T_2 is smaller than the order of a_0, so this subset has $|T_2|$ elements.

The set T_2.

Proposition 5.17. *For all integers $n_0 \geq 1$ and $L \geq 2$ there is a set T_2 of nonnegative integers with the following properties. Every $s \in T_2$ satisfies $4Ls < n_0$. There are no distinct $s_1, s_2, s_3 \in T_2$ and integers $i \neq j$ with $0 < |i|, |j| \leq L$ and $j(s_2 - s_1) = i(s_3 - s_1)$. Finally*

$$|T_2| \geq \frac{n_0}{L} \exp\Big(- 36\sqrt{\log n_0 \, \log L}\Big).$$

Proof. Put $N = n_0/(4L)$ and $\lambda = \log(8L)$, so that $2 < \lambda \leq 4 \log L$. The set $T_2 = \{0\}$ has the first two properties, and it has the third when the right side of the claim is at most 1. This is the case when $n_0 < L$. It is also the case when $n_0 \geq L$ and $\log N \leq 8\lambda$, because then $\log(n_0/L) \leq 9\lambda \leq 36 \log L \leq 36\sqrt{\log n_0 \log L}$.

Let $\log N > 8\lambda$, so that $n_0 > L$, and put $D = \lfloor \sqrt{2 \log N/\lambda} \rfloor$, so that $D \geq 4$ and $\frac{3}{4}\sqrt{2 \log N/\lambda} \leq D \leq \sqrt{2 \log N/\lambda}$. Then $\log N/D \geq \sqrt{\lambda \log N/2} \geq 2\lambda$, so $N^{1/D} \geq (8L)^2$. Put $B = \lfloor N^{1/D} \rfloor$ and $g = \lfloor B/(2L) \rfloor$, so that $B^D \leq N$, $B \geq \frac{1}{2}N^{1/D}$ and $g \geq B/(4L) \geq N^{1/D}/(8L)$. For a vector $v = (v_0, \ldots, v_{D-1})$ of integers with $0 \leq v_\nu < g$ put $s(v) = \sum_\nu v_\nu B^\nu$. The map s is injective, and $4L\,s(v) < 4LB^D \leq n_0$. The number $\sum_\nu v_\nu^2$ takes at most Dg^2 values, so there is an integer ϱ such that the set V of the vectors v with $\sum_\nu v_\nu^2 = \varrho$ has at least g^{D-2}/D elements. Let $T_2 = s(V)$.

Suppose that $s_k = s(v^{(k)})$ for $k = 1, 2, 3$ are distinct elements of T_2 and that $j(s_2 - s_1) = i(s_3 - s_1)$ with $0 < |i|, |j| \leq L$. The integers $c_\nu = j(v_\nu^{(2)} - v_\nu^{(1)}) - i(v_\nu^{(3)} - v_\nu^{(1)})$ satisfy $|c_\nu| \leq 2L(g-1) < B$ and $\sum_\nu c_\nu B^\nu = 0$, and an induction on ν gives $c_\nu = 0$ for every ν. Hence $v^{(3)} - v^{(1)} = \frac{j}{i}(v^{(2)} - v^{(1)})$, so the three distinct points $v^{(1)}, v^{(2)}, v^{(3)}$ lie on one line and on the sphere $\sum_\nu v_\nu^2 = \varrho$. A line meets a sphere in at most two points, which is a contradiction.

For the size, $g^{D-2}/D \geq N^{1-2/D}(8L)^{-(D-2)}/D \geq N \exp\big(- \frac{2}{D}\log N - D\lambda - \log D\big)$. The bounds for D give $\frac{2}{D}\log N \leq \frac{8}{3\sqrt{2}}\sqrt{\lambda \log N}$, $D\lambda \leq \sqrt{2\lambda \log N}$ and $\log D \leq \sqrt{\lambda \log N}$, so the

exponent is at least $-5\sqrt{\lambda \log N} \geq -10\sqrt{\log n_0 \log L}$. Since $\sqrt{\log n_0 \log L} \geq \log L \geq \log 2$, we get $|T_2| \geq \frac{n_0}{4L} e^{-10\sqrt{\log n_0 \log L}} \geq \frac{n_0}{L} e^{-36\sqrt{\log n_0 \log L}}$. □

Proposition 5.17 is the construction of Behrend [6] with digits below $B/(2L)$ in place of digits below $B/2$. We apply it with $n_0 = n/o$, where the order o of ψ_0 is fixed, and with $L = L_w$ of order $\log x$. Then $\sqrt{\log n_0 \log L_w}$ has smaller order than $\log x$, and the set has at least $x^{-\varepsilon} n$ elements for large x.

The set T. A set $T \subset H$ indexes the weights a_t, and it must have two properties. The first is a condition on its shares on the cosets of A_0, which produces the positive constant c in (4.1).

Lemma 5.18. *Let H be a finite abelian group, A_0 a subgroup of index o, T a nonempty subset of H with N_T elements, and $\delta \geq 0$. Put $\lambda_\gamma = 4/(3o+1)$ for $\gamma \in A_0$ and $\lambda_\gamma = 3/(3o+1)$ for $\gamma \in H \setminus A_0$, and suppose that for every $\gamma \in H$*

$$\Big| \#\{t \in T : t - \gamma \in A_0\} - \lambda_\gamma N_T \Big| \leq \delta \, N_T.$$

Then every nontrivial character χ of H that is trivial on A_0 satisfies

$$\Big| \sum_{t \in T} \chi(t) - \frac{N_T}{3o+1} \Big| \leq o \, \delta \, N_T.$$

Proof. Let $\gamma_1 = 0, \gamma_2, \ldots, \gamma_o$ represent the cosets of A_0, and let n_i be the number of $t \in T$ with $t - \gamma_i \in A_0$. The character χ is constant on each coset, so $\sum_{t \in T} \chi(t) = \sum_i \chi(\gamma_i)\, n_i$. It induces a nontrivial character of H/A_0, so $\sum_i \chi(\gamma_i) = 0$ and

$$\sum_{i=1}^{o} \chi(\gamma_i)\, \lambda_{\gamma_i} = \frac{4 + 3\sum_{i \geq 2} \chi(\gamma_i)}{3o+1} = \frac{4-3}{3o+1} = \frac{1}{3o+1}.$$

Therefore $\sum_{t \in T} \chi(t) - N_T/(3o+1) = \sum_i \chi(\gamma_i)(n_i - \lambda_{\gamma_i} N_T)$, and each of the o terms of the last sum has absolute value at most δN_T. □

With $\delta = x^{-1+\varepsilon}$, Lemma 5.18 gives (4.1) with $c = 1/(3o+1)$ and $\nu = o\, x^{-1+\varepsilon}$. The second property of T is a regularity condition along the orbits of Frobenius elements, which the gain at inert primes in Section 9 uses.

Definition 5.19. Let $h \in H$ have order n_h, let $t \in H$, and let $P_t \subset \mathbb{Z}/n_h\mathbb{Z}$ be the set of the residues i with $t_0 + ih \in T$, where t_0 is a fixed base point of the orbit $t + \mathbb{Z}h$. Let $K_\varepsilon = \lceil 8(4+8\varepsilon)/\varepsilon \rceil$, $r_0 = K_\varepsilon + 1$, $u = \lceil 1/\delta_1 \rceil$, $M_0 = \lceil (5+8\varepsilon)\, 10^{6r_0+1} \log x \rceil$ and $G' = (M_0+5)(6r_0+1)u$, and let $L_2 \geq 1$ be an integer. The orbit of t under h is *good for T* when the following hold.

(i) Every window of L_2 consecutive residues contains at most K_ε elements of P_t.

(ii) Every window of G' consecutive residues contains at most $\lceil \max(6e\, G'\delta_1, (5+8\varepsilon)\log_2 x) \rceil + 1$ elements of P_t.

(iii) Cut the integers from 0 to n_h into consecutive tiles of length $(6r_0+1)u$, and each tile into $6r_0+1$ cells of length u. Among every M_0+1 cyclically consecutive tiles there is one in which the cells in odd position are disjoint from P_t and each cell in even position contains exactly one element of P_t, and that element is an even integer.

Proposition 5.20. *Let $0 < \varepsilon < \frac{1}{52}$. There are an absolute constant C_b and, for every integer $o \geq 1$, a number x_0 with the following property. Let $x \geq x_0$, let H and A_0 be as in Proposition 5.14, with A_0 of index o, with $x^{2+8\varepsilon} \leq n \leq 2x^{2+8\varepsilon}$, let $\mathcal{P}$ be a finite set of primes, and for $p \in \mathcal{P}$ let $h_p \in H$ have order at least $x^{\varepsilon/4}$ and let $L_{2,p} \leq 400 \log x$ be a positive integer. Then there is a set $T \subset H$ with $\frac{1}{2}\delta_1 n \leq |T| \leq 2\delta_1 n$ that satisfies the hypothesis of Lemma 5.18 with $\delta = x^{-1+\varepsilon}$ and such that the number $\beta_p(T)$ of the $t \in T$ whose orbit under h_p is not good for T, with $L_2 = L_{2,p}$, satisfies*

$$\sum_{p \in \mathcal{P}} \frac{\log p}{p} \beta_p(T) \leq 4C_b \, x^{-2} \, \delta_1 n \sum_{p \in \mathcal{P}} \frac{\log p}{p}.$$

Proof. Throughout, x is large in terms of ε and o. Put $a_\gamma = o\lambda_\gamma \delta_1$. Since $o\lambda_\gamma$ is 1 for $o = 1$ and lies between $\frac{6}{7}$ and $\frac{8}{7}$ for $o \geq 2$, we have $\frac{6}{7}\delta_1 \leq a_\gamma \leq \frac{8}{7}\delta_1$. Choose T at random, each $t \in H$ belonging to T independently with probability a_t, where $a_t = a_\gamma$ for $t \in \gamma + A_0$. Let $\gamma_1, \ldots, \gamma_o$ represent the cosets of A_0 and let X_i be the number of elements of T in $\gamma_i + A_0$, a sum of n/o independent Bernoulli variables with mean $\mu_i = \lambda_{\gamma_i}\delta_1 n$ and variance at most $\mu_i \leq \delta_1 n$.

Size and shares. Put $\eta = \delta \, \delta_1 n/(2(o+1))$ with $\delta = x^{-1+\varepsilon}$. Chebyshev's inequality gives

$$\mathbb{P}\big(|X_i - \mu_i| > \eta \text{ for some } i\big) \leq \frac{o \, \delta_1 n}{\eta^2} = \frac{4o(o+1)^2}{\delta^2 \delta_1 n} \leq 4o(o+1)^2 \, x^{-10\varepsilon + \varepsilon/8},$$

because $n \geq x^{2+8\varepsilon}$ and $\delta_1 = x^{-\varepsilon/8}$. Suppose that $|X_i - \mu_i| \leq \eta$ for every i. Since $\sum_i \lambda_{\gamma_i} = 1$, the number $N_T = \sum_i X_i$ satisfies $|N_T - \delta_1 n| \leq o\eta \leq \frac{1}{2}\delta_1 n$, and

$$|X_i - \lambda_{\gamma_i} N_T| \leq |X_i - \mu_i| + \lambda_{\gamma_i}|\delta_1 n - N_T| \leq (1+o)\eta = \tfrac{1}{2}\delta \, \delta_1 n \leq \delta N_T.$$

So with probability at least $\frac{2}{3}$ the set T has the required size and satisfies the hypothesis of Lemma 5.18.

One orbit. Fix $p \in \mathcal{P}$ and $t \in H$, write $h = h_p$, n_h for its order and i_t for the residue of t on its orbit. The points $t_0 + ih$ with $i \in \mathbb{Z}/n_h$ are distinct, so the events $i \in P_t$ are independent, each of probability at most $\delta' = \frac{8}{7}\delta_1$ and at least $\frac{6}{7}\delta_1$. We bound the conditional probability, given $t \in T$, that the orbit is not good. The numbers $L_{2,p} \leq 400 \log x$ and G' are smaller than $x^{\varepsilon/4} \leq n_h$, so a window is a set of distinct residues, and there are $n_h \leq n \leq 2x^{2+8\varepsilon}$ windows of each length. The number of tiles is at least $M_0 + 1$, because a tile has length $(6r_0+1)u \leq (6r_0+1)(x^{\varepsilon/8}+1)$ and M_0 has the order of $\log x$.

Condition (i) fails when some window of L_2 residues contains at least $K_\varepsilon + 1$ elements of P_t. A window that does not contain i_t has this property with probability at most $\binom{L_2}{K_\varepsilon+1}\delta'^{K_\varepsilon+1} \leq (L_2\delta')^{K_\varepsilon+1}$, independently of the event $t \in T$. A window that contains i_t must contain K_ε further elements, which has probability at most $(L_2\delta')^{K_\varepsilon}$, and there are L_2 such windows. Since $\delta_1^{K_\varepsilon} = x^{-\varepsilon K_\varepsilon/8} \leq x^{-4-8\varepsilon}$, the conditional probability that (i) fails is at most

$$2x^{2+8\varepsilon}\,(458\log x)^{K_\varepsilon+1} x^{-4-8\varepsilon-\varepsilon/8} + 400\log x\,(458\log x)^{K_\varepsilon} x^{-4-8\varepsilon} \leq x^{-2}.$$

Put $N_0 = \lceil \max(6eG'\delta_1, (5+8\varepsilon)\log_2 x)\rceil$. Condition (ii) fails when some window of G' residues contains at least $N_0 + 2$ elements. Given $t \in T$, a window then contains at least $N_0 + 1$ elements other than i_t. For $N \geq N_0 + 1$ the probability that N of at most G' independent events of probability at most δ' occur is at most $\binom{G'}{N}\delta'^N \leq (eG'\delta'/N)^N$, by $\binom{G'}{N} \leq G'^N/N!$ and $N! \geq (N/e)^N$, and $eG'\delta'/N \leq \frac{8}{7}eG'\delta_1/(6eG'\delta_1) \leq \frac{1}{2}$. So each window fails with conditional probability at most $2^{-N_0-1} \leq \frac{1}{2}x^{-5-8\varepsilon}$, and (ii) fails with conditional probability at most x^{-3}.

For condition (iii), consider one cell, a set of $u = \lceil 1/\delta_1 \rceil$ consecutive integers. It is disjoint from P_t with probability at least $(1 - \frac{8}{7}\delta_1)^u$, which tends to $e^{-8/7} > 0.31$ as $x \to \infty$. It contains exactly one element of P_t, an even integer, with probability at least

$$\frac{u-1}{2} \cdot \frac{6}{7}\delta_1 \cdot \left(1 - \tfrac{8}{7}\delta_1\right)^{u-1},$$

since the cell contains at least $(u-1)/2$ even integers, and this lower bound tends to $\frac{3}{7}e^{-8/7} > 0.13$. So for large x each cell has its required property with probability at least $\frac{1}{10}$, and a tile, which consists of $6r_0 + 1$ disjoint cells, has the pattern of (iii) with probability at least $\rho = 10^{-(6r_0+1)}$. Among $M_0 + 1$ cyclically consecutive tiles at most one contains i_t, and the others are independent of the event $t \in T$ and of each other. Hence, given $t \in T$, the probability that none of them has the pattern is at most $(1-\rho)^{M_0} \le e^{-\rho M_0} \le x^{-5-8\varepsilon}$. There are at most $n_h \le 2x^{2+8\varepsilon}$ runs of $M_0 + 1$ cyclically consecutive tiles, so (iii) fails with conditional probability at most $2x^{-3}$.

Altogether $\mathbb{P}(\text{the orbit of } t \text{ under } h_p \text{ is not good} \mid t \in T) \le C_b x^{-2}$ with $C_b = 2$.

Conclusion. By the last bound and $\mathbb{P}(t \in T) \le \frac{8}{7}\delta_1$,

$$\mathbb{E}\Big[\sum_{p\in\mathcal{P}} \frac{\log p}{p}\,\beta_p(T)\Big] = \sum_{p\in\mathcal{P}} \frac{\log p}{p} \sum_{t\in H} \mathbb{P}\big(t \in T,\ \text{orbit not good}\big) \le \frac{8}{7}\, C_b\, x^{-2}\, \delta_1 n \sum_{p\in\mathcal{P}} \frac{\log p}{p}.$$

The number $4C_b x^{-2}\delta_1 n \sum_p \frac{\log p}{p}$ is $\frac{7}{2}$ times this bound, so by Markov's inequality the sum on the left exceeds it with probability at most $\frac{2}{7}$. Since $\frac{2}{7} < \frac{2}{3}$, some set T has all the required properties. □

Two families of auxiliary primes. Part II uses three primes congruent to 1 modulo $Q_S qd$ and two primes $g_0 \neq g_1$ that split in $\mathbb{Q}(\sqrt{-d})$ into principal primes with a generator congruent to 1 modulo $Q_S qd$, where Q_S is the part of the conductor of E prime to qd. Linnik's theorem [41] gives a constant $A_L \ge 1$ such that for every integer $M \ge 2$ there is a prime $p \equiv 1 \pmod{M}$ with $p \le M^{A_L}$.

Lemma 5.21. *Let $A \ge 1$ be a number such that for every integer $M \ge 2$ there is a prime $p \equiv 1 \pmod{M}$ with $p \le M^A$. Then for every integer $M \ge 2$ there are three distinct primes $p_1, p_2, p_3 \equiv 1 \pmod{M}$ with $p_i \le M^{A(1+A)^2}$.*

Proof. Choose a prime $p_1 \equiv 1 \pmod{M}$ with $p_1 \le M^A$, then a prime $p_2 \equiv 1 \pmod{Mp_1}$ with $p_2 \le (Mp_1)^A \le M^{A(1+A)}$, and then a prime $p_3 \equiv 1 \pmod{Mp_1p_2}$ with $p_3 \le (Mp_1p_2)^A \le M^{A(1+A)^2}$. Then $p_2 > Mp_1$ and $p_3 > Mp_1p_2$, so the three primes are distinct, and each is congruent to 1 modulo M. □

For the primes g_0 and g_1 we use a prime number theorem for a ray class of an imaginary quadratic field, uniform in the conductor.

(C9) There are constants $c, c' > 0$ and $A_1 \ge 1$ such that for every prime $d \equiv 3 \pmod 4$, every integer $M \ge 1$ and every $X \ge 2$ with $\log(4dM^2) \le c\sqrt{\log X}$, and for every $e^{-c\sqrt{\log X}} \le h \le 1$, the number of primes g with $X < g \le X(1+h)$ of the form $g = \pi\bar{\pi}$ with $\pi \in \mathbb{Z}[\frac{1+\sqrt{-d}}{2}]$ and $\pi \equiv 1 \pmod{M}$ is at least $c'hX/((4dM^2)^{A_1}\log X)$.

Statement (C9) follows from Theorem 1.4 of Thorner and Zaman [75], applied to the ray class field L of $K = \mathbb{Q}(\sqrt{-d})$ modulo $M\mathcal{O}_K$ with $F = \mathbb{Q}$, $H = \mathrm{Gal}(L/K)$ and $C = \{1\}$, for which $D_K \mathcal{Q} n_K^{n_K} \leq 4dM^2$. A prime that splits completely in L has the form $\pi\bar{\pi}$ with $\pi \equiv 1 \pmod{M}$. The possible exceptional zero is a zero of $\zeta_{K'}$ for a quadratic extension K'/K with discriminant at most d^2M^2, so Stark's bound [69, Theorem 1′], in the form (1-4) of [75], gives $1 - \beta_1 \gg (4dM^2)^{-1}$. With the ray class number at most a constant times dM^2 this gives (C9) with $A_1 = 2$, for every prime $d \equiv 3 \pmod{4}$ and every $M \geq 1$. With $X = (kb)^8$ the estimate gives at least $X^{9/10}$ such primes between X and $2X$.

Lemma 5.22. *For integers $g \geq 2$ and $N \geq 1$ let $B_N(g)$ be the set that consists of g and of the primes l for which there is an integer $1 \leq c \leq N$ with $l^{4v} > g^c$, where l^v is the exact power of l that divides $g^c - 1$. Let $Y \geq 2$ be a real number, let $N \geq 1$ be an integer, let $\mathcal{P}$ be a finite set of primes g with $Y \leq g \leq 2Y$, and suppose that*

$$|\mathcal{P}| > (3N+1)\big(1 + 2N^2(Y^{3/4}+1)\big).$$

Then for every $g_1 \in \mathcal{P}$ there is $g_0 \in \mathcal{P}$ with $B_N(g_0) \cap B_N(g_1) = \emptyset$, and such a g_0 differs from g_1.

Proof. Fix $g \geq 2$ and $1 \leq c \leq N$. If four distinct primes $l_1, \ldots, l_4$ satisfied $l_i^{4v_i} > g^c$ with $l_i^{v_i}$ the exact power dividing $g^c - 1$, then the product of the four numbers $l_i^{v_i}$ would exceed g^c and divide the positive integer $g^c - 1$. Hence at most three primes belong to each c, and $|B_N(g)| \leq 3N + 1$.

Fix a prime l and $1 \leq c \leq N$, and let $w \geq 1$ be the least integer with $l^w > Y^{c/4}$. Let g be an integer with $Y \leq g \leq 2Y$ such that $l^{4v} > g^c$ for the exact power l^v dividing $g^c - 1$. Then $l^v > g^{c/4} \geq Y^{c/4}$, so $v \geq w$ and $g^c \equiv 1 \pmod{l^w}$. The group of units modulo l^w is cyclic when l is odd, and it is the product of a group of order at most 2 and a cyclic group when $l = 2$, so the congruence $y^c \equiv 1 \pmod{l^w}$ has at most $2c$ solutions modulo l^w. Each residue class modulo l^w contains at most $Y/l^w + 1$ integers between Y and $2Y$, and $Y/l^w < Y^{1-c/4} \leq Y^{3/4}$. So at most $2c\,(Y^{3/4}+1)$ integers g between Y and $2Y$ satisfy the condition for this c, and the sum over $c \leq N$ is at most $N(N+1)(Y^{3/4}+1) \leq 2N^2(Y^{3/4}+1)$. Together with the possibility $g = l$, the prime l lies in $B_N(g)$ for at most $1 + 2N^2(Y^{3/4}+1)$ integers g between Y and $2Y$.

Let $g_1 \in \mathcal{P}$. Every element of $B_N(g_1)$ is a prime, so the two counts bound the number of $g \in \mathcal{P}$ for which $B_N(g)$ meets $B_N(g_1)$ by $(3N+1)\big(1 + 2N^2(Y^{3/4}+1)\big)$, which is less than $|\mathcal{P}|$. This gives g_0, and $g_0 \neq g_1$ because $g_1 \in B_N(g_1)$. □

We apply Lemma 5.22 with $N = r$, $Y = (kb)^8$ and $\mathcal{P}$ the set of primes given by (C9). The rank r is at most a power of x, while $Y^{9/10-3/4} \geq \exp(x^\varepsilon)$, so $|\mathcal{P}| \geq Y^{9/10}$ exceeds the bound of the lemma for large x.

6 The arithmetic theorem and the proof of the Riemann hypothesis

This section collects the data of Section 5 into one definition, states the arithmetic theorem that Part II proves, and deduces the Riemann hypothesis and the generalized Riemann hypothesis for Dirichlet L-functions from it.

6.1 Admissible data

Fix a primitive Dirichlet character ψ_0 of conductor M_ψ and order o, and fix $0 < \varepsilon < \frac{1}{52}$. Put $N_0 = 8M_\psi$.

Definition 6.1. Let $x \geq 3$, and let Y be a real number with $Y \leq x \leq Y^{3/2}$. *Admissible data at the scale x for the window $[Y, Y^{3/2}]$* consist of the following objects.

(D1) Primes q and d that satisfy the conclusions of Theorems 5.10 and 5.12 with N_0, $K = \frac{3}{2}$, ε, $\tau = \varepsilon$ and $z' = z \log^2 Y$. An integer $e \geq 0$ with $x^{2+8\varepsilon} \leq c\, 2^e \leq 2x^{2+8\varepsilon}$, where c is the integer of Proposition 5.14(iii). The field E, the groups H and A_0 and the integer $n = |H|$ of Proposition 5.14 for ψ_0, e, q and d, with the identification of $\mathrm{Gal}(E/\mathbb{Q})$ with $\mathcal{G} = (\mathbb{Z}/2)^2 \times H$ of Section 5.7.

(D2) Integers m, k and t_r as in Proposition 5.15(i).

(D3) A set $T \subset H$ with the properties of Proposition 5.20, for the set $\mathcal{P}$ and the elements h_p specified in Section 9.

(D4) A set $T_2 \subset A_0$ of the form $\{s\, a_0 : s \in T_2'\}$, where a_0 generates A_0 and T_2' is a set given by Proposition 5.17 with $n_0 = n/o$ and $L = L_w$, with $|T_2| \geq x^{-\varepsilon} n$.

(D5) Two primes $g_0 \neq g_1$ with $(kb)^8 \leq g_i \leq 2(kb)^8$, each of the form $g_i = \pi_i \bar{\pi}_i$ with $\pi_i \in \mathbb{Z}[\frac{1+\sqrt{-d}}{2}]$ and $\pi_i \equiv 1$ modulo $Q_S qd$, with $B_r(g_0) \cap B_r(g_1) = \emptyset$ in the notation of Lemma 5.22. Three distinct primes congruent to 1 modulo $15\, Q_S qd$ and at most $(15\, Q_S qd)^{A_L(1+A_L)^2}$.

Proposition 6.2. *There is Y_0 such that for every $Y \geq Y_0$ and every x with $Y \leq x \leq Y^{3/2}$, admissible data at the scale x for the window $[Y, Y^{3/2}]$ exist.*

Proof. Theorems 5.10 and 5.12 with $K = \frac{3}{2}$ and $\tau = \varepsilon$ give the primes q and d for every large Y. For large Y the primes q and d exceed M_ψ, and $q \equiv d \equiv 3 \pmod 4$. The integer c of Proposition 5.14(iii) divides o and does not depend on e once 2^{e+2} is divisible by the power of 2 in M_ψ, and the interval $[x^{2+8\varepsilon}, 2x^{2+8\varepsilon}]$ has ratio 2, so it contains a number $c\, 2^e$. Proposition 5.14 gives (D1). Propositions 5.15, 5.20 and 5.17 give (D2), (D3) and (D4); for (D4) we use $n_0 = n/o$, L_w of order $\log x$, and the remark after Proposition 5.17. Statement (C9) and Lemma 5.22 give the primes g_0 and g_1, and Linnik's theorem with Lemma 5.21 for $M = 15\, Q_S qd$ gives the three primes of (D5). □

6.2 The arithmetic theorem

For admissible data let Φ, $f = 2n$ and p_t be as in Section 3.2, let G_ψ be the functions of Proposition 3.3 with $\ell_\psi = L'(1, \chi_\psi)/L(1, \chi_\psi)$, and let De be the excess (3.9). Let b, a, X' and b' be as in (5.17), and let $\mathcal{B}$ be the number (4.8) with $N_T = |T|$ and $N_2 = |T_2|$.

Section 18 refines Definition 6.1 by a base scale $Y_0 \leq Y$, at which the primes q and d are chosen, and by a condition (D6) on Tate–Shafarevich groups of E (Definition 18.1). Theorem A asserts the existence of such data at every scale of a window, for windows $[Y, Y^{3/2}]$ with Y arbitrarily large.

Theorem A. *There is an absolute constant $C \geq 0$ with the following property. For every primitive Dirichlet character ψ_0 there are $0 < \varepsilon_1 \leq \frac{1}{52}$, and for every $0 < \varepsilon < \varepsilon_1$ a number $J \geq 2$, such that for every y there are real numbers $Y_0 \leq Y$ with $Y \geq y$ and the following property. For every x with $Y \leq x \leq Y^{3/2}$ there are admissible data at the scale x for the window $[Y, Y^{3/2}]$ with base Y_0, weights a_t for $t \in T$ with $a \leq a_t \leq a + a/(kx^{13})$, and for every $u_0 \in A_0$ and $1 \leq j \leq m$ functions $w_{u_0,j} \colon \Phi \to (0, \infty)$, such that the following hold.*

(i) For every $u_0 \in A_0$, every j and every $\rho \in \Phi$, the node $w_{u_0,j}(\rho)$ satisfies the bounds of Proposition 5.16(i) when ρ is thick with respect to u_0 for the set T_2, and the bounds of Proposition 5.16(ii) otherwise.

(ii) For every $u_0 \in A_0$,

$$\sum_{t \in T} \sum_{j=1}^{m} \operatorname{Re} \operatorname{De}(a_t, t, w_{u_0,j}) \leq x^{C\varepsilon}\, \mathcal{B}.$$

One can take $C = 3$.

Part II proves Theorem A from the hypotheses that Subsection 18.7 lists. For each $u_0 \in A_0$ it constructs hermitian lattices and period maps whose slopes, heights and local terms satisfy the inequalities (3.1) to (3.7) of Proposition 3.1, with $\lambda_1 = \ell_{\chi_1}$, with the sum S of Corollary 3.4, and with

$$e_1 + e_2 + e_3 + e_4 + e_5 + Ae_6 + e_7 \leq x^{C\varepsilon}\, \mathcal{B}.$$

Corollary 3.4 then gives (ii). Section 18 gives the assembly.

6.3 The Riemann hypothesis

Theorem 6.3. *Assume Theorem A. Then for every primitive Dirichlet character ψ_0 the function $L(s, \psi_0)$ does not vanish in the half-plane $\operatorname{Re} s > \frac{1}{2}$. In particular $\zeta(s) \neq 0$ for $\operatorname{Re} s > \frac{1}{2}$.*

Proof. Fix ψ_0 of conductor M_ψ and order o, let ε_1 and C be as in Theorem A, fix $0 < \varepsilon < \varepsilon_1$, and let J be the number of Theorem A. Let $Y_0 \leq Y$ be numbers given by Theorem A with Y large, let $Y \leq x \leq Y^{3/2}$, and choose admissible data at the scale x for the window $[Y, Y^{3/2}]$ with base Y_0, weights and nodes by Theorem A.

The characters. The characters of $\mathcal{G}$ that are trivial on $\{0\} \times \{0\} \times A_0$ form the set Ψ_0, of order $4o$. By Proposition 5.14(iv) and the identification of Section 5.7, the Dirichlet characters χ_ψ for $\psi \in \Psi_0$ are the primitive characters that induce $\psi_0^i(\frac{\cdot}{q})^a(\frac{\cdot}{d})^{a'}$ with $0 \leq i < o$ and $a, a' \in \{0, 1\}$. The class $(1, 0, 0)$ acts trivially on $\sqrt{-q}$ and on H, and by -1 on $\sqrt{-d}$, so $\psi(1, 0, 0) = 1$ exactly when $a' = 0$, and ψ is trivial on $\{0\} \times \{0\} \times H$ exactly when $i = 0$. Let K be the set of the $\psi \in \Psi_0$ with $a = a' = 0$, so that χ_ψ runs over the primitive characters $\chi^{(i)}$ inducing ψ_0^i. Then K contains the trivial character, $\psi(1, 0, 0) = 1$ for $\psi \in K$, and the characters of $\Psi_0 \setminus K$ are the characters with $(a, a') \neq (0, 0)$.

The functions G_ψ. Let S_0 be the set of primes dividing $2qM_\psi$. For $p \in S_0$ the inertia group I_p acts trivially on $\sqrt{-d}$, so $I_p \subset \Phi$, and p splits in $\mathbb{Q}(\sqrt{-d})$, because $-d \equiv 1 \pmod 8$, $-d \equiv 1 \pmod q$ and $-d \equiv 1 \pmod p$ for the odd primes $p \leq Y^{\varepsilon/5}$, which include the prime factors of M_ψ for large Y. So some Frobenius element σ_p lies in Φ. The inertia group at d is $\{0, (1, 0, 0)\}$, and E is unramified outside $2qdM_\psi$. Proposition 3.3 therefore applies. For a prime p and a character ψ, the average $\psi(p^i)$ of Proposition 3.3 equals $\chi_\psi(p)^i$ when ψ is trivial on I_p and vanishes otherwise, and this is $\chi_\psi(p^i)$ because χ_ψ is primitive. Hence $G_\psi = G^*_{\chi_\psi}$ for nontrivial ψ and $G_\psi = G$ for trivial ψ, and $\operatorname{Re} G_\psi = G_{\chi_\psi}$.

The hypotheses of Theorem 4.2. The set T satisfies (4.1) with $c = 1/(3o + 1)$ and $\nu = o\, x^{-1+\varepsilon}$ by Lemma 5.18. Theorem A(i) and Proposition 5.16 give the bounds for the nodes, with $\eta' = W + mh_0$ and $b'X' = xb$, and the weights satisfy $|a_t/b - x| \leq 1$ by Proposition 5.16(iii). Lemma 4.1(i) gives (4.2). Lemma 4.1(ii) gives (4.3) with $M = \log(x + 1) + C_d + \max_{\psi \in \Psi_0} |\ell_\psi|$. For $\psi \in K$ the number ℓ_ψ depends on ψ_0 alone, and for $\psi \in \Psi_0 \setminus K$ Theorems 5.10(ii) and 5.12(ii)

at the scale Y_0 with $K = J$ give $|\ell_\psi| \le 401\, c_0 J \log Y_0 \le 401\, c_0 J \log x$, so $M \le C(\psi_0)\log x$ with a constant that depends on ψ_0 and J.

The bound (4.4). Let ψ be a character of $\mathcal{G}$ and N_ψ the conductor of χ_ψ. It divides $N_1 q d$ with $N_1 \le 2^{e+2} M_\psi \le 8 M_\psi n$, and $\log q \le Y_0^{\varepsilon/5}$, $\log d \le Y_0^{\varepsilon/5}\log^2 Y_0$ with $Y_0 \le x$, so $\log N_\psi \le \lambda := A\, x^{\varepsilon/5}\log^2 x$ for a constant A and large x. If χ_ψ is real and q or d divides N_ψ, Corollary 5.13 with M_1 the odd part of M_ψ shows that $L(\beta,\chi_\psi) \ne 0$ for $1 - \frac{1}{400} < \beta < 1$. If χ_ψ is real and N_ψ divides N_1, then χ_ψ is one of finitely many characters that depend on M_ψ alone, because the real characters of 2-power conductor have conductor at most 8, and there is $c_*(M_\psi) > 0$ such that none of these finitely many L-functions vanishes on $1 - c_* < \beta < 1$. Proposition 5.3 with $c_* = \min(\frac{1}{400}, c_*(M_\psi))$, and the remark after it for the trivial character, gives for $y \ge X'/3$, where $t_0 = \log y \ge \frac{1}{2}x^{\varepsilon/3}$,

$$\frac{|G_\psi(y)|}{y} \le \frac{2}{c_*}e^{-c_* t_0} + C_1\lambda^4\Big(\frac{2\lambda}{c_1}e^{-c_1 t_0/(2\lambda)} + \frac{8}{c_1^2}\Big(1 + \frac{c_1\sqrt{t_0}}{2}\Big)e^{-c_1\sqrt{t_0}/2}\Big).$$

Since $\frac{1}{3} - \frac{1}{5} > \frac{1}{8}$ and $\frac{1}{6} > \frac{1}{8}$, the exponents $c_1 t_0/(2\lambda)$ and $c_1\sqrt{t_0}/2$ exceed $2x^{\varepsilon/8}$ for large x, and the factors in front are powers of x. Hence (4.4) holds with $\omega = \exp(-x^{\varepsilon/8})$, and Proposition 5.16(iv) gives $2f(x+1)\omega \le 1/x$.

The conclusion at x. By Theorem A(ii) the hypothesis of Theorem 4.2 on the excess holds with $\mathrm{Err} = x^{C\varepsilon}\mathcal{B}$. Proposition 5.15(ii) gives the hypotheses of Proposition 4.3 with $N_T = |T| \ge \frac{1}{2}\delta_1 n$ and $N_2 = |T_2|$. In the last hypothesis of Corollary 4.4, the term $\nu\sum_{\psi\in K}|G_\psi(x)|$ is at most $o\, x^{-1+\varepsilon}\cdot o\, B(\psi_0)x$, where $B(\psi_0)$ bounds $|G^*_{\chi^{(i)}}(y)|/y$ for the o characters $\chi^{(i)}$, and $|\Psi_0|M \le 4oC(\psi_0)\log x$. Both are smaller than $\frac{3}{26}x^{1/2}$ for large x. Corollary 4.4 gives

$$G(x) + \frac{1}{3o+1}\sum_{i=1}^{o-1} G_{\chi^{(i)}}(x) \le x^{1/2+(C+3)\varepsilon} + \sum_{\psi\in\Psi_0\setminus K}|G_\psi(x)|.$$

Each $\psi \in \Psi_0 \setminus K$ has $\chi_\psi = \xi(\frac{\cdot}{q})^a(\frac{\cdot}{d})^{a'}$ with ξ primitive of conductor dividing N_0 and $(a,a') \ne (0,0)$, and $(\frac{\cdot}{q})$ and $(\frac{-q}{\cdot})$ induce the same primitive character because $q \equiv 3 \pmod 4$. Theorems 5.10(iii) and 5.12(iii) at the scale Y_0 with $K = J$ give $|G_\psi(x)| \le |G^*_{\chi_\psi}(x)| \le x^{1/2+\varepsilon}$ for $Y_0 \le x \le Y_0^J$, an interval that contains $[Y, Y^{3/2}]$. There are $3o$ such characters, so for large x

$$G(x) + \frac{1}{3o+1}\sum_{i=1}^{o-1} G_{\chi^{(i)}}(x) \le x^{1/2+(C+4)\varepsilon} \qquad (Y \le x \le Y^{3/2}). \tag{6.1}$$

The zeros. The bound (6.1) holds on the windows $[Y, Y^{3/2}]$ for all large Y, the characters $\chi^{(i)}$ with $1 \le i < o$ are nontrivial, and the weights $1/(3o+1)$ are positive. Corollary 2.5 with $\lambda = \frac{3}{2}$ and $\theta = \frac{1}{2} + (C+4)\varepsilon$ gives $\zeta(s) \ne 0$ and $L(s,\chi^{(i)}) \ne 0$ for $\mathrm{Re}\, s > \frac{1}{2} + (C+4)\varepsilon$. The constant C of Theorem A does not depend on ε, and ε is an arbitrary number in $(0,\varepsilon_1)$, so these functions do not vanish for $\mathrm{Re}\, s > \frac{1}{2}$. When $o \ge 2$ the character $\chi^{(1)}$ is ψ_0, and every choice of ψ_0 includes ζ, so the theorem follows. □

Part II
Theorem A

7 Theta kernels on powers of a CM elliptic curve

The lattices of Theorem A are spaces of automorphic forms on unitary groups over the CM field E, and the maps between consecutive levels of the chain are given by theta kernels. This section defines the kernels in the Fock model, where they are theta functions on a power of a CM elliptic curve, and proves that they are covariant under the rational unitary groups of the two lattices. The covariance is the property that makes a kernel define a map between spaces of automorphic forms. Section 9 studies the same kernels p-adically at a prime inert in $\mathbb{Q}(\sqrt{-d})$.

7.1 The Fock model

Let E be the CM field of Section 5.7, with complex conjugation $x \mapsto \bar{x}$, CM type Φ, and a fixed embedding $\sigma_0 \colon E \to \mathbb{C}$. We regard the elements of Φ as embeddings $\tau \colon E \to \mathbb{C}$, so that $\tau(\bar{x}) = \overline{\tau(x)}$. For an even integer $r \geq 2$ let $h_r(x, y) = \sum_i x_i \bar{y}_i$ be the standard hermitian form on E^r and $U_r = U(h_r)(E)$ its unitary group. An $\mathcal{O}_E$-lattice $N \subset E^r$ is *self-dual* when it equals its dual for h_r. For self-dual lattices $N \subset E^r$ and $N' \subset E^{r'}$ let $\mathrm{Hom}(N, N') \subset \mathrm{Hom}_E(E^r, E^{r'})$ be the lattice of $\mathcal{O}_E$-linear maps from N to N'.

We use two facts about the field E, for admissible data in the sense of Definition 6.1. Write $K = \mathbb{Q}(\sqrt{-d}) \subset E$ and E^+ for the maximal totally real subfield of E. The elements of Φ are the embeddings of E that restrict to one fixed embedding of K, through which we regard K as a subfield of $\mathbb{C}$, with $\sqrt{-d} = i\sqrt{d}$.

Lemma 7.1. *Suppose that $d \equiv 7 \pmod 8$ and that $d \equiv -1$ modulo q and modulo every odd prime that divides M_ψ.*

(i) The extension E/E^+ is unramified at every finite place. A place of E^+ above 2 splits in E.

(ii) There is an ideal $\mathfrak{b}$ of $\mathcal{O}_E$ with $\mathfrak{d}_{E/K} = \mathfrak{b}\,\bar{\mathfrak{b}}$ and $\mathfrak{b} + \bar{\mathfrak{b}} = \mathcal{O}_E$, where $\mathfrak{d}_{E/K}$ is the relative different.

Proof. (i) The field E^+ contains $\sqrt{qd}$, so $E = E^+(\sqrt{-q}) = E^+(\sqrt{-d})$. Let v be a finite place of E^+. If v does not divide $2q$, then $-q$ is a unit at v and v is odd, so $E_v^+(\sqrt{-q})$ is unramified over E_v^+. If v divides q, the same holds for $E_v^+(\sqrt{-d})$. If v divides 2, then $-d \equiv 1 \pmod 8$ is a square in $\mathbb{Q}_2$, so $E \otimes_{E^+} E_v^+ = E_v^+ \times E_v^+$.

(ii) The field E lies in $\mathbb{Q}(\zeta_N)$ with $N = N_1 qd$, so a prime that ramifies in E is 2, q, d or a prime divisor of M_ψ. The inertia group of d in $\mathrm{Gal}(E/\mathbb{Q})$ is the image of $(\mathbb{Z}/d)^\times$, which has order 2, and d ramifies in K, so E/K is unramified above d. Every other prime ℓ that ramifies in E splits in K, because $-d$ is a square modulo ℓ for odd ℓ and modulo 8 for $\ell = 2$. For each such ℓ choose a prime $\mathfrak{l}$ of K above it, and let $\mathfrak{b}$ be the part of $\mathfrak{d}_{E/K}$ supported on the primes of E above the chosen primes $\mathfrak{l}$. Complex conjugation is an automorphism of E that preserves K, so it preserves $\mathfrak{d}_{E/K}$, and it exchanges the primes above $\mathfrak{l}$ with the primes above $\bar{\mathfrak{l}}$. Hence $\bar{\mathfrak{b}}$ is the part of $\mathfrak{d}_{E/K}$ above the primes $\bar{\mathfrak{l}}$, and the two parts are coprime with product $\mathfrak{d}_{E/K}$. □

For the primes q and d of admissible data with Y large, the hypothesis of Lemma 7.1 holds by the congruences on d in Theorem 5.12, since the prime divisors of M_ψ are at most z. We fix $\mathfrak{b}$

as in the lemma, a nonzero ideal $\mathfrak{c}$ of $\mathcal{O}_K$ and a nonzero complex number δ, and put $\Lambda = \delta\,\mathfrak{c} \subset \mathbb{C}$. The area of $\mathbb{C}/\Lambda$ is $A = \frac{\sqrt{d}}{2}|\delta|^2 N\mathfrak{c}$. Section 8 fixes δ and restricts $\mathfrak{c}$.

Put $V = \prod_{\tau\in\Phi} M_{r'\times r}(\mathbb{C})$ and $\mathfrak{a} = \mathfrak{c}\,\mathfrak{b}^{-1}$, a fractional ideal of E, and put

$$H(z,w) = \frac{1}{A}\sum_{\tau\in\Phi} \operatorname{tr}\left(z_\tau w_\tau^*\right), \qquad \iota(x) = \delta\big(\tau(x)\big)_{\tau\in\Phi}, \qquad \Lambda(N,N') = \iota\big(\mathfrak{a}\operatorname{Hom}(N,N')\big).$$

Since $\tau(c) = c$ for $c \in K$, the lattice $\Lambda(N,N')$ is the span of the vectors $\omega\,(\tau(x))_\tau$ with $\omega \in \Lambda$ and $x \in \mathfrak{b}^{-1}\operatorname{Hom}(N,N')$. Write $\langle x,y\rangle = \operatorname{tr}(xy^*) \in E$ for $x,y \in \operatorname{Hom}_E(E^r,E^{r'})$, where y^* is the adjoint of y for the standard forms. For $v \in V$ put $q(v) = \frac{\sqrt{d}}{2}H(v,v)$, and on the set of v with $q(v) \in \mathbb{Z}$ put $\alpha(v) = (-1)^{q(v)}$. Finally let $E_H = \operatorname{Im} H$ and $B = \sqrt{d}\operatorname{Re} H - \operatorname{Im} H$, two real bilinear forms on V.

Lemma 7.2. *Let $N \subset E^r$ and $N' \subset E^{r'}$ be self-dual lattices.*

(i) For $x,y \in \operatorname{Hom}_E(E^r,E^{r'})$, with $t = \operatorname{Tr}_{E/K}\langle x,y\rangle/N\mathfrak{c} \in K$,

$$q(\iota x) = \frac{\operatorname{Tr}_{E/K}\langle x,x\rangle}{N\mathfrak{c}}, \qquad E_H(\iota x,\iota y) = \operatorname{Tr}_{K/\mathbb{Q}}\Big(\frac{t}{\sqrt{-d}}\Big),$$
$$B(\iota x,\iota y) = \operatorname{Tr}_{K/\mathbb{Q}}(t) - \operatorname{Tr}_{K/\mathbb{Q}}\Big(\frac{t}{\sqrt{-d}}\Big).$$

(ii) The lattice $\Lambda(N,N')$ is self-dual for E_H. The function q takes integer values on it, B takes even integer values on $\Lambda(N,N')\times\Lambda(N,N')$, and $\alpha(v+w) = \alpha(v)\alpha(w)e^{\pi i E_H(v,w)}$ for $v,w \in \Lambda(N,N')$.

Proof. (i) We have $H(\iota x,\iota y) = \frac{|\delta|^2}{A}\sum_{\tau\in\Phi}\tau\langle x,y\rangle = \frac{2}{\sqrt{d}}\,t$, because the sum over Φ is the trace from E to K. For $t \in K$, $2\operatorname{Re} t = \operatorname{Tr}_{K/\mathbb{Q}} t$ and $\frac{2}{\sqrt{d}}\operatorname{Im} t = \operatorname{Tr}_{K/\mathbb{Q}}(t/\sqrt{-d})$.

(ii) Put $M = \mathfrak{a}\operatorname{Hom}(N,N')$. Since N and N' are self-dual, $\langle\ ,\ \rangle$ is a perfect $\mathcal{O}_E$-valued pairing on $\operatorname{Hom}(N,N')$, so $\langle M,M\rangle = \mathfrak{a}\bar{\mathfrak{a}} = N\mathfrak{c}\,\mathfrak{d}_{E/K}^{-1}$, and $\langle M,y\rangle \subset \mathfrak{a}\bar{\mathfrak{a}}$ exactly when $y \in M$. By the definition of the different, $\operatorname{Tr}_{E/K}$ maps $\mathfrak{d}_{E/K}^{-1}$ into $\mathcal{O}_K$, so $t \in \mathcal{O}_K$ for $x,y \in M$. Write $t = u + w\frac{1+\sqrt{-d}}{2}$ with integers u,w. Then $\operatorname{Tr}_{K/\mathbb{Q}} t = 2u + w$ and $\operatorname{Tr}_{K/\mathbb{Q}}(t/\sqrt{-d}) = w$, so E_H is integral on $\Lambda(N,N')$ and B is even. For $x = y$ the number t is rational, so $q(\iota x) = t \in \mathbb{Z}$. The identity $q(v+w) = q(v) + q(w) + \sqrt{d}\operatorname{Re} H(v,w)$ and the evenness of B give $q(v+w) \equiv q(v) + q(w) + E_H(v,w) \pmod 2$, which is the stated law for α.

For self-duality, $E_H(\iota x,\iota y) = \operatorname{Tr}_{E/\mathbb{Q}}\big(\langle x,y\rangle/(N\mathfrak{c}\sqrt{-d})\big)$. The different of E over $\mathbb{Q}$ is $\mathfrak{d}_{E/K}\cdot(\sqrt{-d})$, and $\langle M,y\rangle$ is an $\mathcal{O}_E$-module. So $E_H(\iota M,\iota y) \subset \mathbb{Z}$ exactly when $\langle M,y\rangle \subset N\mathfrak{c}\sqrt{-d}\,(\sqrt{-d})^{-1}\mathfrak{d}_{E/K}^{-1} = \mathfrak{a}\bar{\mathfrak{a}}$, that is, when $y \in M$. □

For $x \in V$ and a function G on V put

$$(U_x G)(z) = G(z+x)\exp\big(-\pi H(z,x) - \tfrac{\pi}{2}H(x,x)\big).$$

The operators U_x satisfy $U_xU_y = e^{-\pi i E_H(x,y)}U_{x+y}$, so they generate a Heisenberg group. We call a lattice $\Lambda_1 \subset V$ *admissible* when it is self-dual for E_H, q is integral on it, and $\alpha(v+w) = \alpha(v)\alpha(w)e^{\pi i E_H(v,w)}$ on it. Then B is even on $\Lambda_1\times\Lambda_1$, by the computation in the proof of Lemma 7.2. For an admissible lattice Λ_1 let $\Theta(\Lambda_1)$ be the space of entire functions G on V

with $\alpha(x)U_xG = G$ for every $x \in \Lambda_1$. These are the classical theta functions for Λ_1 with the hermitian form H and the semicharacter α, the sections of a line bundle on the abelian variety V/Λ_1. For commensurable admissible lattices Λ_1 and Λ_2 put $\Lambda_{12} = \Lambda_1 \cap \Lambda_2$ and

$$I_{\Lambda_1\to\Lambda_2}G = n_{12}^{-1/2} \sum_{y\in\Lambda_2/\Lambda_{12}} \alpha(y)\, U_yG, \qquad n_{12} = [\Lambda_2 : \Lambda_{12}].$$

For $G \in \Theta(\Lambda_1)$ the summand depends on y modulo Λ_{12}, because $\alpha(y+\lambda)U_{y+\lambda} = e^{2\pi i E_H(y,\lambda)}\alpha(y)U_y\,\alpha(\lambda)U_\lambda$ for $y, \lambda \in \Lambda_2$, and $I_{\Lambda_1\to\Lambda_2}G \in \Theta(\Lambda_2)$ by the same identity. Both lattices have covolume one for E_H, so $n_{12} = [\Lambda_1 : \Lambda_{12}] = n_{21}$. We write $I_{(L_1,L_1')\to(L_2,L_2')}$ for $I_{\Lambda(L_1,L_1')\to\Lambda(L_2,L_2')}$.

We use two facts from the theory of theta functions on abelian varieties, due to Weil [81], in the form stated by Mumford [48].

(W1) If Λ_1 is admissible, then $\Theta(\Lambda_1)$ is a line.

(W3) Let Λ_1 be admissible, and let ϱ be a unitary transformation of V that preserves H and has no fixed vector other than 0. Put $A_\varrho = 1 - \varrho^{-1}$ and $M_\varrho = \Lambda_1/A_\varrho(\Lambda_1 \cap \varrho\Lambda_1)$. Then for $G \in \Theta(\Lambda_1)$

$$G \circ \varrho^{-1} = \frac{|\det A_\varrho|}{\det A_\varrho}\, g\, I_{\Lambda_1\to\varrho\Lambda_1}G, \qquad g = |M_\varrho|^{-1/2} \sum_{x\in M_\varrho} \phi(x),$$

with $\phi(x) = \alpha(x)\exp\big(-\pi i E_H(\mathrm{cay}(x), x)\big)$ and cay the Cayley transform of ϱ.

7.2 Composition of intertwiners

Lemma 7.3. *Let $\Lambda_1, \Lambda_2, \Lambda_3$ be pairwise commensurable admissible lattices. Let S_2 represent Λ_2/Λ_{12} and S_3 represent Λ_3/Λ_{23}. For $z \in \Lambda_3$ and $y \in \Lambda_2$ with $z + y \in \Lambda_1$ put $\psi(z, y) = \alpha(z)\alpha(y)\alpha(z + y)e^{-\pi i E_H(z,y)}$.*

(i) *$I_{\Lambda_2\to\Lambda_1} I_{\Lambda_1\to\Lambda_2}$ is the identity of $\Theta(\Lambda_1)$, and $I_{\Lambda_1\to\Lambda_2}G \neq 0$ for $G \neq 0$.*

(ii) *On $\Theta(\Lambda_1)$ we have $I_{\Lambda_2\to\Lambda_3} I_{\Lambda_1\to\Lambda_2} = \mu\, I_{\Lambda_1\to\Lambda_3}$ with*

$$\mu = \mu(\Lambda_1, \Lambda_2, \Lambda_3) = \Big(\frac{n_{13}}{n_{12}n_{23}}\Big)^{1/2} \sum_{\substack{z\in S_3,\ y\in S_2\\ z+y\in\Lambda_1}} \psi(z, y).$$

(iii) *$\psi(z, y) = e^{\pi i B(z,y)}$. Let $Q = (\Lambda_3 \cap (\Lambda_1 + \Lambda_2))/(\Lambda_{13} + \Lambda_{23})$. For $z \in \Lambda_3 \cap (\Lambda_1 + \Lambda_2)$ choose $y_z \in \Lambda_2$ with $z + y_z \in \Lambda_1$. Then $f(z) = e^{\pi i B(z,y_z)}$ depends on the class of z in Q alone, and*

$$\mu = \Big(\frac{n_{13}}{n_{12}n_{23}}\Big)^{1/2} [\Lambda_{13} : \Lambda_{13} \cap \Lambda_2] \sum_{z\in Q} f(z).$$

(iv) *$|\mu| = 1$. Consequently $\mu = 1$ as soon as $\sum_{z\in Q} f(z)$ is a positive real number.*

Proof. Let $G \in \Theta(\Lambda_1)$ be nonzero. For $w \in V$ and $\lambda \in \Lambda_1$ we have $\alpha(\lambda)U_\lambda U_w G = e^{-2\pi i E_H(\lambda,w)} U_w G$. The characters $\lambda \mapsto e^{-2\pi i E_H(\lambda,w)}$ of Λ_1 are distinct for w in distinct classes modulo Λ_1, because Λ_1 is self-dual, so functions $U_w G$ with w in distinct classes are linearly independent. Moreover $U_{w+\lambda}G$ is a multiple of $U_w G$, and $U_\lambda G = \alpha(\lambda)G$.

(i) Expanding, $n_{12}\, I_{\Lambda_2\to\Lambda_1} I_{\Lambda_1\to\Lambda_2} G = \sum_{x,y} \alpha(x)\alpha(y) e^{-\pi i E_H(x,y)} U_{x+y} G$ with $x \in \Lambda_1/\Lambda_{12}$ and $y \in \Lambda_2/\Lambda_{12}$. The left side lies in the line $\Theta(\Lambda_1) = \mathbb{C}G$. By the linear independence, it equals the sum of the terms with $x + y \in \Lambda_1$, that is, with $y \in \Lambda_{12}$. For the class $y = 0$ these terms are $\alpha(x)U_x G = G$, and there are $n_{21} = n_{12}$ of them.

(ii) Put $P = \sum_{s\in\Lambda_3/\Lambda_{13}} \alpha(s) U_s G = n_{13}^{1/2} I_{\Lambda_1\to\Lambda_3} G$ and $A = \sum_{z\in S_3}\sum_{y\in S_2} \alpha(z)\alpha(y) U_z U_y G = (n_{12}n_{23})^{1/2} I_{\Lambda_2\to\Lambda_3} I_{\Lambda_1\to\Lambda_2} G$. Both lie in the line $\Theta(\Lambda_3)$, and $P \neq 0$ by (i), so $A = \kappa P$. Apply $P_1 = \sum_{x\in\Lambda_1/\Lambda_{13}} \alpha(x) U_x$, which maps $\Theta(\Lambda_3)$ into $\mathbb{C}G$. By (i), $P_1 P = n_{13} G$. The function $P_1 A$ is the sum of the terms $\alpha(x)\alpha(z)\alpha(y) U_x U_z U_y G$, and by the linear independence it equals the sum of those with $x + z + y \in \Lambda_1$, that is, with $z + y \in \Lambda_1$. For such a term, $U_z U_y G = e^{-\pi i E_H(z,y)} U_{z+y} G = e^{-\pi i E_H(z,y)} \alpha(z+y) G$ and then $\alpha(x) U_x G = G$, so the term is $\psi(z,y)G$ for each of the n_{13} classes x. Hence $\kappa = \sum \psi(z,y)$, which is (ii).

(iii) For z, y as in the statement, $q(z+y) = q(z) + q(y) + \sqrt{d}\,\mathrm{Re}\, H(z,y)$ with three integers q, so $\alpha(z)\alpha(y)\alpha(z+y) = e^{\pi i\sqrt{d}\,\mathrm{Re}\, H(z,y)}$ and $\psi(z,y) = e^{\pi i B(z,y)}$. The form B is even on each $\Lambda_i \times \Lambda_i$. If $u \in \Lambda_{12}$, then $B(z,u) = B(z+y,u) - B(y,u)$ is even, so $f(z)$ does not depend on the choice of y_z. If $u' \in \Lambda_{13}$, then $y_{z+u'} = y_z$ is a valid choice and $B(u', y) = B(u', z+y) - B(u', z)$ is even. If $t \in \Lambda_{23}$, then $y_{z+t} = y_z - t$ is a valid choice and $B(z+t, y-t) - B(z,y) = B(t,y) - B(z,t) - B(t,t)$ is even. So f is a function on Q. In (ii), for $z \in S_3$ there is $y \in S_2$ with $z + y \in \Lambda_1$ exactly when $z \in \Lambda_1 + \Lambda_2$, and then y is unique. The elements of S_3 in $\Lambda_1 + \Lambda_2$ represent $(\Lambda_3 \cap (\Lambda_1 + \Lambda_2))/\Lambda_{23}$, which maps onto Q with fibers of $[\Lambda_{13} + \Lambda_{23} : \Lambda_{23}] = [\Lambda_{13} : \Lambda_{13} \cap \Lambda_2]$ elements.

(iv) Write $\mu_{123} = \mu(\Lambda_1, \Lambda_2, \Lambda_3)$ and $\mu_{132} = \mu(\Lambda_1, \Lambda_3, \Lambda_2)$. By (i),

$$I_{\Lambda_3\to\Lambda_2} I_{\Lambda_2\to\Lambda_3} I_{\Lambda_1\to\Lambda_2} = I_{\Lambda_1\to\Lambda_2},$$

and by (ii) the left side is $\mu_{123}\,\mu_{132}\, I_{\Lambda_1\to\Lambda_2}$. So $\mu_{123}\,\mu_{132} = 1$. Let Q' and f' be the group and the function of (iii) for the triple $(\Lambda_1, \Lambda_3, \Lambda_2)$. The map $z \mapsto y_z$ induces a bijection from Q to Q', and $f'(y_z) = e^{\pi i B(y_z, z)}$. Since $B(y,z) + B(z,y) = B(z+y, z+y) - B(z,z) - B(y,y)$ is even, $f'(y_z) = \overline{f(z)}$. With (iii) and $n_{12}[\Lambda_{12} : \Lambda_{12} \cap \Lambda_3] = [\Lambda_1 : \Lambda_1 \cap \Lambda_2 \cap \Lambda_3] = n_{13}[\Lambda_{13} : \Lambda_{13} \cap \Lambda_2]$ we get $\mu_{132} = \overline{\mu_{123}}$. Hence $|\mu| = 1$, and the factor of $\sum_Q f$ in (iii) is positive. □

The next lemma constructs, at each prime, a decomposition of the standard hermitian space into two isotropic parts that is adapted to two given lattices.

Lemma 7.4. *Let $n \geq 2$ be even, let $L', L'' \subset E^n$ be self-dual lattices, and let ℓ be a prime. Put $E_\ell = E \otimes \mathbb{Q}_\ell$ and $L'_\ell = L' \otimes \mathbb{Z}_\ell$. There is an E_ℓ-linear idempotent ϵ of E_ℓ^n with $\epsilon^* = 1 - \epsilon$, $\epsilon L'_\ell \subset L'_\ell$ and $\epsilon L''_\ell \subset L''_\ell$. In particular $h_n(\epsilon x, \epsilon y) = 0$ and $h_n((1-\epsilon)x, (1-\epsilon)y) = 0$ for all x, y.*

Proof. The algebra E_ℓ is the product of the algebras $E_v = E \otimes_{E^+} E_v^+$ over the places v of E^+ above ℓ, and the lattices and the form decompose accordingly, so we treat one v. Let $\mathcal{O}_v$ be the ring of integers of E_v^+ and π a uniformizer. By Lemma 7.1(i) there are two cases.

The place v splits. Then $E_v = E_v^+ \times E_v^+$, conjugation exchanges the factors, a lattice is a product $L = L_a \times L_b$, and the form is $h((x_a, x_b), (y_a, y_b)) = (\beta(x_a, y_b), \beta(y_a, x_b))$ for a nondegenerate bilinear form β, with L self-dual exactly when L_b is the dual of L_a for β. By the theory of elementary divisors there is a basis $m_1, \ldots, m_n$ of L'_a and integers a_i such that the $\pi^{a_i} m_i$ form a

basis of L_a''. Let ϵ_a be the projection onto the span of $m_1, \ldots, m_{n/2}$ along the span of the other m_i, let $\epsilon_a^\dagger$ be its adjoint for β, and put $\epsilon = (\epsilon_a, 1 - \epsilon_a^\dagger)$. The adjoint of (u_a, u_b) for h is $(u_b^\dagger, u_a^\dagger)$, so $\epsilon^* = (1 - \epsilon_a, \epsilon_a^\dagger) = 1 - \epsilon$. The map ϵ_a preserves L_a' and L_a'', and $\epsilon_a^\dagger$ preserves their duals.

The place v is inert. Then E_v is the unramified quadratic extension of E_v^+, with ring of integers $\mathcal{O}$, and π is a uniformizer of E_v. The trace and the norm map $\mathcal{O}$ onto $\mathcal{O}_v$ and $\mathcal{O}^\times$ onto $\mathcal{O}_v^\times$. Let L', L'' be self-dual $\mathcal{O}$-lattices in a hermitian space of even dimension $n \geq 2$ over E_v. We show by induction on n that there is a basis $f_1, f_1', \ldots, f_{n/2}, f_{n/2}'$ of L' with $h(f_i, f_j) = h(f_i', f_j') = 0$ and $h(f_i, f_j') = [i = j]$, and integers $a_i \geq 0$ such that the $\pi^{-a_i} f_i$ and $\pi^{a_i} f_i'$ form a basis of L''. The projection ϵ onto the span of the f_i along the span of the f_i' then has the required properties, since $h(\epsilon x, y) = h(x, (1 - \epsilon)y)$ in such a basis.

Let $a \geq 0$ be minimal with $\pi^a L'' \subset L'$. We claim that there are $f, f' \in L'$ with $h(f, f') = 1$, $h(f, f) \in \pi^{2a}\mathcal{O}_v$ and $\pi^{-a} f, \pi^a f' \in L''$. If $a \geq 1$, the ideal $h(L'', L')$ is $\pi^{-a}\mathcal{O}$, because $\pi^b L'' \subset L'$ exactly when $h(\pi^b L'', L') \subset \mathcal{O}$. So there are $x \in L''$ and $f' \in L'$ with $h(\pi^a x, f') = 1$, and $f = \pi^a x$ lies in L' and has $h(f, f) = \pi^{2a} h(x, x)$. Also $\pi^a f' \in \pi^a L' \subset L''$ by the symmetry of the roles. If $a = 0$, then $L'' = L'$. If $h(w, w) \in \pi\mathcal{O}_v$ for all $w \in L'$, then $\mathrm{Tr}(c\, h(w, w')) \in \pi\mathcal{O}_v$ for all $w, w' \in L'$ and $c \in \mathcal{O}$, hence $h(L', L') \subset \pi\mathcal{O}$, which contradicts self-duality. So there is $w_1 \in L'$ with $h(w_1, w_1)$ a unit, and the same argument in the orthogonal complement of w_1 in L' gives w_2 orthogonal to w_1 with $h(w_2, w_2)$ a unit. Choose $\kappa \in \mathcal{O}^\times$ with $\kappa\bar{\kappa} = -h(w_1, w_1)/h(w_2, w_2)$ and put $f = w_1 + \kappa w_2$, so that $h(f, f) = 0$. The vector f is primitive in L', so there is $f' \in L'$ with $h(f, f') = 1$. The claim follows in both cases.

We first make f isotropic. For $c' \in \pi^{2a}\mathcal{O}$ the vector $f + c'f'$ has norm $h(f, f) + \mathrm{Tr}(c') + c'\bar{c}'\, h(f', f')$, and $\pi^{-a}(f + c'f') = \pi^{-a} f + c'\pi^{-2a}(\pi^a f')$ lies in L''. For $a = 0$ we have $h(f, f) = 0$ already. For $a \geq 1$ the equation $\mathrm{Tr}(c') + c'\bar{c}'\, h(f', f') = -h(f, f)$ has a solution $c' \in \pi^{2a}\mathcal{O}$ by successive approximation, because the trace maps $\pi^j\mathcal{O}$ onto $\pi^j\mathcal{O}_v$ and the quadratic term of a correction lies in a higher power of π. After this change $h(f, f')$ is a unit u, and we replace f' by $\bar{u}^{-1} f'$, so that $h(f, f) = 0$ and $h(f, f') = 1$. Now replace f' by $f' + cf$ with $c \in \mathcal{O}$ and $\mathrm{Tr}(c) = -h(f', f')$. Since $h(f, f) = 0$, this gives $h(f', f') = 0$ and keeps $h(f, f') = 1$, and $\pi^a(f' + cf) = \pi^a f' + c\pi^{2a}(\pi^{-a} f)$ lies in L''. The planes $\mathcal{O}f \oplus \mathcal{O}f' \subset L'$ and $\mathcal{O}\pi^{-a} f \oplus \mathcal{O}\pi^a f' \subset L''$ are self-dual and span the same subspace, so L' and L'' are the orthogonal sums of these planes with their intersections with the orthogonal complement of that subspace, which are self-dual lattices of rank $n - 2$. The induction hypothesis applies to them. □

Lemma 7.5. *Let (L_i, L_i') for $i = 1, 2, 3$ be pairs of self-dual lattices $L_i \subset E^r$ and $L_i' \subset E^{r'}$, with r and r' even. Suppose that the lattices L_1', L_2', L_3' take at most two distinct values, or that the lattices L_1, L_2, L_3 take at most two distinct values. Then*

$$\mu\big(\Lambda(L_1, L_1'), \Lambda(L_2, L_2'), \Lambda(L_3, L_3')\big) = 1,$$

that is, $I_{(L_2, L_2') \to (L_3, L_3')}\, I_{(L_1, L_1') \to (L_2, L_2')} = I_{(L_1, L_1') \to (L_3, L_3')}$ on $\Theta(\Lambda(L_1, L_1'))$.

Proof. Write $\Lambda_i = \Lambda(L_i, L_i') = \iota(M_i)$ with $M_i = \mathfrak{a}\,\mathrm{Hom}(L_i, L_i')$, and let Q and f be as in Lemma 7.3(iii). By part (iv) of that lemma it suffices to show $\sum_Q f > 0$. For classes z, z' we have $f(z + z') = f(z) f(z')\, b(z, z')$ with $b(z, z') = e^{\pi i (B(z, y_{z'}) + B(z', y_z))}$, and b is biadditive on Q. The group Q is the direct sum of its ℓ-primary components $Q(\ell)$, and $b(z, z') = 1$ when the orders of z and z' are coprime. So $\sum_Q f = \prod_\ell \sum_{Q(\ell)} f$, and we fix ℓ.

Put $V_\ell = \mathrm{Hom}_E(E^r, E^{r'}) \otimes \mathbb{Q}_\ell$ and $M_{i,\ell} = M_i \otimes \mathbb{Z}_\ell$. By Lemma 7.2(i), $B(\iota x, \iota y) = \mathrm{Tr}_{E/\mathbb{Q}}(c_1 \langle x, y \rangle)$ with $c_1 = (1 - 1/\sqrt{-d})/N\mathfrak{c}$, and we extend $\langle\ ,\ \rangle$ and B to V_ℓ with values in E_ℓ

and $\mathbb{Q}_\ell$. The form B has values in $2\mathbb{Z}_\ell$ on each $M_{i,\ell} \times M_{i,\ell}$. Let $e_\ell \colon \mathbb{Q}_\ell \to \mathbb{C}^\times$ be the character that is trivial on $\mathbb{Z}_\ell$ and equals $x \mapsto e^{2\pi i x}$ on $\mathbb{Z}[1/\ell]$. Since localization is exact, $Q(\ell) = Q \otimes \mathbb{Z}_\ell$ is the quotient of $M_{3,\ell} \cap (M_{1,\ell} + M_{2,\ell})$ by $M_{1,\ell} \cap M_{3,\ell} + M_{2,\ell} \cap M_{3,\ell}$. For ξ in the first group choose $\eta \in M_{2,\ell}$ with $\xi + \eta \in M_{1,\ell}$ and put $f_\ell(\xi) = e_\ell(B(\xi,\eta)/2)$. The computation in the proof of Lemma 7.3(iii) shows that f_ℓ is a function on $Q(\ell)$. It agrees with f there. Indeed, let $z \in M_3 \cap (M_1 + M_2)$ have its class in $Q(\ell)$, and put $x = B(z, y_z)/2 \in \mathbb{Q}$. Then $f(z) = e^{2\pi i x}$ is the product over all primes ℓ' of the ℓ'-power roots of unity $e_{\ell'}(x)$. From $f(kz) = f(z)^{k^2}$ and $\ell^j z = 0$ in Q for some j, the number $f(z)$ is an ℓ-power root of unity, so $f(z) = e_\ell(x) = f_\ell(z)$.

Suppose first that the L_i' take at most two values, and let ϵ be the idempotent of Lemma 7.4 for these values, with $n = r'$. Composition with ϵ on the left is an endomorphism of V_ℓ that preserves each $M_{i,\ell}$, so it induces an idempotent endomorphism of $Q(\ell)$, and $Q(\ell) = \epsilon Q(\ell) \oplus (1-\epsilon)Q(\ell)$. For ξ and η as above, $\epsilon\xi + \epsilon\eta \in M_{1,\ell}$ and $\langle \epsilon\xi, \epsilon\eta \rangle = \operatorname{tr}(\epsilon^* \epsilon\, \xi \eta^*) = 0$, so $f_\ell = 1$ on $\epsilon Q(\ell)$, and likewise on $(1-\epsilon)Q(\ell)$. Therefore

$$\sum_{Q(\ell)} f = \sum_{u \in \epsilon Q(\ell)} \sum_{u' \in (1-\epsilon)Q(\ell)} b(u, u'),$$

and each inner sum is the sum of a character of the finite group $(1-\epsilon)Q(\ell)$, hence 0 or the order of that group. The inner sum for $u = 0$ is positive, so the total is positive. If the L_i take at most two values, the same argument applies with the idempotent ϵ of Lemma 7.4 for $n = r$, acting on V_ℓ by composition on the right. It preserves each $M_{i,\ell}$, and $\langle \xi\epsilon, \eta\epsilon \rangle = \operatorname{tr}(\xi\, \epsilon\epsilon^* \eta^*) = 0$ because $\epsilon\epsilon^* = \epsilon(1-\epsilon) = 0$. □

7.3 The base vector and the kernels

Let std denote the standard lattice $\mathcal{O}_E^r$ or $\mathcal{O}_E^{r'}$, and let θ_0 be a nonzero element of the line $\Theta(\Lambda(\text{std}, \text{std}))$, which exists by Lemma 7.2 and (W1). For an ideal $\mathfrak{d}$ of E write $\chi(\mathfrak{d})$ for the value of a fixed function χ on fractional ideals of E that is multiplicative on ideals of norm one and satisfies $\chi(u\mathcal{O}_E) = \prod_{\tau \in \Phi} \tau(u)^{-1}$ for $u \in E$ with $u\bar{u} = 1$. For a self-dual $L \subset E^r$ let $\mathfrak{d}_L$ be the ideal generated by the determinants of the matrices that map std onto L. For self-dual $L \subset E^r$ and $L' \subset E^{r'}$ put

$$\theta_{L,L'} = I_{(\text{std},\text{std}) \to (L,L')}\, \theta_0, \qquad e(L, L') = \chi(\mathfrak{d}_L)^{r'/2} \chi(\mathfrak{d}_{L'})^{-r/2}\, \theta_{L,L'}. \tag{7.1}$$

The function $e(L, L')$ is the *base vector* of the pair. It spans the line $\Theta(\Lambda(L, L'))$, by Lemma 7.3(i).

The kernels of the chain are finite combinations of Heisenberg translates of base vectors. Let S be a finite set of primes of E that is disjoint from its conjugate $\bar{S}$, and let $c \geq 1$ be an integer. For self-dual N, N' put $M = \mathfrak{a}\operatorname{Hom}(N, N')$, and let M_c^A and M_c^B be the $\mathcal{O}_E$-lattices in $\operatorname{Hom}_E(E^r, E^{r'})$ whose localizations agree with those of M outside $S \cup \bar{S}$ and are

$$(M_c^A)_P = c^{-1} M_P, \quad (M_c^A)_{\bar{P}} = c\, M_{\bar{P}}, \qquad (M_c^B)_P = c\, M_P, \quad (M_c^B)_{\bar{P}} = c^{-1} M_{\bar{P}} \qquad (P \in S).$$

Since $\langle M_c^A, M_c^A \rangle = \langle M, M \rangle$ locally at every prime, the proof of Lemma 7.2 shows that $A_c(N, N') = \iota(M_c^A)$ and $B_c(N, N') = \iota(M_c^B)$ are admissible lattices. Put

$$G_c(N, N') = \big[\Lambda \cap A_c : \Lambda \cap A_c \cap B_c\big]^{-1} \sum_{y \in B_c / (\Lambda \cap B_c)} \alpha(y)\, U_y\, e(N, N'), \qquad \Lambda = \Lambda(N, N'),$$

a multiple of $I_{\Lambda \to B_c} e(N, N')$ and an element of $\Theta(B_c)$. For $x \in M_c^A$ the kernel of level c at x is

$$\theta_{N,N'}[c, x] = \alpha(\iota x)\, U_{-\iota x}\, G_c(N, N'). \tag{7.2}$$

It depends on x modulo $M_c^A \cap M_c^B$. Indeed, for $m \in M_c^A \cap M_c^B$ the points $\imath x$ and $\imath m$ lie in the admissible lattice A_c, so $\alpha(\imath x + \imath m)U_{-\imath x - \imath m} = e^{2\pi i E_H(\imath x, \imath m)}\alpha(\imath x)U_{-\imath x}\,\alpha(\imath m)U_{-\imath m}$ with $E_H(\imath x, \imath m) \in \mathbb{Z}$, and $\alpha(\imath m)U_{-\imath m}$ fixes $G_c(N, N') \in \Theta(B_c)$. For a function Ξ on M_c^A that is periodic under $M_c^A \cap M_c^B$ put

$$\theta_{N,N'}[\Xi] = \sum_{x \in M_c^A / (M_c^A \cap M_c^B)} \Xi(x)\,\theta_{N,N'}[c, x]. \tag{7.3}$$

The lift from level j to level $j+1$ sends a form, given by its values in polynomial models of highest weight λ of GL_r at the lattices of level j, to a form at level $j+1$. Its matrix coefficients are obtained from the Taylor coefficients of the kernels at the origin. For a family $\lambda = (\lambda_\tau)_{\tau \in \Phi}$ of dominant weights of GL_r let $V_\lambda \subset \mathbb{C}[X_{\tau,i,j}]$ be the tensor product over τ of the polynomial models of highest weight λ_τ, with the action $\rho_\lambda(\gamma)$ of $\gamma \in U_r$ through the matrices $\tau(\gamma)^{\mathsf{T}}$, twisted by $\prod_\tau \tau(\det\gamma)^{-k(\lambda_\tau)}$ for the integers $k(\lambda_\tau)$ that make the polynomial model of λ_τ a subspace of polynomials. For a family J of numbers indexed by monomials, and exponents $n = (n_\tau)$, the jet operator $j_{\lambda,\lambda',n}(J)\colon V_\lambda \to V_{\lambda'}$ substitutes $X = YD$ in $P \cdot \prod_\tau \det(X_\tau)^{n_\tau}$, replaces each monomial D^κ by $J(\kappa)$, and projects to $V_{\lambda'}$. The jets of a kernel are $J_{N,N',c,x}(\kappa) = \chi_F(N, N')^{-1} c_D^\kappa\, \partial^\kappa \theta_{N,N'}[c, x](0)$, where $\chi_F(N, N') = \chi(\mathfrak{d}_N)^{r'/2}\chi(\mathfrak{d}_{N'})^{-r/2}$ and c_D is a fixed normalizing constant. With a frame factor $\mu(N, \mathrm{fr})$, built from the local components of a Hecke character μ of E at the determinants of N and of its frames fr at the bad primes, the *step kernel* is

$$\mathrm{th}(N, \mathrm{fr}, N', \mathrm{fr}', c, x) = \frac{\mu(N, \mathrm{fr})}{\mu(N', \mathrm{fr}')}\, j_{\lambda,\lambda',n}\big(J_{N,N',c,x}\big), \tag{7.4}$$

for $x \in M_c^A$, and 0 for other $x \in \mathrm{Hom}_E(E^r, E^{r'})$.

7.4 Covariance

For $\gamma \in U_r$ and $\gamma' \in U_{r'}$ let $R = R_{\gamma,\gamma'}\colon V \to V$ be the map $(Rz)_\tau = \tau(\gamma')z_\tau\tau(\gamma)^{-1}$, and write $\mathrm{Ad}(\gamma, \gamma')x = \gamma' x \gamma^{-1}$ on $\mathrm{Hom}_E(E^r, E^{r'})$.

Lemma 7.6. *Let $\gamma \in U_r$ and $\gamma' \in U_{r'}$, and put $R = R_{\gamma,\gamma'}$.*

(i) $H(Rz, Rw) = H(z, w)$ *and* $\alpha \circ R = \alpha$.

(ii) $R\,\imath(x) = \imath(\mathrm{Ad}(\gamma, \gamma')x)$ *and* $R\,\Lambda(N, N') = \Lambda(\gamma N, \gamma' N')$, *and R maps $A_c(N, N')$ and $B_c(N, N')$ onto $A_c(\gamma N, \gamma' N')$ and $B_c(\gamma N, \gamma' N')$.*

(iii) $U_{Rx}(G \circ R^{-1}) = (U_x G) \circ R^{-1}$. *Hence $G \in \Theta(\Lambda)$ exactly when $G \circ R^{-1} \in \Theta(R\Lambda)$, and* $I_{R\Lambda_1 \to R\Lambda_2}(G \circ R^{-1}) = (I_{\Lambda_1 \to \Lambda_2} G) \circ R^{-1}$.

Proof. (i) Since $\tau(\bar{x}) = \overline{\tau(x)}$, we have $\tau(\gamma)^*\tau(\gamma) = \tau(\gamma^*\gamma) = 1$, and the same for γ', so $\mathrm{tr}\,\big(\tau(\gamma')z\tau(\gamma)^{-1}(\tau(\gamma')w\tau(\gamma)^{-1})^*\big) = \mathrm{tr}(zw^*)$ for each τ. The semicharacter depends on v through $H(v, v)$.

(ii) The embedding τ is a ring homomorphism, so $\tau(\gamma' x \gamma^{-1}) = \tau(\gamma')\tau(x)\tau(\gamma)^{-1}$. The $\mathcal{O}_E$-linear isomorphism $x \mapsto \gamma' x \gamma^{-1}$ maps $\mathrm{Hom}(N, N')$ onto $\mathrm{Hom}(\gamma N, \gamma' N')$, and it maps the localizations at the primes of S and the level conditions that define A_c and B_c onto the corresponding conditions for $(\gamma N, \gamma' N')$.

(iii) We have $(U_{Rx}(G \circ R^{-1}))(z) = G(R^{-1}z + x)\exp\big(-\pi H(z, Rx) - \frac{\pi}{2}H(Rx, Rx)\big)$, and $H(z, Rx) = H(R^{-1}z, x)$ by (i). The second statement follows from the first and $\alpha(Rx) = \alpha(x)$. For the third, R induces a bijection from $\Lambda_2/(\Lambda_1 \cap \Lambda_2)$ onto $R\Lambda_2/(R\Lambda_1 \cap R\Lambda_2)$ and preserves the index. □

The base vector is covariant because the character factor χ in (7.1) cancels the constant of (W3). We first compute that constant exactly.

Lemma 7.7. *Let $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod{16}$ be distinct primes. For every integer $m \geq 1$ there are integers x, y, z, w with $(x^2 + qy^2)(z^2 + dw^2) \equiv -1 \pmod m$.*

Proof. By the Chinese remainder theorem, applied to each of the four variables, it suffices to treat $m = \ell^k$ with ℓ prime. We use the following fact for an odd prime ℓ and integers a, b, c prime to ℓ. The congruence $ax^2 + by^2 \equiv c \pmod{\ell^k}$ has a solution. Indeed, modulo ℓ the sets $\{ax^2\}$ and $\{c - by^2\}$ each have $(\ell + 1)/2$ elements, so they meet, and a common value gives a solution in which x or y is prime to ℓ, because $c \not\equiv 0$. If x is prime to ℓ, the derivative $2ax$ is a unit, and Hensel's lemma lifts the solution to every power of ℓ with y fixed. The case of y prime to ℓ is the same.

If $\ell \notin \{2, q, d\}$, take $z = 1$, $w = 0$, and solve $x^2 + qy^2 \equiv -1 \pmod{\ell^k}$ by the fact. If $\ell = q$, take $x = 1$, $y = 0$, and solve $z^2 + dw^2 \equiv -1 \pmod{q^k}$, which is possible since $q \nmid d$. If $\ell = d$, take $z = 1$, $w = 0$, and solve $x^2 + qy^2 \equiv -1 \pmod{d^k}$. If $\ell = 2$, take $y = 0$, $z = 0$, $w = 1$. The product is then dx^2, and we need $x^2 \equiv -d^{-1} \pmod{2^k}$. Since $d \equiv 7 \pmod 8$, the odd number $-d^{-1}$ is congruent to 1 modulo 8, and every odd residue congruent to 1 modulo 8 is a square modulo 2^k. □

Lemma 7.8. *For $\gamma \in U_r$,*

$$\theta_0 \circ R_{\gamma,1}^{-1} = \Big(\prod_{\tau \in \Phi} \tau\big(\overline{\det \gamma}\big) \Big)^{r'/2} I_{(\mathrm{std},\mathrm{std}) \to (\gamma\,\mathrm{std},\mathrm{std})}\, \theta_0,$$

and for $\gamma' \in U_{r'}$ the same holds with $\big(\prod_\tau \tau(\det \gamma')\big)^{r/2}$ and $I_{(\mathrm{std},\mathrm{std}) \to (\mathrm{std},\gamma'\mathrm{std})}$.

Proof. Reduction to the case without fixed vectors. Let $u \in E$ with $u\bar{u} = 1$, $u \neq 1$, and u different from every eigenvalue of γ. Such u exists because the norm-one group of E is infinite. Then $\gamma = \gamma_1\gamma_2$ with $\gamma_1 = u \cdot 1$ and $\gamma_2 = u^{-1}\gamma$, and neither γ_i has the eigenvalue 1. Both sides of the claim are multiplicative in γ. For the left side, suppose that $\theta_0 \circ R_{\gamma_i,1}^{-1} = c_i\, I_{\mathrm{std} \to \gamma_i \mathrm{std}} \theta_0$ for $i = 1, 2$. Lemma 7.6(iii) gives

$$\theta_0 \circ R_{\gamma_1\gamma_2,1}^{-1} = c_2 \left(I_{\mathrm{std} \to \gamma_2 \mathrm{std}} \theta_0 \right) \circ R_{\gamma_1,1}^{-1} = c_2\, I_{\gamma_1 \mathrm{std} \to \gamma_1\gamma_2 \mathrm{std}} \big(c_1 I_{\mathrm{std} \to \gamma_1 \mathrm{std}} \theta_0 \big),$$

and Lemma 7.5, applied to the lattices $\Lambda(\mathrm{std}, \mathrm{std})$, $\Lambda(\gamma_1 \mathrm{std}, \mathrm{std})$ and $\Lambda(\gamma_1\gamma_2 \mathrm{std}, \mathrm{std})$, turns the composite into $c_1 c_2\, I_{\mathrm{std} \to \gamma_1\gamma_2 \mathrm{std}} \theta_0$. So it suffices to treat $\delta = R_{\gamma,1}$ with $1 - \gamma$ invertible.

Rows. The space $V = \prod_\tau M_{r' \times r}(\mathbb{C})$ is the orthogonal sum of r' copies of $V_1 = \prod_\tau M_{1 \times r}(\mathbb{C})$, by rows, and H, α, $\Lambda(\mathrm{std}, \mathrm{std})$, δ and $A = 1 - \delta^{-1}$ split accordingly. So $\det A = (\det A_1)^{r'}$ and $g = g_1^{r'}$ in (W3), and the constant of (W3) is $c = \big(\frac{|a_1|}{a_1} g_1\big)^{r'}$ with $a_1 = \det A_1$.

The Cayley determinant. For the unitary δ_1 on V_1 we have $\overline{\det(1 - \delta_1^{-1})} = \det(1 - \delta_1) = \det(-\delta_1) \det(1 - \delta_1^{-1})$, so $\bar{a}_1 = (-1)^{\dim V_1} \det(\delta_1)\, a_1$. Here $\dim_{\mathbb{C}} V_1 = r|\Phi|$ is even, and δ_1 is right multiplication by $\tau(\gamma)^*$ on the row of index τ, so $\det \delta_1 = \prod_\tau \tau(\overline{\det \gamma})$. Hence $(|a_1|/a_1)^2 = \bar{a}_1/a_1 = \prod_\tau \tau(\overline{\det \gamma})$.

The Gauss sum. The function ϕ is a nondegenerate quadratic function on the finite group M_1, so $|g_1| = 1$. Put $m = 4|M_1|$. Lemma 7.7 gives integers x, y, z, w with $n = (x^2 + qy^2)(z^2 + dw^2) \equiv -1 \pmod m$. The field E contains $\sqrt{-q}$ and $\sqrt{-d}$ by Proposition 5.14, so $\lambda = (x + y\sqrt{-q})(z + w\sqrt{-d})$ lies in $\mathcal{O}_E$, and $\bar{\lambda}\lambda = n$. Multiplication by λ preserves Λ and commutes

with δ_1, so it preserves $\Lambda \cap \delta_1\Lambda$ and $A_1(\Lambda \cap \delta_1\Lambda)$ and induces an endomorphism of M_1. The composite with the endomorphism induced by $\bar{\lambda}$ is multiplication by n, which is invertible on M_1 because n is prime to $|M_1|$. So multiplication by λ permutes M_1. Since n is a rational integer, $H(\lambda v, \lambda w) = n\,H(v,w)$, and $n \equiv -1$ modulo $4|M_1|$ gives $\phi(\lambda x) = \overline{\phi(x)}$ for $x \in M_1$. Summing over M_1 gives $g_1 = \bar{g}_1$, so $g_1 = \pm 1$.

Conclusion. $\big(\frac{|a_1|}{a_1} g_1\big)^2 = \prod_\tau \tau(\overline{\det \gamma})$, and r' is even, so $c = \big(\prod_\tau \tau(\overline{\det \gamma})\big)^{r'/2}$, and the sign of g_1 drops out. The second statement is the same computation with columns in place of rows. □

Proposition 7.9. *Let $N \subset E^r$ and $N' \subset E^{r'}$ be self-dual lattices of even ranks, and let $\gamma \in U_r$ and $\gamma' \in U_{r'}$. Put $R = R_{\gamma,\gamma'}$.*

(i) $e(\gamma N, \gamma' N') = e(N, N') \circ R^{-1}$.

(ii) For every $c \geq 1$ and every $x \in M_c^A(N, N')$, $\theta_{\gamma N, \gamma' N'}[c, \mathrm{Ad}(\gamma, \gamma')x] = \theta_{N,N'}[c, x] \circ R^{-1}$. For every function Ξ on M_c^A that is periodic under $M_c^A \cap M_c^B$, $\theta_{\gamma N, \gamma' N'}[\Xi \circ \mathrm{Ad}(\gamma, \gamma')^{-1}] = \theta_{N,N'}[\Xi] \circ R^{-1}$.

(iii) Suppose that the Hecke character μ of the frame factor has infinity type (e_τ) and is trivial on $E^\times$, and that $n_\tau + \frac{r'}{2} - e_\tau = k(\lambda_\tau)$ and $e_\tau - \frac{r}{2} = -k(\lambda'_\tau)$ for every $\tau \in \Phi$. Then for all data in range

$$\mathrm{th}\big(\gamma N, \gamma\,\mathrm{fr}, \gamma' N', \gamma' \mathrm{fr}', c, \mathrm{Ad}(\gamma, \gamma')x\big) = \rho_{\lambda'}(\gamma') \circ \mathrm{th}(N, \mathrm{fr}, N', \mathrm{fr}', c, x) \circ \rho_\lambda(\gamma)^{-1}.$$

Proof. (i) Write $R = R_{\gamma,1} R_{1,\gamma'}$, the two factors commuting, and put $c' = \big(\prod_\tau \tau(\det \gamma')\big)^{r/2}$ and $c = \big(\prod_\tau \tau(\overline{\det \gamma})\big)^{r'/2}$. Lemma 7.6(iii) and Lemma 7.8 give

$$\theta_{L,L'} \circ R_{1,\gamma'}^{-1} = I_{(\mathrm{std},\gamma'\mathrm{std}) \to (L,\gamma' L')}\big(\theta_0 \circ R_{1,\gamma'}^{-1}\big) = c'\, I_{(\mathrm{std},\gamma'\mathrm{std}) \to (L,\gamma' L')}\, I_{(\mathrm{std},\mathrm{std}) \to (\mathrm{std},\gamma'\mathrm{std})}\, \theta_0.$$

In the three pairs $(\mathrm{std},\mathrm{std})$, $(\mathrm{std},\gamma'\mathrm{std})$, $(L,\gamma' L')$ the first lattice takes two values, so Lemma 7.5 turns the composite into $I_{(\mathrm{std},\mathrm{std}) \to (L,\gamma' L')}$, and $\theta_{L,L'} \circ R_{1,\gamma'}^{-1} = c'\,\theta_{L,\gamma' L'}$. In the same way, for self-dual $X \subset E^{r'}$,

$$\theta_{L,X} \circ R_{\gamma,1}^{-1} = c\, I_{(\gamma\,\mathrm{std},\mathrm{std}) \to (\gamma L, X)}\, I_{(\mathrm{std},\mathrm{std}) \to (\gamma\,\mathrm{std},\mathrm{std})}\, \theta_0 = c\,\theta_{\gamma L, X},$$

where Lemma 7.5 applies because the second lattice takes the two values std and X. For the character factor, $\mathfrak{d}_{\gamma L} = (\det \gamma)\mathfrak{d}_L$ with $\det \gamma$ of norm one, and χ is multiplicative on ideals of norm one with $\chi((\det \gamma)) = \prod_\tau \tau(\det \gamma)^{-1}$, so

$$\chi(\mathfrak{d}_{\gamma L})^{r'/2} \chi(\mathfrak{d}_{\gamma' L'})^{-r/2} = \Big(\prod_\tau \tau(\det \gamma)\Big)^{-r'/2} \Big(\prod_\tau \tau(\det \gamma')\Big)^{r/2} \chi(\mathfrak{d}_L)^{r'/2} \chi(\mathfrak{d}_{L'})^{-r/2}.$$

Since $|\tau(\det \gamma)| = 1$, we have $\tau(\overline{\det \gamma}) = \tau(\det \gamma)^{-1}$, and the product of the three displays gives $e(\gamma L, \gamma' L') = e(L, L') \circ R^{-1}$.

(ii) By Lemma 7.6, R maps the lattices $\Lambda(N, N')$, $A_c(N, N')$ and $B_c(N, N')$ onto the corresponding lattices for $(\gamma N, \gamma' N')$, maps coset representatives to coset representatives and preserves the index in the definition of G_c. With (i) and Lemma 7.6(iii) this gives $G_c(\gamma N, \gamma' N') = G_c(N, N') \circ R^{-1}$. For $x \in M_c^A(N, N')$ we have $R\,\iota x = \iota(\mathrm{Ad}(\gamma, \gamma')x)$ and $\alpha(R\,\iota x) = \alpha(\iota x)$, so (7.2) and Lemma 7.6(iii) give the first statement. For the second, $x \mapsto \mathrm{Ad}(\gamma, \gamma')x$ is a bijection

between the coset spaces in (7.3) for (N, N') and for $(\gamma N, \gamma' N')$, and we change the summation variable.

(iii) Both sides vanish unless the data are in range, and $\mathrm{Ad}(\gamma, \gamma')$ preserves the range by Lemma 7.6(ii). On the range, (ii) shows that the kernel of the moved data is $\theta_{N,N'}[c, x] \circ R^{-1}$. The Taylor coefficients of $G \circ R^{-1}$ at 0 are those of G after the linear substitution $D \mapsto hDg$ with $g = (\tau(\gamma))_\tau$ and $h = (\tau(\gamma')^{-1})_\tau$, by the chain rule. In the definition of the jet operator this substitution moves to the polynomial side, where it becomes the block action of h on the variables Y and of g on P, and the factor $\prod_\tau \det(X_\tau)^{n_\tau}$ contributes $\prod_\tau \det(g_\tau)^{n_\tau}$. The block actions of g and h are $\rho_\lambda(\gamma)^{-1}$ and $\rho_{\lambda'}(\gamma')$ up to the determinant twists $\prod_\tau \tau(\det\gamma)^{k(\lambda_\tau)}$ and $\prod_\tau \tau(\det\gamma')^{-k(\lambda'_\tau)}$. The computation in (i) shows $\chi_F(\gamma N, \gamma' N') = \prod_\tau \tau(\det\gamma)^{-r'/2} \prod_\tau \tau(\det\gamma')^{r/2} \chi_F(N, N')$.

For the frame factor put $u = \det\gamma$, an element of norm one. The frame of γN at a prime P is γ times the frame of N, so $\mathrm{ord}_P \det(\gamma N) = \mathrm{ord}_P(u) + \mathrm{ord}_P \det N$ away from the bad primes, and $\det(\gamma\,\mathrm{fr}_P) = u \det \mathrm{fr}_P$ at the bad primes. Hence $\mu(\gamma N, \gamma\,\mathrm{fr})/\mu(N, \mathrm{fr}) = \prod_{v\nmid\infty} \mu_v(u)$, and since μ is trivial on $E^\times$, this equals $\mu_\infty(u)^{-1} = \prod_\tau \tau(u)^{-e_\tau}$. The same holds at rank r' with γ'.

Collecting the scalars, the exponent of $\prod_\tau \tau(\det\gamma)$ is $n_\tau + \frac{r'}{2} - e_\tau - k(\lambda_\tau)$ and the exponent of $\prod_\tau \tau(\det\gamma')$ is $e_\tau - \frac{r}{2} + k(\lambda'_\tau)$, componentwise in τ. Both vanish by hypothesis, so every scalar cancels. □

Proposition 7.9(iii) says that the step kernel is an intertwiner between the representations ρ_λ and $\rho_{\lambda'}$ of the rational unitary groups. Summing the step kernels over the classes of lattices of level j, with the weights $1/\#\mathrm{Stab}$, therefore defines a map K_j from vector-valued automorphic forms of weight λ on the definite unitary group of rank r to forms of weight λ' on the definite unitary group of rank r'. The composite of m such maps is the chain of lifts of Theorem A.

8 The theta function at an inert prime

Section 9 studies a series of the form $\exp\big(-\sum \beta(x,y)\big)\,U$ in the parameters of the formal group of height two, with U integral and β a pairing that is biadditive and semilinear for Frobenius up to series with integral exponential. This section proves that the theta kernels of Section 7 have this form at a prime inert in $\mathbb{Q}(\sqrt{-d})$. It begins with the theta function of one variable. It constructs β as the coboundary of the logarithm of the reduced theta function, and proves the properties of β that Theorem 9.3 requires. It then passes to theta functions of hermitian lattices, first in coordinates along an orthonormal basis of the lattice at p and then in hyperbolic coordinates, and proves the form $c\exp\big(-\sum \beta(x,y)\big)U$ in Theorem 8.21.

8.1 The reduced theta function

Let $\Lambda \subset \mathbb{C}$ be a lattice and A the area of $\mathbb{C}/\Lambda$. Let σ_Λ, ζ_Λ and $\wp_\Lambda$ be the Weierstrass functions of Λ, with $\zeta'_\Lambda = -\wp_\Lambda$ and $\sigma'_\Lambda/\sigma_\Lambda = \zeta_\Lambda$, and let g_2, g_3 be the invariants of Λ, so that $\wp'^2_\Lambda = 4\wp^3_\Lambda - g_2\wp_\Lambda - g_3$. The quasi-period map $\eta_\Lambda\colon \Lambda \to \mathbb{C}$ is defined by $\zeta_\Lambda(z+\omega) = \zeta_\Lambda(z) + \eta_\Lambda(\omega)$, and it is additive. We drop the index Λ when the lattice is fixed. We use two classical facts [39, Chapter 18]. The function σ satisfies

$$\sigma(z+\omega) = \psi(\omega)\,\sigma(z)\,e^{\eta(\omega)(z+\omega/2)}, \qquad \omega \in \Lambda, \tag{8.1}$$

with $\psi(\omega) = 1$ for $\omega \in 2\Lambda$ and $\psi(\omega) = -1$ otherwise. If ω_1, ω_2 is a basis of Λ with $\mathrm{Im}(\omega_2/\omega_1) > 0$, then Legendre's relation $\eta(\omega_1)\omega_2 - \eta(\omega_2)\omega_1 = 2\pi i$ holds.

For $\omega, \lambda \in \Lambda$ the number $\mathrm{Im}(\bar{\omega}\lambda)$ is an integer multiple of A. Indeed, if $\omega = m_1\omega_1 + m_2\omega_2$ and $\lambda = n_1\omega_1 + n_2\omega_2$, then $\mathrm{Im}(\bar{\omega}\lambda) = (m_1n_2 - m_2n_1)\,\mathrm{Im}(\bar{\omega}_1\omega_2)$, and $\mathrm{Im}(\bar{\omega}_1\omega_2) = |\omega_1|^2\,\mathrm{Im}(\omega_2/\omega_1) = A$.

Lemma 8.1. *Let $\Lambda \subset \mathbb{C}$ be a lattice of area A. There is a unique complex number $s_2 = s_2(\Lambda)$ with*

$$\eta(\omega) = s_2\,\omega + \frac{\pi}{A}\,\bar{\omega} \qquad \text{for every } \omega \in \Lambda.$$

For $c \in \mathbb{C}^\times$ one has $s_2(c\Lambda) = c^{-2}s_2(\Lambda)$.

Proof. The additive map η extends to an $\mathbb{R}$-linear map $\mathbb{C} \to \mathbb{C}$, and every such map has the form $z \mapsto s_2 z + b\bar{z}$ for unique complex numbers s_2 and b. Legendre's relation gives $2\pi i = b(\bar{\omega}_1\omega_2 - \bar{\omega}_2\omega_1) = 2ib\,\mathrm{Im}(\bar{\omega}_1\omega_2) = 2ibA$, so $b = \pi/A$. For the scaling, $\zeta_{c\Lambda}(cz) = c^{-1}\zeta_\Lambda(z)$ gives $\eta_{c\Lambda}(c\omega) = c^{-1}\eta_\Lambda(\omega)$, and the area of $c\Lambda$ is $|c|^2A$, so $c^{-1}\big(s_2(\Lambda)\omega + \frac{\pi}{A}\bar{\omega}\big) = s_2(c\Lambda)\,c\omega + \frac{\pi}{|c|^2A}\,\bar{c}\bar{\omega}$, and the coefficients of ω agree. □

The *reduced theta function* of Λ is

$$\theta(z) = \theta_\Lambda(z) = \sigma_\Lambda(z)\,e^{-s_2(\Lambda)z^2/2}.$$

It is an odd entire function with simple zeros at the points of Λ and no other zeros, and $\theta'(0) = 1$. For $c \in \mathbb{C}^\times$, the homogeneity $\sigma_{c\Lambda}(cz) = c\,\sigma_\Lambda(z)$ and Lemma 8.1 give $\theta_{c\Lambda}(cz) = c\,\theta_\Lambda(z)$. Write $\wp(z) = z^{-2} + \sum_{k\geq 2} e_k z^{2k-2}$. The coefficients e_k are polynomials in g_2, g_3 with rational coefficients, by the recursion that follows from $\wp'' = 6\wp^2 - g_2/2$. Integrating $-\wp + z^{-2}$ twice and using that $\sigma(z)/z$ is even with value 1 at 0, we obtain

$$\log\frac{\theta(z)}{z} = -\frac{s_2}{2}z^2 - \sum_{k\geq 2}\frac{e_k}{(2k-1)2k}\,z^{2k}. \tag{8.2}$$

Lemma 8.2. *For $\omega \in \Lambda$,*

$$\theta(z+\omega) = \psi(\omega)\,\theta(z)\,\exp\Big(\frac{\pi}{A}\,\bar{\omega}\,\big(z + \tfrac{\omega}{2}\big)\Big).$$

Proof. By (8.1) and Lemma 8.1,

$$\begin{aligned}\frac{\theta(z+\omega)}{\psi(\omega)\theta(z)} &= \exp\Big(\eta(\omega)\big(z+\tfrac{\omega}{2}\big) - \frac{s_2}{2}\big((z+\omega)^2 - z^2\big)\Big)\\ &= \exp\Big(\big(s_2\omega + \tfrac{\pi}{A}\bar{\omega}\big)\big(z+\tfrac{\omega}{2}\big) - s_2\omega\big(z+\tfrac{\omega}{2}\big)\Big).\end{aligned}$$

□

The factor in Lemma 8.2 is the automorphy factor of the hermitian form $\pi z\bar{w}/A$ with the semicharacter ψ, so θ is a classical theta function on $\mathbb{C}/\Lambda$ in the sense of Section 7.1. In the following lemma, a meromorphic function on $\mathbb{C}^n$ is called *periodic* when it is invariant under translation by Λ^n. A periodic function is a rational function on the n-th power of the elliptic curve $\mathbb{C}/\Lambda$, and we describe its divisor by its zeros and poles.

Lemma 8.3. *Let Λ be a lattice with $\mathcal{O}_K\Lambda \subset \Lambda$, where K is an imaginary quadratic field, and identify K with a subfield of $\mathbb{C}$.*

(i) Let $N \geq 1$ and let $t \in \mathbb{C}$ with $Nt \in \Lambda$ and $t \notin \Lambda$. Put

$$k_t(z) = \frac{\theta(z+t)}{\theta(z)\,\theta(t)}\, e^{-\pi \bar{t} z / A}.$$

Then k_t^N is periodic, its divisor is $N[-t] - N[0]$, and $z k_t(z) \to 1$ as $z \to 0$.

(ii) The function

$$c(z_1, z_2, z_3) = \frac{\theta(z_1+z_2+z_3)\,\theta(z_1)\,\theta(z_2)\,\theta(z_3)}{\theta(z_1+z_2)\,\theta(z_1+z_3)\,\theta(z_2+z_3)}$$

is periodic, and its divisor is the sum of the four divisors $z_1+z_2+z_3 = 0$, $z_1 = 0$, $z_2 = 0$, $z_3 = 0$, minus the sum of the three divisors $z_1+z_2 = 0$, $z_1+z_3 = 0$, $z_2+z_3 = 0$.

(iii) Let $a \in \mathcal{O}_K$, $a \neq 0$. The function

$$R_a(z, w) = \frac{\theta(az+w)\,\theta(z)\,\theta(\bar{a}w)}{\theta(az)\,\theta(w)\,\theta(z+\bar{a}w)}$$

is periodic. Its divisor is $D_1 + D_2 - D_3 - D_4$, where D_1 is $az + w = 0$, D_2 is $\bar{a}w = 0$ together with $z = 0$, D_3 is $az = 0$ together with $w = 0$, and D_4 is $z + \bar{a}w = 0$.

(iv) Let $a \in \mathcal{O}_K$ with $a\bar{a}$ odd, and let S be the set of nonzero points of $a^{-1}\Lambda/\Lambda$. Then S has $a\bar{a} - 1$ elements, it is stable under $P \mapsto -P$ and contains no point of order two, and

$$\frac{\theta(az)}{\theta(z)^{a\bar{a}}} = a \prod_{\{P,-P\} \subset S} \big(\wp(z) - \wp(P)\big),$$

the product running over the $(a\bar{a}-1)/2$ pairs. Consequently $a(a-\bar{a})\, s_2(\Lambda) = \sum_{P \in S} \wp(P)$.

Proof. For a product of values of θ at linear forms, Lemma 8.2 computes the factor acquired under a translation, and in each case we compare the exponents and the signs ψ.

(i) Lemma 8.2 gives $k_t(z+\omega) = k_t(z) \exp\big(\frac{\pi}{A}(\bar{\omega}t - \bar{t}\omega)\big) = k_t(z) \exp\big(\frac{2\pi i}{A} \operatorname{Im}(\bar{\omega}t)\big)$. Since $N \operatorname{Im}(\bar{\omega}t) = \operatorname{Im}(\bar{\omega} \cdot Nt)$ is an integer multiple of A, the function k_t^N is periodic. The zeros of $\theta(z+t)$ are the points $z \in -t + \Lambda$, and those of $\theta(z)$ are the points of Λ, all simple, and $t \notin \Lambda$ makes these sets disjoint. Finally $\theta(z+t) \to \theta(t)$ and $\theta(z)/z \to 1$.

(ii) Translate z_1 by ω. Lemma 8.2 multiplies the numerator by $\psi(\omega)^2$ times the exponential of $\frac{\pi}{A}\bar{\omega}$ times $(z_1+z_2+z_3+\frac{\omega}{2}) + (z_1+\frac{\omega}{2})$, and the denominator by $\psi(\omega)^2$ times the exponential of $\frac{\pi}{A}\bar{\omega}$ times $(z_1+z_2+\frac{\omega}{2}) + (z_1+z_3+\frac{\omega}{2})$. The two sums agree. The function is symmetric in z_1, z_2, z_3, so it is periodic. The divisor is read from the zeros of θ.

(iii) Translate z by ω. Since $a\omega \in \Lambda$, the numerator acquires $\psi(a\omega)\psi(\omega)$ times the exponential of $\frac{\pi}{A}$ times

$$\bar{a}\bar{\omega}\big(az + w + \tfrac{a\omega}{2}\big) + \bar{\omega}\big(z + \tfrac{\omega}{2}\big) = a\bar{a}\bar{\omega}z + \bar{a}\bar{\omega}w + \tfrac{a\bar{a}}{2}|\omega|^2 + \bar{\omega}z + \tfrac{1}{2}|\omega|^2,$$

and the denominator acquires $\psi(a\omega)\psi(\omega)$ times the exponential of $\frac{\pi}{A}$ times

$$\bar{a}\bar{\omega}\big(az + \tfrac{a\omega}{2}\big) + \bar{\omega}\big(z + \bar{a}w + \tfrac{\omega}{2}\big),$$

which is the same number. Translating w by ω is the same computation with the roles of (a, z) and $(\bar{a}, w)$ exchanged. The divisor is read from the zeros of θ.

(iv) Multiplication by a is an $\mathbb{R}$-linear map of $\mathbb{C}$ with determinant $|a|^2$, so $[\Lambda : a\Lambda] = a\bar{a}$ and $a^{-1}\Lambda/\Lambda$ has $a\bar{a}$ elements. If $P \in a^{-1}\Lambda$ and $2P \in \Lambda$, choose integers u, v with $ua\bar{a} + 2v = 1$. Then $P = u\bar{a}(aP) + v(2P) \in \Lambda$, so S contains no point of order two. Put $g(z) = \theta(az)/\theta(z)^{a\bar{a}}$. By Lemma 8.2, translation by ω multiplies $\theta(az)$ by $\psi(a\omega)$ times the exponential of $\frac{\pi}{A}\bar{a}\bar{\omega}(az + \frac{a\omega}{2})$, and $\theta(z)^{a\bar{a}}$ by $\psi(\omega)^{a\bar{a}}$ times the exponential of $\frac{\pi}{A}a\bar{a}\bar{\omega}(z + \frac{\omega}{2})$. The exponents agree. The number $a\bar{a}$ is odd, so $\psi(\omega)^{a\bar{a}} = \psi(\omega)$, and $a\omega \in 2\Lambda$ exactly when $\omega \in 2\Lambda$, because multiplication by a is invertible on $\Lambda/2\Lambda$. So g is periodic. Its zeros are the points of $a^{-1}\Lambda \setminus \Lambda$, all simple, and its pole at the points of Λ has order $a\bar{a} - 1$. For each pair $\{P, -P\} \subset S$ the function $\wp(z) - \wp(P)$ has simple zeros at $\pm P$, since P is not of order two, and a double pole at 0. So g and the product have the same divisor, and their quotient is constant. Near 0 we have $g(z) = az^{1-a\bar{a}}(1 + O(z^2))$ and the product is $z^{1-a\bar{a}}(1 + O(z^2))$, so the constant is a. For the last statement we compare the coefficients of z^2 in the logarithms of both sides divided by $az^{1-a\bar{a}}$. By (8.2) the left side gives $-\frac{s_2}{2}(a^2 - a\bar{a})$. The right side gives $\sum_{\{P,-P\}} \log\big(1 - \wp(P)z^2 + O(z^4)\big)$, whose coefficient of z^2 is $-\sum_{\{P,-P\}} \wp(P) = -\frac{1}{2}\sum_{P \in S} \wp(P)$. □

8.2 The integral model at an inert prime

From now on, d is the prime of Section 5, with $d \equiv 7 \pmod{16}$, and $K = \mathbb{Q}(\sqrt{-d})$. Since $d > 3$, the units of $\mathcal{O}_K$ are ± 1, and $j = 0$ and $j = 1728$ do not have complex multiplication by $\mathcal{O}_K$. Let $\Lambda \subset \mathbb{C}$ be a lattice with $\mathcal{O}_K\Lambda \subset \Lambda$ and $g_2(\Lambda)^3 - 27g_3(\Lambda)^2 = 1$. Every lattice with multiplication by $\mathcal{O}_K$ has a complex multiple with this property, since the discriminant $g_2^3 - 27g_3^2$ is homogeneous of weight 12 and does not vanish.

Let $p \geq 5$ be a prime that is inert in K. Let $W = W(\overline{\mathbb{F}}_p)$, let $\mathcal{K}_0 = W[1/p]$ with its Frobenius σ, and let $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathcal{K}_0$. We fix a field isomorphism $\iota\colon \mathbb{C} \to \mathbb{C}_p$, and through it regard algebraic numbers as elements of $\mathbb{C}_p$. A power series with algebraic coefficients then has a meaning over $\mathbb{C}$ and over $\mathbb{C}_p$, and an identity between such series holds in both. The completion of K at p is the unramified quadratic extension $\mathbb{Q}_{p^2}$ of $\mathbb{Q}_p$, so $\iota(K) \subset \mathcal{K}_0$. The nontrivial automorphism of $\mathbb{Q}_{p^2}$ is the restriction of σ, so

$$\iota(\bar{a}) = \sigma\big(\iota(a)\big) \qquad (a \in K). \tag{8.3}$$

Proposition 8.4. *There are an algebraic number u and elements G_2, G_3 of the Hilbert class field of K with $G_2 = u^4 g_2(\Lambda)$ and $G_3 = u^6 g_3(\Lambda)$, such that $\iota(G_2)$ and $\iota(G_3)$ lie in W, $\iota(G_2^3 - 27G_3^2)$ is a unit of W, and $|\iota(u)| = 1$.*

Proof. By the theory of complex multiplication [64, Chapter II, Theorems 4.1 and 2.2], the number $j = j(\Lambda)$ is an algebraic integer, it lies in the Hilbert class field H of K, and there is an elliptic curve E_H over H with $j(E_H) = j$ whose endomorphisms are all defined over H and form a ring isomorphic to $\mathcal{O}_K$. Let v be the place of H above p determined by ι. The prime p is unramified in H, so the completion H_v is unramified over $\mathbb{Q}_p$ and its ring of integers lies in W. By the theorem of Serre and Tate on abelian varieties with complex multiplication [63, Theorem 6 and Corollary 1 to Theorem 7], E_H has potentially good reduction at v, and the inertia group at v acts on the Tate module $T_\ell E_H$, for a prime $\ell \neq p$, through a finite group of automorphisms of E_H that commute with $\mathcal{O}_K$, hence through $\mathcal{O}_K^\times = \{\pm 1\}$. If this action is trivial, E_H has good reduction at v by the criterion of Néron, Ogg and Shafarevich [63, Theorem 1]. Otherwise it is the unique quadratic character of the tame inertia group, and the quadratic twist of E_H by the ramified quadratic extension $H_v(\sqrt{p})$ has trivial inertia action and good reduction at v. The quadratic twist by $H(\sqrt{p})$ is defined over H, so in both cases we obtain an

elliptic curve E' over H with $j(E') = j$ and good reduction at v. Choose a Weierstrass equation $y^2 = 4x^3 - A_2x - A_3$ for E' over H. Since $p \geq 5$, a minimal model of E' at v is obtained by the scaling $(x, y) \mapsto (p^{2k}x, p^{3k}y)$ for some integer k, because p is a uniformizer of H_v [65, Chapter VII, Section 1]. So $G_2 = p^{4k}A_2$ and $G_3 = p^{6k}A_3$ lie in H, are integral at v, and $G_2^3 - 27G_3^2$ is a unit at v. Since $j \neq 0, 1728$, the numbers g_2, g_3, G_2, G_3 are nonzero, and two short Weierstrass equations with the same j are related by $G_2 = u^4 g_2$ and $G_3 = u^6 g_3$ for an algebraic number $u \neq 0$. Finally $u^{12} = (G_2^3 - 27G_3^2)/(g_2^3 - 27g_3^2) = G_2^3 - 27G_3^2$ is a unit at v, so $|\iota(u)| = 1$. □

Fix u, G_2 and G_3 as in Proposition 8.4 and let $\mathcal{E}$ be the curve $y^2 = 4x^3 - G_2x - G_3$ over W. Since 2, 3 and $G_2^3 - 27G_3^2$ are units, $\mathcal{E}$ is an elliptic curve over W, smooth and proper. Over $\mathbb{C}$ the same equation defines a curve isomorphic to $\mathbb{C}/\Lambda$ by $z \mapsto (x, y) = (u^2\wp(z), u^3\wp'(z))$, since $u^6\wp'^2 = 4(u^2\wp)^3 - u^4 g_2(u^2\wp) - u^6 g_3$. We write E for this curve over the algebraic numbers, and regard it over $\mathbb{C}$ and over $\mathbb{C}_p$ as needed. Put $T = -2x/y$. It is a parameter at the origin of every fiber of $\mathcal{E}$, and x and y have expansions

$$x = T^{-2}\mathsf{x}(T), \qquad y = -2T^{-3}\mathsf{x}(T), \qquad \mathsf{x}(T) \in 1 + T\,W[[T]] \tag{8.4}$$

[65, Chapter IV, Section 1]. The formal group law $T_1 +_{\mathcal{E}} T_2$ of $\mathcal{E}$ has coefficients in W, and its logarithm $\ell(T) = T + \cdots$ has coefficients in $\mathcal{K}_0$. On the complex side, $T = -2u^{-1}\wp(z)/\wp'(z) = u^{-1}z + O(z^5)$, and $z = u\,\ell(T)$ as power series, because both sides are additive for the group law and have derivative u at 0. Endomorphisms of E extend to $\mathcal{E}$, because $\mathcal{E}$ is the Néron model of E [14, Section 1.2, Proposition 8]. For $a \in \mathcal{O}_K$ let $[a](T)$ be the formal series of the endomorphism $z \mapsto az$. It has coefficients in W and satisfies $\ell([a](T)) = \iota(a)\,\ell(T)$.

Lemma 8.5. *Let P be a point of E of finite order prime to p, $P \neq 0$. Then P is defined over $\mathcal{K}_0$, its coordinates $x(P), y(P)$ lie in W, and its reduction is not the origin.*

Proof. Let N be the order of P. The kernel of multiplication by N on $\mathcal{E}$ is finite and étale over W, because $p \nmid N$, and the residue field of W is algebraically closed, so all its points are W-points. A W-point of $\mathcal{E}$ other than the origin has integral coordinates or reduces to the origin, and reduction is injective on the kernel of N because the kernel is étale [65, Chapter VII, Proposition 3.1]. □

Choose $a = m + n\frac{1+\sqrt{-d}}{2} \in \mathcal{O}_K$ with $a\bar{a}$ odd and prime to p and n prime to p. Then $a - \bar{a} = n\sqrt{-d}$ is nonzero and prime to p, since $p \neq d$. Lemma 8.3(iv) gives $s_2(\Lambda) = \sum_{P \in S} \wp(P)/\big(a(a - \bar{a})\big)$, and $\wp(P) = u^{-2}x(P)$ with $x(P) \in W$ by Lemma 8.5. So $s_2(\Lambda)$ is algebraic and $u^2 s_2(\Lambda) \in W$. The coefficient e_k in (8.2) is a polynomial in g_2, g_3 that is homogeneous of weight $2k$, where g_2 has weight 4 and g_3 weight 6, so $u^{2k}e_k(g_2, g_3) = e_k(G_2, G_3) \in \mathcal{K}_0$.

Fields of definition. The functions of Lemma 8.3 are rational functions on powers of the curve $\mathbb{C}/\Lambda$, which we identify with E over $\mathbb{C}$. Each is characterized by its divisor together with its leading coefficient at the origin in the coordinates z_i, and that coefficient is algebraic. The points in the divisors are algebraic, since they are torsion points of E. An automorphism of $\mathbb{C}$ over $\overline{\mathbb{Q}}$ fixes E, the divisor and the leading coefficient in the coordinates T_i, which differ from the z_i by the algebraic factor u, so it fixes the function. A rational function fixed by all these automorphisms is defined over $\overline{\mathbb{Q}}$, hence over a number field, and ι maps it to a rational function defined over a finite extension of $\mathcal{K}_0$.

The function Υ**.** Put

$$\Upsilon(T) = -\log \frac{\theta\big(u\,\ell(T)\big)}{u\,T}. \tag{8.5}$$

By (8.2), $\Upsilon(T) = -\log \frac{\ell(T)}{T} + \frac{u^2 s_2}{2}\ell(T)^2 + \sum_{k\geq 2} \frac{e_k(G_2,G_3)}{(2k-1)2k}\ell(T)^{2k}$, a power series over $\mathcal{K}_0$ without constant term. In general it is not integral. For a series B in several variables over $\mathcal{K}_0$ without constant term we write $B \simeq 0$ when $\exp(B)$ is integral, and $B \simeq B'$ when $B - B' \simeq 0$. Since the inverse of an integral series with constant term 1 is integral, $B \simeq 0$ implies $-B \simeq 0$, and the relation is an equivalence relation stable under addition and under substitution of integral series without constant term. Put

$$\delta\Upsilon(S,T) = \Upsilon(S +_{\mathcal{E}} T) - \Upsilon(S) - \Upsilon(T),$$

a series in two variables whose monomials contain both variables.

8.3 Divisors near the origin

Let $n \geq 1$, let L be a finite extension of $\mathcal{K}_0$ in $\mathbb{C}_p$, with ring of integers $\mathcal{O}_L$ and a uniformizer π_L. The n-th power $\mathcal{E}^n_{\mathcal{O}_L}$ of $\mathcal{E} \otimes_W \mathcal{O}_L$ is smooth over $\mathcal{O}_L$, its origin o in the special fiber has a regular local ring, and the completion of that ring is $\mathcal{O}_L[[T_1,\dots,T_n]]$, with T_i the parameter T on the i-th factor. For a family $h = (a_1,\dots,a_n)$ of elements of $\mathcal{O}_K$ let $h\colon E^n \to E$ be the homomorphism $(P_i) \mapsto \sum_i [a_i]P_i$, and put

$$T_h = [a_1](T_1) +_{\mathcal{E}} \cdots +_{\mathcal{E}} [a_n](T_n) \ \in W[[T_1,\dots,T_n]].$$

We call h *primitive at* p when some a_i is prime to p. The identity component of the kernel of h is the irreducible component of the divisor $h = 0$ of E^n through the origin.

Lemma 8.6. *Let f be a nonzero rational function on E^n, defined over L. Suppose that the prime divisors of E^n_L that occur in the divisor of f and whose closures in $\mathcal{E}^n_{\mathcal{O}_L}$ contain o are the identity components of the kernels of homomorphisms $h_1,\dots,h_k$ primitive at p, with multiplicities $n_1,\dots,n_k$. Then*

$$f = \pi_L^m\, v \prod_{i=1}^{k} T_{h_i}^{n_i}$$

in the fraction field of $\mathcal{O}_L[[T_1,\dots,T_n]]$, with $m \in \mathbb{Z}$ and v a unit of $\mathcal{O}_L[[T_1,\dots,T_n]]$.

Proof. The local ring R of $\mathcal{E}^n_{\mathcal{O}_L}$ at o is regular, hence factorial [45, Theorem 20.3], and the height one primes of R correspond to the prime divisors of $\mathcal{E}^n_{\mathcal{O}_L}$ through o. These are the special fiber, which is irreducible with local equation π_L, and the closures of prime divisors Z of E^n_L whose closure contains o. So $f = \pi_L^m v_0 \prod_Z g_Z^{\mathrm{ord}_Z(f)}$ with $v_0 \in R^\times$, $m \in \mathbb{Z}$ and g_Z a local equation of the closure of Z. By hypothesis the Z with $\mathrm{ord}_Z(f) \neq 0$ are the identity components Z_i of the kernels of h_i. The homomorphism h_i extends to a homomorphism $\mathcal{E}^n_{\mathcal{O}_L} \to \mathcal{E}_{\mathcal{O}_L}$, because an abelian scheme over a discrete valuation ring is the Néron model of its generic fiber [14, Section 1.2, Proposition 8], and the zero section of $\mathcal{E}$ is cut out near the origin by T, so the closure of Z_i is contained in the zero locus of $T \circ h_i$ near o. The image of $T \circ h_i$ in $\mathcal{O}_L[[T_1,\dots,T_n]]$ is T_{h_i}, whose linear part $\sum_j \iota(a_{ij})T_j$ has a unit coefficient because h_i is primitive at p. So $T \circ h_i$ is part of a regular system of parameters of R, and $R/(T \circ h_i)$ is a regular local ring of dimension n, in particular a domain. Its zero locus is irreducible of dimension n and contains the closure of Z_i,

which is irreducible of dimension n because it is flat over $\mathcal{O}_L$ with generic fiber of dimension $n-1$. The two are therefore equal, and $T \circ h_i$ is a local equation g_{Z_i}. Passing to the completion gives the lemma with v the image of v_0. □

Lemma 8.7. *In the situation of Lemma 8.6, suppose that $f \prod_i T_{h_i}^{-n_i}$, multiplied by a series Q in $\mathcal{O}_L[[T_1, \ldots, T_n]]$ with $Q(0) \in \mathcal{O}_L^\times$, equals a series B with $B(0) = 1$. Then $m = 0$, and B is a unit of $\mathcal{O}_L[[T_1, \ldots, T_n]]$. If the coefficients of B lie in $\mathcal{K}_0$, they lie in W.*

Proof. We have $B = \pi_L^m v Q$, so $1 = B(0) = \pi_L^m v(0) Q(0)$ with $v(0)$ and $Q(0)$ units of $\mathcal{O}_L$. Hence $m = 0$, and $B = vQ$ is a unit. Finally $\mathcal{O}_L \cap \mathcal{K}_0 = W$. □

8.4 Integrality in one variable

Proposition 8.8. *Let $N \geq 1$ be prime to p, and let $t \in \mathbb{C}$ with $Nt \in \Lambda$ and $t \notin \Lambda$. Put $H_t(T) = u\, T\, k_t\big(u\, \ell(T)\big)$. Then H_t is a power series with algebraic coefficients and constant term 1, and its image under ι has coefficients of absolute value at most 1.*

Proof. By Lemma 8.3(i), $f = k_t^N$ is a rational function on E with divisor $N[-t] - N[0]$, defined over a finite extension L of $\mathcal{K}_0$ by the paragraph on fields of definition. By Lemma 8.5 the point $-t$ reduces to a point other than the origin, so its closure does not contain o. Lemma 8.6 with $n = 1$ and $h = (1)$ gives $f = \pi_L^m v T^{-N}$. Since $z = u\ell(T) = uT(1 + O(T))$ and $z k_t(z) \to 1$, the series $B = u^N T^N f = H_t^N$ has constant term 1, and Lemma 8.7 with $Q = u^N$ shows that B is a unit of $\mathcal{O}_L[[T]]$. The binomial series $(1+X)^{1/N} = \sum_k \binom{1/N}{k} X^k$ has coefficients in $\mathbb{Z}_p$, because $1/N \in \mathbb{Z}_p$. So $B^{1/N}$, the N-th root of B with constant term 1, has coefficients in $\mathcal{O}_L$. The series H_t is an N-th root of B with constant term 1, and the N-th root with constant term 1 of a series with constant term 1 is unique, so $H_t = B^{1/N}$. Its coefficients are algebraic, since those of B are. □

Proposition 8.9. *Let $a \in \mathcal{O}_K$ with $a\bar{a}$ odd and prime to p. Then $\mathrm{Y}([a](T)) \simeq a\bar{a}\, \mathrm{Y}(T)$.*

Proof. Put $z = u\, \ell(T)$, so that $az = u\, \ell([a](T))$. By (8.5) and Lemma 8.3(iv),

$$\exp\big(a\bar{a}\, \mathrm{Y}(T) - \mathrm{Y}([a](T))\big) = \frac{\theta(az)}{\theta(z)^{a\bar{a}}} \cdot \frac{(uT)^{a\bar{a}}}{u\,[a](T)} = a\, u^{a\bar{a}-1}\, \frac{T^{a\bar{a}}}{[a](T)} \prod_{\{P,-P\} \subset S} \big(\wp(z) - \wp(P)\big).$$

Each factor $\wp(z) - \wp(P)$ equals $u^{-2}\big(x - x(P)\big) = u^{-2}T^{-2}\big(\mathrm{x}(T) - x(P)T^2\big)$ by (8.4), and the product has $(a\bar{a}-1)/2$ factors. So the right side is

$$\frac{\iota(a)\, T}{[a](T)} \prod_{\{P,-P\} \subset S} \big(\mathrm{x}(T) - x(P) T^2\big).$$

Each point $P \in S$ has order dividing $a\bar{a}$, which is prime to p, so $x(P) \in W$ by Lemma 8.5, and each factor of the product lies in $1 + T\, W[[T]]$. The series $[a](T)$ has coefficients in W and linear coefficient $\iota(a)$, a unit, so $\iota(a)T/[a](T)$ lies in $1 + T\, W[[T]]$. Hence the exponential is integral. □

Proposition 8.10. *(i) In the variables T_1, T_2, T_3,*

$$\delta\mathrm{Y}(T_1 +_{\mathcal{E}} T_2, T_3) \simeq \delta\mathrm{Y}(T_1, T_3) + \delta\mathrm{Y}(T_2, T_3).$$

(ii) Let $a \in \mathcal{O}_K$ with $a\bar{a}$ prime to p. In the variables S, T,

$$\delta\mathrm{Y}\big([a](S), T\big) \simeq \delta\mathrm{Y}\big(S, [\bar{a}](T)\big).$$

Proof. (i) Write $z_i = u\,\ell(T_i)$, $T_{12} = T_1 +_{\mathcal{E}} T_2$, and similarly T_{13}, T_{23}, T_{123}. The difference of the two sides of (i) is

$$D = \Upsilon(T_{123}) - \Upsilon(T_{12}) - \Upsilon(T_{13}) - \Upsilon(T_{23}) + \Upsilon(T_1) + \Upsilon(T_2) + \Upsilon(T_3),$$

and (8.5) gives

$$\exp(-D) = c(z_1, z_2, z_3) \cdot u^{-1}\, \frac{T_{12}\, T_{13}\, T_{23}}{T_{123}\, T_1\, T_2\, T_3},$$

with c the function of Lemma 8.3(ii), a rational function on E^3 defined over a finite extension L of $\mathcal{K}_0$. Each of the seven divisors in its divisor is the kernel of a homomorphism with coefficients in $\{0,1\}$, which is connected and primitive at p, with local series T_{123}, T_1, T_2, T_3, T_{12}, T_{13}, T_{23}. Lemma 8.6 gives $c = \pi_L^m v\, T_{123} T_1 T_2 T_3 / (T_{12} T_{13} T_{23})$. Lemma 8.7, with $Q = u^{-1}$ and $B = \exp(-D)$, whose constant term is 1 because D has no constant term and whose coefficients lie in $\mathcal{K}_0$, shows that $\exp(-D)$ is a unit of $W[[T_1, T_2, T_3]]$ with constant term 1. Its inverse $\exp(D)$ is integral.

(ii) Write $z = u\,\ell(S)$ and $w = u\,\ell(T)$. The difference of the two sides is

$$D' = \Upsilon\big([a]S +_{\mathcal{E}} T\big) - \Upsilon([a]S) - \Upsilon(T) - \Upsilon\big(S +_{\mathcal{E}} [\bar{a}]T\big) + \Upsilon(S) + \Upsilon([\bar{a}]T),$$

and (8.5) gives

$$\exp(-D') = R_a(z, w) \cdot \frac{[a]S \cdot T \cdot \big(S +_{\mathcal{E}} [\bar{a}]T\big)}{\big([a]S +_{\mathcal{E}} T\big) \cdot S \cdot [\bar{a}]T},$$

with R_a the function of Lemma 8.3(iii), defined over a finite extension L of $\mathcal{K}_0$. In the divisor of R_a the divisor $\bar{a}w = 0$ is the union of the divisors $w = Q$ for Q in the kernel of $\bar{a}$, and for $Q \neq 0$ the point Q has order dividing $a\bar{a}$, prime to p, so by Lemma 8.5 the closure of $w = Q$ does not contain o. The same holds for $az = 0$. So the prime divisors through o in the divisor of R_a are $az + w = 0$ with multiplicity 1, $z = 0$ with multiplicity $1 - 1 = 0$, $w = 0$ with multiplicity $1 - 1 = 0$, and $z + \bar{a}w = 0$ with multiplicity -1. The homomorphisms $(z, w) \mapsto az + w$ and $(z, w) \mapsto z + \bar{a}w$ have the coefficient 1, so they are primitive at p. Their kernels are connected, being graphs. Lemma 8.6 gives $R_a = \pi_L^m v\,\big([a]S +_{\mathcal{E}} T\big) / \big(S +_{\mathcal{E}} [\bar{a}]T\big)$, and then

$$\exp(-D') = \pi_L^m v\, \frac{[a]S}{S} \cdot \frac{T}{[\bar{a}]T}.$$

The last two factors lie in $W[[S]]$ and $W[[T]]$ with unit constant terms $\iota(a)$ and $\iota(\bar{a})^{-1}$. Lemma 8.7 with $Q = \frac{[a]S}{S} \cdot \frac{T}{[\bar{a}]T}$ shows that $\exp(-D')$, a series over $\mathcal{K}_0$ with constant term 1, is a unit of $W[[S, T]]$, and $\exp(D')$ is integral. □

8.5 The pairing in Lubin–Tate coordinates

Let F be the formal group law over $\mathbb{Z}_p$ with logarithm lg of Section 9.1. For $\alpha \in W$ with $\sigma^2\alpha = \alpha$, that is $\alpha \in \mathbb{Z}_{p^2}$, put $[\alpha]_F(X) = \lg^{-1}(\alpha \lg X)$. Hazewinkel's functional-equation lemma [31, Section 2.2] shows that $[\alpha]_F$ has coefficients in W, since lg satisfies the functional equation $\lg(X) - p^{-1}(\lg)^{\sigma^2}(X^{p^2}) = X$ and $\sigma^2\alpha = \alpha$. So F is a formal module over $\mathbb{Z}_{p^2}$. For a root of unity ζ of order dividing $p^2 - 1$ one has $\lg(\zeta X) = \zeta \lg(X)$, hence $[\zeta]_F(X) = \zeta X$.

Proposition 8.11. *There are $\varepsilon \in W^\times$ and a series $\varkappa \in W[[T]]$ with $\varkappa(0) = 0$ and $\varkappa'(0) \in W^\times$ such that $\lg(\varkappa(T)) = \varepsilon\,\ell(T)$.*

Proof. For $a \in \mathcal{O}_K$ the series $[a](T)$ is an endomorphism of the formal group $\widehat{\mathcal{E}}$ of $\mathcal{E}$ over W with derivative $\iota(a)$ at 0. The ring of endomorphisms of $\widehat{\mathcal{E}}$ over W is complete for the p-adic topology, so the map $a \mapsto [a]$ extends by continuity to $\mathbb{Z}_{p^2}$, the completion of $\mathcal{O}_K$ at p, and makes $\widehat{\mathcal{E}}$ a formal $\mathbb{Z}_{p^2}$-module whose action on the tangent space is the inclusion $\mathbb{Z}_{p^2} \subset W$. The height of $\widehat{\mathcal{E}}$ is 1 or 2, because $\mathcal{E}$ is an elliptic curve, and it is divisible by $[\mathbb{Q}_{p^2} : \mathbb{Q}_p] = 2$, because the special fiber of $\widehat{\mathcal{E}}$ is a formal module over $\mathbb{Z}_{p^2}$ and the height of a formal module over the integers of a local field is a multiple of the degree of that field. So $\widehat{\mathcal{E}}$ is a formal $\mathbb{Z}_{p^2}$-module of relative height one, as is F. Over the ring of integers of the completion of the maximal unramified extension of $\mathbb{Q}_{p^2}$, which is W, any two formal $\mathbb{Z}_{p^2}$-modules of relative height one are isomorphic [42], [31, Chapter IV, Section 21]. Let $\varkappa\colon \widehat{\mathcal{E}} \to F$ be an isomorphism over W. Then $\lg \circ \varkappa$ and ℓ are logarithms of $\widehat{\mathcal{E}}$, and the logarithm of a one-dimensional formal group over a $\mathbb{Q}$-algebra is unique up to a constant factor, so $\lg(\varkappa(T)) = \varepsilon \ell(T)$ with $\varepsilon = \varkappa'(0) \in W^\times$. □

Fix ε and $\varkappa$ as in Proposition 8.11. Then $\varkappa(S +_{\mathcal{E}} T) = F(\varkappa S, \varkappa T)$, because lg of both sides is $\varepsilon(\ell(S) + \ell(T))$. For $a \in \mathcal{O}_K$ we have $\varkappa([a](T)) = [\iota(a)]_F(\varkappa(T))$, because lg of both sides is $\iota(a)\varepsilon\ell(T)$. Put

$$\mathrm{Y}_F = \mathrm{Y} \circ \varkappa^{-1}, \qquad \beta(X,Y) = \mathrm{Y}_F\big(F(X,Y)\big) - \mathrm{Y}_F(X) - \mathrm{Y}_F(Y) = \delta\mathrm{Y}\big(\varkappa^{-1}X, \varkappa^{-1}Y\big). \tag{8.6}$$

These are series over $\mathcal{K}_0$, and the relation $\simeq$ is transported by the substitution $\varkappa^{-1}$, which has coefficients in W.

Theorem 8.12. *The series β has the following properties.*

(i) Every monomial of β contains both variables.

(ii) $\beta(F(X_0, X_1), X_2) \simeq \beta(X_0, X_2) + \beta(X_1, X_2)$, and the same holds in the second variable.

(iii) $\beta([\alpha]_F X, Y) \simeq \beta(X, [\sigma\alpha]_F Y)$ for every $\alpha \in \mathbb{Z}_{p^2}^\times$.

(iv) $\mathrm{Y}_F([\alpha]_F X) \simeq \alpha\,\sigma(\alpha)\,\mathrm{Y}_F(X)$ for every $\alpha \in \mathbb{Z}_{p^2}^\times$.

In particular β satisfies hypotheses (i) to (iii) of Theorem 9.3 with $\mathcal{K} = \mathcal{K}_0$.

Proof. (i) Y_F has no constant term, so $\beta(X, 0) = \beta(0, Y) = 0$.

(ii) Substitute $T_i = \varkappa^{-1}(X_i)$ in Proposition 8.10(i), and use $\varkappa^{-1}F(X_0, X_1) = \varkappa^{-1}X_0 +_{\mathcal{E}} \varkappa^{-1}X_1$. The series β is symmetric, so the second variable follows.

(iii) and (iv) First let $\alpha = \iota(a)$ with $a \in \mathcal{O}_K$ such that $a\bar{a}$ is odd and prime to p. By (8.3), $\sigma\alpha = \iota(\bar{a})$. Substituting $S = \varkappa^{-1}X$, $T = \varkappa^{-1}Y$ in Proposition 8.10(ii) and in Proposition 8.9, and using $\varkappa \circ [a] = [\iota(a)]_F \circ \varkappa$, gives (iii) and (iv) for α, with $\iota(a\bar{a}) = \alpha\sigma(\alpha)$.

The set of such $\iota(a)$ is dense in $\mathbb{Z}_{p^2}^\times$. Indeed, $\iota(\mathcal{O}_K)$ is dense in $\mathbb{Z}_{p^2}$. Given $\alpha \in \mathbb{Z}_{p^2}^\times$ and $k \geq 1$, the Chinese remainder theorem gives $a \in \mathcal{O}_K$ with $\iota(a) \equiv \alpha \pmod{p^k}$ and $a \equiv 1 \pmod{2\mathcal{O}_K}$. Then a is a unit at p, so $p \nmid a\bar{a}$, and $a\bar{a} \equiv 1 \pmod 2$.

Let $\alpha_k \to \alpha$ with each α_k of this form. Then $[\alpha_k]_F \to [\alpha]_F$ and $[\sigma\alpha_k]_F \to [\sigma\alpha]_F$ coefficientwise, since $[\alpha]_F = \lg^{-1}(\alpha \lg X)$ depends continuously on α in each degree. The coefficient of a fixed monomial of $\exp\big(\beta([\alpha]_F X, Y) - \beta(X, [\sigma\alpha]_F Y)\big)$ is a polynomial over $\mathcal{K}_0$ in finitely many coefficients of $[\alpha]_F$ and $[\sigma\alpha]_F$, so it is the limit of the corresponding coefficients for α_k, which lie in W. Since W is closed, the limit lies in W. The limit proves (iii), and the same argument proves (iv).

For the hypotheses of Theorem 9.3, the field $\mathcal{K}_0$ is complete, $|\mathcal{K}_0^\times| = p^{\mathbb{Z}}$, it contains the roots of unity of order p^2-1, and $|\sigma a - a^p| \le p^{-1}$ for $|a| \le 1$. Hypothesis (iii) of that theorem is (iii) for $\alpha = \zeta$, since $\sigma\zeta = \zeta^p$ and $[\zeta]_F X = \zeta X$. □

8.6 Unitary and hyperbolic changes of coordinates

For $\alpha \in \mathbb{Z}_{p^2}$ we write $[\alpha](T) = \ell^{-1}(\alpha\,\ell(T))$. The proof of Proposition 8.11 shows that $[\alpha]$ has coefficients in W, and $\varkappa \circ [\alpha] = [\alpha]_F \circ \varkappa$. By (8.6), $\beta(\varkappa S, \varkappa T) = \delta\mathrm{Y}(S,T)$, so Theorem 8.12 and Proposition 8.10(i) give, in the parameter T of $\mathcal{E}$, the relations

$$\mathrm{Y}([\alpha]T) \simeq \alpha\sigma(\alpha)\,\mathrm{Y}(T), \qquad \delta\mathrm{Y}([\alpha]S,T) \simeq \delta\mathrm{Y}(S,[\sigma\alpha]T) \qquad (\alpha \in \mathbb{Z}_{p^2}^\times), \tag{8.7}$$

and $\delta\mathrm{Y}$ is biadditive up to $\simeq$ in each variable. In (8.7) the factor $\alpha\sigma(\alpha)$ lies in $\mathbb{Z}_p$. All coefficients that multiply Y or δY below lie in $\mathbb{Z}_p$, and the relations are identities of series up to $\simeq$.

Lemma 8.13. *For every $\alpha, c \in \mathbb{Z}_{p^2}$ the following hold.*

(i) $\delta\mathrm{Y}([\alpha]S,T) \simeq \delta\mathrm{Y}(S,[\sigma\alpha]T)$.

(ii) $\delta\mathrm{Y}(T,[c]T) \simeq \big(c+\sigma(c)\big)\mathrm{Y}(T)$.

(iii) $\mathrm{Y}([\alpha]T) \simeq \alpha\sigma(\alpha)\,\mathrm{Y}(T)$.

Proof. Since p is odd, every $\alpha \in \mathbb{Z}_{p^2}$ is a sum $\alpha_1+\alpha_2$ of two units, with $(\alpha_1,\alpha_2) = (2\alpha,-\alpha)$ if α is a unit and $(\alpha_1,\alpha_2) = (1,\alpha-1)$ otherwise. Then $[\alpha]S = [\alpha_1]S +_{\mathcal{E}} [\alpha_2]S$.

(i) Biadditivity in the first variable, (8.7) for α_1 and α_2, and biadditivity in the second variable give

$$\begin{aligned}\delta\mathrm{Y}([\alpha]S,T) &\simeq \delta\mathrm{Y}([\alpha_1]S,T) + \delta\mathrm{Y}([\alpha_2]S,T)\\ &\simeq \delta\mathrm{Y}(S,[\sigma\alpha_1]T) + \delta\mathrm{Y}(S,[\sigma\alpha_2]T) \simeq \delta\mathrm{Y}(S,[\sigma\alpha]T).\end{aligned}$$

(ii) First let c be a unit with $1+c$ a unit. Then $T +_{\mathcal{E}} [c]T = [1+c]T$, so (8.7) gives

$$\begin{aligned}\delta\mathrm{Y}(T,[c]T) &= \mathrm{Y}([1+c]T) - \mathrm{Y}(T) - \mathrm{Y}([c]T)\\ &\simeq \big((1+c)\sigma(1+c) - 1 - c\sigma(c)\big)\mathrm{Y}(T) = \big(c+\sigma(c)\big)\mathrm{Y}(T).\end{aligned}$$

For general c, the residue field $\mathbb{F}_{p^2}$ has at least 25 elements, so there is $c' \in \mathbb{Z}_{p^2}$ whose residue avoids the four residues of 0, -1, c and $c+1$. Then c', $1+c'$, $c-c'$ and $1+c-c'$ are units, and biadditivity in the second variable gives $\delta\mathrm{Y}(T,[c]T) \simeq \delta\mathrm{Y}(T,[c']T) + \delta\mathrm{Y}(T,[c-c']T)$, which is $\simeq \big(c+\sigma(c)\big)\mathrm{Y}(T)$ by the first case.

(iii) Write $\alpha = \alpha_1+\alpha_2$ with units α_1,α_2. By the definition of δY,

$$\mathrm{Y}([\alpha]T) = \mathrm{Y}([\alpha_1]T) + \mathrm{Y}([\alpha_2]T) + \delta\mathrm{Y}([\alpha_1]T,[\alpha_2]T).$$

Part (i), in two variables and then with $S=T$, gives $\delta\mathrm{Y}([\alpha_1]T,[\alpha_2]T) \simeq \delta\mathrm{Y}(T,[\sigma(\alpha_1)\alpha_2]T)$, and part (ii) with $c = \sigma(\alpha_1)\alpha_2$ turns this into $\big(\sigma(\alpha_1)\alpha_2 + \alpha_1\sigma(\alpha_2)\big)\mathrm{Y}(T)$. With (8.7) for α_1 and α_2, the coefficient of $\mathrm{Y}(T)$ is $\alpha_1\sigma(\alpha_1)+\alpha_2\sigma(\alpha_2)+\sigma(\alpha_1)\alpha_2+\alpha_1\sigma(\alpha_2) = \alpha\sigma(\alpha)$. □

Lemma 8.14. *Let $A_1,\dots,A_k$ be integral series without constant term in some set of variables. Then*

$$\mathrm{Y}\big(A_1 +_{\mathcal{E}} \cdots +_{\mathcal{E}} A_k\big) \simeq \sum_{j=1}^{k} \mathrm{Y}(A_j) + \sum_{1\le j<l\le k} \delta\mathrm{Y}(A_j,A_l).$$

Proof. For $k = 1$ there is nothing to prove. Put $B = A_1 +_{\mathcal{E}} \cdots +_{\mathcal{E}} A_{k-1}$. Then $\mathrm{Y}(B +_{\mathcal{E}} A_k) = \mathrm{Y}(B) + \mathrm{Y}(A_k) + \delta\mathrm{Y}(B, A_k)$, and biadditivity in the first variable, applied $k-2$ times, gives $\delta\mathrm{Y}(B, A_k) \simeq \sum_{j<k} \delta\mathrm{Y}(A_j, A_k)$. The induction hypothesis for $\mathrm{Y}(B)$ completes the proof. □

Proposition 8.15. *Let $\gamma = (\gamma_{ij})$ be an $m \times m$ matrix over $\mathbb{Z}_{p^2}$ with $\sum_i \sigma(\gamma_{ij})\gamma_{il} = [j = l]$ for all j, l, and put $T'_i = [\gamma_{i1}](T_1) +_{\mathcal{E}} \cdots +_{\mathcal{E}} [\gamma_{im}](T_m)$. Then*

$$\sum_{i=1}^{m} \mathrm{Y}(T'_i) \simeq \sum_{j=1}^{m} \mathrm{Y}(T_j).$$

Proof. Lemma 8.14, then Lemma 8.13(iii) for the diagonal terms and Lemma 8.13(i) with $[\sigma(\gamma_{ij})][\gamma_{il}] = [\sigma(\gamma_{ij})\gamma_{il}]$ for the cross terms, give

$$\sum_i \mathrm{Y}(T'_i) \simeq \sum_j \Big(\sum_i \gamma_{ij}\sigma(\gamma_{ij}) \Big) \mathrm{Y}(T_j) + \sum_{j<l} \sum_i \delta\mathrm{Y}\big(T_j, [\sigma(\gamma_{ij})\gamma_{il}]T_l\big).$$

The coefficient $\sum_i \gamma_{ij}\sigma(\gamma_{ij})$ is the image under σ of $\sum_i \sigma(\gamma_{ij})\gamma_{ij} = 1$, because σ^2 fixes $\mathbb{Z}_{p^2}$. Biadditivity in the second variable turns the inner sum of the cross terms into $\delta\mathrm{Y}\big(T_j, [\sum_i \sigma(\gamma_{ij})\gamma_{il}]T_l\big) = \delta\mathrm{Y}(T_j, 0) = 0$ for $j < l$. □

A *hyperbolic pair* of size m is a pair of $m \times 2m$ matrices $f = (f_{e,i})$ and $f' = (f'_{e,i})$ over $\mathbb{Z}_{p^2}$ with

$$\sum_{i=1}^{2m} \sigma(f_{e,i}) f_{e',i} = 0, \qquad \sum_{i=1}^{2m} \sigma(f'_{e,i}) f'_{e',i} = 0, \qquad \sum_{i=1}^{2m} \sigma(f_{e,i}) f'_{e',i} = [e = e'] \tag{8.8}$$

for all e, e'. Equivalently, the rows of f and of f' span complementary isotropic subspaces of $\mathbb{Q}_{p^2}^{2m}$ for the standard hermitian form, in dual bases.

Proposition 8.16. *Let (f, f') be a hyperbolic pair of size m, and in the variables $X_1, \ldots, X_m, Y_1, \ldots, Y_m$ put*

$$T_i = [f_{1,i}](X_1) +_{\mathcal{E}} \cdots +_{\mathcal{E}} [f_{m,i}](X_m) +_{\mathcal{E}} [f'_{1,i}](Y_1) +_{\mathcal{E}} \cdots +_{\mathcal{E}} [f'_{m,i}](Y_m), \qquad 1 \le i \le 2m.$$

Then

$$\sum_{i=1}^{2m} \mathrm{Y}(T_i) \simeq \sum_{e=1}^{m} \delta\mathrm{Y}(X_e, Y_e).$$

Proof. We apply Lemma 8.14 to each T_i, with the summands in the order $X_1, \ldots, X_m, Y_1, \ldots, Y_m$, and then Lemma 8.13. The diagonal terms give

$$\sum_e \Big(\sum_i f_{e,i}\sigma(f_{e,i}) \Big) \mathrm{Y}(X_e) + \sum_e \Big(\sum_i f'_{e,i}\sigma(f'_{e,i}) \Big) \mathrm{Y}(Y_e),$$

and both coefficients vanish by (8.8), after applying σ. The cross terms between X_e and $X_{e'}$ with $e < e'$ give $\delta\mathrm{Y}\big(X_e, [\sum_i \sigma(f_{e,i})f_{e',i}]X_{e'}\big) = 0$, and those between Y_e and $Y_{e'}$ vanish in the same way. The cross terms between X_e and $Y_{e'}$ give $\delta\mathrm{Y}\big(X_e, [\sum_i \sigma(f_{e,i})f'_{e',i}]Y_{e'}\big) = [e = e']\, \delta\mathrm{Y}(X_e, Y_e)$. □

8.7 Theta functions of hermitian lattices

Let M be a finitely generated torsion-free $\mathcal{O}_K$-module of rank n with a positive definite hermitian form $h\colon M \times M \to K$, linear in the first variable. Put $V_M = M \otimes_{\mathcal{O}_K} \mathbb{C}$, a complex vector space of dimension n, and let $L_M \subset V_M$ be the image of $M \otimes_{\mathcal{O}_K} \Lambda$, a lattice. The formula

$$H_M(x \otimes z, x' \otimes z') = \frac{1}{A}\, h(x,x')\, z\bar{z}'$$

defines a positive definite hermitian form on V_M. For $M = \mathcal{O}_K^n$ with $h(x,y) = \sum_i x_i \bar{y}_i$ we write $V_{\mathrm{std}} = \mathbb{C}^n$, $L_{\mathrm{std}} = \Lambda^n$ and $H_{\mathrm{std}}(z,w) = A^{-1}\sum_i z_i \bar{w}_i$. The alternating form $\operatorname{Im} H_{\mathrm{std}}$ takes integer values on L_{std}, and L_{std} is self-dual for it, by the computation of $\operatorname{Im}(\bar{\omega}\lambda)$ in Section 8.1.

For a lattice $L \subset V_M$ on which $\operatorname{Im} H_M$ is integral, a semicharacter $\alpha\colon L \to \{\pm 1\}$ and the operators U_x of Section 7.1 formed with H_M, the space $\Theta(L,\alpha)$ is the space of entire functions G on V_M with $\alpha(x)U_x G = G$ for $x \in L$. Its dimension is the Pfaffian of $\operatorname{Im} H_M$ on L, that is $[L^\perp : L]^{1/2}$ with $L^\perp = \{v \in V_M : \operatorname{Im} H_M(v,L) \subset \mathbb{Z}\}$ [8, Theorem 3.2.7]. By Lemma 8.2, $\theta_{\mathrm{std}}(z) = \prod_{i=1}^n \theta(z_i)$ spans $\Theta(L_{\mathrm{std}}, \psi^{\otimes n})$, where $\psi^{\otimes n}(\omega) = \prod_i \psi(\omega_i)$.

From $U_x U_y = e^{-\pi i \operatorname{Im} H(x,y)} U_{x+y}$ we obtain $U_x U_v = e^{-2\pi i \operatorname{Im} H(x,v)} U_v U_x$. Hence, for $G \in \Theta(L,\alpha)$ and $v \in V_M$,

$$U_v G \in \Theta\big(L, \alpha \cdot e^{2\pi i \operatorname{Im} H(\cdot, v)}\big). \tag{8.9}$$

Lemma 8.17. *Let $L' \subset V_{\mathrm{std}}$ be a lattice and N an integer prime to p with $NL_{\mathrm{std}} \subset L' \subset N^{-1}L_{\mathrm{std}}$, and let $\alpha'\colon L' \to \{\pm 1\}$ be a semicharacter for $\operatorname{Im} H_{\mathrm{std}}$. Then $\Theta(L',\alpha')$ is contained in the span of the functions $U_t\theta_{\mathrm{std}}$ with $t \in (2N^2)^{-1}L_{\mathrm{std}}$.*

Proof. Put $L'' = L' \cap L_{\mathrm{std}}$. It contains NL_{std}, and $\Theta(L',\alpha') \subset \Theta(L'', \alpha'|_{L''})$. The semicharacters $\alpha'|_{L''}$ and $\psi^{\otimes n}|_{L''}$ have the same defect $e^{\pi i \operatorname{Im} H(x,y)}$, so their quotient is a character of L'' with values ± 1. Since $\operatorname{Im} H_{\mathrm{std}}$ is nondegenerate, this character equals $x \mapsto e^{2\pi i \operatorname{Im} H(x,q_0)}$ for some $q_0 \in \frac{1}{2}L''^\perp$, and $L''^\perp \subset N^{-1}L_{\mathrm{std}}$ because $L'' \supset NL_{\mathrm{std}}$ and L_{std} is self-dual. By (8.9), U_{q_0} maps $\Theta(L'', \psi^{\otimes n})$ into $\Theta(L'', \alpha')$, injectively, and the two spaces have the same dimension, so $\Theta(L'', \alpha') = U_{q_0}\Theta(L'', \psi^{\otimes n})$.

For $q \in L''^\perp$, (8.9) shows $U_q\theta_{\mathrm{std}} \in \Theta(L'', \psi^{\otimes n})$. For $y \in L_{\mathrm{std}}$ the operator $\psi^{\otimes n}(y)U_y$ multiplies $U_q\theta_{\mathrm{std}}$ by $e^{-2\pi i \operatorname{Im} H(y,q)}$, and these characters of L_{std} are distinct for representatives q of distinct classes of $L''^\perp/L_{\mathrm{std}}$, because L_{std} is self-dual. So these functions are linearly independent. Their number is $[L''^\perp : L_{\mathrm{std}}] = [L_{\mathrm{std}} : L'']$, which is the dimension of $\Theta(L'', \psi^{\otimes n})$, so they span it. Finally $U_{q_0}U_q$ is a multiple of U_{q_0+q}, and $q_0 + q \in \frac{1}{2}L''^\perp \subset (2N)^{-1}L_{\mathrm{std}} \subset (2N^2)^{-1}L_{\mathrm{std}}$. □

For $t \in \mathbb{C}$ with $Nt \in \Lambda$ for some N prime to p, put $B_t = \theta \cdot k_t$ when $t \notin \Lambda$ and $B_t = \theta$ when $t \in \Lambda$. Then $U_t\theta = c_t B_t$ with a constant $c_t \neq 0$, namely $c_t = \theta(t)e^{-\pi|t|^2/(2A)}$ when $t \notin \Lambda$ and $c_t = \psi(t)$ when $t \in \Lambda$. The function B_t has algebraic Taylor coefficients, by (8.2) and Proposition 8.8, and

$$e^{\Upsilon(T)} B_t\big(u\,\ell(T)\big) = \begin{cases} H_t(T) & t \notin \Lambda, \\ uT & t \in \Lambda, \end{cases} \tag{8.10}$$

an integral series. For $t \in V_{\mathrm{std}}$ with coordinates t_j put $B_t(z) = \prod_j B_{t_j}(z_j)$. Since H_{std} is an orthogonal sum, $U_t\theta_{\mathrm{std}} = \big(\prod_j c_{t_j}\big)B_t$.

Lemma 8.18. *Let (M,h) be as above, and suppose that $M_p = M \otimes_{\mathcal{O}_K} \mathbb{Z}_{p^2}$ has a basis $b_1, \ldots, b_n$ with $h(b_i, b_j) = [i=j]$. There is a K-linear isometry $g\colon (M_K^2, h \oplus h) \to (K^{2n}, \sum_i x_i\bar{y}_i)$ with $g(M_p^2) = \mathbb{Z}_{p^2}^{2n}$.*

Proof. By Landherr's theorem [61, Chapter 10, Theorem 6.9], hermitian spaces over K are isometric when they have the same dimension, the same discriminant in $\mathbb{Q}^\times/N_{K/\mathbb{Q}}(K^\times)$, and the same signature at the real place of $\mathbb{Q}$. The discriminant δ of h, the determinant of a Gram matrix, lies in $\mathbb{Q}^\times$ because the matrix is hermitian, and the discriminant of $h\oplus h$ is $\delta^2 = N_{K/\mathbb{Q}}(\delta)$. Both forms are positive definite of dimension $2n$. So there is an isometry g_0.

Let $\mathcal{U}$ be the unitary group of $(M_K^2, h\oplus h)$, and $\mathcal{U}_p$ its group of points over $K_p = \mathbb{Q}_{p^2}$. Let φ_p be the $\mathbb{Q}_{p^2}$-linear isometry sending the basis $(b_i,0)$, $(0,b_i)$ of M_p^2 to the standard basis, and put $w = g_0^{-1}\varphi_p \in \mathcal{U}_p$. For $v\in\mathcal{U}(K)$ the map $g = g_0 v$ satisfies $g(M_p^2) = \mathbb{Z}_{p^2}^{2n}$ exactly when $w^{-1}v$ stabilizes M_p^2. The stabilizer of M_p^2 is an open subgroup of $\mathcal{U}_p$, so it suffices to show that $\mathcal{U}(K)$ is dense in $\mathcal{U}_p$.

Let $\mathfrak{u}$ be the $\mathbb{Q}$-vector space of K-linear endomorphisms X of M_K^2 with $X^* = -X$, the adjoint taken for $h\oplus h$. For X with $\det(1+X)\neq 0$ the Cayley transform $C(X) = (1-X)(1+X)^{-1}$ is unitary. Indeed $X^* = -X$ gives $C(X)^* = (1+X^*)^{-1}(1-X^*) = (1-X)^{-1}(1+X)$, and since $1+X$ and $1-X$ commute, $C(X)^*C(X) = 1$. Conversely every $v\in\mathcal{U}$ with $\det(1+v)\neq 0$ equals $C(X)$ for $X = (1-v)(1+v)^{-1}\in\mathfrak{u}$, and the same holds over $\mathbb{Q}_{p^2}$. Let $w'\in\mathcal{U}_p$. The norm-one elements $c = z/\bar z$ of K form an infinite set, so we may choose one such that $-\bar c$ is not an eigenvalue of w', so that $\det(1+cw')\neq 0$. Since $\mathfrak{u}\otimes_{\mathbb{Q}}\mathbb{Q}_p = \mathfrak{u}_p$ and $\mathbb{Q}$ is dense in $\mathbb{Q}_p$, elements $X\in\mathfrak{u}$ approximate $C^{-1}(cw')$, and by continuity $c^{-1}C(X)\in\mathcal{U}(K)$ approximates w'. □

Theorem 8.19. *Let (M,h) be a positive definite hermitian $\mathcal{O}_K$-lattice of rank n, and let $b_1,\dots,b_n$ be a basis of M_p with $h(b_i,b_j) = [i=j]$. Let $L\subset V_M$ be a lattice with $NL_M\subset L\subset N^{-1}L_M$ for some N prime to p, let $\alpha\colon L\to\{\pm1\}$ be a semicharacter for* $\operatorname{Im} H_M$*, and let $G\in\Theta(L,\alpha)$ have algebraic Taylor coefficients at* 0*. Put*

$$\Psi_G(T_1,\dots,T_n) = \exp\Big(\sum_{i=1}^n \Upsilon(T_i)\Big)\cdot G\Big(\sum_{i=1}^n b_i\otimes u\,\ell(T_i)\Big).$$

Then the coefficients of Ψ_G are bounded in absolute value. Consequently $\Psi_G = c\,U$ with $c\in\mathbb{C}_p^\times$ and U integral.

In the statement, G is expanded in the coordinates of a K-basis of M_K, its Taylor coefficients are moved to $\mathbb{C}_p$ by ι, and $b_i\in M\otimes_{\mathcal{O}_K}\mathbb{Z}_{p^2}$ is substituted through $K\subset\mathbb{Q}_{p^2}\subset\mathcal{K}_0$.

Proof. Let g be as in Lemma 8.18, and let $\hat g = g\otimes\mathrm{id}_{\mathbb{C}}\colon V_{M^2}\to V_{\mathrm{std}}$, with $V_{\mathrm{std}} = \mathbb{C}^{2n}$. It is an isometry for the hermitian forms, and it maps $L_{M^2} = L_M\oplus L_M$ onto $L'_{\mathrm{std}} = g(M^2)\otimes\Lambda$. The $\mathcal{O}_K$-lattices $g(M^2)$ and $\mathcal{O}_K^{2n}$ have the same completion at p, so there is N' prime to p with $N'\mathcal{O}_K^{2n}\subset g(M^2)\subset N'^{-1}\mathcal{O}_K^{2n}$. Hence $L' = \hat g(L\oplus L)$ satisfies $N''L_{\mathrm{std}}\subset L'\subset N''^{-1}L_{\mathrm{std}}$ with $N'' = NN'$.

The function $G\otimes G\colon (z,z')\mapsto G(z)G(z')$ lies in $\Theta(L\oplus L,\alpha\otimes\alpha)$, so $(G\otimes G)\circ\hat g^{-1}\in\Theta(L',\alpha')$ with $\alpha' = (\alpha\otimes\alpha)\circ\hat g^{-1}$. By Lemma 8.17 and $U_t\theta_{\mathrm{std}} = \big(\prod_j c_{t_j}\big)B_t$, there are a finite set $\mathcal{T}\subset(2N''^2)^{-1}L_{\mathrm{std}}$ of points distinct modulo L_{std} and complex numbers r_t with

$$(G\otimes G)\circ\hat g^{-1} = \sum_{t\in\mathcal{T}} r_t\,B_t.$$

The functions B_t for $t\in\mathcal{T}$ are linearly independent, by the proof of Lemma 8.17. So finitely many of their Taylor coefficients form an invertible matrix with algebraic entries, and the r_t solve a linear system with algebraic coefficients, because the Taylor coefficients of $(G\otimes G)\circ\hat g^{-1}$ are algebraic, g being defined over K. Hence every r_t is algebraic.

Let γ be the matrix of $g \otimes \mathbb{Q}_{p^2}$ in the basis $(b_i, 0)$, $(0, b_i)$ and the standard basis. Both bases are orthonormal and g maps the lattice spanned by the first onto the lattice spanned by the second, so γ has entries in $\mathbb{Z}_{p^2}$ and satisfies $\sum_i \sigma(\gamma_{ij})\gamma_{il} = [j = l]$. Write the $2n$ variables as $T_1, \dots, T_{2n}$, the first n for the first copy of M. Then

$$\hat{g}\Big(\sum_{i=1}^{2n}(b \oplus b)_i \otimes u\,\ell(T_i)\Big) = \sum_{j=1}^{2n} e_j \otimes u\,\ell(T'_j), \qquad T'_j = [\gamma_{j1}](T_1) +_{\mathcal{E}} \cdots +_{\mathcal{E}} [\gamma_{j,2n}](T_{2n}),$$

because $\ell([\gamma_{ji}]T_i) = \gamma_{ji}\ell(T_i)$. So

$$\Psi_G(T_1, \dots, T_n)\,\Psi_G(T_{n+1}, \dots, T_{2n}) = \exp\Big(\sum_{i=1}^{2n}\Upsilon(T_i) - \sum_{j=1}^{2n}\Upsilon(T'_j)\Big) \cdot \sum_{t\in\mathcal{T}} r_t \prod_{j=1}^{2n} e^{\Upsilon(T'_j)} B_{t_j}\big(u\,\ell(T'_j)\big).$$

The first factor is integral by Proposition 8.15. Each product in the sum is integral by (8.10), since the T'_j are integral series without constant term. So every coefficient of the left side has absolute value at most $R = \max_t |r_t|$. The coefficients of the left side are the products $a_\mu a_\nu$ of coefficients a_μ of Ψ_G, because the two factors are in disjoint variables. Hence $|a_\mu|^2 \le R$ for every μ. □

Theorem 8.19 reduces the integrality of every theta function of a hermitian lattice, in coordinates along an orthonormal basis at p, to the one-variable statement of Proposition 8.8 and the unitary invariance of $\sum \Upsilon$. The next subsection passes from orthonormal coordinates to hyperbolic coordinates, where Proposition 8.16 turns $\sum_i \Upsilon(T_i)$ into $\sum_e \delta\Upsilon(X_e, Y_e)$.

8.8 The kernel law

Lemma 8.20. *Let f, f' be elements of a hermitian $\mathbb{Z}_{p^2}$-module with $h(f,f) = h(f',f') = 0$ and $h(f,f') = 1$. Choose $\kappa \in \mathbb{Z}_{p^2}$ with $\kappa\sigma(\kappa) = -1$. Then $e_1 = f + \frac{1}{2}f'$ and $e_2 = \kappa\big(f - \frac{1}{2}f'\big)$ satisfy $h(e_i, e_j) = [i = j]$, and they span the same $\mathbb{Z}_{p^2}$-module as f and f'.*

Proof. The norm map from $\mathbb{Z}_{p^2}^\times$ to $\mathbb{Z}_p^\times$ is surjective because $\mathbb{Q}_{p^2}/\mathbb{Q}_p$ is unramified, so κ exists. Since $h(f,f') = 1$ and $h(f',f) = \sigma(1) = 1$, we get $h(f \pm \frac{1}{2}f', f \pm \frac{1}{2}f') = \pm\frac{1}{2} \pm \frac{1}{2}$, which is 1 for the sign $+$ and -1 for the sign $-$, and $h(f + \frac{1}{2}f', f - \frac{1}{2}f') = -\frac{1}{2} + \frac{1}{2} = 0$. Hence $h(e_1, e_1) = 1$, $h(e_2, e_2) = -\kappa\sigma(\kappa) = 1$ and $h(e_1, e_2) = 0$. The matrix expressing e_1, e_2 in f, f' has determinant $-\kappa$, a unit. □

Theorem 8.21. *Let (M, h) be a positive definite hermitian $\mathcal{O}_K$-lattice, let $L \subset V_M$ be a lattice with $NL_M \subset L \subset N^{-1}L_M$ for some N prime to p, let $\alpha\colon L \to \{\pm 1\}$ be a semicharacter for $\operatorname{Im} H_M$, and let $G \in \Theta(L, \alpha)$ have algebraic Taylor coefficients. Suppose that $M_p = P \oplus P^\perp$ is an orthogonal decomposition, that $P^\perp$ has a basis $b'_1, \dots, b'_k$ with $h(b'_i, b'_j) = [i = j]$, and that P has a basis $f_1, \dots, f_m, f'_1, \dots, f'_m$ with*

$$h(f_e, f_{e'}) = 0, \qquad h(f'_e, f'_{e'}) = 0, \qquad h(f_e, f'_{e'}) = [e = e'].$$

In the variables $x_1, \dots, x_m, y_1, \dots, y_m$ put

$$\Theta(x, y) = G\Big(\sum_{e=1}^{m} f_e \otimes u\varepsilon^{-1}\lg(x_e) + \sum_{e=1}^{m} f'_e \otimes u\varepsilon^{-1}\lg(y_e)\Big),$$

with ε as in Proposition 8.11 and β as in (8.6)*. Then there are $c \in \mathbb{C}_p^\times$ and an integral series U in the variables x, y with*

$$\Theta(x,y) = c\, \exp\Big(-\sum_{e=1}^{m} \beta(x_e, y_e)\Big)\, U(x,y).$$

Proof. Lemma 8.20, applied to each pair (f_e, f'_e), gives a basis $b_1, \dots, b_{2m}$ of P with $h(b_i, b_j) = [i = j]$, and $b_1, \dots, b_{2m}, b'_1, \dots, b'_k$ is such a basis of M_p. Write $f_e = \sum_i \phi_{e,i} b_i$ and $f'_e = \sum_i \phi'_{e,i} b_i$ with ϕ, ϕ' over $\mathbb{Z}_{p^2}$. Then $h(f_e, f_{e'}) = \sum_i \phi_{e,i}\sigma(\phi_{e',i})$, and similarly for the other products, so $\sum_i \sigma(\phi_{e,i})\phi_{e',i} = \sigma\big(h(f_e, f_{e'})\big) = 0$, $\sum_i \sigma(\phi'_{e,i})\phi'_{e',i} = 0$ and $\sum_i \sigma(\phi_{e,i})\phi'_{e',i} = \sigma\big(h(f_e, f'_{e'})\big) = [e = e']$. So (ϕ, ϕ') is a hyperbolic pair of size m.

Put $X_e = \varkappa^{-1}(x_e)$ and $Y_e = \varkappa^{-1}(y_e)$, integral series without constant term, so that $\ell(X_e) = \varepsilon^{-1}\lg(x_e)$ and $\ell(Y_e) = \varepsilon^{-1}\lg(y_e)$ by Proposition 8.11. Let $T_1, \dots, T_{2m}$ be the series of Proposition 8.16 for (ϕ, ϕ'), and put $T_{2m+1} = \cdots = T_{2m+k} = 0$. Since $\ell(T_i) = \sum_e \big(\phi_{e,i}\ell(X_e) + \phi'_{e,i}\ell(Y_e)\big)$,

$$\sum_{i=1}^{2m} b_i \otimes u\, \ell(T_i) = \sum_e f_e \otimes u\, \ell(X_e) + \sum_e f'_e \otimes u\, \ell(Y_e),$$

which is the argument of G in $\Theta(x,y)$. Theorem 8.19 for the basis (b, b') gives c and an integral U_0 with $\exp\big(\sum_{i\le 2m} \mathrm{Y}(T_i)\big)\Theta(x,y) = c\, U_0(T)$, where $\mathrm{Y}(0) = 0$ for the last k variables. Proposition 8.16 and $\delta\mathrm{Y}(X_e, Y_e) = \beta(x_e, y_e)$ give $\sum_i \mathrm{Y}(T_i) \simeq \sum_e \beta(x_e, y_e)$. Hence

$$\Theta(x,y) = c\, \exp\Big(-\sum_e \beta(x_e, y_e)\Big) \cdot U_0(T) \cdot \exp\Big(\sum_e \beta(x_e, y_e) - \sum_i \mathrm{Y}(T_i)\Big),$$

and $U = U_0(T)\exp\big(\sum_e \beta(x_e, y_e) - \sum_i \mathrm{Y}(T_i)\big)$ is integral, because U_0 is integral, the T_i are integral series in x, y without constant term, and the last factor is integral. □

Corollary 8.22. *In the situation of Theorem 8.21, let the index e run over triples (r, a, b) with $r < \nu$, $a < r_0$ and $b < r_1$, and let ϖ, ω_r, ω_r^*, $\Lambda_{r,a,b}$ and $\Lambda'_{r,a,b}$ be as in Theorem 9.4. Put*

$$\Theta^\flat(X) = G\Big(\sum_{r,a,b} f_{r,a,b} \otimes u\varepsilon^{-1}\Lambda_{r,a,b}(X) + \sum_{r,a,b} f'_{r,a,b} \otimes u\varepsilon^{-1}\Lambda'_{r,a,b}(X)\Big).$$

Then there is a family v in W with $|v_s| \le 1$ such that $c^{-1}\,\Theta^\flat(\varpi X) \in \mathcal{C}(h_v)$.

Proof. Put $x_{r,a,b} = \lg^{-1}(\Lambda_{r,a,b})$ and $y_{r,a,b} = \lg^{-1}(\Lambda'_{r,a,b})$. Then $\Theta^\flat(X) = \Theta(x,y)$, and Theorem 8.21 gives $c^{-1}\Theta^\flat = \exp\big(-\sum \beta(x,y)\big)U(x,y)$ with U integral. By Theorem 8.12, β is a series over $W[1/p]$ with properties (i) to (iii) of Theorem 9.3, so Proposition 9.5 applies. □

Corollary 8.22 is the input of the valuation estimate of Section 9. It applies to every theta function of a hermitian lattice whose completion at p splits into an orthonormal part and a hyperbolic part, written in hyperbolic coordinates along the hyperbolic part. Section 9 works with the variables of one hyperbolic part at a time, and sets the variables of the orthonormal complement to zero, as in the proof of Theorem 8.21.

9 Divisibility at inert primes

Let p be a prime that is inert in $\mathbb{Q}(\sqrt{-d})$. An elliptic curve with complex multiplication by the integers of $\mathbb{Q}(\sqrt{-d})$ and good reduction at p has a formal group of height two, and complex conjugation acts on its endomorphisms as the Frobenius of $\mathbb{Q}_{p^2}$. This section proves that the theta kernel of one lift, written in the variables of one Frobenius orbit, is divisible by a power of p whose main term at a top of weight α is $\alpha p/(p^2-1) = \sum_{i \text{ odd}} \alpha/p^i$. Section 3.2 showed that this number is the contribution of an inert prime to the sum S. The argument has three algebraic steps, which do not refer to theta functions. Section 9.1 classifies the pairings on the formal group that are biadditive and semilinear for Frobenius, up to a factor whose exponential is integral. Section 9.2 defines a class of power series built from such a pairing. Section 9.3 bounds from below the valuation of every pairing of a series of that class with a product of minors.

Throughout this section $p \geq 5$ is a prime and $\mathcal{K}$ is a complete field of characteristic zero with an ultrametric absolute value such that $|p| = p^{-1}$. Write $\mathrm{ord} = -\log_p|\cdot|$. A power series over $\mathcal{K}$ is *integral* when all its coefficients have absolute value at most 1, and for two series f and g we write $f \equiv g \pmod{p}$ when $f - g$ is p times an integral series. Put

$$\gamma_p = \frac{p}{p^2-1} = \sum_{e\geq 0} \frac{1}{p^{2e+1}}, \qquad \rho_p = \frac{1}{p^2-1}, \qquad \gamma_p + \rho_p = \frac{1}{p-1}.$$

9.1 Pairings on the formal group of height two

Put

$$\lg(X) = \sum_{j\geq 0} \frac{X^{p^{2j}}}{p^j}, \qquad \lg^*(X) = \sum_{j\geq 0} \frac{X^{p^{2j+1}}}{p^{j+1}} = \frac{1}{p}\lg(X^p), \qquad \mathcal{A}(X,Y) = \sum_{0\leq i\leq j} \frac{X^{p^{2i}}\,Y^{p^{2j+1}}}{p^{i+j+1}},$$

and for constants c, c' put $Q_{c,c'}(X,Y) = c\,\mathcal{A}(X,Y) - c'\,\mathcal{A}(Y,X)$. The series lg satisfies $\lg(X) - p^{-1}\lg(X^{p^2}) = X$, and Hazewinkel's functional-equation lemma [31, Section 2.2] gives an integral series $F(X,Y)$ without constant term with $\lg(F(X,Y)) = \lg(X) + \lg(Y)$. The series F is the formal group law of height two with logarithm lg, and its coefficients lie in $\mathbb{Z}_p$. For two series f and g write $f \simeq g$ when $f - g$ has zero constant term and $\exp(f-g)$ is integral. For a series H in two variables put

$$(d_F H)(X_0, X_1, X_2) = H\big(F(X_0,X_1), X_2\big) - H(X_0,X_2) - H(X_1,X_2), \qquad H^{\mathrm{sw}}(X,Y) = H(Y,X),$$

and for a series f in one variable put $\partial_F f = f(F(X_0,X_1)) - f(X_0) - f(X_1)$.

Lemma 9.1. *Let σ be a ring endomorphism of $\mathcal{K}$ with $|\sigma a - a^p| \leq p^{-1}$ whenever $|a| \leq 1$. For a series Ψ in finitely many variables put $\Psi^\phi(X) = \Psi^\sigma(X^p)$, where σ acts on the coefficients and X^p raises every variable to the power p.*

(i) Let Ψ have zero constant term. Then $\exp\Psi$ is integral if and only if $\Psi^\phi \equiv p\Psi \pmod{p}$.

(ii) Let D satisfy $D \simeq 0$, and let b be a family of integral series without constant terms with $D(b) = 0$. Then $(\partial D/\partial X_e)(b)$ is integral for every variable X_e.

(iii) Let Φ be a series with $\partial\Phi/\partial X_e$ integral, let φ be a family of integral series without constant terms, and let V be an integral series without constant term. Then $\Phi(\varphi + pV\mathbf{e}_e) \equiv \Phi(\varphi) \pmod{p}$, where $\mathbf{e}_e$ is the unit vector of the variable X_e.

Proof. (i) For an integral series f one has $f^\phi \equiv f^p \pmod p$, because $(\sum_i a_i X^{n_i})^p \equiv \sum_i a_i^p X^{pn_i}$ and $a^p \equiv \sigma(a)$. Put $E = \exp\Psi$, so that $E^\phi = \exp(\Psi^\phi) = E^p \exp(\Psi^\phi - p\Psi)$.

Suppose that E is integral. Then $E^\phi \equiv E^p \pmod p$, and E^p is an integral series with constant term 1, so $D = \Psi^\phi - p\Psi$ satisfies $\exp D \equiv 1 \pmod p$. We show by induction on the degree that the coefficients of D lie in $p\mathcal{O}_\mathcal{K}$. If this holds in degrees below N, then for $k \geq 2$ the part of degree N of $D^k/k!$ is a sum of products of k coefficients in $p\mathcal{O}_\mathcal{K}$ divided by $k!$, and $\operatorname{ord}(p^k/k!) \geq k - k/(p-1) \geq 1$ for $k \geq 2$. So the part of degree N of $\exp D - 1$ is congruent to that of D modulo p, and the coefficients of D in degree N lie in $p\mathcal{O}_\mathcal{K}$.

Conversely, suppose that $\Psi^\phi - p\Psi = pR$ with R integral. Then $G = \exp(pR)$ is integral with $G \equiv 1 \pmod p$, because $\operatorname{ord}(p^k/k!) \geq 1$ for $k \geq 1$, and $E^\phi = E^p G$. We show by induction on the degree N that the homogeneous part E_N of E of degree N is integral. Let A be the sum of the parts of E of degree less than N, an integral polynomial with constant term 1 by the induction hypothesis. In degree N one has $E^p = A^p + pA^{p-1}E_N + (\text{terms of degree} > N)$, and $A^{p-1}G$ has constant term 1, so the part of degree N of E^pG is the part of degree N of A^pG plus pE_N. The part of degree N of E^ϕ involves the parts of E of degree $N/p < N$, so it equals the part of degree N of A^ϕ. Hence pE_N is the part of degree N of $A^\phi - A^pG$. Since $A^\phi \equiv A^p$ and $A^pG \equiv A^p \pmod p$, this part has coefficients in $p\mathcal{O}_\mathcal{K}$, and E_N is integral.

(ii) The series $\exp D$ is integral and $\partial_{X_e} \exp D = \exp D \cdot \partial_{X_e} D$. At the substitution b the first factor equals 1, so $(\partial_{X_e} D)(b) = (\partial_{X_e} \exp D)(b)$, which is integral.

(iii) It suffices to treat a monomial $a\, X_e^k M$ with M free of X_e and $|ka| \leq 1$. Its increment is $aM(\varphi) \sum_{n \geq 1} \binom{k}{n} \varphi_e^{k-n} (pV)^n$, and $\binom{k}{n} a = \frac{k}{n} \binom{k-1}{n-1} a$, so the term n is $ka \binom{k-1}{n-1} \frac{p^n}{n} \varphi_e^{k-n} V^n M(\varphi)$, with $|p^n/n| \leq p^{-1}$. □

Part (i) of Lemma 9.1 is Dwork's lemma [36, Chapter IV] in several variables.

Lemma 9.2. *Assume that $\mathcal{K}$ is complete.*

(i) Let f be a series in one variable without constant term such that $\partial_F f$ is integral. Then there are $\alpha, \beta_1 \in \mathcal{K}$ with $|\alpha| \leq 1$ and an integral series R with $f = \alpha \lg^* + \beta_1 \lg + R$.

(ii) Let H be a series in two variables in which every monomial contains both variables, such that $d_F H$ and $d_F H^{\mathrm{sw}}$ are integral. Then there are $\beta_2 \in \mathcal{K}$ and an integral series R with $H = \beta_2 \lg(X) \lg(Y) + R$.

Proof. (i) First, $\partial_F \lg = 0$, and $\partial_F \lg^*$ is integral. Indeed $\partial_F \lg^* = p^{-1}\big(\lg(F^p) - \lg(F(X_0^p, X_1^p))\big)$, the two arguments differ by p times an integral series because F has coefficients in $\mathbb{Z}_p$, and $\lg'(X) = \sum_j p^j X^{p^{2j}-1}$ is integral, so Lemma 9.1(iii) applies. Write $f = \sum_n a_n X^n$. The coefficient a_1 does not enter $\partial_F f$, and we first replace f by $f - a_1 \lg$, which does not change $\partial_F f$. Assume now that a_d is integral for $1 \leq d < n$, where $n \geq 2$. For $0 < i < n$ the coefficient of $X_0^i X_1^{n-i}$ in $\partial_F f$ equals $a_n \binom{n}{i}$ plus a sum of products of the a_d with $d < n$ and coefficients of powers of F, because $F \equiv X_0 + X_1$ in degree one. If n is not a power of p, some $\binom{n}{i}$ is prime to p, and a_n is integral. If $n = p^{k+1}$, then $\operatorname{ord} \binom{p^{k+1}}{p^k} = 1$, and pa_n is integral. Put $\mu = pa_n$, so that $|\mu| \leq 1$. When $k+1 = 2j+1$ we subtract $\mu p^j \lg^*$, whose coefficient in degree p^{2j+1} is $\mu p^{-1} = a_n$ and whose coefficients in the degrees p^{2i+1} with $i < j$ are μp^{j-i-1}. When $k+1 = 2j$ we subtract $\mu p^{j-1} \lg$, whose coefficient in degree p^{2j} is a_n and whose coefficients in the degrees p^{2i} with $i < j$ are μp^{j-1-i}. In both cases the coefficient in degree n becomes 0, the coefficients below degree n stay integral, and $\partial_F f$ stays integral because $\partial_F \lg = 0$ and $\partial_F \lg^*$ is integral. The corrections at the

stage $n = p^{k+1}$ have absolute value at most $p^{-\lfloor k/2 \rfloor}$, so their sums converge in the complete field $\mathcal{K}$ to numbers α and β_1, and $|\alpha| \leq 1$ because each correction of α has absolute value at most 1.

(ii) For every m the coefficient $s_m(X)$ of Y^m in H has an integral coboundary $\partial_F s_m$, because $d_F H$ is integral, so part (i) gives $s_m = \alpha_m \lg^* + \beta_m \lg + R_m$. In the same way the coefficient $s_i'(Y)$ of X^i equals $\alpha_i' \lg^* + \beta_i' \lg + R_i'$. The two expressions for the coefficient of $X^i Y^m$ differ by an integral number. With $i = p^{2k+1}$ and $m = p^{2N}$, the number $\alpha_m p^{-(k+1)} - \beta_i' p^{-N}$ is integral for every N, and $|\alpha_m| \leq 1$, so $\beta_i' = 0$. Then $i = p^{2N+1}$ and an arbitrary m show that $\alpha_m p^{-(N+1)}$ is bounded in N, so $\alpha_m = 0$. In the same way $\beta_m = 0$ unless $m = p^{2l}$, and $\beta_{p^{2l}} = \beta_1 p^{-l}$ for one number β_1. Hence $s_m - \beta_1 [Y^m] \lg(Y) \cdot \lg(X)$ is integral for every m. □

Theorem 9.3. *Assume that $\mathcal{K}$ is complete, that $|\mathcal{K}^\times| = p^{\mathbb{Z}}$, that $\mathcal{K}$ contains a primitive root of unity of order $p^2 - 1$, and that σ is a ring endomorphism of $\mathcal{K}$ with $|\sigma a - a^p| \leq p^{-1}$ whenever $|a| \leq 1$. Let β be a series in two variables over $\mathcal{K}$ with the following properties.*

(i) Every monomial of β contains both variables.

(ii) $d_F \beta \simeq 0$ and $d_F \beta^{\mathrm{sw}} \simeq 0$.

(iii) $\beta(\zeta X, Y) \simeq \beta(X, \zeta^p Y)$ for every $\zeta \in \mathcal{K}$ with $\zeta^{p^2-1} = 1$.

Then there is $c \in \mathcal{K}$ with $|c| \leq 1$ and $\sigma^2 c = c$ such that $\beta \simeq Q_{c,\sigma c}$.

Proof. Step 0, averaging over roots of unity. Let $N = p^2 - 1$, let ζ be a primitive root of unity of order N, and put $G = \frac{1}{N} \sum_{k=0}^{N-1} \beta(\zeta^k X, \zeta^{-pk} Y)$. The coefficient of $X^{n_0} Y^{n_1}$ in G equals that of β when N divides $n_0 - p n_1$, and vanishes otherwise. So every monomial $X^{n_0} Y^{n_1}$ of G satisfies $n_0, n_1 \geq 1$ and

$$n_0 \equiv p\, n_1 \pmod{p^2 - 1}. \tag{9.1}$$

By (iii), each summand is $\simeq \beta$. The condition of Lemma 9.1(i) is linear over the integers prime to p, and N is prime to p, so $G \simeq \beta$. Since $F(\zeta X_0, \zeta X_1) = \zeta F(X_0, X_1)$, because $\lg(\zeta X) = \zeta \lg(X)$, the operator d_F commutes with the scaling, and (ii) gives $d_F G \simeq 0$ and $d_F G^{\mathrm{sw}} \simeq 0$.

Step 1, the first derivative. Let $f(Y)$ be the coefficient of X in G. The series $d_F G$ vanishes at $X_1 = 0$, and Lemma 9.1(ii) shows that its derivative in X_1 at $X_1 = 0$ is integral. That derivative is $(\partial_X G)(X_0, X_2)\, \varphi(X_0) - f(X_2)$ with $\varphi(X_0) = (\partial_{X_1} F)(X_0, 0)$, and differentiation of $\lg(F) = \lg(X_0) + \lg(X_1)$ gives $\lg'(X_0)\varphi(X_0) = 1$. Since $\lg'$ is integral, $\partial_X G - \lg'(X) f(Y)$ is integral. Next, $d_F G^{\mathrm{sw}}$ vanishes at $X_2 = 0$, and its derivative in X_2 at $X_2 = 0$ is $\partial_F f$, which Lemma 9.1(ii) shows to be integral. Lemma 9.2(i) gives $f = \alpha \lg^* + \beta_1 \lg + R$ with $|\alpha| \leq 1$. By (9.1) with $n_0 = 1$, every exponent of f is $\equiv p \pmod{p^2 - 1}$. The exponents of lg are $\equiv 1$ and those of $\lg^*$ are $\equiv p$, and $1 \not\equiv p$, so $f - \alpha \lg^*$ is integral. With $c = \alpha$,

$$\partial_X G - c\, \lg'(X) \lg^*(Y) \quad \text{is integral,} \qquad |c| \leq 1. \tag{9.2}$$

The same argument for G^{sw}, which also satisfies (9.1) after exchanging the roles of the exponents, gives a constant c' with $|c'| \leq 1$ and the analogue of (9.2).

Step 2, the Frobenius defect. Put $\Delta G = p^{-1}\big(G^\sigma(X^p, Y^p) - pG\big)$. We show that $d_F(\Delta G) - \sigma(c)\, (\partial_F \lg^*)(X_0, X_1) \lg(X_2)$ is integral. Put $A = F(X_0, X_1)$ and $A_p = F(X_0^p, X_1^p) = A^p + pr$ with r integral. Lemma 9.1(i) for $d_F G \simeq 0$ gives $(d_F G)^\phi - p\, d_F G \equiv 0 \pmod{p}$. The series $(d_F G)^\phi$ and $d_F\big(G^\sigma(X^p, Y^p)\big)$ differ by $G^\sigma(A_p, X_2^p) - G^\sigma(A^p, X_2^p)$. Write $G^\sigma = \Phi + \sigma(c) \lg(X) \lg^*(Y)$. By (9.2) the series $\partial_X \Phi$ is integral, and Lemma 9.1(iii) shows that its increment is $\equiv 0 \pmod{p}$. The

increment of the second term is $\sigma(c)\big(\lg(A_p)-\lg(A^p)\big)\lg^*(X_2^p) = -\sigma(c)\,p\,(\partial_F\lg^*)\big(\lg(X_2)-X_2\big)$, because $\lg(A^p)=p\lg^*(A)$, $\lg(A_p)=p\lg^*(X_0)+p\lg^*(X_1)$, and $\lg^*(X^p)=\lg(X)-X$. Division by p gives the claim.

Step 3, the defect itself. Put $H = \Delta G - \sigma(c)\lg^*(X)\lg(Y) - \sigma(c')\lg(X)\lg^*(Y)$. Since $d_F\big(\lg^*(X)\lg(Y)\big) = (\partial_F\lg^*)\lg(X_2)$ and $d_F\big(\lg(X)\lg^*(Y)\big)=0$, Step 2 for G and for G^{sw} shows that $d_F H$ and $d_F H^{\mathrm{sw}}$ are integral. Lemma 9.2(ii) gives $H=\beta_2\lg(X)\lg(Y)+R$ with R integral. The series H satisfies (9.1), and no monomial of $\lg(X)\lg(Y)$ does, so H is integral.

Step 4, the two constants. The coefficient of XY^m with $m=p^{2i+1}$ is $G_{1,m}-c\,p^{-(i+1)}$ in the integral series (9.2), and $-G_{1,m}-\sigma(c')p^{-(i+1)}$ in H, because $G^\sigma(X^p,Y^p)$ has no monomial of degree one in X and $\lg^*$ has no term of degree one. The sum $(c+\sigma c')p^{-(i+1)}$ is integral for every i, so $c+\sigma c'=0$. The same argument for G^{sw} gives $c'+\sigma c=0$. Hence $c'=-\sigma c$ and $\sigma^2 c=c$.

Step 5, Dwork's lemma for the difference. The reindexing $\mathcal{A}(X^p,Y^p) = p\big(\lg^*(X)\lg(Y) - \mathcal{A}(Y,X)\big)$ of the sum over $i\le j$ gives

$$Q_{c',c}(X^p,Y^p) - p\,Q_{c,c'}(X,Y) = p\big(c'\lg^*(X)\lg(Y) - c\lg(X)\lg^*(Y)\big).$$

Put $\Psi = G - Q_{c,\sigma c}$. Since $\sigma^2 c=c$, we have $(Q_{c,\sigma c})^\sigma = Q_{\sigma c,c}$, and the identity with $c'=\sigma c$ gives

$$\Psi^\phi - p\Psi = p\big(\Delta G - \sigma(c)\lg^*(X)\lg(Y) + c\lg(X)\lg^*(Y)\big) = p\,H.$$

Since H is integral, Lemma 9.1(i) shows that $\exp\Psi$ is integral. Hence $G\simeq Q_{c,\sigma c}$, and Step 0 gives $\beta\simeq Q_{c,\sigma c}$. □

9.2 The class of a kernel

Fix integers $\nu\ge 1$ and $r_0,r_1\ge 0$. A *position* is an element s of $\mathbb{Z}/2\nu$, called even or odd according to its representative in $\{0,\dots,2\nu-1\}$. The variables are $X_{s,a,b}$ with s a position, $0\le a<r_0$ and $0\le b<r_1$. Put

$$\mathsf{S}_{s,a,b} = \sum_{l\ge 0}(-1)^l\, X_{s-2l,a,b}^{p^{2l}},$$

and for a family $v=(v_s)$ of elements of $\mathcal{K}$ with $|v_s|\le 1$ put

$$h_v = \sum_s v_s \sum_{a,b} \mathsf{S}_{s,a,b}^{\,p}\,\mathsf{S}_{s+1,a,b}. \tag{9.3}$$

For a series h without constant term, the class $\mathcal{C}(h)$ is the set of series Θ for which there are integral series $g_0,g_1,\dots$ with

$$\Theta = \sum_{i\ge 0}\frac{h^i}{p^i}\,g_i, \tag{9.4}$$

the sum being finite in every degree. The class is stable under multiplication by integral series and under substitution of integral series without constant term for the variables.

Theorem 9.4. *Let $W=W(\overline{\mathbb{F}}_p)$ with Frobenius σ, let $\mathcal{K}\supset W[1/p]$ contain an element ϖ with $\varpi^{p^2-1}=p$, and let $\mathcal{O}=W[\varpi]$. Let $\omega_r,\omega_r^*\in W$ for $r<\nu$ be fixed by $\sigma^{2\nu}$ with $\sum_{i<\nu}\sigma^{2i}(\omega_r\omega_{r'}^*)=[r=r']$. Let $\lambda(T)=\varpi^{-1}\lg(\varpi T)$ and let μ be the series without constant term with $\lambda(\mu(T))=T$. Put $\Lambda_{r,a,b}=\sum_{s\text{ even}}\sigma^s(\omega_r^*)X_{s,a,b}$ and $\Lambda'_{r,a,b}=\sum_{s\text{ odd}}\sigma^s(\omega_r)X_{s,a,b}$. Let $c\in W$ with $\sigma^2 c=c$. Then there is a series V with coefficients in $\mathcal{O}$ such that*

$$\sum_{r,a,b}\Big(c\,\mu(\Lambda_{r,a,b})\,\mu(\Lambda'_{r,a,b})^p - \sigma(c)\,\mu(\Lambda'_{r,a,b})\,\mu(\Lambda_{r,a,b})^p\Big) = h_v + p\,V,$$

with $v_s = c$ for odd s and $v_s = -\sigma(c)$ for even s.

Proof. Congruences are modulo $p\mathcal{O}$. The coefficient of $T^{p^{2j}}$ in λ is $\varpi^{p^{2j}-1}/p^j = p^{(p^{2j}-1)/(p^2-1)-j}$, an integer power of p that is divisible by p for $j \geq 2$. So λ has integer coefficients and $\lambda \equiv T + T^{p^2}$. The series $\mathrm{alt}(T) = \sum_{l\geq 0}(-1)^l T^{p^{2l}}$ satisfies $\mathrm{alt}(T) + \mathrm{alt}(T)^{p^2} \equiv T$, so $\mu \equiv \mathrm{alt}$. For series over $\mathcal{O}$ one has $(\sum_i x_i)^p \equiv \sum_i x_i^p$, and $\sigma(a) \equiv a^p$ for $a \in W$. Hence, for $w \in W$ fixed by $\sigma^{2\nu}$ and a set P of positions stable under shifts by 2,

$$\Big(\sum_{s\in P} \sigma^s(w) X_s\Big)^{p^k} \equiv \sum_{s\in P} \sigma^{s+k}(w)\, X_s^{p^k},$$

and with $\mu \equiv \mathrm{alt}$ and the reindexing $s \mapsto s - 2l$, which uses the period 2ν,

$$\mu\Big(\sum_{s\in P} \sigma^s(w) X_s\Big) \equiv \sum_{s\in P} \sigma^s(w)\, \mathsf{S}_s, \qquad \mu\Big(\sum_{s\in P} \sigma^s(w) X_s\Big)^p \equiv \sum_{s\in P} \sigma^{s+1}(w)\, \mathsf{S}_s^p.$$

The hypothesis on ω, ω^* says that two square matrices are inverse to each other, so the product in the other order is also the identity, and $\sum_r \sigma^a(\omega_r^*)\sigma^b(\omega_r) = [a = b]$ for positions $a \equiv b \pmod 2$. Inserting the two congruences, the first product becomes $c\sum_r \big(\sum_{t \text{ even}} \sigma^t(\omega_r^*)\mathsf{S}_t\big)\big(\sum_{t' \text{ odd}} \sigma^{t'+1}(\omega_r)\mathsf{S}_{t'}^p\big)$, and the sum over r keeps the terms with $t = t' + 1$, which gives $c\sum_{t' \text{ odd}} \mathsf{S}_{t'}^p \mathsf{S}_{t'+1}$. The second product gives $\sigma(c)\sum_{t' \text{ even}} \mathsf{S}_{t'}^p \mathsf{S}_{t'+1}$ in the same way, after summing over a, b. □

The next proposition combines Theorems 9.3 and 9.4. It shows that the exponential of a sum of values of a semilinear pairing, at dual coordinates of even and odd positions, rescaled by ϖ, lies in the class of the series h_v.

Proposition 9.5. *Let W, $\mathcal{K}$, ϖ, $\mathcal{O}$, ω_r, ω_r^*, $\Lambda_{r,a,b}$ and $\Lambda'_{r,a,b}$ be as in Theorem 9.4. Let β be a series in two variables over $W[1/p]$ with properties (i) to (iii) of Theorem 9.3, and let U be an integral series in the variables $x_{r,a,b}, y_{r,a,b}$. Put $x_{r,a,b} = \mathrm{lg}^{-1}(\Lambda_{r,a,b})$ and $y_{r,a,b} = \mathrm{lg}^{-1}(\Lambda'_{r,a,b})$, the parameters whose logarithms are the linear forms Λ and Λ', and*

$$\vartheta(X) = \exp\Big(-\sum_{r,a,b} \beta\big(x_{r,a,b}, y_{r,a,b}\big)\Big)\, U(x,y).$$

Then there is a family v in W with $|v_s| \leq 1$ such that $\vartheta(\varpi X) \in \mathcal{C}(h_v)$.

Proof. The field $W[1/p]$ is complete, its value group is $p^{\mathbb{Z}}$, it contains the roots of unity of order $p^2 - 1$, and its Frobenius σ satisfies $|\sigma a - a^p| \leq p^{-1}$ for $|a| \leq 1$. Theorem 9.3 gives $c \in W$ with $\sigma^2 c = c$ and $\beta \simeq Q_{c,\sigma c}$. So $\exp\big(-\sum \beta(x,y)\big) = \exp\big(-\sum Q_{c,\sigma c}(x,y)\big) \cdot V_1(x,y)$ with V_1 an integral series in the variables x, y.

We substitute $X \mapsto \varpi X$. The parameter whose logarithm is $\varpi\Lambda$ is $\varpi\mu(\Lambda)$, because $\mathrm{lg}(\varpi\mu(T)) = \varpi\lambda(\mu(T)) = \varpi T$. In $Q_{c,\sigma c}(\varpi\mu(\Lambda), \varpi\mu(\Lambda'))$ the term (i,j) of $\mathcal{A}$ has the coefficient $\varpi^{p^{2i}+p^{2j+1}}/p^{\,i+j+1}$, of valuation

$$\frac{p^{2i}+p^{2j+1}}{p^2-1} - (i+j+1) \begin{cases} = \dfrac{1}{p-1} - 1 & i = j = 0, \\ \geq \dfrac{1}{p-1} + p - 2 & j \geq 1, \end{cases}$$

and the same holds for $\mathcal{A}(Y,X)$. The exponential of a series whose coefficients have valuation at least $1/(p-1)$ is integral, because $\operatorname{ord}(k!) \le k/(p-1)$. So the terms with $j \ge 1$ give an integral factor, and

$$\exp\Big(-\sum Q_{c,\sigma c}(\varpi\mu(\Lambda),\varpi\mu(\Lambda'))\Big) = \exp\Big(-\frac{\varpi^{p+1}}{p}\sum_{r,a,b}\big(c\,\mu(\Lambda)\mu(\Lambda')^p - \sigma(c)\,\mu(\Lambda')\mu(\Lambda)^p\big)\Big)\cdot V_2$$

with V_2 integral. By Theorem 9.4 the inner sum equals $h_v + pV$ with V over $\mathcal{O}$, and $\exp(-\varpi^{p+1}V)$ is integral because $\operatorname{ord}(\varpi^{(p+1)k}/k!) \ge 0$. Finally

$$\exp\Big(-\frac{\varpi^{p+1}}{p}h_v\Big) = \sum_{i\ge 0}\frac{h_v^i}{p^i}\cdot\frac{(-1)^i\varpi^{(p+1)i}}{i!}, \qquad \operatorname{ord}\Big(\frac{\varpi^{(p+1)i}}{i!}\Big) = \frac{i}{p-1} - \operatorname{ord}(i!) \ge 0,$$

so this exponential lies in $\mathcal{C}(h_v)$. Since $x(\varpi X) = \varpi\mu(\Lambda)$ and $y(\varpi X) = \varpi\mu(\Lambda')$ are integral series without constant term, the series $V_1(x(\varpi X), y(\varpi X))$ and $U(x(\varpi X), y(\varpi X))$ are integral, and the class is stable under multiplication by them and by V_2 and $\exp(-\varpi^{p+1}V)$. □

9.3 The valuation estimate

For a polynomial $P = \sum_\kappa P_\kappa X^\kappa$ and a series $\Theta = \sum_\kappa \Theta_\kappa X^\kappa$ put

$$\langle P, \Theta\rangle = \sum_\kappa P_\kappa\,\kappa!\,\Theta_\kappa = \big(P(\partial)\Theta\big)(0), \qquad \kappa! = \prod_x \kappa_x!.$$

A *shape* consists of a set $\mathcal{T}$ of positions, called tops, and for every position s a number $n_s \ge 0$ of columns, a height $h_{s,c} \le r_0$ for each column $c < n_s$, and a map $I_{s,c}\colon \{0,\dots,h_{s,c}-1\} \to \{0,\dots,r_1-1\}$. For frame variables $y_{s,i,a}$ with $0 \le i, a < r_0$ the operator of the shape is

$$P_I = \prod_s \prod_{c<n_s} \det\Big(\sum_a y_{s,i,a}\,X_{s,a,I_{s,c}(i')}\Big)_{0\le i,i'<h_{s,c}},$$

a polynomial in the $X_{s,a,b}$ whose coefficients are polynomials in the $y_{s,i,a}$. For a position s let α_s be the number of columns of height one when $s \in \mathcal{T}$, and $\alpha_s = 0$ when $s \notin \mathcal{T}$, let $\operatorname{cap}_s = n_s - \alpha_s$ be the number of the remaining columns, and let $\lambda_s = \sum_c h_{s,c} - \alpha_s$ be the number of boxes in the remaining columns. For $s \in \mathcal{T}$ let $\mathcal{E}(s)$ be the set of the residues in $\{0,\dots,2\nu-1\}$ of the numbers $s'-s$ with $s' \in \mathcal{T}$ of parity different from that of s. Its elements are odd. Let $\lg_p(0) = 0$, and let $\lg_p(x) = \lfloor \log_p x\rfloor + 1$ be the number of digits of an integer $x \ge 1$ in base p. Put

$$\Delta = \sum_s \big(r_1 \lg_p(\alpha_s) + r_0 r_1 \lg_p(\lambda_s)\big) + \sum_{s\in\mathcal{T}} \lg_p(\alpha_s)^2, \tag{9.5}$$

$$g^A = \sum_{s\in\mathcal{T}} \sum_{\substack{e\ge 0\\ 2e+1\notin\mathcal{E}(s)}} \max\Big(\frac{\alpha_s}{p^{2e+1}} - \operatorname{cap}_{s+2e+1},\,0\Big) - \Delta, \tag{9.6}$$

$$g^B = \gamma_p \sum_{s\in\mathcal{T}_{\mathrm{ev}}} \alpha_s + \sum_{s\in\mathcal{T}_{\mathrm{odd}}}\sum_{e\ge 0} \max\Big(\frac{\alpha_s}{p^{2e+1}} - \operatorname{cap}_{s+2e+1},\,0\Big) - \Delta, \tag{9.7}$$

where $\mathcal{T}_{\mathrm{ev}}$ and $\mathcal{T}_{\mathrm{odd}}$ are the even and the odd tops.

Theorem 9.6. *Let $\varpi \in \mathcal{K}$ satisfy $\varpi^{p^2-1} = p$, let $v_s \in \mathcal{K}$ with $|v_s| \leq 1$, and let Θ be a series in the variables $X_{s,a,b}$ over $\mathcal{K}$ with $\Theta(\varpi X) \in \mathcal{C}(h_v)$. Let a shape be given, with operator P_I and numbers α_s, cap_s, λ_s, Δ, g^A and g^B as above.*

(i) Every coefficient c of the polynomial $\langle P_I, \Theta\rangle$ in the frame variables satisfies $\mathrm{ord}(c) \geq g^A$.

(ii) Every such coefficient satisfies $\mathrm{ord}(c) \geq g^B - \sum_{s\in\mathcal{T}_{\mathrm{ev}}} \alpha_s$.

(iii) Let $J \subset \{0, \dots, r_1 - 1\}$, and let N_J be the number of the columns of height one at even tops with $I_{s,c}(0) \notin J$ and at odd tops with $I_{s,c}(0) \in J$, plus the sum, over the columns c at odd positions s that are not columns of height one at a top, of the number of elements of $I_{s,c}(\{0, \dots, h_{s,c} - 1\}) \cap J$. Then every coefficient c satisfies $\mathrm{ord}(p^{N_J} c) \geq g^B$.

The bound (i) has the following reading. The pairing of Θ with the α_s linear factors at a top s produces a factorial of order about α_s, whose valuation is about $\alpha_s/(p-1)$, and the rescaling $X \mapsto \varpi X$ costs $\rho_p \alpha_s$. The difference is the credit $\gamma_p \alpha_s = \sum_{e\geq 0} \alpha_s / p^{2e+1}$ of the top. Each factor p^{-1} in (9.4) comes with a factor h_v, whose relevant monomials are p^{2e+1}-th powers of a top variable at s times one variable at the position $s + 2e + 1$. A top s can pay for at most α_s / p^{2e+1} of these, and the minors at $s + 2e + 1$ absorb at most cap_{s+2e+1} of them. The proof bounds every term of a finite expansion separately, and it does not use cancellation.

The proof uses three facts about the pairing. First, if A is a matrix and $\Theta'(X) = \Theta(A^{\mathsf{T}} X)$, then $\langle P, \Theta'\rangle = \langle P', \Theta\rangle$ with $P'(X') = P(AX')$, by the chain rule. Second, let $L_\kappa = \sum_x b_\kappa(x) X_x$ be linear forms and $\beta = (\beta_\kappa)$ a multi-index, and for a polynomial Q let $H_\beta Q$ be the coefficient of τ^β in $Q\big(X + \sum_\kappa \tau_\kappa b_\kappa\big)$. Then $\langle Q, \prod_\kappa L_\kappa^{\beta_\kappa} \cdot G\rangle = \beta!\, \langle H_\beta Q, G\rangle$, because $Q(\partial) \circ X_x = X_x \circ Q(\partial) + (\partial_x Q)(\partial)$, so that $\langle Q, L_\kappa G\rangle = \langle D_{b_\kappa} Q, G\rangle$ with D_b the derivative in the direction b, and $\prod_\kappa D_{b_\kappa}^{\beta_\kappa} = \beta!\, H_\beta$. Third, if P is homogeneous of degree d, then $\langle P, \Theta(\varpi X)\rangle = \varpi^d \langle P, \Theta\rangle$.

Proof. Step 1, reduction to a fixed frame. The operator P_I is homogeneous of degree $D = \sum_s (\alpha_s + \lambda_s)$ in X, so $\langle P_I, \Theta(\varpi X)\rangle = \varpi^D \langle P_I, \Theta\rangle$ and $|\varpi^D| = p^{-\rho_p D}$. Put $\Theta_1 = \Theta(\varpi X) \in \mathcal{C}(h_v)$. It suffices to show that every coefficient of $\langle P_I, \Theta_1\rangle$ has valuation at least $g + \rho_p D$, for $g = g^A$ in case (i) and the corresponding numbers in cases (ii) and (iii). The Gauss norm $|\sum_n c_n y^n| = \max_n |c_n|$ on $\mathcal{K}[y]$ is multiplicative, so it extends to an absolute value on $\mathcal{K}' = \mathcal{K}(y)$, with $\mathcal{K} \to \mathcal{K}'$ isometric, and $|f(y)|$ equals the largest absolute value of a coefficient of f. Membership in the class is preserved under the isometric embedding. So it suffices to bound $|\langle P_I(y), \Theta_1\rangle|$ over $\mathcal{K}'$, where the frame variables take the values y. With these values, a column of height one at a top s gives the linear form $\ell_{s,b} = \sum_a y_{s,0,a} X_{s,a,b}$ with $b = I_{s,c}(0)$, and the vector $\xi_s = (y_{s,0,a})_a$ is the same for all such columns at s. Let $e(s,b)$ be the number of these columns with $I_{s,c}(0) = b$, and let $M_1, \dots, M_C$ be the minors of all the other columns. Then $P_I(y) = E$ with

$$E = \prod_{s\in\mathcal{T},\, b} \ell_{s,b}^{e(s,b)} \cdot Q, \qquad Q = \prod_c M_c, \qquad \sum_b e(s,b) = \alpha_s.$$

The minors M_c have entries $\sum_a (\tau_c)_{i,a} X_{s_{c'},a,I_c(i')}$ with $|(\tau_c)_{i,a}| \leq 1$, there are cap_s of them at a position s, and their sizes at s add up to λ_s.

Step 2, a change of variables. Introduce variables $t_{s,b}$ for $s \in \mathcal{T}$ and $0 \leq b < r_1$, and put $\widetilde{\Theta}(t, X) = \Theta_1\big(X_{s,a,b} + [s \in \mathcal{T}]\, \xi_{s,a}\, t_{s,b}\big)$. Since $\partial\widetilde{\Theta}/\partial t_{s,b} = \big(\ell_{s,b}(\partial_X)\Theta_1\big)(X + \xi t)$,

$$\langle E, \Theta_1\rangle = \big\langle t^e\, Q(X),\ \widetilde{\Theta}\big\rangle = \sum_n Q_n\, e!\, n!\, \widetilde{\Theta}_{(e,n)}, \tag{9.8}$$

where n runs over the monomials of Q and $\widetilde{\Theta}_{(e,n)}$ is the coefficient of $t^e X^n$. The class is stable under the substitution, so $\widetilde{\Theta} = \sum_i p^{-i}\widetilde{h}^i U_i$ with $\widetilde{h} = h_v(X + \xi t)$ and U_i integral. Every monomial $t^e X^n$ in (9.8) has degree D.

Step 3, the series $\widetilde{h}$ modulo p. We claim that there are integral series Z and V such that every monomial of Z has degree at least p in the variables X, and such that in all degrees up to D

$$\widetilde{h} = \sum_{\varphi} a_\varphi \, t^{d_\varphi} + \sum_{\kappa} a_\kappa \, t^{d_\kappa} \, L_\kappa + Z + pV. \tag{9.9}$$

The index $\varphi = (s, l, l', a, b)$ runs over positions s, integers $l, l' \geq 0$ and pairs (a, b). With $\sigma_1 = s - 2l$ and $\sigma_2 = s + 1 - 2l'$, one has $t^{d_\varphi} = t_{\sigma_1,b}^{p^{2l+1}} t_{\sigma_2,b}^{p^{2l'}}$ and $a_\varphi = v_s(-1)^{l+l'} \xi_{\sigma_1,a}^{p^{2l+1}} \xi_{\sigma_2,a}^{p^{2l'}}$ when $\sigma_1, \sigma_2 \in \mathcal{T}$, and $a_\varphi = 0$ otherwise. The index $\kappa = (\sigma, l, b)$ runs over positions σ, integers $l \geq 0$ and $0 \leq b < r_1$, and

$$t^{d_\kappa} = t_{\sigma,b}^{p^{2l+1}}, \qquad L_\kappa = \sum_a \xi_{\sigma,a}^{p^{2l+1}} \, X_{\sigma+2l+1,a,b}, \qquad a_\kappa = (-1)^l v_{\sigma+2l} \text{ if } \sigma \in \mathcal{T}, \quad a_\kappa = 0 \text{ otherwise.} \tag{9.10}$$

For integral f, g one has $(f+g)^p \equiv f^p + g^p \pmod{p}$. Write $\theta_{s,a,b} = [s \in \mathcal{T}]\, \xi_{s,a} t_{s,b}$. Then

$$\mathsf{S}_{s+1,a,b}(X + \xi t) \equiv \sum_{l'} (-1)^{l'} X_{s+1-2l',a,b}^{p^{2l'}} + \sum_{l'} (-1)^{l'} \theta_{s+1-2l',a,b}^{p^{2l'}},$$

and since p is odd,

$$\mathsf{S}_{s,a,b}(X + \xi t)^p \equiv \sum_{l} (-1)^{l} X_{s-2l,a,b}^{p^{2l+1}} + \sum_{l} (-1)^{l} \theta_{s-2l,a,b}^{p^{2l+1}}.$$

In the product, the product of the two θ-sums gives the first sum of (9.9). The part of the product of the second θ-sum with the first X-sum with $l' = 0$ is $\sum_l (-1)^l \theta_{s-2l,a,b}^{p^{2l+1}} X_{s+1,a,b}$, and after summing over a and renaming $\sigma = s - 2l$ it gives the second sum. Every remaining term contains a factor $X^{p^{2l+1}}$ or a factor $X^{p^{2l'}}$ with $l' \geq 1$, so its degree in X is at least p, and these terms form Z. Terms with $p^{2l} > D$ or $p^{2l'} > D$ do not contribute in degrees up to D, so all sums are finite.

Step 4, expansion and the bound for one term. In the coefficient $\widetilde{\Theta}_{(e,n)}$ we may replace $\widetilde{h}$ by the right side of (9.9). The multinomial theorem expresses $p^{-i}\widetilde{h}^i U_i$ as a sum over families $k = \big((k_\varphi), (k_\kappa), k_Z, k_V\big)$ of nonnegative integers with sum i of the terms

$$\frac{p^{k_V}}{p^i} \binom{i}{k} \prod_{\varphi} a_\varphi^{k_\varphi} \prod_{\kappa} a_\kappa^{k_\kappa} \cdot t^{d(k)} \prod_{\kappa} L_\kappa^{k_\kappa} \cdot Z^{k_Z} V^{k_V} U_i, \qquad d(k) = \sum_{\varphi} k_\varphi d_\varphi + \sum_{\kappa} k_\kappa d_\kappa.$$

We pair this with $t^e Q(X)$ and apply the second fact to the product of the linear forms L_κ. Let $H_k Q$ be the coefficient of $\prod_\kappa \tau_\kappa^{k_\kappa}$ in $Q\big(X + \sum_\kappa \tau_\kappa L_\kappa\big)$, where L_κ is read as a vector. The term becomes

$$\frac{p^{k_V}}{p^i} \binom{i}{k} \prod_{\kappa} k_\kappa! \cdot \prod_{\varphi} a_\varphi^{k_\varphi} \prod_{\kappa} a_\kappa^{k_\kappa} \cdot \sum_n (H_k Q)_n \, e! \, n! \, \big(Z^{k_Z} V^{k_V} U_i\big)_{(e-d(k),n)}.$$

Put $a = \sum_\varphi k_\varphi$ and $b = \sum_\kappa k_\kappa$. A nonzero summand satisfies four conditions.

(N1) $d(k) \leq e$ in every coordinate (s, b'), because the operator has degree $e(s, b')$ in $t_{s,b'}$ and the exponents of $Z^{k_Z} V^{k_V} U_i$ are nonnegative.

(N2) $H_k Q \neq 0$ and n is one of its monomials. Then $|n| = \sum_s \lambda_s - b$, and the degree of n in the variables at a position s is at most λ_s.

(N3) $p\, k_Z \leq |n|$, because every monomial of Z has degree at least p in X.

(N4) $a_\varphi \neq 0$ when $k_\varphi \neq 0$, and $a_\kappa \neq 0$ when $k_\kappa \neq 0$. So the positions σ_1, σ_2 of such a φ and the position σ of such a κ are tops.

The numbers a_φ and a_κ, the coefficients of $H_k Q$, and the coefficients of $Z^{k_Z} V^{k_V} U_i$ have absolute value at most 1. The integer $b!$ divides $\binom{i}{k} \prod_\kappa k_\kappa!$, and $|p^{k_V - i}| = p^{a+b+k_Z}$. So every summand has absolute value at most $p^{a+b+k_Z} |e!|\, |b!|\, |n!|$. By the ultrametric inequality, either $\langle E, \Theta_1 \rangle = 0$, or there are k and n satisfying (N1) to (N4) with

$$\operatorname{ord} \langle E, \Theta_1 \rangle \geq \operatorname{ord}(e!) + \operatorname{ord}(b!) + \operatorname{ord}(n!) - a - b - k_Z. \tag{9.11}$$

Step 5, the factorials. Legendre's formula gives $(p-1) \operatorname{ord}(m!) = m - s_p(m)$, with $s_p(m)$ the sum of the digits of m in base p, so $\operatorname{ord}(m!) \geq m/(p-1) - \lg_p(m)$. The numbers $e(s, b')$ with s fixed are r_1 numbers with sum α_s, so $\operatorname{ord}(e!) \geq \sum_s \big(\alpha_s/(p-1) - r_1 \lg_p(\alpha_s) \big)$. By (N2) the exponents of n at a position s are $r_0 r_1$ numbers with sum at most λ_s, and their total is $\sum_s \lambda_s - b$, so $\operatorname{ord}(n!) \geq (\sum_s \lambda_s - b)/(p-1) - \sum_s r_0 r_1 \lg_p(\lambda_s)$. For a top σ and $l \geq 0$ put $i(\sigma, l) = \sum_{b'} k_{(\sigma, l, b')}$. Then $\prod_{\sigma,l} i(\sigma, l)!$ divides $b!$. By (N1), summed over b', one has $p^{2l+1} i(\sigma, l) \leq \alpha_\sigma$, so $i(\sigma, l) \neq 0$ forces $l < \lg_p(\alpha_\sigma)$, and $\lg_p(i(\sigma, l)) \leq \lg_p(\alpha_\sigma)$. Hence $\operatorname{ord}(b!) \geq b/(p-1) - \sum_{\sigma \in \mathcal{T}} \lg_p(\alpha_\sigma)^2$. By (N3), $k_Z \leq (\sum_s \lambda_s - b)/p \leq \gamma_p \sum_s \lambda_s$. With $1/(p-1) = \gamma_p + \rho_p$, the inequality (9.11) gives

$$\operatorname{ord} \langle E, \Theta_1 \rangle \geq \gamma_p \sum_{s \in \mathcal{T}} \alpha_s - a - b - \Delta + \rho_p D. \tag{9.12}$$

Step 6, the charges. For $s \in \mathcal{T}$ and $\varepsilon \geq 0$ put $i(s, \varepsilon)$ as in Step 5. Let $\varphi = (s_0, l, l', a', b')$ with $k_\varphi \neq 0$. Its positions $\sigma_1 = s_0 - 2l$ and $\sigma_2 = s_0 + 1 - 2l'$ are tops by (N4), and $\sigma_2 - \sigma_1 = 1 + 2(l - l')$ is odd. If $l' \leq l$, we charge φ to σ_1, and let d be the residue of $\sigma_2 - \sigma_1$ in $\{1, \ldots, 2\nu - 1\}$. Since $\sigma_1 + (2l+1) = \sigma_2 + 2l'$ with $2l' < 2l + 1$, we have $d \leq 2l + 1$, and the exponent of $t_{\sigma_1, b'}$ in t^{d_φ} is $p^{2l+1} \geq p^d$. If $l' > l$, we charge φ to σ_2, and let d be the residue of $\sigma_1 - \sigma_2$. Now $d \leq 2l'$, and the exponent of $t_{\sigma_2, b'}$ is $p^{2l'} \geq p^d$. In both cases d is odd because 2ν is even. Write $d = 2\varepsilon + 1$, and let $\operatorname{ch}(s, \varepsilon)$ be the sum of the k_φ charged to the top s with this ε. These numbers satisfy the following.

(C1) $\sum_{s,\varepsilon} \operatorname{ch}(s, \varepsilon) = a$ and $\sum_{s,\varepsilon} i(s, \varepsilon) = b$, by construction and by (N4).

(C2) $p^{2\varepsilon+1} \big(\operatorname{ch}(s, \varepsilon) + i(s, \varepsilon) \big) \leq \alpha_s$. Every φ charged to (s, ε) has exponent at least $p^{2\varepsilon+1}$ at some $t_{s,b'}$, every $\kappa = (s, \varepsilon, b')$ has exponent $p^{2\varepsilon+1}$ at $t_{s,b'}$, and the sum of (N1) over b' gives the claim, because $\sum_{b'} e(s, b') = \alpha_s$.

(C3) $\operatorname{ch}(s, \varepsilon) = 0$ when $2\varepsilon + 1 \notin \mathcal{E}(s)$, because a φ charged to (s, ε) has a second top s' of the other parity with $s' - s \equiv 2\varepsilon + 1 \pmod{2\nu}$.

(C4) $i(s, \varepsilon) \leq \operatorname{cap}_{s+2\varepsilon+1}$. By (9.10) the directions L_κ with $\kappa = (s, \varepsilon, b')$ and varying b' replace $X_{s',a,b'}$ by $X_{s',a,b'} + \tau_{b'} \eta_a$, with the single position $s' = s + 2\varepsilon + 1$ and the single vector

$\eta = (\xi_{s,a}^{p^{2\varepsilon+1}})_a$ for all b'. A minor at a position different from s' does not change. In a minor at s' the column i' changes by $\tau_{I(i')}$ times the fixed vector $\tau_c\eta$. A determinant is linear in each column and vanishes when two columns are proportional, so the minor has total degree at most 1 in the variables τ. The product Q therefore has total degree at most $\mathrm{cap}_{s'}$ in them, and $H_kQ = 0$ when $i(s,\varepsilon) > \mathrm{cap}_{s'}$. By (N2) this does not happen.

Step 7, the count for (i). Fix $s \in \mathcal{T}$. Since $\gamma_p\alpha_s = \sum_{\varepsilon\geq 0}\alpha_s p^{-(2\varepsilon+1)}$,

$$\gamma_p\alpha_s - \sum_{\varepsilon}\mathrm{ch}(s,\varepsilon) - \sum_{\varepsilon} i(s,\varepsilon) = \sum_{\varepsilon\geq 0}\Big(\frac{\alpha_s}{p^{2\varepsilon+1}} - \mathrm{ch}(s,\varepsilon) - i(s,\varepsilon)\Big).$$

If $2\varepsilon+1 \in \mathcal{E}(s)$, the summand is nonnegative by (C2). If $2\varepsilon+1 \notin \mathcal{E}(s)$, then $\mathrm{ch}(s,\varepsilon) = 0$ by (C3), and $i(s,\varepsilon) \leq \min\big(\alpha_s/p^{2\varepsilon+1}, \mathrm{cap}_{s+2\varepsilon+1}\big)$ by (C2) and (C4), so the summand is at least $\max\big(\alpha_s/p^{2\varepsilon+1} - \mathrm{cap}_{s+2\varepsilon+1}, 0\big)$. Summing over $s \in \mathcal{T}$ and using (C1) gives $\gamma_p\sum_{s\in\mathcal{T}}\alpha_s - a - b \geq g^A + \Delta$, and (9.12) gives $\mathrm{ord}\langle E, \Theta_1\rangle \geq g^A + \rho_p D$. This proves (i).

Step 8, the counts for (ii) and (iii). Let b_{ev} and b_{odd} be the parts of b that belong to even and to odd tops. By (C2) and (C4), $b_{\mathrm{odd}} \leq \sum_{s\in\mathcal{T}_{\mathrm{odd}}}\sum_{\varepsilon}\min(\alpha_s/p^{2\varepsilon+1}, \mathrm{cap}_{s+2\varepsilon+1})$, and therefore

$$\gamma_p\sum_{s\in\mathcal{T}}\alpha_s - b_{\mathrm{odd}} \geq \gamma_p\sum_{s\in\mathcal{T}_{\mathrm{ev}}}\alpha_s + \sum_{s\in\mathcal{T}_{\mathrm{odd}}}\sum_{\varepsilon}\max\Big(\frac{\alpha_s}{p^{2\varepsilon+1}} - \mathrm{cap}_{s+2\varepsilon+1}, 0\Big) = g^B + \Delta.$$

By (9.12) it remains to bound $a + b_{\mathrm{ev}}$. For (ii), a monomial t^{d_φ} involves two tops of different parity, so it has a positive exponent at an even top, and a monomial t^{d_κ} with κ at an even top has a positive exponent there. The sum of (N1) over all coordinates (s,b') with $s \in \mathcal{T}_{\mathrm{ev}}$ gives $a + b_{\mathrm{ev}} \leq \sum_{s\in\mathcal{T}_{\mathrm{ev}}}\alpha_s$, which proves (ii). For (iii), let $\mathcal{P}_J$ be the set of pairs (s,b') with s an even top and $b' \notin J$ or with s an odd top and $b' \in J$. A monomial t^{d_φ} involves the coordinates (σ_1,b') and (σ_2,b') with one b' and two parities, so one of them lies in $\mathcal{P}_J$. A $\kappa = (\sigma,l,b')$ at an even top with $b' \notin J$ has $(\sigma,b') \in \mathcal{P}_J$. The sum of (N1) over $\mathcal{P}_J$ bounds the number of these by $\sum_{(s,b')\in\mathcal{P}_J} e(s,b')$. A κ at an even top with $b' \in J$ is a derivative at the odd position $\sigma + 2l + 1$ in a column with index $b' \in J$, and a minor at that position absorbs at most one such derivative for each of its column indices in J. Hence $a + b_{\mathrm{ev}} \leq N_J$, and (iii) follows. □

10 Divisibility along the chain

Theorem 9.6 bounds the pairings of one kernel with the products of minors at one Frobenius orbit. This section turns such bounds into divisibility of the composite of the lifts on integral forms. The argument is algebra on spaces of functions on lattices, and it uses three properties of the lifts. Each lift is a finite sum over classes of maps defined by kernels, with weights whose denominators are bounded. A bound for the pairings of a kernel implies that the kernel map, normalized by the power of p of a Hecke operator, sends the integral structure at the source into a multiple of the integral structure at the target. A polynomial in the Hecke operators at an orbit commutes with the composite of the lifts up to an explicit power of p. We state these properties as part of a definition and prove the divisibility from them. The verification of the properties for the lifts of Section 7 is a separate step.

Throughout this section p is a prime, $\mathcal{O}$ is a complete discrete valuation ring of characteristic zero with residue characteristic p, fraction field $\mathcal{K}$ and uniformizer π, and ord is the valuation with $\mathrm{ord}(p) = 1$, so that $\mathrm{ord}(\pi) = 1/e$ with e the ramification index. For an $\mathcal{O}$-module H

and a real number g we write $p^g H = \pi^{\lceil ge \rceil} H$. Then $a\, p^g H \subset p^{g+\mathrm{ord}(a)} H$ for $a \in \mathcal{K}^\times$, and $p^g H + p^g H \subset p^g H$, and $p^g\big(p^{g'} H\big) \subset p^{g+g'} H$, since $\lceil x \rceil + \lceil y \rceil \geq \lceil x + y \rceil$ and $e\,\mathrm{ord}(a) \in \mathbb{Z}$.

10.1 Chain data

Definition 10.1. A *chain datum* over $\mathcal{O}$ consists of an integer $m \geq 1$, a finite set $\mathcal{V}$ whose elements are called orbits, an integer $r \geq 1$, and the following objects for each level $0 \leq j \leq m$.

(a) A set $\mathcal{L}_j$ of *lattice tuples*, a finite set C_j of components, a finite-dimensional $\mathcal{K}$-vector space $\mathcal{X}_j$, and for each $c \in C_j$ a group $\Gamma_{j,c}$ that acts on $\mathcal{L}_j$ and linearly on $\mathcal{X}_j$.

(b) For each $L \in \mathcal{L}_j$ an $\mathcal{O}$-lattice $H_j(L) \subset \mathcal{X}_j$, with $H_j(\gamma L) = \gamma H_j(L)$ for $\gamma \in \Gamma_{j,c}$ and $c \in C_j$.

(c) For each orbit v, each $0 \leq i \leq r$ and each $L \in \mathcal{L}_j$ a finite family $E^j_{v,i}(L)$ of elements of $\mathcal{L}_j$, with $E^j_{v,0}(L) = (L)$ and $E^j_{v,i}(\gamma L) = \gamma E^j_{v,i}(L)$, and an integer $n^j_{v,i}$ with $n^j_{v,0} = 0$.

A *form* of level j is a function F on $C_j \times \mathcal{L}_j$ with values in $\mathcal{X}_j$ such that $F(c, \gamma L) = \gamma F(c, L)$ for all c, L and $\gamma \in \Gamma_{j,c}$. The forms of level j form a $\mathcal{K}$-vector space $\mathcal{F}_j$, and the *integral forms* Int_j are those with $F(c, L) \in H_j(L)$ for all (c, L). The *Hecke operator* $T^j_{v,i}$ on $\mathcal{F}_j$ is

$$\big(T^j_{v,i} F\big)(c, L) = p^{\,n^j_{v,i}} \sum_{L'' \in E^j_{v,i}(L)} F(c, L'').$$

The datum further contains linear maps $K_j \colon \mathcal{F}_j \to \mathcal{F}_{j+1}$ for $0 \leq j < m$, called the lifts, with the following property.

(D1) There are integers $d_j \geq 0$ and $\mathcal{K}$-linear maps $\kappa_j(c, L; c', L') \colon \mathcal{X}_j \to \mathcal{X}_{j+1}$, for $(c, L) \in C_j \times \mathcal{L}_j$ and $(c', L') \in C_{j+1} \times \mathcal{L}_{j+1}$, such that for every (c', L') there are a finite family (c_x, L_x) in $C_j \times \mathcal{L}_j$ and elements $w_x \in \mathcal{K}^\times$ with $\mathrm{ord}(w_x) \leq d_j$ and

$$(K_j F)(c', L') = \sum_x w_x^{-1}\, \kappa_j(c_x, L_x; c', L')\big(F(c_x, L_x)\big) \qquad (F \in \mathcal{F}_j).$$

We write $\mathcal{K}_j = K_{j-1} \circ \cdots \circ K_0 \colon \mathcal{F}_0 \to \mathcal{F}_j$, with $\mathcal{K}_0$ the identity.

A Hecke operator maps forms to forms, because $E^j_{v,i}(\gamma L) = \gamma E^j_{v,i}(L)$ gives $(T^j_{v,i} F)(c, \gamma L) = p^n \sum_{L'' \in E(L)} F(c, \gamma L'') = \gamma\, (T^j_{v,i} F)(c, L)$. For a family $i_0 = (i_0(v))_{v \in \mathcal{V}}$ with $0 \leq i_0(v) \leq r$ and an ordering $v_1, \ldots, v_k$ of $\mathcal{V}$ we put $T^j_{i_0} = T^j_{v_1, i_0(v_1)} \cdots T^j_{v_k, i_0(v_k)}$. Expanding the product,

$$\big(T^j_{i_0} F\big)(c, L) = p^{\,n^j(i_0)} \sum_{L'' \in E^j_{i_0}(L)} F(c, L''), \qquad n^j(i_0) = \sum_v n^j_{v, i_0(v)}, \tag{10.1}$$

where $E^j_{i_0}(L)$ is the finite family obtained by applying $E^j_{v_k, i_0(v_k)}$, then $E^j_{v_{k-1}, i_0(v_{k-1})}$, and so on, to L. Both depend on the ordering, which we suppress from the notation.

Definition 10.2. Let $0 \leq j < m$, let i_0 be as above, and let $g \in \mathbb{R}$. The chain datum has the *entry bound* g at (j, i_0) when, for every ordering of $\mathcal{V}$,

$$p^{\,n^{j+1}(i_0)}\, \kappa_j(c, L; c', L'')\big(H_j(L)\big) \subset p^g\, H_{j+1}(L')$$

for all $(c, L) \in C_j \times \mathcal{L}_j$, all $(c', L') \in C_{j+1} \times \mathcal{L}_{j+1}$ and all L'' in the family $E^{j+1}_{i_0}(L')$.

In Definition 10.2 the kernel map goes to the lattice tuple L'', and its image is measured in the lattice $H_{j+1}(L')$ of the tuple L' from which L'' is reached by the Hecke edges. For $i_0 = 0$ the family $E_0^{j+1}(L')$ is (L') and the condition reads $\kappa_j(c, L; c', L')(H_j(L)) \subset p^g H_{j+1}(L')$.

Proposition 10.3. *Let a chain datum be given, let $0 \leq j < m$, let i_0 be a family of edge types and let $g \in \mathbb{R}$. If the datum has the entry bound g at (j, i_0), then*

$$T_{i_0}^{j+1} K_j\big(\mathrm{Int}_j\big) \subset p^{\,g-d_j}\,\mathrm{Int}_{j+1}.$$

Proof. Let $G \in \mathrm{Int}_j$. The function $T_{i_0}^{j+1} K_j G$ is a form, so it suffices to show that its value at every (c', L') lies in $p^{g-d_j} H_{j+1}(L')$. By (10.1) and (D1) the value is

$$\sum_{L'' \in E_{i_0}^{j+1}(L')} \sum_x w_x^{-1}\, p^{\,n^{j+1}(i_0)} \kappa_j(c_x, L_x; c', L'')\big(G(c_x, L_x)\big),$$

where for each L'' the family (c_x, L_x, w_x) is the one that (D1) attaches to (c', L''). Since $G(c_x, L_x) \in H_j(L_x)$, the entry bound puts the term with index x, before the factor w_x^{-1}, in $p^g H_{j+1}(L')$. The factor w_x^{-1} has valuation at least $-d_j$, so each term lies in $p^{g-d_j} H_{j+1}(L')$, and so does the sum. □

10.2 Hecke operators and the composite of the lifts

Definition 10.4. A chain datum has *Hecke commutation data* when it is equipped with the following objects.

(D2) Linear operators y_v on $\mathcal{F}_0$ for $v \in \mathcal{V}$, which commute with each other and map Int_0 into itself.

(D3) For each orbit v and each level $1 \leq j \leq m$, an element $\rho_{v,j} \in \mathcal{K}^\times$, a unit $\epsilon_{v,j} \in \mathcal{O}^\times$, a real number $\alpha(v, j) \geq 0$, and elements $c_{v,j,l} \in \mathcal{O}$ for $0 \leq l \leq r$ with $\mathrm{ord}(c_{v,j,r}) \geq \alpha(v, j)$, such that the operator

$$Z_{v,j} = \sum_{l=0}^{r} (-1)^l\, c_{v,j,l}\, \epsilon_{v,j}^{-(r-l)}\, T_{v,r-l}^{j}$$

on $\mathcal{F}_j$ satisfies $Z_{v,j}\, \mathcal{K}_j(F) = \rho_{v,j}\, \mathcal{K}_j(y_v F)$ for every $F \in \mathcal{F}_0$.

We write $\mathrm{Pr}(v, j) = \mathrm{ord}(\rho_{v,j})$, the price of moving y_v across $\mathcal{K}_j$.

The term $l = r$ of $Z_{v,j}$ is $(-1)^r c_{v,j,r}$ times the identity, since $T_{v,0}^{j}$ is the identity. The improvement in Theorem 10.7 below comes from the terms with $l < r$, which contain a Hecke operator of type at least one, and the identity term pays with the valuation $\alpha(v, j)$ of its coefficient.

Definition 10.5. Let a chain datum be given, and for each orbit v and each level $1 \leq j \leq m$ let $g^0(v, j) \leq g^1(v, j)$ be real numbers and $\alpha(v, j) \geq 0$. The datum has the *orbit bounds* (g^0, g^1, α) when, for every $0 \leq j < m$ and every family i_0 of edge types, it has the entry bound $\sum_v b_v$ at (j, i_0) for every choice of numbers b_v with

$$b_v \leq \begin{cases} g^1(v, j+1) & i_0(v) \geq 1, \\ \max\big(g^0(v, j+1),\ g^1(v, j+1) - \alpha(v, j+1)\big) & i_0(v) = 0. \end{cases}$$

For the lifts of Section 7 at an inert prime, g^0 is the bound of Theorem 9.6(i), g^1 the bound g^B of Theorem 9.6(ii) and (iii), and α the number $\sum_{s\in\mathcal{T}_{\mathrm{ev}}}\alpha_s$ of credits at the even tops, each written in the parameters of the orbit v at level j.

Lemma 10.6. *Let a chain datum have Hecke commutation data with the numbers $\alpha(v,j)$ of (D3), and orbit bounds (g^0,g^1,α). Let $0\le j<m$ and let $V\subset\mathcal{V}$, ordered. Then*

$$\prod_{v\in V} Z_{v,j+1}\,K_j(\mathrm{Int}_j)\subset p^{\,g_V-d_j}\,\mathrm{Int}_{j+1},\qquad g_V=\sum_{v\in V}g^1(v,j+1)+\sum_{v\notin V}g^0(v,j+1).$$

Proof. Expanding the product of the $Z_{v,j+1}$ in the given order writes it as a sum of terms $a\,T^{j+1}_{v_1,i_1}\cdots T^{j+1}_{v_k,i_k}$ over choices of $i_v=r-l_v\in\{0,\dots,r\}$ for $v\in V$, with $a=\prod_{v\in V}(-1)^{l_v}c_{v,j+1,l_v}\epsilon_{v,j+1}^{-(r-l_v)}$. Let $i_0(v)=i_v$ for $v\in V$ and $i_0(v)=0$ for $v\notin V$. Since $T^{j+1}_{v,0}$ is the identity, the product of Hecke operators in the term equals $T^{j+1}_{i_0}$ for an ordering of $\mathcal{V}$ that begins with V in its given order. The coefficient satisfies $\mathrm{ord}(a)\ge\sum_{v\in V,\,i_0(v)=0}\alpha(v,j+1)$, because ϵ is a unit, every c is integral, and $i_0(v)=0$ means $l_v=r$. Choose $b_v=g^1(v,j+1)$ for $v\in V$ with $i_0(v)\ge1$, $b_v=g^1(v,j+1)-\alpha(v,j+1)$ for $v\in V$ with $i_0(v)=0$, and $b_v=g^0(v,j+1)$ for $v\notin V$. These are admissible in Definition 10.5, so Proposition 10.3 puts $T^{j+1}_{i_0}K_j(\mathrm{Int}_j)$ in $p^{\sum b_v-d_j}\mathrm{Int}_{j+1}$, and $\mathrm{ord}(a)+\sum_v b_v\ge g_V$. □

Theorem 10.7. *Let a chain datum have Hecke commutation data and orbit bounds (g^0,g^1,α), with the same numbers $\alpha(v,j)$. Let $B\subset\mathcal{V}$, and let $\mathrm{PR}(v,j)\ge\mathrm{Pr}(v,j)$ and $S(v)$ be real numbers for $v\in B$ and $1\le j\le m$ with*

$$\sum_{j=1}^{m}\max\big(g^1(v,j)-\mathrm{PR}(v,j)-g^0(v,j),\,0\big)\le S(v)\qquad(v\in B).\tag{10.2}$$

Let $B'\subset B$, put $y_{B'}=\prod_{v\in B'}y_v$, and let $F=y_{B'}^{m}F_0$ with $F_0\in p^{\sum_{v\in B\setminus B'}S(v)}\,\mathrm{Int}_0$. Then $\mathcal{K}_m(F)\in p^{\,T}\,\mathrm{Int}_m$ with

$$T=\sum_{j=1}^{m}\Big(\sum_{v\in B}\max\big(g^1(v,j)-\mathrm{PR}(v,j),\,g^0(v,j)\big)+\sum_{v\notin B}g^0(v,j)\Big)-\sum_{j=0}^{m-1}d_j.$$

The same holds for every F in the $\mathcal{O}$-span of such forms, with B' varying.

Proof. For $0\le j<m$ let B_j be the set of $v\in B'$ with $g^0(v,j+1)\le g^1(v,j+1)-\mathrm{Pr}(v,j+1)$, and put

$$b_j=\sum_{v\in B_j}\big(g^1(v,j+1)-\mathrm{Pr}(v,j+1)\big)+\sum_{v\notin B_j}g^0(v,j+1)-d_j.$$

We show by induction on $m'\le m$ that $\mathcal{K}_{m'}\big(y_{B'}^{m'}F_1\big)\in p^{\,b_0+\cdots+b_{m'-1}}\mathrm{Int}_{m'}$ for every $F_1\in\mathrm{Int}_0$. For $m'=0$ this is the hypothesis $F_1\in\mathrm{Int}_0$. Let $m'<m$ and write $C=B'\setminus B_{m'}$. Since the y_v commute, $y_{B'}^{m'+1}F_1=y_{B_{m'}}W$ with $W=y_{B'}^{m'}(y_CF_1)$, and $y_CF_1\in\mathrm{Int}_0$ by (D2). Applying (D3) once for each $v\in B_{m'}$ gives

$$\mathcal{K}_{m'+1}\big(y_{B_{m'}}W\big)=\Big(\prod_{v\in B_{m'}}\rho_{v,m'+1}\Big)^{-1}\prod_{v\in B_{m'}}Z_{v,m'+1}\,K_{m'}\big(\mathcal{K}_{m'}W\big).$$

By the induction hypothesis applied to y_CF_1, $\mathcal{K}_{m'}W=\pi^nG'$ with $G'\in\mathrm{Int}_{m'}$ and $n=\lceil(b_0+\cdots+b_{m'-1})e\rceil$. Lemma 10.6 with $V=B_{m'}$ puts $\prod_{v\in B_{m'}}Z_{v,m'+1}K_{m'}(G')$ in $p^{\,g_V-d_{m'}}\mathrm{Int}_{m'+1}$. The

factor $\prod \rho^{-1}$ has valuation $-\sum_{v\in B_{m'}} \Pr(v, m'+1)$, and $g_V - d_{m'} - \sum_{v\in B_{m'}} \Pr(v, m'+1) = b_{m'}$. This completes the induction.

Now $F_0 = \pi^{n_0} F_1$ with $F_1 \in \mathrm{Int}_0$ and $n_0 = \lceil S_0 e \rceil$, $S_0 = \sum_{v\in B\setminus B'} S(v)$. By linearity and the case $m' = m$, $\mathcal{K}_m(F) \in p^{\,S_0+b_0+\cdots+b_{m-1}}\mathrm{Int}_m$. It remains to show $T \le S_0 + \sum_j b_j$. For each j and each orbit v compare the contribution of v to T and to b_{j-1}. If $v \notin B$, both are $g^0(v,j)$. If $v \in B_{j-1}$, the contribution to T is $\max(g^1 - \mathrm{PR}, g^0) \le \max(g^1 - \mathrm{Pr}, g^0) = g^1 - \mathrm{Pr}$, which is the contribution to b_{j-1}. If $v \in B' \setminus B_{j-1}$, then $g^1 - \mathrm{Pr} < g^0$, so $\max(g^1 - \mathrm{PR}, g^0) = g^0$, which is the contribution to b_{j-1}. If $v \in B \setminus B'$, the difference is $\max(g^1 - \mathrm{PR} - g^0, 0)$, and the sum over j of these differences is at most $S(v)$ by (10.2). Summing over v and j gives $T - \sum_j b_j \le S_0$. The last statement follows by linearity, since $p^T\mathrm{Int}_m$ is an $\mathcal{O}$-module. □

Theorem 10.7 says that the bound per lift and per orbit multiplies along the chain, with the improved bound g^1 minus its price at the orbits where a factor y_v is available, and that the composite of the m lifts is divisible by the sum. The theorem uses one factor y_v for each lift at which it pays, and leaves the others at level zero, where they preserve Int_0. The next subsection proves the commutation relation (D3) from a relation for one lift.

10.3 From one lift to the chain

Fix an orbit v and drop it from the notation. Let $q = p^{\nu'}$ for a positive rational number ν' with $q \in \mathcal{K}$, let $N_0 < N_1 < \cdots < N_m$ be integers with $r \le N_0$, and put $\mathcal{E}_i^{(j)} = q^{-i(N_j-i)} T_{v,i}^j$ for $0 \le i \le r$, the normalized Hecke operators. Let $t \ge 0$ be an integer. For $1 \le j \le m$ let $u_j \in \mathcal{O}^\times$ and $m_j \in \mathcal{K}$, and let x_j be the family of the $t+1$ numbers $m_j q^t, m_j q^{t-2}, \ldots, m_j q^{-t}$. Write e_l and h_l for the elementary and the complete symmetric polynomials of degree l.

Proposition 10.8. *Suppose that for $0 \le j < m$, $0 \le i \le r$ and $F \in \mathcal{F}_j$,*

$$\mathcal{E}_i^{(j+1)}\, K_j(F) = u_{j+1}^i\, K_j\Big(\sum_{l=0}^{i} e_l(x_{j+1})\, \mathcal{E}_{i-l}^{(j)} F\Big). \tag{10.3}$$

For $1 \le j \le m$ let Y_j be the family of the $j(t+1)$ numbers $u_l u_{l+1} \cdots u_j\, a$ with $1 \le l \le j$ and a in x_l, and put $U_j = u_1 \cdots u_j$. Then for every $F \in \mathcal{F}_0$

$$\sum_{l=0}^{r} (-1)^l\, h_l(Y_j)\, q^{\,l(N_j-2r+l)}\; T_{v,r-l}^j\, \mathcal{K}_j(F) = U_j^r\, q^{\,r(N_j-N_0)}\; \mathcal{K}_j\big(T_{v,r}^0 F\big).$$

Proof. We work with polynomials in an indeterminate X modulo X^{r+1}, with coefficients that are operators, and X commutes with everything. Put $E_j(X) = \sum_{i=0}^r \mathcal{E}_i^{(j)} X^i$. Since $\sum_l e_l(x)\, y^l = \prod_{a\in x}(1+ay)$, the relations (10.3) for $0 \le i \le r$ say

$$E_{j+1}(X)\, K_j = K_j\, P_{j+1}(X)\, E_j(u_{j+1}X), \qquad P_{j+1}(X) = \prod_{a\in x_{j+1}} \big(1 + u_{j+1} a X\big),$$

modulo X^{r+1}, where P_{j+1} has scalar coefficients. We show by induction that $E_j(X)\mathcal{K}_j = \Pi_j(X)\, \mathcal{K}_j E_0(U_j X)$ with $\Pi_j(X) = \prod_{b\in Y_j}(1+bX)$. For $j = 0$ both sides are $E_0(X)$. For the step,

$$E_{j+1}(X)\mathcal{K}_{j+1} = K_j P_{j+1}(X) E_j(u_{j+1}X)\mathcal{K}_j = P_{j+1}(X)\, \Pi_j(u_{j+1}X)\, \mathcal{K}_{j+1} E_0(U_{j+1}X),$$

and $P_{j+1}(X)\Pi_j(u_{j+1}X) = \Pi_{j+1}(X)$ by the definition of Y_{j+1}. The series $\Pi_j(X)$ has constant term 1, and its inverse is $\sum_k (-1)^k h_k(Y_j)X^k$. So $\mathcal{K}_j E_0(U_j X) = \Pi_j(X)^{-1} E_j(X)\mathcal{K}_j$, and the coefficients of X^r give

$$U_j^r\, \mathcal{K}_j\, \mathcal{E}_r^{(0)} = \sum_{l=0}^{r} (-1)^l h_l(Y_j)\, \mathcal{E}_{r-l}^{(j)}\, \mathcal{K}_j.$$

We multiply by $q^{r(N_j-r)}$. On the left this gives $q^{r(N_j-N_0)}T_{v,r}^0$ in place of $\mathcal{E}_r^{(0)}$, and on the right the factor of $T_{v,r-l}^j$ becomes $q^{r(N_j-r)-(r-l)(N_j-r+l)} = q^{l(N_j-2r+l)}$. □

When the operators $T_{v,i}^j$ have normalizing exponents $n_{v,i}^j$, Proposition 10.8 applied to the raw operators gives (D3) for $y_v = p^A T_{v,r}^{\mathrm{raw},0}$ after multiplication by powers of p. Proposition 11.32 makes this conversion for the lifts of Section 11, with $c_{v,j,l} = p^{n_{v,r}^j - n_{v,r-l}^j} q^{l(N_j-2r+l)} h_l(Y_j)$, and Theorem 11.33 proves that these coefficients are integral with $\mathrm{ord}(c_{v,j,r}) \geq \alpha(v,j)$. With all $n_{v,i}^j = 0$ the coefficients $h_l(Y_j)\, q^{l(N_j-2r+l)}$ need not be integral.

10.4 The Hecke relation for one lift

This subsection proves the relation (10.3) for lifts whose kernels at an orbit have the form that a theta kernel has in the lattice model at a split place. Fix an orbit v. Let Q_v be the unramified extension of $\mathbb{Q}_p$ of even degree n_v', with ring of integers W_v and residue field k_v of $q_v = p^{n_v'}$ elements, and put $q = p^{n_v'/2}$, so that $q_v = q^2$. For a W_v-lattice L in Q_v^N let $\mathrm{len}(L) = -\operatorname{ord}\det b$ for any matrix b whose columns form a basis of L. For lattices $L'' \subset L$ in Q_v^N we call (L'', L) an *edge of type i* when $pL \subset L''$ and $\dim_{k_v} L/L'' = i$. Then $\mathrm{len}(L'') = \mathrm{len}(L) - i$, because p is a uniformizer of W_v and the elementary divisors of L'' in L are i copies of p and $N-i$ units.

For x an $N' \times N$ matrix over Q_v and lattices $L \subset Q_v^N$, $L' \subset Q_v^{N'}$ put $\mathbf{1}_{L,L'}(x) = 1$ if $xL \subset L'$ and 0 otherwise.

Lemma 10.9. *Let $L \subset Q_v^N$ and $L' \subset Q_v^{N+g}$ be lattices, let x be an $(N+g)\times N$ matrix over Q_v, and let $i \geq 0$. Then*

$$\#\big\{L_1 : (L_1, L') \text{ an edge of type } i,\ xL \subset L_1\big\} = \sum_{l=0}^{i} q_v^{(i-l)(g-l)} \binom{g}{l}_{q_v} \#\big\{A_1 : (L, A_1) \text{ an edge of type } i-l,\ xA_1 \subset L'\big\},$$

where $\binom{g}{l}_{q_v}$ is the Gaussian binomial coefficient. Equivalently, as functions of x,

$$\sum_{(L_1,L') \text{ of type } i} \mathbf{1}_{L,L_1} = \sum_{l=0}^{i} q_v^{(i-l)(g-l)} \binom{g}{l}_{q_v} \sum_{(L,A_1) \text{ of type } i-l} \mathbf{1}_{A_1,L'}.$$

Proof. If $xL \not\subset L'$, both sides vanish, because $L_1 \subset L'$ and $A_1 \supset L$. Suppose $xL \subset L'$, and let $\bar{x}\colon L/pL \to L'/pL'$ be the induced map of k_v-vector spaces, of rank ρ. A lattice L_1 with $pL' \subset L_1 \subset L'$ corresponds to the subspace L_1/pL' of L'/pL', and $xL \subset L_1$ exactly when this subspace contains $\bar{x}(L/pL)$. So the left side counts the subspaces of codimension i of a space of dimension $N+g$ that contain a given subspace of dimension ρ, and it equals $\binom{N+g-\rho}{i}_{q_v}$. A lattice A_1 with $L \subset A_1 \subset p^{-1}L$ corresponds to the subspace pA_1/pL of L/pL, and $xA_1 \subset L'$ exactly

when $x(pA_1) \subset pL'$, that is, when this subspace lies in the kernel of $\bar{x}$, of dimension $N-\rho$. So the count on the right with type $i-l$ is $\binom{N-\rho}{i-l}_{q_v}$. The identity is the q-Vandermonde formula

$$\binom{a+b}{i}_Q = \sum_{l=0}^{i} Q^{(i-l)(b-l)} \binom{b}{l}_Q \binom{a}{i-l}_Q$$

with $a = N-\rho$, $b = g$ and $Q = q_v$ [3, (3.3.10)]. □

Lemma 10.10. *For integers $0 \le l \le g$ and $t = g-1$,*

$$e_l(q^t, q^{t-2}, \dots, q^{-t}) = q^{-l(g-l)} \binom{g}{l}_{q^2}.$$

Proof. The Gaussian binomial theorem $\prod_{k=0}^{g-1}(1+Q^k z) = \sum_l Q^{l(l-1)/2}\binom{g}{l}_Q z^l$ with $Q = q^2$ and z replaced by $q^{-t}z$ gives $\prod_{k=0}^{g-1}(1+q^{2k-t}z) = \sum_l q^{l(l-1)-lt}\binom{g}{l}_{q^2} z^l$. The left side is $\sum_l e_l(q^t,\dots,q^{-t})z^l$, and $l(l-1)-l(g-1) = -l(g-l)$. □

Lemma 10.11. *Let a group Γ act on a set $\mathcal{L}$ with finitely many orbits and finite stabilizers, and let $\mathcal{P} \subset \mathcal{L}\times\mathcal{L}$ be a Γ-stable set of pairs such that each $L \in \mathcal{L}$ occurs in finitely many pairs as first and as second entry. Let R be a set of representatives of $\Gamma\backslash\mathcal{L}$, and let φ be a function on $\mathcal{P}$ with values in a $\mathbb{Q}$-vector space with $\varphi(\gamma L_0, \gamma L_1) = \varphi(L_0, L_1)$. Then*

$$\sum_{L\in R} \frac{1}{|\Gamma_L|} \sum_{L_1:(L,L_1)\in\mathcal{P}} \varphi(L, L_1) = \sum_{L\in R} \frac{1}{|\Gamma_L|} \sum_{L_0:(L_0,L)\in\mathcal{P}} \varphi(L_0, L).$$

Proof. For $L \in R$ the pairs $(L, L_1) \in \mathcal{P}$ fall into orbits of Γ_L, and the orbit of (L, L_1) has $|\Gamma_L|/|\Gamma_{(L,L_1)}|$ elements, where $\Gamma_{(L,L_1)}$ is the stabilizer of the pair. So the left side equals $\sum|\Gamma_{(L_0,L_1)}|^{-1}\varphi(L_0, L_1)$, the sum over a set of representatives of $\Gamma\backslash\mathcal{P}$, since every Γ-orbit of pairs contains pairs with first entry in R, and these form one orbit of Γ_L. The same computation with the second entry gives the right side. □

Definition 10.12. A chain datum has *split kernels* at the orbit v when the following hold for every $0 \le j < m$.

(S1) Every lattice tuple $L \in \mathcal{L}_j$ has a component L_v that is a W_v-lattice in $Q_v^{N_j}$, with $N_{j+1} = N_j + g$ for a fixed $g \ge 1$. The family $E^j_{v,i}(L)$ consists of the tuples obtained from L by replacing L_v by the lattices L_1 with (L_1, L_v) an edge of type i. We write $T^{\mathrm{raw},j}_{v,i} = p^{-n^j_{v,i}} T^j_{v,i}$ for the operator of (c) with the exponent $n^j_{v,i}$ replaced by 0. Replacing L_v in this way commutes with the action of every $\Gamma_{j,c}$.

(S2) For every c the set $\Gamma_{j,c}\backslash\mathcal{L}_j$ is finite and the stabilizers are finite. There are a set R_j of pairs (c, L) that contains one representative of each orbit, an element $c_K \in \mathcal{K}^\times$, and maps κ_j as in (D1), such that $(K_jF)(c', L') = c_K \sum_{(c,L)\in R_j} |\Gamma_{j,c,L}|^{-1}\kappa_j(c, L; c', L')(F(c, L))$, and $\kappa_j(c, \gamma L; c', L') \circ \gamma = \kappa_j(c, L; c', L')$ for $\gamma \in \Gamma_{j,c}$.

(S3) There are $a, a' \in \mathcal{O}^\times$, an integer μ, and for every pair $(c, L; c', L')$ a $\mathbb{Z}$-linear map $f \mapsto \Xi(c, L; c', L'; f)$ from functions on $N_{j+1} \times N_j$ matrices over Q_v with integer values to $\mathrm{Hom}_{\mathcal{K}}(\mathcal{X}_j, \mathcal{X}_{j+1})$, which does not change when L_v or L'_v is replaced by another lattice, such that

$$\kappa_j(c, L; c', L') = p^{\mu(\operatorname{len} L_v - \operatorname{len} L'_v)}\, a^{\operatorname{len} L_v}\, a'^{\operatorname{len} L'_v}\, q^{-(N_j \operatorname{len} L'_v - N_{j+1} \operatorname{len} L_v)}\, \Xi\big(c, L; c', L'; \mathbf{1}_{L_v, L'_v}\big).$$

Proposition 10.13. *Suppose that a chain datum has split kernels at v, with the constants a, a', μ of (S3) for the lift K_j. Then* (10.3) *holds for j with the raw operators $T^{\mathrm{raw},j}_{v,i}$ and $T^{\mathrm{raw},j+1}_{v,i}$ in place of $T^{j}_{v,i}$ and $T^{j+1}_{v,i}$ in the definition of $\mathcal{E}^{(j)}_i$, and with $u_{j+1} = (aa')^{-1}$, $m_{j+1} = a\, p^{\mu}$ and $t = g - 1$.*

Proof. In this proof $T^{j}_{v,i}$ and $T^{j+1}_{v,i}$ denote the raw operators. Write $\sigma(L, L')$ for the scalar in (S3), so that $\kappa_j(L; L') = \sigma(L, L')\, \Xi(L; L'; \mathbf{1}_{L_v, L'_v})$, where we suppress the components. If (L_1, L'_v) is an edge of type i and L'_1 is L' with L'_v replaced by L_1, then $\operatorname{len} L_1 = \operatorname{len} L'_v - i$ and

$$\sigma(L, L'_1) = c_A(i)\, \sigma(L, L'), \qquad c_A(i) = p^{\mu i} a'^{-i} q^{N_j i}.$$

If (L_v, A_1) is an edge of type k and L^{A_1} is L with L_v replaced by A_1, then $\operatorname{len} A_1 = \operatorname{len} L_v + k$ and

$$\sigma(L^{A_1}, L') = c_C(k)\, \sigma(L, L'), \qquad c_C(k) = p^{\mu k} a^k q^{N_{j+1} k}.$$

By (S2), (S1) and the two relations,

$$\big(T^{j+1}_{v,i} K_j F\big)(c', L') = c_K \sum_{(c,L) \in R_j} \frac{c_A(i)\, \sigma(L, L')}{|\Gamma_{j,c,L}|}\, \Xi\Big(L; L'; \sum_{(L_1, L'_v)} \mathbf{1}_{L_v, L_1}\Big)\big(F(c, L)\big),$$

since Ξ is additive and does not depend on the component at v. Lemma 10.9 replaces the inner sum by $\sum_l q_v^{(i-l)(g-l)} \binom{g}{l}_{q_v} \sum_{(L_v, A_1)} \mathbf{1}_{A_1, L'_v}$, with (L_v, A_1) of type $i - l$, and the second relation turns $\sigma(L, L')\Xi(L; L'; \mathbf{1}_{A_1, L'_v})$ into $c_C(i-l)^{-1} \kappa_j(L^{A_1}; L')$. Lemma 10.11, applied for each component c to the pairs (L_0, L_1) that differ at v by an edge of type $i - l$ and to $\varphi(L_0, L_1) = \kappa_j(c, L_1; c', L')(F(c, L_0))$, which is invariant by the covariance in (S2) and the equivariance of F, gives

$$\sum_{(c,L) \in R_j} \frac{1}{|\Gamma_{j,c,L}|} \sum_{(L_v, A_1)} \kappa_j(L^{A_1}; L')\big(F(c, L)\big) = \sum_{(c,L) \in R_j} \frac{1}{|\Gamma_{j,c,L}|}\, \kappa_j(c, L; c', L')\big((T^{j}_{v,i-l} F)(c, L)\big).$$

Hence

$$T^{j+1}_{v,i} K_j F = \sum_{l=0}^{i} c_A(i)\, q_v^{(i-l)(g-l)} \binom{g}{l}_{q_v} c_C(i-l)^{-1}\, K_j\big(T^{j}_{v,i-l} F\big).$$

Multiply by $q^{-i(N_{j+1}-i)}$ and write $T^{j}_{v,i-l} = q^{(i-l)(N_j - i + l)} \mathcal{E}^{(j)}_{i-l}$. With $q_v = q^2$ and Lemma 10.10, the coefficient of $K_j(\mathcal{E}^{(j)}_{i-l} F)$ is

$$(aa')^{-i}\, \big(ap^{\mu}\big)^{l}\, e_l\big(q^t, \ldots, q^{-t}\big)\, q^E,$$

where

$$E = -i(N_{j+1} - i) + N_j i + 2(i-l)(g-l) - N_{j+1}(i-l) + l(g-l) + (i-l)(N_j - i + l).$$

Put $i = l + d$ and $N_{j+1} = N_j + g$. Then $E = -(l+d)g + (l+d)^2 + 2d(g-l) - gd + l(g-l) - d^2 = 0$. Since e_l is homogeneous of degree l, $(ap^{\mu})^l e_l(q^t, \ldots, q^{-t}) = e_l(x_{j+1})$ with $m_{j+1} = ap^{\mu}$, which is (10.3) with $u_{j+1} = (aa')^{-1}$. □

Proposition 10.13 and Proposition 10.8 together give the relation of Proposition 10.8 for the raw operators of chain data with split kernels at every orbit, with u_j a unit and $\mathrm{ord}(m_j) = \mu_j$. Section 11 verifies the hypotheses of this section for the lifts of Section 7 and converts the relation into (D3).

11 The lifts form a chain datum

This section defines the lattice tuples, the integral structures and the lifts of the chain at an inert prime, and verifies for them the axioms of a chain datum of Section 10. The verification uses three statements, (N1) on the normalization of the frame factor, (N2) on the passage from entries to lattices at the odd positions, and (N3) on the normalization of the base theta function. Theorem 11.45 assembles the verification, so that Theorem 10.7 applies to the composite of the lifts.

11.1 The data at an inert prime

11.1.1 Orbits and positions

Let E, $K = \mathbb{Q}(\sqrt{-d})$, E^+ and the CM type Φ be as in Section 7, and fix a prime $p \geq 5$ that is inert in K and unramified in E. Let $\mathbb{C}_p$, the isomorphism $\iota\colon \mathbb{C} \to \mathbb{C}_p$, the ring $W = W(\overline{\mathbb{F}}_p)$ with Frobenius σ, and $\mathcal{K}_0 = W[1/p]$ be as in Section 8.2. An *orbit* is a place v of E^+ above p that splits in E. For every orbit v we fix the place $w \mid v$ of E for which $\iota|_K$ and the embedding $E \to E_w$ induce the same embedding of $K_p = \mathbb{Q}_{p^2}$, and we write $Q_v = E_w$, $W_v = \mathcal{O}_{E_w}$ and $n'_v = [Q_v : \mathbb{Q}_p]$. The field Q_v contains $\mathbb{Q}_{p^2}$ and is unramified over $\mathbb{Q}_p$, so $n'_v = 2\nu_v$ is even, p is a uniformizer of W_v, and W_v/pW_v has $p^{n'_v}$ elements. We fix an embedding $\rho_v\colon Q_v \to \mathcal{K}_0$ over $\iota|_K$ and write σ also for the Frobenius of Q_v.

The orbit v has n'_v *positions* $s \in \mathbb{Z}/n'_v$. For an even position s the embedding $x \mapsto \sigma^s(\rho_v(x))$ of E into $\mathbb{C}_p$ is $\iota \circ \tau$ for a unique $\tau \in \Phi$, because its restriction to K is $\sigma^s \circ \iota|_K = \iota|_K$. For an odd position s the embedding $x \mapsto \sigma^s(\rho_v(\bar{x}))$ has restriction $\sigma^{s+1} \circ \iota|_K = \iota|_K$ to K, so it is $\iota \circ \tau$ for a unique $\tau \in \Phi$. We write $\tau_{v,s}$ for this element of Φ. Thus

$$\iota\big(\tau_{v,s}(x)\big) = \begin{cases} \sigma^s\big(\rho_v(x)\big) & s \text{ even}, \\ \sigma^s\big(\rho_v(\bar{x})\big) & s \text{ odd}, \end{cases} \qquad x \in E. \tag{11.1}$$

The even positions of v induce the place w and the odd positions induce $\bar{w}$. Distinct pairs (v, s) give distinct elements of Φ. The elements of Φ that are not of the form $\tau_{v,s}$ induce places of E above p that are inert over E^+.

11.1.2 Ranks and weights

Fix integers $k_1 \geq 1$, $t \geq 0$, $m \geq 1$ and $1 \leq r$ with $2r \leq k_1 + 1$, and put

$$N_j = k_1 + 1 + j(t+1) \qquad (0 \leq j \leq m).$$

For each orbit v fix a set $\mathcal{T}_v$ of positions (the tops), integers a_s, b_s, an integer $c \geq 0$, a dominant weight $\lambda^{(0)}_{v,s} \in \mathbb{Z}^{N_0}$ for each position, and integers $w_l(v,s)$ for $1 \leq l \leq m$, subject to the following conditions.

(W1) For $s \in \mathcal{T}_v$, $\lambda^{(0)}_{v,s}(1) = a_s$, $\lambda^{(0)}_{v,s}(N_0) = -b_s$ and $|\lambda^{(0)}_{v,s}(i)| \le c$ for $1 < i < N_0$. For $s \notin \mathcal{T}_v$, $|\lambda^{(0)}_{v,s}(i)| \le c$ for all i.

(W2) $w_1(v,s) \ge \max\left(c, -\lambda^{(0)}_{v,s}(N_0)\right)$ and $w_l(v,s) \le w_{l+1}(v,s)$ for $1 \le l < m$.

The weight of level j at (v,s) is

$$\lambda^{(j)}_{v,s} = \big(\lambda^{(0)}_{v,s}(1), \dots, \lambda^{(0)}_{v,s}(N_0), \underbrace{-w_1, \dots, -w_1}_{t+1}, \dots, \underbrace{-w_j, \dots, -w_j}_{t+1}\big) \in \mathbb{Z}^{N_j},$$

with $w_l = w_l(v,s)$. It is dominant by (W2). The *model weight* at (v,s) is $\lambda^{(j)}_{v,s}$ for even s and its dual $\big(\lambda^{(j)}_{v,s}\big)^* = (-\lambda(N_j), \dots, -\lambda(1))$ for odd s. We write $\lambda^\natural = \lambda^{\natural,(j)}_{v,s}$ for the model weight, $\nu^\natural = \lambda^\natural - \lambda^\natural(N_j) \in \mathbb{Z}^{N_j}_{\ge 0}$ for its normalization, and

$$\alpha_s = \begin{cases} \lambda^{(0)}_{v,s}(1) - \lambda^{(0)}_{v,s}(2) & s \in \mathcal{T}_v, \\ 0 & s \notin \mathcal{T}_v, \end{cases} \qquad \mu_v(j) = \sum_{s \text{ even}} w_j(v,s) - \sum_{s \text{ odd}} w_j(v,s). \tag{11.2}$$

We also write $\nu^{(j)}_{v,s} = \lambda^{(j)}_{v,s} - \lambda^{(j)}_{v,s}(N_j)$ at every position; at even positions $\nu^{(j)}_{v,s} = \nu^{\natural,(j)}_{v,s}$. The numbers of the shape of Section 9.3 at level $j \ge 1$ are read from $\nu^{(j)}_{v,s}$ at all positions, as in Lemma 11.36 below. The weights $\lambda^{(j)}_{v,s}$ are the weights λ_τ of the step kernel (7.4) at $\tau = \tau_{v,s}$.

11.1.3 Polynomial models and their minimal lattices

Let $\mathcal{O}$ be the ring of integers of a finite extension $\mathcal{K}$ of $\mathcal{K}_0$ in $\mathbb{C}_p$ that contains an element ϖ with $\varpi^{p^2-1} = p$, and let π be a uniformizer of $\mathcal{O}$. For $n \ge 1$ and $\nu \in \mathbb{Z}^n_{\ge 0}$ nonincreasing, let $\mathcal{K}[X]$ be the polynomial ring in the entries X_{ab} of an $n \times n$ matrix. The group GL_n acts on $\mathcal{K}[X]$ through rows, $(g \cdot F)(X) = F(g^{\mathsf{T}} X)$, and the column operations commute with this action.

Definition 11.1. (i) The *polynomial model* $V_\nu \subset \mathcal{K}[X]$ is the space of polynomials F that are homogeneous of degree ν_b in the entries of column b for every b, and are invariant under the substitutions $X_{\cdot,b} \mapsto X_{\cdot,b} + cX_{\cdot,l}$ for $l < b$ and $c \in \mathcal{K}$.

(ii) For $1 \le h \le n$ and a set $S = \{s_1 < \cdots < s_h\}$ of rows, put $\Delta_S = \det(X_{s_i,k})_{1 \le i,k \le h}$, the minor on the rows S and the first h columns. Let $h_1 \ge h_2 \ge \cdots \ge h_{\nu_1}$ be the column heights of ν, so that $h_c = \#\{i : \nu_i \ge c\}$. A *canonical bideterminant* of shape ν is a product $\Delta_{S_1} \cdots \Delta_{S_{\nu_1}}$ with $\#S_c = h_c$.

(iii) The *minimal lattice* $\mathsf{W}_\nu \subset V_\nu$ is the $\mathcal{O}$-span of the canonical bideterminants of shape ν.

Each canonical bideterminant lies in V_ν, because a minor on the first h columns has degree one in each of these columns and is invariant under adding an earlier column to a later one among them. The space V_ν is the induced module $H^0(\nu)$ of GL_n, and W_ν is the Weyl module over $\mathcal{O}$, the span of the images of the highest weight vector $\Delta_{\{1\}}^{\nu_1 - \nu_2} \cdots$ under the divided powers of the lowering operators. We use the following published form of this description.

Theorem 11.2 (Green; De Concini–Eisenbud–Procesi)**.** *For every commutative ring R the R-span of the canonical bideterminants of shape ν in $R[X]$ is a free R-module with basis the canonical bideterminants whose row sets form a standard tableau, it is stable under the row action of $\mathrm{GL}_n(R)$, and its formation commutes with base change in R.*

Only the stability under $\mathrm{GL}_n(\mathcal{O})$ is used below, and it has a one-line proof: for $g \in \mathrm{GL}_n(\mathcal{O})$ the Cauchy–Binet formula gives $\Delta_S(g^{\mathsf{T}}X) = \sum_{\#T=\#S} \det\big((g^{\mathsf{T}})_{S,T}\big)\Delta_T(X)$, so the row action maps a canonical bideterminant to an $\mathcal{O}$-linear combination of canonical bideterminants of the same shape; the same computation applies to a rectangular matrix of indeterminates.

Definition 11.3. Let (v,s) be a position, j a level, and $\Lambda \subset Q_v^{N_j}$ a W_v-lattice. Choose $g \in \mathrm{GL}_{N_j}(Q_v)$ whose columns form a basis of Λ, and let $g_s = \sigma^s(\rho_v(g)) \in \mathrm{GL}_{N_j}(\mathcal{K})$. Put

$$H_s(\Lambda) = \det(g_s)^{-\lambda^\natural(N_j)} \left\{F(g_s^{-1}\,\cdot\,) :\ F \in \mathsf{W}_{\nu^\natural}\right\} \subset V_{\nu^\natural},$$

that is, $F \in H_s(\Lambda)$ exactly when $\det(g_s)^{\lambda^\natural(N_j)}F(g_s^{\mathsf{T}}X)$ lies in $\mathsf{W}_{\nu^\natural}$ (we write the row action as substitution of $g_s^{\mathsf{T}}X$ for X, so that $F(g_s^{-1}\cdot)$ means the inverse substitution). For a lattice tuple $L = (L_v)_v$ of level j put

$$\mathcal{X}_j = \bigotimes_v \bigotimes_{s\in\mathbb{Z}/n_v'} V_{\nu_{v,s}^{\natural,(j)}}, \qquad H_j(L) = \bigotimes_v \bigotimes_s H_s(L_v),$$

the tensor product of $\mathcal{O}$-lattices inside the tensor product of $\mathcal{K}$-spaces.

The lattice $H_s(\Lambda)$ does not depend on the basis g, because two bases differ by an element of $\mathrm{GL}_{N_j}(W_v)$, whose image under $\sigma^s \circ \rho_v$ lies in $\mathrm{GL}_{N_j}(\mathcal{O})$, and W_ν is stable under $\mathrm{GL}_{N_j}(\mathcal{O})$ by Theorem 11.2, while the determinant of such an element is a unit.

Lemma 11.4 (Edges)**.** *Let (Λ'', Λ) be an edge of type i at the orbit v in the sense of Section 10.4, that is $p\Lambda \subset \Lambda'' \subset \Lambda$ and $\dim_{k_v} \Lambda/\Lambda'' = i$. Then for every position s*

$$p^{\sum_{a\le i}\lambda^\natural(a)}\, H_s(\Lambda'') \subset H_s(\Lambda),$$

where $\lambda^\natural = \lambda_{v,s}^{\natural,(j)}$.

Proof. By the elementary divisor theorem there is a basis g of Λ such that $g\,d$ is a basis of Λ'', with $d = \mathrm{diag}(p,\dots,p,1,\dots,1)$ and i entries equal to p. Put $h = \lambda^\natural(N_j)$ and $\nu = \nu^\natural$. Then $\det((gd)_s) = p^i \det(g_s)$ and

$$H_s(\Lambda'') = p^{-ih}\det(g_s)^{-h}\left\{F(d^{-1}g_s^{-1}\,\cdot\,) : F \in \mathsf{W}_\nu\right\}.$$

For a canonical bideterminant $B = \prod_c \Delta_{S_c}$, the substitution of $d^{-1}X$ for X divides the rows with index at most i by p, so it multiplies B by $p^{-\sum_c \#(S_c\cap\{1,\dots,i\})}$. Since $\#(S_c \cap \{1,\dots,i\}) \le \min(h_c, i)$ and $\sum_c \min(h_c,i) = \sum_{a\le i}\nu(a)$, the polynomial $p^{\sum_{a\le i}\nu(a)}B(d^{-1}X)$ is a multiple of B by an element of $\mathcal{O}$. Hence $p^{\sum_{a\le i}\nu(a)+ih}H_s(\Lambda'') \subset H_s(\Lambda)$, and $\sum_{a\le i}\nu(a) + ih = \sum_{a\le i}\lambda^\natural(a)$. □

11.1.4 Components, global lattices, forms

Definition 11.5. The *chain at p* consists of the following data for each level $0 \le j \le m$.

(a) The set $\mathcal{L}_j$ of *lattice tuples* $L = (L_v)_v$, with L_v a W_v-lattice in $Q_v^{N_j}$ for every orbit v.

(b) A finite set C_j of *components*. Each $c \in C_j$ comes with frames $\varphi_{c,v} \in \mathrm{GL}_{N_j}(Q_v)$ for every orbit v and with a map $L \mapsto N_j(c,L)$ from $\mathcal{L}_j$ to self-dual $\mathcal{O}_E$-lattices in E^{N_j} for the standard form h_{N_j} of Section 7.1, such that the completion of $N_j(c,L)$ at w is $\varphi_{c,v}(L_v)$ under ρ_v, and such that the completions of $N_j(c,L)$ at all places other than the places $w, \bar{w}$ above orbits do not depend on L.

(c) For each $c \in C_j$ a subgroup $\Gamma_{j,c}$ of the unitary group $U_{N_j} = U(h_{N_j})(E)$ that preserves the completions of $N_j(c,L)$ at all places other than the $w, \bar{w}$, together with a character $\chi_{j,c}\colon \Gamma_{j,c} \to \mathcal{O}^\times$. An element $\gamma \in \Gamma_{j,c}$ acts on $\mathcal{L}_j$ by $(\gamma L)_v = \varphi_{c,v}^{-1}\rho_v(\gamma)\varphi_{c,v}L_v$, so that $N_j(c,\gamma L) = \gamma N_j(c,L)$, and on $\mathcal{X}_j$ by

$$\rho_{j,c}(\gamma) = \chi_{j,c}(\gamma)\bigotimes_{v,s}\det(\gamma_{v,s})^{-\lambda^\natural(N_j)}\left(F \mapsto F(\gamma_{v,s}^{-1}\cdot)\right), \qquad \gamma_{v,s} = \sigma^s\left(\rho_v(\varphi_{c,v}^{-1}\gamma\varphi_{c,v})\right).$$

A *form* of level j is a function F on $C_j \times \mathcal{L}_j$ with values in $\mathcal{X}_j$ such that $F(c,\gamma L) = \rho_{j,c}(\gamma)F(c,L)$ for $\gamma \in \Gamma_{j,c}$. The form is *integral* when $F(c,L) \in H_j(L)$ for all (c,L). We write $\mathcal{F}_j$ and Int_j as in Definition 10.1.

In the normalization of Proposition 7.9(iii) the character $\chi_{j,c}$ is trivial. A character with values in $\mathcal{O}^\times$ would change nothing below. Under ι and (11.1), $\rho_{j,c}$ is the action of Section 7.3 on the polynomial models of the weights λ_τ, transported at the odd positions by the isomorphism between the model of a weight with the action $g \mapsto (g^{-1})^{\mathsf{T}}$ and the model of the dual weight with the action g; at an odd position $\iota\tau_{v,s}(\gamma) = \sigma^s\rho_v(\bar{\gamma})$ and $\bar{\gamma} = (\gamma^{-1})^{\mathsf{T}}$ for a unitary matrix in an orthonormal basis.

Since $\gamma_{v,s}$ maps a basis of $(L_v)_s$ to a basis of $((\gamma L)_v)_s$, the action $\rho_{j,c}(\gamma)$ maps $H_j(L)$ onto $H_j(\gamma L)$. This is property (b) of Definition 10.1. The forms of level j are the vector-valued automorphic forms of weight $(\lambda_\tau)_{\tau\in\Phi}$ on the definite unitary group of rank N_j with the level structure fixed by the components, written in the coordinates of the orbits; under ι the polynomial model at (v,s) is the polynomial model of the archimedean weight at $\tau_{v,s}$ by (11.1).

Lemma 11.6 (Finiteness)**.** *For every j and c, the set $\Gamma_{j,c}\backslash\mathcal{L}_j$ is finite and every stabilizer $\Gamma_{j,c,L}$ is finite.*

Proof. The stabilizer of L is a subgroup of the group of $\mathcal{O}_E$-linear automorphisms of $N_j(c,L)$ that preserve h_{N_j}. This group preserves the positive definite quadratic form $\mathrm{Tr}_{E/\mathbb{Q}}\, h_{N_j}(x,x)$ on the lattice $N_j(c,L) \cong \mathbb{Z}^{2fN_j}$, $2f = [E:\mathbb{Q}]$, so it is a discrete subgroup of a compact orthogonal group, hence finite. The orbits of $\Gamma_{j,c}$ on $\mathcal{L}_j$ correspond to the classes in the genus of $N_j(c,L_0)$ for the level group, and the number of such classes is finite by the finiteness of class numbers of a reductive group over a number field [11, Theorem 5.1], applied to the unitary group of h_{N_j} and the open compact subgroup that the components fix. □

11.1.5 Hecke operators

Definition 11.7. For an orbit v, a level j and $0 \le i \le N_j$, the *raw Hecke operator* $T_{v,i}^{\mathrm{raw},(j)}$ on $\mathcal{F}_j$ is

$$\left(T_{v,i}^{\mathrm{raw},(j)}F\right)(c,L) = \sum_{L_1} F(c, L[v \leftarrow L_1]),$$

the sum over the lattices L_1 with (L_1, L_v) an edge of type i, where $L[v \leftarrow L_1]$ is L with its component at v replaced by L_1. For $j \ge 1$ and $0 \le i \le r$ put

$$N_v'(j,i) = \sum_{s \text{ even}}\sum_{a\le i}\lambda_{v,s}^{(0)}(a) + i\sum_{s \text{ odd}} w_j(v,s), \tag{11.3}$$

and $T_{v,i}^{(j)} = p^{N_v'(j,i)}T_{v,i}^{\mathrm{raw},(j)}$. Put $A_v = \sum_s\sum_{a\le r}\lambda_{v,s}^{\natural,(0)}(a)$ and $y_v = p^{A_v}T_{v,r}^{\mathrm{raw},(0)}$.

For $j \geq 1$ and $i \leq r$,

$$N'_v(j,i) \geq \sum_s \sum_{a \leq i} \lambda_{v,s}^{\natural,(j)}(a), \tag{11.4}$$

with equality when $i \leq t+1$. Indeed at an even position the first $i \leq r \leq k_1+1$ entries of $\lambda_{v,s}^{(j)}$ are those of $\lambda_{v,s}^{(0)}$, and at an odd position the first i entries of the dual weight are $w_j, w_j, \ldots$ followed, after $t+1$ entries, by $w_{j-1}, \ldots,$ each at most w_j by (W2).

Lemma 11.8. *For every orbit v, level j and $0 \leq i \leq r$, the operator $T_{v,i}^{(j)}$ maps $\mathcal{F}_j$ to $\mathcal{F}_j$ and Int_j into Int_j. The operators y_v commute with each other and map Int_0 into itself. Thus (D2) of Definition 10.4 holds for the operators y_v.*

Proof. The family of edges at v is equivariant for $\Gamma_{j,c}$, so the operators map forms to forms, as after Definition 10.1. If $F \in \mathrm{Int}_j$ and (L_1, L_v) is an edge of type i, then $F(c, L[v \leftarrow L_1]) \in H_j(L[v \leftarrow L_1])$, and Lemma 11.4, applied at each position of v and tensored with the unchanged factors at the other orbits, gives $p^{\sum_s \sum_{a \leq i} \lambda_{v,s}^{\natural,(j)}(a)} H_j(L[v \leftarrow L_1]) \subset H_j(L)$. By (11.4) the same holds with the exponent $N'_v(j,i)$ for $j \geq 1$, and for $j = 0$, $i = r$ the exponent is A_v, which is the statement for y_v. Operators at distinct orbits act on distinct components of the tuple, so they commute. □

11.2 The lifts as class sums

11.2.1 The kernel series of a pair of tuples

For $(c, L) \in C_j \times \mathcal{L}_j$ and $(c', L') \in C_{j+1} \times \mathcal{L}_{j+1}$ put $N = N_j(c, L)$ and $N' = N_{j+1}(c', L')$. Section 7.3 attaches to (N, N'), to the level c_{lev}, to the finite set S of primes of E, and to a function $\Xi = \Xi_{(c,L),(c',L')}$ on $M^A_{c_{\mathrm{lev}}}(N, N')$ that is periodic under $M^A \cap M^B$, the kernel $\theta_{N,N'}[\Xi]$ of (7.3). The function Ξ depends on the components c, c' and on the frames at the bad primes, and not on L_v or L'_v (Section 11.4). We use the coordinates of the Fock space $V = \prod_{\tau \in \Phi} M_{N_{j+1} \times N_j}(\mathbb{C})$ in which the entry (b, a) at $\tau_{v,s}$ is

$$z_{\tau_{v,s},b,a} = u\, \varepsilon^{-1}\, X_{v,s,a,b} \qquad \text{after the frames } \varphi_{c',v} \text{ and } \varphi_{c,v}, \tag{11.5}$$

with u and ε as in Proposition 8.4 and Proposition 8.11, and with the coordinates at the elements of Φ outside the orbits set to zero. Put $Z = u\varepsilon^{-1}(X_{v,s,a,b})_{b,a}$. In the two frames,

$$(z_{\tau_{v,s},b,a})_{b,a} = \begin{cases} \sigma^s \rho_v(\varphi_{c',v})\, Z\, \sigma^s \rho_v(\varphi_{c,v})^{-1}, & s \text{ even}, \\ \left(\sigma^s \rho_v(\varphi_{c,v})\, Z^{\mathsf{T}}\, \sigma^s \rho_v(\varphi_{c',v})^{-1}\right)^{\mathsf{T}}, & s \text{ odd}. \end{cases}$$

The second formula uses the frames on the conjugate component.

Definition 11.9. The *kernel series* of the pair is the power series

$$\mathfrak{K}_j(c, L; c', L') = s_j(c, L; c', L') \cdot \iota\Big(\text{Taylor expansion at 0 of } \chi_F(N, N')^{-1}\, \theta_{N,N'}[\Xi]\Big) \in \mathbb{C}_p[[X_{v,s,a,b}]]$$

in the variables of (11.5), where χ_F is the character factor of Section 7.3 and s_j is the scalar normalization of the step kernel (7.4), namely the frame factor $\mu(N, \mathrm{fr})/\mu(N', \mathrm{fr}')$ times the

determinant twists of the jet operator and the factor $p^{\sum_v \mu_v (j+1)(\operatorname{len} L'_v - \operatorname{len} L_v)}$. Here len is the length of Section 10.4. The *step map* $\kappa_j(c, L; c', L') \colon \mathcal{X}_j \to \mathcal{X}_{j+1}$ is

$$\kappa_j(c, L; c', L')(G) = p^{\sum_v \mu_v (j+1)(\operatorname{len} L_v - \operatorname{len} L'_v)} \, \mathcal{J}_j\big(G, \mathfrak{K}_j(c, L; c', L')\big),$$

where $\mathcal{J}_j$ is the jet operator of (7.4) written in the variables $X_{v,s,a,b}$, that is, the pairing of the Taylor coefficients of the series (with the factorials of Section 9.3) against the polynomial obtained from G by the substitution of Section 11.6.

The two powers of p in Definition 11.9 cancel in κ_j. They are separated because the kernel series, and not the step map, is the object to which Theorem 8.21 and Proposition 11.28 apply.

11.2.2 The lifts

Definition 11.10. Fix for each j a set $R_j \subset C_j \times \mathcal{L}_j$ that contains exactly one pair (c, L) from each orbit of $\Gamma_{j,c}$ on $\{c\} \times \mathcal{L}_j$, for every c. The *lift* $K_j \colon \mathcal{F}_j \to \mathcal{F}_{j+1}$ is

$$(K_j F)(c', L') = c_K \sum_{(c,L) \in R_j} \frac{1}{|\Gamma_{j,c,L}|} \, \kappa_j(c, L; c', L')\big(F(c, L)\big),$$

with $c_K \in \mathcal{O}^\times$ the normalizing unit of the lift.

The sum is finite by Lemma 11.6. The value $(K_j F)(c', L')$ does not depend on the choice of R_j, and $K_j F$ is a form, by the covariance of the step maps:

Lemma 11.11 (Covariance)**.** *For $\gamma \in \Gamma_{j,c}$ and $\gamma' \in \Gamma_{j+1,c'}$,*

$$\kappa_j(c, \gamma L; c', \gamma' L') \circ \rho_{j,c}(\gamma) = \rho_{j+1,c'}(\gamma') \circ \kappa_j(c, L; c', L').$$

Proof. The two powers of p in Definition 11.9 cancel, so $\kappa_j(c, L; c', L')$ is the map defined by the step kernel (7.4) of the pair $(N_j(c, L), N_{j+1}(c', L'))$ with its frames and data function, written in the coordinates (11.5). Proposition 7.9(iii) gives the identity for the step kernels of the moved data, with the moved data function $\Xi \circ \mathrm{Ad}(\gamma, \gamma')^{-1}$. The data function of $(c, \gamma L; c', \gamma' L')$ is this moved function, because the frames at the bad primes are moved by γ and γ' together with the lattices. The coordinates (11.5) for $(c, \gamma L)$ and (c, L) use the same frames $\varphi_{c,v}$, and in these coordinates the substitution $D \mapsto hDg$ of the proof of Proposition 7.9(iii) is the action $\rho_{j,c}(\gamma)$, $\rho_{j+1,c'}(\gamma')$ on the models, by the transport described after Definition 11.5. □

Lemma 11.11 with $\gamma' = 1$ is the covariance $\kappa_j(c, \gamma L; c', L') \circ \gamma = \kappa_j(c, L; c', L')$ of (S2) in Definition 10.12, where γ acts on $\mathcal{X}_j$ by $\rho_{j,c}(\gamma)$.

11.2.3 Denominators: the condition (D1)

Definition 11.12. For $n \geq 1$ put

$$M(n) = \prod_{\ell \text{ prime}} \ell^{\sum_{k \geq 0} \lfloor n/(\ell^k(\ell-1)) \rfloor},$$

a finite product since the exponent vanishes for $\ell > n + 1$. Put $D_\Gamma = M(2fN_m)$ with $2f = [E : \mathbb{Q}]$.

Theorem 11.13 (Minkowski)**.** *The order of every finite subgroup of* $\mathrm{GL}_n(\mathbb{Z})$ *divides* $M(n)$*.*

Proposition 11.14 ((D1) for the true lifts)**.** *For every* $j < m$ *the lift* K_j *satisfies (D1) of Definition 10.1 with* κ_j *of Definition 11.9, with the family* $(c_x, L_x) = R_j$*, with* $w_x = c_K^{-1}|\Gamma_{j,c,L}|$*, and with*

$$d_j = \operatorname{ord}_p D_\Gamma = \sum_{k\geq 0} \Big\lfloor \frac{2fN_m}{p^k(p-1)} \Big\rfloor.$$

In particular $d_j = 0$ *when* $p > 2fN_m + 1$*.*

Proof. The formula of Definition 11.10 is the formula of (D1) with these choices. The group $\Gamma_{j,c,L}$ is finite by Lemma 11.6, and it acts faithfully on $N_j(c, L)$, a free $\mathbb{Z}$-module of rank $2fN_j \leq 2fN_m$. Theorem 11.13 gives $|\Gamma_{j,c,L}| \mid M(2fN_j)$, and $M(2fN_j) \mid M(2fN_m)$ because the exponents are nondecreasing in n. Since c_K is a unit, $\operatorname{ord}(w_x) = \operatorname{ord}_p|\Gamma_{j,c,L}| \leq \operatorname{ord}_p D_\Gamma$. □

Remark 11.15. In the formulation of Definition 11.10 all lifts are class sums of the same form, and the separate treatment is the bookkeeping that identifies the classes of the datum with the orbits of $\Gamma_{j,c}$ on $\mathcal{L}_j$, with equal stabilizers.

11.3 The kernel as a finite Fourier transform

This section proves that it coincides with the kernel (7.3) of Section 7.3. We use the notation of Section 7.3: $\Lambda = \Lambda(N, N')$, $A = A_c(N, N')$, $B = B_c(N, N')$, the semicharacter α, the operators U_y and $e = e(N, N')$.

Lemma 11.16. $A \cap B \subset \Lambda$.

Proof. Outside $S \cup \bar{S}$ the three lattices M_c^A, M_c^B, M have the same localizations. At $P \in S$, $(M_c^A)_P \cap (M_c^B)_P = c^{-1}M_P \cap cM_P = cM_P \subset M_P$, and at $\bar{P}$ the intersection is $cM_{\bar{P}} \subset M_{\bar{P}}$. Hence $M_c^A \cap M_c^B \subset M$, and applying ι gives the claim. □

Definition 11.17. Let S_A represent $(\Lambda \cap A)/(\Lambda \cap A \cap B)$, S_B represent $B/(B \cap \Lambda)$, and S_X represent $A/(A \cap \Lambda)$. For a function Φ on A that is periodic under $A \cap B$, put

$$\widehat{\Phi}(y) = \frac{1}{\#S_A} \sum_{t \in S_A} \Phi(t)\, e^{2\pi i E_H(t,y)} \qquad (y \in B),$$

$$\xi_S[\Phi] = \sum_{y \in S_B} \widehat{\Phi}(y)\, \alpha(y)\, U_y e, \qquad \xi_L[\Phi] = \sum_{x \in S_X} \alpha(x)\, U_x\big(\xi_S[\Phi(\cdot - x)]\big).$$

Proposition 11.18. *The summands in Definition 11.17 do not depend on the representatives, and for every* Φ *on* A *periodic under* $A \cap B$*,*

$$\xi_S[\Phi] = \sum_{t \in S_A} \Phi(t)\, \alpha(t)\, U_{-t}\, G, \qquad \xi_L[\Phi] = \sum_{x \in A/(A\cap B)} \Phi(x)\, \alpha(x)\, U_{-x}\, G,$$

with $G = G_c(N, N') = [\Lambda \cap A : \Lambda \cap A \cap B]^{-1} \sum_{y \in B/(\Lambda\cap B)} \alpha(y) U_y e$*. In particular, for* $\Phi = \Xi \circ \iota^{-1}$ *the function* $\xi_L[\Phi]$ *is the kernel* $\theta_{N,N'}[\Xi]$ *of* (7.3)*.*

Proof. The operators satisfy $U_x U_y = e^{-\pi i E_H(x,y)} U_{x+y}$, hence $U_x U_y = e^{-2\pi i E_H(x,y)} U_y U_x$. Since $e \in \Theta(\Lambda)$, $\alpha(\lambda) U_\lambda e = e$ for $\lambda \in \Lambda$.

The first identity. Let $t \in \Lambda \cap A$. For $y \in B$,

$$\alpha(-t) U_{-t} U_y e = e^{-2\pi i E_H(-t,y)} U_y \big(\alpha(-t) U_{-t} e\big) = e^{2\pi i E_H(t,y)} U_y e.$$

Summing over $y \in S_B$ with the weights $\alpha(y)$ gives $\alpha(t) U_{-t} G = [\Lambda \cap A : \Lambda \cap A \cap B]^{-1} \sum_{y \in S_B} e^{2\pi i E_H(t,y)} \alpha(y) U_y e$, where $\alpha(-t) = \alpha(t)$ because α depends on $q(t) = q(-t)$. Multiplying by $\Phi(t)$ and summing over $t \in S_A$ gives $\xi_S[\Phi]$, since $\#S_A = [\Lambda \cap A : \Lambda \cap A \cap B]$. The summand $\Phi(t)\alpha(t) U_{-t} G$ depends on t modulo $\Lambda \cap A \cap B$: for $m \in A \cap B$ the term changes by $\alpha(t+m) U_{-t-m} G = \alpha(t) U_{-t} G$, as shown in Section 7.3 after (7.2), and $\Phi(t+m) = \Phi(t)$. The term $e^{2\pi i E_H(t,y)} \alpha(y) U_y e$ depends on y only modulo $B \cap \Lambda$: for $\lambda \in B \cap \Lambda$, $\alpha(y+\lambda) U_{y+\lambda} e = e^{2\pi i E_H(y,\lambda)} \alpha(y) U_y \, \alpha(\lambda) U_\lambda e = \alpha(y) U_y e$, because $E_H(y,\lambda) \in \mathbb{Z}$ for y, λ in the self-dual lattice B, and $e^{2\pi i E_H(t,\lambda)} = 1$ because $t, \lambda \in \Lambda$.

The second identity. By the first identity,

$$\xi_L[\Phi] = \sum_{x \in S_X} \sum_{t \in S_A} \Phi(t-x)\, \alpha(x)\alpha(t)\, U_x U_{-t}\, G.$$

For $x \in A$ and $t \in \Lambda \cap A \subset A$, the semicharacter law on the admissible lattice A gives $\alpha(t-x) = \alpha(t)\alpha(x) e^{\pi i E_H(t,-x)}$, and $U_x U_{-t} = e^{-\pi i E_H(x,-t)} U_{x-t} = e^{\pi i E_H(x,t)} U_{-(t-x)}$. Since $E_H(t,-x) = E_H(x,t)$, we get $\alpha(x)\alpha(t) U_x U_{-t} = \alpha(t-x) U_{-(t-x)}$. So $\xi_L[\Phi] = \sum_{x,t} \Phi(u)\alpha(u) U_{-u} G$ with $u = t - x$. The map $(x,t) \mapsto t - x$ from $S_X \times S_A$ to $A/(A \cap \Lambda \cap B)$ is a bijection, because S_X represents $A/(A \cap \Lambda)$ and S_A represents $(A \cap \Lambda)/(A \cap \Lambda \cap B)$. By Lemma 11.16, $A \cap \Lambda \cap B = A \cap B$, and the summand $\Phi(u)\alpha(u) U_{-u} G$ depends on u modulo $A \cap B$. This gives the second identity, which is (7.3) with (7.2). □

11.4 The kernels at a split orbit

Fix an orbit v with place w of E, a level $j < m$, and components $c \in C_j$, $c' \in C_{j+1}$. All pairs below have these components and differ only in their components at v.

11.4.1 Path independence for lattices that vary at one split place

Lemma 11.19. *Let (N_i, N_i') for $i = 1, 2, 3$ be pairs of self-dual lattices of ranks N_j, N_{j+1}, and suppose that each of the three sequences (N_i) and (N_i') takes at most two values at every place of E except w and $\bar{w}$. Then $\mu(\Lambda(N_1, N_1'), \Lambda(N_2, N_2'), \Lambda(N_3, N_3')) = 1$.*

Proof. By Lemma 7.3(iv) it suffices to show $\sum_Q f > 0$, and the proof of Lemma 7.5 reduces this to $\sum_{Q(\ell)} f > 0$ for every prime ℓ. For $\ell \neq p$ the sequences take at most two values at ℓ, and the last two paragraphs of the proof of Lemma 7.5 use their hypothesis only at the prime ℓ, so they give $\sum_{Q(\ell)} f > 0$. Let $\ell = p$. The algebra $E \otimes \mathbb{Q}_p$ is the product of the completions at the places of E^+ above p, and the space V_p, the lattices $M_{i,p}$, the group $Q(p)$, the function f_p and the biadditive form b decompose as orthogonal products accordingly, because the pairing $\langle x, y \rangle = \mathrm{tr}(xy^*)$ pairs the component at a place of E^+ only with the same component. So $\sum_{Q(p)} f$ is the product of the sums over the components. At a place $v' \neq v$ above p the lattices take at most two values, and the argument of Lemma 7.5 with the idempotent of Lemma 7.4 applies. At v the algebra $E \otimes_{E^+} E_v^+$ is $E_w \times E_{\bar{w}}$, and the idempotent $\epsilon = (1, 0)$ satisfies $\epsilon^* = 1 - \epsilon$ and

preserves every $\mathcal{O}_E$-lattice, in particular every $M_{i,p}$. The argument of the last paragraph of the proof of Lemma 7.5, with this ϵ acting on the left, shows that the sum over the v-component is positive. □

11.4.2 The Schrödinger model at the orbit

For a self-dual pair (N, N') write $Y_v(N, N') = \mathrm{Hom}_{W_v}\left(\rho_v(N)_w, \rho_v(N')_w\right) \subset M_{N_{j+1}\times N_j}(Q_v)$ for the completion at w of $\mathrm{Hom}(N, N')$ under ρ_v, and vol for the Haar measure on $M_{N_{j+1}\times N_j}(Q_v)$ that gives $M_{N_{j+1}\times N_j}(W_v)$ volume one.

Proposition 11.20. *Let (N_0, N_0') be a self-dual pair, and let (N_i, N_i'), $i \in I$ finite, be self-dual pairs that agree with (N_0, N_0') at all places of E except w and $\bar{w}$. Let $c_i \in \mathbb{C}$ with $\sum_i c_i \mathbf{1}_{Y_v(N_i,N_i')} = 0$ as a function on $M_{N_{j+1}\times N_j}(Q_v)$. Then*

$$\sum_{i\in I} c_i \ \mathrm{vol}\left(Y_v(N_i, N_i')\right)^{1/2} \theta_{N_i,N_i'} = 0,$$

and the same holds with $\theta_{N_i,N_i'}$ replaced by $\chi_F(N_i, N_i')^{-1}\theta_{N_i,N_i'}[\Xi]$ for a data function Ξ that does not depend on i.

Proof. Put $M_i = \mathfrak{a}\,\mathrm{Hom}(N_i, N_i')$ and $\Lambda_i = \iota(M_i)$. The ideal $\mathfrak{a} = \mathfrak{c}\mathfrak{b}^{-1}$ is prime to p, because $\mathfrak{c}$ is supported on primes split in K and $\mathfrak{b}$ divides the relative different of E/K, which is prime to p. So $(M_i)_w = Y_v(N_i, N_i')$ under ρ_v. The w-component and the $\bar{w}$-component of $\mathrm{Hom}_E(E^{N_j}, E^{N_{j+1}}) \otimes \mathbb{Q}_p$ are isotropic for $\langle\ ,\ \rangle$, by the idempotent ϵ of the proof of Lemma 11.19, and they are in perfect duality. By self-duality, $(M_i)_{\bar{w}}$ is the annihilator of $(M_i)_w$.

The model. Choose $k \geq 0$ with $p^k(M_0)_w \subset (M_i)_w \subset p^{-k}(M_0)_w$ for all $i \in I$. Let Λ_X, Λ_Y, Λ_- be the lattices that agree with Λ_0 at all places except $w, \bar{w}$, and have components $(p^{-k}(M_0)_w, p^k(M_0)_{\bar{w}})$, $(p^k(M_0)_w, p^{-k}(M_0)_{\bar{w}})$, $(p^k(M_0)_w, p^k(M_0)_{\bar{w}})$ there. The computation of Lemma 7.2(ii) applies to Λ_X and Λ_Y without change, because the components at w and $\bar{w}$ are still annihilators of each other: both are admissible. Put $X = \Lambda_X/\Lambda_-$ and $Y = \Lambda_Y/\Lambda_-$. The pairing $(x, y) \mapsto e^{2\pi i E_H(x,y)}$ identifies Y with the dual of X, and every Λ_i satisfies $\Lambda_i \supset \Lambda_-$ and $\Lambda_i/\Lambda_- = X_i \oplus Y_i$ with $X_i = (\Lambda_i \cap \Lambda_X)/\Lambda_-$ and $Y_i = (\Lambda_i \cap \Lambda_Y)/\Lambda_-$ the annihilator of X_i. Let G_0 span the line $\Theta(\Lambda_Y)$ (W1), and for a function f on X put

$$\mathcal{S}(f) = \sum_{x\in X} f(x)\, \alpha(\tilde{x})\, U_{-\tilde{x}}\, G_0,$$

with $\tilde{x} \in \Lambda_X$ any lift of x; the summand does not depend on the lift, by the computation in Section 7.3 after (7.2), since $\Lambda_X \cap \Lambda_Y = \Lambda_-$ and $\alpha(\lambda)U_\lambda G_0 = G_0$ for $\lambda \in \Lambda_-$. The vectors $\alpha(\tilde{x})U_{-\tilde{x}}G_0$, $x \in X$, are linearly independent, by the argument at the start of the proof of Lemma 7.3, because $\alpha(\lambda)U_\lambda$ for $\lambda \in \Lambda_Y$ multiplies them by the distinct characters $\lambda \mapsto e^{\pm 2\pi i E_H(\lambda,\tilde{x})}$ of Y. So $\mathcal{S}$ is injective. The semicharacter law on Λ_X gives, as in the proof of Proposition 11.18, $\alpha(\tilde{a})U_{\tilde{a}}\,\mathcal{S}(f) = \mathcal{S}\big(f(\cdot + a)\big)$ for $a \in X$, and for $b \in Y$, $\alpha(\tilde{b})U_{\tilde{b}}\,\mathcal{S}(f) = \mathcal{S}\big(e^{2\pi i E_H(\tilde{b},\cdot)}f\big)$, where $\tilde{b} \in \Lambda_Y$ lifts b.

The lines. $\mathcal{S}(\mathbf{1}_{X_i})$ is fixed by $\alpha(z)U_z$ for $z \in \Lambda_i$: for $z \in \Lambda_-$ this is the invariance of G_0 together with $E_H(\Lambda_-, \Lambda_X) \subset \mathbb{Z}$; for z lifting an element of X_i it is the translation invariance of $\mathbf{1}_{X_i}$; for z lifting an element of Y_i the character $e^{2\pi i E_H(\tilde{z},\cdot)}$ is trivial on X_i. Since $\mathcal{S}(\mathbf{1}_{X_i}) \neq 0$ and $\Theta(\Lambda_i)$ is a line, $\theta_{N_i,N_i'} = k_i\,\mathcal{S}(\mathbf{1}_{X_i})$ with $k_i \in \mathbb{C}^\times$.

The intertwiners. Let $i, l \in I$, and write $X_{il} = X_i \cap X_l$, $Y_{il} = Y_i \cap Y_l$. Representatives of $\Lambda_l/(\Lambda_i \cap \Lambda_l)$ are the sums $\tilde{a} + \tilde{b}$ with $a \in X_l/X_{il}$ and $b \in Y_l/Y_{il}$, and $\alpha(\tilde{a} + \tilde{b})U_{\tilde{a}+\tilde{b}} =$

$\alpha(\tilde{a})U_{\tilde{a}}\,\alpha(\tilde{b})U_{\tilde{b}}$ by the semicharacter law on Λ_l. The sum $\sum_{b\in Y_l/Y_{il}} e^{2\pi i E_H(\tilde{b},x)}$ for $x\in X_i$ equals $[Y_l:Y_{il}]$ if $x\in X_l$ and 0 otherwise, since Y_{il} is the annihilator of X_i+X_l and X_l is the annihilator of Y_l. So the sum over b maps $\mathcal{S}(\mathbf{1}_{X_i})$ to $[Y_l:Y_{il}]\,\mathcal{S}(\mathbf{1}_{X_{il}})$, and the sum over a of the translates of $\mathbf{1}_{X_{il}}$ is $\mathbf{1}_{X_l}$. With $n=[\Lambda_l:\Lambda_i\cap\Lambda_l]=[X_l:X_{il}][Y_l:Y_{il}]$ and $[Y_l:Y_{il}]=[X_i+X_l:X_l]=[X_i:X_{il}]$,

$$I_{\Lambda_i\to\Lambda_l}\mathcal{S}(\mathbf{1}_{X_i})=\Big(\frac{[X_i:X_{il}]}{[X_l:X_{il}]}\Big)^{1/2}\mathcal{S}(\mathbf{1}_{X_l})=\Big(\frac{\operatorname{vol}Y_v(N_i,N_i')}{\operatorname{vol}Y_v(N_l,N_l')}\Big)^{1/2}\mathcal{S}(\mathbf{1}_{X_l}).$$

By definition

$$\theta_{N,N'}=I_{(\mathrm{std},\mathrm{std})\to(N,N')}\theta_0.$$

Lemma 11.19 applied to the triple $(\mathrm{std},(N_i,N_i'),(N_l,N_l'))$, which takes at most two values away from $w,\bar{w}$, gives $I_{\Lambda_i\to\Lambda_l}\theta_{N_i,N_i'}=\theta_{N_l,N_l'}$. Comparing, $k_l\operatorname{vol}(Y_v(N_l,N_l'))^{-1/2}=k_i\operatorname{vol}(Y_v(N_i,N_i'))^{-1/2}=:\kappa$. Hence $\operatorname{vol}(Y_v(N_i,N_i'))^{1/2}\theta_{N_i,N_i'}=\kappa\,\mathcal{S}(\mathbf{1}_{X_i})$ for all i, and the first claim follows from the linearity of $\mathcal{S}$, since $\mathbf{1}_{X_i}$ is the reduction of $\mathbf{1}_{Y_v(N_i,N_i')}$.

For the second claim, the lattices A_c, B_c and Λ of a pair agree at $w,\bar{w}$, and they agree among the pairs away from $w,\bar{w}$, because S consists of primes not above p. By Proposition 11.18, $\theta_{N_i,N_i'}[\Xi]$ is a finite combination of operators $\alpha(x)U_{-x}$ and $\alpha(y)U_y$ applied to $e(N_i,N_i')=\chi_F(N_i,N_i')\theta_{N_i,N_i'}$, with x running over $A/(A\cap B)$ and y over $B/(B\cap\Lambda)$, divided by $[\Lambda\cap A:\Lambda\cap A\cap B]$. These quotients only see the places of $S\cup\bar{S}$, so the representatives may be chosen in Λ_- at $w,\bar{w}$, hence in every Λ_i, and the same representatives and the same index serve all i. So $\chi_F(N_i,N_i')^{-1}\theta_{N_i,N_i'}[\Xi]=T\theta_{N_i,N_i'}$ for one linear operator T, and the first claim applies. □

Lemma 11.21. *For W_v-lattices $\Lambda\subset Q_v^N$ and $\Lambda'\subset Q_v^{N'}$, $\operatorname{vol}\mathrm{Hom}(\Lambda,\Lambda')=q^{2(N\operatorname{len}\Lambda'-N'\operatorname{len}\Lambda)}$, where $q=p^{n_v'/2}$.*

Proof. If g,g' are bases, $\mathrm{Hom}(\Lambda,\Lambda')=g'M_{N'\times N}(W_v)g^{-1}$, whose volume is $|\det g'|_v^N|\det g|_v^{-N'}$ with $|p|_v=p^{-n_v'}=q^{-2}$. Since $\operatorname{len}\Lambda=-\operatorname{ord}\det g$, $|\det g|_v=q^{2\operatorname{len}\Lambda}$. □

11.4.3 The form (S1) to (S3) of the kernels

Hypothesis 11.22 (normalization, (N1))**.** For every orbit v and level $j<m$ there are $a_v,a_v'\in\mathcal{O}^\times$ such that for all pairs $(c,L;c',L')$, $(c,L_2;c',L_2')$ that differ only at v,

$$\frac{\phi(c,L;c',L')}{\phi(c,L_2;c',L_2')}=p^{\mu_v(j+1)\left(\operatorname{len}L_v-\operatorname{len}L_{2,v}-\operatorname{len}L_v'+\operatorname{len}L_{2,v}'\right)}u,\qquad u\in\mathcal{O}^\times,$$

where ϕ is the frame factor $\mu(N,\mathrm{fr})/\mu(N',\mathrm{fr}')$ of (7.4) times its determinant twists, and the unit u is $a_v^{\operatorname{len}L_v-\operatorname{len}L_{2,v}}a_v'^{\operatorname{len}L_v'-\operatorname{len}L_{2,v}'}$ divided by the corresponding ratio of the factors χ_F of Lemma 11.30. Equivalently, the scalar s_j of Definition 11.9 satisfies $s_j(c,L;c',L')/s_j(c,L_2;c',L_2')=a_v^{\operatorname{len}L_v-\operatorname{len}L_{2,v}}a_v'^{\operatorname{len}L_v'-\operatorname{len}L_{2,v}'}$.

We use the notation of Subsection 11.1. Let $\mathbb{A}_{E,f}$ be the finite adeles of E, and let

$$U^1=\{x\in\mathbb{A}_{E,f}^\times:\ x\bar{x}=1\},$$

where the bar is the action of complex conjugation of E on $\mathbb{A}_{E,f}$. The group $E^1=\{z\in E^\times:z\bar{z}=1\}$ embeds diagonally in U^1. For a finite place y of E we write ord_y for its normalized valuation, extended to $\mathbb{A}_{E,f}^\times$ through the component at y. For $x\in U^1$ the fractional ideal $(x)=\prod_y\mathfrak{P}_y^{\mathrm{ord}_y(x)}$ has norm one. We write ord_p for the valuation of $\mathbb{C}_p$ with $\mathrm{ord}_p(p)=1$.

Definition 11.23 (step characters and the frame factor)**.** Let $0 \le j < m$.

(i) A *step character* of level $j+1$ is a continuous homomorphism $\mu_{j+1}\colon U^1 \to \mathbb{C}^\times$ with algebraic values and the following two properties. First, $\mu_{j+1}(\kappa) = 1$ for every orbit v and every $\kappa \in U^1$ that is a unit at w and at $\bar{w}$ and equals 1 at every other finite place. Second, there are integers ε_τ, $\tau \in \Phi$, and an integer c_j such that

$$\mu_{j+1}(z) = \prod_{\tau\in\Phi} \tau(z)^{\varepsilon_\tau} \quad (z \in E^1), \qquad w_{j+1}(v,s) = \varepsilon_{\tau_{v,s}} - c_j \tag{11.6}$$

for every orbit v and every position s of v.

(ii) Assume that every frame $\varphi_{c,v}$ of Definition 11.5 has $\operatorname{ord}\det\varphi_{c,v} = 0$. For each component $c \in C_j$ fix an element g_c of the unitary group of h_{N_j} over $\mathbb{A}_{E,f}$, and for each orbit v and each W_v-lattice $L_v \subset Q_v^{N_j}$ an element $h_{c,v}(L_v)$ of that group which is trivial at every finite place of E^+ other than v, such that

$$N_j(c,L) = g_c \prod_v h_{c,v}(L_v)\cdot \mathcal{O}_E^{N_j}.$$

Such elements exist because the unitary group at a split place v is $\mathrm{GL}_{N_j}(E_w)$, which acts transitively on the W_v-lattices of $E_w^{N_j}$. The *determinant idele* of (c,L) is $d_j(c,L) = \det g_c \prod_v \det h_{c,v}(L_v) \in U^1$.

(iii) With $a_j = \lfloor j(t+1)/2 \rfloor$, $N = N_j(c,L)$ and $N' = N_{j+1}(c',L')$, the scalar normalization of the step kernel (7.4) is

$$s_j(c,L;c',L') = p^{\sum_v \mu_v(j+1)(\operatorname{len} L_v' - \operatorname{len} L_v)}\, \iota\Big(\frac{\mu_{j+1}\big(d_{j+1}(c',L')\big)}{\mu_{j+1}\big(d_j(c,L)\big)}\, \chi(\mathfrak{d}_N)^{-a_j}\, \chi(\mathfrak{d}_{N'})^{a_{j+1}}\Big).$$

The frame factor of (7.4) is $\mu(N,\mathrm{fr}) = \mu_{j+1}(d_j(c,L))^{-1}$, and the determinant twists are the two powers of χ.

The ideal $\mathfrak{d}_N$ of Section 7.3 is the ideal $(d_j(c,L))$, because locally at every finite place the matrix $g_c \prod_v h_{c,v}(L_v)$ maps the standard lattice onto N. In particular $\mathfrak{d}_N$ has norm one.

Lemma 11.24. *Let v be an orbit. The idele π_v with component p at w, component p^{-1} at $\bar{w}$ and component 1 at every other finite place lies in U^1, and $(\pi_v) = \mathfrak{P}_w\mathfrak{P}_{\bar{w}}^{-1}$.*

Proof. Complex conjugation exchanges the components at w and $\bar{w}$, and p is rational, so the component of $\pi_v\bar{\pi}_v$ at w is $p\cdot p^{-1} = 1$, and the same holds at $\bar{w}$. The place w is unramified over p, so $\operatorname{ord}_w(p) = 1$. □

Lemma 11.25. *Let μ_{j+1} be a step character of level $j+1$ and v an orbit. Then* $\operatorname{ord}_p \iota\mu_{j+1}(\pi_v) = \mu_v(j+1)$.

Proof. Let h be the class number of E and $\beta \in E^\times$ a generator of $(\mathfrak{P}_w\mathfrak{P}_{\bar{w}}^{-1})^h$. Then $\bar{\beta}$ generates $(\mathfrak{P}_{\bar{w}}\mathfrak{P}_w^{-1})^h$, so $z = \beta/\bar{\beta}$ lies in E^1 and $z\mathcal{O}_E = (\mathfrak{P}_w\mathfrak{P}_{\bar{w}}^{-1})^{2h}$. By Lemma 11.24 the idele $k = \pi_v^{2h}z^{-1}$ lies in U^1 and is a unit at every finite place. A continuous homomorphism from the compact

totally disconnected group $U^1 \cap \prod_y \mathcal{O}_y^\times$ to $\mathbb{C}^\times$ has open kernel, hence finite image, so $\mu_{j+1}(k)$ is a root of unity and $\operatorname{ord}_p \iota\mu_{j+1}(k) = 0$. Therefore

$$2h \operatorname{ord}_p \iota\mu_{j+1}(\pi_v) = \operatorname{ord}_p \iota\mu_{j+1}(z) = \sum_{\tau\in\Phi} \varepsilon_\tau \operatorname{ord}_p \iota\tau(z)$$

by (11.6). By (11.1), $\iota\tau_{v,s}(z) = \sigma^s\rho_v(z)$ for even s and $\iota\tau_{v,s}(z) = \sigma^s\rho_v(\bar z)$ for odd s. The field $Q_v = E_w$ is unramified over $\mathbb{Q}_p$ and σ preserves valuations, so $\operatorname{ord}_p \iota\tau_{v,s}(z)$ equals $\operatorname{ord}_w(z) = 2h$ for even s and $\operatorname{ord}_w(\bar z) = \operatorname{ord}_{\bar w}(z) = -2h$ for odd s. Every $\tau \in \Phi$ satisfies $\iota\tau|_K = \iota|_K$. The embeddings of E into $\mathbb{C}_p$ that induce w are the maps $\sigma^s\rho_v$, and those that induce $\bar w$ are the maps $x \mapsto \sigma^s\rho_v(\bar x)$, for $s \in \mathbb{Z}/n'_v$. Their restrictions to K are $\sigma^s \circ \iota|_K$ and $\sigma^{s+1} \circ \iota|_K$, so an embedding $\iota\tau$ with $\tau \in \Phi$ that induces w or $\bar w$ is $\iota\tau_{v,s}$ for some position s. Every other $\tau \in \Phi$ induces a place of E above p different from w and $\bar w$, at which z is a unit, so $\operatorname{ord}_p \iota\tau(z) = 0$. The orbit v has ν_v even and ν_v odd positions, and (11.6) gives

$$\sum_{\tau\in\Phi} \varepsilon_\tau \operatorname{ord}_p \iota\tau(z) = 2h\Big(\sum_{s \text{ even}} \big(w_{j+1}(v,s) + c_j\big) - \sum_{s \text{ odd}} \big(w_{j+1}(v,s) + c_j\big)\Big) = 2h\,\mu_v(j+1).$$

Dividing by $2h$ gives the claim. □

Lemma 11.26. *Let (c, L) and (c, L_2) be lattice tuples of level j that agree at every orbit other than v. Then*

$$d_j(c, L) = d_j(c, L_2)\, \pi_v^{\operatorname{len} L_{2,v} - \operatorname{len} L_v}\, \kappa$$

with $\kappa \in U^1$ a unit at w and $\bar w$ and equal to 1 at every other finite place.

Proof. By Definition 11.23(ii) the quotient $\kappa' = d_j(c, L)/d_j(c, L_2) = \det h_{c,v}(L_v)/\det h_{c,v}(L_{2,v})$ lies in U^1 and equals 1 away from w and $\bar w$. The component at w of $g_c\, h_{c,v}(L_v)$ maps the standard lattice onto the completion of $N_j(c, L)$ at w, which is $\varphi_{c,v}(L_v)$ under ρ_v. Since $\det \varphi_{c,v}$ is a unit and $\operatorname{len} L_v = -\operatorname{ord}\det(\text{basis of } L_v)$, we get $\operatorname{ord}_w \det g_c + \operatorname{ord}_w \det h_{c,v}(L_v) = -\operatorname{len} L_v$, and the same with L_2. Subtracting, $\operatorname{ord}_w(\kappa') = \operatorname{len} L_{2,v} - \operatorname{len} L_v$. The idele $\kappa = \kappa' \pi_v^{\operatorname{len} L_v - \operatorname{len} L_{2,v}}$ lies in U^1, equals 1 away from $w, \bar w$, and has $\operatorname{ord}_w(\kappa) = 0$. From $\kappa\bar\kappa = 1$ we get $\operatorname{ord}_{\bar w}(\kappa) = -\operatorname{ord}_w(\kappa) = 0$. □

Proposition 11.27 ((N1))**.** *Let s_j be as in Definition 11.23. For every orbit v and level $j < m$ put*

$$a_v = p^{-\mu_v(j+1)}\, \iota\mu_{j+1}(\pi_v)\, \iota\chi\big((\pi_v)\big)^{a_j}, \qquad a'_v = p^{\mu_v(j+1)}\, \iota\mu_{j+1}(\pi_v)^{-1}\, \iota\chi\big((\pi_v)\big)^{-a_{j+1}}.$$

Then a_v and a'_v are units of $\mathcal{O}_{\mathbb{C}_p}$, and for all pairs $(c, L; c', L')$ and $(c, L_2; c', L'_2)$ that differ only at v

$$\frac{s_j(c, L; c', L')}{s_j(c, L_2; c', L'_2)} = a_v^{\operatorname{len} L_v - \operatorname{len} L_{2,v}}\, a_v'^{\operatorname{len} L'_v - \operatorname{len} L'_{2,v}}.$$

Proof. Lemma 11.25 gives $\operatorname{ord}_p\big(p^{\mp\mu_v(j+1)} \iota\mu_{j+1}(\pi_v)^{\pm 1}\big) = 0$. The ideal (π_v) is the inverse of $\mathfrak{P}_{\bar w}\mathfrak{P}_w^{-1}$, so $\iota\chi((\pi_v))$ is a unit by Lemma 11.30. Hence a_v and a'_v are units.

Put $\ell = \operatorname{len} L_v$, $\ell_2 = \operatorname{len} L_{2,v}$, $\ell' = \operatorname{len} L'_v$ and $\ell'_2 = \operatorname{len} L'_{2,v}$. By Lemma 11.26 at levels j and $j+1$,

$$d_j(c, L) = d_j(c, L_2)\, \pi_v^{\ell_2 - \ell}\, \kappa, \qquad d_{j+1}(c', L') = d_{j+1}(c', L'_2)\, \pi_v^{\ell'_2 - \ell'}\, \kappa',$$

and $\mu_{j+1}(\kappa) = \mu_{j+1}(\kappa') = 1$ by Definition 11.23(i). Since $\mathfrak{d}_N = (d_j(c, L))$ and χ is multiplicative on ideals of norm one, $\chi(\mathfrak{d}_N) = \chi(\mathfrak{d}_{N_2})\, \chi((\pi_v))^{\ell_2 - \ell}$, where $N_2 = N_j(c, L_2)$, and in the same way

at level $j+1$. The factors of s_j at the orbits other than v are the same for the two pairs. Collecting the factors,

$$\frac{s_j(c,L;c',L')}{s_j(c,L_2;c',L_2')} = p^{\mu_v(j+1)((\ell'-\ell_2')-(\ell-\ell_2))}\,\iota\mu_{j+1}(\pi_v)^{(\ell_2'-\ell')-(\ell_2-\ell)}\,\iota\chi\big((\pi_v)\big)^{-a_j(\ell_2-\ell)+a_{j+1}(\ell_2'-\ell')},$$

which is $(a_v')^{-(\ell_2'-\ell')}\,a_v^{-(\ell_2-\ell)} = a_v^{\ell-\ell_2}\,{a_v'}^{\ell'-\ell_2'}$. □

For the characters of the chain this calculation requires the unit and infinity-type conditions of Definition 11.23, frames of unit determinant, and a coefficient field containing $\iota\mu_{j+1}(\pi_v)$ and $\iota\chi((\pi_v))$. Under these conditions the two scalars belong to its valuation ring and have valuation zero.

Proposition 11.28 (the form (S3))**.** *Assume (N1). For every orbit v, level $j<m$ and components c,c', there is a $\mathbb{Z}$-linear map $f\mapsto \Xi_v(f)$ from integer-valued locally constant compactly supported functions on $M_{N_{j+1}\times N_j}(Q_v)$ to power series in the variables X, depending only on c,c' and the components of the tuples away from v, such that*

$$\mathfrak{K}_j(c,L;c',L') = a_v^{\operatorname{len} L_v}\,{a_v'}^{\operatorname{len} L_v'}\,q^{-(N_j\operatorname{len} L_v'-N_{j+1}\operatorname{len} L_v)}\,\Xi_v\big(\mathbf{1}_{L_v,L_v'}\big),$$

where $\mathbf{1}_{L_v,L_v'}(x)=1$ if $xL_v\subset L_v'$ and 0 otherwise. Consequently the step maps satisfy (S3) of Definition 10.12 with $a=a_v$, $a'=a_v'$ and $\mu=\mu_v(j+1)$.

Proof. Fix the components away from v and a reference pair (L_0,L_0'). For lattice tuples L,L' with these components away from v, the map $L_v\mapsto\varphi_{c,v}(L_v)$ identifies $\mathbf{1}_{L_v,L_v'}$ with $\mathbf{1}_{Y_v(N,N')}$ after the frames, because $Y_v(N,N')=\varphi_{c',v}\operatorname{Hom}(L_v,L_v')\varphi_{c,v}^{-1}$. Put

$$\widetilde{\mathfrak{K}}(L,L') = a_v^{-\operatorname{len} L_v}{a_v'}^{-\operatorname{len} L_v'}q^{N_j\operatorname{len} L_v'-N_{j+1}\operatorname{len} L_v}\,\mathfrak{K}_j(c,L;c',L').$$

By (N1) and Lemma 11.21, $\widetilde{\mathfrak{K}}(L,L')$ is a constant independent of (L_v,L_v') times the Taylor series of $\operatorname{vol}(Y_v(N,N'))^{1/2}\chi_F^{-1}\theta_{N,N'}[\Xi]$ in the fixed coordinates (11.5); the frames do not depend on L_v, so the coordinates are the same for all these pairs. Proposition 11.20 says that every $\mathbb{Z}$-linear relation among the functions $\mathbf{1}_{L_v,L_v'}$ is satisfied by the series $\widetilde{\mathfrak{K}}(L,L')$. Hence the assignment $\mathbf{1}_{L_v,L_v'}\mapsto\widetilde{\mathfrak{K}}(L,L')$ extends to a $\mathbb{Z}$-linear map on the subgroup spanned by these indicators, and further to all integer-valued locally constant compactly supported functions because that subgroup is a direct summand of the free abelian group of such functions on each finite level. This is Ξ_v. For (S3), $\kappa_j=p^{\mu_v(j+1)(\operatorname{len} L_v-\operatorname{len} L_v')}\mathcal{J}_j(\cdot,\mathfrak{K}_j)$, and $\mathcal{J}_j$ is linear in the series, so $\Xi(c,L;c',L';f)=\mathcal{J}_j(\cdot,\Xi_v(f))$ satisfies (S3). □

Remark 11.29. Hypothesis (N1) says that the frame factor and the determinant twists have p-adic valuation $\mu_v(j+1)(\operatorname{len} L_v-\operatorname{len} L_v')$ up to units, and that the character χ of Section 7.3 contributes units at a split place. The second part is proved in Lemma 11.30. The first part depends on the definition of the frame factor of (7.4), which Section 7.3 describes in words.

Lemma 11.30. *Let $\mathfrak{P}_w,\mathfrak{P}_{\bar w}$ be the primes of $\mathcal{O}_E$ at $w,\bar w$. The number $\iota\big(\chi(\mathfrak{P}_{\bar w}\mathfrak{P}_w^{-1})\big)$ is a unit of $\mathcal{O}_{\mathbf{C}_p}$. Consequently $\chi_F(N(L),N'(L'))/\chi_F(N(L_2),N'(L_2'))$ is a product of integer powers of units when the pairs differ only at v.*

Proof. Let h be the class number of E and β a generator of $(\mathfrak{P}_{\bar{w}}\mathfrak{P}_{w}^{-1})^{h}$. The ideal $\mathfrak{P}_{\bar{w}}\mathfrak{P}_{w}^{-1}$ has norm one, so $\beta\bar{\beta}$ is a unit of E^{+}, and replacing h by $2h$ and β by $\beta^{2}/(\beta\bar{\beta})$ we may assume $\beta\bar{\beta}=1$. Then $\chi(\mathfrak{P}_{\bar{w}}\mathfrak{P}_{w}^{-1})^{h}=\prod_{\tau\in\Phi}\tau(\beta)^{-1}$. By (11.1), $\operatorname{ord}_{p}\iota\tau(\beta)$ is -1 times the exponent of $\mathfrak{P}_{w}$ in β if τ is an even position of v, the exponent of $\mathfrak{P}_{\bar{w}}$ if τ is an odd position, and 0 otherwise. The two exponents are $-h$ and h, and there are ν_{v} even and ν_{v} odd positions, so the total valuation is $\nu_{v}h-\nu_{v}h=0$. The ideal $\mathfrak{d}_{N}$ changes by a power of $\mathfrak{P}_{\bar{w}}\mathfrak{P}_{w}^{-1}$ when L_{v} changes, because $N_{\bar{w}}$ is the dual of N_{w}. □

Proposition 11.31 ((S1) and (S2))**.** *For every orbit v the chain of Definition 11.5, with the lifts of Definition 11.10, satisfies (S1) and (S2) of Definition 10.12, with the following change in (S1): the normalizing exponents of the Hecke operators are $n_{v,i}^{j}=N_{v}'(j,i)$ of Definition 11.7, and not* 0*.*

Proof. (S1) is Definition 11.7 with $N_{j+1}=N_{j}+g$, $g=t+1$. Replacing L_{v} by an edge commutes with the action of $\Gamma_{j,c}$ because $(\gamma L)_{v}$ is a W_{v}-linear image of L_{v}. (S2): the finiteness is Lemma 11.6, the class-sum formula is Definition 11.10, and the covariance is Lemma 11.11. □

The change in (S1) does not affect Proposition 10.13, whose proof uses the raw operators: its conclusion (10.3) is a statement about $\mathcal{E}_{i}^{(j)}=q^{-i(N_{j}-i)}T_{v,i}^{\mathrm{raw},(j)}$. Section 11.5 converts the result into (D3) for the normalized operators.

11.5 Hecke commutation data

Fix an orbit v and drop it from the notation. By Proposition 11.28 and Proposition 10.13, the relation (10.3) holds for every $j<m$ with

$$u_{j+1}=(a_{v}a_{v}')^{-1}\in\mathcal{O}^{\times},\qquad m_{j+1}=a_{v}\,p^{\mu_{v}(j+1)},\qquad t=g-1,\qquad q=p^{n_{v}'/2}.$$

Proposition 10.8 then gives, for $F\in\mathcal{F}_{0}$ and $1\le j\le m$,

$$\sum_{l=0}^{r}(-1)^{l}h_{l}(Y_{j})\,q^{l(N_{j}-2r+l)}\,T_{r-l}^{\mathrm{raw},(j)}\,\mathcal{K}_{j}(F)=U_{j}^{r}\,q^{r(N_{j}-N_{0})}\,\mathcal{K}_{j}\big(T_{r}^{\mathrm{raw},(0)}F\big),\tag{11.7}$$

with Y_{j} the family of the numbers $u_{l}\cdots u_{j}\,m_{l}\,q^{t-2i}$, $1\le l\le j$, $0\le i\le t$, and $U_{j}=u_{1}\cdots u_{j}$.

Proposition 11.32. *Put, for $1\le j\le m$ and $0\le l\le r$,*

$$c_{j,l}=p^{N'(j,r)-N'(j,r-l)}\,q^{l(N_{j}-2r+l)}\,h_{l}(Y_{j}),\qquad \rho_{j}=U_{j}^{r}\,p^{N'(j,r)-A}\,q^{r(N_{j}-N_{0})},$$

with $N'=N_{v}'$ and $A=A_{v}$. Then the operator $Z_{j}=\sum_{l=0}^{r}(-1)^{l}c_{j,l}T_{r-l}^{(j)}$ satisfies $Z_{j}\mathcal{K}_{j}(F)=\rho_{j}\,\mathcal{K}_{j}(y_{v}F)$ for every $F\in\mathcal{F}_{0}$. This is (D3) of Definition 10.4 with $\epsilon_{v,j}=1$, and the price is

$$\Pr(v,j)=\operatorname{ord}\rho_{j}=N_{v}'(j,r)-A_{v}+\tfrac{n_{v}'}{2}\,r\,(N_{j}-N_{0}).$$

Proof. Multiply (11.7) by $p^{N'(j,r)}$ and write $T_{r-l}^{\mathrm{raw},(j)}=p^{-N'(j,r-l)}T_{r-l}^{(j)}$ on the left and $T_{r}^{\mathrm{raw},(0)}=p^{-A}y_{v}$ on the right. The units u_{l} have valuation zero, so $\operatorname{ord}U_{j}=0$. □

Written out, by (11.3) and $A_{v}=\sum_{s\text{ even}}\sum_{a\le r}\lambda_{v,s}^{(0)}(a)+\sum_{s\text{ odd}}\sum_{a\le r}\big(-\lambda_{v,s}^{(0)}(N_{0}+1-a)\big)$, which is Definition 11.7 written out with the dual weight at odd positions,

$$\Pr(v,j)=\sum_{s\text{ odd}}\Big(r\,w_{j}(v,s)+\sum_{a=N_{0}+1-r}^{N_{0}}\lambda_{v,s}^{(0)}(a)\Big)+\tfrac{n_{v}'}{2}\,r\,j(t+1),$$

Theorem 11.33 (integrality of the coefficients)**.** *Assume (W1), (W2),* $1 \le r$, $2r \le k_1 + 1$, *and* $\mathrm{ord}(m_l) = \mu_v(l)$ *for* $1 \le l \le m$. *Then for* $1 \le j \le m$ *and* $0 \le l \le r$,

$$c_{j,l} \in \mathcal{O}, \qquad \mathrm{ord}(c_{j,r}) \ge \alpha_{\mathrm{ev}}(v) := \sum_{s \in \mathcal{T}_v \text{ even}} \alpha_s.$$

Hence the Hecke commutation data of Proposition 11.32 satisfy the conditions of (D3) with $\alpha(v,j) = \alpha_{\mathrm{ev}}(v)$.

Proof. Put $\lambda_s(a) = \lambda^{(0)}_{v,s}(a)$ and

$$\mu_j^\flat = \sum_{s \text{ even}} w_1(v,s) - \sum_{s \text{ odd}} w_j(v,s) - \tfrac{n'_v}{2}\, t.$$

Step 1, the elements of Y_j. An element of Y_j is $u\, m_l\, q^{t-2i}$ with u a unit, $1 \le l \le j$ and $0 \le i \le t$. Its valuation is $\mu_v(l) + \frac{n'_v}{2}(t-2i) \ge \mu_v(l) - \frac{n'_v}{2} t$. By (W2), $w_l \ge w_1$ at even positions and $w_l \le w_j$ at odd positions, so $\mu_v(l) \ge \sum_{s \text{ even}} w_1 - \sum_{s \text{ odd}} w_j$. Hence every element of Y_j has valuation at least $\mu_j^\flat$, and $h_l(Y_j)$, a sum of products of l elements of Y_j with integer coefficients, has valuation at least $l\mu_j^\flat$.

Step 2, the exponent. So $\mathrm{ord}\, c_{j,l} \ge E(j,l)$ with

$$E(j,l) = N'(j,r) - N'(j,r-l) + \tfrac{n'_v}{2}\, l\, (N_j - 2r + l) + l\mu_j^\flat.$$

By the definition (11.3), $N'(j,r) - N'(j,r-l) = \sum_{s \text{ even}} \sum_{r-l<a\le r} \lambda_s(a) + l \sum_{s \text{ odd}} w_j$, and the terms $\pm l \sum_{\mathrm{odd}} w_j$ cancel:

$$E(j,l) = \sum_{s \text{ even}} \Big(\sum_{r-l<a\le r} \lambda_s(a) + l\, w_1(v,s) \Big) + \tfrac{n'_v}{2}\, l\, (N_j - 2r + l - t).$$

Step 3, nonnegativity. By dominance and (W2), $\lambda_s(a) \ge \lambda_s(N_0) \ge -w_1(v,s)$ for every a, so each bracket is a sum of l terms $\lambda_s(a) + w_1 \ge 0$. For $j \ge 1$, $N_j - 2r + l - t = k_1 + 1 - 2r + (j-1)(t+1) + 1 + l \ge 0$ since $2r \le k_1 + 1$. Hence $E(j,l) \ge 0$.

Step 4, the top coefficient. For $l = r$ the bracket at an even position s is $\sum_{a \le r}(\lambda_s(a) + w_1)$. Every term is nonnegative, so it is at least the sum of the terms with $a = 1$ and $a = 2$, which is $\lambda_s(1) + \lambda_s(2) + 2w_1 \ge \lambda_s(1) - \lambda_s(2)$, because $\lambda_s(2) \ge -w_1$. For $s \in \mathcal{T}_v$ this is α_s, and for even $s \notin \mathcal{T}_v$ the bracket is $\ge 0 = \alpha_s$. (The case $r = 1$ uses only the term $a = 1$: $\lambda_s(1) + w_1 \ge \lambda_s(1) - \lambda_s(2)$ since $-\lambda_s(2) \le w_1$.) The second summand of $E(j,r)$ is nonnegative. Hence $\mathrm{ord}\, c_{j,r} \ge \alpha_{\mathrm{ev}}(v)$. □

The hypothesis $\mathrm{ord}(m_l) = \mu_v(l)$ is the conclusion $m_{j+1} = a_v p^{\mu_v(j+1)}$ of Proposition 10.13 with Proposition 11.28, so it rests on (N1).

11.6 From entries to lattices

This section proves the passage from bounds for the pairings $\langle P_I, \Theta \rangle$ of Theorem 9.6 to the inclusion of Definition 10.2, at the even positions, including the Hecke edges, and states the corresponding fact at the odd positions as hypothesis (N2).

11.6.1 The transfer at an even position

Fix $j < m$, an orbit v and an even position s, and write $n = N_j$, $n' = N_{j+1}$, $\nu = \nu_{v,s}^{\natural,(j)}$ and $\nu' = \nu_{v,s}^{\natural,(j+1)}$. From the definition of the weights,

$$\nu'(i) = \nu(i) + k \ (1 \le i \le n), \qquad \nu'(i) = 0 \ (n < i \le n'), \qquad k = w_{j+1}(v,s) - w_j(v,s) \ge 0, \tag{11.8}$$

with $w_0 = -\lambda_{v,s}^{(0)}(N_0)$ for $j = 0$. In particular the column heights of ν' are those of ν followed by k columns of height n. Let $D = (D_{ab})$ be an $n \times n'$ matrix of indeterminates (the kernel variables $X_{v,s,a,b}$) and Y an $n' \times n'$ matrix of indeterminates (the variables of the model at level $j+1$). The transfer of a polynomial $G \in V_\nu$ is

$$\mathcal{T}(G) = G\big((DY)_{[n]}\big) \cdot \det\big((DY)_{[n]}\big)^k,$$

where $(DY)_{[n]}$ is the $n \times n$ matrix of the first n columns of DY. This is the substitution of the jet operator $\mathcal{J}_j$ at an even position. The pairing of $\mathcal{T}(G)$ with a power series f in the entries of D is

$$\langle \mathcal{T}(G), f \rangle = \sum_\kappa \kappa!\, f_\kappa \cdot \big[D^\kappa\big]\mathcal{T}(G) \in \mathcal{K}[Y],$$

where $[D^\kappa]$ takes the coefficient of the monomial D^κ.

Lemma 11.34 (Cauchy–Binet transfer)**.** *For $G \in \mathsf{W}_\nu$, $\mathcal{T}(G)$ is an $\mathcal{O}$-linear combination of products $\prod_c \det D_{R_c,S_c} \cdot \prod_c \Delta_{S_c}(Y)$, where c runs over the columns of ν', $R_c \subset \{1,\dots,n\}$ and $S_c \subset \{1,\dots,n'\}$ have h'_c elements, h'_c is the height of the column c of ν', and $\Delta_S(Y)$ is the minor of Y on the rows S and the first $\#S$ columns.*

Proof. By linearity it suffices to treat $G = \prod_c \Delta_{R_c}(X)$, a canonical bideterminant of shape ν. Then $\mathcal{T}(G) = \prod_c \det\big((DY)_{R_c,[h_c]}\big) \cdot \det((DY)_{[n],[n]})^k$, and det of the $n \times n$ block is a minor on all rows and the first n columns. For a minor of a product, the Cauchy–Binet formula gives

$$\det\big((DY)_{R,[h]}\big) = \sum_{\#S=h} \det D_{R,S}\, \det Y_{S,[h]} = \sum_{\#S=h} \det D_{R,S}\, \Delta_S(Y).$$

Taking the product over the columns of ν and the k extra columns of height n (with $R = \{1,\dots,n\}$) gives the claim with integer coefficients. □

Theorem 11.35 (entries to lattices, even positions)**.** *Let L be a lattice tuple of level j and (L'', L') an edge of type $i_0 \le r$ at v for tuples of level $j+1$ (with $i_0 = 0$ meaning $L'' = L'$). Use frames adapted to L_v and L''_v in* (11.5)*, so that $(L_v)_s$ and $(L''_v)_s$ correspond to the standard lattices. Let f be a power series in the entries of D and $g \in \mathbb{R}$, and suppose that for every choice of columns $S_c \subset \{1,\dots,n'\}$ with $\#S_c = h'_c$ and rows R_c,*

$$\operatorname{ord}\Big\langle \prod_c \det D_{R_c,S_c},\, f \Big\rangle + \sum_c \Big(\min(h'_c, i_0) - \#\big(S_c \cap \{1,\dots,i_0\}\big) \Big) \ge g. \tag{11.9}$$

Then $p^{\sum_{a \le i_0} \lambda_{v,s}^{\natural,(j+1)}(a)} \langle \mathcal{T}(G), f \rangle \in p^g\, H_s(L'_v)$ for every $G \in H_s(L_v)$.

Proof. In the adapted frames $H_s(L_v) = \mathsf{W}_\nu$, and $H_s(L''_v) = \mathsf{W}_{\nu'}$ up to the determinant twist, which is 1 since the frame of L''_v is the identity. By Lemma 11.34 and linearity of the pairing,

$$\langle \mathcal{T}(G), f \rangle = \sum \beta \cdot \Big\langle \prod_c \det D_{R_c,S_c}, f \Big\rangle \cdot \prod_c \Delta_{S_c}(Y), \qquad \beta \in \mathcal{O}.$$

Each $\prod_c \Delta_{S_c}(Y)$ is a canonical bideterminant of shape ν', an element of $\mathrm{W}_{\nu'} = H_s(L''_v)$. By the proof of Lemma 11.4, with $L''_v = d\,L'_v$-coordinates, $d = \mathrm{diag}(p,\dots,p,1,\dots,1)$ (i_0 entries p), the element $p^{\sum_{a\le i_0}\lambda^\natural(a)}\prod_c \Delta_{S_c}(Y)$ lies in $p^{\sum_c(\min(h'_c,i_0)-\#(S_c\cap[i_0]))}H_s(L'_v)$: indeed, that proof shows $p^{\sum_{a\le i_0}\lambda^\natural(a)}B \in p^{\sum_c \min(h'_c,i_0)-\sum_c\#(S_c\cap[i_0])}\,H_s(L'_v)$ for the bideterminant B with row sets S_c, because the exact exponent of p produced by $B(d^{-1}\cdot)$ is $-\sum_c\#(S_c\cap[i_0])$. Combining with (11.9) gives the claim term by term. □

Lemma 11.36 (the operators P_I)**.** *In the notation of Section 9.3, with $r_0 = N_j$, $r_1 = N_{j+1}$, the positions of v, the tops $\mathcal{T}_v$, and at each position s the columns of $\nu^{(j+1)}_{v,s}$ with their heights (at odd positions this is $\lambda^{(j+1)}$ normalized, not the model weight), the numbers α_s, cap_s, λ_s of Section 9.3 are*

$$\alpha_s \text{ of (11.2)}, \qquad \mathrm{cap}_s = \nu^\natural(2)\ (s\in\mathcal{T}_v),\ \nu^\natural(1)\ (s\notin\mathcal{T}_v), \qquad \lambda_s = \textstyle\sum_i \nu^\natural(i) - \alpha_s,$$

with $\nu^\natural$ here standing for $\nu^{(j+1)}_{v,s}$. For $J = \{0,\dots,i_0-1\}$, the number N_J of Theorem 9.6(iii) is at most the sum over all columns of the exponents $\min(h,i_0) - \#(I(\cdot)\cap J)$ at even positions and $\#(I(\cdot)\cap J)$ at odd positions, with equality in the terms of the height-one columns at the tops.

Proof. The columns of height one are the last $\nu^{(j+1)}(1) - \nu^{(j+1)}(2)$ columns, and $\nu^{(j+1)}(1) - \nu^{(j+1)}(2) = \lambda^{(0)}_{v,s}(1) - \lambda^{(0)}_{v,s}(2)$, so their number is α_s at a top. The other statements are the definitions. For the comparison with N_J: at an even top a height-one column contributes $\min(1,i_0) - [I(0)\in J] = [I(0)\notin J]$ when $i_0 \ge 1$, which is its term in N_J; other columns at even positions contribute a nonnegative number that N_J does not count; at odd positions both count $\#(I\cap J)$. □

The coefficients in the frame variables $y_{s,i,a}$ of the operator P_I of Section 9.3 are, by the Cauchy–Binet formula $\det\big(\sum_a y_{i,a}D_{a,I(i')}\big) = \sum_R \det y_{\cdot,R}\det D_{R,I}$, the products $\pm\prod_c \det D_{R_c,I_c}$ with arbitrary rows. So the hypotheses of Theorem 11.35 for all (R_c,S_c) are the bounds of Theorem 9.6 for all coefficients of $\langle P_I, f\rangle$, with the exponents of Lemma 11.36.

Lemma 11.37 (change of frames, even positions)**.** *Let $g\in\mathrm{GL}_n(\mathcal{K})$ and $g'\in\mathrm{GL}_{n'}(\mathcal{K})$. For $G\in V_\nu$ put $(g\cdot G)(X) = G(g^{\mathsf T}X)$, and for a series f in D put $(g,g')\cdot f(D) = f(g^{\mathsf T}D(g')^{-\mathsf T})$. Then*

$$\big\langle \mathcal{T}(g\cdot G),\ (g,g')\cdot f\big\rangle(Y) = \det(g)^{-k}\,\langle \mathcal{T}(G), f\rangle\big((g')^{\mathsf T}Y\big),$$

that is, the pairing is equivariant for the row actions on the source model, the kernel variables and the target model, up to the factor $\det(g)^k$.

Proof. Substituting $D\mapsto g^{\mathsf T}D(g')^{-\mathsf T}$ and $Y\mapsto (g')^{\mathsf T}Y$ replaces DY by $g^{\mathsf T}DY$, hence $(DY)_{[n]}$ by $g^{\mathsf T}(DY)_{[n]}$, so $\mathcal{T}(G)(g^{\mathsf T}D(g')^{-\mathsf T},(g')^{\mathsf T}Y) = \det(g)^k\,\mathcal{T}(g\cdot G)(D,Y)$. The pairing satisfies $\langle P(g^{\mathsf T}D(g')^{-\mathsf T}), f\rangle = \langle P, (g,g')^{-1}\cdot f\rangle$ for every polynomial P in D, because $\langle P, f\rangle = (P(\partial)f)(0)$ and the chain rule. Applying this with $P = \mathcal{T}(G)(\cdot,(g')^{\mathsf T}Y)$ and replacing f by $(g,g')\cdot f$ gives the claim. □

At an even position the frame change of (11.5) is the substitution of Lemma 11.37 with $g = \sigma^s\rho_v(\varphi_{c,v}^{-1}\cdot\text{basis of } L_v)$ and g' the same for L'_v. The determinant factors $\det(g)^{-\lambda^\natural(N_j)}$ and $\det(g')^{-\lambda^\natural(N_{j+1})}$ of Definition 11.3 and the factor $\det(g)^k$ of Lemma 11.37 combine, since $-\lambda^{\natural,(j)}(N_j) + k = w_{j+1}(v,s) = -\lambda^{\natural,(j+1)}(N_{j+1})$, to $\big(\det g/\det g'\big)^{w_{j+1}(v,s)}$, whose valuation is $w_{j+1}(v,s)(\mathrm{len}\,L'_v - \mathrm{len}\,L_v)$. These are the even-position terms of the factor $p^{\mu_v(j+1)(\mathrm{len}\,L_v - \mathrm{len}\,L'_v)}$ in κ_j, with the opposite sign, so they cancel.

11.6.2 The odd positions

At an odd position the model weight is the dual of $\lambda^{(j)}$ and the transfer of the jet operator uses the adjugate of Y . We state the needed fact.

Hypothesis 11.38 ((N2))**.** At every odd position s of every orbit, the transfer of the jet operator $\mathcal{J}_j$ has the two properties proved above at even positions: (a) Theorem 11.35, with the exponent $\min(h'_c, i_0) - \#(S_c \cap [i_0])$ in (11.9) replaced by $\#(S_c \cap [i_0])$ and with the columns of $\nu_{v,s}^{(j+1)}$; (b) Lemma 11.37, with the determinant factors combining to $\big(\det g / \det g'\big)^{-w_{j+1}(v,s)}$.

Remark 11.39. The mechanism at odd positions is the same Cauchy–Binet transfer applied to the contragredient model, where minors of the adjugate are complementary minors times powers of the determinant (Jacobi's theorem); the bookkeeping of the determinant powers is what remains to be written. It holds for any power series f, so it does not involve the theta series.

Corollary 11.40 (entries to lattices)**.** *Assume (N2). Let $(c, L) \in C_j \times \mathcal{L}_j$, let $(c', L') \in C_{j+1} \times \mathcal{L}_{j+1}$, let $i_0 = (i_0(v))_v$ with $0 \le i_0(v) \le r$, and let L'' be reached from L' by edges of types $i_0(v)$. Let f be the kernel series of $(c, L; c', L'')$ in frames adapted to L and L'', and suppose that every coefficient in the frame variables of $\langle \prod_v P_{I_v}^{(i_0(v))}, f \rangle$ has valuation at least g, for all column choices I_v, where $P_I^{(i_0)}$ is P_I of Section 9.3 multiplied by p to the exponents of Lemma 11.36. Then*

$$p^{\sum_v N'_v(j+1, i_0(v))}\, \kappa_j(c, L; c', L'')\big(H_j(L)\big) \subset p^g\, H_{j+1}(L').$$

Proof. The jet operator $\mathcal{J}_j$ is the tensor product over the orbits and positions of the transfers, and the pairing of a product of polynomials in disjoint sets of variables with f is computed factor by factor in the frame variables. By Lemma 11.37 at even positions and (N2)(b) at odd positions, the change from the fixed frames (11.5) to frames adapted to L and L'' maps $H_j(L)$ and $H_{j+1}(L'')$ to the standard lattices, and the determinant factors cancel the factor $p^{\sum_v \mu_v(j+1)(\operatorname{len} L_v - \operatorname{len} L''_v)}$ of κ_j, because $\mu_v(j+1) = \sum_{s \text{ even}} w_{j+1} - \sum_{s \text{ odd}} w_{j+1}$. In adapted frames, Theorem 11.35 at even positions and (N2)(a) at odd positions, applied factor by factor, give the inclusion with the exponent $\sum_v \sum_s \sum_{a \le i_0(v)} \lambda_{v,s}^{\natural,(j+1)}(a)$ in place of $\sum_v N'_v(j+1, i_0(v))$, and (11.4) allows the larger exponent. □

11.7 Orbit bounds

11.7.1 Additivity over orbits

For a power series U in the parameters $(x_{v,e}, y_{v,e})$ of all orbits, write $U = \sum_\beta U_\beta \prod_v x_v^{\beta_v} y_v^{\beta'_v}$.

Lemma 11.41 (additivity)**.** *Let ϑ_v, one for each orbit, be power series in the parameters of v with constant term* 1*, and let P_v be polynomials in the variables of v with coefficients in a polynomial ring of frame variables. Suppose that for every v, every integral series U_v in the parameters of v, and every coefficient γ in the frame variables of $\langle P_v, \vartheta_v U_v \rangle$, $\operatorname{ord} \gamma \ge g_v$. Then for every integral series U in all the parameters, every coefficient of $\langle \prod_v P_v, \prod_v \vartheta_v \cdot U \rangle$ has valuation at least $\sum_v g_v$.*

Proof. The pairing with the homogeneous polynomial $\prod_v P_v$ involves the coefficients of $\Theta = \prod_v \vartheta_v \cdot U$ in one degree d, and these involve only the coefficients U_β with $|\beta| \le d$, because the parameters are series without constant term in the variables. So we may replace U by a polynomial and, by linearity and the ultrametric inequality, by one monomial $U_\beta \prod_v x_v^{\beta_v} y_v^{\beta'_v}$ with

$|U_\beta| \leq 1$. Then $\Theta = U_\beta \prod_v \left(\vartheta_v x_v^{\beta_v} y_v^{\beta'_v}\right)$ is a product of series in disjoint sets of variables, and so is $\prod_v P_v$. For such products, $\langle \prod_v P_v, \prod_v \Theta_v \rangle = \prod_v \langle P_v, \Theta_v \rangle$, because $\kappa!$ and the coefficients factor over the disjoint variable sets. Each factor has coefficients of valuation at least g_v by hypothesis, with $U_v = x_v^{\beta_v} y_v^{\beta'_v}$, and the coefficients of a product of polynomials in disjoint frame variables are products of coefficients. □

11.7.2 The kernel law for the true kernels

Hypothesis 11.42 ((N3), the normalized kernel law)**.** For every $j < m$ and every pair $(c, L; c', L')$, in frames adapted to L and L' and in the hyperbolic coordinates of Corollary 8.22, the kernel series satisfies

$$\mathfrak{K}_j(c, L; c', L') = \exp\Big(-\sum_v \sum_e \beta(x_{v,e}, y_{v,e}) \Big) \cdot U$$

with U integral, that is, the constant c of Theorem 8.21 for the normalized kernel series has $|c| \leq 1$.

Lemma 11.43 (hyperbolic frames)**.** *For lattices $L_v \subset Q_v^{N_j}$ and $L'_v \subset Q_v^{N_{j+1}}$, let $\omega_0, \ldots, \omega_{\nu_v - 1}$ be a basis of W_v over $\mathbb{Z}_{p^2}$ and ω_r^* the dual basis for $\mathrm{Tr}_{Q_v/\mathbb{Q}_{p^2}}$. In frames adapted to L_v, L'_v the elements $f_{r,a,b} = \omega_r E_{ba}$ of the w-component and the elements $f'_{r,a,b}$ of the $\bar{w}$-component dual to them form a hyperbolic basis of the completion at v of the hermitian $\mathcal{O}_K$-lattice $\left(\mathfrak{a}\,\mathrm{Hom}(N, N'),\ \mathrm{Tr}_{E/K}\langle\ ,\ \rangle / N\mathfrak{c}\right)$, and the coordinates of Corollary 8.22 in this frame are the variables $X_{v,s,a,b}$ of* (11.5)*, with $\Lambda_{r,a,b} = \sum_{s \text{ even}} \sigma^s(\omega_r^*) X_{v,s,a,b}$ and $\Lambda'_{r,a,b} = \sum_{s \text{ odd}} \sigma^s(\omega_r) X_{v,s,a,b}$. The relation $\sum_{i<\nu_v} \sigma^{2i}(\omega_r \omega_{r'}^*) = [r = r']$ of Theorem 9.4 holds. The completion at a place of E^+ above p inert in E is unimodular and has an orthonormal basis.*

Proof. At v the completion is $\mathrm{Hom}(L_v, L'_v) \oplus \mathrm{Hom}(L_v^\#, L_v'^\#)$, the two summands are isotropic (Lemma 11.19) and in duality, and $N\mathfrak{c}$ and $\mathfrak{a}$ are units at p. In adapted frames the first summand is $M_{N_{j+1} \times N_j}(W_v) = \bigoplus_{r,a,b} \mathbb{Z}_{p^2}\, \omega_r E_{ba}$, and the dual basis of the second summand exists because the trace form of the unramified extension $Q_v/\mathbb{Q}_{p^2}$ is perfect on W_v. The relation for ω, ω^* is the duality of the two bases, since $\mathrm{Tr}_{Q_v/\mathbb{Q}_{p^2}} = \sum_{i<\nu_v} \sigma^{2i}$. The coordinates: by (11.1) the coordinate of $\sum_r \omega_r E_{ba} \otimes u\varepsilon^{-1} \Lambda_{r,a,b}$ at $\tau_{v,s}$, s even, is $u\varepsilon^{-1} \sum_r \sigma^s(\omega_r) \Lambda_{r,a,b} = u\varepsilon^{-1} \sum_{s' \text{ even}} \left(\sum_r \sigma^s(\omega_r) \sigma^{s'}(\omega_r^*) \right) X_{v,s',a,b} = u\varepsilon^{-1} X_{v,s,a,b}$, because the two matrices $(\sigma^s(\omega_r))$ and $(\sigma^{s'}(\omega_r^*))$ are inverse, as in the proof of Theorem 9.4. The odd positions are the same computation on the $\bar{w}$-component. At an inert place the lattice is unimodular because N and N' are self-dual and $\mathfrak{a}$ is a unit there, and a unimodular hermitian lattice over the unramified quadratic extension has an orthonormal basis [33, Section 7]. □

11.7.3 The orbit bounds

Theorem 11.44 (orbit bounds)**.** *Assume (N1), (N2), (N3). For every $j < m$ the chain of Definitions 11.5 and 11.10 has the entry bound $\sum_v b_v$ at (j, i_0), for every family i_0 and every choice of numbers b_v with*

$$b_v \leq \begin{cases} g_v^B(j+1) & i_0(v) \geq 1, \\ \max\left(g_v^A(j+1),\ g_v^B(j+1) - \alpha_{\mathrm{ev}}(v)\right) & i_0(v) = 0, \end{cases}$$

where $g_v^A(j+1)$ and $g_v^B(j+1)$ are the numbers (9.6) *and* (9.7) *for the shape of Lemma 11.36 at level $j+1$. That is, the chain has the orbit bounds $(g^0, g^1, \alpha) = (g^A, g^B, \alpha_{\mathrm{ev}})$ of Definition 10.5.*

Proof. Fix (c, L), (c', L'), i_0 and L'' in the family $E_{i_0}^{j+1}(L')$, and let f be the kernel series of $(c, L; c', L'')$ in frames adapted to L and L''. By (N3) and Lemma 11.43, $f = \prod_v \vartheta_v \cdot U$ with $\vartheta_v = \exp(-\sum_e \beta(x_{v,e}, y_{v,e}))$ and U integral. For each orbit v and each integral series U_v in the parameters of v, Proposition 9.5 gives $(\vartheta_v U_v)(\varpi X) \in \mathcal{C}(h_v)$, with the hypotheses on β given by Theorem 8.12 and those on ω, ω^* by Lemma 11.43. Theorem 9.6 applies to $\vartheta_v U_v$ with the shape of Lemma 11.36:

- if $i_0(v) = 0$ the operator $P_I^{(0)}$ is P_I, and parts (i) and (ii) bound every coefficient of $\langle P_I, \vartheta_v U_v \rangle$ below by g_v^A and by $g_v^B - \alpha_{\mathrm{ev}}(v)$;
- if $i_0(v) \geq 1$ the operator $P_I^{(i_0)}$ is $p^e P_I$ with $e \geq N_J$ by Lemma 11.36, and part (iii) gives the bound g_v^B.

So each b_v in the statement is a lower bound for all coefficients of $\langle P_{I_v}^{(i_0(v))}, \vartheta_v U_v \rangle$. Lemma 11.41 gives the bound $\sum_v b_v$ for $\langle \prod_v P_{I_v}^{(i_0(v))}, f \rangle$, and Corollary 11.40 gives

$$p^{\sum_v N_v'(j+1, i_0(v))} \kappa_j(c, L; c', L'')\big(H_j(L)\big) \subset p^{\sum_v b_v} H_{j+1}(L').$$

Since $n^{j+1}(i_0) = \sum_v N_v'(j+1, i_0(v))$ by Proposition 11.31, this is the entry bound of Definition 10.2. The ordering of the orbits plays no role, because the families $E_{i_0}^{j+1}(L')$ for different orderings coincide: edges at distinct orbits change distinct components. □

11.8 The chain datum

Theorem 11.45. *Assume (N1), (N2), (N3). For a prime $p \geq 5$ inert in K and unramified in E, the data of Definitions 11.5, 11.7 and 11.10, with $\mathcal{V}$ the set of orbits and $\mathcal{O}$ as in Section 11.1, form a chain datum (Definition 10.1) with:*

(i) the lattice tuples, the integral structures $H_j(L)$ of Definition 11.3, the Hecke families of Definition 11.7 with $n_{v,i}^j = N_v'(j, i)$;

(ii) property (D1) with $d_j = \mathrm{ord}_p D_\Gamma$ (Proposition 11.14);

(iii) Hecke commutation data (D2), (D3) with $y_v = p^{A_v} T_{v,r}^{\mathrm{raw},(0)}$, $\epsilon_{v,j} = 1$, the coefficients $c_{v,j,l}$ and the elements $\rho_{v,j}$ of Proposition 11.32, and $\alpha(v, j) = \alpha_{\mathrm{ev}}(v)$ (Lemma 11.8, Theorem 11.33);

(iv) orbit bounds $(g^A, g^B, \alpha_{\mathrm{ev}})$ (Theorem 11.44).

Consequently Theorem 10.7 applies to the composite $\mathcal{K}_m = K_{m-1} \circ \cdots \circ K_0$ of the true lifts, with $\Pr(v, j)$ of Proposition 11.32.

Proof. Items (i) to (iv) are the cited statements. The only change to Section 10 is the normalization in (S1), which affects Proposition 10.13 not at all and Proposition 10.8 only through the conversion in the proof of Proposition 11.32. □

The three hypotheses.

- (N1) Hypothesis 11.22: the scalar normalization of the step kernel varies with the lattice at an orbit by units to the power of the lengths. The character factor χ_F satisfies this by Lemma 11.30; the frame factor and the determinant twists of (7.4) must be shown to have valuation $\mu_v(j+1)(\mathrm{len}\, L_v - \mathrm{len}\, L_v')$ plus units.

- (N2) Hypothesis 11.38: the entry-to-lattice transfer at the odd positions.
- (N3) Hypothesis 11.42: the normalized kernel law, i.e. $|c| \leq 1$ in Theorem 8.21 for the normalized kernel series; this fixes the normalization of θ_0 and of the constant c.

Hypotheses (N1), (N2) and (N3) are not proved here.

12 The loss and the hypotheses (3.1) and (3.2)

This section proves the hypotheses (3.1) and (3.2) of Proposition 3.1 for every $u_0 \in A_0$. Hypothesis (3.1) compares the sum S over primes with the total gain Γ, a number defined below from the valuation estimates of Section 10, the nondegeneracy exponents of the Hecke operators, and a choice of weights. Hypothesis (3.2) compares the slope of a lattice N_u with the slope of the base lattice M_u. The lattice N_u is the sublattice of M_u spanned locally by images of powers of Hecke operators, rescaled at each place by the gain. Its slope exceeds that of M_u by the gain minus the colength of the sublattice, and the colength is controlled by a nondegeneracy theorem for Hecke operators on integral forms of representations of GL_k along the edges of the building.

Subsection 12.1 fixes the classes of primes, the orbits of Frobenius, and the local quantities g^0, g^1, Pr, τ, io and gain. Subsection 12.2 states the nondegeneracy theorem and reduces it to one edge. Subsection 12.3 turns nondegeneracy into a bound for the colength of the sublattice NP_w. Subsection 12.4 defines N_u and proves (3.2). Subsection 12.5 compares the gain with the sum over primes and proves (3.1). Subsection 12.6 lists what this section takes from the other steps of the proof of Theorem A.

12.1 Primes, orbits and the local quantities

Fix admissible data (Definition 6.1) at the scale x, and let b, a, X', b' and δ_1 be as in (5.17). Put

$$w'' = \frac{b'}{xn}, \qquad X'' = \frac{a}{w''} = xnX', \qquad p_* = k^2x^{26}, \qquad \theta = x^{-\varepsilon}, \qquad \gamma_p = \frac{p}{p^2-1}. \tag{12.1}$$

Let $T \subset H$ be the set of tops of the data and $N_T = |T|$. Fix $u_0 \in A_0$, weights a_t for $t \in T$ with $a \leq a_t \leq a + a/(kx^{13})$, and nodes $w_j = w_{u_0,j} \colon \Phi \to (0,\infty)$ for $1 \leq j \leq m$. Let S be the sum of Corollary 3.4, that is, the sum over $t \in T$ and $1 \leq j \leq m$ of the left side of (3.10) with $a = a_t$ and $w = w_j$.

Classes of primes. Let $\mathcal{P}$ be the set of primes $p \notin S_0 \cup \{d\}$ with $\sigma_p \notin \Phi$, the primes that are inert in $\mathbb{Q}(\sqrt{-d})$ and prime to $2qM_\psi$. For $p \in \mathcal{P}$ write $\sigma_p = (1, \epsilon_p, h_p)$ with $\epsilon_p \in \mathbb{Z}/2$ and $h_p \in H$. For odd i,

$$p_t(i\sigma_p) = (0,\, 1+\epsilon_p,\, t + ih_p).$$

Let Φ_0 and Φ_1 be the sets of the elements $(0,0,h)$ and $(0,1,h)$ of Φ. Under the identification of Section 5.7 the places attached to the elements of T are the elements $(0,0,t)$ of Φ_0, which we call the *top places*. We say that $p \in \mathcal{P}$ is of *class I* when $\epsilon_p = 1$, so that $p_t(i\sigma_p) \in \Phi_0$ for odd i, and of *class II* when $\epsilon_p = 0$, so that $p_t(i\sigma_p) \in \Phi_1$ for odd i. For $s \in H$ and odd e put $\pi_p(s,e) = p_s(e\sigma_p)$. For $t \in T$, $1 \leq j \leq m$ and $p \in \mathcal{P}$ put

$$\mathrm{ask}_{j,t}(p) = \sum_{i \text{ odd}} \max\Big(\frac{a_t}{p^i} - w_j\big(\pi_p(t,i)\big), 0\Big), \tag{12.2}$$

and let $\mathrm{dt}_{j,t} = \log d \sum_{i \geq 1} \max\big(a_t/d^i - w_j(p_t g_i), 0\big)$ be the term of d in (3.10). Then

$$S = \sum_{t \in T} \sum_{j=1}^{m} \Big(\sum_{p \in \mathcal{P}} 2 \log p \; \mathrm{ask}_{j,t}(p) + \mathrm{dt}_{j,t} \Big). \tag{12.3}$$

The terms with $p > 2a$ vanish when the nodes are at least 1, because then $a_t/p^i < 1$.

Nodes and bottom weights. The construction of the nodes (Section 18) attaches to each $t \in T$ an integer $b_t \geq b'$, the bottom weight of the base weight at t, and gives the following bounds, with a spread $\eta \geq 1$ that satisfies the size conditions of Subsection 12.6.

(W1) At a top place, $w_j(0,0,t) = b_t + o_j(t)$ with $b'/x \leq o_j(t) \leq b'/x + \eta$.

(W2) At a thick place, $b \leq w_j(\rho) \leq b + \eta + 1$.

(W3) At every other place, $w'' \leq w_j(\rho) \leq w'' + \eta$.

In particular $w_j(\rho) \geq w'' \geq 1$ for every ρ when x is large. In the quantities below the bottom weights b_s are variables, and the value of w_j at a top place is $b_s + o_j(s)$ for the variable b_s. For $1 \leq j \leq m$ put $\kappa_j = \frac{1}{2}\big(k + (2j-1)(t_r+1)\big)$, let $N_m = k + m(t_r+1)$, and let $c_G \geq 0$ be a number with $\kappa_j \leq c_G$ for every j. Put

$$\hat{w}_j(\rho) = w_j(\rho) - \kappa_j, \qquad \mathrm{al}(s) = a_s - 2c_G, \qquad \mathrm{cap}_j(s,e) = \hat{w}_j\big(\pi_p(s,e)\big) + c_G.$$

The number κ_j is the shift between a node and the parameter of the polynomial model of level j in which the valuation estimates are written. The weights a_s, b_s enter the quantities below as variables, and we write $(a,b) = (a_s, b_s)_{s \in T}$ for a vector of weights.

Orbits. Let $p \in \mathcal{P}$. Put $s_p = h_p$ if p is of class I and $s_p = 2h_p$ if p is of class II, let n_p be the order of h_p in the first case and twice the order of $2h_p$ in the second, and fix a set $R_p \subset H$ of representatives of the orbits of the translation by s_p. For $t_0 \in R_p$ let $T(t_0)$ be the set of the elements of T on the orbit of t_0 and $N(t_0) = |T(t_0)|$. If p is of class I, the elements of the orbit of t_0 are $t_0 + i h_p$ with $i \in \mathbb{Z}/n_p$, and we call a top $s = t_0 + i h_p$ *even* when i is even. The orbit of t_0 is *good* when it is good for T in the sense of Definition 5.19, with $h = h_p$ and $L_2 = 2\max_{p'} L_1(p')$, where $L_1(p') = \lceil \log_{p'}(k x^{13})^4 \rceil$ and the maximum runs over the primes of class I. For a top s and odd e we write $\mathrm{Excl}(s,e)$ when p is of class I, $\pi_p(s,e)$ is the top place of a top $s' \neq s$, and e is the least positive integer with $s + e h_p = s'$.

Lemma 12.1. *Let $p \in \mathcal{P}$ be of class I with $n_p > o$. Then n_p is even.*

Proof. By Proposition 5.14(iii) the group H maps onto a cyclic group of order 2^e with a kernel of order c dividing o. If n_p were odd, the image of h_p would be an element of odd order in a group of order 2^e, hence trivial, so h_p would lie in the kernel and $n_p \leq c \leq o$. □

When $x^{\varepsilon/4} > o$, every prime of class I whose h_p has order at least $x^{\varepsilon/4}$ therefore has an even orbit length, and on each of its orbits the even and odd tops are well defined.

Lemma 12.2. *Let p be of class I with $n_p \geq x^{\varepsilon/4}$, and let $t_0 \in R_p$ have a good orbit. Let u, r_0, M_0 be as in Definition 5.19. Then*

$$1 \leq N(t_0), \qquad n_p \leq (6r_0+1)\,u\,(M_0+1)\,\big(N(t_0)+1\big).$$

In particular $n_p\,\delta_1 \leq C_{31}\,N(t_0)\log x$ for a constant C_{31} depending on ε and x large.

Proof. Cut the residues modulo n_p into $\lfloor n_p/((6r_0+1)u)\rfloor$ tiles as in Definition 5.19(iii). Condition (iii) gives, in each run of M_0+1 cyclically consecutive tiles, a tile whose $3r_0+1$ even cells each contain an element of P_t, so $N(t_0)\geq 1$, and the disjoint runs of M_0+1 consecutive tiles give at least $\lfloor n_p/((6r_0+1)u(M_0+1))\rfloor$ distinct elements of P_t. Since $u \leq x^{\varepsilon/8}+1$ and $M_0 \leq C\log x$, the second claim gives the third. □

The valuation bounds at an orbit. Fix $p \in \mathcal{P}$ with $p \leq X''$, an orbit $t_0 \in R_p$, a lift $1 \leq j \leq m$ and weights (a,b). The *digit loss* at the orbit is

$$\mathrm{dl}(t_0) = K_{d1}\,n_p\,N_m^2\log_p(2a) + K_{d2}\,N(t_0)\log_p(2a)^2,$$

where K_{d1}, K_{d2} are the constants of the term Δ of Theorem 9.6 in global parameters. The *base bound* and the *improved bound* are

$$g_j^0(t_0) = \sum_{s\in T(t_0)} \sum_{\substack{e \text{ odd}\\ \text{not Excl}(s,e)}} \max\Big(0, \frac{\mathrm{al}(s)}{p^e} - \mathrm{cap}_j(s,e)\Big) - \mathrm{dl}(t_0), \tag{12.4}$$

$$g_j^1(t_0) = \gamma_p \sum_{s\in T(t_0)} \mathrm{al}(s) - \sum_{\substack{s\in T(t_0)\\ s \text{ odd}}} \sum_{e \text{ odd}} \min\Big(\frac{\mathrm{al}(s)}{p^e}, \mathrm{cap}_j(s,e)\Big) - \mathrm{dl}(t_0), \tag{12.5}$$

where for p of class II no top is called odd. The *price* is

$$\mathrm{Pr}_j(t_0) = \begin{cases} \displaystyle\sum_{\substack{s\in T(t_0)\\ s \text{ odd}}} \big(\hat{w}_j(0,0,s) - b_s\big) + K_{\mathrm{Pr}}\,n_p\,(w''+\eta+c_G+N_m), & p \geq p_*,\\ r_0\big(N(t_0)(2b'+c_G) + n_p(w''+\eta+c_G+N_m)\big), & p < p_*. \end{cases} \tag{12.6}$$

These are the numbers g^0, g^1 and PR of Theorem 10.7 at the orbit of t_0, written in the parameters of the chain (Section 11).

Nondegeneracy exponents and shares. Subsection 12.2 attaches to p and t_0 a type r_p of Hecke operator, $r_p = r_0$ for $p < p_*$ and $r_p = 1$ for $p \geq p_*$, a depth D_p, a denominator $t_{t_0} \geq 0$ and a share $\kappa_{t_0}(a,b) \geq 0$ (Theorem 12.11). Put

$$\mathrm{Sig}(p) = m \sum_{t_0 \text{ good}} \kappa_{t_0}(a,b),$$

the sum over the representatives $t_0 \in R_p$ with a good orbit. We call p a *candidate* when $p < X'$, p is of class I, $n_p \geq x^{\varepsilon/4}$, and, if $p \geq p_*$, no power p^i with $i \geq 1$ satisfies $|p^i - X'| \leq \rho_B X'$, where $\rho_B \leq x^{-2}$ is the band width of Subsection 12.6. The set of *served orbits* at p is

$$\mathcal{V}_p = \begin{cases} \{t_0 \in R_p : \text{ the orbit of } t_0 \text{ is good}\}, & p \text{ a candidate and } \mathrm{Sig}(p) \leq \theta,\\ \emptyset & \text{otherwise.} \end{cases}$$

Definition 12.3. For $p \in \mathcal{P}$ with $p \leq X''$ and weights (a, b) put

$$S_{t_0} = \Big\lceil \sum_{j=1}^{m} \max\big(0,\ g_j^1(t_0) - \mathrm{Pr}_j(t_0) - \max(g_j^0(t_0), 0)\big) \Big\rceil \qquad (t_0 \in \mathcal{V}_p),$$

$$\tau(p) = e_p + \sum_{t_0 \in \mathcal{V}_p} m\big(\lceil t_{t_0} \rceil + D_p + 1\big) \quad \text{if } \mathcal{V}_p \neq \emptyset, \qquad \tau(p) = 0 \quad \text{if } \mathcal{V}_p = \emptyset,$$

$$\mathrm{io}(p) = 2\,\mathrm{Sig}(p) \max_{t_0 \in \mathcal{V}_p} S_{t_0} \quad \text{if } \mathcal{V}_p \neq \emptyset, \qquad \mathrm{io}(p) = 0 \quad \text{if } \mathcal{V}_p = \emptyset,$$

$$\mathrm{gain}_j(p) = \sum_{t_0 \in \mathcal{V}_p} \max\big(g_j^1(t_0) - \mathrm{Pr}_j(t_0),\ \max(g_j^0(t_0), 0)\big) + \sum_{t_0 \in R_p \setminus \mathcal{V}_p} \max\big(g_j^0(t_0), 0\big),$$

where e_p is the exponent of the comparison of integral structures in (L2) of Subsection 12.6. Put $\mathrm{gain}_j(p) = 0$ when $p > X''$, or when p is of class I and $h_p = 0$.

The number S_{t_0} is the exponent $S(v)$ of Theorem 10.7 at the served orbit $v = t_0$, and $\mathrm{gain}_j(p)$ is the contribution of the orbits at p to the exponent T of that theorem at the level j, with $B = \mathcal{V}_p$ and $\mathrm{PR} = \mathrm{Pr}$.

Dropped primes and the total gain. A prime $p \in \mathcal{P}$ *has no gain* when $p > X''$, or when p is of class I with h_p of odd order and $h_p \neq 0$, or when p is of class I with $p < X'(1 + C_r\rho_B)$ and one of the following holds: $n_p < x^{\varepsilon/4}$; $p \geq p_*$ and $|p^i - X'| \leq \rho_B X'$ for some $i \geq 1$; p is a candidate with $\mathrm{Sig}(p) > \theta$. Here $C_r \geq 1$ is a constant. For the auxiliary primes T_u of Section 12.4 let $\mathcal{D}$ be the set of the primes that have no gain and differ from every T_u. Put $\mathrm{gain}_j^{\mathcal{D}}(p) = 0$ for $p \in \mathcal{D}$ and $\mathrm{gain}_j^{\mathcal{D}}(p) = \mathrm{gain}_j(p)$ otherwise. If p has no gain then $\mathcal{V}_p = \emptyset$, because a prime with $\mathcal{V}_p \neq \emptyset$ is a candidate with $\mathrm{Sig}(p) \leq \theta$ and with n_p even by Lemma 12.1. So $\tau(p) = \mathrm{io}(p) = 0$ for $p \in \mathcal{D}$. The *total gain* is

$$\Gamma = \sum_{p \in \mathcal{P}} 2\log p \Big(\sum_{j=1}^{m} \mathrm{gain}_j^{\mathcal{D}}(p) - \tau(p) \Big), \tag{12.7}$$

and the *total correction* is $\mathrm{IO} = \sum_{p \in \mathcal{P}} \log p\ \mathrm{io}(p)$. Both depend on u_0 through the nodes, and on the weights (a, b).

Remark 12.4. Let $\mathbf{K}$ be the coefficient field of the hermitian lattices of Proposition 3.1, of degree $[\mathbf{K} : \mathbb{Q}]$. For a finite place w of $\mathbf{K}$ with prime ideal $\mathfrak{p}_w$, ramification index e_w and residue characteristic p_w put $\ell_w = \log N\mathfrak{p}_w / [\mathbf{K} : \mathbb{Q}]$ and $\sigma(w, y) = 2\ell_w e_w y$. Then $\sum_{w|p} \sigma(w, y) = 2y \log p$, because $\sum_{w|p} e_w f_w = [\mathbf{K} : \mathbb{Q}]$. So $\Gamma = \sum_w \sigma\big(w, \sum_j \mathrm{gain}_j^{\mathcal{D}}(p_w) - \tau(p_w)\big)$, which is the form in which Γ enters the slope of N_u in Subsection 12.4.

12.2 Nondegeneracy of Hecke operators

This subsection states the nondegeneracy theorem for Hecke operators at one Frobenius orbit, and reduces it to a statement on one edge of the building.

The local setting. Let p be a prime, $n' \geq 2$ an even integer, $q = p^{n'}$, $W = W(\mathbb{F}_q)$ the ring of Witt vectors with fraction field $\mathbb{Q}_q$ and Frobenius σ. Let $\mathcal{O}$ be a complete discrete valuation ring finite over W, with a uniformizer ϖ such that $\varpi^2 = p$, fraction field $\mathcal{K}$ and an embedding

$\iota\colon \mathbb{Q}_q \to \mathcal{K}$ compatible with $W \to \mathcal{O}$. Let $k \geq 2$. A *weight* is a family $\lambda = (\lambda_s)_{s \in \mathbb{Z}/n'}$ of dominant weights of GL_k. Put $\lambda_s^* = \lambda_s$ for even s and $\lambda_s^*(i) = -\lambda_s(k-1-i)$ for odd s, and let

$$V = \bigotimes_{s \in \mathbb{Z}/n'} \nabla_{\mathcal{K}}(\lambda_s^*),$$

on which $\mathrm{GL}_k(\mathbb{Q}_q)$ acts through $\iota \circ \sigma^s$ on the factor s. For a W-lattice $L \subset \mathbb{Q}_q^k$ let $L^{(s)}$ be the $\mathcal{O}$-span of $\iota\sigma^s(L)$, and let $H(L) = V \cap \bigotimes_s \nabla_{\mathcal{O}}(\lambda_s^*; L^{(s)})$ be the tensor product of the integral dual Weyl modules attached to the lattices $L^{(s)}$. Then $H(gL) = gH(L)$ for $g \in \mathrm{GL}_k(\mathbb{Q}_q)$. We normalize the action so that $\mathrm{diag}(c_1, \ldots, c_k)$ multiplies a vector of weight μ by $\prod_i c_i^{-\mu_i}$, and we use that $H(\mathcal{O}^k)$ has an $\mathcal{O}$-basis of weight vectors. These conventions are those of the integral structures of Section 11.

Let $1 \leq r < k$. An *r-edge* is a pair $e = (L'' \subset L)$ of lattices with $pL \subset L''$ and $L/L'' \cong \mathbb{F}_q^r$. We write $\mathrm{src}(e) = L''$ and $\mathrm{tgt}(e) = L$. The *midpoint* of e is the family of $\mathcal{O}$-lattices $M_e^{(s)} = L''^{(s)} + \varpi L^{(s)}$, and $H(M_e)$ is defined like $H(L)$. Put $A_r = \sum_s \sum_{i<r} \lambda_s^*(i)$,

$$R(e) = \varpi^{-A_r} H(M_e), \qquad R'(e) = \varpi^{-A_r} H(M_e)^\vee, \qquad \mathrm{be}(F, G) = G(p^{A_r} F) \quad (F \in V,\ G \in V^*),$$

where $^\vee$ is the $\mathcal{O}$-dual lattice in V^*. For $t \in \mathbb{N}$ the *enlarged lattices* are

$$N_t^{\mathrm{src}}(L'') = \varpi^{-t} H(L'') \cap \bigcap_{\mathrm{src}(e) = L''} R(e), \qquad N_t^{\mathrm{tgt}}(L) = \varpi^{-t} H(L)^\vee \cap \bigcap_{\mathrm{tgt}(e) = L} R'(e),$$

the intersections over the r-edges with the given source or target.

Lemma 12.5. *For every r-edge e the lattices $R(e)$ and $R'(e)$ are dual under* be*, and $H(\mathrm{src}\, e) \subset R(e)$ and $H(\mathrm{tgt}\, e)^\vee \subset R'(e)$.*

Proof. All three statements are invariant under $\mathrm{GL}_k(\mathbb{Q}_q)$, which acts transitively on r-edges, so we may take $L'' = \mathcal{O}^k$ and $L = d^2\mathcal{O}^k$ with $d = \mathrm{diag}(\varpi^{-1} I_r, I_{k-r})$, so that $M_e = d\mathcal{O}^k$ in every factor. Let (v_μ) be a basis of $H(\mathcal{O}^k)$ of weight vectors and (v_μ^*) the dual basis. Writing $|\mu|_r = \sum_{i<r} \mu_i$, the lattice $H(M_e)$ has the basis $\varpi^{|\mu|_r} v_\mu$ and $H(L)$ has the basis $p^{|\mu|_r} v_\mu$. Every weight μ of V is a sum over s of weights of $\nabla(\lambda_s^*)$, each of which is below λ_s^* in the dominance order, so $|\mu|_r \leq A_r$. Hence $R(e)$ has the basis $\varpi^{|\mu|_r - A_r} v_\mu$, which contains $H(\mathcal{O}^k)$, and $R'(e)$ has the basis $\varpi^{-|\mu|_r - A_r} v_\mu^*$, which contains the basis $p^{-|\mu|_r} v_\mu^*$ of $H(L)^\vee$. Finally $\mathrm{be}(\varpi^{|\mu|_r - A_r} v_\mu, \varpi^{-|\nu|_r - A_r} v_\nu^*) = \delta_{\mu\nu}$. □

Definition 12.6. Let $t, D, c^* \in \mathbb{N}$ and $\kappa \in \mathbb{R}$. The statement $\mathrm{LP}(r, \lambda; t, D, \kappa, c^*)$ holds when $4(D+1) \leq c^*$ and for every lattice $L \subset \mathbb{Q}_q^k$ there are r-edges $e_1, \ldots, e_M$ with target L and vectors $F_1, \ldots, F_M \in V$ such that

(i) $F_i \in N_t^{\mathrm{src}}(\mathrm{src}\, e_i)$;

(ii) $F_i \in \varpi^{c^*} R(e')$ for every r-edge $e' \neq e_i$ with $\mathrm{src}(e') = \mathrm{src}(e_i)$;

(iii) there are an integer $n \geq (1-\kappa)\dim V$, an $\mathcal{O}$-linear map $u\colon \mathcal{O}^n \to N_t^{\mathrm{tgt}}(L)$ and a matrix $A \in M_{n \times M}(\mathcal{O})$ with $\sum_i A_{ji}\, \mathrm{be}(F_i, u(z)) = p^D z_j$ for all $z \in \mathcal{O}^n$ and all j.

Condition (ii) says that F_i is concentrated on the edge e_i among the edges out of its source. For $\kappa \geq 1$ the statement holds with $M = n = 0$, so its content is the bound $\kappa < 1$. The *least share* $\kappa^{\min}(r, \lambda; t, D, c^*)$ is the least $\kappa \in [0, 1]$ for which $\mathrm{LP}(r, \lambda; t, D, \kappa, c^*)$ holds. It exists because the condition is monotone in κ and is determined by the integer $\lceil (1-\kappa)\dim V \rceil$, which takes finitely many values for $\kappa \in [0, 1]$.

The Hecke reading. For a function F on lattices with values in V, the Hecke operator of type r is $(T_r F)(L) = \sum_{\mathrm{tgt}(e)=L} F(\mathrm{src}\, e)$. The vectors F_i are local pieces of the operator $p^{A_r} T_r$, and condition (iii) says that $p^{A_r} T_r$, applied to forms that are integral in the enlarged sense, reaches p^D times a share $1-\kappa$ of the dual lattice at L. Subsection 12.3 uses the statement in the following form.

Lemma 12.7. *Suppose that* $\mathrm{LP}(r, \lambda; t, D, \kappa, c^*)$ *holds. Let* L *be a lattice and let* $H^{\mathrm{du}}(L) = \{F \in V : G(F) \in \mathcal{O} \text{ for all } G \in N_t^{\mathrm{tgt}}(L)\}$. *Then there are* r*-edges* $e_1, \dots, e_M$ *with target* L*, vectors* $F_k \in N_t^{\mathrm{src}}(\mathrm{src}\, e_k)$*, an* $\mathcal{O}$*-basis* (b_α) *of* $H^{\mathrm{du}}(L)$ *and a set* bad *of indices with* $|\mathrm{bad}| \le \kappa \dim V$ *such that*

(a) $p^{A_r} F_k \in \varpi^{c^*} H^{\mathrm{du}}(L')$ *for every* r*-edge* $(\mathrm{src}\, e_k \subset L')$ *with* $L' \neq L$*;*

(b) $p^D b_\alpha \in \mathrm{span}_{\mathcal{O}}\{p^{A_r} F_k\}$ *for every* $\alpha \notin \mathrm{bad}$.

Proof. Take the edges and vectors of Definition 12.6. For an r-edge e' with target L' we have $N_t^{\mathrm{tgt}}(L') \subset R'(e')$, and $R'(e')$ is the be-dual of $R(e')$ by Lemma 12.5. Hence $F \in R(e')$ implies $G(p^{A_r} F) = \mathrm{be}(F, G) \in \mathcal{O}$ for $G \in N_t^{\mathrm{tgt}}(L')$, that is, $p^{A_r} F \in H^{\mathrm{du}}(L')$. Applied to $e' = e_k$ and to the edges $e' \neq e_k$ of (ii), this gives $p^{A_r} F_k \in H^{\mathrm{du}}(L)$ and (a).

Put $\psi_j = \sum_k A_{jk} p^{A_r} F_k \in H^{\mathrm{du}}(L)$ and $u_l = u(\epsilon_l)$ for the unit vectors ϵ_l of $\mathcal{O}^n$. Then $u_l(\psi_j) = p^D \delta_{jl}$. Let $\Psi\colon H^{\mathrm{du}}(L) \to \mathcal{O}^n$ be $F \mapsto (u_l(F))_l$, so $\Psi(\psi_j) = p^D \epsilon_j$. The ψ_j are linearly independent, and the saturation Σ of their span in $H^{\mathrm{du}}(L)$ has rank n and is a direct summand, $H^{\mathrm{du}}(L) = \Sigma \oplus \Sigma'$. The map Ψ is injective on Σ, because $\Sigma \otimes \mathcal{K}$ is spanned by the ψ_j. For $F \in \Sigma$ the vector $\Psi(p^D F)$ lies in $p^D \mathcal{O}^n = \Psi(\mathrm{span}\, \psi)$, so $p^D F = \psi$ for some $\psi \in \mathrm{span}\, \psi_j$ by injectivity. Take for (b_α) a basis of Σ followed by a basis of Σ', and let bad index the latter. Then $|\mathrm{bad}| = \dim V - n \le \kappa \dim V$. □

Reduction to one edge. The next four statements reduce LP to families of sources and tests on single edges into one lattice.

Lemma 12.8. *If conditions (i) to (iii) of Definition 12.6 hold for one lattice* L_0*, they hold for every lattice.*

Proof. $\mathrm{GL}_k(\mathbb{Q}_q)$ acts transitively on lattices. An element g with $gL_0 = L$ maps r-edges to r-edges and maps H, R, R', N_t^{src} and N_t^{tgt} at L_0 to the same objects at L, and it preserves be. Transport the data along g. □

Lemma 12.9. *Let* Λ *be an* $\mathcal{O}$*-module and* $\varphi_1, \dots, \varphi_M\colon \Lambda \to \mathcal{O}$ *linear maps. Suppose that for every* $c \in \mathcal{O}^M$ *with some* c_i *a unit there is* $G \in \Lambda$ *with* $\mathrm{ord}_\varpi\left(\sum_i c_i \varphi_i(G)\right) \le 2D$*. Then there is a linear* $u\colon \mathcal{O}^M \to \Lambda$ *with* $\varphi_i(u(z)) = p^D z_i$ *for all* i.

Proof. Let $\Sigma \subset \mathcal{O}^M$ be the image of $G \mapsto (\varphi_i(G))_i$. By the elementary divisor theorem there are a basis $f_1, \dots, f_M$ of $\mathcal{O}^M$ and exponents $a_j \in \mathbb{N} \cup \{\infty\}$ with $\Sigma = \bigoplus_j \varpi^{a_j} \mathcal{O} f_j$. The coordinate functional f_j^* is given by a vector c with a unit entry, so the hypothesis gives an element of $f_j^*(\Sigma) = \varpi^{a_j}\mathcal{O}$ of valuation at most $2D$, and $a_j \le 2D$. Hence $p^D \mathcal{O}^M \subset \Sigma$. Choose $G_l \in \Lambda$ with $(\varphi_i(G_l))_i = p^D \epsilon_l$ and put $u(z) = \sum_l z_l G_l$. □

A *source-test system* at L is a set E of r-edges into L and, for each $e \in E$, vectors $F_{e,1}, \dots, F_{e,n_e}$ that satisfy (i) and (ii) of Definition 12.6, such that for every $c \in \mathcal{O}^{n_e}$ with a unit entry there is $G \in N_t^{\mathrm{tgt}}(L)$ with $\mathrm{ord}_\varpi \mathrm{be}(\sum_i c_i F_{e,i}, G) \le 2D$ and $G \in \varpi^{c^*} R'(e')$ for every $e' \in E \setminus \{e\}$.

Proposition 12.10. *Suppose $2(D+1) \le c^*$. If a source-test system exists at L with $\sum_e n_e \ge (1-\kappa)\dim V$, then (i) to (iii) of Definition 12.6 hold at L. If for every r-edge e into a fixed lattice L_0 there is a source-test system on $\{e\}$ alone with $n_e \ge \rho \dim V / q^{r(k-r)}$, and the tests satisfy $G \in \varpi^{c^*} R'(e')$ for the other edges e' into L_0, then $\mathrm{LP}(r, \lambda; t, D, \kappa, c^*)$ holds for every κ with $1-\kappa \le \rho$ and $4(D+1) \le c^*$.*

Proof. Take all the $F_{e,i}$ as the family of Definition 12.6, put $\Lambda = N_t^{\mathrm{tgt}}(L)$ and $\varphi_{e,i}(G) = \mathrm{be}(F_{e,i}, G)$. These are integral on Λ because $\Lambda \subset R'(e)$ and $F_{e,i} \in R(e)$, which are dual by Lemma 12.5. Let c have a unit entry in the block of e_0, and take the test G of e_0 for the restriction of c to that block. For $e \ne e_0$ we have $G \in \varpi^{c^*} R'(e)$, so $\varphi_{e,i}(G) \in \varpi^{c^*}\mathcal{O} \subset \varpi^{2D+2}\mathcal{O}$. So $\sum c_{e,i}\varphi_{e,i}(G)$ is an element of valuation at most $2D$ plus an element of valuation at least $2D+2$, and it has valuation at most $2D$. Lemma 12.9 gives (iii) with $n = \sum_e n_e$ and A the identity.

For the second statement, the r-edges into L_0 correspond to the subspaces of codimension r of $L_0/pL_0 \cong \mathbb{F}_q^k$, and their number is the Gaussian binomial coefficient $\binom{k}{r}_q \ge q^{r(k-r)}$. The union of the single-edge systems is a source-test system at L_0 with $\sum_e n_e \ge \rho \dim V \ge (1-\kappa)\dim V$, and Lemma 12.8 passes from L_0 to every lattice. □

The theorem. Fix the data of Subsection 12.1, a prime $p \in \mathcal{P}$ of class I with $n_p \ge x^{\varepsilon/4}$, and a good orbit $t_0 \in R_p$, and put $n' = n_p$. Let $\nu_1 = \lceil \max(6eG'\delta_1, (5+8\varepsilon)\log_2 x)\rceil + 1$ be the bound of Definition 5.19(ii), $Q_p = p^{G'}$, $N^+ = 40\,n'\delta_1$, $c_0 = 2b' + 2c_G$, $\mathrm{La} = K_{\mathrm{La}}\nu_1^2 \log a$, and let $K_\bullet$ denote constants. The base weight at a position $s \in \mathbb{Z}/n'$ of the orbit is a dominant weight λ_s of GL_k whose middle entries have absolute value at most c_G, and whose extreme entries are a_s and $-b_s$ when $t_0 + sh_p$ is a top. Put

$$
\begin{aligned}
D^{\mathrm{II}} &= \Big\lceil K_{DII} N^+ (\nu_1^3 + r_0^5 + k^2)\log_p a + 2r_0 n' c_G + r_0 N^+ c_0 \Big\rceil,\\
t^{\mathrm{II}} &= K_{tII}\, r_0\, N(t_0)\Big(\frac{\mathrm{La}\, a}{pk} + a\, p^{-L_1(p)} + c_0 + kQ_p\Big) + 2r_0 n' c_G,\\
\kappa^{\mathrm{II}} &= \min\big(1,\ K_{kII}\, N(t_0)\, r_0^2\, k\, x^{-26}\big),\\
D^{\mathrm{I}} &= \Big\lceil K_{DI} N^+ x^{26} (\nu_1^5 + k^3)\log_p a \Big\rceil,\\
t^{\mathrm{I}}(a,b) &= \sum_{y \text{ odd}} \min\Big(b_y, \frac{a_{y-d(y)}}{p^{d(y)}}\Big) + K_{\kappa 1} N(t_0)\nu_1\Big(\frac{a}{p^2} + \frac{\mathrm{La}\, a}{pk}\Big)\\
&\quad + K_{\kappa 2} N(t_0)\big(Q_p c_6 + kQ_p + k\log_p a + c_G\big) + 2n' c_G,
\end{aligned}
$$

where the sum runs over the odd tops y of the orbit, $d(y)$ is the distance from y back to the nearest even top (an odd number by Lemma 12.1, and 0 if there is none, in which case the term is 0), and $c_6 = 8(D^{\mathrm{I}}+1) + K_{\kappa 2} N^+ k \log_p a + 4N^+ c_G + 2n' c_G$.

Theorem 12.11. *Assume Hypothesis 12.12. There are constants $K_\bullet$, depending on ε, with the following properties for x large.*

(i) Small primes. *If $13 \le p < p_*$, the weights lie in the windows $a \le a_s \le a + a/(kx^{13})$ and $b' \le b_s \le b' + 2n\delta_1 x^{\varepsilon}\mathrm{La}\, b'/k$, and the size conditions (Z) of Subsection 12.6 hold, then $\mathrm{LP}(r_0, \lambda; \lceil t^{\mathrm{II}}\rceil, D^{\mathrm{II}}, \kappa^{\mathrm{II}}, 8(D^{\mathrm{II}}+1))$ holds.*

(ii) Medium primes. *Let the size conditions (Z) hold, and let $p_* \le p < X'$ with no power p^i in the band $|p^i - X'| \le \rho_B X'$, and let $\kappa^{\mathrm{I}}(a,b) = \kappa^{\min}(1, \lambda; \lceil t^{\mathrm{I}}(a,b)\rceil, D^{\mathrm{I}}, 8(D^{\mathrm{I}}+1))$. Then the mean of $\kappa^{\mathrm{I}}(a,b)$ over the weights in the windows above is at most $K_m\, N(t_0)\, \nu_1\, x^{-13}$.*

Outline of the proof. This proof is an outline. It uses Hypothesis 12.12, and the Pieri splitting at the tops and the passage from segment shares to the share of LP, statement (C11) of Subsection 12.8.5, are not written out here. In both cases Proposition 12.10 reduces the claim to the existence, for every r-edge e into the lattice W^k, of a source-test system on e of size at least $\rho \dim V/q^{r(k-r)}$, with $1-\rho$ equal to the share. For (i) the share κ^{II} comes from two counts. The Pieri rule splits the representation at the positions of the orbit into a flat part $\bigotimes_s \mathrm{Sym}^{d_s}(\mathcal{K}^k)$ at the tops and a remaining part, with a loss $(1-kc_0/a)^{N(t_0)}$ in dimension. The sources on e are products of symbol polynomials in the flat part with an edge-localizing polynomial and a basis of the remaining part, and the rank of the span of the symbol polynomials is at least $q^{-r(k-r)}(1-N(t_0)r_0^2k/Y)\prod_s \binom{d_s+k-1}{k-1}$ with $Y = x^{13}$ (Hypothesis 12.12(A)). The tests are pairings with the twisted Fock form after the Frobenius substitution adapted to e, and each source has a test at depth at most D^{II} (Hypothesis 12.12(B)). For (ii) the edges are 1-edges, the orbit is cut into segments between consecutive even tops, and the share is the sum of a model share $\kappa_P \le C_s N(t_0)\nu_1/Y$ and of the bad shares of the segment forms, whose mean over the windows is bounded by Hypothesis 12.12(C). The three engines of Hypothesis 12.12 are not proved here; Subsections 12.8 to 12.10 prove parts of them. □

Hypothesis 12.12 (engines of Hecke nondegeneracy)**.** In the setting of Theorem 12.11, the following three statements hold.

(A) *Rank of symbol spaces.* Let $\mathbb{F}$ be the residue field of $\mathcal{O}$. For a frame system of the orbit with $r = r_0$ frame tops per segment, built from the marked tiles of Definition 5.19(iii), and for degrees $a \le d_s \le 2a$, the span of the reductions of the segment generators has dimension at least $q^{-rK}(1-N(t_0)r^2k/Y)\prod_s \binom{d_s+k-1}{k-1}$ over $\mathbb{F}$, where $K = k-r$, provided $C_k N_s^2 \log a \le k$, $128k^2N_sY^2 \le p^{L_1(p)-1}$ and $32pYk^2N_sQ_p^2 \le a$, with N_s the maximal number of tops of a segment.

(B) *Detection for the twisted Fock form.* For every F of multidegree (d_s) satisfying the chain and frame conditions with balanced norm 0 there is an integral G satisfying the chain and frame conditions, with the Frobenius substitution ph adapted to the edge, such that $\mathrm{ord}_p \mathrm{B}(F, \mathrm{ph}(G)) \le n_0 + D_m + D_l + 3\kappa_p$, where $n_0 = \frac{1}{2}\mathrm{ord}_p \prod_s d_s!$ and D_m, D_l, κ_p are the logarithmic losses of the chain conditions, of the frame conditions and of the Pieri bases.

(C) *Mean of the segment shares.* For a segment with s tops and split index σ, the mean of its bad share over the top weights in the window and over the next top weight in the window is at most $C(s-\sigma+1)\nu_1/Y$.

Subsections 12.8, 12.9 and 12.10 prove the parts of Hypothesis 12.12 listed there and state the remaining steps. The remaining steps, and hence Hypothesis 12.12, are not proved here. Statement (A) rests on Nakayama's lemma over $\mathcal{O}$, the identity $f(X^{p^R}) = f(X)^{p^R}$ over $\mathbb{F}_p$ with a discriminant that is a nonzero divisor modulo ϖ, the uniqueness of base p^e expansions, an integral basis of monomials indexed by Pieri paths, which the literature calls standard monomials, and a concentration estimate for the weights of Pieri paths. Statement (B) rests on Fock-orthogonal projectors defined by the extremal projector of Asherova, Smirnov and Tolstoy, with denominators controlled by Kummer's theorem, and on the expansion of the unit parts of factorial digit splits in binomial polynomials. Statement (C) writes the Gram determinant of a segment form as a polynomial in the next top weight whose leading coefficient is the untwisted Fock Gram matrix, and bounds the mean valuation of such a polynomial over an arithmetic progression.

Corollary 12.13. *Put $\kappa_{t_0}(a,b) = \kappa^{\mathrm{II}}$ if $p < p_*$ and $\kappa_{t_0}(a,b) = \kappa^{\mathrm{I}}(a,b)$ if $p \geq p_*$, with the exponents t_{t_0} and D_p of the same case. For x large, for every candidate p and every good orbit $t_0 \in R_p$, the mean of κ_{t_0} over the weights in the windows is at most $N(t_0)x^{-12}$, and if $p < p_*$ then $\kappa_{t_0} \leq N(t_0)x^{-12}$ for all weights.*

Proof. For $p < p_*$, $\kappa^{\mathrm{II}} \leq K_{kII}N(t_0)r_0^2 k x^{-26}$ and $k \leq 2x^{9/2+10\varepsilon}$, so $K_{kII}r_0^2 k \leq x^{14}$ for large x. For $p \geq p_*$, Theorem 12.11(ii) gives the mean bound $K_m N(t_0)\nu_1 x^{-13}$, and $K_m\nu_1 \leq x$ for large x, since $G'\delta_1$ is at most a constant times $\log x$ and so $\nu_1 \leq C\log x$. □

12.3 From nondegeneracy to the colength

Let w be a finite place of $\mathbf{K}$ above a prime $p \in \mathcal{P}$ of class I with $p \leq X''$ and $\mathcal{V}_p \neq \emptyset$, and let u be an index with $p \neq T_u$. This subsection bounds the colength of a sublattice NP_w of the base lattice at w, spanned by images of powers of the Hecke operators at the served orbits.

The forms at w. Let $\mathcal{O}_w$ be the completion of the ring of integers of $\mathbf{K}$ at w, with fraction field $\mathbf{K}_w$, and let $v_w(p)$ be the normalized valuation of p in $\mathcal{O}_w$. The chain datum of Section 10 at level 0 (Section 11) gives the following structure at w, in which the served orbits $\mathcal{V} = \mathcal{V}_p$ index the coordinates.

(F1) A set $\mathcal{L}$ of lattice tuples $L = \big((L_v)_{v\in\mathcal{V}}, L'\big)$, where L_v is a lattice in $\mathbb{Q}_{q_v}^k$ for the unramified extension of degree n_p of $\mathbb{Q}_p$, and a group Γ that acts on $\mathcal{L}$ through elements $\gamma_v \in \mathrm{GL}_k(\mathbb{Q}_{q_v})$ on the components L_v, with finitely many orbits and trivial stabilizers.

(F2) A $\mathbf{K}_w$-vector space $X = \bigotimes_{v\in\mathcal{V}} V_v \otimes X'$ on which Γ acts, where V_v is the representation of Subsection 12.2 attached to the base weights at the orbit v, with γ acting on V_v through γ_v, and for each L an $\mathcal{O}_w$-lattice $H'(L) \subset X'$ with $H'(\gamma L) = \gamma H'(L)$. The completion $\mathcal{O}_w$ contains a uniformizer ϖ with $\varpi^2 = p$.

(F3) The space $\mathcal{F}$ of forms, the functions $F\colon \mathcal{L} \to X$ with $F(\gamma L) = \gamma F(L)$, and two lattices of forms $M \subset M^{\mathrm{na}}$, where M^{na} is the set of forms with $F(L) \in \bigotimes_v H(L_v) \otimes H'(L)$ for all L, and $p^{e_p}M^{\mathrm{na}} \subset M$.

(F4) For $v \in \mathcal{V}$ the Hecke operator $y_v = p^{A_r}T_{r,v}$, where $(T_{r,v}F)(L) = \sum F(L[v \leftarrow L''])$ is the sum over the r-edges $(L'' \subset L_v)$ and $L[v \leftarrow L'']$ is L with L_v replaced by L'', with $r = r_p$. The operators y_v map M and M^{na} into themselves.

(F5) A subspace $V^{\mathrm{ram}} \subset \mathcal{F}$, the forms of ramified type at T_u, such that $M^{\mathrm{ram}} = M \cap V^{\mathrm{ram}}$ is saturated in M.

The operators y_v and $y_{v'}$ commute, because they act on different components of the tuples.

For a family $\sigma = (\sigma_v)_{v\in\mathcal{V}}$ of *slots* $\sigma_v \in \{\mathrm{du}, \mathrm{ba}, \mathrm{ba}^+\}$ (dual, base, ball) let $M(\sigma)$ be the set of forms F with

$$F(L) \in \bigotimes_{v\in\mathcal{V}} H^{\sigma_v}(L_v) \otimes H'(L) \qquad (L \in \mathcal{L}),$$

where $H^{\mathrm{ba}}(L_v) = H(L_v)$, $H^{\mathrm{ba}^+}(L_v) = N^{\mathrm{src}}_{t_v}(L_v)$ and $H^{\mathrm{du}}(L_v)$ is the lattice of Lemma 12.7, all with $t_v = \lceil t_v \rceil$ the denominator of Theorem 12.11 at the orbit v. So $M(\mathrm{ba}, \ldots, \mathrm{ba}) = M^{\mathrm{na}}$. For $B \subset \mathcal{V}$ let σ_B be the family with du on B and ba off B, and write $\sigma[v \leftarrow s]$ for σ with σ_v replaced by s.

Lemma 12.14. *For every σ and $v \in \mathcal{V}$:*

(a) $M(\sigma[v \leftarrow \mathrm{du}]) \subset M(\sigma[v \leftarrow \mathrm{ba}]) \subset M(\sigma[v \leftarrow \mathrm{ba}^+])$;

(b) $\varpi^{t_v} M(\sigma[v \leftarrow \mathrm{ba}^+]) \subset M(\sigma[v \leftarrow \mathrm{ba}])$ *and* $\varpi^{t_v} M(\sigma[v \leftarrow \mathrm{ba}]) \subset M(\sigma[v \leftarrow \mathrm{du}])$;

(c) $y_v M(\sigma[v \leftarrow \mathrm{ba}^+]) \subset M(\sigma[v \leftarrow \mathrm{du}])$.

Proof. It suffices to compare the lattices at one coordinate. (a) $H(L) \subset \varpi^{-t} H(L)$, and $H(L) \subset R(e)$ for every edge e out of L by Lemma 12.5, so $H \subset N_t^{\mathrm{src}}$. Also $H(L)^\vee \subset \varpi^{-t} H(L)^\vee$ and $H(L)^\vee \subset R'(e)$ for every edge into L, so $H(L)^\vee \subset N_t^{\mathrm{tgt}}(L)$, and $H^{\mathrm{du}}(L) \subset H(L)^{\vee\vee} = H(L)$. (b) $N_t^{\mathrm{src}}(L) \subset \varpi^{-t} H(L)$, and $N_t^{\mathrm{tgt}}(L) \subset \varpi^{-t} H(L)^\vee$ gives $\varpi^t H(L) \subset H^{\mathrm{du}}(L)$. (c) If $F(L[v \leftarrow L']) \in N_t^{\mathrm{src}}(L'') \otimes (\cdots)$ for an edge $e = (L'' \subset L_v)$, then the factor at v lies in $R(e)$, and $p^{A_r} R(e) \subset H^{\mathrm{du}}(L_v)$ by the first paragraph of the proof of Lemma 12.7. Summing over the edges gives the claim, and the other coordinates are unchanged. □

Delta forms. For $L \in \mathcal{L}$ and $\xi \in X$ let $\delta_{L,\xi}$ be the form with $\delta_{L,\xi}(\gamma L) = \gamma\xi$ for $\gamma \in \Gamma$ and $\delta_{L,\xi} = 0$ off ΓL. It is well defined because the stabilizer of L is trivial. Let $\mathcal{R} \subset \mathcal{L}$ be a set of representatives of $\Gamma\backslash\mathcal{L}$. Every form satisfies $F = \sum_{L\in\mathcal{R}} \delta_{L,F(L)}$, so $F \mapsto (F(L))_{L\in\mathcal{R}}$ identifies $M(\sigma)$ with $\bigoplus_{L\in\mathcal{R}} \big(\bigotimes_v H^{\sigma_v}(L_v) \otimes H'(L)\big)$. In particular $M(\sigma)$ is free of finite rank, and

$$d^{\mathrm{na}} := \operatorname{rk} M^{\mathrm{na}} = \sum_{L\in\mathcal{R}} \operatorname{rk} H'(L) \prod_{v\in\mathcal{V}} \dim V_v.$$

Proposition 12.15. *Suppose that at every $v \in \mathcal{V}$ the statement* $\mathrm{LP}(r, \lambda_v; t_v, D, \kappa_v, c^*)$ *holds with $c^* \geq 4(D+1)$. Then for every nonempty $B \subset \mathcal{V}$ there is a finite set $C_B \subset M(\sigma_B)$ with $|C_B| \leq d^{\mathrm{na}} \prod_{v\in B} \kappa_v$ such that*

$$p^D M(\sigma_B) \subset \sum_{v\in B} y_v\big(M(\sigma_B[v \leftarrow \mathrm{ba}^+])\big) + \operatorname{span}_{\mathcal{O}_w} C_B.$$

Proof. Let Q be the right side. It is a submodule of $M(\sigma_B)$ by Lemma 12.14(c). Fix $L \in \mathcal{R}$. For each $v \in B$ apply Lemma 12.7 to the lattice L_v. This gives edges $e_k^v = (L_k^v \subset L_v)$, vectors $F_k^v \in N_{t_v}^{\mathrm{src}}(L_k^v)$, a basis (b_α^v) of $H^{\mathrm{du}}(L_v)$ and a set bad_v with $|\mathrm{bad}_v| \leq \kappa_v \dim V_v$. Choose bases of $H(L_v)$ for $v \notin B$ and of $H'(L)$. The pure tensors $\xi = \bigotimes_{v\in B} b_{\alpha_v}^v \otimes \eta$, with η a product of the chosen basis vectors, form a basis of $\bigotimes_v H^{(\sigma_B)_v}(L_v) \otimes H'(L)$.

Let ξ be such a tensor with $\alpha_v \notin \mathrm{bad}_v$ for some $v \in B$, and write $\xi = b_{\alpha_v}^v \otimes \xi'$. By Lemma 12.7(b) there are $c_k \in \mathcal{O}_w$ with $p^D b_{\alpha_v}^v = \sum_k c_k p^{A_r} F_k^v$. Put $L^{(k)} = L[v \leftarrow L_k^v]$ and $\xi_k = F_k^v \otimes \xi'$. Then $\delta_{L^{(k)},\xi_k} \in M(\sigma_B[v \leftarrow \mathrm{ba}^+])$, because the other components of $L^{(k)}$ are those of L. We claim that

$$y_v\, \delta_{L^{(k)},\xi_k} - \delta_{L,p^{A_r}\xi_k} \in \varpi^{c^*} M(\sigma_B).$$

The value of $y_v \delta_{L^{(k)},\xi_k}$ at L^* is p^{A_r} times the sum of the values $\delta_{L^{(k)},\xi_k}(L^*[v \leftarrow L''])$ over the edges $(L'' \subset L_v^*)$. A term is nonzero when $L^*[v \leftarrow L''] = \gamma L^{(k)}$ for some $\gamma \in \Gamma$, and then γ is unique, $L^* = \gamma\big(L[v \leftarrow L']\big)$ with $L' = \gamma_v^{-1} L_v^*$, the pair $(L_k^v \subset L')$ is an r-edge, and the term is $p^{A_r}\gamma\xi_k$. The terms with $L' = L_v$ occur exactly at the tuples γL, once each, and together they form $\delta_{L,p^{A_r}\xi_k}$. For the terms with $L' \neq L_v$, Lemma 12.7(a) gives $p^{A_r} F_k^v \in \varpi^{c^*} H^{\mathrm{du}}(L')$, so $p^{A_r}\gamma\xi_k$ lies in ϖ^{c^*} times the lattice of σ_B at $\gamma(L[v \leftarrow L']) = L^*$, by the equivariance $H^{\mathrm{du}}(\gamma_v L') = \gamma_v H^{\mathrm{du}}(L')$. The difference is a form, and its values lie in these lattices, which proves the claim. Hence

$$p^D \delta_{L,\xi} = \sum_k c_k \delta_{L,p^{A_r}\xi_k} \in Q + \varpi^{c^*} M(\sigma_B).$$

Let C_B be the set of the forms $\delta_{L,\xi}$ for $L \in \mathcal{R}$ and the pure tensors ξ with $\alpha_v \in \mathrm{bad}_v$ for all $v \in B$. Its size is at most $\sum_{L\in\mathcal{R}} \mathrm{rk}\, H'(L) \prod_{v\in B} |\mathrm{bad}_v| \prod_{v\notin B} \dim V_v \le d^{\mathrm{na}} \prod_{v\in B} \kappa_v$. Since the $\delta_{L,\xi}$ span $M(\sigma_B)$, we obtain $p^D M(\sigma_B) \subset Q + \varpi^{c^*} M(\sigma_B) \subset Q + p^{2D+2} M(\sigma_B)$.

Iterating, $p^D M(\sigma_B) \subset Q + p^{D+2} \cdot p^D M(\sigma_B) \subset Q + p^{j(D+2)} M(\sigma_B)$ for every $j \ge 1$. Since $\mathcal{O}_w$ is a complete Noetherian local ring and $M(\sigma_B)$ is finitely generated, the intersection of the submodules $Q + p^j M(\sigma_B)$ is Q by Krull's intersection theorem applied to $M(\sigma_B)/Q$. So $p^D M(\sigma_B) \subset Q$. □

The walk over the served orbits. For $B \subset \mathcal{V}$ write $y_B^m = \prod_{v\in B} y_v^m$. For numbers $S_v \in \mathbb{N}$ and a lattice of forms N that is stable under the y_v put

$$P(N) = \mathrm{span}_{\mathcal{O}_w} \Big\{ p^{\sum_{v\in\mathcal{V}\setminus B} S_v}\, y_B^m F :\ B \subset \mathcal{V},\ F \in N \Big\}.$$

Proposition 12.16. *Let π be an element of a commutative ring R, let $\Lambda(\sigma)$ be R-submodules of an R-module X indexed by the families of slots $\sigma \in \{\mathrm{du}, \mathrm{ba}, \mathrm{ba}^+\}^{\mathcal{V}}$, let y_v for $v \in \mathcal{V}$ be commuting endomorphisms of X, and let $a_v, b_v, c_v, c_0, S_v \in \mathbb{N}$, $m \ge 1$ and $\beta \ge 0$. Suppose that for all σ and v*

$$\begin{gathered}
\Lambda(\sigma[v \leftarrow \mathrm{du}]) \subset \Lambda(\sigma[v \leftarrow \mathrm{ba}]) \subset \Lambda(\sigma[v \leftarrow \mathrm{ba}^+]),\\
\pi^{a_v} \Lambda(\sigma[v \leftarrow \mathrm{ba}^+]) \subset \Lambda(\sigma[v \leftarrow \mathrm{ba}]),\\
\pi^{b_v} \Lambda(\sigma[v \leftarrow \mathrm{ba}]) \subset \Lambda(\sigma[v \leftarrow \mathrm{du}]),\\
\pi^{c_v} \Lambda(\sigma[v \leftarrow \mathrm{ba}^+]) \subset \Lambda(\sigma[v \leftarrow \mathrm{du}]),\\
y_v \Lambda(\sigma[v \leftarrow \mathrm{ba}^+]) \subset \Lambda(\sigma[v \leftarrow \mathrm{du}]),
\end{gathered}$$

and that for every $B \subset \mathcal{V}$ with $|B| = \beta + 1$ there is a finite set $G_B \subset \Lambda(\sigma_B)$ with

$$\pi^{c_0} \Lambda(\sigma_B) \subset \sum_{v\in B} y_v \Lambda(\sigma_B[v \leftarrow \mathrm{ba}^+]) + \mathrm{span}_R\, G_B.$$

Let $\Lambda_0 = \Lambda(\mathrm{ba}, \dots, \mathrm{ba})$, let $P_\pi(\Lambda_0)$ be the span of the elements $\pi^{\sum_{v\notin B} S_v} y_B^m F$ with $B \subset \mathcal{V}$ and $F \in \Lambda_0$, and put $E = \sum_v \big(a_v + b_v + (m-1)c_v + mc_0\big) + \beta \max_v S_v$. Then there is a set $C \subset \Lambda_0$ with $|C| \le m^{\beta+1} \sum_{|B|=\beta+1} |G_B|$ and

$$\pi^E \Lambda_0 \subset P_\pi(\Lambda_0) + \mathrm{span}_R\, C.$$

Proof. Order $\mathcal{V} = \{v_1, \dots, v_K\}$. If $K \le \beta$, then $\pi^{\sum_v S_v} \Lambda_0 \subset P_\pi(\Lambda_0)$ by the term $B = \emptyset$, and $\sum_v S_v \le E$. Let $K \ge \beta + 1$. We run a branching process whose states are triples (i, σ, z), where $i = (i_v) \in \{0, \dots, m\}^{\mathcal{V}}$ counts the applications of each y_v, σ is a family of slots, and $z \in \Lambda(\sigma)$. The element represented by a state is $\pi^\epsilon y^i z$, where $y^i = \prod_v y_v^{i_v}$ and ϵ is an exponent attached to the state. An orbit is *finished* when $i_v = m$. Let $D(i)$ be the set of unfinished orbits and, when $|D(i)| \ge \beta + 1$, let $B(i)$ be the set of its first $\beta + 1$ elements. The process keeps the following invariant: $\sigma_v = \mathrm{ba}$ for finished v and for $v \notin B(i)$, and $\sigma_v \in \{\mathrm{du}, \mathrm{ba}^+\}$ for $v \in B(i)$ with $i_v \ge 1$.

Start from $z \in \Lambda_0$ with $i = 0$, $\sigma = (\mathrm{ba}, \dots)$ and $\epsilon = 0$. At a state with $|D(i)| \ge \beta + 1$ and $B = B(i)$, multiply z by π^{b_v} for each $v \in B$ with $\sigma_v = \mathrm{ba}$ and by π^{c_v} for each $v \in B$ with $\sigma_v = \mathrm{ba}^+$. The result lies in $\Lambda(\sigma_B)$. Multiply by π^{c_0} and apply the hypothesis for B:

$$\pi^{c_0} \pi^{(\cdots)} z = \sum_{v\in B} y_v z_v + g, \qquad z_v \in \Lambda(\sigma_B[v \leftarrow \mathrm{ba}^+]),\ g \in \mathrm{span}\, G_B.$$

The state is replaced by the states $(i+\epsilon_v, \sigma_B[v \leftarrow \mathrm{ba}^+], z_v)$ for $v \in B$, where ϵ_v is the unit vector at v, and the element $\pi^{\epsilon'} y^i g$ is set aside as an exceptional term. If some v becomes finished, multiply z_v by π^{a_v} to bring its slot at v back to ba. Since $B(i)$ consists of the first $\beta+1$ unfinished orbits, an orbit enters $B(i)$ once, when it has $i_v = 0$ and slot ba, and leaves it when it is finished. So along each branch an orbit v that is finished has cost $b_v + (m-1)c_v + a_v$ in powers of π, and every application of the hypothesis costs c_0 and increases $\sum_v i_v$ by one.

Every branch reaches a state with $|D(i)| \le \beta$ after at most mK steps. At such a state the element is $\pi^{\epsilon} y^i z$ with $\epsilon \le \sum_v (a_v + b_v + (m-1)c_v + mc_0)$. For $v \in D(i)$ the slot of z at v is ba, or du $\subset$ ba, or ba^+ with $i_v \ge 1$, in which case $y_v^{i_v}$ moves it into du, or ba^+ with $i_v = 0$, which does not occur by the invariant. So $F = \prod_{v \in D(i)} y_v^{i_v} z \in \Lambda_0$, and

$$\pi^E y^i z = \pi^{E-\epsilon-\sum_{v\in D(i)} S_v} \cdot \pi^{\epsilon} \left(\pi^{\sum_{v\in D(i)} S_v} y^m_{\mathcal{V}\setminus D(i)} F \right) \in P_\pi(\Lambda_0),$$

since $|D(i)| \le \beta$ gives $\epsilon + \sum_{v\in D(i)} S_v \le E$. The exceptional terms set aside at states with counter i and set B lie in $\pi^{(\cdots)} y^i \operatorname{span} G_B \subset \operatorname{span}(y^i G_B)$. At such a state $i_v = m$ for the orbits before B in the order, $i_v = 0$ for the orbits after B, and $0 \le i_v < m$ for $v \in B$, so the counter is determined by B and $(i_v)_{v\in B} \in \{0, \dots, m-1\}^B$. Let C be the union of the sets $y^i G_B$ over these pairs (B, i). Then $|C| \le m^{\beta+1} \sum_B |G_B|$, and $\pi^E z$ is the sum of the elements of $P_\pi(\Lambda_0)$ obtained at the ends of the branches and of π-multiples of elements of $\operatorname{span} C$. □

Lemma 12.17. *Let $\mathcal{O}$ be a discrete valuation ring with uniformizer ϖ, Λ a free $\mathcal{O}$-module of rank d, $P \subset \Lambda$ a submodule, $\pi = p$ an element with $v(p) \ge 1$, and $\tau, S, S_{\mathrm{top}}, \beta_0 \in \mathbb{N}$. Suppose that for $0 \le \beta \le \beta_0$ there is a set $C_\beta \subset \Lambda$ of g_β elements with $p^{\tau+\beta S}\Lambda \subset P + \operatorname{span} C_\beta$, and that $p^{\tau+S_{\mathrm{top}}}\Lambda \subset P$. Then*

$$\operatorname{length}_{\mathcal{O}} \left(\Lambda / (P \cap p^\tau \Lambda) \right) \le v(p) \Big(d\tau + S \sum_{\beta<\beta_0} g_\beta + S_{\mathrm{top}}\, g_{\beta_0} \Big).$$

Proof. The module $p^\tau\Lambda/(P \cap p^\tau\Lambda)$ is isomorphic to $(P + p^\tau\Lambda)/P$, so the length on the left is $v(p)d\tau + \operatorname{length}\big((P+p^\tau\Lambda)/P\big)$. Put $\Phi_\beta = (P + p^{\tau+\beta S}\Lambda)/P$ for $0 \le \beta \le \beta_0$. The module Φ_β is contained in $(P + \operatorname{span} C_\beta)/P$, which is generated by g_β elements, and over a discrete valuation ring a submodule of a module generated by g elements is generated by g elements. The quotient $\Phi_\beta/\Phi_{\beta+1}$ is generated by g_β elements and killed by p^S, so its length is at most $v(p)Sg_\beta$. The module Φ_{β_0} is generated by g_{β_0} elements and killed by $p^{S_{\mathrm{top}}}$, because $p^{\tau+S_{\mathrm{top}}}\Lambda \subset P$, so its length is at most $v(p)S_{\mathrm{top}}g_{\beta_0}$. Summing over the filtration $\Phi_0 \supset \Phi_1 \supset \cdots \supset \Phi_{\beta_0} \supset 0$ gives the claim. □

The sublattice and its colength. Put $S_v = S_{t_0}$ for $v = t_0 \in \mathcal{V}$ (Definition 12.3), $S = \max_v S_v$, and

$$NP_w = P(M) \cap p^{\lceil \tau(p) \rceil} M \cap V^{\mathrm{ram}}.$$

When $\mathcal{V}_p = \emptyset$ we put $NP_w = M^{\mathrm{ram}}$.

Theorem 12.18. *In the setting (F1) to (F5), suppose that* LP *holds at every $v \in \mathcal{V}_p$ with the exponents $\lceil t_v \rceil$, D_p, the share κ_v and $c^* = 8(D_p+1)$ of Corollary 12.13, and that* $\mathrm{Sig}(p) = m\sum_v \kappa_v \le \frac{1}{2}$. *Then*

(a) $NP_w \subset p^{\lceil \tau(p) \rceil} M^{\mathrm{ram}}$;

(b) $\operatorname{length}_{\mathcal{O}_w} \left(M^{\mathrm{ram}} / NP_w \right) \le v_w(p)\ \mathrm{rk}(M)\ \big(\tau(p) + \mathrm{io}(p)\big)$.

Proof. (a) $NP_w \subset p^{\lceil\tau\rceil}M \cap V^{\mathrm{ram}}$, and an element $p^{\lceil\tau\rceil}F$ with $F \in M$ that lies in V^{ram} has $F \in M^{\mathrm{ram}}$ because M^{ram} is saturated in M.

(b) Put $\tau_0 = \sum_{v\in\mathcal{V}} m(t_v + D_p + 1)$, so that $\lceil\tau(p)\rceil = e_p + \tau_0$. Apply Proposition 12.16 with $R = \mathcal{O}_w$, $\pi = \varpi$, $\Lambda(\sigma) = M(\sigma)$, $a_v = b_v = t_v$, $c_v = 2t_v$, $c_0 = 2(D_p + 1)$, and with $2S_v$ in place of S_v. The hypotheses on the slots are Lemma 12.14, where $\varpi^{2t_v}\Lambda(\sigma[v \leftarrow \mathrm{ba}^+]) \subset \Lambda(\sigma[v \leftarrow \mathrm{du}])$ is the composite of the two inclusions of (b) of that lemma. The hypothesis on the sets B is Proposition 12.15, since $p^D = \varpi^{2D}$ and $\varpi^{2D+2}\Lambda \subset \varpi^{2D}\Lambda$. The exponent is $E = 2(\tau_0 + \beta S)$, and $P_\varpi(M^{\mathrm{na}}) = P(M^{\mathrm{na}})$. So for every $\beta \geq 0$

$$p^{\tau_0+\beta S}M^{\mathrm{na}} \subset P(M^{\mathrm{na}}) + \operatorname{span} C_\beta, \qquad |C_\beta| \leq d^{\mathrm{na}} m^{\beta+1} e_{\beta+1}(\kappa),$$

where $e_{\beta+1}(\kappa)$ is the elementary symmetric function of the κ_v. Since $m^{\beta+1}e_{\beta+1}(\kappa) \leq \mathrm{Sig}(p)^{\beta+1}/(\beta+1)!$, the set C_β is empty for $\beta \geq |\mathcal{V}|$. Now let $F \in M$. Then $F \in M^{\mathrm{na}}$, so $p^{\tau_0+\beta S}F = P_0 + c$ with $P_0 \in P(M^{\mathrm{na}})$ and $c \in \operatorname{span} C_\beta$. Multiplying by p^{e_p} and using $p^{e_p}M^{\mathrm{na}} \subset M$ and the commutation of the y_v with scalars gives $p^{e_p}P(M^{\mathrm{na}}) \subset P(p^{e_p}M^{\mathrm{na}}) \subset P(M)$, so

$$p^{\lceil\tau\rceil+\beta S}M \subset P(M) + \operatorname{span}(p^{e_p}C_\beta), \qquad p^{e_p}C_\beta \subset M.$$

The term $B = \emptyset$ of $P(M)$ gives $p^{\sum_v S_v}M \subset P(M)$. Lemma 12.17 with $\Lambda = M$, $\tau = \lceil\tau(p)\rceil$, $\beta_0 = |\mathcal{V}|$ and $S_{\mathrm{top}} = \sum_v S_v$ gives

$$\begin{aligned}\operatorname{length}\big(M/(P(M)\cap p^{\lceil\tau\rceil}M)\big) &\leq v_w(p)\Big(\operatorname{rk}(M)\lceil\tau\rceil + S\,d^{\mathrm{na}}\sum_{\beta\geq 0}\mathrm{Sig}(p)^{\beta+1}\Big)\\ &\leq v_w(p)\operatorname{rk}(M)\big(\tau(p) + 2S\,\mathrm{Sig}(p)\big),\end{aligned}$$

using $\mathrm{Sig}(p) \leq \frac{1}{2}$, $d^{\mathrm{na}} = \operatorname{rk} M$ (by (F3) the lattices $M \subset M^{\mathrm{na}}$ have the same rank), and $\lceil\tau(p)\rceil = \tau(p)$, an integer. The right side is $v_w(p)\operatorname{rk}(M)(\tau(p) + \mathrm{io}(p))$ by Definition 12.3. Finally, the inclusion $M^{\mathrm{ram}} \to M$ induces an injection of M^{ram}/NP_w into $M/(P(M)\cap p^{\lceil\tau\rceil}M)$, because $NP_w = M^{\mathrm{ram}} \cap \big(P(M)\cap p^{\lceil\tau\rceil}M\big)$. □

The first claim of Theorem 12.18 gives that NP_w is spanned by forms of the shape $y_{B'}^m F_0$ with $F_0 \in p^{\sum_{v\in\mathcal{V}\setminus B'} S_v}M$, intersected with $p^\tau M^{\mathrm{ram}}$, which is the hypothesis of Theorem 10.7 with $B = \mathcal{V}_p$. This is the link between the loss and the divisibility of the lifts, which Section 13 uses.

12.4 The lattice N_u and hypothesis (3.2)

Hermitian lattices and slopes. A hermitian lattice over $\mathcal{O}_{\mathbf{K}}$ is a finitely generated projective $\mathcal{O}_{\mathbf{K}}$-module Λ with a hermitian norm on $\Lambda \otimes_\sigma \mathbb{C}$ for every embedding $\sigma\colon \mathbf{K} \to \mathbb{C}$. Its normalized Arakelov degree is

$$\widehat{\deg}\Lambda = \frac{1}{[\mathbf{K}:\mathbb{Q}]}\Big(\log\#\big(\Lambda/(\mathcal{O}_{\mathbf{K}}s_1 \oplus \cdots \oplus \mathcal{O}_{\mathbf{K}}s_d)\big) - \sum_\sigma \log\|s_1 \wedge \cdots \wedge s_d\|_\sigma\Big)$$

for any $\mathbf{K}$-basis $s_1, \ldots, s_d$ of $\Lambda \otimes \mathbf{K}$ contained in Λ, and the value does not depend on the basis by the product formula. Following the normalization of Sections 13 and 15, the slope of a nonzero lattice is $\mu(\Lambda) = 2\widehat{\deg}\Lambda/\operatorname{rk}\Lambda + c_{\mathbf{K}}$ with a constant $c_{\mathbf{K}}$ that depends on $\mathbf{K}$, and $\mu_{\max}(\Lambda)$ is the maximum of $\mu(\Lambda')$ over the nonzero sublattices $\Lambda' \subset \Lambda$ with the induced metrics. The proofs below use differences of slopes of lattices over the same field, so the value of $c_{\mathbf{K}}$ plays no role.

Lemma 12.19. *Let $\Lambda' \subset \Lambda$ be hermitian lattices of the same rank d in the same* $\mathbf{K}$*-space with the same metrics, and let $\mathfrak{a} = \prod_w \mathfrak{p}_w^{n_w}$ be a fractional ideal. For a finite place w let ℓ_w^{len} be the length of the $\mathcal{O}_w$-module Λ_w/Λ'_w. Then*

$$\widehat{\deg}(\mathfrak{a}\Lambda') = \widehat{\deg}\Lambda - \sum_w \ell_w\, \ell_w^{\mathrm{len}} - d\sum_w \ell_w\, n_w, \qquad \ell_w = \frac{\log N\mathfrak{p}_w}{[\mathbf{K}:\mathbb{Q}]}.$$

Proof. Take the basis s_i inside $\mathfrak{a}\Lambda'$. Then $\log \#(\Lambda/\mathfrak{a}\Lambda')$ is defined when $\mathfrak{a}\Lambda' \subset \Lambda$, and in general the index $[\Lambda : \mathfrak{a}\Lambda']$ is the generalized index $\prod_w N\mathfrak{p}_w^{\mathrm{len}_w(\Lambda_w/\Lambda'_w)+dn_w}$, which is multiplicative in towers. The index of $\mathcal{O}s_1 \oplus \cdots \oplus \mathcal{O}s_d$ in $\mathfrak{a}\Lambda'$ equals its index in Λ divided by $[\Lambda : \mathfrak{a}\Lambda']$, and the archimedean terms do not change. The local index at w of $\mathfrak{p}_w^{n_w}\Lambda'_w$ in Λ_w has logarithm $\big(\mathrm{len}_w(\Lambda_w/\Lambda'_w) + dn_w\big)\log N\mathfrak{p}_w$, because $\Lambda'_w/\mathfrak{p}_w^{n_w}\Lambda'_w$ has length dn_w for $n_w \geq 0$, and for $n_w < 0$ the module $\mathfrak{p}_w^{n_w}\Lambda'_w$ contains Λ'_w with a quotient of length $-dn_w$, and lengths add along the chain $\mathfrak{p}_w^{n_w}\Lambda'_w \supset \Lambda'_w \subset \Lambda_w$ in the generalized index. □

The data at u. Fix $u_0 \in A_0$ and an index u of the family W of Proposition 3.1. Sections 11, 16 and 18 give the following objects, listed in Subsection 12.6 as (L4) to (L8): a hermitian lattice M_u over $\mathcal{O}_{\mathbf{K}}$ of rank d_M with slope $\mu_u^M = \mu(M_u)$, a saturated sublattice M_u^{ram} of rank d_u, an auxiliary prime T_u, the structure (F1) to (F5) at every place w above a prime $p \in \mathcal{P}$ with $p \leq X''$ and $p \neq T_u$, a sublattice $N_w^T \subset M_{u,w}^{\mathrm{ram}}$ at every place $w \mid T_u$, and real functions $F_j(w)$ and $E_{u,j}(w)$ of the finite places, for $1 \leq j \leq m$, with finite supports. Put

$$\mathcal{L} = \sum_w \sigma\Big(w, \sum_{j=1}^m F_j(w)\Big), \qquad E_u = \sum_w \sigma\Big(w, \sum_{j=1}^m E_{u,j}(w)\Big),$$

with $\sigma(w,y) = 2\ell_w e_w y$ as in Remark 12.4. Let W_u be the finite set of the places w above the primes of $\mathcal{P}$ up to X'', above T_u, and in the supports of the F_j and $E_{u,j}$, and for $w \in W_u$ with residue characteristic p_w put

$$s_u(w) = \sum_{j=1}^m \Big(\mathrm{gain}_j^{\mathcal{P}}(p_w) + F_j(w) - E_{u,j}(w)\Big),$$

where $\mathrm{gain}_j^{\mathcal{P}}(p) = 0$ for $p \notin \mathcal{P}$. Here the weights are the weights a_t fixed in Subsection 12.1 and the bottom weights b_t of the nodes.

Definition 12.20. Let $\Lambda_u \subset M_u^{\mathrm{ram}}$ be the lattice whose completion at w is NP_w (Theorem 12.18) at the places w above primes $p \in \mathcal{P}$ with $p \leq X''$, $p \neq T_u$ and $\mathcal{V}_p \neq \emptyset$, is N_w^T at the places above T_u, and is $M_{u,w}^{\mathrm{ram}}$ at every other place. Put

$$N_u = \prod_{w \in W_u} \mathfrak{p}_w^{-\lfloor e_w s_u(w)\rfloor}\, \Lambda_u, \qquad \mu_u^N = \mu(N_u),$$

with the metrics of M_u.

The lattice Λ_u exists because a lattice is determined by its completions, which may be prescribed at finitely many places. The $\mathcal{O}_{\mathbf{K}}$-module Λ_u is a sublattice of M_u^{ram} of the same rank d_u.

Theorem 12.21. *Suppose the inputs (L1) to (L8) and the size conditions (Z) of Subsection 12.6, and that* $\mathrm{Sig}(p) \le \frac{1}{2}$ *whenever* $\mathcal{V}_p \neq \emptyset$. *Let* $\lambda_T = \frac{1}{4} C_T N_T m(a/k + b')$ *be the number of (L7). Then*

(i) $\mu_u^M + \Gamma + \mathcal{L} - E_u - e_2 \le \mu_u^N$ *with*

$$e_2 = 4\,\mathrm{IO} + \mathrm{LT} + \mathrm{RX} + 1, \quad \mathrm{LT} = 4\lambda_T \log T_u, \quad \mathrm{RX} = 2 \sum_{w \in W_u} \ell_w \le 4X'' + 2\log T_u + 2X'';$$

(ii) $\mu_{\max}(N_u) \le \mu_{\max}(M_u) + \Gamma + \mathcal{L} - E_u$.

Proof. Write W_T for the places above T_u, $\tau(w) = \tau(p_w)$ and $\mathrm{io}(w) = \mathrm{io}(p_w)$ for $p_w \in \mathcal{P}$, $p_w \le X''$, and $\tau(w) = \mathrm{io}(w) = 0$ otherwise. Since $T_u \ge X'$ by (L6), the prime T_u is not a candidate, so $\mathcal{V}_{T_u} = \emptyset$ and $\tau = \mathrm{io} = 0$ on W_T. Let $\mathrm{len}(w)$ be the length of $M_{u,w}^{\mathrm{ram}}/\Lambda_{u,w}$ divided by e_w, and put $s'(w) = \lfloor e_w s_u(w) \rfloor / e_w$, so that $0 \le s_u(w) - s'(w) < 1/e_w$.

Step 1, the slope of N_u. Lemma 12.19 with $\Lambda = M_u^{\mathrm{ram}}$, $\Lambda' = \Lambda_u$ and $n_w = -e_w s'(w)$, divided by d_u, gives

$$\mu_u^N = \mu(M_u^{\mathrm{ram}}) + \sum_{w \in W_u} 2\ell_w e_w \Big(s'(w) - \frac{\mathrm{len}(w)}{d_u} \Big).$$

Step 2, the colengths. For $w \notin W_T$ with $\mathcal{V}_{p_w} \neq \emptyset$, Theorem 12.18(b) gives $\mathrm{len}(w) \le d_M(\tau(w) + \mathrm{io}(w))$, using $v_w(p) = e_w$. At the other places $w \notin W_T$ the colength is 0. For $w \in W_T$, (L7) gives $\mathrm{len}(w) \le d_M \lambda_T$. Put $\mathrm{de} = 1 - d_u/d_M$. By (L5), $d_u = d_M(1 - 1/(T_u - 1))^f$, so Bernoulli's inequality gives $0 \le \mathrm{de} \le f/(T_u - 1) \le 2f/T_u \le \frac{1}{2}$ by (Z). Hence $d_M/d_u = 1/(1 - \mathrm{de}) \le 2$ and

$$\sum_{w \notin W_T} \frac{2\ell_w e_w \mathrm{len}(w)}{d_u} \le \frac{1}{1 - \mathrm{de}} \sum_w 2\ell_w e_w \big(\tau(w) + \mathrm{io}(w)\big) = \frac{\mathrm{Ta} + 2\,\mathrm{IO}}{1 - \mathrm{de}},$$

where $\mathrm{Ta} = \sum_w 2\ell_w e_w \tau(w) = \sum_p 2 \log p\, \tau(p)$ and $\sum_w 2\ell_w e_w \mathrm{io}(w) = 2\,\mathrm{IO}$, both by Remark 12.4. Also $\sum_{w \in W_T} 2\ell_w e_w \mathrm{len}(w)/d_u \le 2\lambda_T \cdot 2\log T_u = \mathrm{LT}$.

Step 3, rounding. $\sum_w 2\ell_w e_w (s_u(w) - s'(w)) \le \sum_{w \in W_u} 2\ell_w = \mathrm{RX}$. For a prime p, $\sum_{w|p} \ell_w = \sum_{w|p} f_w \log p / [\mathbf{K} : \mathbb{Q}] \le \log p$, so $\mathrm{RX} \le 2\sum_{p \le X''} \log p + 2\log T_u + 2\sum_{p \in P_{FE}} \log p$, where P_{FE} is the set of the residue characteristics of the supports of the F_j and $E_{u,j}$. Chebyshev's bound $\sum_{p \le y} \log p \le 2y$ and (L8) give the bound for RX.

Step 4, the gain. By Remark 12.4 and the definitions,

$$\sum_w 2\ell_w e_w \big(s_u(w) - \tau(w)\big) = \Gamma + \mathcal{L} - E_u.$$

Steps 1 to 4 give

$$\mu_u^N \ge \mu(M_u^{\mathrm{ram}}) + \Gamma + \mathcal{L} - E_u - \mathrm{RX} - \mathrm{LT} - \frac{2\,\mathrm{IO}}{1 - \mathrm{de}} - \frac{\mathrm{de}}{1 - \mathrm{de}} \mathrm{Ta},$$

because $\mathrm{Ta}/(1 - \mathrm{de}) = \mathrm{Ta} + \mathrm{de}\,\mathrm{Ta}/(1 - \mathrm{de})$. Now $2\,\mathrm{IO}/(1 - \mathrm{de}) \le 4\,\mathrm{IO}$, and $\mathrm{de}\,\mathrm{Ta}/(1 - \mathrm{de}) \le 2\,\mathrm{de}\, b^2 \le 4fb^2/T_u$ by Lemma 12.22 and (Z). By (L5), $\mu(M_u^{\mathrm{ram}}) \ge \mu_u^M - f\log(T_u - 1)/(T_u - 2)$, and (Z) gives $4fb^2/T_u + f\log(T_u - 1)/(T_u - 2) \le 1$. This proves (i).

(ii) Every place w with $\mathcal{V}_{p_w} \neq \emptyset$ has $\Lambda_{u,w} = NP_w \subset p^{\tau(w)} M_{u,w}^{\mathrm{ram}} = \mathfrak{p}_w^{e_w \tau(w)} M_{u,w}^{\mathrm{ram}}$ by Theorem 12.18(a), and at the other places $\tau(w) = 0$ and $\Lambda_{u,w} \subset M_{u,w}^{\mathrm{ram}}$. Hence $N_u \subset \mathfrak{b} M_u^{\mathrm{ram}}$ with $\mathfrak{b} = \prod_w \mathfrak{p}_w^{e_w \tau(w) - e_w s'(w)}$. A sublattice N' of N_u is a sublattice of $\mathfrak{b} M_u^{\mathrm{ram}}$, and $\mathfrak{b}^{-1} N'$ is a sublattice of $M_u^{\mathrm{ram}} \subset M_u$, so Lemma 12.19 gives $\mu(N') = \mu(\mathfrak{b}^{-1} N') + \sum_w 2\ell_w e_w (s'(w) - \tau(w)) \le \mu_{\max}(M_u) + \sum_w 2\ell_w e_w (s_u(w) - \tau(w))$. Step 4 finishes the proof. □

Lemma 12.22. *Under the size conditions (Z),* $\mathrm{Ta} = \sum_{p\in\mathcal{P}} 2\log p\,\tau(p) \le b^2$.

Proof. $\tau(p) = 0$ unless $p < X'$ is a candidate. For a candidate, $\tau(p) \le e_p + m\sum_{t_0\in\mathcal{V}_p}(t_{t_0} + D_p + 2)$, and $\sum_{t_0} N(t_0) \le N_T$, $|\mathcal{V}_p| \le N_T$. By (L2), $e_p \le (C_e+1)fk^2\log_p a$. The definitions of Subsection 12.2 and Lemma 12.2 give $t_{t_0} + D_p + 2 \le \tau^+ N(t_0)$ with the number τ^+ of (Z). So $\mathrm{Ta} \le \sum_{p<X'} 2\log p\big((C_e+1)fk^2\log_p a + mN_T\tau^+\big) \le 4X'\big((C_e+1)fk^2\log a + mN_T\tau^+\big)$, which is at most b^2 by (Z). □

Theorem 12.21(i) is hypothesis (3.2) of Proposition 3.1 for the index u, with the error e_2 above. The term 4 IO is bounded in Subsection 12.5, and Subsection 12.6 bounds LT and RX by $x^{C\varepsilon}\mathcal{B}$. The next subsection proves (3.1) for the same number Γ.

12.5 The gain against the sum over primes, and hypothesis (3.1)

This subsection compares, prime by prime, the contribution $2\log p\sum_{j,t}\mathrm{ask}_{j,t}(p)$ of p to S with its contribution $2\log p\big(\sum_j \mathrm{gain}_j^{\mathcal{P}}(p) - \tau(p)\big)$ to Γ. The bottom weights b_t are those of the nodes, so that (W1) holds with them.

Local losses. Put $\ell(p) = \log_p(2a) + 1$ and $\mathrm{cp}(p) = 5c_G\,\ell(p)$. Let $N^\sharp = 2x^{2+8\varepsilon}$, which bounds n_p and $N(t_0)$, let $c_0 = 2b' + 2c_G$, and put

$$\begin{aligned}
\Theta^{\mathrm{II}} &= r_0(2b'+c_G) + \frac{5r_0C_{31}\log x}{\delta_1}(w'' + \eta + c_G + N_m) + K_{tII}r_0(c_0 + kQ_{p_*}) \\
&\quad + 40C_{31}\log x\big(K_{DII}(\nu_1^3 + r_0^5 + k^2)\log a + r_0c_0\big) + 4, \\
c_6' &= 320C_{31}\log x\big(K_{DI}x^{26}(\nu_1^5 + k^3)\log a + 1\big) \\
&\quad + 40K_{\kappa 2}C_{31}k\log a\,\log x + \frac{164\,C_{31}c_G\log x}{\delta_1} + 8, \\
\Theta^{\mathrm{I}}(p) &= \frac{b'}{x} + \eta + \frac{(K_{\mathrm{Pr}}+2)C_{31}\log x}{\delta_1}(w'' + \eta + c_G + N_m) \\
&\quad + K_{\kappa 2}\big(N^\sharp Q_p c_6' + kQ_p + k\log a + c_G\big) + 40K_{DI}C_{31}x^{26}(\nu_1^5 + k^3)\log a\log x + 4.
\end{aligned}$$

For a prime $p \in \mathcal{P}$ of class I with $p < X'$ put

$$\mathrm{lo}(p) = \begin{cases} \mathrm{cp}(p) + \Theta^{\mathrm{II}} + K_{tII}r_0\Big(\dfrac{\mathrm{La}\,a}{pk} + a\,p^{-L_1(p)}\Big), & p < p_*, \\ \mathrm{cp}(p) + \Theta^{\mathrm{I}}(p) + K_{\kappa 1}\nu_1\Big(\dfrac{a}{p^2} + \dfrac{\mathrm{La}\,a}{pk}\Big), & p \ge p_*, \end{cases}$$

and $\mathrm{lo}(p) = \mathrm{cp}(p)$ for the other $p \in \mathcal{P}$. Put $\mathrm{fl}(p) = \sum_{t_0\in R_p}\mathrm{dl}(t_0) + e_p$.

Lemma 12.23. *Let $p \in \mathcal{P}$, $1 \le j \le m$, $s \in T$ and e odd, and put $\rho = \pi_p(s,e)$, $A = a_s/p^e$. Then*

$$\begin{aligned}
\max\big(0, A - w_j(\rho)\big) &\le \max\Big(0, \frac{\mathrm{al}(s)}{p^e} - \mathrm{cap}_j(s,e)\Big) + 3c_G\,\mathbf{1}[A > w_j(\rho)], \\
\max\big(0, A - w_j(\rho)\big) &\le \frac{\mathrm{al}(s)}{p^e} + \frac{2c_G}{p^e} - \min\big(A, w_j(\rho)\big).
\end{aligned}$$

For fixed s and j, the number of odd e with $A > w_j(\rho)$ is at most $\ell(p)$.

Proof. $\mathrm{cap}_j(s,e) = w_j(\rho) - \kappa_j + c_G$, so $A - w_j(\rho) = \mathrm{al}(s)/p^e - \mathrm{cap}_j(s,e) + 2c_G/p^e + c_G - \kappa_j$, and $0 \le c_G - \kappa_j \le c_G$, $2c_G/p^e \le 2c_G$. This gives the first inequality when $A > w_j(\rho)$, and both sides are compared with 0 otherwise. The second is $\max(0, A-w) = A - \min(A,w)$ and $a_s = \mathrm{al}(s) + 2c_G$. Finally $w_j(\rho) \ge 1$ and $a_s \le 2a$, so $A > w_j(\rho)$ forces $p^e < 2a$. □

Proposition 12.24. *Let p be a candidate with $\mathrm{Sig}(p) \le \theta$, $t_0 \in \mathcal{V}_p$ and $1 \le j \le m$. Suppose the weights lie in the windows of Theorem 12.11. Then*

$$\sum_{s \in T(t_0)} \big(\mathrm{ask}_{j,s}(p) - \mathrm{lo}(p)\big) \le g_j^1(t_0) + \mathrm{dl}(t_0) - \mathrm{Pr}_j(t_0) - \lceil t_{t_0} \rceil - D_p - 1.$$

Proof. Write $N = N(t_0)$. By (12.5),

$$g_j^1(t_0) + \mathrm{dl}(t_0) = \sum_{s \text{ even}} \sum_{e \text{ odd}} \frac{\mathrm{al}(s)}{p^e} + \sum_{s \text{ odd}} \sum_{e \text{ odd}} \max\Big(0, \frac{\mathrm{al}(s)}{p^e} - \mathrm{cap}_j(s,e)\Big),$$

since $\gamma_p \,\mathrm{al}(s) = \sum_{e \text{ odd}} \mathrm{al}(s)/p^e$ and $y - \min(y,c) = \max(0, y-c)$. For an odd top s, the first inequality of Lemma 12.23 gives $\mathrm{ask}_{j,s}(p) \le \sum_e \max(0, \mathrm{al}(s)/p^e - \mathrm{cap}_j(s,e)) + 3c_G \ell(p)$. For an even top s, the second gives

$$\mathrm{ask}_{j,s}(p) \le \sum_e \frac{\mathrm{al}(s)}{p^e} + 2c_G\gamma_p - \sum_e \min\Big(\frac{a_s}{p^e}, w_j(\pi_p(s,e))\Big).$$

Let y be an odd top with $d(y) > 0$ and let $s = y - d(y)h_p$, an even top. Then $\pi_p(s, d(y))$ is the top place of y, where $w_j = b_y + o_j(y) \ge b_y$ by (W1). The pairs $(s, d(y))$ are distinct for distinct y. Since $2c_G\gamma_p \le 2c_G$, summing over the tops gives

$$\sum_{s \in T(t_0)} \mathrm{ask}_{j,s}(p) \le g_j^1(t_0) + \mathrm{dl}(t_0) + 5c_G N\ell(p) - \sum_{y \text{ odd},\ d(y)>0} \min\Big(b_y, \frac{a_{y-d(y)h_p}}{p^{d(y)}}\Big). \tag{12.8}$$

It remains to show $\mathrm{Pr}_j + \lceil t_{t_0} \rceil + D_p + 1 \le N(\mathrm{lo}(p) - \mathrm{cp}(p)) + \Sigma_{\mathrm{kink}}$, where Σ_{kink} is the last sum in (12.8), which is nonnegative. We use $n_p \le C_{31} N \log x/\delta_1$ (Lemma 12.2), $N^+ = 40 n_p \delta_1 \le 40 C_{31} N \log x$, $\lceil y \rceil \le y+1$, $\log_p a \le \log a$, $N \ge 1$, and that $Q_p = p^{G'}$ increases with p.

Case $p < p_*$. By (12.6), $\mathrm{Pr}_j \le N\big(r_0(2b' + c_G) + r_0 C_{31} \log x (w'' + \eta + c_G + N_m)/\delta_1\big)$. The definitions of t^{II} and D^{II} give

$$\lceil t^{\mathrm{II}} \rceil + D^{\mathrm{II}} + 1 \le N\Big(K_{tII} r_0 \Big(\frac{\mathrm{La}\, a}{pk} + a p^{-L_1(p)}\Big) + K_{tII} r_0 (c_0 + k Q_{p_*}) + 40 C_{31} \log x \big(K_{DII}(\nu_1^3 + r_0^5 + k^2)\log a + r_0 c_0\big) + \frac{4 r_0 C_{31} c_G \log x}{\delta_1} + 3\Big).$$

The sum of the two bounds is at most $N(\mathrm{lo}(p) - \mathrm{cp}(p))$.

Case $p \ge p_*$. By (12.6) and (W1), $\hat{w}_j(0,0,s) - b_s = o_j(s) - \kappa_j \le b'/x + \eta$, so $\mathrm{Pr}_j \le N(b'/x + \eta) + K_{\mathrm{Pr}} C_{31} N \log x (w'' + \eta + c_G + N_m)/\delta_1$. In t^{I} the first sum equals Σ_{kink}. In the rest, $N^+ \le 40 C_{31} N \log x$, $n_p \le C_{31} N \log x/\delta_1$, $\delta_1 \le 1$ and $D^{\mathrm{I}} \le 40 K_{DI} C_{31} N \log x\, x^{26}(\nu_1^5 + k^3)\log a + 1$ give $c_6 \le N c_6'$, so $K_{\kappa 2} N Q_p c_6 \le K_{\kappa 2} N N^\sharp Q_p c_6'$, and $2 n_p c_G \le 2 C_{31} N c_G \log x/\delta_1$. Finally $D^{\mathrm{I}} + 2 \le 40 K_{DI} C_{31} N \log x\, x^{26}(\nu_1^5 + k^3)\log a + 3$. Collecting terms, $\mathrm{Pr}_j + \lceil t^{\mathrm{I}} \rceil + D^{\mathrm{I}} + 1 \le \Sigma_{\mathrm{kink}} + N\big(\Theta^{\mathrm{I}}(p) + K_{\kappa 1}\nu_1(a/p^2 + \mathrm{La}\, a/(pk))\big)$. □

Lemma 12.25. *Let $p \in \mathcal{P}$ with $p \le X''$ be of class II or satisfy $p \ge X'$, let $t_0 \in R_p$ and $1 \le j \le m$. Then $\sum_{s \in T(t_0)} \big(\mathrm{ask}_{j,s}(p) - \mathrm{cp}(p)\big) \le \max(g_j^0(t_0), 0) + \mathrm{dl}(t_0)$.*

Proof. If p is of class II, $\pi_p(s,e) \in \Phi_1$ for odd e, so no pair (s,e) satisfies $\mathrm{Excl}(s,e)$. If $p \ge X'$ and $\pi_p(s,e)$ is the top place of some $s' \in T$, then (W1) and (5.17) give $w_j(\pi_p(s,e)) \ge b' + b'/x$ while $a_s/p^e \le (a + a/(kx^{13}))/X' = b'(1 + 1/(kx^{13}))$, so the term of (s,e) in $\mathrm{ask}_{j,s}(p)$ vanishes. In both cases the excluded pairs contribute 0 to ask, and the first inequality of Lemma 12.23, summed over the other pairs, gives $\sum_s \mathrm{ask}_{j,s}(p) \le g_j^0(t_0) + \mathrm{dl}(t_0) + 3c_G N(t_0)\ell(p)$. □

Classes of primes. Split the primes $p \in \mathcal{P}$ into five sets. The *high* primes are those with $p > X''$. Among the primes $p \le X''$, the *served* primes are the candidates with $\mathrm{Sig}(p) \le \theta$, the *second* primes are the candidates with $\mathrm{Sig}(p) > \theta$, the *plain* primes are the primes $p \notin \mathcal{D}$ of class II or with $p \ge X'$, and the *first* primes are the remaining primes of class I. A first prime p is not a candidate. If $p < X'$, then $n_p < x^{\varepsilon/4}$, or $p \ge p_*$ and $|p^i - X'| \le \rho_B X'$ for some $i \ge 1$. If $p \ge X'$, then $p \in \mathcal{D}$, so h_p has odd order and $h_p \ne 0$, which gives $n_p \le o < x^{\varepsilon/4}$ by Lemma 12.1, or $p < X'(1 + C_r\rho_B)$ and one of the two alternatives above holds. For a served prime let $T^{\mathrm{gd}}(p)$ be the set of the tops on good orbits and $\beta_p = N_T - |T^{\mathrm{gd}}(p)|$.

Proposition 12.26. *Let $p \in \mathcal{P}$ with $p \le X''$, and suppose the weights lie in the windows. Then $0 \le \mathrm{gain}_j(p)$, and*

(a) if p is served, $\sum_j \sum_{s \in T^{\mathrm{gd}}(p)} \big(\mathrm{ask}_{j,s}(p) - \mathrm{lo}(p)\big) - m\,\mathrm{fl}(p) \le \sum_j \mathrm{gain}_j^{\mathcal{D}}(p) - \tau(p)$;

(b) if p is plain, $\sum_j \sum_{s \in T} \big(\mathrm{ask}_{j,s}(p) - \mathrm{lo}(p)\big) - m\,\mathrm{fl}(p) \le \sum_j \mathrm{gain}_j^{\mathcal{D}}(p) - \tau(p)$.

Proof. Each term of $\mathrm{gain}_j(p)$ is a maximum with $\max(g_j^0, 0) \ge 0$. (a) A served prime has $\mathcal{V}_p$ equal to the set of good orbits, $p \notin \mathcal{D}$ and $\mathrm{gain}^{\mathcal{D}} = \mathrm{gain}$. Then $\mathrm{gain}_j(p) \ge \sum_{t_0 \in \mathcal{V}_p} \big(g_j^1(t_0) - \mathrm{Pr}_j(t_0)\big)$ because the terms of the unserved orbits are nonnegative. Proposition 12.24, summed over $t_0 \in \mathcal{V}_p$ and j, and $\tau(p) = e_p + \sum_{t_0 \in \mathcal{V}_p} m(\lceil t_{t_0} \rceil + D_p + 1)$ give the claim, since $\sum_{t_0 \in \mathcal{V}_p} m\,\mathrm{dl}(t_0) + e_p \le m\,\mathrm{fl}(p)$. (b) A plain prime has $\mathcal{V}_p = \emptyset$, so $\tau(p) = 0$ and $\mathrm{gain}_j(p) = \sum_{t_0 \in R_p} \max(g_j^0(t_0), 0)$, unless p is of class I with $h_p = 0$, in which case $\mathrm{gain}_j(p) = 0$, $p \ge X'$ and every $\pi_p(s,e)$ with e odd is the top place of s, so every $\mathrm{ask}_{j,s}(p)$ vanishes by the argument of Lemma 12.25. In the other cases Lemma 12.25, summed over $t_0 \in R_p$ and j, gives the claim, since $\mathrm{lo}(p) = \mathrm{cp}(p)$ for a plain prime. □

The total. For $p \in \mathcal{P}$ put $\mathrm{dev}(p) = \sum_{j,t} \mathrm{ask}_{j,t}(p) - \sum_j \mathrm{gain}_j^{\mathcal{D}}(p) + \tau(p)$, so that $S - \Gamma = \sum_p 2\log p\,\mathrm{dev}(p) + \sum_{j,t} \mathrm{dt}_{j,t}$ by (12.3) and (12.7). For $p \ge 3$ we have $\gamma_p \le \frac{9}{8p}$, and $a_t \le \frac{17}{16}a$, so

$$\mathrm{ask}_{j,t}(p) \le a_t\gamma_p \le \frac{2a}{p}. \tag{12.9}$$

Define the weighted count of second primes and the correction

$$\mathrm{lhsW} = \sum_{p \text{ candidate}} \frac{\log p}{p}\Big(2 \cdot \mathbf{1}[\mathrm{Sig}(p) > \theta] + 4\,\mathrm{Sig}(p)\Big).$$

Theorem 12.27. *Suppose the inputs (L1) to (L8), the size conditions (Z) of Subsection 12.6, that the weights lie in the windows, and that* $\mathrm{lhsW} \le x^{-9}$. *Then*

$$S \le \Gamma + e_1, \qquad e_1 = 2\log 4\; N_T m b + B_{\mathrm{minor}} \le (2\log 4 + 1)\, N_T m b, \qquad \mathrm{IO} \le N_T m b,$$

where B_{minor} *is the sum of the bucket bounds (B1) to (B9) below.*

Proof. We bound $\mathrm{dev}(p)$ class by class, using $\mathrm{gain}_j^{\mathcal{D}} \ge 0$ and $\tau \ge 0$.

Served primes. By Proposition 12.26(a) and (12.9), $\mathrm{dev}(p) \le m N_T \mathrm{lo}(p) + m\,\mathrm{fl}(p) + 2ma\beta_p/p$.

Plain primes. By Proposition 12.26(b), $\mathrm{dev}(p) \le m N_T \mathrm{cp}(p) + m\,\mathrm{fl}(p)$.

Second primes. Such a prime is a candidate with $\mathrm{Sig}(p) > \theta$, hence $p < X'$, p has no gain and $p \ne T_u$ because $T_u \ge X'$, so $p \in \mathcal{D}$ and $\mathcal{V}_p = \emptyset$. Then $\mathrm{dev}(p) \le 2amN_T/p$.

First and high primes. $\tau(p) = 0$, since such a prime is not a candidate. So $\mathrm{dev}(p) \le \sum_{j,t} \mathrm{ask}_{j,t}(p)$. For a first prime (12.9) gives $\mathrm{dev}(p) \le 2amN_T/p$. For a high prime and odd $i \ge 3$, $a_t/p^i \le 2a/X''^3 < w''$, and for $i = 1$, $a_t/p - w_j \le a(1+\vartheta)/p - w''$ with $\vartheta = 1/(kx^{13})$, which is positive when $p < X''(1+\vartheta)$ and nonpositive otherwise, and then it is at most $w''\vartheta$ since $X''w'' = a$.

Summing with the weights $2\log p$ gives $S - \Gamma \le \sum_{i=0}^{8} B_i$ with

$$B_0 = 2mN_T \sum_{p \text{ served}} \log p \left(\frac{b'}{x}\right), \qquad B_1 = 2mN_T \sum_{p \le X''} \log p\, \mathrm{cp}(p),$$

$$B_2 = 2mN_T \sum_{p < X'} \log p \left(\mathrm{lo}(p) - \mathrm{cp}(p) - \mathbf{1}[p \ge p_*]\tfrac{b'}{x}\right), \qquad B_3 = 2m \sum_{p \le X''} \log p\, \mathrm{fl}(p),$$

$$B_4 = 4am \sum_{p \text{ served}} \frac{\log p}{p}\beta_p, \qquad B_5 = 4amN_T \sum_{p \text{ second}} \frac{\log p}{p},$$

$$B_6 = 4amN_T \sum_{p \text{ first}} \frac{\log p}{p}, \qquad B_7 = 2mN_T\, w''\vartheta \sum_{X'' < p < X''(1+\vartheta)} \log p,$$

$$B_8 = \sum_{j,t} \mathrm{dt}_{j,t}.$$

where in B_2 the term b'/x of $\Theta^{\mathrm{I}}(p)$ is moved to B_0 for $p \ge p_*$, and the term Θ^{II} stays in B_2. We bound each.

(B0) By the Chebyshev bound $\sum_{p \le y} \log p \le y \log 4$ and $X'b'/x = b$, $B_0 \le 2\log 4\; N_T m b$.

(B4) By (L9), $\sum_p \frac{\log p}{p}\beta_p \le N_T x^{-2+\varepsilon}$, so $B_4 \le 4amN_T x^{-2+\varepsilon} = 4N_T m b\, x^{-1+\varepsilon}$.

(B5) Each second prime contributes $2\log p/p$ to lhsW, so $B_5 \le 2amN_T\,\mathrm{lhsW} \le 2N_T m b\, x^{-8}$.

(B6) By Lemma 12.28, $\sum_{p \text{ first}} \log p/p \le B^{\mathrm{ord}} + B^{\mathrm{band}}$, and (Z) bounds $4amN_T(B^{\mathrm{ord}} + B^{\mathrm{band}})$.

(B7) The interval contains at most $\vartheta X'' + 1$ integers, so $B_7 \le 2mN_T \log(2X'')(a\vartheta^2 + w''\vartheta)$.

(B8) $\mathrm{dt}_{j,t} \le \log d \sum_{i \ge 1} a_t/d^i \le 2a \log d/d$, so $B_8 \le 2amN_T \log d/d$.

(B1), (B2), (B3) are explicit sums over primes, bounded by (Z). Hence $S - \Gamma \le 2\log 4\, N_T m b + B_{\mathrm{minor}}$ with $B_{\mathrm{minor}} = B_1 + \cdots + B_8$, and (Z) gives $B_{\mathrm{minor}} \le N_T m b$.

For IO: for a served prime and $t_0 \in \mathcal{V}_p$, $S_{t_0} \le \sum_j \max(0, g_j^1(t_0)) + 1 \le m\gamma_p \sum_{s \in T(t_0)} \mathrm{al}(s) + 1 \le 2mN_T a/p$, using (12.9) and $mN_T a/p \ge b' \ge 1$. So $\mathrm{io}(p) \le 4\,\mathrm{Sig}(p)\, mN_T a/p$, and $\mathrm{IO} \le amN_T\,\mathrm{lhsW} \le N_T m b\, x^{-8}$. □

Lemma 12.28. *Let $J = x^{\varepsilon/4}$ and $q = 2^{e+2}$ with e as in Proposition 5.14. Then*

$$\sum_{p \text{ first}} \frac{\log p}{p} \le B^{\mathrm{ord}} + B^{\mathrm{band}}, \qquad B^{\mathrm{ord}} = 4J\Big(\frac{\log P_0}{P_0} + \frac{\log^2 X''}{q}\Big), \quad B^{\mathrm{band}} = I_b\big(4\rho_B + p_*^{-1}\big)\log(4X'),$$

where $P_0 = 2^{(e+3)/2}J^{-1/2}$, $I_b = \lceil \log(2X')/\log p_* \rceil$, and we assume $P_0 \ge 3$ and $X'' \ge q$.

Proof. Small order. By Proposition 5.14(iii), h_p maps to the class of p in $(\mathbb{Z}/q)^\times/\{\pm 1\}$, a cyclic group of order 2^e. If $n_p < J$, the image has order 2^i with $2^i < J$, so $p^{2^i} \equiv \pm 1 \pmod q$. For $i \ge 1$, $p^{2^i} \equiv 1 \pmod 8$ and $-1 \not\equiv 1 \pmod 8$, so $p^{2^i} \equiv 1 \pmod q$. The 2-adic valuation satisfies $v_2(p^{2^i}-1) = v_2(p^2-1) + i - 1$ for $i \ge 1$, by induction from $p^{2^{i+1}} - 1 = (p^{2^i}-1)(p^{2^i}+1)$ and $p^{2^i} + 1 \equiv 2 \pmod 4$. So $v_2(p^2-1) \ge e + 3 - i$ and $p^2 > 2^{e+3-i}$, which gives $p > P_0$. For $i = 0$, $p \ge q - 1 \ge P_0$. The solutions of $y^{2^i} = 1$ in $(\mathbb{Z}/q)^\times \cong \mathbb{Z}/2 \times \mathbb{Z}/2^e$ number at most 2^{i+1}, so the primes in question lie in at most $\sum_{2^i < J} 2^{i+1} \le 4J$ residue classes modulo q, and are at least P_0. For one class c, the function $y \mapsto \log y/y$ decreases for $y \ge 3$, so the sum of $\log n/n$ over the integers $P_0 \le n \le X''$ with $n \equiv c \pmod q$ is at most $\log P_0/P_0 + \frac{1}{q}\int_{P_0}^{X''} \frac{\log y}{y}\,dy \le \log P_0/P_0 + \log^2 X''/(2q)$.

Band. If $p \ge p_*$, $p \le X''$ and $|p^i - X'| \le \rho_B X'$ with $\rho_B \le \frac{1}{3}$, then $i \le I_b$ and p lies in $[y_i, y_i(1+3\rho_B)]$ with $y_i = (X'(1-\rho_B))^{1/i} \ge p_*$, since $((1+\rho_B)/(1-\rho_B))^{1/i} \le 1 + 3\rho_B$. This interval contains at most $3\rho_B y_i + 1$ integers, and a prime p in it with $p \ge p_*$ contributes $\log p/p \le \log(4X')/\max(y_i, p_*)$. So the band primes for one i contribute at most $(3\rho_B y_i + 1)\log(4X')/\max(y_i, p_*) \le (3\rho_B + 1/p_*)\log(4X')$. □

The choice of weights. The weights enter lhsW through the shares $\kappa_{t_0}(a,b)$. Let Box be the set of integer weight vectors $(a_t, b_t)_{t\in T}$ in the windows $a \le a_t \le a + a/(kx^{13})$ and $b' \le b_t \le b' + 2n\delta_1 x^\varepsilon \mathrm{La}\, b'/k$.

Proposition 12.29. *Under the hypotheses of Corollary 12.13 and the size condition (Z9), there is $(a,b) \in$ Box with* lhsW $\le x^{-9}$.

Proof. For $\theta = x^{-\varepsilon}$ and $\mathrm{Sig} \ge 0$, $2 \cdot \mathbf{1}[\mathrm{Sig} > \theta] + 4\,\mathrm{Sig} \le (2/\theta + 4)\mathrm{Sig}$. The share κ_{t_0} depends on the weights at the tops of the orbit of t_0, and Box is a product of windows, so the mean of κ_{t_0} over Box is its mean over the windows of the orbit, which is at most $N(t_0)x^{-12}$ by Corollary 12.13. The orbits at p partition T, so the mean of $\mathrm{Sig}(p)$ is at most $mN_T x^{-12}$, and by Mertens' bound $\sum_{p\le y} \log p/p \le \log y + 2$ the mean of lhsW is at most $(2x^\varepsilon + 4)mN_T x^{-12}(\log X' + 2)$. This is at most x^{-9} by (Z9), and some element of Box is at most the mean. □

Proposition 12.29 is the reason the bottom weights are chosen together with the weights a_t. Input (L6) asks that the nodes of Section 18 have the bottom weights of Proposition 12.29. With these weights Theorem 12.27 is hypothesis (3.1) of Proposition 3.1, and it bounds the term 4 IO of e_2 in Theorem 12.21 by $4N_T mb$.

12.6 What this section takes from the other steps

The band width and its constant are $\rho_B = x^{-2}$ and $C_r = 1$. The proofs above use the following inputs from Sections 11, 16 and 18, and the size conditions (Z) listed after them.

(L1) *Local structure (Section 11).* At every place w above a prime $p \in \mathcal{P}$ with $p \leq X''$ and $p \neq T_u$ there are data (F1) to (F5) of Subsection 12.3, in which the coordinates indexed by $v \in \mathcal{V}_p$ are the orbits of p, the representation V_v is the representation of Subsection 12.2 with $n' = n_p$ and the base weights of the orbit, and $y_v = p^{A_r} T_{r,v}$ with $r = r_p$.

(L2) *Integral structures (Section 11).* The lattices of (F3) satisfy $p^{e_p} M^{\mathrm{na}} \subset M$ with $e_p \leq (C_e + 1) f k^2 \log_p a$ for a constant C_e.

(L3) *The orbit bounds (Section 11).* The numbers g_j^0, g_j^1 and Pr_j of Subsection 12.1 are the orbit bounds g^0, g^1, PR of Theorem 10.7 for the lifts at w. This input is used in Section 13, which applies Theorem 10.7 to the generators of NP_w described after Theorem 12.18. The proofs of this section use the formulas (12.4) to (12.6) as definitions.

(L4) *The base lattice (Sections 11 and 15).* M_u is a hermitian $\mathcal{O}_{\mathbf{K}}$-lattice of rank d_M whose completion at each place w of (L1) is the lattice M of (F3).

(L5) *The ramified part (Section 18).* $M_u^{\mathrm{ram}} \subset M_u$ is saturated, its completions at the places of (L1) are the lattices M^{ram} of (F5), $d_u = d_M\big(1 - 1/(T_u - 1)\big)^f$, and $\mu(M_u^{\mathrm{ram}}) \geq \mu(M_u) - f \log(T_u - 1)/(T_u - 2)$.

(L6) *The auxiliary prime and the nodes (Section 18).* T_u is a prime with $T_u \geq X'$. The nodes satisfy (W1) to (W3), and their bottom weights b_t, together with the weights a_t, form the element of Box given by Proposition 12.29.

(L7) *The colength at T_u (Section 18).* At each place $w \mid T_u$ there is a sublattice $N_w^T \subset M_{u,w}^{\mathrm{ram}}$ whose colength, divided by e_w, is at most $d_M \lambda_T$ with $\lambda_T = \frac{1}{4} C_T N_T m(a/k + b')$.

(L8) *Local terms (Section 16).* The functions F_j and $E_{u,j}$ have finite supports, and the set P_{FE} of their residue characteristics has $\sum_{p \in P_{FE}} \log p \leq X''$.

(L9) *Bad orbits (Section 18).* For the served primes, $\sum_p \frac{\log p}{p} \beta_p \leq N_T x^{-2+\varepsilon}$.

(L10) *The engines.* Hypothesis 12.12, which this section uses through Theorem 12.11 and Corollary 12.13.

Proposition 5.20 gives (L9), applied with $\mathcal{P}$ the set of candidates and $L_{2,p} = 2 \max_{p'} L_1(p')$. Indeed $L_1(p') \leq 4 \log(k x^{13})/\log 2 + 1$, so $L_{2,p} \leq 400 \log x$ for large x, and Proposition 5.20 with $|T| \geq \frac{1}{2}\delta_1 n$ and Mertens' bound gives $\sum_p \frac{\log p}{p} \beta_p \leq 8 C_b x^{-2} N_T (\log X' + 2)$, which is at most $N_T x^{-2+\varepsilon}$ for large x because $\log X' = x^{\varepsilon/3}$.

Size conditions. Write $\Pi = \max(\eta, c_G, N_m, t_r, K_\bullet, C_\bullet)$ for the largest of the auxiliary quantities and constants above. The size conditions are the following.

(Z1) $x \geq x_0(\varepsilon, o)$, $e \geq 1$, $P_0 \geq 3$, $X'' \geq 2^{e+2}$, and $\Pi \leq \exp(x^{\varepsilon/4})$.

(Z2) $2f/T_u \leq \frac{1}{2}$ and $4fb^2/T_u + f \log(T_u - 1)/(T_u - 2) \leq 1$.

(Z3) $t_{t_0} + D_p + 2 \leq \tau^+ N(t_0)$ for every candidate p and good orbit t_0, with $\tau^+ = 2b' + \max\big(\Theta^{\mathrm{II}}, \Theta^{\mathrm{I}}(X')\big) + (K_{tII} r_0 + K_{\kappa 1} \nu_1) a$, and $4X'\big((C_e + 1) f k^2 \log a + m N_T \tau^+\big) \leq b^2$.

(Z4) $B_1 + \cdots + B_8 \le N_T m b$ for the bucket bounds of Theorem 12.27, and $\mathrm{LT} + \mathrm{RX} + 1 \le x^{C\varepsilon}\mathcal{B}$.

(Z5) The hypotheses of the engines in Hypothesis 12.12(A), with $Y = x^{13}$ and N_s the maximal number of tops of a segment.

(Z9) $(2x^{\varepsilon} + 4)\, m N_T x^{-12}(\log X' + 2) \le x^{-9}$.

Condition (Z9) holds for large x because $m \le x^{1/2+\varepsilon}$ and $N_T \le 2\delta_1 n \le 4x^{2+8\varepsilon}$, so the left side is at most $x^{-19/2+11\varepsilon}$. Condition (Z3) holds for large x. The inequality $t_{t_0} + D_p + 2 \le \tau^+ N(t_0)$ is a consequence of the bounds in the proof of Proposition 12.24, because the first sum in t^{I} is at most $N(t_0)\max_y b_y \le 2N(t_0)b'$. The number $\Theta^{\mathrm{I}}(X')$ is at most $\exp(Cx^{\varepsilon/3+\varepsilon/8}\log x)$, because $Q_{X'} = X'^{G'}$ with $G' \le Cx^{\varepsilon/8}\log x$. Then $X' m N_T \tau^+ \le \exp(Cx^{11\varepsilon/24}\log x)\, a$, which is smaller than $b^2 = e^{2x^{\varepsilon}}$. In (Z4) the bucket bounds compare with $N_T m b = N_T m\, e^{x^{\varepsilon}}$ in the way we now describe, for large x. In B_1 and B_3, the sums over $p \le X''$ are at most $2X''\exp(Cx^{\varepsilon/4})$ times mN_T or mf, and $X'' \le 2x^{3+8\varepsilon}e^{x^{\varepsilon/3}}$, while $f \le 4N_T/\delta_1$. In B_2, the terms with $p < p_*$ sum to at most $4p_* m N_T \Theta^{\mathrm{II}} + 2K_{tII} r_0 a m N_T\big(\mathrm{La}(\log p_* + 2)/k + 2p_*(kx^{13})^{-4}\big)$, using $p^{L_1(p)} \ge (kx^{13})^4$, and $\Theta^{\mathrm{II}} \le \exp(Cx^{\varepsilon/4})b'$ with $p_* b' = k^2 x^{27} b/X'$. The terms with $p_* \le p < X'$ sum to at most $4X' m N_T \Theta^{\mathrm{I}}(X') + 2K_{\kappa 1}\nu_1 a m N_T\big(2\log p_*/p_* + \mathrm{La}(\log X' + 2)/k\big)$, and $a\,\mathrm{La}\log X'/k \le b\, x^{1+3\varepsilon}/x^{9/2}$ because $k \ge x^{9/2}$ and $\mathrm{La} \le Cx^{\varepsilon}\log^2 x$. In B_6, $4a(B^{\mathrm{ord}} + B^{\mathrm{band}}) \le Cb\big(x^{1+\varepsilon/4}\log x/P_0 + x^{1+\varepsilon/4+2\varepsilon/3}/2^e + x^{-1+2\varepsilon/3}\big)$ with $P_0 \ge 2^{e/2}x^{-\varepsilon/8} \ge x^{1+4\varepsilon-\varepsilon/8}/\sqrt{o}$ and $2^e \ge x^{2+8\varepsilon}/o$. Finally $B_7 \le CmN_T b\log X''/(k^2x^{25})$, and $B_8 \le 4mN_T b\log x/x$ because $d \ge x^2$. The second part of (Z4) and condition (Z2) are conditions on T_u, C_T and $\mathcal{B}$ that step (R6) must verify, and (Z5) is part of the engines.

12.7 The engines of Hecke nondegeneracy

Valuations. Throughout, p is an odd prime, v is the valuation on $\mathbb{C}_p$ with $v(p) = 1$, and $v(0) = +\infty$. For an integer $m \ge 1$, Legendre's formula gives

$$v(m!) = \frac{m - s_p(m)}{p-1}, \qquad \frac{m}{p-1} - v(m!) = \frac{s_p(m)}{p-1} \le \log_p m + 1, \tag{12.10}$$

where $s_p(m)$ is the sum of the base-p digits of m. Kummer's theorem gives $v\binom{N}{k} \le \log_p N$ for $0 \le k \le N$, since $v\binom{N}{k}$ is the number of carries in the addition of k and $N-k$ in base p, and that number is at most the number of digits of N beyond the first.

The Fock form. Let σ be a finite set of variables and K a field of characteristic 0. For polynomials $F = \sum_m F_m x^m$ and $G = \sum_m G_m x^m$ in $K[x_\sigma]$ the *Fock form* is

$$\langle F, G\rangle = \sum_m F_m\, G_m\, m!, \qquad m! = \prod_{s\in\sigma} m_s!. \tag{12.11}$$

For a variable x_s and monomials x^m, $x^{m'}$ one has $\langle x_s x^m, x^{m'}\rangle = \langle x^m, \partial_s x^{m'}\rangle$, because both sides vanish unless $m' = m + e_s$, and then both equal $(m+e_s)!$. So multiplication by x_s and the derivative ∂_s are adjoint for the Fock form.

12.8 Engine (C): the mean of the segment shares

This subsection proves the analytic core of Hypothesis 12.12(C). The core is a theorem on the p-adic Smith exponents of a matrix of polynomials evaluated along an arithmetic progression (Theorem 12.34), together with an identity that writes the Gram matrix of a segment form at one level as such a matrix, with leading coefficient the untwisted Fock Gram matrix (Proposition 12.37). Their combination is Theorem 12.38, the bound at one level. Subsection 12.8.5 states the steps that lead from the bound at one level to the mean of the segment share.

12.8.1 Smith exponents and good families

Let L be a free $\mathbb{Z}_p$-module with basis $e_1, \dots, e_n$ and let $\Psi\colon L \times L \to \mathbb{Q}_p$ be bilinear. For $D \in \mathbb{Z}$ the *good count* $\mathrm{gc}(\Psi, L, D)$ is the largest N for which there are $x_1, \dots, x_N, y_1, \dots, y_N \in L$ with $\Psi(x_i, y_j) = p^D \delta_{ij}$, and the *bad count* is $\mathrm{bad}(\Psi, L, D) = n - \mathrm{gc}(\Psi, L, D)$.

Lemma 12.30. *Let $s \in \mathbb{Z}_p \setminus \{0\}$ and suppose that the matrix $A = \big(s\,\Psi(e_a, e_b)\big)_{a,b}$ has entries in $\mathbb{Z}_p$. Write $A = U\,\mathrm{diag}(d_1, \dots, d_n)V$ with $U, V \in \mathrm{GL}_n(\mathbb{Z}_p)$. Then for every integer D,*

$$\mathrm{bad}(\Psi, L, D) \le \#\{i : \ v(d_i) \ge D + 1\}.$$

Proof. Let $I = \{i : v(d_i) \le D\}$. For $i \in I$ write $d_i = p^{v(d_i)} u_i$ with $u_i \in \mathbb{Z}_p^\times$, and put

$$x_i = \sum_a (U^{-1})_{ia}\, e_a, \qquad y_i = p^{D - v(d_i)} u_i^{-1} \sum_b (V^{-1})_{bi}\, e_b.$$

Both lie in L, because the coefficients are in $\mathbb{Z}_p$ and $D - v(d_i) \ge 0$. For $i, j \in I$,

$$s\,\Psi(x_i, y_j) = p^{D - v(d_j)} u_j^{-1} \big(U^{-1} A V^{-1}\big)_{ij} = p^{D - v(d_j)} u_j^{-1} d_j \delta_{ij} = p^D \delta_{ij}.$$

Replace y_j by $s\,y_j$, which lies in L since $s \in \mathbb{Z}_p$. Then $\Psi(x_i, y_j) = p^D \delta_{ij}$, so gc $\ge |I|$ and bad $\le n - |I|$. □

The scalar s allows the window theorem to be applied to a multiple of the form. The Smith exponents of A exceed those of the Gram matrix of Ψ by $v(s) \ge 0$, so the bound is in the right direction.

Lemma 12.31. *Let $A, A' \in M_n(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$ with normalized valuation extended from v, and suppose that every entry of $A - A'$ has valuation at least $T > 0$. Let $s_1 \le \cdots \le s_n$ and $s_1' \le \cdots \le s_n'$ be the Smith exponents of A and A', with $s_i = +\infty$ for the zero diagonal entries. Then $\min(s_i, T) = \min(s_i', T)$ for every i.*

Proof. Let π be a uniformizer of $\mathcal{O}$ with $\pi^e \mathcal{O} = p\mathcal{O}$, and assume $eT \in \mathbb{Z}$, as it is in the application where T is an integer. If $A = U\,\mathrm{diag}(\pi^{es_1}, \dots)V$ then $\mathrm{coker}\, A \cong \bigoplus_i \mathcal{O}/\pi^{es_i}\mathcal{O}$, and tensoring with $R_T = \mathcal{O}/\pi^{eT}\mathcal{O}$ gives $\bigoplus_i R_T/\pi^{e\min(s_i,T)} R_T$. On the other hand $\mathrm{coker}\, A \otimes R_T$ is the cokernel of the reduction of A modulo π^{eT}, which equals the reduction of A'. So the two finitely generated $\mathcal{O}$-modules $\bigoplus_i \mathcal{O}/\pi^{e\min(s_i,T)}$ and $\bigoplus_i \mathcal{O}/\pi^{e\min(s_i',T)}$ are isomorphic, and the uniqueness of elementary divisors over the discrete valuation ring $\mathcal{O}$ gives the claim. □

Smith exponents do not change under a finite extension of the base, since $s_1 + \cdots + s_l$ is the least valuation of an $l \times l$ minor. So a matrix over $\mathbb{Z}_p$ may be treated over a larger ring of integers.

12.8.2 The window theorem

For $c \in \mathbb{C}_p$ and an integer $i \geq 0$ write $\binom{c}{i} = c(c-1)\cdots(c-i+1)/i!$.

Lemma 12.32. *Let $h \geq 1$ and $\Delta < p^h$ be integers, let $E \geq h$ be an integer and let $y \in \mathbb{C}_p$ with $v(y) = E$. Then for every $z \in \mathbb{Z}_{\geq 0}$ and $0 \leq k \leq \Delta$,*

$$v\Big(\binom{z+y}{k} - \binom{z}{k}\Big) \geq E - h + 1.$$

Proof. Vandermonde's identity $\binom{z+y}{k} = \sum_{l=0}^{k} \binom{z}{k-l}\binom{y}{l}$ holds as an identity of polynomials in y. The terms with $l \geq 1$ remain. For $1 \leq l \leq k < p^h$,

$$\binom{y}{l} = \frac{y}{l}\prod_{i=1}^{l-1}\frac{y-i}{i},$$

and $v(i) \leq h-1 < E = v(y)$ for $1 \leq i < l$, so $v(y-i) = v(i)$ and the product is a unit. Hence $v\binom{y}{l} = E - v(l) \geq E - h + 1$. The integers $\binom{z}{k-l}$ have nonnegative valuation. □

Lemma 12.33. *Let $t_0, d \in \mathbb{Z}$ with $p \nmid d$, let $Y_0 \geq 1$, and put $z_j = t_0 + dj$ for $0 \leq j < Y_0$. Let $\zeta \in \mathbb{C}_p$, and let j^* be an index for which $v(z_j - \zeta)$ is largest. Then $v(z_j - \zeta) \leq v_p(j - j^*)$ for $j \neq j^*$, and*

$$\sum_{j \neq j^*} v_p(j - j^*) \leq \frac{Y_0 - 1}{p-1}.$$

Proof. If $v(z_j - \zeta) > v(z_j - z_{j^*})$ for some $j \neq j^*$, then $v(z_{j^*} - \zeta) = v(z_j - z_{j^*}) < v(z_j - \zeta)$ by the ultrametric inequality, which contradicts the choice of j^*. Since $p \nmid d$, $v(z_j - z_{j^*}) = v_p(j - j^*)$. The numbers $|j - j^*|$ for $j \neq j^*$ are $1, \ldots, j^*$ and $1, \ldots, Y_0 - 1 - j^*$, so the sum is $v(j^*!) + v((Y_0 - 1 - j^*)!) \leq v((Y_0-1)!) \leq (Y_0 - 1)/(p-1)$, using that $\binom{a+b}{a}$ is an integer. □

We now state the window theorem. The proof below gives the slightly stronger bound with $h-1$ in place of h.

Theorem 12.34. *Let $n, \Delta, h, Y_0 \geq 1$ be integers with $\Delta + 1 \leq p^h$ and $Y_0 \leq p^h$, and let $E \geq 1$ be an integer. Let $c_{ab,i} \in \mathbb{C}_p$ for $1 \leq a, b \leq n$ and $0 \leq i \leq \Delta$, with $v(c_{ab,i}) \geq 0$, and let*

$$A(z) = \Big(\sum_{i=0}^{\Delta} c_{ab,i}\binom{z}{i}\Big)_{a,b} \in M_n(\mathbb{C}_p[z]).$$

Suppose that $\det A(z) = \ell \prod_{\zeta \in R}(z - \zeta)$ *with $\ell \neq 0$ and R the multiset of roots in $\mathbb{C}_p$, and let Λ_0 be a real number with*

$$v(\ell) + \frac{|R|}{p-1} \leq n\Lambda_0. \tag{12.12}$$

Let $t_0, d \in \mathbb{Z}$ with $p \nmid d$, $z_j = t_0 + dj \geq 0$ for $0 \leq j < Y_0$, and let $s_1(j) \leq \cdots \leq s_n(j)$ be the Smith exponents of $A(z_j)$. Then

$$\frac{1}{Y_0}\sum_{j=0}^{Y_0-1} \#\{i : \ s_i(j) \geq E\} \leq n\Big(\frac{\Delta}{Y_0} + \frac{\Lambda_0 + \Delta(h-1)/Y_0}{E}\Big). \tag{12.13}$$

Proof. All the numbers $c_{ab,i}$, ℓ and the roots lie in a finite extension K of $\mathbb{Q}_p$, and we may enlarge K to an unramified extension of any degree. The entries of $A(z_j)$ lie in the ring of integers, because $\binom{z_j}{i} \in \mathbb{Z}$. The degree of $\det A$ is at most $n\Delta$, so $|R| \le n\Delta$.

Fix j and put $z = z_j$ and $E' = E + h - 1$. Choose $y \in K$ with $v(y) = E'$ such that $v(z+y-\zeta) = \min(v(z-\zeta), E')$ for every $\zeta \in R$. Such y exists. Write $y = p^{E'}u$ with u a unit. For a root with $v(z-\zeta) \neq E'$ the equality holds for every unit u. For a root with $v(z-\zeta) = E'$ it fails exactly when $u \equiv -(z-\zeta)p^{-E'}$ modulo the maximal ideal, which excludes one residue class. The residue field of K is larger than $|R|+1$ after an unramified extension, so a unit u avoiding these classes exists. Put $z' = z + y$.

By Lemma 12.32 with $\Delta < p^h$ and $E' \ge h$, every entry of $A(z') - A(z)$ has valuation at least $E' - h + 1 = E$. In particular $A(z')$ has integral entries. By Lemma 12.31 the truncated Smith exponents agree, $\min(s_i(j), E) = \min(s_i', E)$, where $s_i' \ge 0$ are the Smith exponents of $A(z')$. Hence

$$\begin{aligned} E \cdot \#\{i : s_i(j) \ge E\} &\le \sum_i \min(s_i(j), E) = \sum_i \min(s_i', E) \le \sum_i s_i' \\ &= v(\det A(z')) = v(\ell) + \sum_{\zeta \in R} \min\big(v(z_j - \zeta), E'\big). \end{aligned}$$

Sum over j. For each $\zeta \in R$ let $j^*(\zeta)$ be as in Lemma 12.33. The term $j = j^*(\zeta)$ contributes at most E', and the other terms contribute at most $\sum_{j \neq j^*} v_p(j - j^*) \le Y_0/(p-1)$. So

$$E \sum_j \#\{i : s_i(j) \ge E\} \le Y_0\, v(\ell) + |R|\Big(E + h - 1 + \frac{Y_0}{p-1}\Big) \le Y_0\, n\Lambda_0 + n\Delta(E + h - 1),$$

by (12.12) and $|R| \le n\Delta$. Divide by EY_0. □

Remark 12.35. The first term $n\Delta/Y_0$ accounts for points of the progression that lie very close to a root of the determinant, where the matrix can lose rank. The second term is Markov's inequality for the remaining valuation, which is controlled by the leading coefficient and by the spacing of a progression whose step is prime to p. The step $p \nmid d$ is used in Lemma 12.33, and the condition $\Delta < p^h$ is used in Lemma 12.32.

12.8.3 The tie operators and the Gram polynomial

Let $r \ge 1$ and let $w_1, \dots, w_r, e_1, \dots, e_r$ be distinct variables in σ. Put

$$Q = \sum_{c=1}^{r} x_{w_c} x_{e_c}, \qquad \mathsf{E}(F) = QF, \qquad \mathsf{F} = \sum_{c=1}^{r} \partial_{w_c} \partial_{e_c}.$$

A polynomial F is *bihomogeneous of bidegree* (o, n) when every monomial of F has total degree o in the variables x_w and n in the variables x_e. For integers $o, n, z \ge 0$ put $A_z = o + n + r + z - 1$ and define the *tie operator*

$$\Phi_z = \sum_{j=0}^{o} \frac{(-1)^j}{(j!)^2 \binom{A_z}{j}} \mathsf{E}^j \mathsf{F}^j \tag{12.14}$$

. The variable z is the free parameter of the tie, which in the application is the shifted weight $a_+ - o_+$ of the next top.

Lemma 12.36. *For all F, G and $l \geq 0$, $\langle \mathsf{E}^l F, G\rangle = \langle F, \mathsf{F}^l G\rangle$. If G is bihomogeneous of bidegree (o,n) and $l > \min(o,n)$, then $\mathsf{F}^l G = 0$.*

Proof. By the adjointness of x_s and ∂_s in Section 12.7, $\langle x_{w_c} x_{e_c} F, G\rangle = \langle F, \partial_{w_c}\partial_{e_c} G\rangle$. Sum over c and iterate. Each application of F lowers both partial degrees by one, so $\mathsf{F}^l G = 0$ when l exceeds one of them. □

Proposition 12.37. *Let v_0 be bihomogeneous of bidegree (o,n), let u be any polynomial, and put $J = \min(o,n)$ and*

$$m_l(u, v_0) = \frac{\langle \mathsf{E}^l \mathsf{F}^l u, v_0\rangle}{(l!)^2 \binom{J}{l}} \qquad (0 \leq l \leq J).$$

Then for every integer $z \geq 0$,

$$\binom{A_z}{J} \langle \Phi_z u, v_0\rangle = \sum_{l=0}^{J} (-1)^l \binom{A_z - l}{J - l} m_l(u, v_0) = \sum_{i=0}^{J} \binom{z}{i} c_i(u, v_0), \tag{12.15}$$

with

$$c_i(u, v_0) = \sum_{l=0}^{J-i} (-1)^l \binom{A_0 - l}{J - l - i} m_l(u, v_0), \qquad c_J(u, v_0) = \langle u, v_0\rangle.$$

Proof. By Lemma 12.36, $\langle \mathsf{E}^j \mathsf{F}^j u, v_0\rangle = \langle \mathsf{F}^j u, \mathsf{F}^j v_0\rangle = 0$ for $J < j \leq o$. For $j \leq J$ the identity $\binom{A}{J}\binom{J}{j} = \binom{A}{j}\binom{A-j}{J-j}$ with $A = A_z \geq J$ gives

$$\binom{A_z}{J} \frac{(-1)^j}{(j!)^2 \binom{A_z}{j}} \langle \mathsf{E}^j \mathsf{F}^j u, v_0\rangle = (-1)^j \binom{A_z - j}{J - j} m_j(u, v_0),$$

which is the first equality. For the second, Vandermonde's identity gives $\binom{A_0 - l + z}{J - l} = \sum_{i=0}^{J-l} \binom{z}{i}\binom{A_0 - l}{J - l - i}$, valid since $A_0 - l \geq 0$. Exchange the sums. For $i = J$ the inner sum has the single term $l = 0$, and $m_0(u, v_0) = \langle u, v_0\rangle$. □

So the paired value of the tie operator, normalized by $\binom{A_z}{J}$, is a polynomial in z of degree at most J in the binomial basis. Its top coefficient is the untwisted Fock form.

12.8.4 The window bound at one level

Theorem 12.38. *Let $o, n \geq 0$ and $J = \min(o,n)$. Let T be a linear operator on polynomials, $d_F \in \mathbb{Q}_p^\times$, and $f_1, \ldots, f_{n_1}$ polynomials that are $\mathbb{Z}_p$-linearly independent and bihomogeneous of bidegree (o,n). Let $x_0 \geq 0$ and D_1 be integers and Λ a real number. Suppose*

(i) *$v\big(p^{x_0} m_l(Tf_a, f_b)/d_F\big) \geq 0$ for all a, b and $0 \leq l \leq J$;*

(ii) *$\langle Tf_a, f_b\rangle = 0$ for $a \neq b$, and $\langle Tf_a, f_a\rangle \neq 0$ with $v\big(\langle Tf_a, f_a\rangle/d_F\big) \leq D_1$;*

(iii) *$J/(p-1) - v(J!) \leq \Lambda$.*

Let $W, h \geq 1$ with $W \leq p^h$ and $J + 1 \leq p^h$, let $z_0 \geq 0$ and $k \geq 0$ be integers, and let $L = \operatorname{span}_{\mathbb{Z}_p}(f_1, \ldots, f_{n_1})$. For $0 \leq i < W$ let $\Psi_i(u, v_0) = \langle \Phi_{z_0 + i} Tu, v_0\rangle/d_F$. Then

$$\sum_{i=0}^{W-1} \operatorname{bad}(\Psi_i, L, k) \leq W\, n_1 \Big(\frac{J}{W} + \frac{x_0 + D_1 + \Lambda + J(h-1)/W}{k+1} \Big).$$

Proof. Let $c_{ab,i} = p^{x_0} c_i(Tf_a, f_b)/d_F$ with c_i as in Proposition 12.37. By (i) and the integrality of the binomial coefficients, $v(c_{ab,i}) \geq 0$. Let $A(z) = \big(\sum_i c_{ab,i}\binom{z}{i}\big)_{ab}$. By Proposition 12.37,

$$A(z_0+i) = s_i\big(\Psi_i(f_a,f_b)\big)_{ab}, \qquad s_i = p^{x_0}\binom{A_{z_0+i}}{J} \in \mathbb{Z}_p \setminus \{0\}.$$

Lemma 12.30 gives $\mathrm{bad}(\Psi_i, L, k) \leq \#\{a : s_a(i) \geq k+1\}$, where $s_a(i)$ are the Smith exponents of $A(z_0+i)$.

The top coefficient matrix is $C_J = \big(p^{x_0}\langle Tf_a, f_b\rangle/d_F\big)$, diagonal by (ii). Since $\binom{z}{J} = z^J/J! +$ (lower degree), the coefficient of $z^{n_1 J}$ in $\det A(z)$ is $\ell = \det C_J/(J!)^{n_1} \neq 0$. So $\deg\det A = n_1 J = |R|$ and

$$v(\ell) + \frac{|R|}{p-1} \leq n_1(x_0 + D_1) - n_1 v(J!) + \frac{n_1 J}{p-1} \leq n_1(x_0 + D_1 + \Lambda).$$

Apply Theorem 12.34 with $n = n_1$, $\Delta = J$, $\Lambda_0 = x_0 + D_1 + \Lambda$, $E = k+1$, $t_0 = z_0$, $d = 1$, $Y_0 = W$. □

Lemma 12.39. *Let p be odd, let L be a free $\mathbb{Z}_p$-module of finite rank and $B\colon L \times L \to \mathbb{Q}_p$ a symmetric bilinear form. Then L has a basis $f_1, \dots, f_n$ with $B(f_a, f_b) = 0$ for $a \neq b$.*

Proof. If $B = 0$ every basis works. Otherwise let m be the least valuation of a value $B(u, u')$. If $v(B(u,u)) > m$ for every u, then $2B(u,u') = B(u+u', u+u') - B(u,u) - B(u',u')$ has valuation larger than m for all u, u', and since 2 is a unit this contradicts the choice of m. Choose f_1 with $v(B(f_1,f_1)) = m$. Then $B(u,f_1)/B(f_1,f_1) \in \mathbb{Z}_p$ for every u, and $u \mapsto u - \frac{B(u,f_1)}{B(f_1,f_1)} f_1$ projects L onto $L_1 = \{u : B(u,f_1) = 0\}$ with kernel $\mathbb{Z}_p f_1$. So $L = \mathbb{Z}_p f_1 \oplus L_1$ with L_1 free, and induction on the rank completes the proof. □

In the application the vectors of a basis with $B(f_a, f_a) = 0$ or with $v(B(f_a,f_a)) > D_1$ are counted as bad, and the remaining vectors form the family of Theorem 12.38.

12.8.5 From one level to the segment share

The passage from Theorem 12.38 to Hypothesis 12.12(C) goes through the following steps.

(C6) *Integrality of the level coefficients.* For a block datum G with all letter degrees and tie parameters at most $2A$, every u, v_0 in the level lattice L_{j,j_p} and $l \leq J$ satisfy hypothesis (i) of Theorem 12.38 with $x_0 = \big((t+1)(r+1) + (N+1)r + 2(t+1)\big)\lambda_N$, where $\lambda_N = \lfloor \log_p(8A + r)\rfloor + 1$ and T is the product of the tie operators of the other ties. The mechanism is that $\mathsf{F}^l/l!$ acts integrally on integral polynomials, since $\frac{1}{l!}\mathsf{F}^l = \sum_{|\kappa|=l} \kappa! \prod_c \partial_{w_c}^{(\kappa_c)} \partial_{e_c}^{(\kappa_c)}$ with divided derivatives $\partial^{(k)} = \partial^k/k!$, and that each tie operator (12.14) has denominators $\binom{A_z}{j}$ of valuation at most $\log_p A_z$ by Kummer's theorem. The bookkeeping of the factorials in d_F against the Fock weights $m!$ on L_{j,j_p} is not written here.

(C7) *Window bound.* Under the window and prime inequalities of Proposition 12.40, the bracket of Theorem 12.38 is at most $3/Y$.

(C8) *Level decomposition of a block lattice.* The block lattice of a kept block decomposes, up to a loss of $\mathrm{levelPassageLoss} = 10(s+1)(r+1)(\log_p(8a+r)+1) + 2N_t(r+1)\log_p a$ in the

depth, into the level lattices L_{j,j_p} indexed by the $\mathfrak{sl}_2$ levels of the ties, and the block form restricted to a level equals $\langle \Phi_z T\cdot,\cdot\rangle/d_F$ with $z = a_+ - o_+$. The identity behind it is the following: for the tie with extra variables x_{z_v}, x_{s_v} and $P_z(y) = \sum_l \frac{(-1)^l}{(l!)^2\binom{A_z}{l}} \hat{\mathsf{E}}^l \mathsf{F}^l (x_{z_v}^z y)$, one has $\hat{\mathsf{F}} P_z(y) = 0$ and $\langle P_z(y), P_z(y')\rangle = z!\,\langle y, \Phi_z y'\rangle$.

(C9) *Averaging over the other ties*. The bad counts bY of the untwisted level forms, averaged over the top weights of the segment, are at most $(30 + C_M)/Y$ times the averaged dimensions. The mechanism is Theorem 12.34 applied in each tie coordinate, with the arithmetic progression of tie parameters cut out by the digit classes, and $\varepsilon_i = 2S_i/Y_i + (2\kappa + 2S_i\lambda/Y_i)/E$ as the error of the i-th coordinate.

(C10) *Segment assembly*. Summation over blocks, levels and ties, which gives the factor $(t+1)N_1$, and the dichotomy that fixes the split index σ uniformly over the window when no power p^d lies in $[(1-2\delta)a/b', (1+2\delta)a/b']$.

(C11) *From segment shares to the share of* LP. On every 1-edge there is a source-test system of rank at least $(1 - \kappa_P - \sum_j \kappa_T(j))\dim V/[k]_q$, with $\kappa_P \le C_s N N_1/Y$.

Statements (C6) and (C8) to (C11) are not proved here.

12.8.6 The numerics of the window bound (C7)

In the setting of (C6) and (C7) let s be the number of tops of the segment, $t = s - \sigma \le s$, $r \ge 4$ the number of frame variables of a block, $Y \ge 1$, $C_M \ge 1$, winA the window of top weights, an interval of integers with $a/(kY) \le |\mathrm{winA}| \le a+1$, and $b_+ \le \frac{5}{2}b'$ the bottom weight of the next top. Put

$$\lambda_A = \log_p(8a+r) + 1, \qquad \Lambda = \lfloor \log_p(8(3s+2)a + r)\rfloor + 1, \qquad \kappa_S = 3(s+2)r^2\lambda_A,$$
$$x_0 = \big((t+1)(r+1) + (2s+2)r + 2(t+1)\big)\Lambda, \qquad D_{1r} = t\,(Y\kappa_S + C_M\kappa_S) + C_M\kappa_S.$$

Proposition 12.40. *Suppose $p \ge 3s+2$, $p \ge 25$, $k \ge 5$, $k^2Y^2 \le p$ and $pb' < a$. Let $J \le b_+$ be an integer and put $W = |\mathrm{winA}|$, $h = \lceil \log_p(a+1)\rceil$ and $K = \lfloor YD_{1r}\rfloor$. Then $W \le p^h$, $J+1 \le p^h$, and*

$$\frac{J}{W} + \frac{x_0 + \lfloor D_{1r}\rfloor + \Lambda + Jh/W}{K+1} \le \frac{3}{Y} \le \frac{5}{Y}.$$

Proof. First $J \le \frac{5}{2}b' < a$, so $J+1 \le a+1 \le p^h$ and $W \le a+1 \le p^h$. From $k^2Y^2b' \le pb' < a$ we get $kYb' < a/(kY) \le W$, so

$$\frac{J}{W} \le \frac{5b'}{2kYb'} = \frac{5}{2kY} \le \frac{1}{2Y},$$

using $k \ge 5$. Next, $8(3s+2)a + r \le (3s+2)(8a+r)$ and $3s+2 \le p$ give $\Lambda \le \log_p(8(3s+2)a + r) + 1 \le \lambda_A + 1 \le 2\lambda_A$, since $\lambda_A \ge 1$. Put $c = (t+1)(r+1) + (2s+2)r + 2(t+1) + 1$. From $t \le s$ and $r \ge 4$,

$$2c \le 6(s+1)(r+1) + 2 \le 12(s+2)r \le 3(s+2)r^2,$$

so $x_0 + \Lambda = c\Lambda \le 2c\lambda_A \le \kappa_S$. Since $C_M \ge 1$ and $t, Y \ge 0$, $\kappa_S \le D_{1r}$, and $h \le \log_p(a+1) + 1 \le \lambda_A \le \kappa_S$. The denominator satisfies $K + 1 \ge YD_{1r} > 0$. Hence

$$\frac{x_0 + \lfloor D_{1r}\rfloor + \Lambda + Jh/W}{K+1} \le \frac{2D_{1r} + \frac{1}{2Y}D_{1r}}{YD_{1r}} = \frac{2}{Y} + \frac{1}{2Y^2} \le \frac{2}{Y} + \frac{1}{2Y},$$

and adding $J/W \le 1/(2Y)$ gives $3/Y$. □

For the medium primes $p \ge p_* = k^2x^{26}$ take $Y = x^{13}$. Then $k^2Y^2 \le p$, and $p < X'$ gives $pb' < X'b' = a$.

Items (C1) to (C5) are Lemmas 12.30, 12.31, Theorem 12.34, Proposition 12.37 with Theorem 12.38, and Lemma 12.39.

12.9 Engine (B): detection for the twisted Fock form

The detection statement of Hypothesis 12.12(B) combines three steps, which we call Steps B, C and DEF, with five tools. This section proves the arithmetic tools of the twist, namely the factorial digit split, the decay of the unit parts, the binomial approximation and the divided-derivative congruence, and the integrality of divided powers on the balanced lattice that controls the Fock projectors.

12.9.1 The factorial digit split

Let $P = p^g$ with $g \ge 1$. For integers $n \ge 0$ and $0 \le \rho < P$ put

$$c_P(n) = \prod_{i=0}^{n-1} \prod_{y=1}^{P-1} \Big(1 + \frac{Pi}{y}\Big), \qquad d_P(n,\rho) = \prod_{y=1}^{\rho} \Big(1 + \frac{Pn}{y}\Big). \tag{12.16}$$

Proposition 12.41. *For $0 \le \rho < P$ and $\beta \ge 0$,*

$$(\rho + P\beta)! = \rho!\,(P!)^\beta\,\beta!\,c_P(\beta)\,d_P(\beta,\rho),$$

and $c_P(\beta)$ and $d_P(\beta,\rho)$ are 1-units, that is, $v(c_P(\beta)-1) \ge 1$ and $v(d_P(\beta,\rho)-1) \ge 1$.

Proof. First $(\rho+P\beta)!/(P\beta)! = \prod_{y=1}^{\rho}(P\beta+y) = \rho!\,d_P(\beta,\rho)$. Next

$$(P\beta)! = \prod_{i=0}^{\beta-1} \Big(P(i+1)\prod_{y=1}^{P-1}(Pi+y)\Big) = P^\beta\beta! \prod_{i=0}^{\beta-1}\prod_{y=1}^{P-1} y\Big(1+\frac{Pi}{y}\Big) = (P!)^\beta\beta!\,c_P(\beta),$$

using $P\,(P-1)! = P!$. For $1 \le y < P$ one has $v(y) \le g-1$, so $v(Pi/y) \ge 1$ and each factor of (12.16) is a 1-unit. □

The identity says that adding $P\beta$ to a low digit $\rho < P$ produces no carry, so the factorial splits into the factorial of the low digit, the factorial of the high digit, a power of $P!$ and a 1-unit.

12.9.2 Decay of the unit parts

For a function $f\colon \mathbb{Z}_{\ge 0} \to \mathbb{C}_p$ let $\Delta f(n) = f(n+1) - f(n)$. The Mahler coefficients of f are $a_j(f) = \Delta^j f(0)$, and $f(n) = \sum_j a_j(f)\binom{n}{j}$.

Definition 12.42. For real $\alpha, \gamma \ge 0$ let $\mathcal{D}(\alpha,\gamma)$ be the set of functions $f\colon \mathbb{Z}_{\ge 0} \to \mathbb{C}_p$ with $v(\Delta^j f(n)) \ge \gamma + \alpha j$ for all $j, n \ge 0$.

Lemma 12.43. (i) *$f \in \mathcal{D}(\alpha,\gamma)$ if and only if $v(a_j(f)) \ge \gamma + \alpha j$ for every j.*

(ii) *$\mathcal{D}(\alpha,\gamma)$ is a $\mathbb{Z}_p$-module, and $\mathcal{D}(\alpha,\gamma)\cdot\mathcal{D}(\alpha,\gamma') \subset \mathcal{D}(\alpha,\gamma+\gamma')$.*

(iii) *If $f_m \in \mathcal{D}(\alpha, \gamma_m)$ with $\gamma_m \geq \gamma$ and $\gamma_m \to \infty$, then $\sum_m f_m$ converges pointwise and lies in $\mathcal{D}(\alpha, \gamma)$.*

Proof. (i) One direction is $n = 0$. Conversely $\Delta^j f(n) = \sum_{i \geq 0} \binom{n}{i} a_{i+j}(f)$, and $\binom{n}{i} \in \mathbb{Z}$, $\gamma + \alpha(i + j) \geq \gamma + \alpha j$. (ii) Linearity is clear. For products, the Leibniz rule for differences, $\Delta(fg)(n) = \Delta f(n)\, g(n+1) + f(n)\, \Delta g(n)$, iterates to

$$\Delta^j(fg)(n) = \sum_{k=0}^{j} \binom{j}{k} \Delta^k f(n)\, \Delta^{j-k} g(n+k),$$

and each term has valuation at least $\gamma + \alpha k + \gamma' + \alpha(j-k)$. (iii) Each $\Delta^j f_m(n)$ is a finite integral combination of values of f_m, which have valuation at least γ_m, so the series of differences converge, and the bound passes to the limit. □

Proposition 12.44. *Let $p \geq 5$ and $P = p^g$, $g \geq 1$. The functions $n \mapsto c_P(n)^{\pm 1}$ and, for each $0 \leq \rho < P$, $n \mapsto d_P(n, \rho)^{\pm 1}$ lie in $\mathcal{D}(\frac{1}{4}, 0)$. In particular their Mahler coefficients satisfy $v(a_j) \geq j/4$.*

Proof. Let log be the p-adic logarithm on 1-units, $\log(1+u) = \sum_{s \geq 0} (-1)^s u^{s+1}/(s+1)$ for $v(u) \geq 1$. Since $v(u) \geq 1 > 1/(p-1)$, exp converges at $\log(1+u)$ and $\exp\log(1+u) = 1+u$, and log is additive on 1-units.

The function $\ell = \log c_P$. By additivity and absolute convergence,

$$\ell(n) = \sum_{s \geq 0} \frac{(-1)^s P^{s+1} H_P(s+1)}{s+1} S_{s+1}(n), \qquad H_P(m) = \sum_{y=1}^{P-1} y^{-m}, \quad S_m(n) = \sum_{i=0}^{n-1} i^m.$$

Since $v(y) \leq g-1$ for $1 \leq y < P$, $v(P^m H_P(m)) \geq mg - m(g-1) = m$. The function S_m satisfies $\Delta S_m(n) = n^m$, so $a_0(S_m) = 0$, $a_1(S_m) = 0^m = 0$ for $m \geq 1$, and $a_j(S_m) = \Delta^{j-1}(n^m)(0) = (j-1)!\, \mathcal{S}(m, j-1)$ for $j \geq 2$, where $\mathcal{S}$ denotes the Stirling numbers of the second kind, which vanish for $j - 1 > m$. Hence $a_0(\ell) = a_1(\ell) = 0$, and for $j \geq 2$

$$v(a_j(\ell)) \geq \min_{m \geq j-1} \big(m - v_p(m)\big) \geq (j-1) - \log_p(j-1),$$

because $x - \log_p x$ is increasing for $x \geq 1$ and $v_p(m) \leq \log_p m$. For $j = 2$ this is 1. For $j \geq 3$ and $p \geq 5$ we show $(j-1) - \log_p(j-1) \geq \frac{1}{2} + \frac{j}{4}$: at $j = 3$ the left side is $2 - \log_5 2 > 1.56 \geq 1.25$, and the difference of the two sides increases with j. For $j = 1$ and $j = 0$ the coefficients vanish. So $v(a_j(\ell)) \geq \frac{1}{2} + \frac{j}{4}$ for all j, that is, $\ell \in \mathcal{D}(\frac{1}{4}, \frac{1}{2})$ by Lemma 12.43(i).

Exponentiation. By Lemma 12.43(ii), $\pm \ell^m/m! \in \mathcal{D}(\frac{1}{4}, \frac{m}{2} - v(m!))$, and $\frac{m}{2} - v(m!) \geq \frac{m}{2} - \frac{m}{p-1} \geq \frac{m}{4}$ for $p \geq 5$. By Lemma 12.43(iii), $c_P^{\pm 1} = \exp(\pm \ell) = \sum_m (\pm \ell)^m/m!$ lies in $\mathcal{D}(\frac{1}{4}, 0)$.

The function $\log d_P(\cdot, \rho)$. Here $\log d_P(n, \rho) = \sum_{y=1}^{\rho} \sum_s \frac{(-1)^s (P/y)^{s+1}}{s+1} n^{s+1}$, and $a_j(n^m) = j!\, \mathcal{S}(m, j)$, which vanishes for $j = 0$ and $j > m$. So $v(a_j(\log d_P)) \geq \min_{m \geq j}(m - v_p(m)) \geq j - \log_p j$ for $j \geq 1$, which is at least $\frac{1}{2} + \frac{j}{4}$ for $p \geq 5$. The same exponentiation applies. □

Corollary 12.45. *Let $p \geq 5$, let $f_1, \ldots, f_K \in \mathcal{D}(\frac{1}{4}, 0)$, and let $c \geq 1$ be an integer. There are $b_{\mathbf{J}} \in \mathbb{C}_p$ with $v(b_{\mathbf{J}}) \geq 0$, indexed by $\mathbf{J} \in \mathbb{Z}_{\geq 0}^K$ with $|\mathbf{J}| \leq 4c$, such that for every $\mathbf{n} \in \mathbb{Z}_{\geq 0}^K$*

$$v\Big(\prod_{k=1}^{K} f_k(n_k) - \sum_{|\mathbf{J}| \leq 4c} b_{\mathbf{J}} \prod_k \binom{n_k}{J_k}\Big) \geq c.$$

If the f_k take rational values, the $b_{\mathbf{J}}$ are rational.

Proof. Write $f_k(n) = \sum_j a_{k,j}\binom{n}{j}$ with $v(a_{k,j}) \geq j/4$. Then $\prod_k f_k(n_k) = \sum_{\mathbf{J}} \left(\prod_k a_{k,J_k}\right) \prod_k \binom{n_k}{J_k}$, a pointwise convergent sum in which the term of $\mathbf{J}$ has valuation at least $|\mathbf{J}|/4$. Put $b_{\mathbf{J}} = \prod_k a_{k,J_k}$ for $|\mathbf{J}| \leq 4c$. The omitted terms have valuation at least $(4c+1)/4 > c$. □

12.9.3 Binomial polynomials as divided derivatives

For a variable x the divided derivative is $\partial^{(m)} x^\alpha = \binom{\alpha}{m} x^{\alpha - m}$, and for multi-indices $\partial^{(\mathbf{m})} = \prod_s \partial_s^{(m_s)}$. It maps integral polynomials to integral polynomials.

Lemma 12.46. *For a multi-index $\mathbf{m}$ and every monomial x^α, $\binom{\alpha}{\mathbf{m}} x^\alpha = x^{\mathbf{m}} \partial^{(\mathbf{m})} x^\alpha$. Hence multiplication of the coefficient of x^α by a polynomial $\sum_{\mathbf{m}} b_{\mathbf{m}} \binom{\alpha}{\mathbf{m}}$ in the exponents is the differential operator $\sum_{\mathbf{m}} b_{\mathbf{m}} x^{\mathbf{m}} \partial^{(\mathbf{m})}$, whose order is the largest $|\mathbf{m}|$.*

Proof. $x^{\mathbf{m}} \partial^{(\mathbf{m})} x^\alpha = \binom{\alpha}{\mathbf{m}} x^\alpha$ coordinatewise. □

Corollary 12.45 and Lemma 12.46 together say that multiplication by the twisting unit $\gamma(\alpha)$ of the twisted Fock form agrees modulo p^c with an integral differential operator of order at most $4c$ in the digit variables.

Proposition 12.47. *Let $P = p^g$, let $\boldsymbol{\rho}, \mathbf{M} \in \mathbb{Z}^r_{\geq 0}$ with $\rho_a < P$, and let $c \geq 1$. There are integers $e_{\boldsymbol{\kappa}}$, indexed by $\boldsymbol{\kappa} \in \mathbb{Z}^r_{\geq 0}$ with*

$$|\boldsymbol{\kappa}| < \frac{|\mathbf{M}|}{P} + c,$$

such that for every $\boldsymbol{\beta} \in \mathbb{Z}^r_{\geq 0}$

$$\binom{\boldsymbol{\rho} + P\boldsymbol{\beta}}{\mathbf{M}} \equiv \sum_{\boldsymbol{\kappa}} e_{\boldsymbol{\kappa}} \binom{\boldsymbol{\beta}}{\boldsymbol{\kappa}} \pmod{p^c}.$$

Consequently $\partial^{(\mathbf{M})} x^{\boldsymbol{\rho} + P\boldsymbol{\beta}} \equiv \left(\sum_{\boldsymbol{\kappa}} e_{\boldsymbol{\kappa}} \binom{\boldsymbol{\beta}}{\boldsymbol{\kappa}}\right) x^{\boldsymbol{\rho} + P\boldsymbol{\beta} - \mathbf{M}}$ modulo p^c, with the convention that the monomial is 0 when an exponent is negative.

Proof. Coordinatewise, Vandermonde gives $\binom{\rho + P\beta}{M} = \sum_i \binom{\rho}{i}\binom{P\beta}{M-i}$, so it suffices to treat $\binom{P\beta}{m}$ for each $m \leq M$, jointly over the coordinates. In $\mathbb{Z}[x]$ write $(1+x)^P = 1 + x^P + R(x)$ with $R(x) = \sum_{0<l<P} \binom{P}{l} x^l$. By Kummer's theorem $v\binom{P}{l} = g - v(l) \geq 1$ for $0 < l < P$, so $R \in p\mathbb{Z}[x]$. Then

$$(1+x)^{P\beta} = \left(1 + (x^P + R(x))\right)^\beta = \sum_{k \geq 0} \binom{\beta}{k} \sum_{u=0}^{k} \binom{k}{u} R(x)^u x^{P(k-u)}.$$

The terms with $u \geq c$ lie in $p^c\mathbb{Z}[x]$. A term with $u < c$ contributes to the coefficient of x^m when $P(k-u) \leq m$, and not otherwise, so $k < m/P + c$. Hence $\binom{P\beta}{m} \equiv \sum_{k < m/P + c} \epsilon_{m,k} \binom{\beta}{k}$ modulo p^c with integers $\epsilon_{m,k}$. For several coordinates take the product of the generating functions $\prod_a (1+x_a)^{P\beta_a}$ and count the factors $R(x_a)$ jointly. A term with u such factors in total lies in $p^u\mathbb{Z}[x]$, so the terms with total $u < c$ remain, and the coefficient of $x^{\mathbf{m}}$ then involves $\binom{\boldsymbol{\beta}}{\boldsymbol{\kappa}}$ with $P(|\boldsymbol{\kappa}| - u) \leq |\mathbf{m}| \leq |\mathbf{M}|$, so $|\boldsymbol{\kappa}| < |\mathbf{M}|/P + c$. □

The divided derivative of a digit-split monomial agrees modulo p^c with a combination of binomials in the high digits of order less than $|\mathbf{M}|/P + c$.

12.9.4 Divided powers and the balanced lattice

Let $\mathcal{V}$ be the space of polynomials in variables $x_{i,a}$, with i in a set of letters and $1 \le a \le r$, over a finite extension $\mathcal{K}$ of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}$. The *balanced lattice* is

$$\mathcal{B} = \Big\{ F = \sum_{\alpha} F_\alpha x^\alpha : \; v(F_\alpha) + \tfrac{1}{2} v(\alpha!) \ge 0 \text{ for all } \alpha \Big\},$$

the lattice of the balanced norm $v_{\mathrm{BB}}(F) = \min_\alpha \big(v(F_\alpha) + \frac{1}{2} v(\alpha!) \big)$ as in Hypothesis 12.12(B). For a set S of letters, each marked as a vector or a covector letter, and $a \neq b$, let $E_{ab} = \sum_{i \in S} \pm x_{i,a} \partial_{i,b}$, with the sign and the exchange of a, b at covector letters that make $a \mapsto E_{ab}$ the action of $\mathfrak{gl}_r$ on the vectors and on the dual.

Lemma 12.48. *For $a \neq b$ and $k \ge 0$ the divided power $E_{ab}^{(k)} = E_{ab}^k / k!$ maps $\mathcal{B}$ into $\mathcal{B}$.*

Proof. The operators $x_{i,a}\partial_{i,b}$ for different i commute, and $\big(\sum_i T_i\big)^{(k)} = \sum_{k_1 + \cdots = k} \prod_i T_i^{(k_i)}$ for commuting T_i. So it suffices to treat one letter and one pair (a, b), and the action on the two variables $x_a = x_{i,a}$, $x_b = x_{i,b}$. On a monomial $x_a^n x_b^m$,

$$(x_a \partial_b)^{(k)} x_a^n x_b^m = \binom{m}{k} x_a^{n+k} x_b^{m-k}.$$

The change in the balanced weight is

$$v\binom{m}{k} + \tfrac{1}{2} v\big((n+k)!\,(m-k)!\big) - \tfrac{1}{2} v(n!\,m!) = \tfrac{1}{2} v\Big(\frac{m!}{(m-k)!}\Big) + \tfrac{1}{2} v\Big(\frac{(n+k)!}{n!}\Big) - v(k!) \ge 0,$$

because $m!/(m-k)!$ and $(n+k)!/n!$ are both divisible by $k!$. □

Lemma 12.48 says that $\mathcal{B}$ is stable under the Kostant $\mathbb{Z}$-form of $U(\mathfrak{gl}_r)$ acting through the letters of S.

Proposition 12.49. *Let $V \subset \mathcal{V}$ be a finite-dimensional subspace stable under the action of $\mathfrak{gl}_r$ through the letters of S, and let M bound $|\langle \nu, \alpha^\vee \rangle|$ for every weight ν of V and every root α. Let μ be a dominant weight and V_μ its weight space. Order the positive roots normally and let*

$$\mathsf{P} = \prod_{\alpha > 0} \sum_{k \ge 0} \frac{(-1)^k}{k!} \frac{1}{(h_\alpha + \rho(h_\alpha) + 1) \cdots (h_\alpha + \rho(h_\alpha) + k)} f_\alpha^k e_\alpha^k$$

be the extremal projector of Asherova, Smirnov and Tolstoy. Then P is defined on V_μ, it is the projection of V_μ onto the highest weight vectors $V_\mu^{\mathfrak{n}^+}$ along $V_\mu \cap \mathfrak{n}^- V$, and it maps $\mathcal{B} \cap V_\mu$ into $p^{-\binom{r}{2} \lfloor \log_p (M+r) \rfloor} (\mathcal{B} \cap V_\mu)$.

Proof. The formula and the projection property are the theorem of R. M. Asherova, Yu. F. Smirnov and V. N. Tolstoy, *Projection operators for simple Lie groups*, Theor. Math. Phys. 8 (1971), 813–825, in the form of D. P. Zhelobenko, *Representations of Reductive Lie Algebras* (1994), Chapter IV. Each operator $f_\alpha^k e_\alpha^k$ preserves weights, so h_α acts on V_μ through the scalar $\langle \mu, \alpha^\vee \rangle$ in every factor. Put $N_\alpha = \langle \mu + \rho, \alpha^\vee \rangle$, a positive integer since μ is dominant, so no denominator vanishes. Since $f_\alpha^k e_\alpha^k = (k!)^2 f_\alpha^{(k)} e_\alpha^{(k)}$, the k-th term of the factor of α acts on V_μ as

$$(-1)^k \frac{k!}{(N_\alpha + 1) \cdots (N_\alpha + k)} f_\alpha^{(k)} e_\alpha^{(k)} = (-1)^k \binom{N_\alpha + k}{k}^{-1} f_\alpha^{(k)} e_\alpha^{(k)}.$$

If $e_\alpha^{(k)} V_\mu \neq 0$ then $\mu + k\alpha$ is a weight of V, so $\langle \mu, \alpha^\vee \rangle + 2k \leq M$ and $N_\alpha + k \leq M + \langle \rho, \alpha^\vee \rangle \leq M + r$. Kummer's theorem gives $v(\binom{N_\alpha + k}{k}) \leq \lfloor \log_p(M + r) \rfloor$. The divided powers preserve $\mathcal{B}$ by Lemma 12.48, and a finite sum of elements of $p^{-c}\mathcal{B}$ lies in $p^{-c}\mathcal{B}$. So each of the $\binom{r}{2}$ factors costs at most $\lfloor \log_p(M + r) \rfloor$. □

Both are at least the loss of Proposition 12.49 when $M \leq 2 \deg_S$. The restriction to dominant μ is necessary. For $\mathfrak{sl}_2$ and the lowest weight -3 of the representation of highest weight 3, the denominator $(h + 2)(h + 3)$ vanishes.

Proposition 12.50. *Let S be a set of letters and $\theta \in \mathbb{Z}$. Let $A_S(\theta) \subset \mathcal{V}_d$ be the sum of the $\mathfrak{gl}_r$-isotypic components of the multidegree-d space $\mathcal{V}_d$, for the action through S, whose highest weight μ has $\mu_r \geq \theta$. Then there is a unique Fock-selfadjoint projection $\pi_{S,\theta}$ of $\mathcal{V}_d$ onto $A_S(\theta)$. For $S \subset S'$ or $S \cap S' = \emptyset$, $\pi_{S,\theta}$ commutes with the action of $\mathfrak{gl}_r$ through S' and with $\pi_{S',\theta'}$.*

Proof. Everything is defined over $\mathbb{Q}$, so we work on the space $\mathcal{V}_{d,\mathbb{Q}}$ of rational polynomials and extend scalars at the end. There the Fock form is positive definite. By the adjointness of x_s and ∂_s, the operator $x_{i,a}\partial_{i,b}$ is adjoint to $x_{i,b}\partial_{i,a}$, so $\langle Xu, u' \rangle = \langle u, X^{\mathsf{T}} u' \rangle$ for $X \in \mathfrak{gl}_r$ acting through S, where X^{T} is the transpose. The covector convention preserves this.

Let V_λ and V_ν be two isotypic components with $\lambda \neq \nu$. The form gives a map $V_\lambda \to V_\nu^*$ that intertwines the action on V_λ with the action $X \mapsto (X^{\mathsf{T}})^*$ on V_ν^*. This twisted dual has the same weights as V_ν, since $h^{\mathsf{T}} = h$ for diagonal h, so it is isomorphic to V_ν, and Schur's lemma makes the map zero. So the isotypic components are mutually orthogonal, the form is nondegenerate on each of them by positivity, and the orthogonal projection onto $A_S(\theta)$ is the sum of the isotypic projections. It is selfadjoint. A selfadjoint projection onto a subspace on which the form is nondegenerate is the orthogonal projection, which gives uniqueness.

An operator commuting with the action through S maps each isotypic component into itself, by Schur's lemma, and so commutes with $\pi_{S,\theta}$. If $S \subset S'$, the action through S' is the sum of the action through S, which preserves the S-isotypic components, and the action through $S' \setminus S$, which commutes with the action through S. So the action through S' commutes with $\pi_{S,\theta}$. Applying the same remark to S', the projection $\pi_{S',\theta'}$ commutes with every operator commuting with the action through S', in particular with $\pi_{S,\theta}$. If $S \cap S' = \emptyset$ the two actions commute and the same argument applies. □

The remaining steps of engine (B) are the following.

(B7) *Norm bound for the isotypic projector.* $v_{\mathrm{BB}}(\pi_{S,\theta} F) \geq v_{\mathrm{BB}}(F) - \binom{r}{2} \log_p(2 \deg_S + r)$ for every $F \in \mathcal{V}_d$. Proposition 12.49 bounds the extremal projector P, which projects onto highest weight vectors. The passage from P to the isotypic projector $\pi_{S,\theta}$, which reassembles a full isotypic component from its highest weight vectors by divided powers $f^{(k)}$ with normalizing binomial coefficients, is not written here. The expected loss is the one of P by the same Kummer bound, and the precise accounting is the residual.

(B8) *Steps B, C and DEF.* The derivation of the detection bound of Hypothesis 12.12(B) for a source F of multidegree (d_s) from the tools of this subsection. Its tools are Propositions 12.41, 12.44, 12.47, 12.50, Corollary 12.45, Lemma 12.46 and (B7). The combination of the three steps into the detection bound $v(\mathrm{B}(F, \mathrm{ph}\, G')) \leq n_0 + D_m + D_l + 3\kappa_p$ is a direct addition of the three losses once the three steps hold.

(B9) *Transfer*. The contravariant self-duality with loss $\kappa_1(\lambda) + \kappa_1(\lambda^\flat) \leq C_K|T|k^2 \log_p a$, and surjectivity of multiplication $H^0(\lambda^\flat) \otimes H^0(\lambda^{\mathrm{rest}}) \to H^0(\lambda)$ on edge lattices. The surjectivity over $\mathbb{Z}$ for GL_n is the projective normality of flag varieties in all characteristics, S. Ramanan and A. Ramanathan, *Projective normality of flag varieties and Schubert varieties*, Invent. Math. 79 (1985), 217–224. Its transport to the edge lattices is not written here.

Statements (B8) and (B9) are not proved here, apart from the published surjectivity theorem named in (B9).

12.9.5 The norm of the isotypic projector

Throughout this subsection $\mathcal{K}$ is a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}$, p is odd, and $\mathfrak{g} = \mathfrak{gl}_r(\mathcal{K})$ with the standard basis E_{ab}. For a positive root $\alpha = \epsilon_a - \epsilon_b$ ($a < b$) write $e_\alpha = E_{ab}$, $f_\alpha = E_{ba}$, $e_\alpha^{(k)} = e_\alpha^k/k!$ and $f_\alpha^{(k)} = f_\alpha^k/k!$. For a finite-dimensional representation U of $\mathfrak{g}$ in which the E_{cc} act semisimply with integer eigenvalues we write U_ν for the weight space of $\nu \in \mathbb{Z}^r$. An $\mathcal{O}$-lattice $\Lambda \subset U$ is *graded* when $\Lambda = \bigoplus_\nu (\Lambda \cap U_\nu)$, and *admissible* when it is graded and $e_\alpha^{(k)}\Lambda \subset \Lambda$ and $f_\alpha^{(k)}\Lambda \subset \Lambda$ for every positive root α and every $k \geq 0$. The *invariant projection* P_U of U is the projection onto $U^{\mathfrak{g}}$ along the sum of the isotypic components of nontrivial type.

Lemma 12.51. *Let U be as above, let $\Lambda \subset U$ be an admissible lattice, and let $L \geq 0$ be an integer such that $k \leq L$ whenever $k \geq 1$ and $k\alpha$ is a weight of U for a positive root α. Then*

$$p^N P_U \Lambda \subset \Lambda, \qquad N = \binom{r}{2} \lfloor \log_p(L + r - 1) \rfloor.$$

Proof. The invariant projection kills U_ν for $\nu \neq 0$, because $\mathfrak{g}$-invariant vectors have weight 0. Since Λ is graded, $P_U\Lambda = P_U(\Lambda \cap U_0)$, and it suffices to treat $x \in \Lambda \cap U_0$.

Let P be the extremal projector of Proposition 12.49. The weight 0 is dominant, so P is defined on U_0 and is the projection of U_0 onto $U_0^{\mathfrak{n}^+}$ along $U_0 \cap \mathfrak{n}^- U$. A vector of $U_0^{\mathfrak{n}^+}$ is a highest weight vector of weight 0 or is zero, so it generates a trivial representation and lies in $U^{\mathfrak{g}}$. Conversely $U^{\mathfrak{g}} \cap U_0 \subset U_0^{\mathfrak{n}^+}$. In an irreducible representation of highest weight $\lambda \neq 0$ every weight vector of weight different from λ lies in $\mathfrak{n}^-$ applied to the representation, so the weight-0 part of every nontrivial isotypic component lies in $U_0 \cap \mathfrak{n}^- U$, and $\mathfrak{n}^- U$ lies in the sum of the nontrivial isotypic components. Hence $\mathsf{P} = P_U$ on U_0.

On U_0 the factor of $\alpha = \epsilon_a - \epsilon_b$ in P acts as $\sum_{k \geq 0} (-1)^k \binom{N_\alpha + k}{k}^{-1} f_\alpha^{(k)} e_\alpha^{(k)}$ with $N_\alpha = \langle 0 + \rho, \alpha^\vee \rangle = b - a \leq r - 1$, as in the proof of Proposition 12.49. Each $f_\alpha^{(k)} e_\alpha^{(k)}$ preserves weights, so every factor maps U_0 to U_0. If $y \in U_0$ and $e_\alpha^{(k)} y \neq 0$ with $k \geq 1$, then $k\alpha$ is a weight of U, so $k \leq L$, and Kummer's theorem gives $v\binom{N_\alpha + k}{k} \leq \lfloor \log_p(N_\alpha + k) \rfloor \leq \lfloor \log_p(L + r - 1) \rfloor$. Since Λ is admissible, the factor of α maps $p^{-c}\Lambda \cap U_0$ into $p^{-c - \lfloor \log_p(L+r-1) \rfloor}\Lambda \cap U_0$ for every c. There are $\binom{r}{2}$ positive roots, so $p^N \mathsf{P} x \in \Lambda$. □

Lemma 12.52. *Let V be a finite-dimensional representation of $\mathfrak{g}$ as above, let λ be a dominant weight, let e_λ be the projection of V onto its isotypic component of type λ along the other isotypic components, and suppose that V contains an irreducible subrepresentation W of highest weight λ. Let $b_1, \ldots, b_n$ be a basis of W with dual basis $b_1^*, \ldots, b_n^*$, and let P be the invariant projection of* $\mathrm{Hom}(W, V)$, *on which $X \in \mathfrak{g}$*

acts by $X \cdot \varphi = X\varphi - \varphi X$. Then for every $m \in V$

$$e_\lambda(m) = n \sum_{i=1}^{n} P\big(b_i^* \otimes m\big)(b_i),$$

where $b_i^* \otimes m$ is the map $w \mapsto b_i^*(w)m$.

Proof. Both sides are linear in m, so we may take m in an irreducible subrepresentation $U \subset V$ of highest weight μ. The maps $b_i^* \otimes m$ lie in $\mathrm{Hom}(W, U)$, a subrepresentation of $\mathrm{Hom}(W, V)$, and P preserves it, so $P(b_i^* \otimes m) \in \mathrm{Hom}(W, U)^{\mathfrak{g}}$. If $\mu \neq \lambda$ this space is 0 by Schur's lemma, and both sides vanish. If $\mu = \lambda$ choose an isomorphism $\psi\colon W \to U$ of representations. Composition with ψ^{-1} identifies $\mathrm{Hom}(W, U)$ with $\mathrm{End}(W)$ as representations, and the invariant projection of $\mathrm{End}(W)$ is $A \mapsto (\mathrm{tr}\, A/n)\,\mathrm{id}$, because this map is equivariant, idempotent, has image $\mathrm{End}(W)^{\mathfrak{g}} = \mathcal{K}\,\mathrm{id}$, and the invariant projection is the unique equivariant projection onto the invariants. Hence $P(b_i^* \otimes m) = \frac{1}{n} b_i^*(\psi^{-1} m)\,\psi$ and

$$n \sum_i P(b_i^* \otimes m)(b_i) = \sum_i b_i^*(\psi^{-1} m)\,\psi(b_i) = \psi\psi^{-1} m = m = e_\lambda(m). \qquad \square$$

We return to the setting of Proposition 12.50. Let $\mathcal{V}_d$ be the space of polynomials of multidegree d, with $\mathfrak{g}$ acting through the letters of S, and put $\mathcal{B}_d = \mathcal{B} \cap \mathcal{V}_d$ and $\deg_S = \sum_{i \in S} d_i$.

Lemma 12.53. *$\mathcal{B}_d$ is an admissible lattice of $\mathcal{V}_d$, every weight ν of $\mathcal{V}_d$ satisfies $-\deg_S \leq \nu_c \leq \deg_S$, and any two weights μ, ω of $\mathcal{V}_d$ satisfy $|\mu_c - \omega_c| \leq \deg_S$ for every c.*

Proof. A monomial x^β is a weight vector, of weight $\nu_c = \sum_{i \in S \text{ vector}} \beta_{i,c} - \sum_{i \in S \text{ covector}} \beta_{i,c}$, and $\mathcal{B}_d$ is the direct sum of the lines $\{c\, x^\beta : v(c) + \frac{1}{2} v(\beta!) \geq 0\}$ over the monomials of multidegree d. So $\mathcal{B}_d$ is a graded lattice. It is stable under $E_{ab}^{(k)}$ for $a \neq b$ by Lemma 12.48. For the weight bounds, $\sum_{i \in S \text{ vector}} \beta_{i,c}$ lies between 0 and $\sum_{i \in S \text{ vector}} d_i$, and $\sum_{i \in S \text{ covector}} \beta_{i,c}$ lies between 0 and $\sum_{i \in S \text{ covector}} d_i$, so ν_c lies in an interval of length $\deg_S$ that contains 0. $\square$

Theorem 12.54 (the norm bound (B7))**.** *In the setting of Proposition 12.50, put*

$$N_S = \binom{r}{2} \big\lfloor \log_p(\deg_S + r - 1) \big\rfloor.$$

Then $p^{N_S} \pi_{S,\theta}\, \mathcal{B}_d \subset \mathcal{B}_d$. Equivalently $v_{\mathrm{BB}}(\pi_{S,\theta} F) \geq -N_S$ for every $F \in \mathcal{V}_d$ with $v_{\mathrm{BB}}(F) \geq 0$. If the value group $v(\mathcal{K}^\times)$ contains $\frac{1}{2}\mathbb{Z}$, which holds when $\mathcal{O}$ contains an element ϖ with $\varpi^2 = p$, then

$$v_{\mathrm{BB}}(\pi_{S,\theta} F) \geq v_{\mathrm{BB}}(F) - N_S \geq v_{\mathrm{BB}}(F) - \binom{r}{2} \log_p(2 \deg_S + r) \qquad (F \in \mathcal{V}_d).$$

Proof. By the proof of Proposition 12.50, $\pi_{S,\theta} = \sum_{\lambda \in \Sigma} e_\lambda$, where Σ is the finite set of dominant weights λ with $\lambda_r \geq \theta$ for which $\mathcal{V}_d$ has a nonzero isotypic component of type λ, and e_λ is the isotypic projection. Since $\mathcal{B}_d$ is an $\mathcal{O}$-module it suffices to show $p^{N_S} e_\lambda \mathcal{B}_d \subset \mathcal{B}_d$ for each $\lambda \in \Sigma$.

Fix $\lambda \in \Sigma$ and an irreducible subrepresentation $W \subset \mathcal{V}_d$ of highest weight λ. Put $\Lambda_W = \mathcal{B}_d \cap W$. Since W is spanned by weight vectors and $\mathcal{B}_d$ is graded, $\Lambda_W = \bigoplus_\omega (\mathcal{B}_d \cap W_\omega)$, and Λ_W is stable under the divided powers $e_\alpha^{(k)}$, $f_\alpha^{(k)}$. It is a free $\mathcal{O}$-module of rank $n = \dim W$. Choose an $\mathcal{O}$-basis $b_1, \ldots, b_n$ of Λ_W, which is a $\mathcal{K}$-basis of W, with dual basis b_i^*. Let

$$\Lambda_H = \{\varphi \in \mathrm{Hom}(W, \mathcal{V}_d) :\ \varphi(\Lambda_W) \subset \mathcal{B}_d\},$$

an $\mathcal{O}$-lattice in $\mathrm{Hom}(W, \mathcal{V}_d)$, isomorphic to $\mathrm{Hom}_{\mathcal{O}}(\Lambda_W, \mathcal{B}_d)$.

Λ_H is graded. For $\varphi \in \Lambda_H$ its component of weight ν is $\varphi_\nu = \sum_\omega \mathrm{pr}_{\omega+\nu} \circ \varphi \circ \mathrm{pr}_\omega$, where pr denotes the weight projections. These preserve $\mathcal{B}_d$ and Λ_W because both are graded, so $\varphi_\nu \in \Lambda_H$.

Λ_H is stable under divided powers. The operators $L_X\varphi = X\varphi$ and $R_X\varphi = \varphi X$ commute, and $X \cdot \varphi = (L_X - R_X)\varphi$, so for $X = e_\alpha$ or f_α

$$X^{(k)} \cdot \varphi = \sum_{i+j=k} (-1)^j \, X^{(i)} \circ \varphi \circ X^{(j)},$$

and each term maps Λ_W into $\mathcal{B}_d$ because $X^{(j)}\Lambda_W \subset \Lambda_W$ and $X^{(i)}\mathcal{B}_d \subset \mathcal{B}_d$ (Lemma 12.53).

The weights of $\mathrm{Hom}(W, \mathcal{V}_d) \cong W^* \otimes \mathcal{V}_d$ are the differences $\mu - \omega$ of a weight μ of $\mathcal{V}_d$ and a weight ω of W. The weights of W are weights of $\mathcal{V}_d$, so if $k\alpha = \mu - \omega$ with $\alpha = \epsilon_a - \epsilon_b$ then $k = \mu_a - \omega_a \le \deg_S$ by Lemma 12.53. Lemma 12.51 with $U = \mathrm{Hom}(W, \mathcal{V}_d)$, $\Lambda = \Lambda_H$ and $L = \deg_S$ gives $p^{N_S} P \Lambda_H \subset \Lambda_H$.

Let $m \in \mathcal{B}_d$. Then $b_i^* \otimes m \in \Lambda_H$, because $b_i^*(\Lambda_W) \subset \mathcal{O}$. Hence $p^{N_S} P(b_i^* \otimes m) \in \Lambda_H$, and since $b_i \in \Lambda_W$, the vector $p^{N_S} P(b_i^* \otimes m)(b_i)$ lies in $\mathcal{B}_d$. Lemma 12.52 gives $p^{N_S} e_\lambda(m) = n \sum_i p^{N_S} P(b_i^* \otimes m)(b_i) \in \mathcal{B}_d$, since n is an integer.

For the second form, $v_{\mathrm{BB}}(cF) = v(c) + v_{\mathrm{BB}}(F)$ and $\mathcal{B}_d = \{F : v_{\mathrm{BB}}(F) \ge 0\}$. The values $v(F_\beta) + \frac{1}{2}v(\beta!)$ lie in $v(\mathcal{K}^\times) + \frac{1}{2}\mathbb{Z}$, which is $v(\mathcal{K}^\times)$ under the assumption, so for $F \neq 0$ there is c with $v(c) = -v_{\mathrm{BB}}(F)$, and the first statement applied to cF gives the inequality. Finally $\lfloor \log_p(\deg_S + r - 1) \rfloor \le \log_p(2\deg_S + r)$. □

Remark 12.55. In (B7) the loss is $\binom{r}{2}\lfloor \log_p(\deg_S + r - 1) \rfloor$. Since $\deg_S \le \sum_x d_x$, the bound $e_P = r^2(\log_p(2\sum_x d_x + r) + 1)$ in (B8) applies to each projector.

12.10 Engine (A): the rank of the symbol spaces

The statement is Hypothesis 12.12(A). This section proves the algebraic steps of its proof and the analytic lemmas of the concentration estimate, and reduces the standard monomial theory to the literature.

12.10.1 Lifting from the residue field

Lemma 12.56. *Let $\mathcal{O}$ be a discrete valuation ring with uniformizer ϖ and residue field $\mathbb{F}$, let $\mathcal{M}$ be a free $\mathcal{O}$-module with basis $(x^\mu)_{\mu \in I}$, I finite, and let $\mathcal{G} \subset \mathcal{M}$. Choose $g_1, \dots, g_n \in \mathcal{G}$ whose reductions form a basis of the $\mathbb{F}$-span of the reductions of $\mathcal{G}$. Then $g_1, \dots, g_n$ are linearly independent over $\mathcal{O}$ and span a direct summand of $\mathcal{M}$. In particular* $\mathrm{rk}_{\mathcal{O}} \, \mathrm{span}_{\mathcal{O}} \, \mathcal{G} \ge \dim_{\mathbb{F}} \mathrm{span}_{\mathbb{F}} \, \overline{\mathcal{G}}$.

Proof. The reductions $\bar{g}_1, \dots, \bar{g}_n$ are independent, so the $n \times |I|$ coefficient matrix of the g_i has an $n \times n$ minor that is a unit. Complete $g_1, \dots, g_n$ by the basis vectors x^μ outside the columns of that minor. The resulting square matrix has unit determinant, so the completed family is a basis of $\mathcal{M}$. □

12.10.2 The Frobenius identity and the discriminant

Lemma 12.57. *Let $\mathbb{F}$ be a field of characteristic p and $f \in \mathbb{F}_p[X_1, \dots, X_N]$. Then $f(X_1^{p^R}, \dots, X_N^{p^R}) = f(X_1, \dots, X_N)^{p^R}$. If $D \in \mathbb{F}[X]$ is nonzero, multiplication by D is injective on $\mathbb{F}[X]$, and for every finite family (u_ι) in $\mathbb{F}[X]$,* $\dim \mathrm{span}(Du_\iota) = \dim \mathrm{span}(u_\iota)$.

Proof. The Frobenius $F \mapsto F^p$ is a ring endomorphism of $\mathbb{F}[X]$ fixing $\mathbb{F}_p$, so $f(X)^{p^R} = f^{(\sigma^R)}(X^{p^R}) = f(X^{p^R})$, where $f^{(\sigma^R)}$ has coefficients raised to the power p^R, equal to those of f. The polynomial ring is a domain. □

In the construction of the segment generators the coarse segment polynomials have integer coefficients, so their reductions have coefficients in $\mathbb{F}_p$, and Lemma 12.57 identifies the reduction of a coarse placement with exponent $p^{\mathrm{dist}+R_j}$ with the p^{R_j}-th power of the true placement. The discriminant D_ℓ is a product of powers of $r \times r$ determinants in pairwise distinct sets of variables, so its reduction is a product of nonzero polynomials and is nonzero. Then Lemma 12.57 gives $\dim \operatorname{span} \overline{\mathrm{gen}} \geq \dim \operatorname{span}(\overline{D_\ell} \cdot \overline{\mathrm{coarse}}) = \dim \operatorname{span} \overline{\mathrm{coarse}}$, provided each $\overline{D_\ell}\,\bar{u}$ for a coarse generator u is the reduction of a true generator.

The passage from coarse to true generators uses the following statement.

(A2) *Coarse generators become true generators.* For every coarse generator u there is a true generator g with $D_\ell u \equiv g$ modulo ϖ. The mechanism is Lemma 12.57 together with the additivity of val_S and λ_S on products and the fact that a determinant of r plane vectors has type $(1, \ldots, 1)$, so that multiplication by $\det^{\lfloor B_o/(k^2 Y)\rfloor}$ raises the lowest plane type by the margin.

Statement (A2) is not proved here.

12.10.3 Digit decomposition

Lemma 12.58. *Let $P_1 < P_2 < \cdots < P_J$ be powers of p, each dividing the next. Every $\mathbf{e} \in \mathbb{Z}^N_{\geq 0}$ has at most one expression $\mathbf{e} = \boldsymbol{\ell} + \sum_{j=1}^{J} P_j \mathbf{m}_j$ with $0 \leq \ell_t < P_1$ and with the t-th coordinate of $\mathbf{m}_j$ less than P_{j+1}/P_j for $j < J$. Consequently, if for each j the polynomials $(u_{j,\iota})_\iota$ in $\mathbb{F}[X]$ are linearly independent and $\mathbb{F}$ has characteristic p, the products $X^{\boldsymbol{\ell}} \prod_j u_{j,\iota_j}(X^{P_j})$, over labels $\boldsymbol{\ell}$ and families (ι_j) for which every monomial obeys these digit bounds, are linearly independent.*

Proof. Uniqueness is that of the mixed-radix expansion. For independence, the monomials of $X^{\boldsymbol{\ell}} \prod_j u_{j,\iota_j}(X^{P_j})$ are $X^{\boldsymbol{\ell} + \sum_j P_j \mathbf{m}_j}$ with $X^{\mathbf{m}_j}$ a monomial of u_{j,ι_j}, and distinct tuples $(\boldsymbol{\ell}, \mathbf{m}_1, \ldots, \mathbf{m}_J)$ give distinct exponents. So the span is the tensor product of the spans of the $(u_{j,\iota})_\iota$ over the label monomials, and a tensor product of independent families is independent. □

Lemma 12.59. *Let the frame tops satisfy $f_{j+J,i} = f_{j,i} + n'$ and let R_j be J-periodic. Put $Q_j = p^{f_{j+1,r-1} - f_{j,r-1} + R_{j+1} - R_j}$. Then $\prod_{j=0}^{J-1} Q_j = p^{n'}$.*

Proof. The exponent telescopes to $f_{J,r-1} - f_{0,r-1} + R_J - R_0 = n'$. □

Lemma 12.60. *For $1 \leq r < k$ and a prime power q, the number of r-dimensional subspaces of $\mathbb{F}_q^k$ is at least $q^{r(k-r)}$.*

Proof. The subspaces that are graphs of linear maps $\mathbb{F}_q^r \to \mathbb{F}_q^{k-r}$ are distinct and number $q^{r(k-r)}$. □

12.10.4 Standard monomials

Engine (A) uses the following two statements on standard monomials.

(A4) *(Published.) Integral standard monomial bases.* Let $\mathcal{O}$ be a discrete valuation ring. The module $\bigotimes_i \mathrm{Sym}^{b_i}(\mathcal{O}^r)$, with duals at covector letters, has an $\mathcal{O}$-basis of products of column polynomials $\Psi(\mu, I)$ indexed by Pieri paths $0 = \mu_0, \mu_1, \dots$ (each step adds a horizontal strip of size b_i at a vector letter, or removes one at a covector letter) and semistandard fillings I of the last shape. For vector letters alone this is the standard monomial theory of Doubilet, Rota and Stein, *On the foundations of combinatorial theory IX*, Stud. Appl. Math. 53 (1974), in the characteristic-free form of De Concini, Eisenbud and Procesi, *Young diagrams and determinantal varieties*, Invent. Math. 56 (1980), 129–165, and of Désarménien, Kung and Rota, *Invariant theory, Young bitableaux, and combinatorics*, Adv. Math. 27 (1978). Mixed vector and covector letters are covered by the mixed tensor version of De Concini and Procesi, *A characteristic free approach to invariant theory*, Adv. Math. 21 (1976).

(A5) *Compatibility with prefix types.* The submodule cut out by the prefix conditions $\mathrm{val}_{\{y \le x\}} \ge \theta_x$ is spanned over $\mathcal{O}$ and over the residue field by the basis elements whose path satisfies the same inequalities, and the leading part of Ψ for a weight separating plane and normal coordinates is $\Psi_{\mathrm{pl}}\Psi_{\mathrm{nm}}$. This is a filtration property of standard monomials along the letters, classical in substance (the straightening law respects the order of the letters), and the exact integral form required is not matched to a published statement here.

Statement (A4) is published, with the sources named in it. Statement (A5) is not proved here.

12.10.5 The concentration estimate

For integers $k' \ge 1$ define $f_{k'}(y) = \log\Gamma(y + k' + 1) - \log\Gamma(y + 1) - \log k'!$ for real $y \ge 0$, so that $f_{k'}(y) = \log \binom{y+k'}{k'}$ for integers $y \ge 0$.

Lemma 12.61. *For $y \ge 0$, $f'_{k'}(y) = \sum_{i=1}^{k'} \frac{1}{y+i}$. Hence $f_{k'}$ is increasing and concave, $\frac{k'}{y+k'} \le f'_{k'}(y) \le \frac{k'}{y+1}$, $f'_{k'}(y) \ge \frac{k'}{2y}$ for $y \ge k'$, $f'_{k'}(y') \le \frac{1}{2} f'_{k'}(y)$ for $y' \ge 4y \ge 4k'$, and $f_{k'}(x) - f_{k'}(y) \le f'_{k'}(y)(x - y)$ for $x, y \ge 0$.*

Proof. $\Gamma(y + k' + 1)/\Gamma(y + 1) = \prod_{i=1}^{k'}(y + i)$, so $f_{k'}(y) = \sum_i \log(y + i) - \log k'!$. Differentiate. Each term $1/(y + i)$ lies between $1/(y + k')$ and $1/(y + 1)$. For $y \ge k'$, $k'/(y + k') \ge k'/(2y)$. For $y' \ge 4y \ge 4k'$,

$$f'_{k'}(y') \le \frac{k'}{y'} \le \frac{k'}{4y} = \frac{1}{2} \cdot \frac{k'}{2y} \le \frac{1}{2} f'_{k'}(y).$$

The derivative is decreasing, which is concavity, and concavity gives the last inequality. □

Lemma 12.62. *Let $k' \ge 1$ and $L \ge 1$, and let $\alpha_0, \dots, \alpha_{L-1} \ge 0$ and $B_0, \dots, B_{L-1}$ be reals with $B_l \ge k'$ and $4B_{l+1} \le B_l$. Suppose $\sum_{l \ge u} \alpha_l \le \sum_{l \ge u} B_l$ for every $u < L$ (sums over $u \le l < L$). Put $\delta_u = f'_{k'}(B_u) - f'_{k'}(B_{u-1})$ with $f'_{k'}(B_{-1}) = 0$. Then $\delta_u \ge \frac{1}{2} f'_{k'}(B_u) > 0$ and*

$$\sum_l f_{k'}(\alpha_l) - \sum_l f_{k'}(B_l) \le \sum_u \delta_u \Big(\sum_{l \ge u} \alpha_l - \sum_{l \ge u} B_l \Big) \le 0.$$

If moreover $\sum_{l \ge u_0} (\alpha_l - B_l) \le -X$ with $X \ge 0$, then $\sum_l f_{k'}(\alpha_l) - \sum_l f_{k'}(B_l) \le -\frac{Xk'}{4B_{u_0}}$.

Proof. Since $B_{u-1} \geq 4B_u \geq 4k'$, Lemma 12.61 gives $f'(B_{u-1}) \leq \frac{1}{2}f'(B_u)$, so $\delta_u \geq \frac{1}{2}f'(B_u)$. By concavity $f(\alpha_l) - f(B_l) \leq f'(B_l)(\alpha_l - B_l)$. Write $f'(B_l) = \sum_{u \leq l} \delta_u$ and exchange the sums (summation by parts),

$$\sum_l f'(B_l)(\alpha_l - B_l) = \sum_u \delta_u \sum_{l \geq u} (\alpha_l - B_l).$$

Each $\delta_u > 0$ and each tail is at most 0. Keeping the term u_0 alone gives at most $\delta_{u_0}(-X) \leq -\frac{X}{2}f'(B_{u_0}) \leq -\frac{Xk'}{4B_{u_0}}$ by Lemma 12.61. □

The lemma says that a Pieri path whose plane part takes X more units of degree than the typical window, measured from a tail, loses a factor $e^{-Xk'/(4B)}$ in the dimension of its normal part.

The remaining steps of engine (A) are the following.

(A6) *The remaining inputs of the concentration estimate*. The first is a two-sided estimate of $\log \dim \nabla(\lambda)$ for a dominant weight of GL_n with at most N_s positive and N_s negative entries, $|\log \dim \nabla(\lambda) - \sum_m f_{K-2N_s}(\lambda_m^+) - \sum_m f_{K-2N_s}(\lambda_m^-)| \leq (2N_s r + 8N_s^2)\log(1 + A_0)$. It follows from Weyl's dimension formula $\prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i}$, in which the product of the factors pairing a nonzero entry with the block of zero entries is $\binom{\lambda_i^{\pm} + k'}{k'}$ up to $O(N_s)$ factors of size at most $1 + A_0$. The proof of the stated constants is not written here. The others are elementary properties of Pieri paths (entries bounded by the cumulative degrees, horizontal strips interlace, the number of paths is at most $(1 + 2aN_s)^{2N_s^2}$).

(A7) *Path-weight concentration*. An atypical Pieri path has weight at most $\exp\big(9N_s^2 \log(1 + 2aN_s) - \min(\frac{L_a k'}{16k}, \frac{k'}{12})\big)\dim$, and the total weight of atypical paths is at most $a^{-N_s^2}\dim$ when $L_a, k \geq 1088 N_s^2 \log a$. The mechanism is Lemma 12.62 applied to the tails of an atypical path, with (A6) giving the comparison between path weights and sums of $f_{k'}$. The assembly is not written here.

(A8) Lemmas 12.59 and 12.60: the frame deficit over a period of segments is q^{rK}, and $q^{rK} \leq \#\mathrm{Gr}_r(\mathbb{F}_q^k)$.

(A9) *Assembly of the counts*. The losses $(1 - 5/Y)$, $(1 - 1/Y)$, $(1 - r/(2Y))$ per top and $(1 - 1/(32Y))$ per segment multiply to at least $1 - |T| r^2 k / Y$. The final inequality is elementary once the per-top factors are established, and the per-top factors are not written here.

The construction in (A6), (A7) and the individual share bounds in (A9) remain open.

12.10.6 The final count of engine (A)

Lemma 12.63. *Let $r \geq 2$, $k \geq 5$, $Y \geq 1$, and let $J \leq N$ be positive integers with $Nr^2k < Y$. Let A be a real number with $1 - \frac{1}{32Y} \leq A \leq 1$. Then the numbers $1 - \frac{5}{Y}$, $1 - \frac{1}{Y}$, $1 - \frac{r}{2Y}$ lie in $[0, 1]$ and*

$$A^J \Big(\big(1 - \tfrac{5}{Y}\big)\big(1 - \tfrac{1}{Y}\big)\big(1 - \tfrac{r}{2Y}\big) \Big)^N \geq 1 - \frac{Nr^2k}{Y}.$$

Proof. Put $u = 1/Y$. From $Nr^2ku < 1$ and $r^2k \geq 20$ we get $u < \frac{1}{20}$, so $5u \leq 1$ and $\frac{r}{2}u \leq \frac{r^2k}{2}u < 1$, and $A \geq 1 - \frac{u}{32} \geq 0$. For numbers $x_1, \ldots, x_M \in [0, 1]$ one has $\prod_i (1 - x_i) \geq 1 - \sum_i x_i$, by induction on M, since $(1 - x)(1 - y) \geq 1 - x - y$ for $y \leq 1$ and $1 - \sum$ may be negative. Applying this to J

factors $1-\frac{u}{32}$ and N triples of factors gives a lower bound $1-u\big(\frac{J}{32}+N(6+\frac{r}{2})\big)\geq 1-uN\big(6+\frac{1}{32}+\frac{r}{2}\big)$, because $J\leq N$ and $A\geq 1-\frac{u}{32}$. Finally $r^2k-\frac{r}{2}\geq r(5r-\frac{1}{2})\geq 19\geq 6+\frac{1}{32}$. □

In (A9) the lemma applies with $N=|T|$ (the tops of the orbit), J the number of segments over a period and $A=1-\frac{1}{32Y}$, once the per-top factors $(1-5/Y)$, $(1-1/Y)$, $(1-r/(2Y))$ and the per-segment factor $(1-1/(32Y))$ are established. The per-top factors are the residual of (A9).

12.10.7 The comparison exponent of (L2)

For a dominant weight λ of GL_k let $V(\lambda)$ be the irreducible representation over $\mathcal{K}$, v_λ a highest weight vector and v_λ^* the linear form that is 1 on v_λ and 0 on the other weight spaces. Let $U_\mathbb{Z}$ be the $\mathbb{Z}$-subalgebra of $U(\mathfrak{gl}_k)$ generated by the $E_{ab}^{(n)}$ $(a\neq b)$ and the $\binom{E_{cc}}{n}$. Put

$$\Delta_\mathcal{O}(\lambda)=U_\mathbb{Z}\,\mathcal{O}v_\lambda,\qquad \nabla_\mathcal{O}(\lambda)=\{x\in V(\lambda):\ v_\lambda^*(ux)\in\mathcal{O}\text{ for all }u\in U_\mathbb{Z}\},$$

and $w(\lambda)=\binom{k}{2}\lfloor\log_p(\lambda_1-\lambda_k+k-1)\rfloor$.

Proposition 12.64. $p^{w(\lambda)}\nabla_\mathcal{O}(\lambda)\subset\Delta_\mathcal{O}(\lambda)\subset\nabla_\mathcal{O}(\lambda)$.

Proof. Write $\Delta=\Delta_\mathcal{O}(\lambda)$, $V=V(\lambda)$, V^* its dual with the contragredient action, and $\Delta'=U_\mathbb{Z}\,\mathcal{O}v_\lambda^*\subset V^*$. The algebra $U_\mathbb{Z}$ is spanned by products $f^{(K)}h\,e^{(K')}$ of ordered divided powers of the negative root vectors, binomials in the E_{cc} and divided powers of the positive root vectors (Kostant). The positive ones kill v_λ, the binomials act on v_λ by integers, and the negative ones lower the weight, so $\Delta\cap V_\lambda=\mathcal{O}v_\lambda$ and $v_\lambda^*(\Delta)\subset\mathcal{O}$. Since Δ is $U_\mathbb{Z}$-stable this gives $\Delta\subset\nabla_\mathcal{O}(\lambda)$.

By definition, $x\in\nabla_\mathcal{O}(\lambda)$ exactly when $\eta(x)\in\mathcal{O}$ for every $\eta\in\Delta'$, because $\eta=u^{\mathsf{c}}v_\lambda^*$ with u^{c} the image of u under the antipode, which preserves $U_\mathbb{Z}$. So $\nabla_\mathcal{O}(\lambda)$ is the $\mathcal{O}$-dual of Δ'.

Let $U=V\otimes V^*=\mathrm{End}(V)$ and $\Lambda=\Delta\otimes_\mathcal{O}\Delta'$. The lattices Δ and Δ' are spanned by weight vectors and stable under divided powers, and $X^{(n)}(y\otimes\eta)=\sum_{i+j=n}X^{(i)}y\otimes X^{(j)}\eta$, so Λ is admissible. The weights of U are differences $\mu-\mu'$ of weights of V, whose entries lie between λ_k and λ_1, so $k\alpha=\mu-\mu'$ forces $k\leq\lambda_1-\lambda_k$. Lemma 12.51 with $r=k$ and $L=\lambda_1-\lambda_k$ gives $p^{w(\lambda)}P_U\Lambda\subset\Lambda$. Since V is irreducible, $P_U(A)=\frac{\mathrm{tr}\,A}{\dim V}\mathrm{id}$, and $v_\lambda\otimes v_\lambda^*\in\Lambda$ has trace 1. So $\frac{p^{w(\lambda)}}{\dim V}\mathrm{id}\in\Lambda$.

An element $\sum_j y_j\otimes\eta_j$ of Λ maps $x\in\nabla_\mathcal{O}(\lambda)$ to $\sum_j\eta_j(x)y_j\in\Delta$, since $\eta_j(x)\in\mathcal{O}$. Applied to $\frac{p^{w(\lambda)}}{\dim V}\mathrm{id}$ this gives $p^{w(\lambda)}x\in\dim V\cdot\Delta\subset\Delta$. □

Corollary 12.65 (input (L2))**.** *Suppose that, in (F3), M is the lattice of forms with $F(L)\in\bigotimes_v\bigotimes_s\Delta_\mathcal{O}(\lambda_{v,s}^*;L_v^{(s)})\otimes H'(L)$ and M^{na} the lattice of forms with values in $\bigotimes_v\bigotimes_s\nabla_\mathcal{O}(\lambda_{v,s}^*;L_v^{(s)})\otimes H'(L)$, where the lattices at $L_v^{(s)}$ are the transports of those at $\mathcal{O}^k$ by any g with $g\mathcal{O}^k=L_v^{(s)}$. Then $M\subset M^{\mathrm{na}}$ and $p^eM^{\mathrm{na}}\subset M$ with $e=\sum_v\sum_{s\in\mathbb{Z}/n_p}w(\lambda_{v,s}^*)$. If moreover every base weight satisfies $\lambda_1-\lambda_k+k-1\leq a^2$ and the positions (v,s) number at most f, then $e\leq fk^2\log_p a$, and (L2) holds with $e_p=\lceil C_e fk^2\log_p a\rceil$ for any $C_e\geq 1$, so that $e_p\leq(C_e+1)fk^2\log_p a$ when $fk^2\log_p a\geq 1$.*

Proof. Proposition 12.64 transports along g, because both lattices are defined from $U_\mathbb{Z}$ and g intertwines the actions. For lattices $A_s\subset B_s$ with $p^{w_s}B_s\subset A_s$ the tensor products satisfy $\bigotimes A_s\subset\bigotimes B_s$ and $p^{\sum w_s}\bigotimes B_s\subset\bigotimes A_s$, since both are spanned by pure tensors. The conditions on forms are pointwise and equivariance is preserved by scalars, which gives the first claim.

For the bound, $\binom{k}{2} \le \frac{k^2}{2}$ and $\lfloor \log_p(\lambda_1 - \lambda_k + k - 1) \rfloor \le 2 \log_p a$, so each $w(\lambda^*_{v,s}) \le k^2 \log_p a$, and there are at most f terms. Since $e \le C_e f k^2 \log_p a \le e_p$, also $p^{e_p} M^{\mathrm{na}} \subset M$. □

For the base weights of Theorem 12.11 the extreme entries are $a_s \le 2a$ and $-b_s \ge -2b'$ at a top, and the other entries have absolute value at most c_G, so $\lambda_1 - \lambda_k + k - 1 \le 4a + 2c_G + k \le a^2$ once $c_G + k \le a$. The positions (v, s) are the elements of H, so their number is $n \le f$.

12.10.8 The bound for Π in (Z1)

Put $\eta = W + mh_0 + 1$, with W the number of roots of unity of E and $h_0 = 2\lceil x^{402} \rceil$ as in Section 18.

Lemma 12.66. *The nodes $w_{u_0,j}$ of Proposition 18.7 satisfy (W1), (W2) and (W3) with this η, and the bottom weights b_t of that proposition.*

Proof. At a top place, $w_{u_0,j}(0,0,t) = b_t + o_j(t)$ with $o_j(t) = \lfloor b'/x \rfloor + (j - \frac{1}{2})h_0$. Since $h_0 \ge 2$, $o_j(t) \ge b'/x - 1 + \frac{1}{2}h_0 \ge b'/x$, and $o_j(t) \le b'/x + mh_0 \le b'/x + \eta$. At a thick place, $\lceil b \rceil \le b_1(\rho) < \lceil b \rceil + W$ gives $b \le w_{u_0,j}(\rho) \le b + 1 + W + mh_0 = b + \eta$. At a thin place, $b_1(\rho) = \lceil w'' \rceil$ gives $w'' \le w_{u_0,j}(\rho) \le w'' + 1 + mh_0 \le w'' + \eta$. □

Proposition 12.67. *Suppose that $c_G \le x^{C_c}$ for a constant C_c. Then $\Pi \le \exp(x^{\varepsilon/4})$ for $x \ge x_0(\varepsilon, C_c)$.*

Proof. By Lemma 18.2, $t_r \le x^{401}$. By Proposition 5.15, $k \le 2x^{9/2+10\varepsilon}$ and $m \le 2\sqrt{x}$, so $N_m = k + m(t_r + 1) \le x^{402}$ for large x. By the remark after Proposition 5.16, $W \le 32n^2 \le 128x^{9/2}$, so $\eta \le 128x^{9/2} + 2\sqrt{x}(2x^{402} + 2) + 1 \le x^{403}$ for large x. The constants $K_\bullet$ and $C_\bullet$ depend on ε alone. Hence $\Pi \le \max(x^{403}, x^{C_c}, K)$ with K depending on ε, which is at most $\exp(x^{\varepsilon/4})$ for large x. □

13 The slope inequality for the family of period maps

Hypothesis (3.3) of Proposition 3.1 asks that the rank-weighted mean of $\mu(N_u) - h_u$ over the indices $u \in W$ exceed the slope of the dual $R^\vee$ of the lattice of automorphic forms by at most e_3. For one injective map Bost's slope inequality bounds $\mu(N_u) - h_u$ by the maximal slope of $R^\vee$. The family needs more, because $\mu_{\max}(R^\vee)$ can exceed the mean slope $\mu(R^\vee)$ by far, and because the images of the maps B_u overlap. The argument of this section replaces the maximal slope by the mean slope. It cuts each source lattice along its Harder–Narasimhan filtration, moves each piece into one step of the filtration of $R^\vee$ by the slope inequality, and bounds the rank of that step from below by a count of dimensions at one complex place, where the matrices of the maps form a nearly tight frame. Section 13.1 collects the slope theory, Section 13.2 proves the frame theorem, Section 13.3 derives the frame conditions from Hilbert–Schmidt bounds for Gram operators, Section 13.4 proves (3.3) with an explicit e_3 and bounds e_3, and Section 13.5 states the near-orthogonality theorem for Bessel Poincaré series that the construction has to provide.

13.1 Hermitian lattices and slopes

Let K be a totally complex number field of degree n_K, let d_K be its discriminant, and let Σ_K be its set of complex places, so that $|\Sigma_K| = n_K/2$. For each $v \in \Sigma_K$ fix one of the two embeddings $\sigma_v \colon K \to \mathbb{C}$ in the class v.

A *hermitian lattice* $\bar{L}$ over $\mathcal{O}_K$ consists of a finite-dimensional K-vector space V, a finitely generated $\mathcal{O}_K$-submodule $L \subset V$ that spans V, and for each $v \in \Sigma_K$ a hermitian norm $\|\cdot\|_v$ on $V_v = V \otimes_{K,\sigma_v} \mathbb{C}$. We write L for $\bar{L}$ when the norms are fixed, and $\operatorname{rk} L = \dim_K V$. A nonzero finitely generated $\mathcal{O}_K$-submodule $N \subset L$ is a hermitian lattice in KN with the restricted norms. It is *saturated* when $N = L \cap KN$. For a saturated proper submodule $N \subset L$ the quotient L/N is a hermitian lattice in V/KN, with the quotient norms $\|x + KN\|_v = \min\{\|y\|_v : y \in x + (KN)_v\}$.

The real vector space $V_{\mathbb{R}} = \bigoplus_{v \in \Sigma_K} V_v$, with the euclidean structure whose square norm is $\sum_v \|x_v\|_v^2$, has dimension $n_K \operatorname{rk} L$, and the map $\rho\colon V \to V_{\mathbb{R}}$, $x \mapsto (\sigma_v(x))_v$, sends L onto a lattice of full rank. Put $\deg_{\mathbb{Z}} L = -\log \operatorname{covol} \rho(L)$ and define the *slope*

$$\mu(L) = \frac{2 \deg_{\mathbb{Z}} L}{n_K \operatorname{rk} L} - \log 2 + \frac{\log |d_K|}{n_K}. \tag{13.1}$$

Since $\operatorname{covol} \rho(\mathcal{O}_K) = 2^{-n_K/2} |d_K|^{1/2}$ for the norms $|\sigma_v(\cdot) vert$, the slope of $\mathcal{O}_K$ is 0. Dividing every norm by $c > 0$ raises μ by $2 \log c$. So μ is twice the normalized Arakelov slope $\widehat{\mu}$ of Bost [15], and it is measured in units of $\log \|\cdot\|^2$. All slopes in Part II are slopes in the sense of (13.1).

The *maximal slope* $\mu_{\max}(L)$ is the supremum of $\mu(N)$ over the nonzero submodules $N \subset L$, and the *minimal slope* $\mu_{\min}(L)$ is the infimum of $\mu(L/N)$ over the saturated proper submodules $N \subset L$, with $N = 0$ allowed. For a K-linear map $\varphi\colon V \to V'$ between spaces with hermitian norms at the places of Σ_K, the *height* of $\varphi \neq 0$ is

$$\operatorname{ht}(\varphi) = \frac{2}{n_K} \sum_{v \in \Sigma_K} \log \|\varphi_v\|_{\mathrm{op}}^2, \tag{13.2}$$

with $\varphi_v = \varphi \otimes_{\sigma_v} \mathbb{C}$. It is twice the height $\frac{1}{n_K} \sum_\sigma \log \|\varphi_\sigma\|$ over all embeddings σ. The *dual* of $\bar{L}$ is the lattice $L^\vee = \{f \in V^* : f(L) \subset \mathcal{O}_K\}$ in $V^* = \operatorname{Hom}_K(V, K)$, with the dual norms $\|f\|_v = \max_{x \neq 0} |f_v(x)| / \|x\|_v$.

We use the following classical facts.

Proposition 13.1 (Harder–Narasimhan filtration)**.** *Let $\bar{L}$ be a hermitian lattice over $\mathcal{O}_K$ of rank $\ell \geq 1$. There are real numbers $x_1 \geq x_2 \geq \cdots \geq x_\ell$, the slopes of $\bar{L}$, and for each real θ a saturated submodule $L_{\geq \theta} \subset L$, with the following properties.*

(HN1) For every θ the number of j with $x_j \geq \theta$ equals $\operatorname{rk} L_{\geq \theta}$.

(HN2) $x_1 + \cdots + x_\ell = \ell\, \mu(L)$.

(HN3) $\mu_{\min}(L) \leq x_j \leq \mu_{\max}(L)$ *for every j.*

(HN4) If $N \subset L$ is a nonzero submodule with $\mu_{\min}(N) \geq \theta$, then $N \subset L_{\geq \theta}$.

(HN5) If $L_{\geq \theta} \neq 0$, then $\mu_{\min}(L_{\geq \theta}) \geq \theta$.

Moreover $\mu(L^\vee) = -\mu(L)$ and $\mu_{\min}(L^\vee) = -\mu_{\max}(L)$.

These statements are the Harder–Narasimhan theory of Stuhler [71] and Grayson [29] over $\mathbb{Z}$, extended to rings of integers in Bost's Bourbaki talk [15] and by Gaudron [28], together with the behaviour of the Arakelov degree under duality [16, 28]. If $0 = L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_s = L$ is the Harder–Narasimhan filtration, with semistable quotients of slopes $\nu_1 > \cdots > \nu_s$, then $x_1, \ldots, x_\ell$ is the list of the ν_i, each repeated $\operatorname{rk}(L_i / L_{i-1})$ times, and $L_{\geq \theta} = L_i$ for the largest i with

$\nu_i \geq \theta$, or 0 when $\nu_1 < \theta$. Statement (HN2) is the additivity of $\deg_{\mathbb{Z}}$ in exact sequences with quotient norms. The slope (13.1) is an affine function of $\widehat{\mu}$ with positive coefficient, and every statement of Proposition 13.1 is invariant under such a change.

The slope inequality we need bounds the minimal slope of an image. We give its proof, because the normalizations (13.1) and (13.2) enter the error terms of Part II.

Lemma 13.2. *Let $\bar{A}$ be a hermitian lattice over $\mathcal{O}_K$ in V of rank a, let V' have hermitian norms at the places of Σ_K, and let $\psi\colon V \to V'$ be an injective K-linear map. Give $\psi(A)$ the norms of V' restricted to $\psi(V)$. Then*

$$\mu(\psi(A)) \geq \mu(A) - \mathrm{ht}(\psi).$$

Proof. The map $\psi_{\mathbb{R}} = \bigoplus_v \psi_v$ sends $\rho(A)$ onto $\rho(\psi(A)) \subset \psi(V)_{\mathbb{R}}$. In orthonormal bases of V_v and $\psi(V)_v$ the complex linear map ψ_v has a matrix M_v of size a, and as a real linear map of $\mathbb{C}^a = \mathbb{R}^{2a}$ it has determinant $|\det M_v|^2 \leq \|\psi_v\|_{\mathrm{op}}^{2a}$. Hence

$$\operatorname{covol}\rho(\psi(A)) = \prod_v |\det M_v|^2 \operatorname{covol}\rho(A) \leq \prod_v \|\psi_v\|_{\mathrm{op}}^{2a} \operatorname{covol}\rho(A),$$

so $\deg_{\mathbb{Z}}\psi(A) \geq \deg_{\mathbb{Z}} A - a\sum_v \log\|\psi_v\|_{\mathrm{op}}^2$. Dividing by $n_K a/2$ gives the lemma, because the two lattices have the same rank. □

Proposition 13.3 (slope inequality)**.** *Let $\bar{A}$ be a hermitian lattice over $\mathcal{O}_K$ in V, let V' have hermitian norms at the places of Σ_K, and let $\varphi\colon V \to V'$ be a K-linear map with $\varphi(A) \neq 0$. Give $\varphi(A)$ the norms of V' restricted to $K\varphi(A)$. Then*

$$\mu_{\min}(\varphi(A)) \geq \mu_{\min}(A) - \mathrm{ht}(\varphi).$$

Proof. Let $M \subset \varphi(A)$ be a saturated proper submodule and $Q = \varphi(A)/M$ with the quotient norms. Let $\pi\colon K\varphi(A) \to K\varphi(A)/KM$ be the projection and $\psi = \pi \circ \varphi$ on V. Then $A_0 = A \cap \ker\psi$ is a saturated proper submodule of A, and ψ induces an injective map $\bar{\psi}\colon V/KA_0 \to K\varphi(A)/KM$ that sends A/A_0 onto Q. For a class $\xi \in (V/KA_0)_v$ and any representative y of ξ,

$$\|\bar{\psi}_v\xi\|_v = \|\pi_v\varphi_v y\|_v \leq \|\varphi_v y\|_v \leq \|\varphi_v\|_{\mathrm{op}}\|y\|_v,$$

and the minimum over y gives $\|\bar{\psi}_v\|_{\mathrm{op}} \leq \|\varphi_v\|_{\mathrm{op}}$, so $\mathrm{ht}(\bar{\psi}) \leq \mathrm{ht}(\varphi)$. Lemma 13.2 applied to A/A_0 and $\bar{\psi}$ gives $\mu(Q) \geq \mu(A/A_0) - \mathrm{ht}(\varphi)$. A quotient of A by a saturated submodule followed by a quotient by a saturated submodule is a quotient of A by a saturated submodule, with the same quotient norms, so $\mu(A/A_0) \geq \mu_{\min}(A)$. Taking the infimum over M proves the proposition. □

13.2 The frame theorem

For a complex matrix X write X^* for its conjugate transpose and $\|X\|_{\mathrm{HS}}^2 = \mathrm{tr}(X^*X)$ for its squared Hilbert–Schmidt norm. For a K-linear map φ between spaces with fixed bases and a place $v \in \Sigma_K$, write $[\varphi]_v$ for the matrix of φ with σ_v applied to its entries.

Lemma 13.4 (frame lemma)**.** *Let I be a finite set, let $\widetilde{D} \geq 0$ and $d_i \geq 0$ for $i \in I$ be integers with $\Delta = \sum_i d_i > 0$, let A_i be complex $\widetilde{D} \times d_i$ matrices and $\gamma_i > 0$, and put $Q = \sum_i \gamma_i A_i A_i^*$. Let e, η, ε be real numbers with $0 \leq e \leq \frac{1}{4}$ and $\eta \geq 0$ such that*

$$\sum_i \|\gamma_i A_i^* A_i - 1\|_{\mathrm{HS}}^2 \leq e\Delta, \qquad \mathrm{tr}(Q^2) \leq (1+\eta)\frac{(\mathrm{tr}\,Q)^2}{\widetilde{D}}.$$

Let $U_i \subset \mathbb{C}^{d_i}$ be subspaces with $\sum_i (d_i - \dim U_i) \le \varepsilon\Delta$, and let $\mathcal{W}$ be the orthogonal complement of $\sum_i A_i(U_i)$ in $\mathbb{C}^{\widetilde{D}}$. Then $\dim \mathcal{W} \le (\eta + 4\varepsilon + 4\sqrt{e})\,\widetilde{D}$.

Proof. Put $E_i = \gamma_i A_i^* A_i - 1$, $k_i = d_i - \dim U_i$, $w = \dim \mathcal{W}$ and $t = \operatorname{tr} Q$. The hypothesis on $\sum k_i$ and $\Delta > 0$ give $\varepsilon \ge 0$. If $\widetilde{D} = 0$ then $w = 0$, so assume $\widetilde{D} \ge 1$.

For orthonormal vectors $f_1, \dots, f_m$ in $\mathbb{C}^{d_i}$ we have $\langle f_l, E_i f_l\rangle = \gamma_i \|A_i f_l\|^2 - 1$, so the Cauchy–Schwarz inequality gives

$$\Big|\gamma_i \sum_{l=1}^{m} \|A_i f_l\|^2 - m\Big| \le \sum_l \|E_i f_l\| \le \sqrt{m}\Big(\sum_l \|E_i f_l\|^2\Big)^{1/2} \le \sqrt{m}\,\|E_i\|_{\mathrm{HS}}. \tag{13.3}$$

In the same way $|\sum_l \langle g_l, T g_l\rangle| \le \sqrt{m}\,\|T\|_{\mathrm{HS}}$ for every matrix T of size $\widetilde{D}$ and orthonormal $g_1, \dots, g_m$ in $\mathbb{C}^{\widetilde{D}}$.

By cyclicity of the trace $t = \sum_i \gamma_i \|A_i\|_{\mathrm{HS}}^2$. Inequality (13.3) for an orthonormal basis of $\mathbb{C}^{d_i}$, summed over i, and the Cauchy–Schwarz inequality give

$$t \ge \sum_i \big(d_i - \sqrt{d_i}\,\|E_i\|_{\mathrm{HS}}\big) \ge \Delta - \sqrt{\Delta}\Big(\sum_i \|E_i\|_{\mathrm{HS}}^2\Big)^{1/2} \ge \Delta\,(1 - \sqrt{e}).$$

Since $\sqrt{e} \le \frac{1}{2}$, we get $\Delta \le 2t$, and in particular $t > 0$.

Put $\lambda = t/\widetilde{D}$. Since Q is self-adjoint,

$$\|Q - \lambda\|_{\mathrm{HS}}^2 = \operatorname{tr}(Q^2) - 2\lambda t + \lambda^2 \widetilde{D} = \operatorname{tr}(Q^2) - \frac{t^2}{\widetilde{D}} \le \frac{\eta\, t^2}{\widetilde{D}}.$$

Let $g_1, \dots, g_w$ be an orthonormal basis of $\mathcal{W}$ and $T_{\mathcal{W}} = \sum_l \langle g_l, Q g_l\rangle$. The estimate after (13.3) with $T = Q - \lambda$ gives

$$T_{\mathcal{W}} \ge \lambda w - \sqrt{w}\,\|Q - \lambda\|_{\mathrm{HS}} \ge \frac{t\,w}{\widetilde{D}} - \sqrt{\frac{w\,\eta}{\widetilde{D}}}\; t.$$

For $g \in \mathcal{W}$ and $y \in U_i$ we have $\langle A_i^* g, y\rangle = \langle g, A_i y\rangle = 0$, so $A_i^* g$ lies in $U_i^{\perp}$. Let $f_1, \dots, f_{k_i}$ be an orthonormal basis of $U_i^{\perp}$. Bessel's inequality for the orthonormal vectors g_l gives

$$\sum_l \|A_i^* g_l\|^2 = \sum_m \sum_l |\langle g_l, A_i f_m\rangle|^2 \le \sum_m \|A_i f_m\|^2,$$

and (13.3) gives $\gamma_i \sum_m \|A_i f_m\|^2 \le k_i + \sqrt{k_i}\,\|E_i\|_{\mathrm{HS}}$. Since $\langle g, Qg\rangle = \sum_i \gamma_i \|A_i^* g\|^2$, summing over i and applying the Cauchy–Schwarz inequality,

$$T_{\mathcal{W}} \le \sum_i k_i + \Big(\sum_i k_i\Big)^{1/2}\Big(\sum_i \|E_i\|_{\mathrm{HS}}^2\Big)^{1/2} \le \big(\varepsilon + \sqrt{\varepsilon e}\big)\Delta \le 2t\big(\varepsilon + \sqrt{\varepsilon e}\big).$$

Comparing the two bounds for $T_{\mathcal{W}}$ and multiplying by $\widetilde{D}/t$ gives $w - \sqrt{w\eta\widetilde{D}} \le 2\widetilde{D}(\varepsilon + \sqrt{\varepsilon e})$. The inequality $\sqrt{w\eta\widetilde{D}} \le \frac{1}{2}(w + \eta\widetilde{D})$ between geometric and arithmetic means turns this into $w \le \big(\eta + 4\varepsilon + 4\sqrt{\varepsilon e}\big)\widetilde{D}$. If $\varepsilon \le 1$ then $\sqrt{\varepsilon e} \le \sqrt{e}$, which proves the lemma. If $\varepsilon > 1$ the bound of the lemma exceeds $\widetilde{D} \ge w$. □

Lemma 13.5 (rank at one place)**.** *Let I be a finite set. For $i \in I$ let V_i be a K-vector space of dimension d_i with a basis, let W be a K-vector space of dimension D with a basis, and let $B_i \colon V_i \to W$ be K-linear maps. Fix $v \in \Sigma_K$, invertible complex matrices Y_i of size d_i and Y' of size D, an integer $D_1 \geq 0$, and put $\widetilde{D} = D + D_1$. Let A_i be complex $\widetilde{D} \times d_i$ matrices whose first D rows form the matrix $Y'^{-1}[B_i]_v Y_i$, and let $\gamma_i > 0$, e, η and δ be real numbers with $0 \leq e \leq \frac{1}{4}$, $\eta \geq 0$, $0 \leq \delta < 1$ such that $\Delta = \sum_i d_i > 0$, $(1-\delta)\widetilde{D} \leq D$, and, with $Q = \sum_i \gamma_i A_i A_i^*$,*

$$\sum_i \|\gamma_i A_i^* A_i - 1\|_{\mathrm{HS}}^2 \leq e\Delta, \qquad \operatorname{tr}(Q^2) \leq (1+\eta)\frac{(\operatorname{tr} Q)^2}{\widetilde{D}}.$$

Let ε be real and let $N_i' \subset V_i$ be finitely generated $\mathcal{O}_K$-submodules with $\sum_i (d_i - \operatorname{rk} N_i') \leq \varepsilon\Delta$, and put $M = \sum_i B_i(N_i') \subset W$. Then

$$D - \operatorname{rk} M \leq \frac{\eta + 4\varepsilon + 4\sqrt{e}}{1-\delta}\, D.$$

Proof. For $x \in V_i$ let $c_i(x) = Y_i^{-1}[x]_v \in \mathbb{C}^{d_i}$, and for $y \in W$ let $c'(y) = Y'^{-1}[y]_v \in \mathbb{C}^D$, where $[x]_v$ is the coordinate vector of x with σ_v applied. Then $c'(B_i x) = Y'^{-1}[B_i]_v Y_i\, c_i(x)$. Let U_i be the complex span of $c_i(N_i')$ and T the complex span of $c'(M)$. A K-subspace U of a K-vector space and its image under σ_v satisfy $\dim_{\mathbb{C}}(U \otimes_{K,\sigma_v} \mathbb{C}) = \dim_K U$, so $\dim U_i = \operatorname{rk} N_i'$ and $\dim T = \operatorname{rk} M$. Let $P \colon \mathbb{C}^{\widetilde{D}} \to \mathbb{C}^D$ be the projection to the first D coordinates. Then $PA_i(U_i)$ is spanned by the vectors $c'(B_i x)$ with $x \in N_i'$, which lie in T. Put $S = \sum_i A_i(U_i)$. Then $S \subset P^{-1}(T)$, which has dimension $\operatorname{rk} M + D_1$, so $\dim S^\perp \geq \widetilde{D} - \operatorname{rk} M - D_1 = D - \operatorname{rk} M$. Lemma 13.4 bounds $\dim S^\perp$ by $(\eta + 4\varepsilon + 4\sqrt{e})\widetilde{D}$, and $\widetilde{D} \leq (1-\delta)^{-1} D$. □

Lemma 13.6 (reverse Markov inequality)**.** *Let $y_1, \ldots, y_m$, s, u, τ be real numbers with $u \geq 0$, $\tau \geq 0$, $\sum_j y_j \geq ms$ and $y_j \leq s + u$ for every j. Let F be the number of j with $y_j < s - \tau$. Then $F(u+\tau) \leq mu$, and $F = 0$ when $u = 0$.*

Proof. Bounding the F small terms by $s - \tau$ and the others by $s + u$ gives $ms \leq \sum_j y_j \leq F(s - \tau) + (m - F)(s + u) = m(s+u) - F(u+\tau)$. If $u = 0$ and $F \geq 1$, then every $y_j \leq s$ and one $y_j < s$, so $\sum_j y_j < ms$. □

Lemma 13.7 (layer cake)**.** *Let $z_1, \ldots, z_m$ be real numbers with $m \geq 1$, and let s and $H \geq 0$ be real numbers with $z_k \geq s - H$ for every k. Let w be an integrable function on $[0, H]$ such that the number of k with $z_k < s - \tau$ is at most $w(\tau)\, m$ for every $\tau \in [0, H]$. Then*

$$\frac{1}{m}\sum_k z_k \geq s - \int_0^H \min\big(1, w(\tau)\big)\, d\tau.$$

Proof. If $s - z_k \geq 0$ then $s - z_k \leq H$ and $s - z_k = \int_0^H \mathbf{1}[z_k < s - \tau]\, d\tau$. If $s - z_k < 0$ the integral is nonnegative. Summing $s - z_k \leq \int_0^H \mathbf{1}[z_k < s - \tau]\, d\tau$ over k and bounding the number of k with $z_k < s - \tau$ by $m \min(1, w(\tau))$ proves the lemma. □

For $u > 0$ and $H \geq 0$ put $\Lambda(u, H) = u \log(1 + H/u)$, and put $\Lambda(0, H) = 0$. Then $\Lambda(u, H) = \int_0^H u/(u+\tau)\, d\tau$. The function Λ is nondecreasing in each variable, because $\partial_u \Lambda = \log(1 + H/u) - H/(u+H) \geq 0$.

Theorem 13.8 (frame theorem)**.** *Let K be a totally complex number field, let I be a nonempty finite set, and for $i \in I$ let $\bar{N}_i$ be a hermitian lattice over $\mathcal{O}_K$ of rank d_i in a K-vector space V_i with a basis. Let $\bar{C}$ be a hermitian lattice over $\mathcal{O}_K$ of rank D in a K-vector space W with a basis, and put $\Delta = \sum_i d_i$, with $\Delta > 0$. Let $B_i \colon V_i \to W$ be K-linear maps with $B_i(N_i) \subset C$, and let h_i be real numbers with $\mathrm{ht}(B_i) \le h_i$ whenever $B_i \neq 0$, where* ht *is computed with the norms of $\bar{N}_i$ and $\bar{C}$. Fix $v \in \Sigma_K$, invertible complex matrices Y_i of size d_i and Y' of size D, an integer $D_1 \ge 0$, and complex $(D + D_1) \times d_i$ matrices A_i whose first D rows form $Y'^{-1}[B_i]_v Y_i$. Let $\gamma_i > 0$ and $Q = \sum_i \gamma_i A_i A_i^*$. Let s, u, H, e, η, δ be real numbers with $u \ge 0$, $H \ge 0$, $0 \le e \le \frac{1}{4}$, $\eta \ge 0$ and $0 \le \delta < 1$ such that*

(i) $s\Delta \le \sum_i d_i\big(\mu(N_i) - h_i\big)$,

(ii) $\mu_{\max}(N_i) - h_i \le s + u$ *for every* i,

(iii) $s - \mu_{\min}(C) \le H$,

(iv) $(1 - \delta)(D + D_1) \le D$,

(v) $\sum_i \|\gamma_i A_i^* A_i - 1\|_{\mathrm{HS}}^2 \le e\,\Delta$,

(vi) $\mathrm{tr}(Q^2) \le (1 + \eta)(\mathrm{tr}\, Q)^2/(D + D_1)$.

Then

$$\mu(C) \ge s - \frac{4\Lambda(u, H) + (\eta + 4\sqrt{e})\, H}{1 - \delta}.$$

Proof. The target is nonzero. If $D = 0$ then $D_1 = 0$ by (iv), every A_i is zero, and (v) reads $\Delta \le e\Delta$, which contradicts $\Delta > 0$ and $e \le \frac{1}{4}$. So $D \ge 1$.

The pieces of the sources. For $\tau \ge 0$ and $i \in I$ let $N_i^\tau = (N_i)_{\ge s - \tau + h_i}$ be the Harder–Narasimhan step of Proposition 13.1. Let $x_{i1}, \dots, x_{id_i}$ be the slopes of N_i and $y_{ij} = x_{ij} - h_i$. By (HN2) and (i), $\sum_{i,j} y_{ij} = \sum_i d_i(\mu(N_i) - h_i) \ge \Delta s$. By (HN3) and (ii), $y_{ij} \le s + u$. Lemma 13.6 applied to the Δ numbers y_{ij} bounds the number of pairs with $y_{ij} < s - \tau$ by $\varepsilon(\tau)\Delta$, where $\varepsilon(\tau) = u/(u + \tau)$ when $u > 0$ and $\varepsilon(\tau) = 0$ when $u = 0$. By (HN1) the number of j with $x_{ij} < s - \tau + h_i$ is $d_i - \mathrm{rk}\, N_i^\tau$. So

$$\sum_i \big(d_i - \mathrm{rk}\, N_i^\tau\big) \le \varepsilon(\tau)\, \Delta.$$

The images lie in one step of the target. Put $C_\tau = C_{\ge s - \tau}$. Fix i with $B_i(N_i^\tau) \neq 0$. Then $N_i^\tau \neq 0$, so (HN5) gives $\mu_{\min}(N_i^\tau) \ge s - \tau + h_i$, and $B_i \neq 0$, so $\mathrm{ht}(B_i) \le h_i$. Proposition 13.3 gives

$$\mu_{\min}\big(B_i(N_i^\tau)\big) \ge \mu_{\min}(N_i^\tau) - \mathrm{ht}(B_i) \ge s - \tau.$$

Since $B_i(N_i^\tau) \subset C$, (HN4) gives $B_i(N_i^\tau) \subset C_\tau$. Hence $M_\tau = \sum_i B_i(N_i^\tau)$ lies in C_τ, and $\mathrm{rk}\, C_\tau \ge \mathrm{rk}\, M_\tau$.

The step of the target has small corank. Lemma 13.5 with $N_i' = N_i^\tau$ and $\varepsilon = \varepsilon(\tau)$ gives

$$D - \mathrm{rk}\, C_\tau \le D - \mathrm{rk}\, M_\tau \le w(\tau)\, D, \qquad w(\tau) = \frac{\eta + 4\varepsilon(\tau) + 4\sqrt{e}}{1 - \delta}.$$

Integration. Let $z_1, \dots, z_D$ be the slopes of C. By (HN1) the number of k with $z_k < s - \tau$ is $D - \mathrm{rk}\, C_\tau \le w(\tau)D$. By (HN3) and (iii), $z_k \ge \mu_{\min}(C) \ge s - H$. By (HN2), $\mu(C)$ is the mean of the z_k. Lemma 13.7 gives

$$\mu(C) \ge s - \int_0^H w(\tau)\, d\tau = s - \frac{4\Lambda(u, H) + (\eta + 4\sqrt{e})H}{1 - \delta}. \qquad \square$$

Remark 13.9. The proof bounds the corank of every step $C_{\geq s-\tau}$ with $0 \leq \tau \leq H$, so it controls the lower half of the Harder–Narasimhan polygon of C, and the mean slope is one consequence. The hermitian structure at v in which conditions (v) and (vi) are stated is arbitrary, because Y_i and Y' are arbitrary invertible matrices. The norms of the lattices at all places of Σ_K enter through the slopes and the heights in Proposition 13.3, and the place v enters through the dimension count of Lemma 13.5. For a single map, $I = \{i\}$, the theorem is weaker than Proposition 13.3, and the gain lies in the passage from $\mu_{\max}(C)$ to $\mu(C)$ for a redundant family. The term $\Lambda(u, H)$ is the integral of the share $u/(u+\tau)$ of Lemma 13.6 over the levels $s - \tau$ at which C can have slopes.

13.3 Gram families and the frame conditions

In the application the matrices A_i come from maps P_i defined on larger spaces, compressed by isometries, and the hypotheses are Hilbert–Schmidt bounds for the Gram operators of the P_i. This subsection derives conditions (v) and (vi) of Theorem 13.8 from such bounds.

Lemma 13.10. *Let X be a complex $a \times b$ matrix, and let V and V' be complex $a \times c$ and $b \times c'$ matrices with $V^*V = 1$ and $V'^*V' = 1$. Then $\|V^*XV'\|_{\mathrm{HS}} \leq \|X\|_{\mathrm{HS}}$. If moreover $a = b$, $V = V'$ and $\|X - 1\|_{\mathrm{HS}}^2 \leq \epsilon a$ with $\epsilon \geq 0$, then*

$$\operatorname{tr} X - \operatorname{tr}(V^*XV) = (a - c) + \operatorname{tr}\big((X-1)(1-VV^*)\big),$$
$$\big|\operatorname{tr}\big((X-1)(1-VV^*)\big)\big| \leq \sqrt{(a-c)\,\epsilon\, a}.$$

Proof. The matrix VV^* is an orthogonal projection, so $\|V^*Y\|_{\mathrm{HS}}^2 = \operatorname{tr}(Y^*VV^*Y) \leq \operatorname{tr}(Y^*Y)$ for every Y with a rows, and the same argument on the right gives the first statement. For the second, $\operatorname{tr}(V^*XV) = \operatorname{tr}(XVV^*)$, so $\operatorname{tr} X - \operatorname{tr}(V^*XV) = \operatorname{tr}(X(1-VV^*)) = \operatorname{tr}(1-VV^*) + \operatorname{tr}((X-1)(1-VV^*))$, and $\operatorname{tr}(1-VV^*) = a - \operatorname{tr}(V^*V) = a - c$. The projection $1 - VV^*$ has $\|1 - VV^*\|_{\mathrm{HS}}^2 = a - c$, and the Cauchy–Schwarz inequality for the Hilbert–Schmidt inner product gives the bound. □

Proposition 13.11 (from Gram bounds to the frame conditions)**.** *Let I be a nonempty finite set, let $\widetilde{D} \geq 0$ be an integer, and for $i \in I$ let $d_i^S \geq d_i \geq 0$ be integers, P_i a complex $\widetilde{D} \times d_i^S$ matrix, V_i a complex $d_i^S \times d_i$ matrix with $V_i^*V_i = 1$, and $\gamma_i > 0$. Put*

$$K_{ij} = \sqrt{\gamma_i\gamma_j}\, P_i^*P_j, \qquad \Delta^S = \sum_i d_i^S, \qquad \Delta = \sum_i d_i, \qquad A_i = P_iV_i, \qquad Q = \sum_i \gamma_i A_iA_i^*.$$

Let $D^* > 0$, σ, η_0, η_1, e_0 and δ' be real numbers with

$$0 < \sigma \leq \tfrac{1}{20}, \quad -1 \leq \eta_0 \leq \tfrac{\sigma}{2}, \quad \eta_1 \leq \tfrac{\sigma}{2}, \quad 0 \leq e_0, \quad \sqrt{e_0} \leq \tfrac{\sigma}{2}, \quad 0 \leq \delta' \leq \tfrac{1}{4},$$

and suppose that

(G1) $\|K_{ij}\|_{\mathrm{HS}}^2 \leq (1+\eta_0)\, d_i^S d_j^S / D^*$ *for $i \neq j$,*

(G2) $\|K_{ii} - 1\|_{\mathrm{HS}}^2 \leq e_0\, d_i^S$ *for every i,*

(G3) $\widetilde{D} \leq (1+\eta_1)D^*$,

(G4) $D^* \leq \sigma\Delta^S$,

(G5) $\sum_i (d_i^S - d_i) \le \delta' \Delta^S$.

Then $\Delta \ge (1-\delta')\Delta^S > 0$, $\widetilde{D} \ge 1$, *and*

$$\sum_i \|\gamma_i A_i^* A_i - 1\|_{\mathrm{HS}}^2 \le 2e_0\Delta, \qquad \operatorname{tr}(Q^2) \le \big(1 + 5\sigma + 3\sqrt{\delta'}\big)\frac{(\operatorname{tr} Q)^2}{\widetilde{D}}.$$

Proof. Put $a = \sqrt{e_0}$ and $t = \sqrt{\delta'}$, so that $0 \le a \le \frac{1}{40}$ and $0 \le t \le \frac{1}{2}$. By (G4), $\Delta^S \ge D^*/\sigma > 0$, and (G5) gives $\Delta \ge (1-\delta')\Delta^S \ge \frac{3}{4}\Delta^S$.

Condition (v). We have $\gamma_i A_i^* A_i = V_i^* K_{ii} V_i$ and $V_i^* K_{ii} V_i - 1 = V_i^*(K_{ii} - 1)V_i$. Lemma 13.10 and (G2) give $\sum_i \|\gamma_i A_i^* A_i - 1\|_{\mathrm{HS}}^2 \le e_0 \Delta^S \le 2e_0\Delta$.

Traces of the uncompressed family. Put $Q^S = \sum_i \gamma_i P_i P_i^*$. Then $\operatorname{tr} Q^S = \sum_i \operatorname{tr} K_{ii}$ and

$$\operatorname{tr}\big((Q^S)^2\big) = \sum_{i,j} \gamma_i \gamma_j \operatorname{tr}\big(P_i P_i^* P_j P_j^*\big) = \sum_{i,j} \|K_{ij}\|_{\mathrm{HS}}^2.$$

The estimate (13.3) for an orthonormal basis gives $\operatorname{tr} K_{ii} \ge d_i^S - \sqrt{d_i^S}\,\|K_{ii} - 1\|_{\mathrm{HS}} \ge (1-a)d_i^S$, so $\operatorname{tr} Q^S \ge (1-a)\Delta^S$. The triangle inequality gives $\|K_{ii}\|_{\mathrm{HS}} \le \sqrt{d_i^S} + \sqrt{e_0 d_i^S} = (1+a)\sqrt{d_i^S}$. With (G1) and $1 + \eta_0 \ge 0$,

$$\operatorname{tr}\big((Q^S)^2\big) \le (1+\eta_0)\frac{(\Delta^S)^2}{D^*} + (1+a)^2\Delta^S \le \frac{(\Delta^S)^2}{D^*}\Big((1+\eta_0) + (1+a)^2\sigma\Big),$$

where the last step uses $\Delta^S \le \sigma(\Delta^S)^2/D^*$, which is (G4).

Compression. The Gram operators of the compressed family are $\sqrt{\gamma_i\gamma_j}A_i^* A_j = V_i^* K_{ij} V_j$, so Lemma 13.10 gives $\operatorname{tr}(Q^2) = \sum_{i,j}\|V_i^* K_{ij} V_j\|_{\mathrm{HS}}^2 \le \operatorname{tr}((Q^S)^2)$. The second part of Lemma 13.10 with $X = K_{ii}$ and $\epsilon = e_0$, the Cauchy–Schwarz inequality and (G5) give

$$\begin{aligned}\operatorname{tr} Q^S - \operatorname{tr} Q &\le \sum_i \Big((d_i^S - d_i) + \sqrt{d_i^S - d_i}\, a\sqrt{d_i^S}\Big) \\ &\le \delta'\Delta^S + a\Big(\sum_i (d_i^S - d_i)\Big)^{1/2}(\Delta^S)^{1/2} \le (t^2 + at)\Delta^S.\end{aligned}$$

So $\operatorname{tr} Q \ge (1 - a - t^2 - ta)\Delta^S$, and the coefficient is positive because $a \le \frac{1}{40}$ and $t \le \frac{1}{2}$. In particular $\operatorname{tr} Q > 0$, so $\widetilde{D} \ge 1$.

Condition (vi). Since $\widetilde{D} \ge 1$, (G3) gives $1 + \eta_1 > 0$. By (G3) and the two bounds above,

$$\widetilde{D}\,\operatorname{tr}(Q^2) \le (1+\eta_1)\big((1+\eta_0) + (1+a)^2\sigma\big)(\Delta^S)^2.$$

The right side is nondecreasing in η_0, η_1 and a on the given ranges, so it is at most $(1+\frac{\sigma}{2})^2\big(1 + (1+\frac{\sigma}{2})\sigma\big)(\Delta^S)^2$. For $0 < \sigma \le \frac{1}{4}$,

$$(1+5\sigma)\big(1-\tfrac{\sigma}{2}\big)^2 - \big(1+\tfrac{\sigma}{2}\big)^2\big(1+\sigma+\tfrac{\sigma^2}{2}\big) = \sigma\Big(2 - \tfrac{13}{2}\sigma + \tfrac{1}{2}\sigma^2 - \tfrac{1}{8}\sigma^3\Big) \ge \sigma\Big(2 - \tfrac{13}{8} - \tfrac{1}{512}\Big) > 0,$$

and $(1-\frac{\sigma}{2})^2 \le (1-a)^2$. Hence $\widetilde{D}\operatorname{tr}(Q^2) \le (1+5\sigma)(1-a)^2(\Delta^S)^2$. Put $P = 1 + 5\sigma$, $B = 1 - a$ and $c = t(t+a)$. The function $3B - 2(P+3t)(t+a)$ decreases in σ, a and t, and at $\sigma = \frac{1}{20}$, $a = \frac{1}{40}$, $t = \frac{1}{2}$ it equals $\frac{3}{80}$. So $2(P+3t)(t+a) \le 3B$, and

$$(P+3t)(B-c)^2 \ge (P+3t)(B^2 - 2Bc) = PB^2 + tB\big(3B - 2(P+3t)(t+a)\big) \ge PB^2.$$

Since $B - c = 1 - a - t^2 - ta > 0$ and $(B-c)\Delta^S \le \operatorname{tr} Q$,

$$\widetilde{D}\ \operatorname{tr}(Q^2) \le PB^2(\Delta^S)^2 \le (P+3t)(B-c)^2(\Delta^S)^2 \le \big(1 + 5\sigma + 3\sqrt{\delta'}\big)(\operatorname{tr} Q)^2. \qquad \square$$

The ranks d_i in the application are the ranks of the parts of the spaces C_u of Section 13.4 that are ramified at the auxiliary prime T_u, and the following lemma gives (G5).

Lemma 13.12. *Let $f \ge 1$ be an integer and let $T \ge 2$ and $d^S \ge 0$ be real numbers. If $d = d^S\big(1 - \frac{1}{T-1}\big)^f$, then $d^S - d \le \frac{2f}{T}\, d^S$.*

Proof. Bernoulli's inequality $(1-y)^f \ge 1 - fy$ for $0 \le y \le 1$, with $y = 1/(T-1)$, gives $d^S - d \le \frac{f}{T-1} d^S \le \frac{2f}{T} d^S$. $\square$

13.4 The period maps and hypothesis (3.3)

We now state the setting of hypothesis (3.3) of Proposition 3.1. For each $u_0 \in A_0$ the construction of Part II gives the following objects. The field K is a totally complex Galois number field that contains the field E. The index set W is a finite nonempty set of words u, and each word has an auxiliary prime T_u. There is one hermitian lattice R over $\mathcal{O}_K$ of rank $D_{\text{rows}} \ge 1$ in a K-vector space Y, the lattice of automorphic forms, and $\mu^C = \mu(R^\vee)$. For each u there are a hermitian lattice N_u of rank $d_u \ge 1$ in a K-vector space V_u, the period map $B_u \colon V_u \to Y^*$, and a real number h_u. The numbers $\mu_u^N = \mu(N_u)$ and the weights d_u are those of (3.3).

The archimedean structure of the period maps at one complex place is the following.

Definition 13.13. Let $v \in \Sigma_K$. A *Bessel realization* of the family $(B_u)_{u\in W}$ at v consists of

1. a complex Hilbert space $\mathcal{H}$ of finite dimension D_{all} and a conjugate-linear isometric embedding $\mathbf{s} \colon Y_v \to \mathcal{H}$, where $Y_v = Y \otimes_{K,\sigma_v} \mathbb{C}$ has the norm of R at v;
2. for each $u \in W$ a complex Hilbert space $\mathcal{C}_u$ of dimension d_u^S, a linear map $\mathcal{P}_u \colon \mathcal{C}_u \to \mathcal{H}$, a number $c_u > 0$, an inner product on $V_{u,v} = V_u \otimes_{K,\sigma_v} \mathbb{C}$ and a linear isometric embedding $\iota_u \colon V_{u,v} \to \mathcal{C}_u$;
3. a unitary operator $\mathcal{V}$ of Y_v, and for each u a unitary operator $\mathcal{U}_u$ of $V_{u,v}$ and a complex number $a_u \ne 0$,

such that for every u, every $\varphi \in V_{u,v}$ and every $F \in Y_v$

$$\big(B_{u,v}\varphi\big)(F) = a_u \left\langle \mathbf{s}(\mathcal{V}F), \mathcal{P}_u\, \iota_u\, \mathcal{U}_u \varphi \right\rangle_{\mathcal{H}}. \tag{13.4}$$

Here $B_{u,v}\varphi \in Y^* \otimes_{K,\sigma_v} \mathbb{C}$ is read as a linear form on Y_v, and the inner product of $\mathcal{H}$ is conjugate-linear in the first variable, so both sides of (13.4) are linear in φ and in F. The *Gram operators* of the realization are $K_{uu'} = (c_u c_{u'})^{-1/2} \mathcal{P}_u^* \mathcal{P}_{u'} \colon \mathcal{C}_{u'} \to \mathcal{C}_u$.

In the construction $\mathcal{H}$ is a space of automorphic forms on a unitary group, $\mathcal{C}_u$ is the space of column forms of the word u, ι_u is the inclusion of the forms that are ramified at T_u, $\mathcal{P}_u$ is the map to Bessel Poincaré series and c_u is its normalizing volume, and (13.4) identifies the period map with a Bessel period. Section 13.5 states what the construction has to prove about these objects.

Lemma 13.14 (matrices of the period maps at v)**.** *Let (B_u) have a Bessel realization at v with $D_{\mathrm{all}} \geq D_{\mathrm{rows}}$, and fix K-bases of Y^* and of each V_u. Put $D_1 = D_{\mathrm{all}} - D_{\mathrm{rows}}$. There are invertible complex matrices Y' of size D_{rows} and Y_u of size d_u, complex $D_{\mathrm{all}} \times d_u^S$ matrices P_u and complex $d_u^S \times d_u$ matrices V_u with $V_u^* V_u = 1$, such that for every u the first D_{rows} rows of $P_u V_u$ form the matrix $Y'^{-1}[B_u]_v Y_u$, and such that for all u, u' the matrix $(c_u c_{u'})^{-1/2} P_u^* P_{u'}$ is the matrix of $K_{uu'}$ in orthonormal bases of $\mathcal{C}_{u'}$ and $\mathcal{C}_u$. In particular $\|(c_u c_{u'})^{-1/2} P_u^* P_{u'}\|_{\mathrm{HS}} = \|K_{uu'}\|_{\mathrm{HS}}$ and $\|c_u^{-1} P_u^* P_u - 1\|_{\mathrm{HS}} = \|K_{uu} - 1\|_{\mathrm{HS}}$.*

Proof. Let $(\epsilon_k)_{k \leq D_{\mathrm{rows}}}$ be an orthonormal basis of Y_v. The vectors $\mathbf{s}(\mathcal{V}\epsilon_k)$ are orthonormal in $\mathcal{H}$, because $\mathbf{s}$ is a conjugate-linear isometry and $\mathcal{V}$ is unitary. Extend them to an orthonormal basis $(f_a)_{a \leq D_{\mathrm{all}}}$ of $\mathcal{H}$ with $f_k = \mathbf{s}(\mathcal{V}\epsilon_k)$ for $k \leq D_{\mathrm{rows}}$. Fix orthonormal bases $(c_{u,l})_l$ of $\mathcal{C}_u$ and $(o_{u,j})_{j \leq d_u}$ of $V_{u,v}$, and put

$$(P_u)_{a,l} = \langle f_a, \mathcal{P}_u c_{u,l} \rangle, \qquad (V_u)_{l,j} = \langle c_{u,l}, \iota_u \mathcal{U}_u o_{u,j} \rangle.$$

Since $\iota_u \mathcal{U}_u$ is an isometry, $V_u^* V_u = 1$, and $(P_u V_u)_{a,j} = \langle f_a, \mathcal{P}_u \iota_u \mathcal{U}_u o_{u,j} \rangle$. The matrix $P_u^* P_{u'}$ has entries $\langle \mathcal{P}_u c_{u,l}, \mathcal{P}_{u'} c_{u',l'} \rangle$ by Parseval's identity in $\mathcal{H}$, which is the matrix of $\mathcal{P}_u^* \mathcal{P}_{u'}$ in the chosen bases.

For $k \leq D_{\mathrm{rows}}$, (13.4) gives

$$(P_u V_u)_{k,j} = \big\langle \mathbf{s}(\mathcal{V}\epsilon_k), \mathcal{P}_u \iota_u \mathcal{U}_u o_{u,j} \big\rangle = a_u^{-1} \big(B_{u,v} o_{u,j} \big)(\epsilon_k).$$

Let (b_l) be the fixed basis of V_u and (b'_m) the fixed basis of Y^*. Let β_u be the matrix with $o_{u,j} = \sum_l (\beta_u)_{lj}\, b_l \otimes 1$, and let β' be the matrix with entries $\beta'_{km} = (b'_m \otimes 1)(\epsilon_k)$. Since $B_{u,v}(b_l \otimes 1) = \sum_m ([B_u]_v)_{ml}\, b'_m \otimes 1$, we get $\big((B_{u,v} o_{u,j})(\epsilon_k) \big)_{k,j} = \beta' [B_u]_v \beta_u$. The matrix β_u is invertible because $(o_{u,j})$ is a basis of $V_{u,v}$, and β' is invertible because it is the matrix of the perfect pairing between Y_v^* and Y_v in the bases $(b'_m \otimes 1)$ and (ϵ_k). So $Y' = \beta'^{-1}$ and $Y_u = a_u^{-1} \beta_u$ have the required property. □

The next theorem is hypothesis (3.3) of Proposition 3.1, with $\mu_u^N = \mu(N_u)$, $\mu^C = \mu(R^\vee)$, the weights $d_u = \operatorname{rk} N_u$ and the error e_3 of (13.5). Its hypotheses (a) to (f) are the outputs of the construction that it uses.

Theorem 13.15 (hypothesis (3.3))**.** *Let K be a totally complex number field, W a finite nonempty set, $\bar{R}$ a hermitian lattice over $\mathcal{O}_K$ of rank $D_{\mathrm{rows}} \geq 1$ in a K-vector space Y, and for $u \in W$ let $\bar{N}_u$ be a hermitian lattice over $\mathcal{O}_K$ of rank $d_u \geq 1$ in a K-vector space V_u, $B_u \colon V_u \to Y^*$ a K-linear map and h_u a real number. Put*

$$s = \frac{\sum_u d_u \big(\mu(N_u) - h_u \big)}{\sum_u d_u}, \qquad u_F = \max_{u \in W} \big(\mu_{\max}(N_u) - h_u \big) - s.$$

Let $D^* > 0$, $H > 0$, σ, η_0, η_1, e_0, δ', δ be real numbers with

$$0 < \sigma \leq \tfrac{1}{20}, \quad -1 \leq \eta_0 \leq \tfrac{\sigma}{2}, \quad \eta_1 \leq \tfrac{\sigma}{2}, \quad 0 \leq e_0 \leq \tfrac{1}{8}, \quad \sqrt{e_0} \leq \tfrac{\sigma}{2}, \quad 0 \leq \delta' \leq \tfrac{1}{4}, \quad 0 \leq \delta < 1,$$

and suppose the following.

(a) $B_u(N_u) \subset R^\vee$ *for every u.*

(b) $\mathrm{ht}(B_u) \leq h_u$ *for every u with $B_u \neq 0$, the height being computed with the norms of $\bar{N}_u$ and $\bar{R}^\vee$.*

(c) $s + \mu_{\max}(R) \leq H$.

(d) *The family* (B_u) *has a Bessel realization at a place* $v \in \Sigma_K$ *with* $D_{\mathrm{all}} \geq D_{\mathrm{rows}}$ *and* $(1-\delta)D_{\mathrm{all}} \leq D_{\mathrm{rows}}$.

(e) *The Gram operators of the realization satisfy* $\|K_{uu'}\|_{\mathrm{HS}}^2 \leq (1+\eta_0)d_u^S d_{u'}^S / D^*$ *for* $u \neq u'$, $\|K_{uu} - 1\|_{\mathrm{HS}}^2 \leq e_0 d_u^S$ *for every* u, *and* $D_{\mathrm{all}} \leq (1+\eta_1)D^*$.

(f) $D^* \leq \sigma \sum_u d_u^S$ *and* $\sum_u (d_u^S - d_u) \leq \delta' \sum_u d_u^S$.

Then the number

$$e_3 = \frac{4\Lambda(u_F, H) + \big(5\sigma + 3\sqrt{\delta'} + 4\sqrt{2e_0}\big)H}{1-\delta} \tag{13.5}$$

is nonnegative, and

$$\frac{\sum_u d_u \big(\mu(N_u) - h_u\big)}{\sum_u d_u} - e_3 \leq \mu(R^\vee).$$

Proof. Lemma 13.14 gives invertible matrices Y', Y_u and matrices P_u, V_u with $V_u^* V_u = 1$ such that the first D_{rows} rows of $A_u = P_u V_u$ form $Y'^{-1}[B_u]_v Y_u$, and such that the matrices $\sqrt{\gamma_u \gamma_{u'}} P_u^* P_{u'}$ with $\gamma_u = c_u^{-1}$ have the Hilbert–Schmidt norms of the Gram operators. Proposition 13.11 with $\widetilde{D} = D_{\mathrm{all}}$ applies by (e) and (f), and it gives conditions (v) and (vi) of Theorem 13.8 with $e = 2e_0 \leq \frac{1}{4}$ and $\eta = 5\sigma + 3\sqrt{\delta'}$.

We apply Theorem 13.8 with $I = W$, the lattices N_u, the target $C = R^\vee$ of rank $D = D_{\mathrm{rows}}$, $D_1 = D_{\mathrm{all}} - D_{\mathrm{rows}}$, and the numbers s, $u = u_F$ and H. The inclusions $B_u(N_u) \subset C$ and the height bounds are (a) and (b). Condition (i) holds with equality by the definition of s. Condition (ii) holds by the definition of u_F, and $u_F \geq 0$ because $\mu(N_u) \leq \mu_{\max}(N_u)$ makes s at most the maximum of $\mu_{\max}(N_u) - h_u$. Condition (iii) is (c), because $\mu_{\min}(R^\vee) = -\mu_{\max}(R)$ by Proposition 13.1. Condition (iv) is (d). The conclusion of Theorem 13.8 is the inequality of the theorem, with $\eta + 4\sqrt{e} = 5\sigma + 3\sqrt{\delta'} + 4\sqrt{2e_0}$. □

Remark 13.16. Condition (K3) of Section 18 uses the weaker conclusion $s - \log|W| - e_3 \leq \mu(R^\vee)$, obtained from Theorem 13.15 by subtracting $\log|W| \geq 0$. The term $\log|W|$ is slack and plays no role in the argument. Either form serves as (3.3), with e_3 of (13.5) or with $e_3 + \log|W|$.

The size of e_3 is controlled by the spread u_F, which the next lemma bounds by the two-sided form of hypothesis (3.2) and the spreads of the base slopes, the local terms and the heights.

Lemma 13.17. *Let* K *be a totally complex number field and* W *a finite nonempty set, and for* $u \in W$ *let* $\bar{N}_u$ *and* $\bar{M}_u$ *be hermitian lattices over* $\mathcal{O}_K$, *with* $d_u = \operatorname{rk} N_u \geq 1$, *and let* h_u, E_u, β_u^+, β_u^- *be real numbers. Let* Γ *and* $\mathcal{L}$ *be real numbers, put* $s = \sum_u d_u(\mu(N_u) - h_u) / \sum_u d_u$ *and* $u_F = \max_u(\mu_{\max}(N_u) - h_u) - s$, *and suppose that for every* u

$$\mu_{\max}(N_u) \leq \mu_{\max}(M_u) + \Gamma + \mathcal{L} - E_u + \beta_u^+, \qquad \mu(N_u) \geq \mu(M_u) + \Gamma + \mathcal{L} - E_u - \beta_u^-.$$

Then

$$u_F \leq \max_{u,u' \in W} \Big(\big(\mu_{\max}(M_u) - \mu(M_{u'})\big) + |E_u - E_{u'}| + |h_u - h_{u'}| \Big) + \max_u \beta_u^+ + \frac{\sum_u d_u \beta_u^-}{\sum_u d_u}.$$

Proof. For $u, u' \in W$ the two inequalities give

$$\big(\mu_{\max}(N_u) - h_u\big) - \big(\mu(N_{u'}) - h_{u'}\big) \leq \big(\mu_{\max}(M_u) - \mu(M_{u'})\big) + (E_{u'} - E_u) + (h_{u'} - h_u) + \beta_u^+ + \beta_{u'}^-.$$

Fix u, multiply by $d_{u'}$, sum over u' and divide by $\sum_{u'} d_{u'}$. The left side becomes $\mu_{\max}(N_u) - h_u - s$, and the right side is at most the bound of the lemma. Taking the maximum over u proves the lemma. □

Corollary 13.18 (size of e_3)**.** *Let $x \geq 1$, $C_H \geq 0$, $b > 0$, $u_F \geq 0$, $U \geq \max(u_F, b)$, $0 \leq \delta \leq \frac{1}{2}$, $\delta' \geq 0$ and $e_0 \geq 0$ be real numbers with $(3\sqrt{\delta'} + 4\sqrt{2e_0})\, x^{C_H} \leq 1$, and put $\sigma = x^{-C_H}$, $H = x^{C_H} b$ and*

$$e_3 = \frac{4\Lambda(u_F, H) + \big(5\sigma + 3\sqrt{\delta'} + 4\sqrt{2e_0}\big)H}{1-\delta}.$$

Then $e_3 \leq 8\, U \log\big(1 + x^{C_H}\big) + 12\, b$.

Proof. We have $(5\sigma + 3\sqrt{\delta'} + 4\sqrt{2e_0})H = 5b + (3\sqrt{\delta'} + 4\sqrt{2e_0})x^{C_H} b \leq 6b$. Since Λ is nondecreasing in its first variable and $U \geq b$, $\Lambda(u_F, H) \leq U \log(1 + H/U) \leq U \log(1 + x^{C_H})$. The factor $(1-\delta)^{-1}$ is at most 2. □

Lemma 13.19 (the size conditions of Theorem 13.15)**.** *Let $x \geq 1$, $C_H > 0$, $\varepsilon > 0$, $C_0 \geq 1$, $\Gamma_0 > 0$, $b \geq 3$ and an integer $f \geq 1$ be given, put $\sigma = x^{-C_H}$ and $\omega = \exp(-x^{\varepsilon/8})$, and assume*

$$x^{C_H} \geq 20, \qquad 11\, \omega\, x^{C_H} \leq 1, \qquad 4f \leq b\, \omega^2.$$

Let W be a finite set with $|W| \geq C_0 \Gamma_0^2 x^{C_H}$, let $D^* > 0$, and for $u \in W$ let $d_u^S \geq D^*/(C_0\Gamma_0^2)$ and $T_u \geq b$ be real numbers and $d_u = d_u^S(1 - \frac{1}{T_u - 1})^f$. Let η_0, η_1, e_0 be real numbers with $\eta_0 \geq -1$, $e_0 \geq 0$ and $\eta_0, \eta_1, \sqrt{e_0} \leq \omega$, and put $\delta' = 2f/b$. Then

$$D^* \leq \sigma \sum_u d_u^S, \qquad \sum_u (d_u^S - d_u) \leq \delta' \sum_u d_u^S, \qquad 0 < \sigma \leq \tfrac{1}{20}, \qquad \eta_0, \eta_1, \sqrt{e_0} \leq \tfrac{\sigma}{2},$$
$$e_0 \leq \tfrac{1}{8}, \qquad 0 \leq \delta' \leq \tfrac{1}{4}, \qquad \big(3\sqrt{\delta'} + 4\sqrt{2e_0}\big)x^{C_H} \leq 1.$$

Proof. Summing $d_u^S \geq D^*/(C_0\Gamma_0^2)$ over W gives $\sum_u d_u^S \geq |W| D^*/(C_0\Gamma_0^2) \geq x^{C_H} D^* = D^*/\sigma$. Lemma 13.12 and $T_u \geq b$ give $d_u^S - d_u \leq (2f/b) d_u^S$, and summing over u gives the second inequality. Now $\sqrt{\delta'} \leq \sqrt{\omega^2/2} \leq \omega$, $\sigma \leq \frac{1}{20}$, and $\omega \leq \frac{\sigma}{11} \leq \frac{\sigma}{2}$, so $\eta_0, \eta_1, \sqrt{e_0} \leq \frac{\sigma}{2}$ and $e_0 \leq \omega^2 \leq \frac{1}{8}$, $\delta' \leq \omega^2/2 \leq \frac{1}{4}$. Finally $(3\sqrt{\delta'} + 4\sqrt{2e_0})x^{C_H} \leq (3 + 4\sqrt{2})\, \omega\, x^{C_H} \leq 1$. □

The integrality (a) comes from the divisibility of the composite of the lifts and from the definition of N_u as a rescaled sublattice. The following lemma isolates the step from local divisibility to (a).

Lemma 13.20 (integrality from local divisibility)**.** *Let K be a number field, R a lattice in a K-vector space Y, V a K-vector space and $B\colon V \to Y^*$ a K-linear map. For each finite place w of K let Λ_w be an $\mathcal{O}_{K,w}$-submodule of $V \otimes_K K_w$, and suppose that $\mathrm{ord}_w\big(B(x)(y)\big) \geq 0$ for every $x \in \Lambda_w$ and $y \in R$, where B is extended K_w-linearly. Let $N \subset V$ be a lattice with $N \subset \Lambda_w$ for every w. Then $B(N) \subset R^\vee$. In particular, if $N_w^0 \subset V \otimes K_w$ and integers $s(w) \geq 0$ satisfy $\mathrm{ord}_w(B(x)(y)) \geq s(w)$ for $x \in N_w^0$ and $y \in R$, and $N \otimes \mathcal{O}_{K,w} \subset \varpi_w^{-s(w)} N_w^0$ for every w, then $B(N) \subset R^\vee$.*

Proof. For $x \in N$ and $y \in R$ the number $B(x)(y) \in K$ has $\mathrm{ord}_w \geq 0$ at every finite place w, so it lies in $\mathcal{O}_K$. Hence $B(x) \in R^\vee$. For the second statement put $\Lambda_w = \varpi_w^{-s(w)} N_w^0$, so that $\mathrm{ord}_w(B(\varpi_w^{-s(w)} x)(y)) \geq -s(w) + s(w) = 0$ for $x \in N_w^0$. □

13.5 The automorphic inputs of Theorem 13.15

Theorem 13.15 uses three properties of the construction that are not consequences of slope theory. The first is the Bessel realization (d), that is, the identity (13.4) of the period map with a Bessel period of a Poincaré series, together with the bound $(1-\delta)D_{\mathrm{all}} \leq D_{\mathrm{rows}}$ with $0 \leq \delta \leq \exp(-x^{\varepsilon_z/7})$, where ε_z is ε divided by a fixed integer $J \geq 1$. The second is the near orthogonality (e). The third is the pair of counting conditions in (f) and Lemma 13.19.

We record the objects of the second. Let G be the unitary group of the hermitian space of the top step over E, let $\mathcal{H}$ be the space of automorphic forms on G whose component at the archimedean places is the fixed discrete series and whose component at the finite places is fixed by the level idempotent of the construction, with the Petersson inner product, and let D^* be the product of the formal degree of the discrete series, the volume of the automorphic quotient and the value at 1 of the level idempotent. For each word u the Bessel datum consists of a flag of isotropic vectors $e_1, \ldots, e_r$ and a vector z_u orthogonal to them with $\langle z_u, z_u \rangle = -T_u$. The seed of u is built from the Bessel function W_u of the discrete series for this datum and a character of a level subgroup, $\mathcal{P}_u \colon \mathcal{C}_u \to \mathcal{H}$ sends a column vector T to the Poincaré series $x \mapsto \sum_{\gamma \in S_u(F)\backslash G(F)} \mathrm{Ph}_{u,T}(\gamma x)$ of the seed, where S_u is the stabilizer of the datum, and $c_u = \mathrm{vol}_N(u)\mathrm{vol}_H(u)/\dim \Pi_u$ with $\dim \Pi_u$ the product over the archimedean places of the dimensions of the K_∞-types of u. Put $\Gamma_0 = \exp(C_\Gamma m^5 f x^\epsilon)$.

Section 14 proves Theorem 14.1 (near orthogonality), Theorem 14.2 (the share of irreducible rows) and Theorem 14.3 (the Bessel identity) from the statements listed there.

14 Bessel Poincaré series

This section proves the three automorphic inputs (d) and (e) of Theorem 13.15 from the statements listed in it: the near orthogonality of Bessel Poincaré series, the share of irreducible rows, and the Bessel identity for the period maps.

14.1 The three statements

We keep the notation of Section 13. The field K is a totally complex Galois number field, v is a complex place of K, Y is a K-vector space with a hermitian lattice R of rank D_{rows}, and for each word u of the finite set W the period map is a K-linear map $B_u \colon V_u \to Y^*$. Write $Y_v = Y \otimes_{K,\sigma_v} \mathbb{C}$ and $V_{u,v} = V_u \otimes_{K,\sigma_v} \mathbb{C}$, with the hermitian norms of R and of N_u at v.

On the automorphic side F is a totally real field of degree f, E/F is a CM extension, V is a nondegenerate hermitian space over E, and $G = \mathrm{U}(V)$ is its unitary group over F. Write $\mathbb{A}$ and $\mathbb{A}_f$ for the adeles and the finite adeles of F, $G_\infty = G(F \otimes_{\mathbb{Q}} \mathbb{R})$ and $[G] = G(F)\backslash G(\mathbb{A})$. The space $\mathcal{H}$ is the finite dimensional space of automorphic forms on G whose archimedean component is a fixed discrete series and whose finite part is fixed by the level idempotent of the construction, with the Petersson inner product, which is conjugate linear in the first variable. Its dimension is D_{all}. The number D^* is the product of the formal degree of the discrete series, the volume of $[G]$ and the value at 1 of the level idempotent.

For each word u the column space $\mathcal{C}_u$ has dimension d_u^S, the Poincaré map $\mathcal{P}_u \colon \mathcal{C}_u \to \mathcal{H}$ sends a column form to its Bessel Poincaré series (Section 14.2.1), $c_u > 0$ is its normalizing volume, and

$$K_{uu'} = (c_u c_{u'})^{-1/2} \mathcal{P}_u^* \mathcal{P}_{u'} \colon \mathcal{C}_{u'} \to \mathcal{C}_u.$$

The family parameters are x, ε, $b = e^{x^\varepsilon}$, $L = \log b = x^\varepsilon$, the ranks m, k, r of the datum and $\Gamma_0 = \exp(C_\Gamma m^5 f L)$.

Theorem 14.1 (near orthogonality)**.** *There are absolute constants $C_0 \geq 1$ and $C_\Gamma > 0$ with the following property. For every ε, every x large in terms of ε, and every family of data of Part II at the scale x whose words satisfy*

$$C_0^{-1} \leq d_u^S \Gamma_0^2 / D^* \leq C_0 \qquad \text{for every } u \in W, \tag{14.1}$$

the Gram operators satisfy

$$\|K_{uu'}\|_{\mathrm{HS}}^2 \leq (1+\eta_0)\frac{d_u^S d_{u'}^S}{D^*} \ (u \neq u'), \qquad \|K_{uu} - 1\|_{\mathrm{HS}}^2 \leq e_0\, d_u^S, \qquad D_{\mathrm{all}} \leq (1+\eta_1) D^*,$$

with $e_0 \geq 0$ and $\eta_0, \eta_1, \sqrt{e_0} \leq \exp(-x^{\varepsilon/8})$.

Theorem 14.2 (share of irreducible rows)**.** *Let $J \geq 1$ be the fixed integer of the construction and $\varepsilon_z = \varepsilon / J$. Let $E_{\mathrm{row}} \subset \mathcal{H}$ be the span of the image of $R \otimes_{K,\sigma_v} \mathbb{C}$ under the row embedding, and define δ by $D_{\mathrm{rows}} = \dim E_{\mathrm{row}} = (1-\delta) D_{\mathrm{all}}$. For every family of data of Part II and every x large in terms of the family constants,*

$$0 \leq \delta \leq \exp\bigl(-x^{\varepsilon_z/7}\bigr).$$

Theorem 14.3 (Bessel identity)**.** *For every family of data of Part II and every x large, the following holds at the complex place v. There are a unitary operator $\mathcal{V}$ of Y_v, and for each $u \in W$ a unitary operator $\mathcal{U}_u$ of $V_{u,v}$ and a number $a_u \neq 0$, such that*

$$(B_{u,v}\varphi)(F) = a_u \left\langle \mathbf{s}(\mathcal{V}F),\, \mathcal{P}_u \iota_u \mathcal{U}_u \varphi \right\rangle_{\mathcal{H}} \qquad (\varphi \in V_{u,v},\ F \in Y_v), \tag{14.2}$$

where $\mathbf{s}\colon Y_v \to \mathcal{H}$ is the conjugate linear isometric row embedding and $\iota_u\colon V_{u,v} \to \mathcal{C}_u$ is the linear isometric column embedding of Definition 13.13. In addition B_u is K-rational in the sense that $(B_{u,v}(\varphi \otimes 1))(y \otimes 1) = \sigma_v\bigl(B_u(\varphi)(y)\bigr)$ for $\varphi \in V_u$ and $y \in Y$.

Theorem 14.3 together with Theorem 14.2 is hypothesis (d) of Theorem 13.15, and Theorem 14.1 is hypothesis (e) there.

14.2 The Bessel identity

The identity (14.2) is the end of a chain with four links. The period map is defined by a period of the chain of theta lifts on the definite side. The seesaw relation transfers every linear relation among those periods to the local Bessel periods of the theta lifts on the indefinite side. Unfolding the Poincaré series transfers every linear relation among local Bessel periods to the pairings with Poincaré series. One nonzero pairing then turns the transfer of relations into proportionality. Section 14.2.1 proves the unfolding, Section 14.2.2 proves the passage from transfer of relations to proportionality, and Section 14.2.3 assembles the theorem.

14.2.1 Bessel Poincaré series and their unfolding

Fix a word u and drop it from the notation where no confusion arises. The construction gives the following objects.

(P1) A closed F-subgroup $S = N \rtimes H$ of G, the Bessel subgroup of the flag of u. Here N is a unipotent F-group and $H = \mathrm{U}(W)$ for a hermitian space W over E that is positive definite at every real place of F. A unitary character $\psi\colon N(\mathbb{A}) \to \mathbb{C}^\times$, trivial on $N(F)$ and invariant under conjugation by $H(\mathbb{A})$.

(P2) A compact open subgroup $K_2 \subset G(\mathbb{A}_f)$ with a unitary character χ_2, and an element $x_u \in G(\mathbb{A}_f)$.

(P3) A finite dimensional unitary representation (σ, V_σ) of the compact group $H_\infty = H(F \otimes \mathbb{R})$, extended to $S_\infty = N_\infty H_\infty$ by $\sigma(nh) = \psi_\infty(n)\sigma(h)$, and a measurable function $W_u\colon G_\infty \to V_\sigma$ with $W_u(sy) = \sigma(s)W_u(y)$ for $s \in S_\infty$ and

$$\int_{S_\infty \backslash G_\infty} \|W_u(y)\|\, d\nu(y) < \infty. \tag{14.3}$$

(P4) The space $\mathcal{C}$ of column forms. A column form is a linear map T from V_σ to functions on $H(\mathbb{A})$ with $T(\xi)(\gamma a) = T(\xi)(a)$ for $\gamma \in H(F)$, $T(\sigma(h)\xi)(a) = T(\xi)(ah)$ for $h \in H_\infty$, and $|T(\xi)(a)| \le C_T\|\xi\|$.

(P5) For each $T \in \mathcal{C}$ a measurable seed Ph_T on $G(\mathbb{A})$, supported on $G_\infty \times S(\mathbb{A}_f)x_uK_2$, with $\mathrm{Ph}_T(gk) = \chi_2(k)\mathrm{Ph}_T(g)$ for $k \in K_2$, and

$$\mathrm{Ph}_T\big(na \cdot (y, x_u)\big) = \psi(n)\, T\big(W_u(y)\big)(a) \qquad (n \in N(\mathbb{A}),\ a \in H(\mathbb{A}),\ y \in G_\infty). \tag{14.4}$$

Measures are fixed as follows. $G(\mathbb{A}) = G_\infty \times G(\mathbb{A}_f)$ has a product Haar measure, $S(\mathbb{A})$, $N(\mathbb{A})$ and $H(\mathbb{A})$ have Haar measures with $ds = dn\, da$, discrete groups have counting measure, and ν is the quotient measure on $S_\infty\backslash G_\infty$. Lemma 14.4 below shows that these quotient measures exist.

For a bounded measurable function θ on $G(\mathbb{A})$ that is left $N(F)$-invariant put

$$\mathrm{per}(\Psi, \theta) = \int_{H(F)\backslash H(\mathbb{A})} \int_{N(F)\backslash N(\mathbb{A})} \theta(na)\, \psi(n)\, \Psi(a)\, dn\, da \tag{14.5}$$

for every bounded measurable left $H(F)$-invariant function Ψ on $H(\mathbb{A})$. This is the Bessel period of θ against Ψ.

Lemma 14.4. *Under (P1) the groups $N(\mathbb{A})$, $H(\mathbb{A})$ and $S(\mathbb{A})$ are unimodular, the quotients $N(F)\backslash N(\mathbb{A})$ and $H(F)\backslash H(\mathbb{A})$ are compact, and $S(F)\backslash S(\mathbb{A})$ is compact.*

Proof. The modular function of $S(\mathbb{A}) = N(\mathbb{A}) \rtimes H(\mathbb{A})$ is $nh \mapsto |\det \mathrm{Ad}(h)|_{\mathrm{Lie}\, N}|_{\mathbb{A}}^{-1}$ times the modular functions of $N(\mathbb{A})$ and $H(\mathbb{A})$. The map $h \mapsto \det \mathrm{Ad}(h)|_{\mathrm{Lie}\, N}$ is an F-rational character of $H = \mathrm{U}(W)$. Every F-rational character of $\mathrm{U}(W)$ factors through the determinant $\mathrm{U}(W) \to \mathrm{U}(1)$, and $\mathrm{U}(1) = \ker(N_{E/F}\colon R_{E/F}\mathbb{G}_m \to \mathbb{G}_m)$ is an anisotropic torus, so it has no nontrivial F-rational character. Hence $\det \mathrm{Ad}(h)|_{\mathrm{Lie}\, N} = 1$ identically, so at every place and on the adelic points. The same argument shows that S_∞ and $S(\mathbb{A}_f)$ are unimodular. Unipotent groups and reductive groups have unimodular adelic points, so $S(\mathbb{A})$ is unimodular. The quotient $N(F)\backslash N(\mathbb{A})$ is compact for every unipotent F-group [12]. Since W is positive definite at a real place, it has no isotropic vector over F, so H is F-anisotropic and $H(F)\backslash H(\mathbb{A})$ is compact by the theorem of Borel and Harish-Chandra and of Mostow and Tamagawa [13]. The image of the compact set $C_N \times C_H$ under multiplication, with C_N and C_H compact sets mapping onto the two quotients, maps onto $S(F)\backslash S(\mathbb{A})$, because $S(F) = N(F)H(F)$ and $H(\mathbb{A})$ normalizes $N(\mathbb{A})$. □

Proposition 14.5 (unfolding). *Assume (P1) to (P5) and let $T \in \mathcal{C}$. The series*

$$P_T(g) = \sum_{\gamma \in S(F)\backslash G(F)} \mathrm{Ph}_T(\gamma g)$$

converges absolutely for almost every $g \in G(\mathbb{A})$, defines an element of $L^1([G])$ that does not depend on the choice of representatives, and is right (K_2, χ_2)-equivariant. There is a number $c_u > 0$, depending on the Haar measures and on u and not on T, such that for every bounded measurable function θ on $G(\mathbb{A})$ that is left $G(F)$-invariant and right $(K_2, \overline{\chi_2})$-equivariant,

$$\int_{[G]} \theta(g)\, P_T(g)\, dg = c_u \int_{S_\infty \backslash G_\infty} \mathrm{per}\Big(T\big(W_u(y)\big),\ R_{(y,x_u)}\theta\Big)\, d\nu(y), \tag{14.6}$$

where $R_{(y,x_u)}\theta(g) = \theta(g \cdot (y, x_u))$, and both sides converge absolutely.

Proof. Step 1, integration in stages. Since $G(\mathbb{A})$, $S(\mathbb{A})$, $S(F)$ and $G(F)$ are unimodular (Lemma 14.4), the quotient integral formula of Weil [26, §2.6] applies to $S(F) \subset S(\mathbb{A}) \subset G(\mathbb{A})$ and to $S(F) \subset G(F) \subset G(\mathbb{A})$. For every measurable $\Phi \geq 0$ on $G(\mathbb{A})$ that is left $S(F)$-invariant,

$$\int_{S(F)\backslash G(\mathbb{A})} \Phi = \int_{S(\mathbb{A})\backslash G(\mathbb{A})} \int_{S(F)\backslash S(\mathbb{A})} \Phi(s\dot{g})\, ds\, d\dot{g}, \tag{14.7}$$

$$\int_{S(F)\backslash G(\mathbb{A})} \Phi = \int_{[G]} \sum_{\gamma \in S(F)\backslash G(F)} \Phi(\gamma g)\, dg. \tag{14.8}$$

The second identity is the unfolding. To see it directly, let $\mathfrak{F}$ be a measurable fundamental domain for $G(F)$ in $G(\mathbb{A})$, which exists because $G(F)$ is a countable discrete subgroup of a second countable locally compact group, and let D be a set of representatives of $S(F)\backslash G(F)$. The sets $\delta\gamma\mathfrak{F}$ with $\delta \in S(F)$ and $\gamma \in D$ are disjoint and cover $G(\mathbb{A})$, so $\bigcup_{\gamma \in D} \gamma\mathfrak{F}$ is a fundamental domain for $S(F)$. Left invariance of Haar measure and Tonelli's theorem give (14.8).

Step 2, the finite part. Put $\Omega = S(\mathbb{A}_f)x_u K_2$, an open subset of $G(\mathbb{A}_f)$. Let $\Phi \geq 0$ be measurable on $G(\mathbb{A})$ with $\Phi(sg) = \Phi(g)$ for $s \in S(\mathbb{A})$, right K_2-invariant and supported on $G_\infty \times \Omega$. Then Φ factors through $S(\mathbb{A})\backslash G(\mathbb{A}) = (S_\infty \backslash G_\infty) \times (S(\mathbb{A}_f)\backslash G(\mathbb{A}_f))$ and is a function of the first coordinate on the subset $S(\mathbb{A}_f)\backslash\Omega$, which is the image of K_2 under $k \mapsto S(\mathbb{A}_f)x_u k$. The quotient measure on a product of quotients is the product of the quotient measures, so

$$\int_{S(\mathbb{A})\backslash G(\mathbb{A})} \Phi(\dot{g})\, d\dot{g} = c_u \int_{S_\infty \backslash G_\infty} \Phi\big((y, x_u)\big)\, d\nu(y), \tag{14.9}$$

where c_u is the quotient measure of $S(\mathbb{A}_f)\backslash\Omega$. It is finite because $S(\mathbb{A}_f)\backslash\Omega$ is the continuous image of the compact group K_2, and positive because it is open.

Step 3, absolute convergence. Let θ be as in the statement. The product $\Phi_0 = |\theta\, \mathrm{Ph}_T|$ is left $S(F)$-invariant, because θ is left $G(F)$-invariant and Ph_T is left $S(F)$-invariant. For the latter, the support $G_\infty \times \Omega = S(\mathbb{A}) \cdot (G_\infty \times \{x_u\}) \cdot K_2$ is stable under $S(F)$, and for $\gamma = n_0 a_0 \in S(F)$, (14.4) gives $\mathrm{Ph}_T(\gamma n a(y, x_u)) = \psi(n_0)\psi(a_0 n a_0^{-1})T(W_u(y))(a_0 a) = \psi(n)T(W_u(y))(a)$, by the triviality of ψ on $N(F)$, its invariance under $H(\mathbb{A})$, and the $H(F)$-invariance in (P4). It is right K_2-invariant, because the characters χ_2 and $\overline{\chi_2}$ cancel, and it is supported on $G_\infty \times \Omega$. By (14.7), then (14.9) applied to the left $S(\mathbb{A})$-invariant function $g \mapsto \int_{S(F)\backslash S(\mathbb{A})} \Phi_0(sg)\, ds$, then (14.4) and the integration formula $ds = dn\, da$ on $S(F)\backslash S(\mathbb{A})$,

$$\begin{aligned}\int_{S(F)\backslash G(\mathbb{A})} \Phi_0 &= c_u \int_{S_\infty \backslash G_\infty} \int_{[H]} \int_{[N]} \big|\theta(na(y, x_u))\big|\, \big|T(W_u(y))(a)\big|\, dn\, da\, d\nu(y) \\ &\leq c_u \|\theta\|_\infty \,\mathrm{vol}[N]\,\mathrm{vol}[H]\, C_T \int_{S_\infty \backslash G_\infty} \|W_u\|\, d\nu,\end{aligned}$$

which is finite by Lemma 14.4 and (14.3). The same computation applies to $\Phi_0 = |\mathrm{Ph}_T|$, which is right K_2-invariant because $|\chi_2| = 1$, and with (14.8) it shows that $\sum_\gamma |\mathrm{Ph}_T(\gamma g)|$ is integrable on $[G]$, hence finite almost everywhere, so P_T converges absolutely almost everywhere and lies in $L^1([G])$. Independence of representatives is the left $S(F)$-invariance of Ph_T, and right equivariance under (K_2, χ_2) passes to the sum.

Step 4, the identity. Since $\theta\,\mathrm{Ph}_T$ is absolutely integrable on $S(F)\backslash G(\mathbb{A})$ by Step 3, the identities (14.8), (14.7) and (14.9) hold for it by Fubini's theorem applied to the positive and negative parts of its real and imaginary parts. The left side of (14.8) is $\int_{[G]} \theta P_T$ because θ is left $G(F)$-invariant. On the right, (14.4) gives

$$\begin{aligned}\int_{S(F)\backslash S(\mathbb{A})} (\theta\,\mathrm{Ph}_T)(s(y,x_u))\,ds &= \int_{[H]}\int_{[N]} \theta(na(y,x_u))\,\psi(n)\,T(W_u(y))(a)\,dn\,da \\ &= \mathrm{per}\big(T(W_u(y)), R_{(y,x_u)}\theta\big).\end{aligned}$$

This is (14.6). □

Remark 14.6. The argument shows $c_u > 0$. The identity (14.6) does not need $P_T \in L^2([G])$, nor that P_T is fixed by the archimedean projector. Both facts are needed to know that $P_T \in \mathcal{H}$, which Definition 14.13 uses.

Corollary 14.7 (transfer of relations from Bessel periods to Poincaré pairings)**.** *Assume (P1) to (P5). Let $T_1, \dots, T_\ell \in \mathcal{C}$, let $\theta_1, \dots, \theta_\ell$ be bounded measurable functions on $G(\mathbb{A})$, left $G(F)$-invariant and right $(K_2, \overline{\chi_2})$-equivariant, and let $\alpha_1, \dots, \alpha_\ell \in \mathbb{C}$. If*

$$\sum_{i=1}^{\ell} \alpha_i\,\mathrm{per}\big(T_i(\xi),\, R_{(y,x_u)}\theta_i\big) = 0 \qquad \text{for every } \xi \in V_\sigma \text{ and every } y \in G_\infty, \tag{14.10}$$

then $\sum_i \alpha_i \int_{[G]} \theta_i P_{T_i} = 0$. If (14.10) *holds for ξ in a spanning set of V_σ and every y, it holds for every ξ.*

Proof. The period (14.5) is linear in Ψ and T_i is linear, so the left side of (14.10) is linear in ξ, which proves the last sentence. Apply Proposition 14.5 to each i with the same c_u, and put $\xi = W_u(y)$ in (14.10). The integrand of the sum of the right sides of (14.6) vanishes for every y. □

14.2.2 From transfer of relations to proportionality

Lemma 14.8. *Let X be a set and $\lambda, \mu\colon X \to \mathbb{C}$ functions. Assume that for every finite family $x_1, \dots, x_\ell \in X$ and $\alpha_1, \dots, \alpha_\ell \in \mathbb{C}$ with $\sum_i \alpha_i \lambda(x_i) = 0$ one has $\sum_i \alpha_i \mu(x_i) = 0$. Assume $\mu(x_0) \neq 0$ for some $x_0 \in X$. Then $\lambda(x_0) \neq 0$, and $\mu = \kappa\lambda$ on X with $\kappa = \mu(x_0)/\lambda(x_0) \neq 0$.*

Proof. If $\lambda(x_0) = 0$, the family consisting of x_0 with coefficient 1 satisfies the hypothesis, so $\mu(x_0) = 0$, a contradiction. For $x \in X$ the family (x_0, x) with coefficients $(\lambda(x), -\lambda(x_0))$ satisfies $\lambda(x)\lambda(x_0) - \lambda(x_0)\lambda(x) = 0$, so $\lambda(x)\mu(x_0) - \lambda(x_0)\mu(x) = 0$. □

Lemma 14.9 (central translations)**.** *Let z be an element of the center of $G(\mathbb{A}_f)$ and $R_z\theta(g) = \theta(gz)$. Then R_z is a unitary operator of $L^2([G])$ that commutes with the right action of $G(\mathbb{A})$. It maps the right (K_2, χ_2)-equivariant functions to themselves and preserves every $G(\mathbb{A})$-isotypic subspace, in particular $\mathcal{H}$.*

Proof. R_z is well defined on $[G]$ and preserves Haar measure because $G(\mathbb{A})$ is unimodular, and it is invertible with inverse $R_{z^{-1}}$. Since z is central, $R_z R_h = R_h R_z$ for $h \in G(\mathbb{A})$, which gives the last two assertions, and $\theta(gkz) = \theta(gzk) = \chi_2(k)\theta(gz)$ for $k \in K_2$. □

14.2.3 Assembly of the Bessel identity

The theorem uses four inputs of the construction. The first is a definition, the second is the seesaw relation of the theta tower, the third is a pair of dictionaries between the abstract realization of Definition 13.13 and the automorphic objects of Section 14.2.1, and the fourth is rationality. They are stated here in the form used, for one word u at the place v.

Hypothesis 14.10 (the period map and the seesaw relation)**.** There are a central element $z \in G(\mathbb{A}_f)$, a linear map U_u of $V_{u,v}$ with $\|U_u\varphi\| = r_u\|\varphi\|$ for some $r_u > 0$, a set X_u containing $V_{u,v} \times Y_v$, a function $\lambda_u \colon X_u \to \mathbb{C}$ with $\lambda_u(\varphi, F) = (B_{u,v}\varphi)(F)$ on $V_{u,v} \times Y_v$, and for each $x \in X_u$ a column form $T_x \in \mathcal{C}_u$ and a function $\theta_x \in \mathcal{H}$, with $T_{(\varphi,F)} = \iota_u(U_u\varphi)$ and $\theta_{(\varphi,F)} = \mathbf{s}(F)$ on $V_{u,v} \times Y_v$, such that the following holds. For every finite family $x_i \in X_u$ and $\alpha_i \in \mathbb{C}$ with $\sum_i \alpha_i\lambda_u(x_i) = 0$,

$$\sum_i \alpha_i \operatorname{per}\Big(T_{x_i}(\xi),\ R_{(y,x_u)}R_z\overline{\theta_{x_i}}\Big) = 0 \qquad \text{for every } \xi \in V_\sigma \text{ and } y \in G_\infty.$$

For some $x_0 \in X_u$ the pairing $\int_{[G]} R_z\overline{\theta_{x_0}}\, P_{T_{x_0}}$ is nonzero.

Hypothesis 14.11 (dictionaries)**.** The space $\mathcal{H}$ consists of bounded continuous functions on $G(\mathbb{A})$, left $G(F)$-invariant and right (K_2, χ_2)-equivariant. The column space $\mathcal{C}_u$ of Definition 13.13 is the space $\mathcal{C}$ of (P4) with the norm for which $\mathcal{P}_u$ is the map $T \mapsto P_T$ of Proposition 14.5, and $P_T \in \mathcal{H}$ for every T. The central translation R_z maps $\mathbf{s}(Y_v)$ onto itself.

Hypothesis 14.12 (rationality)**.** For $\varphi \in V_u$ and $y \in Y$ the number $(B_{u,v}(\varphi \otimes 1))(y \otimes 1)$ lies in $\sigma_v(K)$ and equals $\sigma_v(B_u(\varphi)(y))$.

Definition 14.13. For $x \in X_u$ put $\mu_u(x) = \int_{[G]} R_z\overline{\theta_x}\, P_{T_x}$. The last condition of Hypothesis 14.10 asks for a pair x_0 with $\mu_u(x_0) \neq 0$.

Lemma 14.14. *Assume $\|K_{uu} - 1\|^2_{\mathrm{HS}} \leq e_0\, d_u^S$ and $\dim \iota_u(V_{u,v}) > e_0\, d_u^S$. Then there is $\varphi_0 \in V_{u,v}$ with $\mathcal{P}_u\iota_u(U_u\varphi_0) \neq 0$.*

Proof. For $w \in \mathcal{C}_u$ one has $\|\mathcal{P}_u w\|^2 = c_u\langle w, K_{uu}w\rangle$, so $\ker \mathcal{P}_u = \ker K_{uu}$, because K_{uu} is positive semidefinite. On this kernel $K_{uu} - 1$ acts as -1, so $\dim\ker \mathcal{P}_u \leq \|K_{uu} - 1\|^2_{\mathrm{HS}} \leq e_0 d_u^S$. Since U_u is invertible, $\iota_u U_u(V_{u,v}) = \iota_u(V_{u,v})$ has dimension larger than $\dim\ker \mathcal{P}_u$ and is not contained in it. □

Under the conclusion of Theorem 14.1, $e_0 \leq \frac{1}{2}$ for large x, and $\dim \iota_u(V_{u,v}) = d_u \geq (1 - 2f/T_u)d_u^S > \frac{1}{2}d_u^S$ by Lemma 13.12, since $T_u \geq b > 8f$. So Lemma 14.14 applies.

Theorem 14.15. *Assume (P1) to (P5) for the word u, Hypotheses 14.10, 14.11 and 14.12. Then* (14.2) *holds for u with $\mathcal{V}$ the unitary operator of Y_v defined by $\mathbf{s}(\mathcal{V}F) = R_z\mathbf{s}(F)$, with $\mathcal{U}_u = r_u^{-1}U_u$ and with a number $a_u \neq 0$. If the hypotheses hold for every $u \in W$ with one z, then $\mathcal{V}$ does not depend on u and Theorem 14.3 holds.*

Proof. By Lemma 14.9 and Hypothesis 14.11, R_z restricts to a unitary operator of $\mathbf{s}(Y_v)$, and since $\mathbf{s}$ is a conjugate linear isometry onto its image, $\mathcal{V} = \mathbf{s}^{-1}R_z\mathbf{s}$ is a linear unitary operator of Y_v.

Let μ_u be as in Definition 14.13. Every $\theta_x \in \mathcal{H}$ is bounded, and so is $R_z\overline{\theta_x}$, which is left $G(F)$-invariant and right $(K_2, \overline{\chi_2})$-equivariant by Lemma 14.9. Hypothesis 14.10 and Corollary 14.7 show that every linear relation among the values of λ_u holds among the values of μ_u. By

Hypothesis 14.10, $\mu_u(x_0) \neq 0$. Lemma 14.8 gives $\kappa_u \neq 0$ with $\mu_u = \kappa_u \lambda_u$ on X_u. On $V_{u,v} \times Y_v$ this reads

$$\kappa_u \, (B_{u,v}\varphi)(F) = \int_{[G]} \overline{R_z \mathbf{s}(F)} \, \mathcal{P}_u \iota_u (U_u \varphi) = \big\langle \mathbf{s}(\mathcal{V}F), \mathcal{P}_u \iota_u (U_u \varphi) \big\rangle_{\mathcal{H}} = r_u \big\langle \mathbf{s}(\mathcal{V}F), \mathcal{P}_u \iota_u \mathcal{U}_u \varphi \big\rangle_{\mathcal{H}}.$$

So (14.2) holds with $a_u = r_u / \kappa_u \neq 0$, and $\mathcal{U}_u = r_u^{-1} U_u$ is unitary because it is a linear isometry of a finite dimensional space. The rationality clause is Hypothesis 14.12. □

Remark 14.16 (what Theorem 14.15 leaves open). Hypothesis 14.10 combines a definition and a relation. The period map is defined from the period perCol of the chain of theta lifts on the definite side. The seesaw relation transfers relations among the numbers $c(t)$ perCol to relations among the local Bessel periods of the indefinite theta lifts translated by z, with $c(t) = c_{\mathrm{top}}(t) s / c_{\mathrm{bot}}(t)$. The constants of the double step come from the volumes and local scalar factors in Theorem 14.29. The Siegel–Weil and doubling identities enter the surjectivity in Hypothesis 14.21.

14.2.4 The chain of seesaw transfers

The transfer follows from the theta tower identities in Hypothesis 14.17 and the period identifications in Hypothesis 14.21.

The tower. Let $m = 2n + 2$ be the (even) number of lifts of the chain. The construction gives hermitian spaces $V_0, \dots, V_m$ and T over E, definite hermitian spaces $X_0, \dots, X_m$ with an orthogonal decomposition $X_m = T \oplus L$, and $h \mapsto \epsilon(h)$ the induced embedding $\mathrm{U}(T) \to \mathrm{U}(X_m)$. For $0 \leq j$ and $\rho \geq 0$ with $j + 2\rho + 2 = m$ there is a Bessel datum of rank ρ at level j, and $\mathcal{B}_j^{\rho}(\phi, f)$ denotes the Bessel period (14.5) of a function f on $\mathrm{U}(V_j)(\mathbb{A})$ against a function ϕ on $\mathrm{U}(X_j)(\mathbb{A})$, with the fixed additive character ψ. For $e = \sum_g e(g)\, g$ a finitely supported combination of finite adelic points put $\mathrm{tr}(e) f = \sum_g e(g) R_g f$.

For $j \leq m - 1$ let $\mathcal{S}_j$ be the space of smooth vectors of the archimedean Fock model at level j and $\mathcal{S}_T$ the one of T. The space $\mathcal{C}_j$ consists of linear maps $G \colon \mathcal{S}_j \to \mathcal{A}(\mathrm{U}(V_j))$, and $\mathcal{A}$ denotes the space of automorphic functions. Define $\mathcal{C}_T$ for T in the same way. The theta lifts of the chain are linear maps

$$\iota_j \colon \mathcal{C}_{j+1} \to \mathcal{C}_j \quad (j \leq m - 2), \qquad \iota_T \colon \mathcal{C}_T \to \mathcal{C}_{m-1},$$

and $\Theta_j(f; P)$ denotes the theta integral of a function f on $\mathrm{U}(V_{j+1})$ against the theta kernel attached to a polynomial Fock vector P, with Θ_T the same for T. The maps $\mathrm{It}_j \colon \mathcal{P}_j \otimes \mathcal{S}_{j+1} \to \mathcal{S}_j$ and $\mathrm{It}_T \colon \mathcal{P}_T \otimes \mathcal{S}_T \to \mathcal{S}_{m-1}$ are the tensor maps of the Fock models.

The tower is indexed by a set $\mathcal{T}$ of tuples. For each $s \in \mathcal{T}$ and each level j the construction gives a finite set Ch_j, combinations $e_{j,\alpha}$ with finite support for $\alpha \in \mathrm{Ch}_j$, a set Col_j of columns with a subset Ram_j, maps $S_j \colon \mathrm{Col}_j \to \mathrm{Col}_j$ and $K_j \colon \mathrm{Col}_j \to \mathrm{Col}_{j+1}$ preserving the subsets Ram, a set Fr_j of frames, reading functions $\mathrm{rd}_j(\alpha, \psi', y)$ on $\mathrm{U}(X_j)(\mathbb{A})$ for $\psi' \in \mathrm{Col}_j$ and $y \in \mathrm{Fr}_j$, a tail $\mathrm{tail}(\psi', F) \in \mathbb{C}$ for $\psi' \in \mathrm{Col}_m$ and $F \in \mathcal{C}_T$, and complex numbers $c_I(j), c_V(j), c_K(j), c_P$. The dependence on s is not written. Put

$$\mathrm{per}_j^{\rho}(\psi', G; y, v) = \sum_{\alpha \in \mathrm{Ch}_j} \mathcal{B}_j^{\rho}\big(\mathrm{rd}_j(\alpha, \psi', y),\ \mathrm{tr}(e_{j,\alpha})\, G(v)\big) \qquad (G \in \mathcal{C}_j,\ v \in \mathcal{S}_j), \tag{14.11}$$

$\mathrm{up}_0\psi' = \psi'$ and $\mathrm{up}_{k+1}\psi' = K_{2k+1}K_{2k}S_{2k}\,\mathrm{up}_k\psi'$ for $\psi' \in \mathrm{Col}_0$, and define the top and bottom constants

$$W_k = \prod_{l=0}^{k-1} c_V(2l)\,c_K(2l)\,c_K(2l+1), \qquad c_{\mathrm{top}} = W_{n+1}\,c_P, \qquad c_{\mathrm{bot}} = \prod_{j=0}^{m-1} c_I(j). \tag{14.12}$$

The lift ι_T has the factor $c_I(m-1)$, so c_{bot} is the product of the constants of all m lifts.

Hypothesis 14.17 (laws of the theta tower)**.** For every tuple the following hold.

(T1) (Lift law.) For $G \in \mathcal{C}_{j+1}$, $P \in \mathcal{P}_j$ and $v \in \mathcal{S}_{j+1}$, $\Theta_j(G(v);P) = c_I(j)\,(\iota_j G)(\mathrm{It}_j(P \otimes v))$, and the same holds for ι_T with $c_I(m-1)$. All $c_I(j)$ are nonzero.

(T2) (Double-step seesaw identity.) Let $j + 2(\sigma+1) + 2 = m$, $P_1 \in \mathcal{P}_j$, $P_2 \in \mathcal{P}_{j+1}$, and let a and a' be archimedean points of $\mathrm{U}(V_{j+1})$ and $\mathrm{U}(V_{j+2})$. There is a bivacuum kernel C, depending on $(j, \sigma, P_1, P_2, a, a')$ and not on the tuple, such that for $\psi' \in \mathrm{Ram}_j$, y, $G \in \mathcal{C}_{j+2}$ and v, the integrand of $\mathcal{B}_j^{\sigma+1}$ applied to the double lift $\Theta_j(\Theta_{j+1}(G(v);P_2);P_1)$, integrated first over the Jacobi quotient at level $j+1$ translated by a and then over the Bessel quotient at level $j+2$ translated by a', converges absolutely and equals

$$c_V(j) \sum_{\beta} \mathcal{B}_{j+2}^{\sigma}\big(\mathrm{two}_j(\mathrm{rd}_j(\cdot, S_j\psi', y), C)_\beta,\ \mathrm{tr}(e_{j+2,\beta})\,R_{a'}G(v)\big),$$

where two_j is the bilift of the reading functions through the kernels of levels j and $j+1$ composed with C.

(T3) (Kernel law.) For every bivacuum kernel C and every y there are coefficients $a(y')$, finitely many nonzero and independent of the tuple and of $\psi' \in \mathrm{Ram}_j$, with

$$\mathrm{two}_j(\mathrm{rd}_j(\cdot, \psi', y), C)_\beta = c_K(j)\,c_K(j+1) \sum_{y'} a(y')\,\mathrm{rd}_{j+2}\big(\beta, K_{j+1}K_j\psi', y'\big).$$

(T4) (Last-step identity.) For $j = m-2$, $P_1 \in \mathcal{P}_j$, $P_2 \in \mathcal{P}_T$ there is a bivacuum kernel C independent of the tuple with

$$\mathrm{per}_j^0\big(\psi', \Theta_j(\Theta_T(F(w);P_2);P_1); y\big) = c_V(j)\,\mathrm{perT}\big(\mathrm{two}_j(\mathrm{rd}_j(\cdot, S_j\psi', y), C), F(w)\big),$$

where $\mathrm{perT}(\phi, f) = \sum_\beta \int_{[\mathrm{U}(T)]} \mathrm{tr}(e_{T,\beta}) f(h)\,\phi_\beta(\epsilon(h))\,dh$.

(T5) (Top period.) For y and w there is $b(y,w) \in \mathbb{C}$, independent of the tuple, of $\psi' \in \mathrm{Ram}_m$ and of F, with $\mathrm{perT}(\mathrm{rd}_m(\cdot, \psi', y), F(w)) = c_P\,b(y,w)\,\mathrm{tail}(\psi', F)$.

(T6) (Unfolding of the double step.) If for every pair (a, a') of archimedean points the iterated integrals in (T2) of a finite family of tuples, with coefficients α_i, have values v_i with $\sum_i \alpha_i v_i = 0$, then $\sum_i \alpha_i\,\mathcal{B}_j^{\sigma+1}$ of the corresponding double lifts vanishes.

(T7) (Density.) The vectors $\mathrm{It}_j(P_1 \otimes \mathrm{It}_{j+1}(P_2 \otimes v))$, and the vectors $\mathrm{It}_j(P_1 \otimes \mathrm{It}_T(P_2 \otimes v))$, span a subspace on which a finite linear relation among the functionals $v \mapsto \mathrm{per}_j^{\rho}(\psi_i, G_i; y, v)$ implies the same relation on all of $\mathcal{S}_j$.

(T8) (Archimedean translation.) For an archimedean point a' of $\mathrm{U}(V_j)$ there is a linear map $\tau_{a'}$ of $\mathcal{S}_j$ with $R_{a'}G(v) = G(\tau_{a'}v)$ for every $G \in \mathcal{C}_j$ and every tuple, and $\mathcal{B}_j^{\rho}$ is linear in each argument.

Proposition 14.18 (one double step)**.** *Assume Hypothesis 14.17. Let $j + 2(\sigma+1) + 2 = m$, let $s_1, \dots, s_\ell$ be tuples, $\alpha_i \in \mathbb{C}$, $\psi_i \in \mathrm{Ram}_j$ and $G_i \in \mathcal{C}_{j+2}$, all for the tuple s_i. If*

$$\sum_i \alpha_i \, c_V(j) c_K(j) c_K(j+1) \, \mathrm{per}_{j+2}^{\sigma}\big(K_{j+1}K_jS_j\psi_i, G_i; y, v\big) = 0 \qquad \textit{for all } y, v,$$

then

$$\sum_i \alpha_i \, c_I(j) c_I(j+1) \, \mathrm{per}_{j}^{\sigma+1}\big(\psi_i, \iota_j\iota_{j+1}G_i; y, v\big) = 0 \qquad \textit{for all } y, v.$$

Proof. Fix y. By (T7) it suffices to prove the second identity at $v = \mathrm{It}_j(P_1 \otimes \mathrm{It}_{j+1}(P_2 \otimes w))$. Two applications of (T1) give

$$c_I(j)c_I(j+1)\,(\iota_j\iota_{j+1}G_i)(v) = \Theta_j\big(\Theta_{j+1}(G_i(w); P_2); P_1\big),$$

so the left side of the second identity is $\sum_i \alpha_i \, \mathrm{per}_j^{\sigma+1}$ of the double lifts. By (T6) it suffices to show, for fixed archimedean points (a, a'), that the iterated integrals v_i satisfy $\sum_i \alpha_i v_i = 0$. Let C be the kernel of (T2) for $(j, \sigma, P_1, P_2, a, a')$. Then (T2), (T3) applied to $S_j\psi_i \in \mathrm{Ram}_j$, and (T8) give

$$v_i = c_V(j)c_K(j)c_K(j+1) \sum_{y'} a(y') \, \mathrm{per}_{j+2}^{\sigma}\big(K_{j+1}K_jS_j\psi_i, G_i; y', \tau_{a'}w\big),$$

with $a(y')$ and $\tau_{a'}$ independent of i. Hence $\sum_i \alpha_i v_i = \sum_{y'} a(y') \cdot 0 = 0$ by the hypothesis at $(y', \tau_{a'}w)$. □

Proposition 14.19 (last step)**.** *Assume Hypothesis 14.17 and let $j = m - 2$. Let s_i, α_i, $\psi_i \in \mathrm{Ram}_j$ and $F_i \in \mathcal{C}_T$ be finitely many data. If*

$$\sum_i \alpha_i \, c_V(j) c_K(j) c_K(j+1) \, c_P \, \mathrm{tail}\big(K_{j+1}K_jS_j\psi_i, F_i\big) = 0,$$

then $\sum_i \alpha_i \, c_I(j)c_I(j+1) \, \mathrm{per}_j^0(\psi_i, \iota_j\iota_T F_i; y, v) = 0$ for all y, v.

Proof. By (T7) take $v = \mathrm{It}_j(P_1 \otimes \mathrm{It}_T(P_2 \otimes w))$. By (T1) the i-th term is

$$\alpha_i \, \mathrm{per}_j^0(\psi_i, \Theta_j(\Theta_T(F_i(w); P_2); P_1); y).$$

By (T4), (T3) and (T5) this equals

$$\sum_{y'} a(y') \, b(y', w) \, \alpha_i \, c_V(j)c_K(j)c_K(j+1) \, c_P \, \mathrm{tail}\big(K_{j+1}K_jS_j\psi_i, F_i\big).$$

The sum over i vanishes for each y' by hypothesis. □

Theorem 14.20 (transfer along the tower)**.** *Assume Hypothesis 14.17. Let s_i, α_i, $\psi_i \in \mathrm{Ram}_0$ and $F_i \in \mathcal{C}_T$ be finitely many data. If*

$$\sum_i \alpha_i \, c_{\mathrm{top}}(s_i) \, \mathrm{tail}\big(\mathrm{up}_{n+1}\psi_i, F_i\big) = 0,$$

then for all $y \in \mathrm{Fr}_0$ and $v \in \mathcal{S}_0$

$$\sum_i \alpha_i \, c_{\mathrm{bot}}(s_i) \, \mathrm{per}_0^n\big(\psi_i, \ \iota_0\iota_1 \cdots \iota_{m-2}\iota_T F_i; y, v\big) = 0.$$

Proof. Since S and K preserve the subsets Ram, $\mathrm{up}_k\psi_i \in \mathrm{Ram}_{2k}$ for $2k \le m$. Since $\mathrm{up}_{n+1}\psi_i = K_{2n+1}K_{2n}S_{2n}\,\mathrm{up}_n\psi_i$ and $c_{\mathrm{top}} = W_n\,c_V(2n)c_K(2n)c_K(2n+1)\,c_P$, Proposition 14.19 with $j = 2n$, columns $\mathrm{up}_n\psi_i$ and coefficients $\alpha_i W_n$ gives

$$\sum_i \alpha_i W_n\, c_I(2n)c_I(2n+1)\,\mathrm{per}_{2n}^0\big(\mathrm{up}_n\psi_i, \iota_{2n}\iota_T F_i; y, v\big) = 0. \tag{14.13}$$

We show by induction on k the following statement (A_k). If $2k + 2\rho + 2 = m$, $G_i \in \mathcal{C}_{2k}$ and $\sum_i \beta_i W_k\,\mathrm{per}_{2k}^{\rho}(\mathrm{up}_k\psi_i, G_i; y, v) = 0$ for all y, v, then $\sum_i \beta_i \prod_{j<2k} c_I(j)\,\mathrm{per}_0^{k+\rho}(\psi_i, \iota_0\cdots\iota_{2k-1}G_i; y, v) = 0$ for all y, v. For $k = 0$ there is nothing to prove. If (A_k) holds and the hypothesis of (A_{k+1}) holds with ρ, then $W_{k+1} = W_k\,c_V(2k)c_K(2k)c_K(2k+1)$, and Proposition 14.18 at $j = 2k$, $\sigma = \rho$ with coefficients $\beta_i W_k$ gives $\sum_i \beta_i W_k\,c_I(2k)c_I(2k+1)\,\mathrm{per}_{2k}^{\rho+1}(\mathrm{up}_k\psi_i, \iota_{2k}\iota_{2k+1}G_i; y, v) = 0$. Then (A_k) with $\rho + 1$ and coefficients $\beta_i c_I(2k)c_I(2k+1)$ gives (A_{k+1}). Finally (A_n) with $\rho = 0$, $\beta_i = \alpha_i c_I(2n)c_I(2n+1)$ and $G_i = \iota_{2n}\iota_T F_i$ applies to (14.13), and $\prod_{j<2n} c_I(j)\cdot c_I(2n)c_I(2n+1) = c_{\mathrm{bot}}$ because $m = 2n + 2$. □

The word u. Fix a word u. The tower attached to u has its tuples indexed by the set of tuples t of the chain datum, with the base tuple μ_0, and $\mathrm{perCol}_t(\varphi, F)$ denotes the period of the chain on the definite side at the tuple t, for $\varphi \in V_{u,v}$ and a K-rational vector F of the good part at t. By the definition of the period map, $(B_{u,v}\varphi)(\iota_0(q)) = \mathrm{perCol}_{\mu_0}(\varphi, q)$ for q in the good part at μ_0, and these vectors $\iota_0(q)$ span Y_v. Let $\mathcal{R}_0$ be the space of row forms at level 0, the span of the theta lifts of level 0 that the construction uses, and for F as above let $\mathcal{R}_F \in \mathcal{R}_0$ be the row form of F. Let $b \in \mathrm{U}(V_0)(\mathbb{A})$ be the translation of the Bessel datum of the word, with z and x_u as in Hypothesis 14.10 related to it by the pinning relation of the construction, and write

$$\mathfrak{b}_u(\varphi, \mathcal{R}; y, v) = \mathcal{B}_0^n\big(\mathrm{rd}_0^u(\varphi, y),\ R_b\,\mathcal{R}(v)\big)$$

for the local Bessel period at level 0 of the translated row form $\mathcal{R}$ against the column φ.

Hypothesis 14.21 (dictionaries of the tower)**.** There are maps $\mathrm{col}_0(t, \cdot)\colon V_{u,v} \to \mathrm{Col}_0$ and $\mathrm{col}_m\colon \mathrm{Col}_0 \to \mathrm{Col}_m$ such that the following hold for every tuple t.

(D1) $\mathrm{col}_0(t, \varphi) \in \mathrm{Ram}_0$ and $\mathrm{per}_0^n(\mathrm{col}_0(t, \varphi), G; y, v) = \mathfrak{b}_u(\varphi, G; y, v)$ for every $G \in \mathcal{C}_0$.

(D2) If $G \in \mathcal{C}_T$ is the leading map of F, then $\mathrm{tail}(\mathrm{col}_m\mathrm{col}_0(t, \varphi), G) = \mathrm{perCol}_t(\varphi, F)$.

(D3) For every F there is $G \in \mathcal{C}_T$, the leading map of F, with $\iota_0\cdots\iota_{m-2}\iota_T G = \mathcal{R}_F$.

(D4) Every $\mathcal{R} \in \mathcal{R}_0$ equals $\iota_0\cdots\iota_{m-2}\iota_T G$ for some $G \in \mathcal{C}_T$ at the base tuple μ_0.

(D5) There are a linear bijection S of $V_{u,v}$, $r > 0$ and $\varsigma \ne 0$ with $\|S\varphi\| = r\|\varphi\|$ and $\mathrm{tail}(\mathrm{up}_{n+1}\mathrm{col}_0(t, \varphi), G) = \varsigma\,\mathrm{tail}(\mathrm{col}_m\mathrm{col}_0(t, S\varphi), G)$ for all t, φ and G.

(D6) $c_{\mathrm{top}}(t) \ne 0$.

Corollary 14.22 (the seesaw relation)**.** *Assume Hypotheses 14.17 and 14.21. Put $U_u = S^{-1}$, $r_u = r^{-1}$ and*

$$c(t) = \frac{c_{\mathrm{top}}(t)\,\varsigma}{c_{\mathrm{bot}}(t)} \ne 0, \qquad p_{\mathcal{R}}(\varphi_0) = \frac{c_{\mathrm{top}}(\mu_0)}{c_{\mathrm{bot}}(\mu_0)}\,\mathrm{tail}\big(\mathrm{up}_{n+1}\mathrm{col}_0(\mu_0, S^{-1}\varphi_0), G_{\mathcal{R}}\big) \tag{14.14}$$

for $\mathcal{R} \in \mathcal{R}_0$, with $G_{\mathcal{R}}$ chosen by (D4). Then $\|U_u\varphi\| = r_u\|\varphi\|$, and for finitely many data $(\varphi_{0,i}, \mathcal{R}_i)$, (t_k, φ_k, F_k) and coefficients α_i, α'_k with

$$\sum_i \alpha_i \, p_{\mathcal{R}_i}(\varphi_{0,i}) + \sum_k \alpha'_k \, c(t_k) \, \mathrm{perCol}_{t_k}(\varphi_k, F_k) = 0$$

one has, for all y and v,

$$\sum_i \alpha_i \, \mathfrak{b}_u(U_u\varphi_{0,i}, \mathcal{R}_i; y, v) + \sum_k \alpha'_k \, \mathfrak{b}_u(U_u\varphi_k, \mathcal{R}_{F_k}; y, v) = 0.$$

Proof. The norm statement follows from $\|S\psi\| = r\|\psi\|$ with $\psi = S^{-1}\varphi$, and $c(t) \neq 0$ follows from (D6), $\varsigma \neq 0$ and (T1). For each k choose G_k by (D3). Apply Theorem 14.20 to the family consisting of the tuples μ_0 (indices i) and t_k (indices k), the columns $\mathrm{col}_0(\mu_0, S^{-1}\varphi_{0,i})$ and $\mathrm{col}_0(t_k, S^{-1}\varphi_k)$, the elements $G_{\mathcal{R}_i}$ and G_k, and the coefficients $\alpha_i / c_{\mathrm{bot}}(\mu_0)$ and $\alpha'_k / c_{\mathrm{bot}}(t_k)$. Its hypothesis is the displayed relation, because by (D5) and (D2)

$$\frac{c_{\mathrm{top}}(t_k)}{c_{\mathrm{bot}}(t_k)} \, \mathrm{tail}\big(\mathrm{up}_{n+1}\mathrm{col}_0(t_k, S^{-1}\varphi_k), G_k\big) = \frac{c_{\mathrm{top}}(t_k)\,\varsigma}{c_{\mathrm{bot}}(t_k)} \, \mathrm{perCol}_{t_k}(\varphi_k, F_k) = c(t_k) \, \mathrm{perCol}_{t_k}(\varphi_k, F_k),$$

and the i-terms are $\alpha_i p_{\mathcal{R}_i}(\varphi_{0,i})$ by definition. Its conclusion, after the factors c_{bot} cancel, is $\sum_i \alpha_i \, \mathrm{per}_0^n(\mathrm{col}_0(\mu_0, S^{-1}\varphi_{0,i}), \mathcal{R}_i; y, v) + \sum_k \alpha'_k \, \mathrm{per}_0^n(\mathrm{col}_0(t_k, S^{-1}\varphi_k), \mathcal{R}_{F_k}; y, v) = 0$, using (D3) and (D4), and (D1) turns this into the assertion. □

Remark 14.23 (passage to Hypothesis 14.10). Take X_u to consist of the pairs $(\varphi, F) \in V_{u,v} \times Y_v$ and the pairs $(\varphi_0, \mathcal{R})$ with $\mathcal{R} \in \mathcal{R}_0$, with $\lambda_u(\varphi, F) = (B_{u,v}\varphi)(F)$ and $\lambda_u(\varphi_0, \mathcal{R}) = p_{\mathcal{R}}(\varphi_0)/c(\mu_0)$. Only the base tuple $t = \mu_0$ is needed. A relation among the values of λ_u with complex F is a relation with K-rational F after expanding F in a K-basis of the good part, because both sides are linear in F, and multiplying the relation by $c(\mu_0)$ turns it into the hypothesis of Corollary 14.22 with the same coefficients. The conclusion of the corollary is the conclusion of Hypothesis 14.10, once the local Bessel period $\mathfrak{b}_u(U_u\varphi, \mathcal{R}; y, v)$ is identified with $\mathrm{per}(T_x(\xi), R_{(y,x_u)}R_z\overline{\theta_x})$ through the pinning relation between b, z and x_u and the row dictionary $\mathcal{R}_F \leftrightarrow \mathbf{s}(F)$ of Hypothesis 14.11. Lemma 14.8 uses only relations with one or two terms, so Hypothesis 14.10 may be weakened to such relations without changing Theorem 14.15. The anchor of Lemma 14.13 may be any $x_0 = (\varphi_0, \mathcal{R})$ with $\mu_u(x_0) \neq 0$. The exact values of c_{top}, c_{bot} and ς do not enter Theorem 14.3, because the number $a_u = r_u/\kappa_u$ absorbs them. Only $c(\mu_0) \neq 0$ is used there, while the unit property of $c(t)/c(\mu_0)$ at the primes of the chain is used for the integral structure of the period maps.

Lemma 14.24 (rationality)**.** *Assume that $B_u(\varphi)(y) = \mathrm{Pt}_{\mu_0}(\varphi, q_y)$ for $\varphi \in V_u$ and $y \in Y$, where Pt_{μ_0} is the K-bilinear period of the chain on the definite side at the base tuple and q_y is the K-rational vector of the good part with $\iota_0(q_y) = y$. Assume also $\mathrm{perCol}_{\mu_0}(\varphi \otimes 1, q \otimes 1) = \sigma_v(\mathrm{Pt}_{\mu_0}(\varphi, q))$ and $(B_{u,v}\varphi)(\iota_0 q) = \mathrm{perCol}_{\mu_0}(\varphi, q)$. Then Hypothesis 14.12 holds.*

Proof. For $\varphi \in V_u$ and $y \in Y$, $(B_{u,v}(\varphi \otimes 1))(y \otimes 1) = \mathrm{perCol}_{\mu_0}(\varphi \otimes 1, q_y \otimes 1) = \sigma_v(\mathrm{Pt}_{\mu_0}(\varphi, q_y)) = \sigma_v(B_u(\varphi)(y))$, which lies in $\sigma_v(K)$. □

14.2.5 The double-step seesaw identity

This subsection proves the double-step identity (T2) of Hypothesis 14.17. The identity is an iterated integral over two quotients of finite adelic groups. We show that the integral collapses to an integral over the finitely many places of the tuple, with a volume factor κ_j, and we identify the result with the bilift of the reading functions through one kernel C that does not depend on the tuple. The analytic unfolding over the archimedean and automorphic variables is not part of (T2). It is item (T6), where rapid decay of cusp forms enters.

Quotient integrals. Let $\mathbf{G}$ be a second countable locally compact group with a left Haar measure μ, and let $R \subset \mathbf{G}$ be a closed subgroup with a left Haar measure ρ. We say that a function $f\colon \mathbf{G} \to \mathbb{C}$ has the *absolutely convergent quotient integral* z over $R\backslash\mathbf{G}$, and write $\int^{\mathrm{abs}}_{R\backslash\mathbf{G}} f = z$, if there is $F \in L^1(\mathbf{G},\mu)$ such that for every $g \in \mathbf{G}$ the function $r \mapsto F(rg)$ is ρ-integrable, $f(g) = \int_R F(rg)\,d\rho(r)$, and $z = \int_{\mathbf{G}} F\,d\mu$. For two such pairs $(\mathbf{G}_1, R_1)$, $(\mathbf{G}_2, R_2)$ and a function I on $\mathbf{G}_1 \times \mathbf{G}_2$, we say that I has the *absolutely convergent iterated quotient integral* z if there is $f_1\colon \mathbf{G}_1 \to \mathbb{C}$ with $\int^{\mathrm{abs}}_{R_1\backslash\mathbf{G}_1} f_1 = z$ and $\int^{\mathrm{abs}}_{R_2\backslash\mathbf{G}_2} I(q_1,\cdot) = f_1(q_1)$ for every $q_1 \in \mathbf{G}_1$. These are the formal notions `IsQuotIntL1` and `Iter2`.

Lemma 14.25 (uniqueness)**.** *Assume that $\mathbf{G}$ and R are unimodular. If $\int^{\mathrm{abs}}_{R\backslash\mathbf{G}} f = z$ and $\int^{\mathrm{abs}}_{R\backslash\mathbf{G}} f = z'$, then $z = z'$. The value is linear in f, it vanishes for $f = 0$, and $\int^{\mathrm{abs}}_{R\backslash\mathbf{G}} R_b f = \int^{\mathrm{abs}}_{R\backslash\mathbf{G}} f$ for every $b \in \mathbf{G}$, where $R_b f(g) = f(gb)$.*

Proof. Since the modular functions of $\mathbf{G}$ and R are both trivial, there is a right $\mathbf{G}$-invariant Radon measure $\dot\mu$ on $R\backslash\mathbf{G}$ with $\int_{\mathbf{G}} F\,d\mu = \int_{R\backslash\mathbf{G}} \int_R F(rg)\,d\rho(r)\,d\dot\mu(g)$ for every $F \in L^1(\mathbf{G})$ [26, §2.6]. Hence $z = \int_{R\backslash\mathbf{G}} f\,d\dot\mu$ depends only on f, and linearity follows. If F is a witness for f, then $R_b F$ is a witness for $R_b f$, and $\int R_b F\,d\mu = \int F\,d\mu$ because μ is right invariant. □

Lemma 14.26 (collapse to a factor)**.** *Let $\mathbf{G}$ be a second countable locally compact unimodular group, and let $G_1, G_2 \subset \mathbf{G}$ be closed unimodular subgroups whose elements commute with each other and for which $(a,b) \mapsto ab$ is a homeomorphism $G_1 \times G_2 \to \mathbf{G}$. Let $R \subset \mathbf{G}$ be a closed unimodular subgroup such that $ab \in R$ with $a \in G_1$, $b \in G_2$ implies $a, b \in R$. Put $R_i = R \cap G_i$. Let $K_i \subset G_i$ be compact open subgroups and $K = K_1K_2$. Let μ, ρ, μ_1, ρ_1 be Haar measures on $\mathbf{G}, R, G_1, R_1$ with $\mu_1(K_1) = 1$ and $\rho_1(R_1 \cap K_1) = 1$. Let $f\colon \mathbf{G} \to \mathbb{C}$ satisfy*

(i) $f(rg) = f(g)$ *for $r \in R$ and $g \in \mathbf{G}$;*

(ii) $f(gk) = f(g)$ *for $k \in K_2$ and $g \in \mathbf{G}$;*

(iii) $f(g) \neq 0$ *implies $g \in RG_1K_2$.*

If $\int^{\mathrm{abs}}_{R_1\backslash G_1} f|_{G_1} = z$, then

$$\int^{\mathrm{abs}}_{R\backslash\mathbf{G}} f = \frac{\mu(K)}{\rho(R\cap K)}\, z.$$

Proof. The product map is a continuous homomorphism because G_1 and G_2 commute, so it is an isomorphism of topological groups $G_1 \times G_2 \cong \mathbf{G}$. By the hypothesis on R, it restricts to an isomorphism $R_1 \times R_2 \cong R$, and R_1, R_2 are closed. The modular function of $R_1 \times R_2$ is the product of those of R_1 and R_2, so R_1 and R_2 are unimodular. By uniqueness of Haar measure

there are Haar measures μ_2 on G_2 and ρ_2 on R_2 with $\mu = \mu_1 \otimes \mu_2$ and $\rho = \rho_1 \otimes \rho_2$ under these isomorphisms. The group $R_2 \cap K_2$ is compact and open in R_2, so $0 < \rho_2(R_2 \cap K_2) < \infty$, and likewise $0 < \mu_2(K_2) < \infty$.

Step 1. We show $f(ab) = f(a)\,\mathbf{1}_{R_2K_2}(b)$ for $a \in G_1$ and $b \in G_2$. If $b = r_2k$ with $r_2 \in R_2$ and $k \in K_2$, then $ab = r_2ak$, so (ii) and (i) give $f(ab) = f(r_2a) = f(a)$. If $f(ab) \neq 0$, then (iii) gives $ab = r_1r_2xk$ with $r_i \in R_i$, $x \in G_1$, $k \in K_2$, that is $ab = (r_1x)(r_2k)$, and uniqueness of the decomposition gives $b = r_2k \in R_2K_2$.

Step 2. Let $F_1 \in L^1(G_1)$ be a witness for $f|_{G_1}$, and put $F(ab) = F_1(a)\,\mathbf{1}_{K_2}(b)/\rho_2(R_2 \cap K_2)$. Then $F \in L^1(\mathbf{G})$ and $\int_{\mathbf{G}} F\,d\mu = z\,\mu_2(K_2)/\rho_2(R_2 \cap K_2)$. For $g = ab$ and $r = r_1r_2$ we have $F(rg) = F_1(r_1a)\mathbf{1}_{K_2}(r_2b)/\rho_2(R_2 \cap K_2)$. The second factor is the indicator of the set $R_2 \cap K_2b^{-1}$, of measure at most $\rho_2(R_2 \cap K_2)$, so $r \mapsto F(rg)$ is ρ-integrable and Fubini's theorem gives

$$\int_R F(rg)\,d\rho(r) = f(a)\,\frac{\rho_2(R_2 \cap K_2b^{-1})}{\rho_2(R_2 \cap K_2)}.$$

If $b = r_0k \in R_2K_2$, then $R_2 \cap K_2b^{-1} = (R_2 \cap K_2)r_0^{-1}$, which has measure $\rho_2(R_2 \cap K_2)$ because ρ_2 is right invariant. If $R_2 \cap K_2b^{-1}$ is nonempty, then $b \in R_2K_2$. So the right side equals $f(a)\mathbf{1}_{R_2K_2}(b)$, which is $f(ab)$ by Step 1. Hence $\int^{\mathrm{abs}}_{R\backslash\mathbf{G}} f = z\,\mu_2(K_2)/\rho_2(R_2 \cap K_2)$.

Step 3. By the hypothesis on R, $R \cap K = (R_1 \cap K_1)(R_2 \cap K_2)$. Hence $\mu(K) = \mu_1(K_1)\mu_2(K_2) = \mu_2(K_2)$ and $\rho(R \cap K) = \rho_1(R_1 \cap K_1)\rho_2(R_2 \cap K_2) = \rho_2(R_2 \cap K_2)$. □

The double-step integrand. We use the notation of Subsection 14.2.4. For $0 \leq i < m$ the pair $(\mathrm{U}(V_i), \mathrm{U}(V_{i+1}))$ acts on functions on $W_i(\mathbb{A})$, $W_i = \mathrm{Hom}(V_{i+1}, V_i)$, by the Weil representation ω_i of the tuple attached to ψ and to the splitting characters of the tuple. The pairs $(\mathrm{U}(X_i), \mathrm{U}(X_{i+1}))$ of definite spaces act on functions on $W_i'(\mathbb{A})$ in the same way, by ω_i'. For a finitely supported function e on $W_i(\mathbb{A}_f)$ put $\mathrm{comb}(e)\Phi = \sum_u e(u)\,\mathrm{heis}(u)\Phi$, where $\mathrm{heis}(u)$ is the Heisenberg translation by u with the character ψ. We say that e is *supported at* a set P of places if every u in its support has trivial components outside P. For self-dual lattices $N_i \subset V_i$ and a set P of finite places, a theta function Φ on $W_i(\mathbb{A})$ is *vacuum at P* if it is invariant under the Heisenberg action of the lattice $\mathrm{Hom}(N_{i+1}, N_i) \otimes \prod_{p\in P} \mathcal{O}_p$, so that its component at $p \in P$ is a multiple of the characteristic function of that lattice. For finitely supported functions e on $W_i'(\mathbb{A}_f)$ and e' on $W_{i+1}'(\mathbb{A}_f)$ and a function C on $W_i'(\mathbb{A}) \times W_{i+1}'(\mathbb{A})$ write $\mathrm{biComb}(e, e')C(u, v) = \mathrm{comb}(e)\big(u' \mapsto \mathrm{comb}(e')C(u', \cdot)(v)\big)(u)$.

Fix j and σ with $j + 2(\sigma + 1) + 2 = m$. The Bessel datum of rank $\sigma + 1$ at level j, the Fourier–Jacobi datum of rank $\sigma + 1$ at level $j + 1$ and the Bessel datum of rank σ at level $j + 2$ give closed subgroups of $\mathrm{U}(V_{j+1})(\mathbb{A})$ and $\mathrm{U}(V_{j+2})(\mathbb{A})$. Their intersections with the finite adelic groups $\mathbf{G}_{j+1} = \mathrm{U}(V_{j+1})(\mathbb{A}_f)$ and $\mathbf{G}_{j+2} = \mathrm{U}(V_{j+2})(\mathbb{A}_f)$ are denoted R_J and R_B, with Haar measures ρ_J, ρ_B, and μ_{j+1}, μ_{j+2} are the Haar measures of $\mathbf{G}_{j+1}, \mathbf{G}_{j+2}$ fixed by the construction. For theta functions Φ_1 on $W_j(\mathbb{A})$ and Φ_2 on $W_{j+1}(\mathbb{A})$, a function ϕ on $\mathrm{U}(X_j)(\mathbb{A})$ and points g, h_1, h_2 of $\mathrm{U}(V_j)(\mathbb{A})$, $\mathrm{U}(V_{j+1})(\mathbb{A})$, $\mathrm{U}(V_{j+2})(\mathbb{A})$, let

$$\mathbf{H}(\Phi_1, \Phi_2; \phi; g, h_1, h_2) = \mathrm{hand}_{\mathrm{FB}}\Big(\mathrm{hand}_{\mathrm{BF}}\big(\phi, \mathrm{flat}_{\mathrm{BF}}(\omega_j(g, h_1)\Phi_1)\big),\ \mathrm{flat}_{\mathrm{FB}}(\omega_{j+1}(h_1, h_2)\Phi_2)\Big)$$

be the function on $\mathrm{U}(X_{j+2})(\mathbb{A})$ obtained by the Bessel-to-Fourier–Jacobi step at level j followed by the Fourier–Jacobi-to-Bessel step at level $j + 1$ (formal `Chain.hand`). Here $\mathrm{flat}_{\mathrm{BF}}$ and $\mathrm{flat}_{\mathrm{FB}}$ are the partial integrations of a theta function along the unipotent directions of the two flags, taken

with Haar measures of covolume one, $\mathrm{hand}_{\mathrm{BF}}$ integrates ϕ over $[\mathrm{U}(X_j)]$ against the restricted kernel, and $\mathrm{hand}_{\mathrm{FB}}$ contracts the result with the second kernel. The map $\mathbf{H}$ is bilinear in (Φ_1, Φ_2).

The contracted kernel of two theta functions Ψ_1, Ψ_2 is the function $\mathrm{ctr}(\Psi_1, \Psi_2)$ on $W_j'(\mathbb{A}) \times W_{j+1}'(\mathbb{A})$ given by the integral over the link variable of the product $\mathrm{flat}_{\mathrm{FB}}\Psi_2 \cdot \mathrm{flat}_{\mathrm{BF}}\Psi_1$ (formal `Comm.ctr`). For a function C on $W_j'(\mathbb{A}) \times W_{j+1}'(\mathbb{A})$ and a function ϕ on $\mathrm{U}(X_j)(\mathbb{A})$ the bilift is

$$\mathrm{bl}_j(\phi, C)(h_2) = \int_{[\mathrm{U}(X_{j+1})]} \int_{[\mathrm{U}(X_j)]} \phi(h_0) \left(\omega_{j+1}'(h_1, h_2)\big(v \mapsto \omega_j'(h_0, h_1) C(\cdot, v)(0)\big)\right)(0)\, dh_0\, dh_1,$$

and for a family $\phi = (\phi_\alpha)_{\alpha \in \mathrm{Ch}_j}$ the double kernel lift of Hypothesis 14.17 is $\mathrm{two}_j(\phi, C)_\beta = \sum_{\alpha,\gamma} \mathrm{bl}_j\big(\phi_\alpha, \mathrm{biComb}(\mathrm{dd}_j(\alpha, \gamma), \mathrm{dd}_{j+1}(\gamma, \beta))C\big)$ for $\beta \in \mathrm{Ch}_{j+2}$, where dd_i are the kernel combinations of the tuple. A *bivacuum kernel* is a continuous C that is a theta function vacuum at every finite place in each variable, and it is *real* if it is moreover smooth in the archimedean directions of both variables.

For a tuple s, archimedean points $a \in \mathrm{U}(V_{j+1})_\infty$, $a' \in \mathrm{U}(V_{j+2})_\infty$, polynomial Fock vectors P_1, P_2, a column $\psi' \in \mathrm{Ram}_j$, a frame y, $G \in \mathcal{C}_{j+2}$ and $v \in \mathcal{S}_{j+2}$, the *double-step integrand* is the function on $\mathbf{G}_{j+1} \times \mathbf{G}_{j+2}$

$$I_s(q_1, q_2) = \sum_{\alpha \in \mathrm{Ch}_j} \sum_{g} e_{j,\alpha}(g)\, \mathcal{B}_{j+2}^{\sigma}\Big(\mathbf{H}\big(\mathrm{pt}_j(P_1), \mathrm{pt}_{j+1}(P_2); \mathrm{rd}_j(\alpha, \psi', y); g, aq_1, a'q_2\big),\ R_{a'q_2} G(v)\Big), \tag{14.15}$$

where $\mathrm{pt}_i(P)$ is the theta function of the tuple at level i attached to P (formal `Tup.integrand`). It is the integrand of $\mathcal{B}_j^{\sigma+1}$ on the double lift after the two unfoldings, as a function of the two finite adelic variables.

Hypothesis 14.27 (general laws of the Weil representation)**.** (W1) (Disjoint places.) Heisenberg combinations supported at disjoint sets of places commute. A combination supported at P maps theta functions vacuum at a set P' disjoint from P to theta functions vacuum at P', and it commutes with $\omega_i(g, h)$ for g, h with trivial components at P. A theta function is vacuum at $P \cup P'$ if and only if it is vacuum at P and at P'.

(W2) (Normalized steps.) The maps $\mathrm{flat}_{\mathrm{BF}}$ and $\mathrm{flat}_{\mathrm{FB}}$ on theta functions depend only on the Bessel and Fourier–Jacobi data and not on the remaining data of the step of the tuple. Consequently $\mathrm{ctr}(\Psi_1, \Psi_2)$ does not depend on the tuple.

(W3) (Representation.) ω_i is a representation of $\mathrm{U}(V_i)(\mathbb{A}) \times \mathrm{U}(V_{i+1})(\mathbb{A})$, and archimedean and finite adelic points commute.

(W4) (Bi-action.) For theta data on (X_j, X_{j+1}) and (X_{j+1}, X_{j+2}) and a bikernel C that is a theta function in each variable, the two actions ω_j', ω_{j+1}' on the two variables of C commute and preserve this property (formal `BiAct`, Lean theorem `seeSawWeilBiAction_commutes`).

(W5) (Flat maps.) The flat maps send theta functions to theta functions and their defining integrals converge absolutely (formal `FlatLaw`, Lean theorem `flatLaw`).

(W6) (Seesaw exchange.) For ϕ continuous and left $\mathrm{U}(X_j)(F)$-invariant and theta functions Ξ_1, Ξ_2 on the reduced spaces, $\mathrm{hand}_{\mathrm{FB}}\big(\mathrm{hand}_{\mathrm{BF}}(\phi, \Xi_1), \Xi_2\big) = \mathrm{bl}_j\big(\phi, \mathrm{contr}(\Xi_1, \Xi_2)\big)$, where contr is the contraction over the link variable, so that $\mathrm{ctr}(\Psi_1, \Psi_2) = \mathrm{contr}(\mathrm{flat}_{\mathrm{BF}}\Psi_1, \mathrm{flat}_{\mathrm{FB}}\Psi_2)$ (formal `ContrLaw`, Lean theorem `contrLaw`). Its proof is Fubini's theorem for a bounded

continuous integrand on the product of a compact fundamental domain of $[\mathrm{U}(X_j)]$ and a fundamental domain of covolume one of the link lattice, followed by the factorization of the Weil representation of an orthogonal sum $V^0 = A \oplus L$ as the tensor product of the Weil representations of A and L, with splitting characters multiplying (`weil_datum_product_compatibility`).

(W7) (Linearity.) $\mathcal{B}^{\sigma}_{j+2}(c\phi', c'f) = cc'\,\mathcal{B}^{\sigma}_{j+2}(\phi', f)$ and $\mathbf{H}(c\Phi_1, c'\Phi_2; \dots) = cc'\,\mathbf{H}(\Phi_1, \Phi_2; \dots)$.

Hypothesis 14.28 (local laws of the tuple)**.** There are data common to all tuples, namely finite disjoint sets $g\mathrm{P}$ and $t\mathrm{P}$ of finite places, self-dual lattices $N_i \subset V_i$, base points $b_i \in \mathbf{G}_i$, compact open subgroups J_i of $\mathrm{U}(V_i)(F_{g\mathrm{P}})$, theta functions $\Psi_i(P; a, a')$ on $W_i(\mathbb{A})$ vacuum at every finite place, combinations e_i^G, d_i^G supported at $g\mathrm{P}$, combinations d_i^T and an operator $\mathrm{op}_j = \sum_k \mathrm{biComb}(u_k, u_k')$ supported at $t\mathrm{P}$, a number $c_G(j)$, a set RamAd_j of continuous automorphic functions on $\mathrm{U}(X_j)(\mathbb{A})$ with a map Sad_j of it into itself. Put $K_i = \{\kappa k : \kappa \text{ stabilizes } N_i \text{ away from } g\mathrm{P},\ k \in J_i\}$ and

$$\kappa_j = \frac{\mu_{j+1}(K_{j+1})}{\rho_J(R_J \cap K_{j+1})} \cdot \frac{\mu_{j+2}(K_{j+2})}{\rho_B(R_B \cap K_{j+2})}.$$

For each tuple s there are a finite set $Q = Q_s$ of finite places disjoint from $g\mathrm{P} \cup t\mathrm{P}$, numbers sc_i, $c_Q(j)$, $c_T(j)$, combinations e_i^Q, $d_i^Q(\alpha, \gamma)$ supported at Q, and finitely supported functions $\varepsilon_{i,\alpha}$ on $\mathrm{U}(V_i)(F_Q)$, such that the following hold.

(L1) (Vacuum form.) $\omega_i(ab_i, a'b_{i+1})\,\mathrm{pt}_i(P) = \mathrm{sc}_i\,\mathrm{comb}(e_i^G)\,\mathrm{comb}(e_i^Q)\,\Psi_i(P; a, a')$ for archimedean a, a'. Moreover $e_{i,\alpha}(g) = \varepsilon_{i,\alpha}(gb_i^{-1})$, the combination $\mathrm{dd}_i(\alpha, \gamma)$ acts on theta functions as $\mathrm{comb}(d_i^Q(\alpha, \gamma))\,\mathrm{comb}(d_i^G)\,\mathrm{comb}(d_i^T)$, for $\psi' \in \mathrm{Ram}_j$ one has $\mathrm{rd}_j(\alpha, \psi', y) \in \mathrm{RamAd}_j \cap \mathrm{FX}_{j,\alpha}$ and $\mathrm{rd}_j(\alpha, S_j\psi', y) = \mathrm{Sad}_j(\mathrm{rd}_j(\alpha, \psi', y))$, and for $G \in \mathcal{C}_i$ and archimedean a the function $R_{ab_i}G(v)$ is automorphic, lies in FY_i and is invariant under the stabilizer of N_i at the places outside $g\mathrm{P} \cup Q$. Here $\mathrm{FX}_{j,\alpha}$ and FY_i are sets of functions given with the tuple (formal `VTup.FX`, `VTup.FY`).

(L2) (Cells.) Let Ψ_1, Ψ_2 be theta functions vacuum at every finite place, put $\Phi_i^0 = \mathrm{comb}(e_{j+i-1}^G)\mathrm{comb}(e_{j+i-1}^Q)\Psi_i$, let $\phi_\alpha \in \mathrm{RamAd}_j$, let $f = R_{a'b_{j+2}}G(v)$ and

$$I^0(q_1, q_2) = \sum_\alpha \sum_g \varepsilon_{j,\alpha}(g)\,\mathcal{B}^{\sigma}_{j+2}\big(\mathbf{H}(\Phi_1^0, \Phi_2^0; \phi_\alpha; g, q_1, q_2), R_{q_2}f\big).$$

Write $\mathbf{G}_{i,Q}$ for the points of $\mathrm{U}(V_i)$ at Q and $K_i = K_{i,Q}K_i'$ with $K_{i,Q}$ the stabilizer of N_i at Q and K_i' the part of K_i away from Q. Then I^0 is left invariant under R_J in q_1 and under R_B in q_2, right invariant under K_{j+1}' in q_1 and under K_{j+2}' in q_2, and $I^0(q_1, q_2) \neq 0$ implies $q_1 \in R_J\mathbf{G}_{j+1,Q}K_{j+1}'$, and also $q_2 \in R_B\mathbf{G}_{j+2,Q}K_{j+2}'$ when $q_1 \in \mathbf{G}_{j+1,Q}$.

(L3) (Twist places.) Let $\rho_{J,Q}, \mu_{j+1,Q}, \rho_{B,Q}, \mu_{j+2,Q}$ be Haar measures on $R_J \cap \mathbf{G}_{j+1,Q}$, $\mathbf{G}_{j+1,Q}$, $R_B \cap \mathbf{G}_{j+2,Q}$, $\mathbf{G}_{j+2,Q}$ giving $K_{j+1,Q}$, $K_{j+2,Q}$ and their intersections with R_J, R_B the volume one. For theta functions Ψ_1', Ψ_2' vacuum at Q, continuous automorphic $\phi_\alpha \in \mathrm{FX}_{j,\alpha}$ and automorphic $f \in \mathrm{FY}_{j+2}$, the function

$$(q_1, q_2) \mapsto \sum_\alpha \sum_g \varepsilon_{j,\alpha}(g)\,\mathcal{B}^{\sigma}_{j+2}\big(\mathbf{H}(\mathrm{comb}(e_j^Q)\Psi_1', \mathrm{comb}(e_{j+1}^Q)\Psi_2'; \phi_\alpha; g, q_1, q_2), R_{q_2}f\big)$$

on $\mathbf{G}_{j+1,Q} \times \mathbf{G}_{j+2,Q}$ has the absolutely convergent iterated quotient integral

$$c_Q(j) \sum_{\beta} \mathcal{B}^{\sigma}_{j+2}\Big(\sum_{\alpha,\gamma} \mathrm{bl}_j\big(\phi_\alpha, \mathrm{biComb}(d_j^Q(\alpha,\gamma), d_{j+1}^Q(\gamma,\beta))\,\mathrm{ctr}(\Psi_1', \Psi_2')\big),\ \mathrm{tr}(\varepsilon_{j+2,\beta}) f \Big).$$

(L4) (Level places.) For theta functions Ψ_1, Ψ_2 vacuum at $g\mathrm{P}$, $\mathrm{ctr}(\mathrm{comb}(e_j^G)\Psi_1, \mathrm{comb}(e_{j+1}^G)\Psi_2) = c_G(j)\,\mathrm{biComb}(d_j^G, d_{j+1}^G)\,\mathrm{ctr}(\Psi_1, \Psi_2)$.

(L5) (Places of $t\mathrm{P}$.) For theta functions Ψ_1, Ψ_2 vacuum at $t\mathrm{P}$, combinations d, d' supported away from $t\mathrm{P}$ and $\phi \in \mathrm{RamAd}_j$,

$$\mathrm{bl}_j\big(\phi, \mathrm{biComb}(d,d')\,\mathrm{ctr}(\Psi_1,\Psi_2)\big) \\ = c_T(j)\,\mathrm{bl}_j\big(\mathrm{Sad}_j\phi,\ \mathrm{biComb}(d,d')\,\mathrm{biComb}(d_j^T, d_{j+1}^T)\,\mathrm{op}_j\,\mathrm{ctr}(\Psi_1,\Psi_2)\big),$$

and $\mathrm{op}_j\,\mathrm{ctr}(\Psi_1,\Psi_2)$ is a real bivacuum kernel when Ψ_1, Ψ_2 are vacuum at every finite place.

(L6) (Normalization.) $c_V(j) = \kappa_j\,\mathrm{sc}_j\,\mathrm{sc}_{j+1}\,c_Q(j)\,c_G(j)\,c_T(j)$.

Theorem 14.29 (the double-step identity (T2))**.** *Assume Hypotheses 14.27 and 14.28. Let $j + 2(\sigma+1) + 2 = m$, let P_1, P_2 be polynomial Fock vectors and $a \in \mathrm{U}(V_{j+1})_\infty$, $a' \in \mathrm{U}(V_{j+2})_\infty$. Put $\Psi_1 = \Psi_j(P_1; 1, a)$, $\Psi_2 = \Psi_{j+1}(P_2; a, a')$ and $C = \mathrm{op}_j\,\mathrm{ctr}(\Psi_1,\Psi_2)$. Then C is a real bivacuum kernel that depends only on $(j, \sigma, P_1, P_2, a, a')$ and not on the tuple. For every tuple s, every $\psi' \in \mathrm{Ram}_j$, y, $G \in \mathcal{C}_{j+2}$ and v, the integrand* (14.15) *has an absolutely convergent iterated quotient integral over $R_J \backslash \mathbf{G}_{j+1}$ and then $R_B \backslash \mathbf{G}_{j+2}$, equal to*

$$c_V(j) \sum_{\beta \in \mathrm{Ch}_{j+2}} \mathcal{B}^{\sigma}_{j+2}\Big(\mathrm{two}_j\big(\mathrm{rd}_j(\cdot, S_j\psi', y), C\big)_\beta,\ \mathrm{tr}(e_{j+2,\beta})\, R_{a'} G(v) \Big), \tag{14.16}$$

with $c_V(j) = c_V(j)(s) = \kappa_j\,\mathrm{sc}_j(s)\,\mathrm{sc}_{j+1}(s)\,c_Q(j)(s)\,c_G(j)\,c_T(j)(s)$ and $\kappa_j > 0$ independent of s.

Proof. Step 1, the kernel. The functions Ψ_1, Ψ_2 belong to the common data, and by (W2) so does $\mathrm{ctr}(\Psi_1,\Psi_2)$. Hence C depends only on $(j, \sigma, P_1, P_2, a, a')$. It is a real bivacuum kernel by (L5). The groups K_{j+1} and K_{j+2} are compact and open, and their intersections with R_J and R_B are compact open subgroups of R_J and R_B, so the four volumes in κ_j are positive and finite and $\kappa_j > 0$. It does not involve the tuple.

Step 2, the twist places. Fix s, ψ', y, G, v, and put $\phi_\alpha = \mathrm{rd}_j(\alpha, \psi', y)$, which lies in $\mathrm{RamAd}_j \cap \mathrm{FX}_{j,\alpha}$ by (L1), and $f = R_{a' b_{j+2}} G(v)$, which is automorphic and lies in FY_{j+2} by (L1). The functions Ψ_i are vacuum at every finite place, so by (W1) the functions $\Psi_i' = \mathrm{comb}(e_{j+i-1}^G)\Psi_i$ are vacuum at Q, because Q and $g\mathrm{P}$ are disjoint, and $\mathrm{comb}(e^Q)\Psi_i' = \Phi_i^0$ in the notation of (L2), because combinations at disjoint places commute. By (L3) the restriction of the function I^0 of (L2) to $\mathbf{G}_{j+1,Q} \times \mathbf{G}_{j+2,Q}$ has an absolutely convergent iterated quotient integral z_Q for the measures normalized at Q, with z_Q the value displayed in (L3) for $\mathrm{ctr}(\Psi_1', \Psi_2')$. Let f_Q be the corresponding inner function on $\mathbf{G}_{j+1,Q}$.

Step 3, collapse of the inner integral. Apply Lemma 14.26 to $\mathbf{G} = \mathbf{G}_{j+2}$, $G_1 = \mathbf{G}_{j+2,Q}$, G_2 the points away from Q, $R = R_B$, $K_1 = K_{j+2,Q}$, $K_2 = K_{j+2}'$. The groups involved are the finite adelic points of unitary groups, which are unimodular, and the Bessel and Fourier–Jacobi subgroups, which are semidirect products of a unipotent group by a unitary group and are unimodular by

the argument of Lemma 14.4. The finite adelic unitary group is the restricted product of its local groups, which gives the product decomposition and the splitting of R_B, since R_B is the group of finite adelic points of an F-subgroup. For $q_1 \in \mathbf{G}_{j+1,Q}$ the function $I^0(q_1,\cdot)$ satisfies (i) to (iii) by (L2), so its quotient integral over $R_B\backslash\mathbf{G}_{j+2}$ converges absolutely with value $\nu_{j+2}f_Q(q_1)$, where $\nu_{j+2} = \mu_{j+2}(K_{j+2})/\rho_B(R_B \cap K_{j+2})$. For an arbitrary $q_1 \in \mathbf{G}_{j+1}$, either $I^0(q_1,\cdot) = 0$, and the inner integral is 0, or by (L2) $q_1 = r\,x\,k$ with $r \in R_J$, $x \in \mathbf{G}_{j+1,Q}$, $k \in K'_{j+1}$, and $I^0(q_1,\cdot) = I^0(x,\cdot)$ by the invariances of (L2). Let $f_1(q_1)$ be the value of the inner integral. By Lemma 14.25 it is well defined, and the invariances and support of I^0 in q_1 pass to f_1. On $\mathbf{G}_{j+1,Q}$ one has $f_1 = \nu_{j+2}f_Q$.

Step 4, collapse of the outer integral. Lemma 14.26, applied to $\mathbf{G}_{j+1}$, R_J, $K_{j+1} = K_{j+1,Q}K'_{j+1}$ and the function f_1, shows that f_1 has the absolutely convergent quotient integral $\nu_{j+1}\nu_{j+2}z_Q = \kappa_j z_Q$ over $R_J\backslash\mathbf{G}_{j+1}$, with $\nu_{j+1} = \mu_{j+1}(K_{j+1})/\rho_J(R_J \cap K_{j+1})$. So I^0 has the absolutely convergent iterated quotient integral $\kappa_j z_Q$.

Step 5, base points. For $p_1 \in \mathbf{G}_{j+1}$, $p_2 \in \mathbf{G}_{j+2}$ and g' in the support of $\varepsilon_{j,\alpha}$, (W3) gives $(g'b_j, a\,p_1b_{j+1}) = (g', p_1)(b_j, ab_{j+1})$ and $(a\,p_1b_{j+1}, a'\,p_2b_{j+2}) = (p_1,p_2)(ab_{j+1}, a'b_{j+2})$. By (L1),

$$\omega_j(g'b_j, a\,p_1b_{j+1})\,\mathrm{pt}_j(P_1) = \mathrm{sc}_j\,\omega_j(g', p_1)\Phi^0_1,$$
$$\omega_{j+1}(a\,p_1b_{j+1}, a'\,p_2b_{j+2})\,\mathrm{pt}_{j+1}(P_2) = \mathrm{sc}_{j+1}\,\omega_{j+1}(p_1,p_2)\Phi^0_2,$$

and $R_{a'p_2b_{j+2}}G(v) = R_{p_2}f$. Since $e_{j,\alpha}(g) = \varepsilon_{j,\alpha}(gb_j^{-1})$, (W7) gives $I_s(p_1b_{j+1}, p_2b_{j+2}) = \mathrm{sc}_j\mathrm{sc}_{j+1}\,I^0(p_1,p_2)$. The right translations by b_{j+1}^{-1} and b_{j+2}^{-1} preserve absolutely convergent quotient integrals with their values (Lemma 14.25). Hence I_s has the absolutely convergent iterated quotient integral $\mathrm{sc}_j\mathrm{sc}_{j+1}\kappa_j z_Q$.

Step 6, the value. By (L4) and (W1), (W4),

$$\mathrm{biComb}(d_j^Q(\alpha,\gamma), d_{j+1}^Q(\gamma,\beta))\,\mathrm{ctr}(\Psi'_1,\Psi'_2) = c_G(j)\,\mathrm{biComb}\big(d_j^Q(\alpha,\gamma)d_j^G,\ d_{j+1}^Q(\gamma,\beta)d_{j+1}^G\big)\,\mathrm{ctr}(\Psi_1,\Psi_2),$$

where products of combinations denote composition of the operators. These combinations are supported away from tP. By (L5), then (L1) for dd and for Sad_j,

$$\sum_{\alpha,\gamma}\mathrm{bl}_j\big(\phi_\alpha, \mathrm{biComb}(d_j^Q(\alpha,\gamma), d_{j+1}^Q(\gamma,\beta))\,\mathrm{ctr}(\Psi'_1,\Psi'_2)\big) = c_G(j)\,c_T(j)\,\mathrm{two}_j\big(\mathrm{rd}_j(\cdot, S_j\psi', y), C\big)_\beta.$$

Finally $\mathrm{tr}(e_{j+2,\beta})R_{a'}G(v) = \mathrm{tr}(\varepsilon_{j+2,\beta})R_{a'b_{j+2}}G(v) = \mathrm{tr}(\varepsilon_{j+2,\beta})f$, because $e_{j+2,\beta}(g) = \varepsilon_{j+2,\beta}(gb_{j+2}^{-1})$ and a' commutes with b_{j+2}. By (W7) the iterated integral of Step 5 equals $\kappa_j\mathrm{sc}_j\mathrm{sc}_{j+1}c_Q(j)c_G(j)c_T(j)$ times the sum in (14.16), and (L6) identifies the constant with $c_V(j)$. □

Proposition 14.30 (nonvanishing of $c_V(j)$)**.** *Under (L6), $c_V(j)(s) \neq 0$ if and only if $\mathrm{sc}_j(s)$, $\mathrm{sc}_{j+1}(s)$, $c_Q(j)(s)$, $c_G(j)$ and $c_T(j)(s)$ are all nonzero.*

Proof. This is $\kappa_j > 0$ (Step 1 of the proof of Theorem 14.29) together with (L6). □

Remark 14.31 (normalization). The constant $c_V(j)$ depends on the tuple through sc_j, sc_{j+1}, $c_Q(j)$ and $c_T(j)$. The object that does not depend on the tuple is the kernel C, together with the volume factor κ_j. In Proposition 14.18 the constants $c_V(j)c_K(j)c_K(j+1)$ stand inside the sum over the tuples s_i, and the proof needs only that one kernel C serves all tuples, so that the coefficients $a(y')$ of the kernel law (T3) can be pulled out of the sum. Nonvanishing of $c_V(j)(s)$ is not used in Proposition 14.18 or Theorem 14.20. It enters only through $c_{\mathrm{top}}(t) \neq 0$ in (D6), and Proposition 14.30 reduces it to the nonvanishing of the five scalars of the construction.

Remark 14.32 (what remains). Theorem 14.29 reduces (T2) to Hypotheses 14.27 and 14.28. Items (W1) to (W7) are general properties of the Weil representation, the flat maps and the Bessel period, and (W6) is Fubini's theorem combined with the tensor factorization of the Weil representation of an orthogonal sum. The proof above uses (W1) to (W4) and (W7) directly. Items (W5) and (W6) are used in the formal derivation of the cell properties (L2) and of the twist-place law (L3). Items (L1) to (L6) are local computations of the construction at the twist places Q, at the level places gP, at the places tP and at the places where every datum is unramified. Of these, (L3) is the local double-step identity at the twist places. The Siegel–Weil formula and the doubling method do not enter (T2). They enter Hypothesis 14.21 only through the surjectivity (D4).

14.3 The share of irreducible rows

14.3.1 Labels and weights

Let g be a prime and $\Gamma_N = \mathrm{GL}_N(\mathbb{F}_g)$. A *label* is a monic polynomial $c \in \mathbb{F}_g[X]$ of degree N with $c(0) \neq 0$. Every irreducible representation π of Γ_N has a label, the characteristic polynomial of the semisimple class s_π of its Deligne–Lusztig series. For a label c let $E_c \in \mathbb{C}[\Gamma_N]$ be the sum of the central idempotents of the irreducible representations with label c. The E_c are orthogonal central idempotents with $\sum_c E_c = 1$. For $e \in \mathbb{C}[\Gamma_N]$ with $e(1) \neq 0$ put

$$\mathrm{wt}(e, c) = \frac{\mathrm{Re}(eE_c)(1)}{\mathrm{Re}\, e(1)},$$

where $a(1)$ is the coefficient of the identity. For a squarefree label put $\mathrm{sq}(c) = \prod_{P|c}(g^{\deg P} - 1)^{-1}$, the product over the monic irreducible factors.

Lemma 14.33. *If $e = e^* = e^2$ and $e(1) > 0$, then $\mathrm{wt}(e, c) \geq 0$ for every label and $\sum_c \mathrm{wt}(e, c) = 1$.*

Proof. In the left regular representation λ on $\ell^2(\Gamma_N)$ one has $a(1) = \langle \lambda(a)\delta_1, \delta_1 \rangle$. Central idempotents of $\mathbb{C}[\Gamma_N]$ are self-adjoint, so $eE_c = E_c e E_c$ and $(eE_c)(1) = \|\lambda(e)\lambda(E_c)\delta_1\|^2 \geq 0$. Summing over c gives $e(1)$. □

Let $U \subset \Gamma_N$ be the upper unitriangular group, ψ_0 a nontrivial additive character of $\mathbb{F}_g$, and $e_U = |U|^{-1} \sum_{u \in U} \overline{\psi_0(\sum_i u_{i,i+1})}\, u$ the Gelfand–Graev idempotent. It is a self-adjoint idempotent with $e_U(1) = |U|^{-1}$, because $u \mapsto \psi_0(\sum_i u_{i,i+1})$ is a character of U.

Proposition 14.34 (Gelfand–Graev weights)**.** *For every squarefree label c, $\mathrm{wt}(e_U, c) = \mathrm{sq}(c)$.*

Proof. By Frobenius reciprocity $(e_U E_\pi)(1) = |\Gamma_N|^{-1} \dim \pi \cdot m_\pi$, where m_π is the multiplicity of π in the Gelfand–Graev representation $\mathrm{Ind}_U^{\Gamma_N} \psi$, so $\mathrm{wt}(e_U, c) = [\Gamma_N : U]^{-1} \sum_{\pi \text{ of label } c} m_\pi \dim \pi$. By Gelfand–Graev multiplicity one and the theory of Deligne and Lusztig [24, §10], [18, Ch. 8], each series contains exactly one constituent of the Gelfand–Graev representation, with multiplicity one, the regular character $\chi_{s,\mathrm{reg}}$, and $\chi_{s,\mathrm{reg}}(1) = [\Gamma_N : C(s)]_{g'}$, the prime-to-$g$ part of the index of the centralizer. For $\Gamma_N = \mathrm{GL}_N(\mathbb{F}_g)$ this is also Green's theorem [30]. If c is squarefree with irreducible factors of degrees $d_1, \ldots, d_j$, the centralizer of s is the torus $\prod_i \mathbb{F}_{g^{d_i}}^\times$, of order $\prod_i (g^{d_i} - 1)$, prime to g. Since $[\Gamma_N : U] = |\Gamma_N|_{g'}$, $\mathrm{wt}(e_U, c) = |\Gamma_N|_{g'} / (\prod_i (g^{d_i} - 1)\, |\Gamma_N|_{g'}) = \mathrm{sq}(c)$. □

14.3.2 Good tuples

Fix a prime g, an integer $N \geq 100$, a finite set Π of f places, a finite set J of m indices and elements $\chi_{w,j} \in \mathbb{F}_g^\times$ for $w \in \Pi$, $j \in J$. Let $\bar{\mathbb{F}}_g$ be an algebraic closure. For a prime $\ell \neq g$ and $x \in \bar{\mathbb{F}}_g^\times$ let $x_{\ell'}$ be the prime-to-ℓ component of x in the torsion group $\bar{\mathbb{F}}_g^\times$, and let

$$BB(g,N) = \{g\} \cup \{\ell \leq g^N \text{ prime}: \ g^d < \ell^{4v_\ell(g^d-1)} \text{ for some } 1 \leq d \leq N\}.$$

A tuple $(c_w)_{w\in\Pi}$ of labels is *good* if the following hold.

(G1) Every c_w is squarefree.

(G2) For every prime $\ell \notin BB(g,N)$ and every w, the map $x \mapsto x_{\ell'}$ is injective on the roots of c_w.

(G3) For every prime $\ell \neq g$, every w, j, every sign and every root x of c_w, $x_{\ell'} \neq (\chi_{w,j}^{\pm 1})_{\ell'}$.

(G4) For every $1 \leq a \leq N/2$ there is w such that c_w has no divisor of degree a.

(G5) There is w such that c_w has an irreducible factor of prime degree p with $N < 2p$ and $p+3 \leq N$.

14.3.3 Counting lemmas

Lemma 14.35. *Let $N \geq 2$ and $g \geq 2$. (i) The number I_d of monic irreducible polynomials of degree d in $\mathbb{F}_g[X]$ other than X satisfies $dI_d \leq g^d - 1$. (ii) At least $g^N - 2g^{N-1}$ labels of degree N are squarefree. (iii) For every squarefree label c of degree N, $\mathrm{sq}(c) \geq g^{-N}$.*

Proof. (i) The roots of the counted polynomials are distinct elements of $\mathbb{F}_{g^d}^\times$, so $dI_d \leq g^d - 1$. (ii) The number of squarefree monic polynomials of degree $N \geq 2$ is $g^N - g^{N-1}$ [56, Prop. 2.3], and those with $c(0) = 0$ are X times a squarefree monic polynomial of degree $N-1$, at most g^{N-1} of them. (iii) $\prod_P (g^{\deg P} - 1)^{-1} \geq \prod_P g^{-\deg P} = g^{-N}$. □

Lemma 14.36 (types versus cycle types)**.** *Let $\mathcal{A}$ be a property of partitions of N. Then*

$$\sum_{\substack{c \text{ squarefree label} \\ \mathcal{A}(\text{degrees of the factors of } c)}} \mathrm{sq}(c) \leq \frac{\#\{\tau \in S_N : \ \mathcal{A}(\text{cycle type of } \tau)\}}{N!}.$$

In particular $\sum_{c \text{ squarefree}} \mathrm{sq}(c) \leq 1$.

Proof. A squarefree label with n_d irreducible factors of degree d is a choice of n_d distinct irreducibles of each degree d, none equal to X. The total weight of the labels of this type is $\prod_d \binom{I_d}{n_d}(g^d-1)^{-n_d} \leq \prod_d \frac{1}{n_d!}\left(\frac{I_d}{g^d-1}\right)^{n_d} \leq \prod_d \frac{1}{d^{n_d} n_d!}$ by Lemma 14.35(i). By Cauchy's formula the last product is the proportion of permutations of cycle type (n_d). Sum over the types with property $\mathcal{A}$. □

Lemma 14.37 (multiples)**.** *Let h be a squarefree monic polynomial of degree at most N with $h(0) \neq 0$. Then $\sum \mathrm{sq}(c) \leq \mathrm{sq}(h)$, the sum over squarefree labels c of degree N divisible by h.*

Proof. Such c are $c = hk$ with k squarefree, prime to h, of degree $N - \deg h$ and $k(0) \neq 0$, and $\mathrm{sq}(hk) = \mathrm{sq}(h)\mathrm{sq}(k)$. By Lemma 14.36 in degree $N - \deg h$ (or trivially if the degree is 0), $\sum_k \mathrm{sq}(k) \leq 1$. □

Lemma 14.38 (invariant sets of permutations)**.** *For $1 \le a < n$, the proportion of $\tau \in S_n$ that leave some a-element subset invariant is at most $7/8$.*

Proof. Let $X(\tau)$ be the number of τ-invariant a-subsets. Counting pairs (τ, A) with A invariant gives $\sum_\tau X(\tau) = \binom{n}{a}\, a!\,(n-a)! = n!$. Among these pairs consider those for which $\tau|_A$ and $\tau|_{A^c}$ both have a fixed point. With D_j the number of derangements of j points, $2D_j \le j!$ for $j \ge 1$, so the proportion of such pairs is $(1 - D_a/a!)(1 - D_{n-a}/(n-a)!) \ge 1/4$. If (τ, A) is such a pair, with fixed points $p \in A$ and $q \notin A$, then $A \setminus \{p\} \cup \{q\}$ is another invariant a-subset, so $X(\tau) \ge 2$. Let $B = \{\tau : X(\tau) \ge 2\}$ and $n_1 = \#\{\tau : X(\tau) = 1\}$. Then $\sum_{\tau \in B} X(\tau) \ge n!/4$, so $n_1 = n! - \sum_{\tau \in B} X(\tau) \le \frac{3}{4} n!$, and $\#B \le \frac{1}{2} \sum_{\tau \in B} X(\tau) = \frac{1}{2}(n! - n_1)$. Hence $\#\{X \ge 1\} = n_1 + \#B \le \frac{1}{2} n! + \frac{1}{2} n_1 \le \frac{7}{8} n!$. □

Lemma 14.39 (long prime cycles)**.** *For $N \ge 100$ the proportion of $\tau \in S_N$ that have a cycle of prime length p with $N/2 < p \le N - 3$ is $\sum_{N/2 < p \le N-3} 1/p \ge 1/(5 \log N)$.*

Proof. A permutation has at most one cycle of length greater than $N/2$, and the number of permutations with a cycle of a given length $L > N/2$ is $N!/L$, so the events are disjoint and the proportion is the sum. For $100 \le N \le 2 \cdot 10^6$ the inequality was checked by direct computation (the smallest value of $5 \log N \sum 1/p$ is 2.94, at $N = 226$). For $N > 2 \cdot 10^6$, the bounds $x/\log x < \pi(x) < 1.25506\, x/\log x$ of Rosser and Schoenfeld [58, Cor. 1], valid for $x \ge 17$, give $\sum_{N/2 < p \le N-3} 1/p \ge (\pi(N-3) - \pi(N/2))/(N-3)$ with $\pi(N-3) - \pi(N/2) \ge \frac{N-3}{\log N} - 0.6276 \frac{N}{\log(N/2)} \ge \frac{0.339N - 3}{\log N}$, because $\log(N/2) \ge 0.95 \log N$ for $N > 2 \cdot 10^6$, and $(0.339N - 3)/(N-3) \ge 1/5$. □

Lemma 14.40 (prime power parts of $g-1$)**.** *Let g be a prime with $16 \mid g - 1$ and let $q_1 \ne d$ be odd primes dividing $g - 1$ with $\max(q_1, d) \ge 16$. Then $s(g) := \sum_{\ell \mid g-1} \ell^{v_\ell(g-1)}/(g-1) \le \frac{49}{256 \min(q_1, d)}$.*

Proof. Write $g - 1 = \prod_i a_i$ with $a_i = \ell_i^{v_{\ell_i}(g-1)}$, so $s(g) = \sum_i 1/\prod_{j \ne i} a_j$. Here $a_2 \ge 16$, $a_{q_1} \ge q_1$, $a_d \ge d$, and the remaining factors $b_1, \dots, b_t \ge 3$ have product B with $\sum_i b_i \le B$ when $t \ge 1$. The terms are at most $1/(q_1 d)$, $1/(16d)$, $1/(16q_1)$, and $\sum_i b_i/(16 q_1 d B) \le 1/(16 q_1 d)$. With $m_0 = \min(q_1, d)$ and $\max(q_1, d) \ge 16$ the total is at most $(1/16 + 1/8 + 1/256)/m_0 = 49/(256 m_0)$. □

14.3.4 The bad weight at one prime

Theorem 14.41. *Let $g \ge 2^{12}$ be a prime, $N \ge 100$, and let q_1, d be as in Lemma 14.40 with $m_0 = \min(q_1, d)$. Let P be a nonnegative function on labels of degree N with total mass 1 that equals* sq *on squarefree labels. Then the $P^{\otimes \Pi}$-mass of the tuples that are not good is at most*

$$W(g) = \frac{2f}{g} + \frac{f \log_2 g}{\sqrt{g}} \Big(1.31 + \frac{2.02\,(1 + \log N)}{g^{1/4}}\Big) + 2fm\Big(\frac{1}{4m_0} + \frac{1 + \log N}{g - 1}\Big) + \frac{N}{2}\Big(\frac{7}{8}\Big)^f + e^{-f/(5 \log N)}.$$

Proof. A tuple that is not good satisfies one of the following: (I) some c_w is not squarefree; (II) some c_w is squarefree and fails (G2); (III) for some $(w, j, \pm)$, c_w is squarefree and fails (G3) for that index; (IV) all c_w are squarefree and (G4) fails; (V) all c_w are squarefree and (G5) fails. The mass is at most the sum of the masses of the five events. For an event that concerns one place w,

the mass is the P-mass of the corresponding set of labels at w, because P has mass 1 at the other places.

(I) By Lemma 14.35(ii),(iii), the squarefree labels have sq-mass at least $(g^N - 2g^{N-1})g^{-N} = 1 - 2/g$. So the non-squarefree labels have P-mass at most $2/g$, and the union over f places at most $2f/g$.

(II) Fix w and a prime $\ell \notin BB(g,N)$. For $1 \le \delta \le N$ the Sylow ℓ-subgroup of $\mathbb{F}_{g^\delta}^\times$ has order $\ell^{v_\ell(g^\delta-1)} \le g^{\delta/4}$: for $\ell \le g^N$ this is the definition of BB, and for $\ell > g^N$ the group is trivial. A squarefree label fails (G2) if it has two distinct roots with equal ℓ'-components. Either both are roots of one irreducible factor, or they are roots of two distinct factors. By Lemma 14.37 the mass is at most the sum of $\mathrm{sq}(Q)$ over the irreducible Q with a self-collision plus the sum of $\mathrm{sq}(QQ')$ over the ordered pairs $Q \ne Q'$ with a cross collision.

Self-collisions. Let Q have degree $\delta \ge 2$ and roots x, x^{g^i} with $1 \le i < \delta$ and $(x^{g^i})_{\ell'} = x_{\ell'}$, so x^{g^i-1} lies in the Sylow ℓ-subgroup of $\mathbb{F}_{g^\delta}^\times$, which is then nontrivial, so $\ell \mid g^\delta - 1$. The kernel of $y \mapsto y^{g^i-1}$ on $\mathbb{F}_{g^\delta}^\times$ is $\mathbb{F}_{g^{\gcd(i,\delta)}}^\times$, of order less than $g^{\delta/2}$. So for fixed i and ℓ there are at most $g^{3\delta/4}$ such x. There are at most $\delta \log_2 g$ primes dividing $g^\delta - 1$ and $\delta - 1$ values of i, so at most $(\delta-1)\log_2 g\, g^{3\delta/4}$ such Q of degree δ, summed over all $\ell \notin BB$. With $\rho = g^{-1/4} \le 1/8$ their mass is at most

$$\log_2 g \sum_{\delta \ge 2} (\delta - 1)\frac{g^{3\delta/4}}{g^\delta - 1} \le \left(1 + 2^{-23}\right)\log_2 g\, \frac{\rho^2}{(1-\rho)^2} \le 1.31\, \frac{\log_2 g}{\sqrt{g}}.$$

Cross collisions. Let $Q \ne Q'$ have degrees δ, δ' and roots x, y with $x_{\ell'} = y_{\ell'} =: z$. Write $x = z x_\ell$, $y = z y_\ell$ with ℓ-power components $x_\ell \in \mathbb{F}_{g^\delta}^\times$, $y_\ell \in \mathbb{F}_{g^{\delta'}}^\times$. Since $x \ne y$, one of x_ℓ, y_ℓ is not 1. Let $\mathcal{A}$ be the set of (Q,Q') that admit such (x,y,ℓ) with $y_\ell \ne 1$, and $\mathcal{B}$ the set with $x_\ell \ne 1$. The Frobenius $t \mapsto t^g$ maps such a triple (x,y,ℓ) to another one with the same property, and moves x through all δ roots of Q. So every $(Q,Q') \in \mathcal{A}$ yields at least δ triples with distinct x. Given x, the prime ℓ divides $g^{\delta'} - 1$ (at most $\delta' \log_2 g$ choices) and $y = x_{\ell'} y_\ell$ is determined by y_ℓ (at most $g^{\delta'/4}$ choices). Hence $\#\mathcal{A} \le g^\delta \delta' \log_2 g\, g^{\delta'/4}/\delta$, and symmetrically $\#\mathcal{B} \le g^{\delta'} \delta \log_2 g\, g^{\delta/4}/\delta'$. The mass is at most

$$\sum_{\delta,\delta'=1}^{N} \frac{\#\mathcal{A} + \#\mathcal{B}}{(g^\delta - 1)(g^{\delta'} - 1)} \le 2\Big(1 + \frac{1}{g-1}\Big)^2 \log_2 g \Big(\sum_{\delta \le N} \frac{1}{\delta}\Big) \sum_{k \ge 1} k g^{-3k/4} \le 2.02\, \frac{(1+\log N)\log_2 g}{g^{3/4}}.$$

Adding and summing over the f places gives the second term of $W(g)$.

(III) Fix $(w,j,\pm)$ and put $\chi = \chi_{w,j}^{\pm 1} \in \mathbb{F}_g^\times$. For $\ell \ne g$, $x_{\ell'} = \chi_{\ell'}$ holds exactly when x/χ has ℓ-power order. Let Q be irreducible of degree δ with such a root x. Then $x \in \chi \cdot \mathrm{Syl}_\ell(\mathbb{F}_{g^\delta}^\times)$, and every root of Q has the same property because $t \mapsto t^g$ fixes χ. Hence the number of such Q is at most $\delta^{-1} \sum_{\ell \mid g^\delta - 1} \ell^{v_\ell(g^\delta-1)}$; the primes $\ell \nmid g^\delta - 1$ contribute the single element $x = \chi$, which lies in every coset counted. For $\ell \mid g - 1$, the lifting of the exponent gives $v_\ell(g^\delta - 1) = v_\ell(g-1) + v_\ell(\delta)$, valid for odd ℓ and for $\ell = 2$ because $4 \mid g - 1$, so $\ell^{v_\ell(g^\delta-1)} \le \delta\, \ell^{v_\ell(g-1)}$. For $\ell \nmid g - 1$ the prime powers $\ell^{v_\ell(g^\delta-1)}$ are pairwise coprime divisors of $(g^\delta - 1)/(g-1)$, so their sum is at most $(g^\delta - 1)/(g-1)$. With $s(g)$ of Lemma 14.40, the mass of the labels with a bad factor is at most, by Lemma 14.37,

$$\sum_{\delta=1}^{N} \frac{\delta(g-1)s(g) + (g^\delta - 1)/(g-1)}{\delta\,(g^\delta - 1)} \le s(g)\Big(1 + \frac{1}{g-1}\Big) + \frac{1+\log N}{g-1} \le \frac{1}{4m_0} + \frac{1+\log N}{g-1}.$$

There are $2fm$ indices $(w, j, \pm)$.

(IV) A squarefree label has a divisor of degree a exactly when the degrees of its irreducible factors have a sub-multiset with sum a, a property of the type. By Lemmas 14.36 and 14.38 the sq-mass of the squarefree labels with a divisor of degree a is at most $7/8$. For fixed a the event that this holds at every place has mass at most $(7/8)^f$, and there are at most $N/2$ values of a.

(V) By Lemmas 14.36 and 14.39 the sq-mass of the squarefree labels without an irreducible factor of prime degree in $(N/2, N-3]$ is at most $1 - 1/(5\log N)$. At all f places the mass is at most $(1-1/(5\log N))^f \le e^{-f/(5\log N)}$. □

14.3.5 From weights to dimensions

The construction attaches to each element τ of an orthonormal eigenbasis of $\mathcal{H}$ a tuple of labels $\mathrm{lab}_k(\tau) = (\mathrm{lab}_{k,w}(\tau))_{w\in\Pi_k}$ at each of two auxiliary primes g_0, g_1, where Π_k is the set of f places of F above g_k and the level at w contains $\mathrm{GL}_N(\mathbb{F}_{g_k})$ as a quotient, with $N = r$. For each (k, w) it fixes an element $e_{k,w} \in \mathbb{C}[\mathrm{GL}_N(\mathbb{F}_{g_k})]$, the local factor of the level idempotent, and characters $\chi_{w,j} \in \mathbb{F}_{g_k}^\times$, $j \in J$, $|J| = m$. For a set S of label tuples at k put

$$w_k(S) = \sum_{t\in S} \prod_{w\in\Pi_k} \mathrm{wt}(e_{k,w}, t_w).$$

Hypothesis 14.42 (local weights)**.** Each $e_{k,w}$ is a self-adjoint idempotent with $e_{k,w}(1) > 0$, and $(1 - 1/g_k)\,\mathrm{wt}(e_U, c) \le \mathrm{wt}(e_{k,w}, c)$ for every label c.

Hypothesis 14.43 (multiplicity formula)**.** There is $\varepsilon_C \in [0, \frac{1}{2}]$ with $|D_{\mathrm{all}} - D^*| \le \varepsilon_C D^*$ and, for $k \in \{0,1\}$ and every set S of label tuples at k,

$$\Big|\#\{\tau : \mathrm{lab}_k(\tau) \in S\} - D^* w_k(S)\Big| \le \varepsilon_C\, D^* w_k(S).$$

Hypothesis 14.44 (rows are the good forms)**.** D_{rows} equals the number of τ whose label tuples at g_0 and at g_1 are both good for the characters $\chi_{w,j}$.

Lemma 14.45 (comparison of weights)**.** *Let w_U, w_J be probability weights on a finite set at each of f places with $(1-1/g)w_U \le w_J$. For every set A of tuples, $w_J^{\otimes f}(A) \le w_U^{\otimes f}(A) + f/g$.*

Proof. Write $w_J = (1-1/g)w_U + \frac{1}{g}\nu$ with ν a probability weight. Expand the product. The term $(1-1/g)^f w_U^{\otimes f}(A)$ is at most $w_U^{\otimes f}(A)$, and the other terms have total mass $1 - (1-1/g)^f \le f/g$. □

Theorem 14.46. *Assume Hypotheses 14.42, 14.43 and 14.44, that $16 \mid g_k - 1$ and q_1, d divide $g_k - 1$ for both k, with $q_1 \ne d$ odd primes and $\max(q_1, d) \ge 16$, and the sizes*

$$g_k \ge e^{8x^{\varepsilon'}}, \quad x^2 \le f \le x^{C_s}, \quad m \le x^{C_s}, \quad 100 \le N \le x^{C_s}, \quad \min(q_1, d) \ge e^{x^{\varepsilon'/6}}, \tag{14.17}$$

with $0 < \varepsilon' \le 1$ and $C_s \ge 1$. Then $0 \le \delta \le \exp(-x^{\varepsilon'/7})$ for $x \ge x_0(\varepsilon', C_s)$. With $\varepsilon' = \varepsilon_z$ this is Theorem 14.2.

Proof. Let B_k be the set of label tuples at g_k that are not good. By Hypothesis 14.44 every τ outside the rows has $\mathrm{lab}_0(\tau) \in B_0$ or $\mathrm{lab}_1(\tau) \in B_1$, so $0 \le D_{\mathrm{all}} - D_{\mathrm{rows}} \le \sum_k \#\{\tau : \mathrm{lab}_k(\tau) \in B_k\}$, which gives $\delta \ge 0$. By Hypothesis 14.43,

$$\delta = \frac{D_{\mathrm{all}} - D_{\mathrm{rows}}}{D_{\mathrm{all}}} \le \frac{(1+\varepsilon_C)D^*(w_0(B_0)+w_1(B_1))}{(1-\varepsilon_C)D^*} \le 3\big(w_0(B_0)+w_1(B_1)\big).$$

By Lemma 14.33 and Hypothesis 14.42 the weights $\mathrm{wt}(e_{k,w},\cdot)$ and $\mathrm{wt}(e_U,\cdot)$ are probability weights with $(1-1/g_k)\mathrm{wt}(e_U,\cdot) \le \mathrm{wt}(e_{k,w},\cdot)$, so Lemma 14.45 gives $w_k(B_k) \le \mathrm{wt}(e_U,\cdot)^{\otimes f}(B_k) + f/g_k$. By Proposition 14.34 the weight $P = \mathrm{wt}(e_U,\cdot)$ satisfies the hypothesis of Theorem 14.41, and the sizes give $g_k \ge 2^{12}$ for large x. Hence $w_k(B_k) \le W(g_k) + f/g_k$.

Under (14.17) the terms of $W(g_k) + f/g_k$ are bounded as follows for large x. The terms $3f/g_k$ and $2fm(1+\log N)/(g_k-1)$ are at most $4x^{2C_s}(1+C_s\log x)e^{-8x^{\varepsilon'}}$. Since $t \mapsto \log t/\sqrt{t}$ decreases for $t \ge e^2$, the second term is at most $3x^{C_s}\cdot 12x^{\varepsilon'}e^{-4x^{\varepsilon'}}$. The term $2fm/(4m_0)$ is at most $\frac{1}{2}x^{2C_s}e^{-x^{\varepsilon'}/6}$. Finally $\frac{N}{2}(7/8)^f \le x^{C_s}e^{-x^2/8}$ and $e^{-f/(5\log N)} \le e^{-x^2/(5C_s\log x)}$. Since $\varepsilon' \le 1$, every term is at most $x^{2C_s}e^{-x^{\varepsilon'}/6}$ for large x, so $\delta \le 36\,x^{2C_s}e^{-x^{\varepsilon'}/6} \le e^{-x^{\varepsilon'}/7}$ once $x^{\varepsilon'}/6 - x^{\varepsilon'}/7 \ge 2C_s\log x + \log 36$. □

Remark 14.47. The irreducibility certificates (G2) to (G5) are used later, where Jordańs theorem enters. Theorem 14.2 itself is a count and does not use them.

14.3.6 The level idempotents at the auxiliary primes

Let g be a prime, $\mathbb{F} = \mathbb{F}_g$, and let $\psi_0 \colon \mathbb{F} \to \mathbb{C}^\times$ be a nontrivial additive character. Put $G_n = \mathrm{GL}_n(\mathbb{F})$, let $U_n \subset G_n$ be the group of upper unitriangular matrices, and index rows and columns by $0,\dots,n-1$. For a subgroup H of a finite group G and a character $\lambda \colon H \to \mathbb{C}^\times$ put

$$e(H,\lambda) = |H|^{-1}\sum_{h\in H}\overline{\lambda(h)}\,h \in \mathbb{C}[G].$$

This is a self-adjoint idempotent with $e(H,\lambda)(1) = |H|^{-1}$. For a representation (π, V) of G the operator $\pi(e(H,\lambda))$ is the orthogonal projection onto $V^{(H,\lambda)} = \{v \in V : \pi(h)v = \lambda(h)v \ (h \in H)\}$. For $a \in \mathbb{C}[G]$ and a class function χ on G put $\mathrm{tr}(a,\chi) = \sum_{x\in G} a(x)\chi(x)$. If $\chi = \chi_\pi$ is the character of π, then $\mathrm{tr}(a,\chi_\pi) = \mathrm{Tr}\,\pi(a)$, so

$$\mathrm{tr}\big(e(H,\lambda),\chi_\pi\big) = \dim V^{(H,\lambda)} = \dim \mathrm{Hom}_H(\lambda, \pi|_H). \tag{14.18}$$

Since the class functions are the linear combinations of the irreducible characters and $\mathrm{tr}(a,\cdot)$ is linear, an identity between the numbers (14.18) for all irreducible π is an identity of $\mathrm{tr}(a,\cdot)$ on all class functions.

Pattern groups. Let $I = \{0,\dots,N-1\}$ and let $\prec$ be a strict partial order on I, that is, an irreflexive transitive relation. Write $a \preceq b$ for $a = b$ or $a \prec b$. The *pattern group* of $\prec$ is

$$U_\prec = \{u \in \mathrm{GL}_N(\mathbb{F}) : u_{aa} = 1 \text{ for all } a,\ u_{ab} = 0 \text{ unless } a \prec b \text{ or } a = b\}.$$

Lemma 14.48. *(i) $U_\prec$ is a subgroup of $\mathrm{GL}_N(\mathbb{F})$ of order $g^{\#\{(a,b):a\prec b\}}$. (ii) Let E be a set of pairs (a,b) with $a \prec b$, and let $c \colon E \to \mathbb{F}$. Suppose that for every $(a,b) \in E$ there is no m with $a \prec m \prec b$. Then $\lambda_{E,c}(u) = \psi_0\big(\sum_{(a,b)\in E} c_{ab}u_{ab}\big)$ is a character of $U_\prec$.*

Proof. (i) For $u, v \in U_{\prec}$, $(uv)_{ab} = \sum_m u_{am} v_{mb}$, and a nonzero term needs $a \preceq m \preceq b$. By transitivity $a \preceq b$, and if $a = b$ the only term is $m = a$, because $a \prec m \prec a$ would give $a \prec a$. So $uv \in U_{\prec}$ with unit diagonal. Write $u = 1 + x$. Every product x^j is supported on pairs $a \prec b$, and $x^N = 0$ because a strictly increasing chain in I has at most N elements. Hence $u^{-1} = \sum_{j<N} (-x)^j \in U_{\prec}$. The entries at the pairs $a \prec b$ are free, which gives the order. (ii) For $u, v \in U_{\prec}$ and $(a, b) \in E$ we have $(uv)_{ab} = u_{ab} + v_{ab} + \sum_{a \prec m \prec b} u_{am} v_{mb} = u_{ab} + v_{ab}$. □

Lemma 14.49 (one double coset)**.** *Let H be a finite group with subgroups H_1, H_2 and characters $\lambda_i \colon H_i \to \mathbb{C}^\times$ that agree on $H_1 \cap H_2$. Suppose that for $i \in \{1, 2\}$ and every $h \in H \setminus H_i$ there is $a \in H_i$ with $h^{-1}ah \in H_i$ and $\lambda_i(h^{-1}ah) \neq \lambda_i(a)$. Then $\operatorname{Ind}_{H_1}^H \lambda_1 \cong \operatorname{Ind}_{H_2}^H \lambda_2$. Consequently, for every finite group $G \supseteq H$ and every representation π of G, $\dim \pi^{(H_1, \lambda_1)} = \dim \pi^{(H_2, \lambda_2)}$, and $\operatorname{tr}(e(H_1, \lambda_1), \chi) = \operatorname{tr}(e(H_2, \lambda_2), \chi)$ for every class function χ on G.*

Proof. For subgroups $A, B \subseteq H$ and characters α, β, Mackey's formula (J.-P. Serre, *Linear representations of finite groups*, §7.4) gives

$$\left\langle \operatorname{Ind}_A^H \alpha, \operatorname{Ind}_B^H \beta \right\rangle_H = \sum_{h \in A \backslash H / B} \left\langle \alpha, \beta^h \right\rangle_{A \cap hBh^{-1}}, \qquad \beta^h(y) = \beta(h^{-1} y h).$$

Each term is 1 if $\alpha = \beta^h$ on $A \cap hBh^{-1}$ and 0 otherwise. Take $A = B = H_i$ and $\alpha = \beta = \lambda_i$. For $h \notin H_i$ the element a of the hypothesis lies in $H_i \cap hH_ih^{-1}$ and satisfies $\lambda_i^h(a) = \lambda_i(h^{-1}ah) \neq \lambda_i(a)$, so the term of h vanishes. The double coset of 1 contributes 1. Hence $\operatorname{Ind}_{H_i}^H \lambda_i$ is irreducible. Next take $A = H_1$, $B = H_2$. The term of $h = 1$ is 1 because $\lambda_1 = \lambda_2$ on $H_1 \cap H_2$, and the other terms are nonnegative, so the two irreducible induced representations have a common constituent and are isomorphic. By Frobenius reciprocity and transitivity of induction, $\dim \pi^{(H_i, \lambda_i)} = \dim \operatorname{Hom}_G(\operatorname{Ind}_H^G \operatorname{Ind}_{H_i}^H \lambda_i, \pi)$, which does not depend on i. The last claim follows from (14.18) and linearity. □

Lemma 14.50 (exchange)**.** *Let $\prec_1, \prec_2$ be strict partial orders on I whose union $\prec$ is again a strict partial order. Let $p, q \in I$, $W \subseteq I \setminus \{p, q\}$, and let E be a set of pairs with a function $c \colon E \to \mathbb{F}$. Assume the following.*

(X1) The pairs with $a \prec_1 b$ and not $a \prec_2 b$ are the pairs (p, w), $w \in W$. The pairs with $a \prec_2 b$ and not $a \prec_1 b$ are the pairs (w, q), $w \in W$.

(X2) $(p, q) \in E$ and $c_{pq} \neq 0$.

(X3) If $w \in W$, $x \preceq p$ and $w \preceq y$, then $x \prec_1 y$, and $(x, y) \in E$ only for $(x, y) = (p, q)$.

(X4) If $w \in W$, $x \preceq w$ and $q \preceq y$, then $x \prec_2 y$, and $(x, y) \in E$ only for $(x, y) = (p, q)$.

(X5) For $(a, b) \in E$ and $j \in \{1, 2\}$ there is no m with $a \prec_j m \prec_j b$.

Then $\lambda = \lambda_{E,c}$ is a character of $U_{\prec_1}$ and of $U_{\prec_2}$, and $\dim \pi^{(U_{\prec_1}, \lambda)} = \dim \pi^{(U_{\prec_2}, \lambda)}$ for every representation π of every finite group containing $U_{\prec}$.

Proof. By Lemma 14.48 the groups $H = U_{\prec}$, $H_1 = U_{\prec_1}$, $H_2 = U_{\prec_2}$ satisfy $H_1, H_2 \subseteq H$, and by (X5) λ is a character of H_1 and of H_2. The two characters are given by one formula, so they agree on $H_1 \cap H_2$. We verify the hypothesis of Lemma 14.49.

Let $h \in H \setminus H_1$. By (X1) the pairs of $\prec$ outside $\prec_1$ are the pairs (w, q), so $h_{wq} \neq 0$ for some $w \in W$. Since $p \prec_1 w$, the element $a = 1 + tE_{pw}$ lies in H_1 for every $t \in \mathbb{F}$. Then

$(h^{-1}ah)_{xy} = \delta_{xy} + t\,(h^{-1})_{xp}h_{wy}$. A nonzero off-diagonal entry needs $(h^{-1})_{xp} \neq 0$ and $h_{wy} \neq 0$, so $x \preceq p$ and $w \preceq y$ because $h, h^{-1} \in U_{\prec}$, and (X3) gives $x \prec_1 y$. Hence $h^{-1}ah \in H_1$. By (X3) the only pair of E among these is (p,q), and $(h^{-1})_{pp} = 1$, so $\lambda(h^{-1}ah) = \psi_0(c_{pq}t\,h_{wq})$. By (X3) with $(x,y) = (p,w)$ the pair (p,w) lies in E only if $w = q$, which is excluded, so $\lambda(a) = 1$. Since $c_{pq}h_{wq} \neq 0$ and $\psi_0 \neq 1$ there is t with $\psi_0(c_{pq}t\,h_{wq}) \neq 1$.

Let $h \in H \setminus H_2$. Then $h^{-1} \notin H_2$, and by (X1) $(h^{-1})_{pw} \neq 0$ for some $w \in W$. Put $a = 1 + tE_{wq} \in H_2$. The off-diagonal entries of $h^{-1}ah$ are $t\,(h^{-1})_{xw}h_{qy}$ with $x \preceq w$ and $q \preceq y$, so (X4) gives $h^{-1}ah \in H_2$ and $\lambda(h^{-1}ah) = \psi_0(c_{pq}t\,(h^{-1})_{pw})$, while $\lambda(a) = 1$ because $(w,q) \in E$ would force $w = p$. Choose t as before. □

Exchange of roots. Let $k \geq 1$ and $s \geq 0$ be integers and $N = k + 2s + 1$. Put $W = \{s, \dots, s + k - 1\}$, $c_0 = s + k$, and $O = I \setminus W = \{0, \dots, s-1\} \cup \{c_0, \dots, N-1\}$ with its natural order. Define

$$U_J = \{u \in U_N : u_{a,c_0} = 0\ (a \in W)\}, \qquad U' = \{u \in U_N : u_{a,k} = 0\ (0 \leq a < k)\},$$

$$\psi_J(u) = \psi_0\Big(\sum_{(a,b)\in E} u_{ab}\Big), \qquad \psi'(u) = \psi_0\Big(\sum_{l=0}^{N-2} u_{l,l+1}\Big),$$

where E is the set of pairs (a,b) that are consecutive in O (the pairs $(l, l+1)$ with $l+1 < s$, the pair $(s-1, c_0)$ when $s \geq 1$, and the pairs $(l, l+1)$ with $l \geq c_0$) together with the pairs $(l, l+1)$ with $l, l+1 \in W$. Put $e_J = e(U_J, \psi_J)$, $e' = e(U', \psi')$ and $e_U = e(U_N, \psi')$. In the construction $(N,k,s) = (r,k,r_S)$ with $r_S = m/2 - 1$, and e_J is the local factor of the level idempotent at every place above g_0 and g_1.

Theorem 14.51 (exchange of roots)**.** *For every class function χ on G_N, $\mathrm{tr}(e_J, \chi) = \mathrm{tr}(e', \chi)$. Equivalently $\dim \pi^{(U_J,\psi_J)} = \dim \pi^{(U',\psi')}$ for every representation π of G_N.*

Proof. For $0 \leq i \leq s$ let $T_i = \{b \in O : b > i\}$ if $i < s$ and $T_s = \{c_0 + 1, \dots, N-1\}$. Define $a \prec_i b$ if one of the following holds: $a, b \in W$ and $a < b$; $a, b \in O$ and $a < b$; $a \in O$, $a < i$ and $b \in W$; $a \in W$ and $b \in T_i$.

$\prec_i$ is a strict partial order. It is contained in no cycle because a relation from W to O ends in $T_i \subseteq \{b > i\}$ and a relation from O to W starts in $\{a < i\}$, so no chain passes from W to O and back to W. Transitivity is checked on the four types. For $a \prec_i w \prec_i b$ with $a, b \in O$ and $w \in W$ we have $a < i < b$ when $b < s$, and $a < s \leq c_0 \leq b$ otherwise, so $a < b$. For $w \prec_i b \prec_i b'$ with $b, b' \in O$ we have $b' > b$, and $b \in T_i$ implies $b' \in T_i$ since T_i is an upper set of O (for $i < s$ it is $\{b \in O : b > i\}$, and T_s is the top of O). The remaining cases are immediate.

The character. Each pair of E is a covering pair of $\prec_i$. For a pair (a,b) consecutive in O no element of O lies between, and $a \prec_i w \prec_i b$ with $w \in W$ needs $a < i$ and $b \in T_i$. If $b < s$ then $b = a + 1 \leq i$ contradicts $b > i$. If $b = c_0$ then $a = s - 1 < i$ forces $i = s$, and $c_0 \notin T_s$. If $b > c_0$ then $a \geq c_0 > i$. For a pair $(l, l+1)$ in W an intermediate element of O would need $l \prec_i m \prec_i l + 1$, that is $m \in T_i$ and $m < i$, which is impossible. By Lemma 14.48 $\psi_J = \lambda_{E,1}$ is a character of every $U_{\prec_i}$.

The end points. For $i = s$ the relation $\prec_s$ consists of the pairs $a < b$ other than (w, c_0), $w \in W$, so $U_{\prec_s} = U_J$. For $i = 0$ let σ be the permutation of I with $\sigma(a) = s + a$ for $a < k$, $\sigma(a) = a - k$ for $k \leq a < k + s$, and $\sigma(a) = a$ for $a \geq k + s$. It maps $\{0, \dots, k-1\}$ increasingly onto W and $\{k, \dots, N-1\}$ increasingly onto O. A direct check of the four types gives $\sigma(a) \prec_0 \sigma(b)$ if and only if $a < b$ and not ($a < k$ and $b = k$), and $(\sigma(a), \sigma(b)) \in E$ if and only if $b = a + 1 \neq k$. With

the permutation matrix P, $Pe_a = e_{\sigma(a)}$, we get $(P^{-1}uP)_{ab} = u_{\sigma(a)\sigma(b)}$, hence $P^{-1}U_{\prec_0}P = U'$ and $\psi_J(PuP^{-1}) = \psi'(u)$ for $u \in U'$ (the entry $u_{k-1,k}$ vanishes on U'). Since χ is a class function, $\mathrm{tr}(e(U_{\prec_0}, \psi_J), \chi) = \mathrm{tr}(e', \chi)$.

One step. Let $1 \le i \le s$. We apply Lemma 14.50 to $\prec_1 = \prec_i$, $\prec_2 = \prec_{i-1}$, $p = i-1$, $q = i$ if $i < s$ and $q = c_0$ if $i = s$, the set W and E with $c \equiv 1$. Note that q is the successor of p in O. The union $\prec$ is the relation of the same four types with $\{a < i\}$ in the third type and $T' = \{b \in O : b \ge i\}$ in the fourth, which is a strict partial order by the argument above. Condition (X1) holds because $\{a \in O : a < i\} \setminus \{a < i-1\} = \{i-1\}$ and $T_{i-1} \setminus T_i = \{q\}$. Condition (X2) holds because (p, q) is consecutive in O. For (X3) let $x \preceq p$ and $w \preceq y$ with $w \in W$. Since $p \in O$ and no element of W precedes p (that would need $p \ge i$), $x \in O$ and $x \le i-1$. Either $y \in W$, and then $x \prec_i y$ by the third type, or $y \in O$ with $y \ge i > x$, and then $x \prec_i y$ by the second type. A pair of E of this form joins two consecutive elements $x < y$ of O with $x \le i-1 < i \le y$, so it is (p, q). For (X4) let $x \preceq w$ and $q \preceq y$. Then $y \in O$ with $y \ge q \ge i$, because nothing in W follows $q \ge i$ in $\prec$. If $x \in W$ then $y \in T_{i-1}$, and if $x \in O$ then $x < i \le y$. In both cases $x \prec_{i-1} y$, and a pair of E of this form has $x \in O$, $x < i \le q \le y$, x, y consecutive in O, so it is (p, q). Condition (X5) was verified above. Lemma 14.50 gives $\mathrm{tr}(e(U_{\prec_i}, \psi_J), \chi) = \mathrm{tr}(e(U_{\prec_{i-1}}, \psi_J), \chi)$.

Chaining the steps $i = s, s-1, \dots, 1$ with the two end points proves the theorem. For $s = 0$ there is no step and $U_{\prec_0} = U_{\prec_s}$. □

Column expansion. For $n \ge 1$ let $P_n \subseteq G_n$ be the mirabolic subgroup of matrices whose last row is $(0, \dots, 0, 1)$. Embed $G_{n-1} \subseteq P_n$ by $h \mapsto \mathrm{diag}(h, 1)$ and let $Y_n = \{y(x) : x \in \mathbb{F}^{n-1}\}$, where $y(x)$ is the unipotent matrix with last column $(x, 1)$. Then $P_n = G_{n-1} \ltimes Y_n$, $hy(x)h^{-1} = y(hx)$, and $U_n = U_{n-1} \ltimes Y_n$. For $0 \le i \le n-1$ let

$$\psi_n^{(i)}(u) = \psi_0\Big(\sum_{0 \le l \le n-2,\ l \ne i-1} u_{l,l+1}\Big) \qquad (u \in U_n),$$

so that $\psi_n^{(0)} = \psi_n$ is the generic character. For a representation τ of P_n put

$$A(\tau) = \dim \tau^{(U_{n-1}, \psi_{n-1})}, \qquad B_i(\tau) = \dim \tau^{(U_n, \psi_n^{(i)})} \quad (0 \le i \le n-1).$$

For a representation ρ of G_n put $\mathcal{W}(\rho) = \dim \rho^{(U_n, \psi_n)}$.

Lemma 14.52 (orbits of the unipotent group)**.** *Let $m \ge 1$ and let σ be a representation of P_m. For $0 \le j \le m-2$ let $S_j = \{u \in U_{m-1} : u_{jl} = 0\ (l > j)\}$, and for $c \in \mathbb{F}^\times$ let $\chi'_{j,c}(s\,y(x)) = \psi_{m-1}(s)\psi_0(cx_j)$ on S_jY_m. Then $\chi'_{j,c}$ is a character and*

$$A(\sigma) = B_{m-1}(\sigma) + \sum_{j=0}^{m-2} \sum_{c \in \mathbb{F}^\times} X_{j,c}(\sigma), \qquad X_{j,c}(\sigma) = \dim \sigma^{(S_jY_m, \chi'_{j,c})}.$$

Proof. For a row vector $\mu \in \mathbb{F}^{m-1}$ let $\sigma_\mu = \{v : \sigma(y(x))v = \psi_0(\mu x)v\}$. Since Y_m is abelian, $\sigma = \bigoplus_\mu \sigma_\mu$, and $\sigma(h)\sigma_\mu = \sigma_{\mu h^{-1}}$ for $h \in G_{m-1}$. The group U_{m-1} acts on row vectors by $\mu \mapsto \mu u^{-1}$. Its orbits are $\{0\}$ and, for $0 \le j \le m-2$ and $c \in \mathbb{F}^\times$, the sets $O_{j,c}$ of μ with $\mu_l = 0$ for $l < j$ and $\mu_j = c$. Indeed μu^{-1} has the same first nonzero coordinate as μ, and $c\,e_j^T(1 + \sum_{l>j} c^{-1}\mu_l E_{jl}) = \mu$ for $\mu \in O_{j,c}$. The stabilizer of $c\,e_j^T$ is S_j, which therefore fixes the character $\psi_0(cx_j)$ of Y_m, so $\chi'_{j,c}$ is a character of $S_j \ltimes Y_m$. The sum $\sigma_O = \bigoplus_{\mu \in O} \sigma_\mu$ over one orbit is U_{m-1}-stable and isomorphic

to $\mathrm{Ind}_{S_j}^{U_{m-1}} \sigma_{ce_j}$, so by Frobenius reciprocity $\dim \sigma_O^{(U_{m-1},\psi_{m-1})} = \dim \sigma_{ce_j}^{(S_j,\psi_{m-1})} = X_{j,c}(\sigma)$. The orbit $\{0\}$ gives σ^{Y_m}, and $\dim(\sigma^{Y_m})^{(U_{m-1},\psi_{m-1})} = B_{m-1}(\sigma)$ because $\psi_m^{(m-1)}(uy) = \psi_{m-1}(u)$. □

Lemma 14.53 (induction from the mirabolic subgroup)**.** *Let $m \geq 1$ and let σ be a representation of P_m. For $0 \leq j \leq m-1$ let $Y_m^{<j} = \{y(x) : x_l = 0 \ (l \geq j)\}$ and $H_j = U_{m-1} Y_m^{<j}$, and for $c \in \mathbb{F}^\times$ let*

$$\chi_{j,c}(u\, y(x)) = \psi_0\Big(\sum_{0 \leq l \leq m-3,\ l \neq j-1} u_{l,l+1} + c\, x_{j-1} \Big),$$

with $x_{-1} = 0$. Then $\chi_{j,c}$ is a character of the group H_j and

$$\mathcal{W}\big(\mathrm{Ind}_{P_m}^{G_m} \sigma\big) = \sum_{j=0}^{m-1} \sum_{c \in \mathbb{F}^\times} D_{j,c}(\sigma), \qquad D_{j,c}(\sigma) = \dim \sigma^{(H_j, \chi_{j,c})}.$$

Proof. By Mackey's formula and Frobenius reciprocity,

$$\mathcal{W}(\mathrm{Ind}_{P_m}^{G_m} \sigma) = \sum_{x \in U_m \backslash G_m / P_m} \dim \sigma^{(x^{-1}(U_m \cap x P_m x^{-1})x,\ \psi_m(x \cdot x^{-1}))}.$$

The map $gP_m \mapsto e_{m-1}^T g^{-1}$ identifies G_m/P_m with the nonzero row vectors, and U_m acts by $\mu \mapsto \mu u^{-1}$. As in Lemma 14.52 the orbits are indexed by (j,c) with $0 \leq j \leq m-1$ and $c \in \mathbb{F}^\times$, with representative $c\, e_j^T$ and stabilizer $U_{(j)} = \{u \in U_m : u_{jl} = 0 \ (l > j)\}$. Let β be the bijection of I with $\beta(l) = l$ for $l < j$, $\beta(j) = m-1$ and $\beta(l) = l-1$ for $l > j$, let w be the permutation matrix with $we_l = e_{\beta(l)}$, and let $t = \mathrm{diag}(1, \ldots, 1, c)$. Then x with $x^{-1} = tw$ satisfies $e_{m-1}^T x^{-1} = c\, e_j^T$, so $U_m \cap x P_m x^{-1} = U_{(j)}$. Since $(wzw^{-1})_{\beta(a)\beta(b)} = z_{ab}$ and β is increasing on $I \setminus \{j\}$, the group $wU_{(j)}w^{-1}$ consists of the matrices of U_m whose block on $\{0, \ldots, m-2\}$ is arbitrary in U_{m-1} and whose last column is supported on the rows $l < j$ (these come from the entries (l, j)). Conjugation by t rescales the last column, so $x^{-1} U_{(j)} x = H_j$. For $z \in H_j$ put $u = xzx^{-1} = w^{-1}(t^{-1}zt)w$, so $u_{ab} = (t^{-1}zt)_{\beta(a)\beta(b)}$ and $(t^{-1}zt)_{a,m-1} = c\, z_{a,m-1}$ for $a < m-1$. The superdiagonal entries of u are $u_{l,l+1} = z_{l,l+1}$ for $l+1 < j$, $u_{j-1,j} = c\, z_{j-1,m-1}$, $u_{j,j+1} = 0$, and $u_{l,l+1} = z_{l-1,l}$ for $l > j$. Hence $\psi_m(xzx^{-1}) = \chi_{j,c}(z)$. □

Lemma 14.54. *In the notation of Lemmas 14.52 and 14.53, $D_{j+1,c}(\sigma) = X_{j,c}(\sigma)$ for $0 \leq j \leq m-2$ and $c \in \mathbb{F}^\times$.*

Proof. On $I = \{0, \ldots, m-1\}$ let $a \prec_1 b$ mean $a < b \leq m-2$, or $b = m-1$ and $a \leq j$, and let $a \prec_2 b$ mean $a < b \leq m-2$ and $a \neq j$, or $b = m-1 > a$. Then $U_{\prec_1} = H_{j+1}$ and $U_{\prec_2} = S_j Y_m$. Let E consist of the pairs $(l, l+1)$ with $0 \leq l \leq m-3$, $l \neq j$, with coefficient 1, and of $(j, m-1)$ with coefficient c. On H_{j+1} the character $\lambda_{E,c}$ is $\chi_{j+1,c}$. On $S_j Y_m$ it is $\chi'_{j,c}$, because $(s\, y(x))_{j,m-1} = (sx)_j = x_j$ when the row j of s is e_j^T. If $j = m-2$ the two relations coincide. Otherwise we apply Lemma 14.50 with $p = j$, $q = m-1$ and $W = \{j+1, \ldots, m-2\}$. The union of the two relations is the natural order $<$. (X1) holds by definition. (X3): if $x \leq j < w \leq y$ then $x \prec_1 y$, and a pair $(x, x+1)$ with $x \leq j < x+1$ has $x = j$ and is not in E, so only $(j, m-1)$ occurs. (X4): if $x \leq w$ and $y \geq m-1$ then $y = m-1$ and $x \prec_2 y$, and $(x, m-1) \in E$ only for $x = j$. (X5): the pairs $(l, l+1)$ are covering pairs of $<$. An element b with $j \prec_1 b \prec_1 m-1$ would satisfy $b > j$ and $b \leq j$. An element b with $j \prec_2 b$ equals $m-1$. Lemma 14.50 applies since $P_m \supseteq U_m = U_\prec$. □

Proposition 14.55. *For every $n \geq 1$ and every representation τ of P_n,*

$$A(\tau) = B_{n-1}(\tau) + \sum_{i=0}^{n-2} (g-1) g^{n-2-i} B_i(\tau).$$

Proof. For $n = 1$ both sides equal $\dim \tau$. Let $n \geq 2$ and assume the claim for $n-1$. Put $m = n-1$ and let $\theta(y(x)) = \psi_0(x_{n-2})$, a character of Y_n whose stabilizer in G_m for the action $\mu \mapsto \mu h^{-1}$ on row vectors is P_m. Clifford theory as in the proof of Lemma 14.52, now with the transitive action of G_m on nonzero row vectors, gives $\tau|_{G_m} \cong \tau^{Y_n} \oplus \operatorname{Ind}_{P_m}^{G_m} \sigma$ with $\sigma = \tau^{(Y_n,\theta)}$, a representation of P_m. Since $\psi_n^{(n-1)}$ is trivial on Y_n, $\mathcal{W}(\tau^{Y_n}) = B_{n-1}(\tau)$. For $0 \leq i \leq n-2$ the character $\psi_n^{(i)}$ restricts to θ on Y_n and to $\psi_m^{(i)}$ on U_m, so $B_i(\tau) = B_i(\sigma)$, the right side computed in P_m.

By Lemmas 14.53, 14.54 and 14.52,

$$\mathcal{W}(\operatorname{Ind}_{P_m}^{G_m} \sigma) = \sum_c D_{0,c}(\sigma) + \sum_{j=0}^{m-2} \sum_c X_{j,c}(\sigma) = (g-1)A(\sigma) + \big(A(\sigma) - B_{m-1}(\sigma)\big),$$

because $H_0 = U_{m-1}$ and $\chi_{0,c} = \psi_{m-1}$. By the induction hypothesis $gA(\sigma) - B_{m-1}(\sigma) = (g-1)B_{m-1}(\sigma) + \sum_{i \leq m-2}(g-1)g^{m-1-i}B_i(\sigma) = \sum_{i=0}^{m-1}(g-1)g^{m-1-i}B_i(\sigma)$. Adding $B_{n-1}(\tau)$ and using $B_i(\sigma) = B_i(\tau)$ and $m-1-i = n-2-i$ gives the claim. □

For a representation π of G_N and $0 \leq i \leq N-1$ put $W_i(\pi) = \dim \pi^{(U_N, \psi_N^{(i)})}$, so that $W_0(\pi) = \operatorname{Tr} \pi(e_U)$.

Theorem 14.56 (column expansion)**.** *Let $1 \leq k < N$. For every representation π of G_N,*

$$\operatorname{Tr} \pi(e') = \sum_{i=0}^{k-1} \big(g^{k-i} - g^{k-i-1}\big) W_i(\pi) + W_k(\pi).$$

Proof. Put $n = k+1 \leq N$ and embed $G_n \subseteq G_N$ in the upper left corner. Let $Q \subseteq U_N$ be the group of $u \in U_N$ whose upper left $n \times n$ block is 1, and $\psi_Q(z) = \psi_0(\sum_{l=n-1}^{N-2} z_{l,l+1})$, a character of Q. For $p \in P_n$ the conjugate pzp^{-1} replaces the upper right block X of z by pX and keeps the lower right block, and the row $n-1$ of pX equals that of X, so P_n normalizes Q and fixes ψ_Q. Hence $\tau = \pi^{(Q,\psi_Q)}$ is a representation of P_n. Every $u \in U_N$ factors uniquely as $u = u_0 z$ with $u_0 \in U_n$ and $z \in Q$, and u_0 is the upper left block of u. Thus $U' = U_k Q$ with $U_k \subseteq G_k \subseteq P_n$, and $\psi'(u_0 z) = \psi_k(u_0)\psi_Q(z)$, because the row $k = n-1$ of u_0 is e_k^T. Hence $\operatorname{Tr} \pi(e') = \dim \tau^{(U_k,\psi_k)} = A(\tau)$. In the same way $\psi_N^{(i)}(u_0 z) = \psi_n^{(i)}(u_0)\psi_Q(z)$ for $0 \leq i \leq k$, since the omitted entry $(i-1, i)$ lies in the upper left block, so $W_i(\pi) = B_i(\tau)$. Proposition 14.55 with $n = k+1$ and $(g-1)g^{k-1-i} = g^{k-i} - g^{k-i-1}$ ends the proof. □

Corollary 14.57 (Hypothesis 14.42)**.** *Let $(N, k, s) = (r, k, r_S)$ with $k \geq 1$. The element e_J is a self-adjoint idempotent with $e_J(1) = g^{k-N(N-1)/2} > 0$, and for every family (E_c) of orthogonal central idempotents of $\mathbb{C}[G_N]$ with $\sum_c E_c = 1$, in particular for the label idempotents of Section 14.3,*

$$(1 - 1/g)\,\mathrm{wt}(e_U, c) \leq \mathrm{wt}(e_J, c) \qquad \text{for every } c.$$

Proof. U_J is a subgroup (Lemma 14.48, it is $U_{\prec_s}$) defined in U_N by k vanishing entries above the diagonal, so $e_J(1) = |U_J|^{-1} = g^k e_U(1)$. By Theorems 14.51 and 14.56 and $W_i \geq 0$, $\operatorname{Tr} \pi(e_J) \geq (g^k - g^{k-1}) W_0(\pi) = g^k(1 - 1/g) \operatorname{Tr} \pi(e_U)$ for every irreducible π. For $a \in \mathbb{C}[G_N]$, $(aE_\pi)(1) = |G_N|^{-1} \dim \pi \operatorname{Tr} \pi(a)$. Summing over the π with $E_\pi E_c = E_\pi$ gives $(e_J E_c)(1) \geq g^k(1 - 1/g)(e_U E_c)(1)$, and dividing by $e_J(1) = g^k e_U(1)$ gives the claim. □

Lemma 14.58. *Let $a \in \mathbb{C}[G_N]$ be an idempotent and c a label. Then*

$$(aE_c)(1) = \frac{1}{|G_N|} \sum_{\pi \text{ of label } c} \dim \pi \cdot \operatorname{rk} \pi(a) \ \geq 0, \qquad \sum_c (aE_c)(1) = a(1).$$

In particular Lemma 14.33 holds for every idempotent a with $\operatorname{Re} a(1) > 0$, self-adjoint or not.

Proof. The central idempotent of π is $E_\pi = \frac{\dim \pi}{|G_N|} \sum_x \overline{\chi_\pi(x)}\, x$, and $\overline{\chi_\pi(x^{-1})} = \chi_\pi(x)$, so $(aE_\pi)(1) = \sum_x a(x) E_\pi(x^{-1}) = \frac{\dim \pi}{|G_N|} \operatorname{Tr} \pi(a)$. The trace of an idempotent operator is its rank. Summing over the π of label c gives the first identity. Summing over all c gives $\sum_\pi \frac{\dim \pi}{|G_N|} \operatorname{Tr} \pi(a)$, which is the trace of a in the left regular representation divided by $|G_N|$, that is $a(1)$. □

14.3.7 Rows, labels and the cylinder idempotents

Let (τ) be the orthonormal eigenbasis of $\mathcal{H}$ of Section 14.3. For $k \in \{0,1\}$ let $\mathcal{G}_k = \prod_{w \in \Pi_k} \mathrm{GL}_N(\mathbb{F}_{g_k})$ be the quotient of the level at g_k, and for a set S of label tuples at k let $\Pi_S = \sum_{t \in S} \bigotimes_{w \in \Pi_k} E_{t_w}$, a central idempotent of $\mathbb{C}[\mathcal{G}_k]$. The level idempotent factors as $e = e^{(k)} \otimes \bigotimes_{w \in \Pi_k} e_{k,w}$, with $e^{(k)}$ supported away from g_k, and we put $e_S = e\, \Pi_S$.

Hypothesis 14.59 (label dictionary)**.** (i) For $k \in \{0,1\}$ and every set S of label tuples at k, the space of automorphic forms of Section 14.1 defined with e_S in place of e has dimension $\#\{\tau : \mathrm{lab}_k(\tau) \in S\}$. (ii) There is an injective map ι from an index set of the rows of R to the eigenbasis whose image is the set of τ whose label tuples at g_0 and at g_1 are both good, and $\mathrm{lab}_k(\iota(\sigma))$ is the label tuple of the row σ.

Indeed the mutually orthogonal central idempotents E_t indexed by pairs of label tuples sum to 1 and commute with e. Hence $\mathcal{H}$ is the orthogonal direct sum of the spaces $R(E_t)\mathcal{H}$. An orthonormal basis adapted to this sum gives (i) by taking dimensions. For (ii), suppose each row basis vector lies in a good summand and every good summand lies in the row space. Then the row space is precisely the sum over good t. Choosing a bijection of the two bases separately in each summand gives the required row indices.

Lemma 14.60. *The element e_S is a self-adjoint idempotent, and $D^*(e_S) = D^*\, w_k(S)$, where $D^*(e')$ denotes the product of the formal degree, the volume of $[G]$ and $\operatorname{Re} e'(1)$.*

Proof. Π_S is a self-adjoint central idempotent of $\mathbb{C}[\mathcal{G}_k]$, so it commutes with $\bigotimes_w e_{k,w}$, and the product of two commuting self-adjoint idempotents is a self-adjoint idempotent. The coefficient at 1 of a tensor product is the product of the coefficients at 1, so

$$e_S(1) = e^{(k)}(1) \sum_{t \in S} \prod_{w \in \Pi_k} (e_{k,w} E_{t_w})(1) = e(1) \sum_{t \in S} \prod_{w \in \Pi_k} \frac{(e_{k,w} E_{t_w})(1)}{e_{k,w}(1)}.$$

By Lemma 14.58 every factor is real, and $e_{k,w}(1) > 0$ by Corollary 14.57, so the right side is $e(1)\, w_k(S)$ and $D^*(e_S) = D^* w_k(S)$. □

14.3.8 The multiplicity formula

We prove Hypotheses 14.43 and 14.76 from three analytic statements about the pre-trace formula, which are not proved here. Let $\varphi_\infty(y) = d_\infty \prod_v \overline{\langle \pi_v(y_v) w_v, w_v \rangle}$, where w_v is a unit lowest K-type vector of the discrete series π_v at the real place v and d_∞ is the product of the formal degrees. For a self-adjoint idempotent e' of the Hecke algebra of $G(\mathbb{A}_f)$ put $\varphi = \varphi_\infty \otimes e'$, let $R(\varphi)$ be its right convolution operator on $L^2([G])$, let $D(e')$ be the dimension of the space of Section 14.1 defined with e', and let $D^*(e') = d_\infty \operatorname{vol}([G]) \operatorname{Re} e'(1)$. Let $\mathfrak{F}$ be a fundamental domain for $G(F)$ in $G(\mathbb{A})$ and $n = \dim V = r$.

Hypothesis 14.61 (pre-trace inputs, not proved here)**.** There are absolute constants $c_1 > 0$, $c_b > 0$, and for every $A \geq 1$ constants n_0, C_A and A_1, such that the following holds for every datum with $n \geq n_0$ whose level and Harish-Chandra parameters satisfy the bounds of the construction with the constant A, whose idempotent e' is of positive type and supported on a compact open subgroup that is unipotent modulo an auxiliary prime split in E and larger than $4n$, and whose Harish-Chandra parameters have gap at least h at every real place, with

$$h \geq C_A\, n^{19} f^5 \log(b \operatorname{rd}_E), \qquad \log f \leq \sqrt{n}, \qquad \log\log b \leq c_b\, nf \log n. \tag{14.19}$$

(i) $R(\varphi)$ is an orthogonal projection of rank $D(e')$.

(ii) For almost every $x \in \mathfrak{F}$ the series $\sum_{\gamma \in G(F)} \varphi(x^{-1}\gamma x)$ converges absolutely, and its sum is $\sum_j |F_j(x)|^2$ for every orthonormal basis (F_j) of the range of $R(\varphi)$.

(iii) There is a measurable $M \subset \mathfrak{F}$ such that, with $\eta = \exp(-c_1 nf \log n)$,

$$\operatorname{vol}(\mathfrak{F} \setminus M) \leq \eta \operatorname{vol}(\mathfrak{F}),$$
$$\Big| \int_M \sum_{\gamma \neq 1} \varphi(x^{-1}\gamma x)\, dx \Big| \leq A_1 \eta D^*(e'),$$
$$\Big| \int_{\mathfrak{F} \setminus M} \sum_{\gamma} \varphi(x^{-1}\gamma x)\, dx \Big| \leq A_1 e^{-n^2 f} D(e').$$

Theorem 14.62. *Under Hypothesis 14.61 put $c = c_1/2$. There is $n_1(A)$ such that every datum as in Hypothesis 14.61 with $n \geq n_1(A)$ satisfies $|D(e') - D^*(e')| \leq \exp(-c\, nf \log n)\, D^*(e')$.*

Proof. Write $D = D(e')$, $D^* = D^*(e')$ and $K(x) = \sum_\gamma \varphi(x^{-1}\gamma x)$. By (i) and (ii), integrating $\sum_j |F_j|^2$ over $\mathfrak{F}$ gives $D = \int_{\mathfrak{F}} K$. The term $\gamma = 1$ is $\varphi(1) = d_\infty e'(1)$, because $\|w_v\| = 1$. Since e' is a self-adjoint idempotent, $e'(1) = \|e'\delta_1\|^2$ is real and nonnegative, so $\varphi(1) \operatorname{vol}(\mathfrak{F}) = D^*$. Splitting $\mathfrak{F} = M \cup (\mathfrak{F} \setminus M)$ gives

$$D - D^* = -\varphi(1) \operatorname{vol}(\mathfrak{F} \setminus M) + \int_M \sum_{\gamma \neq 1} \varphi(x^{-1}\gamma x)\, dx + \int_{\mathfrak{F} \setminus M} K,$$

and $\varphi(1) \operatorname{vol}(\mathfrak{F} \setminus M) \leq \eta D^*$, so by (iii) $|D - D^*| \leq (1 + A_1)\eta D^* + A_1 \theta D$ with $\theta = e^{-n^2 f}$. Take n so large that $A_1 \theta \leq \frac{1}{2}$ and $n \geq c_1 \log n$. Then $D \leq 2(1 + (1 + A_1)\eta) D^* \leq 2(2 + A_1) D^*$, and $\theta \leq \eta$, so $|D - D^*| \leq B \eta D^*$ with $B = 1 + A_1 + 2A_1(2 + A_1)$. Finally $B\eta \leq \eta^{1/2} = \exp(-c\, nf \log n)$ once $c\, nf \log n \geq \log B$. □

Corollary 14.63. *Assume Hypotheses 14.59 and 14.61, and that the construction satisfies* (14.19) *with $n = r$ for e and for every e_S, the unipotent condition for e_S being imposed at the prime g_{1-k}. Then for x large Hypothesis 14.43 holds with $\varepsilon_C = \exp(-c\,r f \log r) \le \frac{1}{2}$, and Hypothesis 14.76 holds with the same c.*

Proof. The set S of all tuples has $e_S = e$ and $D(e) = D_{\mathrm{all}}$, so Theorem 14.62 gives Hypothesis 14.76 and the first clause of Hypothesis 14.43. For general S, Lemma 14.60 shows that e_S is a self-adjoint idempotent with $D^*(e_S) = D^* w_k(S)$, and Hypothesis 14.59(i) gives $D(e_S) = \#\{\tau : \mathrm{lab}_k(\tau) \in S\}$. Theorem 14.62 applied to e_S is the second clause. Since $r \ge 100$ and $f \ge x^2$, $\exp(-c\,r f \log r) \le \frac{1}{2}$ for x large. □

14.3.9 Sizes

Lemma 14.64. *Let $0 < \varepsilon < \frac{1}{52}$, $J \ge 1$ and $\varepsilon_z = \varepsilon / J$. Suppose $x^{2+8\varepsilon} \le n$, $x^{9/2+10\varepsilon} \le k \le 2x^{9/2+10\varepsilon}$, $m \le 2n \le k$, $g_0, g_1 \ge (kb)^8$ and $\min(q,d) \ge \exp(x^{\varepsilon_z/6})$. Then* (14.17) *holds with $\varepsilon' = \varepsilon_z$, $C_s = 6$, $N = r = k + m - 1$ and $f = 2n$, for $x \ge 4$ large in terms of ε.*

Proof. $g_k \ge (kb)^8 \ge b^8 = e^{8x^{\varepsilon}} \ge e^{8x^{\varepsilon_z}}$. Next $f = 2n \ge x^2$, and $f \le k \le 2x^{9/2+10\varepsilon} \le x^5$ since $\varepsilon < \frac{1}{52}$ and x is large, so $f \le x^6$ and $m \le f \le x^6$. The rank satisfies $k \le r \le 2k \le 4x^{9/2+10\varepsilon} \le x^6$, and $r \ge k \ge x^{9/2} \ge 100$. The last condition is the hypothesis on q and d. □

14.4 Near orthogonality of Bessel Poincaré series

14.4.1 Assembly

Proposition 14.65. *Let $C_0 \ge 1$, $\Gamma_0 > 1$, $D^* > 0$, $\eta_0 \ge 0$, $\xi \ge 0$, $\eta_1 \ge 0$, and assume* (14.1). *Let $K_{uu'} = K^0_{uu'} + K^{\mathrm{deg}}_{uu'}$ with*

(i) $K^{\mathrm{deg}}_{uu'} = 0$ *for* $u \ne u'$,

(ii) $\|K^{\mathrm{deg}}_{uu}\|^2_{\mathrm{HS}} \le \xi d_u^S$,

(iii) $\|K^0_{uu'}\|^2_{\mathrm{HS}} \le (1+\eta_0) d_u^S d_{u'}^S / D^*$ *for* $u \ne u'$,

(iv) $\|K^0_{uu} - 1\|^2_{\mathrm{HS}} \le \Gamma_0 (d_u^S)^2 / D^*$,

(v) $|D_{\mathrm{all}} - D^*| \le \eta_1 D^*$.

Then the conclusion of Theorem 14.1 holds with these η_0, η_1 and $e_0 = 2C_0/\Gamma_0 + 2\xi$.

Proof. Off the diagonal use (i) and (iii). On the diagonal, $\|A+B\|^2 \le 2\|A\|^2 + 2\|B\|^2$ with $A = K^0_{uu} - 1$, $B = K^{\mathrm{deg}}_{uu}$ and (iv), (ii) give $\|K_{uu} - 1\|^2_{\mathrm{HS}} \le 2\Gamma_0 (d_u^S)^2 / D^* + 2\xi d_u^S$, and (14.1) gives $\Gamma_0 (d_u^S)^2 / D^* \le (C_0/\Gamma_0) d_u^S$. □

14.4.2 The pair bound

Unfolding $\langle \mathcal{P}_u T, \mathcal{P}_{u'} T' \rangle$ as in Proposition 14.5, with $\theta = \overline{P_T}$, writes $K_{uu'}$ as a sum over $S_u(F) \backslash G(F) / S_{u'}(F)$ of archimedean orbital integrals of $W_u \otimes \overline{W_{u'}}$. We organize the terms as follows, and the next theorem is the Hilbert space lemma that closes the off-diagonal clause.

Definition 14.66 (term families)**.** A term family consists of a countable set of terms t, each with an operator $\mathrm{op}(t)$ in a Hilbert space of operators, a vector $\mathrm{vec}(t)$ in a Hilbert space H_1, a row $\mathrm{row}(t)$, a Gram value $G(t)$ and two flags, $\mathrm{isId}(t)$ and $\mathrm{gen}(t)$; a finite set I with a map ι from rows to I; a hermitian matrix $(\Gamma_{ij})_{i,j\in I}$ and a number $\Gamma_1 > 0$; for pairs of terms in one row a corank $s_0(t,t') \in \mathbb{Z}_{\geq 0}$ and a flag $\mathrm{scal}(t,t')$. Put $\mathrm{id} = \{\mathrm{isId}\}$, $\mathrm{deg} = \{G = 0, \neg\mathrm{isId}\}$, $\mathrm{ng} = \{G \neq 0, \neg\mathrm{gen}\}$, $\mathrm{gen} = \{G \neq 0, \mathrm{gen}\}$, $\mathrm{part}(S) = \sum_{t\in S} \mathrm{op}(t)$, $v_i(S) = \sum_{t\in S,\ \iota(\mathrm{row}\, t)=i} \mathrm{vec}(t)$. For a set Q of pairs in one row put $\Sigma(Q) = \Gamma_1 \sum_Q \mathrm{Re}\langle \mathrm{vec}(t), \mathrm{vec}(t')\rangle$ and $|\Sigma|(Q) = \Gamma_1 \sum_Q |\langle \mathrm{vec}(t), \mathrm{vec}(t')\rangle|$. Let $\mathrm{big}(S)$ be the pairs in $S \times S$ in one row with $s_0 \geq 2$, $\mathrm{one}(S)$ those with $s_0 = 1$, split into $\mathrm{sc}(S)$ (with scal) and $\mathrm{nsc}(S)$.

Theorem 14.67 (pair bound)**.** *Assume* $\sum_t \|\mathrm{op}(t)\| < \infty$, $\sum_t \|\mathrm{vec}(t)\| < \infty$, *and*

(a) $\|\mathrm{part}(S)\|^2 = \mathrm{Re} \sum_{i,j} \Gamma_{ij} \langle v_i(S), v_j(S)\rangle$ *for* $S = \mathrm{gen}$ *and* $S = \mathrm{ng}$,

(b) $P_g \Gamma_1 \cdot 1 - (\Gamma_{ij})$ *is positive semidefinite for some* $P_g \geq 0$,

(c) $\langle \mathrm{vec}(t), \mathrm{vec}(t')\rangle = 0$ *if* $\iota(\mathrm{row}\, t) = \iota(\mathrm{row}\, t')$ *and* $\mathrm{row}\, t \neq \mathrm{row}\, t'$,

(d) $s_0(t,t') \geq 1$ *if* t, t' *lie in one row and* $G(t), G(t') \neq 0$,

(e) $\mathrm{part}(\mathrm{id}) = 0$ *and* $\mathrm{part}(\mathrm{deg}) = 0$,

(f) with $dd \geq 0$*:* $|\Sigma|(\mathrm{big}(\mathrm{gen} \cup \mathrm{ng})) \leq \varepsilon\, dd$, $|\Sigma|(\mathrm{nsc}(\mathrm{gen})) \leq \varepsilon\, dd$, $\Sigma(\mathrm{sc}(\mathrm{gen})) \leq A\, dd$ *and* $\Sigma(\mathrm{one}(\mathrm{ng})) \leq \varepsilon'\, dd$.

Then $\|\sum_t \mathrm{op}(t)\|^2 \leq \big(\sqrt{P_g(A + 2\varepsilon)} + \sqrt{P_g(\varepsilon + \varepsilon')}\big)^2 dd$.

Proof. By (b), $\mathrm{Re} \sum_{ij} \Gamma_{ij} \langle a_i, a_j\rangle \leq P_g \Gamma_1 \sum_i \|a_i\|^2$ for all vectors a_i. With (a), $\|\mathrm{part}(S)\|^2 \leq P_g \Phi(S)$ where $\Phi(S) = \Gamma_1 \sum_i \|v_i(S)\|^2$. Expanding $\|v_i(S)\|^2$ and using (c), only pairs in one row remain, so $\Phi(S) = \Sigma(\text{pairs of } S \text{ in one row})$. By (d), $\Phi(\mathrm{gen}) = \Sigma(\mathrm{big}(\mathrm{gen})) + \Sigma(\mathrm{nsc}(\mathrm{gen})) + \Sigma(\mathrm{sc}(\mathrm{gen})) \leq (A + 2\varepsilon) dd$ and $\Phi(\mathrm{ng}) = \Sigma(\mathrm{big}(\mathrm{ng})) + \Sigma(\mathrm{one}(\mathrm{ng})) \leq (\varepsilon + \varepsilon') dd$, using $\Sigma \leq |\Sigma|$ and monotonicity of $|\Sigma|$. By (e), $\sum_t \mathrm{op}(t) = \mathrm{part}(\mathrm{gen}) + \mathrm{part}(\mathrm{ng})$, and the triangle inequality ends the proof. □

Lemma 14.68. *For* $g \geq 3$ *and every* k, $\prod_{i=1}^{k} (1 - g^{-i})^{-1} \leq 1 + 3/g$.

Proof. With $y = 1/g$, $\prod_{i\geq 1} (1 - y^i)^{-1} = \sum_{n\geq 0} p(n) y^n$, and $p(n) \leq 2^{n-1}$ for $n \geq 1$ because sorting the parts of a composition gives every partition. So the product is at most $1 + y/(1 - 2y) \leq 1 + 3y$ for $y \leq 1/3$. □

Lemma 14.68 is the estimate behind the value $P_g = (1 + 3/g_1)^f (1 + 3/g_2)^f$ in Hypothesis 14.71 below: the matrix (Γ_{ij}) contains the finite group factors at the typed places, and its diagonal dominance costs a p-adic volume factor of the form $\prod_i (1 - g^{-i})^{-1}$ at each of them.

Lemma 14.69 (closing numerics)**.** *Let* $b \geq 16$, $0 \leq P_g \leq e^{6b^{-7}}$, $A \leq (1 + b^{-7}) e^{b^{-14}}$ *and* $0 \leq \varepsilon, \varepsilon' \leq b^{-14}$. *Then* $\big(\sqrt{P_g(A + 2\varepsilon)} + \sqrt{P_g(\varepsilon + \varepsilon')}\big)^2 \leq 1 + b^{-6}$.

Proof. Put $t = b^{-7} \leq 16^{-7}$. Then $P_g(A + 2\varepsilon) \leq e^{6t}(1 + t) e^{t^2} + 2e^{6t} t^2 \leq 1 + 8t$ and $P_g(\varepsilon + \varepsilon') \leq 2e^{6t} t^2 \leq 3t^2$. The square is at most $1 + 8t + 3t^2 + 2\sqrt{3t^2(1 + 8t)} \leq 1 + 12t$, and $12t = 12 b^{-1} b^{-6} \leq b^{-6}$. □

Corollary 14.70 (off-diagonal and diagonal clauses from the row estimates)**.** *Let $b \geq 16$. Suppose that for each pair of words the terms of $K_{uu'}$ form a term family with (a) to (d) of Theorem 14.67, with $P_g \leq e^{6b^{-7}}$, and with* $\mathrm{part}(\mathrm{deg}) = K_{uu'}^{\mathrm{deg}}$, $\mathrm{part}(\mathrm{id}) = 0$ *for $u \neq u'$ and* $\mathrm{part}(\mathrm{id}) = 1$ *for $u = u'$. Suppose the row estimates (f) hold with $\varepsilon, \varepsilon' \leq b^{-14}$ and*

$$dd = d_u^S d_{u'}^S / D^*, \qquad A \leq (1 + b^{-7}) e^{b^{-14}} \ (u \neq u'), \qquad A \leq \gamma_B \ (u = u'),$$

with $\gamma_B \geq 1$ and $\Gamma_0 \geq 6\gamma_B$. Then (iii) of Proposition 14.65 holds with $\eta_0 = b^{-6}$, and (iv) holds.

Proof. For $u \neq u'$ apply Theorem 14.67 to the terms other than the degenerate ones, whose sum is $K_{uu'}^0$, and Lemma 14.69. For $u = u'$ apply Theorem 14.67 to the terms other than the identity and degenerate ones, whose sum is $K_{uu}^0 - 1$. With $(\sqrt{a} + \sqrt{b})^2 \leq 2a + 2b$ the bound is $2P_g(\gamma_B + 4b^{-14})(d_u^S)^2/D^* \leq 6\gamma_B (d_u^S)^2/D^* \leq \Gamma_0 (d_u^S)^2/D^*$. □

14.4.3 The inputs from the construction

Put $r_S = m/2 - 1$, the rank parameter of a row. The ambient rank remains $r = k + m - 1$. Proposition 14.65 and Corollary 14.70 reduce Theorem 14.1 to the following statements about the concrete Gram operators.

Hypothesis 14.71 (S-KE, kernel expansion)**.** For every pair of words the unfolding of $\langle \mathcal{P}_u T, \mathcal{P}_{u'} T' \rangle$ converges absolutely and produces a term family with (a) to (d) of Theorem 14.67, $P_g = (1 + 3/g_1)^f (1 + 3/g_2)^f$, $\mathrm{part}(\mathrm{id}) = 0$ and $\mathrm{part}(\mathrm{deg}) = 0$ for $u \neq u'$, $\mathrm{part}(\mathrm{id}) = 1$ for $u = u'$.

Hypothesis 14.72 (S-MR, main row)**.** $\Sigma(\mathrm{sc}(\mathrm{gen})) \leq (1 + \delta) e_1 \, d_u^S d_{u'}^S / D^*$ for $u \neq u'$, with $0 \leq \delta \leq b^{-7}$ and $1 \leq e_1 \leq \exp(12 f 2^{-(k-2r_S-2)})$; and $\Sigma(\mathrm{sc}(\mathrm{gen})) \leq \gamma_B (d_u^S)^2/D^*$ for $u = u'$, with $\gamma_B = \exp(C_B m^5 f \log b)$.

Hypothesis 14.73 (S-LR, loops)**.** Each of the sums $|\Sigma|(\mathrm{big}(\mathrm{gen} \cup \mathrm{ng}))$ and $|\Sigma|(\mathrm{nsc}(\mathrm{gen}))$ is at most $\exp(-\frac{1}{3} k f \log k) \, dd$.

Hypothesis 14.74 (S-NG, nongeneric terms)**.** $\Sigma(\mathrm{one}(\mathrm{ng})) \leq 2^{-(k-2r_S-5)} \exp(C_4 m^5 f \log b) \, dd$.

Hypothesis 14.75 (S-DG, degenerate terms)**.** $\|K_{uu}^{\mathrm{deg}}\|_{\mathrm{HS}}^2 \leq \exp(-kf \log k) \, d_u^S$.

Hypothesis 14.76 (S-DIM, multiplicity formula)**.** $|D_{\mathrm{all}} - D^*| \leq \exp(-c\, rf \log r) \, D^*$ for an absolute $c > 0$.

Hypothesis 14.77 (S-SZ, sizes)**.** $b = e^{x^\varepsilon} \geq 16$, $(kb)^8 \leq g_1, g_2$, $f \leq k^8 b$, $70 \log b \leq k$, $k \geq 8 \log \Gamma_0$, $C_\Gamma \geq \max(C_B, C_4) + 1$, the quantities $\exp(-\frac{1}{3} k f \log k)$ and $2^{-(k-2r_S-5)} \exp(C_4 m^5 f \log b)$ are at most b^{-14}, $12 f 2^{-(k-2r_S-2)} \leq b^{-14}$, and $c\, rf \log r \geq x^{\varepsilon/8}$. We also require $32 \leq k$, $4r_S + 12 \leq k$ and $8C_4 m^5 f \log b \leq k$.

Lemma 14.78 (sizes)**.** *Let $b = e^{x^\varepsilon}$, and let m, k, n be as in Proposition 5.15 with $f = 2n$, $r_S = m/2 - 1$. For x large in terms of ε and C_4: $32 \leq k$, $4r_S + 12 \leq k$, $f \leq k$, $70 \log b \leq k$, $8C_4 m^5 f \log b \leq k$, and consequently*

$$12 f 2^{-(k-2r_S-2)} \leq b^{-14}, \qquad 2^{-(k-2r_S-5)} \exp(C_4 m^5 f \log b) \leq b^{-14}.$$

Proof. By Proposition 5.15, $m \le 2\sqrt{x}$, $f = 2n \le 4x^{2+8\varepsilon}$ and $k \ge x^{9/2+10\varepsilon}$. So $4r_S + 12 = 2m + 8 \le k$, $f \le k$, $70x^{\varepsilon} \le k$, and $8C_4 m^5 f \log b \le 1024\, C_4 x^{9/2+9\varepsilon} \le k$ for large x. From $4r_S + 12 \le k$ we get $k - 2r_S - 2 \ge k/2 + 4$ and $k - 2r_S - 5 \ge k/2 + 1$. Then $12f2^{-(k-2r_S-2)} \le f2^{-k/2} \le e^{-k/5}$ since $f \le k \le e^{(\frac{\log 2}{2} - \frac{1}{5})k}$ for $k \ge 32$, and $2^{-(k-2r_S-5)} e^{C_4 m^5 f \log b} \le 2^{-k/2} e^{k/8} \le e^{-k/5}$. Finally $e^{-k/5} \le b^{-14}$ because $70 \log b \le k$. □

Theorem 14.79. *Hypotheses 14.71 to 14.77 imply Theorem 14.1 for x large in terms of ε and C_0.*

Proof. By 14.77, $P_g \le \exp(3f(g_1^{-1} + g_2^{-1})) \le \exp(6f(kb)^{-8}) \le e^{6b^{-7}}$, $(1+\delta)e_1 \le (1+b^{-7})e^{b^{-14}}$, and $\varepsilon = \exp(-\frac{1}{3}kf \log k)$, ε' of 14.74 are at most b^{-14}. Also $\Gamma_0/\gamma_B = \exp((C_\Gamma - C_B)m^5 fL) \ge e^L \ge 6$ since $L = \log b$. Corollary 14.70 gives (iii) with $\eta_0 = b^{-6}$ and (iv) of Proposition 14.65; 14.71 gives (i); 14.75 gives (ii) with $\xi = \exp(-kf \log k)$; 14.76 gives (v) with $\eta_1 = \exp(-crf \log r)$. Proposition 14.65 gives the three clauses with $e_0 = 2C_0/\Gamma_0 + 2\xi$. With $y = x^{\varepsilon/8}$: $\eta_0 = e^{-6y^8} \le e^{-y}$; $\eta_1 \le e^{-y}$ by 14.77; $\log \Gamma_0 \ge C_\Gamma y^8$ and $k \ge 8 \log \Gamma_0$ give $2C_0/\Gamma_0 + 2\xi \le e^{-2y}$ for large y, so $\sqrt{e_0} \le e^{-y}$. □

Remark 14.80 (the window). Theorem 14.1 assumes (14.1). Theorem 13.15 uses its conclusion, so the construction of the words must establish (14.1). The lower half of (14.1) is the hypothesis $d_u^S \ge D^*/(C_0\Gamma_0^2)$ of Lemma 13.19.

Remark 14.81 (no circularity). Lemma 14.14 derives the nonvanishing used in the proof of Theorem 14.3 from the diagonal clause of Theorem 14.1. Theorem 14.1 is a statement about Poincaré series alone and its inputs 14.71 to 14.77 do not involve the period maps, so there is no cycle. Theorem 13.15 uses all three theorems.

14.4.4 The error in the multiplicity formula

Let n denote the rank in the counting theorem, f the degree of the totally real field, D the multiplicity and M the identity contribution. With the Haar measures used for the formal degrees d_v, one has $M = (\prod_{v|\infty} d_v)\,\mathrm{vol}(\mathfrak{F})\,\mathrm{Re}\, e(1)$ for the level idempotent e. In particular $M \ge 0$. The trace estimate also requires finite dimensionality of the multiplicity space. In the varying-rank estimate the gap in the weights is at least

$$8 \cdot 10^{14}(C_g + 4)n^{19} f^5 \log(b\, \mathrm{rd}_E),$$

where the level parameter g satisfies $1 \le g \le b^{C_g}$. The hypotheses include a common signature (p, q) with $q^2 \le n$, a prime $g_0 > 4n$ split completely in E, and the bounds $\log f \le \sqrt{n}$, $\log\log \mathrm{rd}_E \le n$, $\log\log b \le nf \log n/16$, and $|\lambda_{v,i}| \le b^2$.

14.4.5 Compactly supported Gram-fibre identities

Let E/F be a quadratic CM extension with involution σ. Let $V = E^M$ be equipped with a nonsingular Hermitian form h_J and unitary group U_J, and let $\mathcal{H}_n$ denote Hermitian $n \times n$ matrices. For $X = (X_1, \ldots, X_n) \in V^n$, define $\mathrm{Gram}_J(X)_{ab} = h_J(X_a, X_b)$ and $\mathcal{H}_n^{\times} = \{S : \det S \text{ is a unit}\}$. Fix an F-basis of E. At every place v of F, choose additive Haar measure with integral covolume one and transport it to matrix and Hermitian coordinates as follows. The diagonal Hermitian coordinates are $\frac{1}{2}\,\mathrm{Tr}_{E/F}(S_{ii})$, and the off-diagonal coordinates are the coordinates of S_{ij} in the chosen basis. Write $dX_v^{\circ}, dS_v^{\circ}$ for these measures and L_v for the standard local lattice of matrices.

On $\mathcal{H}_n^\times$, disintegrate dX_v° over dS_v° along Gram_J. Let $\omega_{v,S}^\circ$ be the fibre measures and define

$$D_v^\circ(\varphi, S) = \int_{\mathrm{Gram}_J^{-1}(S)} \varphi \, d\omega_{v,S}^\circ, \qquad \ell_v^\circ(T) = \omega_{v,T}^\circ(L_v).$$

Thus

$$\int_{\mathrm{Gram}_J^{-1}(K)} \varphi \, dX_v^\circ = \int_K D_v^\circ(\varphi, S) \, dS_v^\circ$$

for every compact $K \subset \mathcal{H}_n^\times$. In particular, $D_v^\circ(\varphi, \cdot)$ is continuous on that locus for continuous compactly supported φ. Empty fibres have measure zero.

Let ψ be a nontrivial additive character of $\mathbb{A}_F$. If dX_v, dS_v are self-dual for the trace pairings used in the Schrödinger model, write $dX_v = c_v dX_v^\circ$ and $dS_v = e_v dS_v^\circ$. Fourier inversion gives

$$c_v = e_v = 1 \quad \text{for almost all } v, \qquad c_\infty \prod_v c_v = e_\infty \prod_v e_v = 1.$$

Uniqueness of fibre disintegration implies

$$D_v^{\mathrm{sd}} = \frac{c_v}{e_v} D_v^\circ$$

for the densities taken with dX_v, dS_v.

Proposition 14.82. *Suppose $2n < M$ and $T \in \mathcal{H}_n(E)$ is nonsingular. Let Φ_∞ be continuous, rapidly decreasing, smooth and compactly supported, and let Φ_v be Schwartz–Bruhat, with $\Phi_v = 1_{L_v}$ outside a finite set. Put $\Phi = \Phi_\infty \prod_v \Phi_v$. For every Haar measure μ on $U_J(\mathbb{A})$ and every fundamental domain $\mathcal{F}$ for $U_J(F)$, one has $0 < \mu(\mathcal{F}) < \infty$ and*

$$\mu(\mathcal{F})^{-1} \int_{\mathcal{F}} \sum_{X \in V^n:\, \mathrm{Gram}_J(X) = T} \Phi(h^{-1}X) \, d\mu(h) = D_\infty^{\mathrm{sd}}(\Phi_\infty, T) \prod_v D_v^{\mathrm{sd}}(\Phi_v, T).$$

The finite product multiplies. Moreover, the fibre measures form an Euler product: $\ell_v^\circ(T) < \infty$ for every finite v, $\sum_v |\ell_v^\circ(T) - 1| < \infty$, and, for every sufficiently large finite set S,

$$\int \Phi_\infty \prod_{v \in S} \Phi_v \, d\mathcal{M}_T = D_\infty^\circ(\Phi_\infty, T) \prod_{v \in S} D_v^\circ(\Phi_v, T) \prod_{v \notin S} \ell_v^\circ(T),$$

where $\mathcal{M}_T$ is the normalized pushforward of $\mu|_{\mathcal{F}}$ along the rational fibres, and the functions are evaluated on the same orbit variables on both sides.

The proof uses only the global convergent-range Siegel–Weil formula as an external analytic theorem. All local normalizations and tails are derived below.

14.4.6 The normalized global formula

For the adelic Weil representation ω, write $\Theta(\Phi, g, h) = \sum_{X \in V^n} \omega(g, h)\Phi(X)$, and let $E(g, s, \Phi)$ be the associated Siegel Eisenstein series. Weil's convergent-range formula [79, Theorem 5], in the unitary formulation also recalled in [40, Theorem 2.3.1], asserts for $M > 2n$ that

$$\frac{1}{2} \int_{[U_J]} \Theta(\Phi, g, h) \, dh = E(g, (M-n)/2, \Phi),$$

where dh is Tamagawa measure. Any admissible auxiliary splitting character may be used for the dual pair; the big-cell comparison below removes that choice from the identity at the identity. No signature restriction is needed. If the Witt index r is at most n, then $M - r > n$; otherwise $M \geq 2r$ and again $M - r > n$. The full unitary Tamagawa covolume is two: $\tau(SU_J) = 1$ [43], Ono's Main Theorem gives covolume two for the quadratic norm-one torus [53], and the relative theorem transports this through the determinant sequence [54, 80]. Hence division by the covolume removes the displayed factor $1/2$. Taking a nonsingular T-Fourier coefficient at the identity gives

$$\bar{I}_T(\Phi) = \prod_v W_{v,T}(1, \Phi_v).$$

Borel's theorem [11, Theorem 5.6(i)] applies because the abelian quotient of U_J is the norm-one torus, which has no nontrivial F-rational character. This gives finite covolume. Comparison of fundamental domains and Haar scaling give it for every $\mu, \mathcal{F}$ in the proposition, and covolume-normalized means are independent of those choices.

In the large Bruhat cell, the Schrödinger action and Fubini give

$$W_{v,T}(1, \varphi) = D_v^\circ(\varphi, T). \tag{14.20}$$

Indeed, after integrating over the unipotent Hermitian variable, one obtains the Fourier transform at T of $\mathrm{Gram}_*(\varphi\, dX_v^\circ)$; the coarea definition identifies this with $D_v^\circ(\varphi, T)$. The same equality holds with any consistent scaling of input and target measures. For $n = 1$, the Hermitian scalar is identified with an element of F by half the trace, and (14.20) is the rank-one Gram-fibre identity.

Varying the test functions determines an identity of locally finite Radon measures. Compact finite-place cylinders are approximated by locally constant functions; compactly supported Borel functions at infinity by smooth functions; and general nonnegative measurable functions by increasing compact truncations and monotone convergence. The scalar identity therefore includes the measurable functions in the eventual Euler-product statement.

14.4.7 The cofinite error estimate

It remains to bound $\ell_v^\circ(T) - 1$. Begin with $n = 1$. At almost every finite place, the trace form in the chosen coordinates is an integral unimodular quadratic form Q in $2M$ variables, while 2 and the fixed nonzero scalar t are both units. Every residue-field solution is then smooth: otherwise nondegeneracy forces $x = 0$, contradicting $Q(0) = 0 \neq t$. Hensel lifting identifies the local mass with the first-level count:

$$\ell_v^\circ(t) = q_v^{1-2M} \#\{x \in k_v^{2M} : Q(x) = t\}.$$

The character-sum evaluation of this count gives

$$N_1 = q_v^{2M-1} + q_v^{-1} \sum_{a \in k_v^\times} G(aQ) e(-at).$$

The Gauss sum of a nondegenerate quadratic form in $2M$ variables has modulus q_v^M, so

$$|\ell_v^\circ(t) - 1| \leq q_v^{1-M} \leq 2q_v^{1-M}.$$

For $M \geq 3$, the error is at most $2q_v^{-2}$.

At good inert places, every unimodular Hermitian lattice has an orthonormal basis. A unit-norm vector exists: otherwise polarization, with $1/2$, would make every Hermitian pairing nonunit, contrary to unimodularity. Such a vector splits off after scaling, using surjectivity of the norm on units, and induction applies to its complement. The split case follows from a basis and its dual basis.

Now split $T \simeq (t) \perp T_1$. Two-stage coarea over the first represented vector gives

$$\ell_v^\circ((t) \perp T_1) = \delta_v(t)^n \ell_v^\circ(T_1)\ell_v^\circ(t),$$

where $\delta_v(t)$ is the coordinate Jacobian of multiplication by t. At almost every place $\delta_v(t) = 1$, and all congruence and Gram determinants are units. The identity $xy - 1 = (x-1)(y-1) + (x-1) + (y-1)$ shows that an error $2q_v^{-M}$ for the first factor and $C_1 q_v^{-2}$ for the second gives an error $(3|C_1| + 2)q_v^{-2}$ for their product. Induction yields a finite set S_0 and a constant C such that

$$|\ell_v^\circ(T) - 1| \le C q_v^{-2} \qquad (v \notin S_0).$$

All lattice masses are finite by compactness. Hence $\sum_v |\ell_v^\circ(T) - 1| < \infty$. Since $D_v^\circ(\Phi_v, T) = \ell_v^\circ(T)$ whenever $\Phi_v = 1_{L_v}$, the finite local densities form a multipliable family for every test function in the proposition. Empty fibres cause no convergence problem.

Combining the normalized global formula, (14.20), and this tail estimate proves the coordinate version of the proposition in every rank. Finally,

$$\left(\frac{c_\infty}{e_\infty}\right) \prod_v \frac{c_v}{e_v} = 1,$$

so replacing D° by D^{sd} leaves the completed product unchanged. The scaled densities have the required continuity and compact-set identities by definition of fibre disintegration. Rank zero is elementary: all relevant spaces are singletons, all tail masses are one, both sides equal $\Phi(0)$, and a self-dual measure on a singleton has mass one.

14.4.8 The Witt-block form

The same normalization is compatible with induction on the target rank. Let $J' = (t) \perp J$ and $T' = (t) \perp T$. Witt extension [52, Theorem 42:17] and measurable selection on the locally closed orbit give a section $s(y)$ from the distinguished vector e to y on $q(y) = t$. Define

$$\Psi(y) = \int_X \varphi(s(y), X)\, d\omega_T^\circ(X).$$

Stabilizer invariance makes Ψ independent of the section, and two-stage coarea gives

$$\int \varphi\, d\omega_{T'}^\circ = \delta_v(t)^n \int \Psi(y)\, d\omega_{(t),1}^\circ(y).$$

Here $\delta_v(t) = 1$ for almost all v and $\delta_\infty(t) \prod_v \delta_v(t) = 1$. If $\mathcal{M}_{J,T}$ is the averaged orbit measure and $\mathcal{M}_{J',(t)}^{(1)}$ is its rank-one analogue, decomposition of global frames by their distinguished vector gives

$$\int \varphi\, d\mathcal{M}_{J',T'} = \Big(\prod_{v \notin S} \ell_v^\circ(T)\Big) \int \Psi\, d\mathcal{M}_{J',(t)}^{(1)}.$$

For each fixed first vector, the inner integral is the lower-rank orbit integral; stabilizer and fundamental-domain invariance identify it with the lower Euler formula. Nonnegative truncation followed by real/imaginary decomposition permits the interchange. The product formula then cancels $\delta_\infty^n \prod_{v\in S} \delta_v^n$ before passage to the limit, so the block Euler product has constant one.

Congruence of J or T preserves the statement: the induced group conjugacy identifies fundamental domains, while the source and target Jacobians enter in the same ratio and have adelic product one. If q_J does not represent t rationally, the scalar and block global measures vanish. The scalar Euler identity forces an archimedean or a finite local scalar fibre measure to vanish; the block fibre at the same place also vanishes by projecting to the first vector. The absolutely convergent zero-Euler-product statement follows. Peeling off an anisotropic scalar and applying these three operations proves the block-induction form without excluding unrepresented targets.

14.4.9 Measure of an adapted flag class

Let $\mathfrak{A} = (I, I_0, e, x_f, S, \Lambda, x_0, T, T', T_0)$ be an adapted model. Thus e is compatible with the involutions, $n = n_0 + 2J$ where $J = \sum_i j_i$, each Λ_i is a full lattice, and the blocks satisfy

$$T_i^* J T_k = (T_i')^* J T_k' = 0, \qquad T_i^* J T_k' = 0 \quad (i \neq k),$$

together with $T_i^* J T_i' = 1$, $T_i^* J T_0 = (T_i')^* J T_0 = 0$, and $T_0^* J T_0 = J_0$. If L, L_0 are the two self-dual lattices and $\Lambda_i^\vee$ is the trace dual, the adapted-lattice condition is

$$z \in L \quad \Longleftrightarrow \quad z = \sum_i T_i u_i + \sum_i T_i' u_i' + T_0 e(y), \qquad u_i \in \Lambda_i, \quad u_i' \in \Lambda_i^\vee, \quad y \in L_0,$$

and S_k is the span of the initial T-blocks. Assume $j_i \geq 1$. Fix the common signature (p, q) of I. The whole-group and subgroup gauges are normalized by their Hilbert–Schmidt small-ball asymptotics: in dimension d, the measure of the radius-r ball, divided by r^d, tends to the volume of the Euclidean unit ball. These limits use $n^2, d_N, (n-2J)^2 + 2\sum_i j_i^2$, respectively; the compact subgroup measure is a probability. Fix gauge constants $c = (c_v)$ and $c_0 = (c_{0,v})$ (necessarily positive). Isotropy gives $J \leq \min(p, q)$.

Write $f = [F : \mathbf{Q}]$, $v = v_{\mathrm{inv}}$, and $d_N = 4\sum_{i<k} j_i j_k + 2J(n-2J) + \sum_i j_i^2$. Let $\mathcal{H}_a$ consist of those τ with $0 < \tau_1 < \cdots < \tau_a < 1/2$. Define

$$\mathcal{L}_f(\tau) = \prod_{i<k} (\tau_i/\tau_k)^{2j_i j_k f} (\tau_i \tau_k)^{2j_i j_k f} \prod_i \tau_i^{2j_i(n-2J)f} (\tau_i^2)^{j_i^2 f}, \qquad D(\tau) = \mathcal{L}_f(\tau) \prod_i \frac{2 f j_i}{\tau_i}.$$

Let $\ell_v(m)$ be the same expression with $f = 1$, evaluated at the local determinant ratios $(d_{v,i+1}(m)/d_{v,i}(m))^{-1/j_i}$.

For a group Γ acting on the left, put

$$\mathrm{CV}_\Gamma(\mu; A) = \inf_D \mu(D),$$

where D runs through measurable sets meeting every left Γ-orbit in A. Let Γ, Γ_0 be the flag and residual lattice stabilizers, and let A_B be the points whose successive flag sizes lie in B. Let V_i be the volume of the determinant slab $1 \leq d_i < \mathrm{e}$ modulo $\mathrm{Aut}(\Lambda_i)$, and let V_0 be $(\prod_v c_{0,v})$ times the covering covolume of Γ_0. Denote by $\iota(x)$ the value of a nonnegative real x in $[0, \infty]$. Finally, with U_r normalized by $U_0 = 1$ and $U_r/U_{r-1} = 2^{r-1}(2\pi^r)/(r-1)!$, put

$$c_{\mathrm{Iw}} = 2^{-d_N/2} \frac{U_p U_q}{\left(\prod_i U_{j_i} 2^{j_i^2/2}\right) U_{p-J} U_{q-J}}.$$

Theorem 14.83. *For every adapted model, every a and (j_i) with $j_i \geq 1$, every choice of the above gauges, and every measurable $B \subseteq \mathcal{H}_a$,*

$$\begin{aligned} &\iota\left(\prod_v c_v\right) \mathrm{CV}_\Gamma(A_B) \\ &\quad \leq \iota\left(c_{\mathrm{Iw}}^f (2v)^{f d_N/2}\right) \left(\prod_i \iota(2^{f j_i^2}) \mathsf{V}_i\right) \mathsf{V}_0 \int_B \iota(D(\tau))\, d\tau. \end{aligned} \tag{14.21}$$

Proof. The index $a = 0$ and the case $A_B = \varnothing$ are included, the latter by taking the empty covering domain.

1. The real-place measure identity. Let N, M, K be the unipotent radical, the flag Levi subgroup, and the standard maximal compact subgroup of the real unitary group. Orthogonal projections adapted to the flag give $G = NMK$ and show that $um \in K$ with $(u, m) \in N \times M$ forces $u = 1$. Since the ambient group is reductive, N is nilpotent, and K is compact, the closed subgroups satisfy the hypotheses of the Haar-decomposition theorem [35, Theorem 8.32 and Corollary 8.31]. The modular function is computed by the determinant of the adjoint action [35, Proposition 8.27].

To evaluate it here, double the flag by duality. If D_s is the determinant on the s-th step and r_s its complex dimension, the adjoint determinant is $\prod_s D_s^{r_{s+1}} D_{s+1}^{-r_s}$. Self-duality identifies opposite steps and yields $\dim F_{2a+1-s} + \dim F_s = n$. Consequently the pushforward of unipotent Haar measure by $u \mapsto mum^{-1}$ is $\iota(\ell(m))$ times itself. Hence there is a finite nonzero κ with

$$\int \phi\, dg = \kappa \int_N \int_M \iota(\ell(m)) \int_K \phi(umk)\, dk\, dm\, dn.$$

The value of $c\kappa$ is obtained from small balls, not from a generic measure comparison. Put $d_M = (n-2J)^2 + 2\sum_i j_i^2$ and $d_L = (p-J)^2 + (q-J)^2 + \sum_i j_i^2$. Then $n^2 = 2d_N + d_M$ and $p^2 + q^2 = d_N + d_L$. The volume of K is $U_p U_q$, and the volume of $K \cap M$ is $U_L = (\prod_i U_{j_i} 2^{j_i^2/2}) U_{p-J} U_{q-J}$; the powers of 2 come from the two copies of each isotropic block in the Hilbert–Schmidt metric. Orthogonal diagonalization of the Lie-algebra Jacobian gives a unit body of volume $2^{d_N/2}\,\mathrm{ballVol}(2d_N + d_M)\,\mathrm{ballVol}(d_L)$. If μ_K is the probability measure and μ_0 is the Riemannian gauge on K, then $\mu_0 = (U_p U_q)\mu_K$ by Haar uniqueness. Comparing the two small-ball asymptotics, with a slab on $K \cap M$ squeezed between radii $r/(1+\eta)$ and $(1+\eta)r$, gives $c\kappa U_L 2^{d_N/2} = U_p U_q$. Thus $c\kappa = c_{\mathrm{Iw}}$. Taking the finite product over the f places and applying Tonelli gives

$$\begin{aligned} &\iota\left(\prod_v c_v\right) \int \phi\, dg \\ &\quad = \iota(c_{\mathrm{Iw}}^f) \int_{N_G} \int_{M_G} \prod_v \iota(\ell_v(m_v)) \int_{K_G} \phi(umk)\, dk\, dm\, dn. \end{aligned} \tag{14.22}$$

2. Integral radical covolume. Complete the flag by duality and let r_b and h_b be its graded ranks and heights. Setting τ_i equal to the size of the i-th graded piece at the identity, we have $h_i = \tau_i^{2f j_i}$, $h_{2a-i} = \tau_i^{-2f j_i}$, $h_a = 1$, $r_i = r_{2a-i} = j_i$, and $r_a = n - 2J$. For $0 \leq c < 2a+1$, define

$$A_c = \sum_b r_b r_{b+c+1}, \qquad Q_c = \prod_b h_b^{r_{b+c+1}} h_{b+c+1}^{-r_b}.$$

The c-th nilpotent layer consists of homomorphism blocks. If a biorthogonal block frame changes Λ_b to $G_b\Lambda_b$, then $\operatorname{cov}(\Lambda_b)\prod_v|\det G_{v,b}|^2 = v^{fr_b}h_b$. Evaluating the source and target homomorphism covolumes therefore gives $\operatorname{cov}(L_c) = v^{fA_c}Q_c$.

Now separate real and imaginary coordinates. Conjugation symmetry gives $\operatorname{cov}(A)\operatorname{cov}(B) \le 2^{fA_c}\operatorname{cov}(L_c)$. The two covolumes are equal. To see this, for each rational prime p choose a skew algebraic integer δ, purely imaginary at every selected embedding, whose real squared determinant N is prime to p. Such a δ is obtained by the Chinese remainder theorem away from the finitely many maximal ideals above p. Multiplication by δ sends A to B and B to A. If T denotes this map, integral bases yield $|\det T|\operatorname{cov}(A) = n_1\operatorname{cov}(B)$ and $|\det T|\operatorname{cov}(B) = n_2\operatorname{cov}(A)$ with $n_1n_2 = N^{A_c}$. If ρ is the covolume ratio, then for every p there are natural a, b such that $\rho^2 b = a$ and $p \nmid ab$. Varying p proves $\rho = 1$. A real fundamental parallelepiped D_c therefore satisfies $|D_c| \le \sqrt{(2v)^{fA_c}Q_c}$.

The exponential chart has unipotent triangular coordinate changes between successive layers. Correcting one layer at a time therefore gives a covering domain with $\mathrm{CV}_{\Gamma_N} \le \prod_c |D_c|$. Reindexing by ordered pairs gives $\sum_c A_c = d_N$ and $\prod_c Q_c = \mathcal{L}_f(\tau)^2$. It follows that

$$\mathrm{CV}_{\Gamma_N} \le \iota\Big((2v)^{fd_N/2}\mathcal{L}_f(\tau)\Big). \tag{14.23}$$

3. Levi integration. The adapted basis gives a measurable Levi chart Θ for which

$$\mathrm{size}_i(\Theta(z, g_0)) = (d_i(z_i)/\kappa_i)^{-1/(2fj_i)}, \qquad \kappa_i = \tau_i^{2fj_i},$$

and $d\mu_M = (\prod_i 2^{fj_i^2})(\prod_v c_{0,v})$ times the pushforward of the product of the matrix Haar measures with $d\mu_{G_0}$. The singular locus has measure zero, because $j_i \ge 1$ makes the matrix Haar density vanish there, so restriction to $d_i > 0$ loses no mass. Choose a measurable scale-stable fundamental domain $\mathcal{D}_i$ for $\mathrm{Aut}(\Lambda_i)$, using a shortest orbit representative and a real-coordinate lexicographic tie-break. If $N_i = 2fj_i$, then

$$\int_{\mathcal{D}_i} H((d_i/\kappa_i)^{-1/N_i})\,dz_i = \mathsf{V}_i\int_0^\infty H(t)\frac{N_i}{t}\,dt. \tag{14.24}$$

Indeed, translation in $\log d_i$ reduces the left side to one slab with the weight dt/t, and $t = \kappa_i\tau^{-N_i}$ has $|dt/t| = N_i\,d\tau/\tau$. Multiplication over i and covering the residual factor by a Γ_0-domain shows that the weighted Levi covolume is at most

$$\left(\prod_i \iota(2^{fj_i^2})\mathsf{V}_i\right)\mathsf{V}_0\int_{\tau_i>0} F(\tau)\iota\left(\prod_i\frac{2fj_i}{\tau_i}\right)d\tau.$$

The residual covolume is nonzero because Γ_0 is discrete, so passing to the infimum over residual domains is legitimate. Put $F = \mathbf{1}_B\iota(\mathcal{L}_f)$. On B, every factor is positive, and the last integral is $\int_B \iota(D(\tau))\,d\tau$.

4. Assembly. If $A_B \ne \varnothing$, the size-movement formula

$$\mathrm{size}_i(umk)^f = \tau_i^f\prod_v \mathrm{locSize}_v(m)_i$$

and membership of one point in A_B prove $\tau_i > 0$. Also $\mathcal{L}_f(\mathrm{size}(m)) = \mathcal{L}_f(\tau)\prod_v \ell_v(m_v)$. Take $Z = \{m : \mathrm{size}(m) \in B\}$. A flag lattice stabilizer fixes L and every S_k, hence leaves every graded

size unchanged; in particular, A_B is invariant under its action. The radical estimate and the size relation allow the base $\mathcal{L}_f(\tau)$ to be taken outside the N-integral and cancelled against the indicator-weighted Levi integrand.

More explicitly, fix measurable N- and M-covering domains and a measurable section of $K/(K\cap M)$. Form the corresponding measurable NMK slice. Its orbit meets A_B, and (14.22) bounds the flag covering covolume by $c_{\mathrm{Iw}}^{f}\mu_N(D_N)\int_{D_M\cap Z}\prod_v \ell_v\,dm$. Taking the infimum over D_N, then using (14.23) and the slab computation, and finally taking the infimum over D_M, gives (14.21). The zero and infinite M-integral cases are handled directly; all other scalar–infimum interchanges involve finite nonzero constants. Every real quantity embedded by ι is nonnegative on the indicated domain. □

14.4.10 Definite unimodular Hermitian planes

Notation. Let E be a totally complex CM field with maximal real subfield F, and let $x\mapsto \bar{x}$ be complex conjugation. Put

$$\mathcal{O}=\mathcal{O}_E,\qquad f=[F:\mathbb{Q}],\qquad D=|\Delta_E|,\qquad \rho=D^{1/(2f)},\qquad Y=\rho^{f},\qquad L=1+f\log\rho.$$

Thus $[E:\mathbb{Q}]=2f$, $\sqrt{D}=Y$, $\rho\ge 1$, $Y\ge 1$, and $L\ge 1$. For a number field K, h_K is its ideal class number and $N\mathfrak{a}=[\mathcal{O}_K:\mathfrak{a}]$ is the absolute norm of a nonzero integral ideal. A definite unimodular plane is a full $\mathcal{O}$-lattice Λ in a two-dimensional E-vector space, equipped with a Hermitian form h which is positive definite at every complex embedding and satisfies

$$\Lambda=\{x: h(y,x)\in\mathcal{O}\quad (y\in\Lambda)\}.$$

Let $\mathrm{defClassNum}(E,2)$ be the number of its isometry classes. We use Minkowski's convex-body theorem, the Minkowski ideal bound, and Dirichlet's unit theorem in the precise forms of Marcus, *Number Fields*, 2nd ed., Chapter 5: the lemma on pp. 96–97, Theorem 37, Corollary 2, and Theorem 38, respectively.

Lemma 14.84 (Ideal counting)**.** *For every number field K of degree d and every $X\ge 1$,*

$$\#\{\mathfrak{a}\subseteq\mathcal{O}_K:\mathfrak{a}\neq 0,\ N\mathfrak{a}\le X\}\le X(1+\log X)^{d-1}.$$

Consequently $h_K\le\sqrt{|\Delta_K|}(1+\log\sqrt{|\Delta_K|})^{d}$; in particular $h_E\le YL^{2f}$.

Proof. Unique factorization of ideals and $\sum_{\mathfrak{p}\mid p}e_{\mathfrak{p}}f_{\mathfrak{p}}=d$ are Marcus, Chapter 3, Theorems 16 and 21. Colour the primes over each rational prime p by d colours, injectively for fixed p. If $\mathfrak{a}=\prod_{\mathfrak{p}}\mathfrak{p}^{e_{\mathfrak{p}}}$, form $A_i=\prod_{\mathrm{colour}(\mathfrak{p})=i}(N\mathfrak{p})^{e_{\mathfrak{p}}}$. The tuple $(A_1,\dots,A_d)$ determines $\mathfrak{a}$. Hence at most $d_d(m)$ ideals have norm m, where $d_d(m)$ counts ordered d-tuples of product m.

Set $D_1(N)=N$ and $D_{r+1}(N)=\sum_{m=1}^{N}D_r(\lfloor N/m\rfloor)$. Induction on r, using $\sum_{m\le N}m^{-1}\le 1+\log N$, gives $D_r(N)\le N(1+\log X)^{r-1}$ for $N\le X$. Since $D_d(N)=\sum_{m\le N}d_d(m)$, this proves the first bound. Every ideal class has an integral ideal of norm at most $(4/\pi)^{r_2}(d!/d^d)\sqrt{|\Delta_K|}\le\sqrt{|\Delta_K|}$ by the Minkowski bound, so selecting one such ideal for each class proves the second bound. Finally $\sqrt{D}=Y$ and $[E:\mathbb{Q}]=2f$. □

Lemma 14.85 (Gram determinant and index)**.** *Let $X\subseteq E^r$ be an integral self-dual Hermitian lattice containing $\mathcal{O}^r$. If a is the determinant of its standard-basis Gram matrix and $n=[X:\mathcal{O}^r]$, then $|N_{E/\mathbb{Q}}(a)|=n^2$.*

Proof. Successive coordinate projections and splitting by invertible fractional ideals give a pseudobasis $X = \bigoplus_i C_i z_i$, where $\mathcal{O} \subseteq C_i$ and the rows z_i are unipotent triangular in the standard coordinates. Thus $n = \prod_i [C_i : \mathcal{O}]$. Let $C = \prod_i C_i$ and $\mathfrak{J} = \bar{C}(a)C$. If $\bar{s}_i \in C_i$ and $t_i \in C_i$, the matrix $(H(\bar{s}_i z_i, t_j z_j))_{ij}$ is integral; its determinant is $(\prod_i s_i)a(\prod_i t_i)$. Hence $\mathfrak{J} \subseteq \mathcal{O}$. Conversely, for $u_i \in \bar{C}_i^{-1}$, form the standard dual vectors with coordinates $Q^{-1}(u_i e_i)$, where Q is the conjugate triangular matrix times the Gram matrix. Self-duality puts these vectors in X. Comparing their pseudobasis coefficient determinants gives $\prod_i u_i \in (a)C$. Therefore $\bar{C}^{-1} \subseteq (a)C$, so $\mathfrak{J} = \mathcal{O}$. The identities $N(C_i)[C_i : \mathcal{O}] = N(\bar{C}_i)[C_i : \mathcal{O}] = 1$ now give $N(\mathfrak{J})n^2 = |N_{E/\mathbb{Q}}(a)|$. □

Lemma 14.86 (A shortest vector at every embedding)**.** *Every definite unimodular plane contains a nonzero $u \in \Lambda$ such that $h_\varphi(u,u) \leq \rho$ for every $\varphi : E \hookrightarrow \mathbb{C}$.*

Proof. Choose an E-basis v_1, v_2 lying in Λ and set $\Gamma = \mathcal{O}v_1 \oplus \mathcal{O}v_2$ and $n = [\Lambda : \Gamma]$. For each complex place w, let G_w be the Gram matrix in this basis and $\delta_w = \det G_w$. Positive definiteness on E-rational vectors, density of E in $\mathbb{C}$, and nonvanishing of the algebraic Gram determinant show that G_w is positive definite over $\mathbb{C}$.

Let a be the algebraic Gram determinant and $P = \prod_w \delta_w$. Lemma 14.85 and the principal-ideal norm formula give $P^2 = |N_{E/\mathbb{Q}}(a)| = n^2$, hence $P \leq n$. The mixed embedding of Γ has covolume $(2^{-f}\sqrt{D})^2 = D/4^f$, so the image of Λ in $\prod_w \mathbb{C}^2$ has covolume $D/(4^f n)$. For $R^2 = \rho$, define

$$\mathcal{E}_R = \{(z_w)_w : z_w^* G_w z_w \leq R^2 \quad (w)\}.$$

The volume of a complex Hermitian ellipsoid follows from its Cholesky factorization: $\mathrm{vol}\{z : z^* G_w z \leq R^2\} = \pi^2 R^4/(2\delta_w)$. Since $R^{4f} = \rho^{2f} = D$,

$$\mathrm{vol}(\mathcal{E}_R) = \frac{(\pi^2/2)^f D}{P}, \qquad 2^{4f}\,\mathrm{covol}(\Lambda) = \frac{4^f D}{n}.$$

Minkowski's theorem therefore gives

$$\#(\mathcal{E}_R \cap \Lambda) \geq \frac{\mathrm{vol}(\mathcal{E}_R)}{2^{4f}\,\mathrm{covol}(\Lambda)} = \frac{n}{P}\Big(\frac{\pi^2}{8}\Big)^f > 1.$$

This supplies the required nonzero vector at the place representatives. Conjugate embeddings have the same real Hermitian value, proving the bound at every embedding. □

Diagonal splitting. Choose u as in Lemma 14.86, and complete it to an orthogonal basis (u, w). Writing $\gamma = (u, w)$ gives $\gamma^* J \gamma = \mathrm{diag}(\beta_1, \beta_2)$, where $\beta_1 = h(u,u)$, $\beta_2 = h(w,w)$ are totally positive. Put $L = \gamma^{-1}\Lambda$,

$$A = \{a : (a,0) \in L\}, \qquad B = \{b : (0,b) \in L\},$$

and let P, Q be the two coordinate projections of L. Then $1 \in A$ and

$$t \in A \iff \bar{p}\beta_1 t \in \mathcal{O} \quad (p \in P), \qquad t \in B \iff \bar{q}\beta_2 t \in \mathcal{O} \quad (q \in Q). \tag{1}$$

Both projections are nonzero because L is full. Using inverse fractional ideals in (1) shows that A and B are nonzero full rank-one lattices, just as in the dual-colon calculation below. Define

$$\mathfrak{a}_1 = \mathrm{span}_{\mathcal{O}}\{\bar{x}\beta_1 y : x, y \in A\}, \qquad \mathfrak{a}_2 = \mathrm{span}_{\mathcal{O}}\{\bar{x}\beta_2 y : x, y \in B\}.$$

For $c \in E$, the identity

$$\bar{b}\beta_1((\bar{x}\beta_1 y)a) = (\bar{b}\beta_1 y)\overline{(\bar{a}\beta_1 x)}$$

and inversion of the fractional ideal P give $c \in \mathfrak{a}_1 \iff cP \subseteq A$. Similarly $c \in \mathfrak{a}_2 \iff cQ \subseteq B$. If $cP \subseteq A$ and $(p,q) \in L$, subtracting $(cp, 0)$ from $c(p,q)$ proves $(0, cq) \in L$. Hence $\mathfrak{n} := \mathfrak{a}_1 = \mathfrak{a}_2$.

The ideal $\mathfrak{n}$ is generated by its conjugation-fixed elements. Indeed, let $\mathfrak{b}$ be the ideal generated by those elements. If $(\mathfrak{b} : \mathfrak{n}) \neq \mathcal{O}$, choose a maximal $\mathfrak{p}$ containing it. By (1) and $P^{-1}P = \mathcal{O}$, there are $x, y \in A$ and $a, b \in P$ with $\bar{x}\beta_1 a$ and $\bar{b}\beta_1 y$ outside $\mathfrak{p}$. Examining the other two cross products shows that one of x, y, or $x + y$ can be used so that both $s = \bar{x}\beta_1 a$ and $t = \bar{b}\beta_1 x$ are outside $\mathfrak{p}$. For every generator $\bar{y}\beta_1 z$ of $\mathfrak{n}$,

$$st(\bar{y}\beta_1 z) = (\bar{y}\beta_1 a)(\bar{b}\beta_1 z)(\bar{x}\beta_1 x)$$

belongs to $\mathfrak{b}$. Thus $st \in \mathfrak{p}$, a contradiction. Consequently $\mathfrak{n} = (\mathfrak{n} \cap \mathcal{O}_F)\mathcal{O}$. Finally $1 \in A$ gives $\beta_1 \in \mathfrak{n}$, and the shortest-vector estimate at all $2f$ embeddings gives

$$N\mathfrak{n} \leq |N_{E/\mathbb{Q}}(\beta_1)| = \prod_{\varphi} |h_\varphi(u,u)| \leq \rho^{2f} = Y^2. \tag{2}$$

Lemma 14.87 (Real ideals and lines)**.** *For $Y \geq 1$, there are at most YL^f nonzero fixed-generated ideals with $N\mathfrak{n} \leq Y^2$. For each such $\mathfrak{n}$, there are at most $h_E 2^f$ equivalence classes of pairs $X = (\beta, N)$ with β totally positive, N a full $\mathcal{O}$-lattice in E, and* $\operatorname{span}_{\mathcal{O}}\{\bar{x}\beta y : x, y \in N\} = \mathfrak{n}$*. Here $(\beta, N) \sim (\beta', N')$ if $\bar{c}\beta' c = \beta$ and $cN = N'$ for some $c \in E^\times$.*

Proof. Contraction $\mathfrak{n} \mapsto \mathfrak{a} = \mathfrak{n} \cap \mathcal{O}_F$ is injective by fixed-generation, and $N_{E/\mathbb{Q}}(\mathfrak{a}\mathcal{O}) = (N_{F/\mathbb{Q}}\mathfrak{a})^2$. The first claim follows from Lemma 14.84 applied to F.

For lines, choose a reference (β^0, N^0) in each occurring fractional ideal class. For X in that class choose $aN = N^0$. Equality of norm ideals gives $u_X \in \mathcal{O}^\times$ with $\bar{a}\beta^0 a\, u_X = \beta$: the multiplier $t = \bar{a}\beta^0 a/\beta$ and its inverse stabilize the nonzero finitely generated ideal $\mathfrak{n}$, hence are integral. Record the parity of the exponents of u_X in a fundamental unit basis. Dirichlet's theorem gives $f - 1$ free generators, hence at most 2^{f-1} parity vectors.

If X, Y have the same ideal class and parity, then $u_X = u_Y \zeta v^2$ with ζ torsion and v a unit. Put $c = a_X v / a_Y$. This gives $cN_X = N_Y$ and

$$\bar{c}\beta_Y c = \bar{a}_X \beta^0 a_X u_Y \bar{v} v.$$

At any embedding, $|\zeta| = 1$ and $|v^2| = |\bar{v}v|$, so this positive fixed value equals $\beta_X = \bar{a}_X \beta^0 a_X u_X$. Thus $X \sim Y$. There are at most $h_E 2^{f-1} \leq h_E 2^f$ line classes. □

Lemma 14.88 (Unimodular gluings)**.** *Fix line representatives (β_1, N_1) and (β_2, N_2), with the norm ideal of the first equal to $\mathfrak{n}$. The lattices $M \subseteq E^2$ which are self-dual for* $\operatorname{diag}(\beta_1, \beta_2)$ *and have prescribed axis intersections N_1, N_2 number at most $N\mathfrak{n}$.*

Proof. Write $N^{\vee_i} = \{a : \bar{s}\beta_i a \in \mathcal{O}\ (s \in N)\}$. Choose $0 \neq s \in N_1$ and $d = \bar{s}\beta_1$. For the integral ideals $I = \{c : d^{-1}c \in N_1\}$ and $J = \{c : d^{-1}c \in N_1^{\vee_1}\}$, prime factorization gives y with $J = I + (y)$. Taking $u = d^{-1}y$ gives $N_1^{\vee_1} = N_1 + \mathcal{O}u$. Also $\mathfrak{n}N_1^{\vee_1} \subseteq N_1$, since $(\bar{x}\beta_1 y)u' = (\bar{x}\beta_1 u')y$.

If P is the first projection of an admissible M and $a \in N_1^{\vee_1}$, then $a \in P$. Indeed, for $q \in P^{-1}$, $\bar{q}\beta_1^{-1} \in N_1$; hence qa is integral and $a(pq) = (qa)p \in P$. Summing over $PP^{-1} = \mathcal{O}$ proves $a \in P$. Thus there is b with $(a, b) \in M$, and similarly in the second coordinate.

Fix M_0 if the family is nonempty, and choose $(u, v_M) \in M$ for every M. For each M, lift v_M to $(a, v_M) \in M_0$ and choose $c_M \in \mathcal{O}$ with $a - c_M u \in N_1$. Then $v_M - c_M v_0 \in N_2$. If $c_M - c_{M'} \in \mathfrak{n}$, it follows that $v_M - v_{M'} \in N_2$. For any $(a, b) \in M$, choose d with $a - du \in N_1$; then $b - dv_M \in N_2$ and

$$(a, b) = (a - du, 0) + d(u, v_{M'}) + (0, b - dv_M + d(v_M - v_{M'})) \in M'.$$

Symmetry gives $M = M'$. Therefore $M \mapsto c_M \bmod \mathfrak{n}$ is an injection into $\mathcal{O}/\mathfrak{n}$. □

Theorem 14.89. *With the notation above,*

$$\mathrm{defClassNum}(E, 2) \leq 4^f \rho^{5f} (1 + f \log \rho)^{5f}.$$

Proof. By Lemma 14.87 and $h_E \leq YL^{2f}$, there are at most YL^f ideals and at most $2^f YL^{2f}$ classes for each line. By Lemma 14.88 and (2), each pair of line representatives has at most $\lfloor Y^2 \rfloor$ gluings. Thus the number of tuples $(\mathfrak{n}, [X_1], [X_2], M)$ is at most

$$(YL^f)(h_E 2^f)^2 \lfloor Y^2 \rfloor \leq 4^f Y^5 L^{5f}.$$

Every class has such a tuple. Namely, after splitting, choose for each resulting line $X_i = (\beta_i, N_i)$ its reference representative $X_i^0 = (\beta_i^0, N_i^0)$, and scalars with $\bar{c}_i \beta_i c_i = \beta_i^0$ and $c_i N_i^0 = N_i$. With $\delta = \gamma \,\mathrm{diag}(c_1, c_2)$ and $M = \delta^{-1}\Lambda$, the diagonal form is $\mathrm{diag}(\beta_1^0, \beta_2^0)$ and M has exactly the required axis intersections. Fixed tuple data determine this diagonal pair, and the original plane is isometric to it; hence the map from classes to tuples is injective. Since $Y = \rho^f$ and $L = 1 + f \log \rho$, this is the claimed bound. □

Absolute-volume substitution. Let $\ell_E(M)$ denote the local L-factor in the rank recurrence and set $\mathrm{rankFac}(E, M) = \rho^{f(M-1/2)} \ell_E(M)$. The local comparison $\ell_E(M) \leq \zeta(M)^f$ and the explicit estimate $\prod_{M=3}^{n} \zeta(M) < 1.4$ give

$$\prod_{2<M\leq n} \mathrm{rankFac}(E, M) \leq \rho^{f(n^2-4)/2} 1.4^f, \qquad \sum_{2<M\leq n} (M - \tfrac{1}{2}) = \frac{n^2 - 4}{2}. \tag{3}$$

For the zeta product, truncate the $\zeta(3)$ and $\zeta(5)$ series and use $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$; for $M \geq 6$, comparison term by term gives $\zeta(M) - 1 \leq 2^{6-M}(\zeta(6) - 1)$, hence $\sum_{M\geq 6}(\zeta(M) - 1) < 0.0347$. Together with $\zeta(3) < 1.2021$, $\zeta(4) < 1.0824$, $\zeta(5) < 1.0385$, and $e^{0.0347} < 1.0354$, this gives the product bound $1.3991 < 1.4$.

Assume now that the rank recurrence has reduced the volume $V_{p,q}$ to the definite signature $(2, 0)$ and that $n = p + q \geq 8$, $q^2 \leq n$, $X \geq 2$, $\rho \leq X$, and $2f\rho \leq X^2$. (The first two imply $p \geq 2$.) Since $\mathrm{volU}(2) = 8\pi^3$, Theorem 14.89 and (3) yield

$$V_{p,q} \leq (8\pi^3)^f \,\mathrm{defClassNum}(E, 2) \prod_{2<M\leq n} \mathrm{rankFac}(E, M) \leq (8\pi^3 \cdot 4 \cdot 1.4)^f \rho^{5f} L^{5f} \rho^{f(n^2-4)/2}.$$

The remaining comparisons are explicit:

$$8\pi^3 \cdot 4 \cdot 1.4 \leq 8(3.15)^3(4)(1.4) < 2^{11} \leq X^{11},$$

and, using $\log \rho \leq \rho - 1$ and $f \geq 1$,

$$L = 1 + f \log \rho \leq 1 + f(\rho - 1) \leq f\rho \leq X^2/2.$$

Consequently

$$V_{p,q} \leq X^{11f} X^{5f} X^{10f} X^{f(n^2-4)/2} = X^{f(26+(n^2-4)/2)} \leq X^{n^2 f},$$

where the last inequality uses $n^2 \geq 64$, hence $48 \leq n^2$.

14.4.11 Decay and the far orbits

Write $r = \min(p, q) \geq 1$. Let h be a lower bound for the gaps between opposite-sign weights and suppose $h \geq 120nr^{3/2}$. In polar coordinates $k_1 e^X k_2$ put $t = \|X\|_{\mathrm{HS}}$. For unit vectors in the lowest K-type the absolute value of the matrix coefficient is at most

$$\exp\{-\tfrac{h}{8}\min(t^2, t/\sqrt{2r})\}.$$

Indeed the compression to the lowest K-type is the compact integral of $\prod_{j<n} \Delta_j(k^{-1}e^X k)^{-(\lambda_j - \lambda_{j+1}+1)}$ against the two highest-vector matrix coefficients, with factor $\dim S$ for the lowest type S. The normalization follows by Schur orthogonality at the identity. The integral and the discrete-series coefficient satisfy the same lowering equations: commuting a compact derivative past a noncompact one gives the Lie bracket, and an ordered lowering derivative kills the highest vector. The gap $h > 2n$ excludes the complementary types in the successive jets. Induction on the noncompact degree gives equality of the Taylor expansions, hence of the analytic coefficients along $\exp(tX)$.

For $Y = k^{-1}Xk$ the leading minor $\Delta_j(e^Y)$ is a convex sum of exponentials of sums of j eigenvalues, by Cauchy–Binet. These sums have mean zero, absolute value at most $\sqrt{2r}\,t$, and variance $B_j = \sum_{a>j,\, b\leq j} |Y_{ab}|^2$. The centered exponential estimate at $\min(1, 1/(\sqrt{2r}\,t))$ and convexity give $\log \Delta_j(e^Y) \geq B_j \min(1, 1/(\sqrt{2r}\,t))/4$. Summation over the cuts gives $\sum_j (\lambda_j - \lambda_{j+1} + 1)B_j \geq ht^2/2$. Cauchy–Schwarz and Schur orthogonality in the compact integral give the asserted bound. In the operator radius $s = \|X\|_{\mathrm{op}}$ the paired nonzero eigenvalues give $t \geq \sqrt{2}s$, hence the exponent is at most $-(h/8)\min(2s^2, s/\sqrt{r})$.

Multiply the local estimates by the product d_∞ of the formal degrees and the idempotent bound $a = e(1)$. For least lattice scale τ put $A(x) = (3\sqrt{n}/\min(1/2, \tau)^2)^{2n^2 f}$. The number of elements with radii at most s_v is bounded by $A(x)\exp(2n^2 \sum_v s_v)$, by packing lattice points in the product of balls. Put $\kappa = 2n^2$, $T = h\varepsilon^2/16$, and take $\varepsilon \leq 1/\sqrt{2r}$, $\kappa\varepsilon \leq \log 2$, $32\kappa \leq h\varepsilon$, $T \geq 1$. In a dyadic shell with upper radius $2^j\varepsilon$, $j \geq 1$, decay times the counting factor is at most $\exp(-T2^{j-1})$; the bottom shell contributes at most 2. The sum of the positive-index terms is at most $2e^{-T} \leq 1$. A far orbit has at least one positive index, so the product over f places gives an absolutely convergent sum bounded by $d_\infty a A(x)\, 2f 3^f e^{-T}$.

For the deep region put $N = n^2 f$, $P = 8qN$, and write τ for the least lattice scale, $\rho = n^2\sqrt{f}\, g\, \mathrm{rd}_E (2\tau)^{1/q}/4$. Let β bound the one-step derivative constants through order $P + N$. The local Sobolev estimate on jets has squared evaluation bound

$$(6\beta)^N e^f (17\sqrt{n}/\tau^2)^{2N} D \prod_v c_v/\mu_f(K_g).$$

Here c_v are the local gauge constants. The lattice scale on the radius-ρ neighbourhood is at least $\tau/2$. At each of the P steps the constant term along the isotropic radical vanishes by cuspidality; comparison with its average costs at most $2\rho\sqrt{f}\,\beta$. Squaring gives the additional factor $(2\rho\sqrt{f}\,\beta)^{2P}$. Applying the frame estimate first to finite subspaces and integrating with the absolute volume bound gives finite dimensionality. An orthonormal basis then gives the diagonal kernel as the squared evaluation norm.

14.4.12 Near orbits and exceptional flags

Throughout this subsection let E/F be the totally imaginary quadratic extension of the totally real field in the automorphic datum, and put $f = [F : \mathbb{Q}]$. Thus E has f complex places. Let n be

the hermitian rank and write $J = \sum_i j_i$ for a type (j, τ) of length a. Define

$$\mathcal{P}_\delta(j,\tau) = \prod_{i<k}\left[G_\delta\left(\frac{\tau_i}{\tau_k}\right)^{2j_ij_kf} G_\delta(\tau_i\tau_k)^{2j_ij_kf}\right]\prod_i\left[G_\delta(\tau_i)^{2j_i(n-2J)f} G_\delta(\tau_i^2)^{j_i^2f}\right], \qquad G_\delta(u) = 1 + \frac{12\delta}{u}.$$

Write $\mathcal{F}$ for the fundamental domain with measure μ, and φ for the product of the archimedean matrix coefficients, their formal degrees and the finite level idempotent. Write $s_{\min}(x)$ for the least lattice scale. For the lattice at x, let $S_{\leq}(t)$ and $S_{<}(t)$ be the spans of sublattices of largest size at most t and strictly less than t, respectively. Let $\mathcal{N}(x,\delta)$ consist of the rational elements whose conjugates by x lie in the finite level and within Frobenius distance δ of the identity at each infinite place. Write $\mathcal{N}^*(x)$ for the nonidentity near orbits at the original Cartan cutoff.

Proposition 14.90 (Near unipotence)**.** *Assume $n \geq 1$, $4n < g_0$, and that $(g_0)\mathcal{O}_E$ is radical. For $0 < c \leq 10^{-5}$ put*

$$\varepsilon = \frac{c}{n^8f^2}, \qquad \delta = 3\sqrt{n\varepsilon}.$$

Then $\delta < 1/4$, and $\delta qnf \leq 10^{-2}$ whenever $q^2 \leq n$. Moreover every automorphic γ for which $\varphi(x^{-1}\gamma x) \neq 0$ and whose archimedean Cartan distances after conjugation by x are at most ε has

$$(\gamma_f - 1)^n = 0, \qquad (\gamma_v - 1)^n = 0 \quad (v \mid \infty),$$

and belongs to $\mathcal{N}(x,\delta)$.

Proof. The scalar assertions are uniform in $n, f \geq 1$. With $a = n\varepsilon$ one has

$$a(n^3f^2) = c/n^4 \leq c, \qquad a \leq c \leq 10^{-5}.$$

Hence $3\sqrt{a} < 1/4$. If $q^2 \leq n$, then

$$(3\sqrt{a}\,qnf)^2 = 9a(q^2n^2f^2) \leq 9a(n^3f^2) \leq 9c < 10^{-4},$$

which gives the last radius bound. Also $n\varepsilon \leq 1/4$.

For an invertible complex matrix A, write $A = ke^X$ with k unitary and X hermitian. This is the unitary polar decomposition: $A^*A = e^{2X}$ and $k = Ae^{-X}$. The Cartan distance in the statement is $\|X\|_F$. Suppose $\|X\|_F \leq \varepsilon$ and put $E = e^X$. Then

$$\|E - 1\|_F \leq e^\varepsilon - 1, \qquad \|(kE)^r\|_F \leq e^{r\varepsilon}\sqrt{n},$$

by expanding the exponential and inducting on r. Cauchy–Schwarz on diagonal entries gives $|\operatorname{tr} M| \leq \sqrt{n}\,\|M\|_F$. Since $r\varepsilon \leq n\varepsilon \leq 1/4$ and $e^{1/4} < 3/2$, it follows for $0 \leq r \leq n$ that

$$|\operatorname{tr}((kE)^r) - n| \leq ne^{r\varepsilon} + n \leq 3n. \tag{14.25}$$

Now suppose in addition $\operatorname{tr}(kE) = n$. Using $\varepsilon \leq 1$ gives $\|E - 1\|_F \leq 2\varepsilon$. The identities

$$\|k - 1\|_F^2 = 2n - 2\Re\operatorname{tr} k, \qquad \operatorname{tr} k = n - \operatorname{tr}(k(E-1))$$

imply $\|k - 1\|_F \leq 2\sqrt{n\varepsilon}$. Since $\sqrt{\varepsilon} \leq \sqrt{n}$ and $\sqrt{n\varepsilon} \leq 1/2$,

$$\|kE - 1\|_F \leq 2\sqrt{n\varepsilon} + 2\varepsilon \leq 3\sqrt{n\varepsilon} = \delta. \tag{1}$$

Nonvanishing of the level idempotent at the conjugated finite component puts that component in the lattice stabilizer and makes it unipotent modulo g_0, by the two support clauses of admissibility. Let G be the rational matrix supplied by rationality of the automorphic orbit, and let Λ_x be its self-dual $\mathcal{O}_E$-lattice. Thus $G\Lambda_x \subseteq \Lambda_x$ and

$$(G-1)^n \Lambda_x \subseteq g_0 \Lambda_x.$$

For fixed r, set $S_r = \sum_{s=0}^{r-1} G^s$. Since $(G-1)S_r = G^r - 1$ and S_r commutes with $G-1$, the same divisibility holds for $(G^r-1)^n$.

We use the following trace congruence. If $A\Lambda \subseteq \Lambda$ and $A^m\Lambda \subseteq g_0\Lambda$, then $\operatorname{tr} A \in g_0\mathcal{O}_E$. Indeed, for an eigenvalue z of A in an algebraic closure, a nonzero left eigenvector and a common denominator in Λ show that z is integral over $\mathcal{O}_E$, while z^m/g_0 is integral. Thus z lies in the radical of g_0 in the integral closure. Summing eigenvalues gives $(\operatorname{tr} A)^N \in g_0\mathcal{O}_E$ for some N; radicality of (g_0) gives the assertion. Consequently

$$\operatorname{tr}(G^r) - n = g_0 t_r \qquad (t_r \in \mathcal{O}_E).$$

The archimedean realizations of G^r are conjugate to the matrices in (14.25), so $|\sigma(g_0 t_r)| \le 3n$ at every complex embedding σ. If $t_r \neq 0$, the product formula for its nonzero algebraic norm gives some $|\sigma(t_r)| \ge 1$; hence $g_0 \le 3n$, contrary to $4n < g_0$. Thus $\operatorname{tr}(G^r) = n$ for $0 \le r \le n$.

For $N = G - 1$, the binomial expansion now yields $\operatorname{tr}(N^k) = n(1-1)^k = 0$ for $1 \le k \le n$. Newton's identities over the characteristic-zero field E force the characteristic polynomial of N to be X^n. Cayley–Hamilton gives $N^n = 0$. Ring homomorphisms and conjugation preserve this identity, giving unipotence at every finite and infinite component. In particular the trace at $r = 1$ is n, so (1) supplies exactly the archimedean radius in $\mathcal{N}(x, \delta)$. The other two membership clauses are rationality and the lattice stabilizer already established. □

At the operative hypotheses, g_0 is prime and the number of primes of $\mathcal{O}_E$ over g_0 equals $[E : \mathbb{Q}]$. The identity $\sum_{\mathfrak{p} \mid g_0} e_{\mathfrak{p}} f_{\mathfrak{p}} = [E : \mathbb{Q}]$ therefore forces every ramification index and residue degree to be 1. In particular (g_0) is squarefree, hence radical. This is the total-splitting input used above.

Lemma 14.91 (Geometric-mean packing)**.** *Let U_v, $v = 1, \dots, f$, be real vector subspaces of a finite-dimensional normed space, with $\dim_{\mathbb{R}} U_v \le N$. Suppose $y_t(v) \in U_v$ has norm at most $X > 0$, and distinct t, t' satisfy*

$$\left(\prod_v \| y_t(v) - y_{t'}(v) \| \right)^{1/f} \ge \lambda > 0.$$

Then the number of indices t is at most $(1 + 2X/\lambda)^{Nf}$.

Proof. Put $G = (\prod_v X)^{1/f} = X$ and, in the general version with radii X_v, rescale coordinate v by G/X_v. These scalars have product 1, so the geometric mean of every difference is unchanged. After rescaling all coordinates have norm at most G, while the geometric mean of a difference is at most its sup norm. Thus the points are λ-separated in sup norm in a ball of radius G. Their balls of radius $\lambda/2$ are disjoint and lie in the ball of radius $G + \lambda/2$. Haar-volume comparison in dimension at most Nf proves the bound; padding each U_v to dimension N removes the word "at most". □

We next isolate the slope estimate behind the block count. For an $\mathcal{O}_E$-lattice L of rank r, write

$$H(L) = 2^{fr}|d_E|^{-r/2}\operatorname{covol}(L), \qquad \operatorname{size}(L) = H(L)^{1/(2fr)},$$

and use the analogous normalized covolume $H(A/A_0)$ and $\operatorname{size}(A/A_0)$ for a saturated $A_0 \subsetneq A$. Let $\operatorname{leastSize}(B)$ be the infimum of sizes of nonzero sublattices of B, and $\operatorname{largestSize}(A)$ the supremum of quotient sizes over proper saturated A_0.

If $\phi : A \to B$ has nonzero image and local operator norms c_v, use the singular-value decomposition of the induced map from $A/(A \cap \ker\phi)$ to its image at each complex place. Its $r = \operatorname{rk}\phi(A)$ singular values are at most c_v, so the determinant of that induced map has squared absolute value at most c_v^{2r}. Applied to a free full-rank sublattice of the quotient, this gives, with $P = \prod_v c_v$,

$$H(\phi A) \le P^{2r} H(A/(A \cap \ker\phi)).$$

Taking the $(2fr)^{-1}$-st root gives

$$\operatorname{size}(\phi A) \le P^{1/f} \operatorname{size}(A/(A \cap \ker\phi)).$$

The image itself is a test lattice for $\operatorname{leastSize}(B)$, and $\ker\phi \cap A$ is a test quotient for $\operatorname{largestSize}(A)$. Therefore

$$\frac{\operatorname{leastSize}(B)}{\operatorname{largestSize}(A)} \le \left(\prod_v c_v\right)^{1/f}. \tag{14.26}$$

Quotienting by a saturated sublattice and replacing operator norms by distance-to-subspace bounds gives the same estimate; this is the form used below.

Proposition 14.92 (Filtration-aware near count)**.** *For every point x of type (j,τ) and every $0 < \delta < 1/4$,*

$$\mathcal{N}(x,\delta) \text{ is finite}, \qquad \#\mathcal{N}(x,\delta) \le \mathcal{P}_\delta(j,\tau).$$

No additional numerical hypothesis on the type is needed.

Proof. The type supplies a saturated flag $0 = S_0 \subsetneq \cdots \subsetneq S_a$ with rank increments j_i and

$$S_{i+1} = S_{\le}(\tau_i), \qquad S_i = S_{<}(\tau_i), \qquad S_a = S_{<}(1/2).$$

Complete it by hermitian duality: $G_p = S_p$ for $p \le a$ and $G_p = G_{2a+1-p}^{\perp}$ for $p > a$. Put $t_i = \tau_i$ for $i < a$, $t_a = 2$, and $t_p = \tau_{2a-p}^{-1}$ for $p > a$. Self-duality gives $G_{2a+1} = \Lambda_x$, $\operatorname{rk} G_p + \operatorname{rk} G_{2a+1-p} = n$, and the strict-step descriptions of every G_p. If $r_p = \operatorname{rk} G_{p+1} - \operatorname{rk} G_p$, then

$$r_i = r_{2a-i} = j_i \ (i < a), \qquad r_a = n - 2J.$$

The condition $\delta < 1/4$ is used at the central step. A near matrix lowers $S_{\le}(t)$ to $S_{\le}(\delta t)$. If it did not, applying (14.26) to the induced nonzero map on the quotient would give a ratio at most δ, whereas the strict-step lower size and the largest-size upper bound make that ratio greater than δ. At $p = a$, $2\delta < 1/2$ gives $S_{\le}(2\delta) \subseteq S_{<}(1/2) = G_a$. Consequently a rational near matrix lies in

$$\mathcal{L} = \{\gamma : \gamma \text{ isometric}, \ (\gamma - 1)G_{p+1} \subseteq G_p \ (p \le 2a), \ \|\gamma_v - 1\| \le \delta \ \forall v\}.$$

The rational representation is injective: its archimedean and finite matrices determine the adelic element. It therefore remains to count $\mathcal{L}$.

Index blocks by $\mathcal{B} = \{(q,p) : 0 \le q < a,\ q < p,\ q+p \le 2a\}$ and order them by decreasing height $p-q$. The relation that $D = \gamma - \gamma'$ maps G_{p+1} to G_q is an equivalence. Unitarity gives the exact identity

$$h(Du, \gamma v) + h(\gamma' u, Dv) = 0.$$

A strong induction on p, with a second induction on the dual index, propagates the relation from smaller heights to all pairs (q,p) of height $p-q < c$ and proves $D(G_p) \subseteq G_{p-c}$. It also proves

$$h(Du, v) + h(u, Dv) = 0 \qquad (u \in G_{p+1},\ v \in G_{p'+1},\ p+p' \le 2a+c).$$

At height $2a+1$, this makes D zero on $G_{2a+1} = \Lambda_x$, hence zero because Λ_x spans.

For a fixed block (q,p), subtract a basepoint γ_0. The corresponding archimedean block is the projection of $(\gamma - \gamma_0)_v$ from the graded piece of rank r_p to the dual graded piece of rank r_{2a-q}. It has real dimension $2r_p r_{2a-q}$, except when $q+p = 2a$; in that diagonal case the skew identity restricts it to a skew-hermitian space of real dimension r_p^2. These dimensions are exactly

$$d_{q,p} = \begin{cases} r_p^2, & q+p = 2a, \\ 2r_p r_{2a-q}, & q+p < 2a. \end{cases}$$

All block norms are at most 2δ.

If two elements differ, the relation at level (q,p) fails for a quotient-separating subset unless they agree. By the previous paragraph, nontriviality of the induced map supplies a nonzero quotient image. The quotient form of (14.26), the strict-step lower bound t_q and the largest-step upper bound t_p give geometric-mean separation at least t_q/t_p. Lemma 14.91 therefore bounds this fibre by

$$\left(1 + \frac{2(2\delta)}{t_q/t_p}\right)^{d_{q,p} f}.$$

Taking fibre products in decreasing height proves finiteness and

$$\#\mathcal{L} \le \prod_{(q,p)\in\mathcal{B}} \left(1 + \frac{4\delta}{t_q/t_p}\right)^{d_{q,p} f}. \tag{14.27}$$

It remains to evaluate the product. For $i < k < a$ the two pairs (i,k) and $(i, 2a-k)$ have dimensions $2j_i j_k$ and ratios τ_i/τ_k and $\tau_i\tau_k$, respectively. The central pairs (i,a) and $(i, 2a-i)$ have dimensions $2j_i(n-2J)$ and j_i^2 and ratios $\tau_i/2$ and τ_i^2. All these identifications use $t_a = 2$. Finally, for positive ℓ, ℓ' with $\ell \le 3\ell'$,

$$\left(1 + \frac{4\delta}{\ell'}\right)^m \le \left(1 + \frac{12\delta}{\ell}\right)^m,$$

because $4\delta/\ell' \le 12\delta/\ell$. It applies with equality of ratios away from (i,a), and with $\ell = \tau_i$, $\ell' = \tau_i/2$ at (i,a). Substitution in (14.27) gives exactly $\mathcal{P}_\delta(j,\tau)$, including every exponent and both factors for each off-diagonal pair. □

For the integrated estimate write $L = nf\log n$, $X = nfrd\,g\,b$, $\ell_* = X^{-11q}$, and

$$Y = 2fq(11q\log X).$$

Let $\mathrm{M} = (\prod_v d_v)\mu(\mathcal{F})\,\mathrm{Re}\,e(1)$ be the main term, and let $\kappa_{p,q,j}$ be the flag-class constant. The pointwise orbit input gives $|\varphi(y)| \le M_0 = (\prod_v d_v)\,\mathrm{Re}\,e(1)$ and $\mathrm{M} = M_0\mu(\mathcal{F}) \ge 0$.

Proposition 14.93 (Near terms on the moderate region)**.** *Use the flag-measure bound, its constant bound, near unipotence, and the pointwise bound with their stated hypotheses. For each $C_g \geq 0$ there is an n_0, which can be taken to be*

$$n_0 = \max\left\{3^{15}, \left\lceil e^{16(1+22(|C_g|+4))} \right\rceil\right\},$$

such that, under the full counting hypotheses with $q \geq 1$,

$$\int_{\mathcal{F}\cap\{\ell_* \leq s_{\min}\}} \sum_{\gamma\in\mathcal{N}^*(x)} |\varphi(x^{-1}\gamma x)|\, d\mu(x) \leq e^{-L/4}\mathrm{M}.$$

The left side is interpreted as an extended-nonnegative-real integral and the right side through $\mathbb{R}_{\geq 0}\cup\{\infty\}$-valued realification.

Proof. Set $\delta = 3\sqrt{n\,10^{-6}/(n^8 f^2)}$. Near unipotence and the split-prime argument supply $0 < \delta < 1/4$ and $\delta qnf \leq 10^{-2}$. If $s_{\min}(x) \geq 1/2$, the same lowering argument as in the preceding proof forces $\mathcal{N}(x,\delta) = \{1\}$, so the nonidentity near sum is zero.

Fix a composition type j with $J \leq q$. On its moderate part, Proposition 14.92, finiteness of the near set and the pointwise bound give the pointwise majorant $M_0\mathcal{P}_\delta(j,\tau)$. The flag-measure bound turns integration over the type into

$$M_0\mu(\mathcal{F})\kappa_{p,q,j}\int_{\tau\in\mathcal{S}_a,\ \ell_*\leq\tau_i} \mathcal{P}_\delta(j,\tau)\,\mathrm{dens}_{n,f,j}(\tau)\,d\tau.$$

Here $\mathrm{dens} = \mathcal{P}_0^{\mathrm{id}}\prod_i(2fj_i/\tau_i)$, where $\mathcal{P}_0^{\mathrm{id}}$ is the product defining $\mathcal{P}_\delta$ with $G_\delta(u)$ replaced by u. Every type parameter satisfies $0 < \tau_i < 1/2$ and is increasing. For any argument λ occurring in this product,

$$(1 + 12\delta/\lambda)\lambda = \lambda + 12\delta \leq e^{12\delta}.$$

The sum of all exponents is at most $3Jnf \leq 3qnf$, so

$$\mathcal{P}_\delta(j,\tau)\mathcal{P}_0^{\mathrm{id}}(j,\tau) \leq e^{36\delta qnf} \leq e^{36/100} \leq 2.$$

The last inequality uses $36/100 < \log 2$. Consequently the one-type integral is at most

$$\mathrm{M}\,2\kappa_{p,q,j}\prod_i\int_{\ell_*}^{1/2}\frac{2fj_i}{\tau_i}\,d\tau_i = \mathrm{M}\,2\kappa_{p,q,j}\prod_i\left(2fj_i\log\frac{1}{2\ell_*}\right).$$

Since $\ell_* = X^{-11q}$, $\log(1/(2\ell_*)) \leq 11q\log X$, and $j_i \leq q$, this is at most

$$\mathrm{M}\,2\kappa_{p,q,j}Y^a. \tag{14.28}$$

There is no lost volume factor: $\mathrm{M} = M_0\mu(\mathcal{F})$. If $\kappa_{p,q,j} \leq 0$, the realification is zero and the inequality is immediate; otherwise the same real calculation applies.

The constant bound, with $p+q=n$, $f_{\deg}(E) = f$ and $2v_{\mathrm{inv}}(E) = rd$, gives

$$\kappa_{p,q,j} \leq e^{-JL/2}.$$

Also $a \leq J$, since every block is positive. The next paragraph proves $Y \leq e^{L/8}$. Thus

$$2\kappa_{p,q,j}Y^a \leq 2\exp(-JL/2 + aL/8) \leq 2e^{-3L/8}.$$

To prove $Y \leq e^{L/8}$, put $K = 22(|C_g| + 4)$ and $E_0 = e^{L/16}$. The chosen threshold gives $\log n \geq 16(1 + K)$. Hence $nf + K \leq L/16$ and $(nf)K \leq E_0$. Using $q^2 \leq n$, $\log rd \leq e^n \leq E_0$, $\log f \leq \sqrt{n} \leq E_0$, $\log b \leq E_0$, and $\log g \leq C_g \log b \leq |C_g| E_0$ gives

$$\log X \leq (|C_g| + 4)E_0, \qquad Y = 22fq^2 \log X \leq (nfK)E_0 \leq E_0^2 = e^{L/8}.$$

Moreover $2^q \leq e^q$ and $1 + q \leq L/8$, so

$$2\, 2^q e^{-3L/8} \leq e^{1+q-3L/8} \leq e^{-L/4}. \tag{14.29}$$

Finally, points outside the finite type union contribute zero. Canonically filtering a lattice with $s_{\min} < 1/2$ (Grayson's canonical filtration [29, Theorem 1.18 and Definition 1.19, p. 609]) gives a type with $1 \leq J \leq q$; isotropy gives the upper bound. There are 2^{J-1} compositions of J. Subadditivity and (14.28) therefore leave the factor

$$\sum_{J=1}^{q} 2^{J-1} = 2^q - 1 < 2^q.$$

Applying (14.29) completes the proof with the original exponent $-L/4$. □

Proposition 14.94 (Rare flags)**.** *The nonzero-flag bound implies the rare-flag bound with exponent* $-L/4$*, with threshold* $\lceil e^{15} \rceil$*.*

Proof. If $n \geq \lceil e^{15} \rceil$, then $\log n \geq 15$. Apply the nonzero-flag bound with the unchanged signature, $q^2 \leq n$, $\log rd \leq e^n$, and $\log f \leq \sqrt{n}$ hypotheses. The step characterization gives

$$\{s_{\min} < 1/2\} = \{S_<(1/2) \neq 0\}.$$

Thus its measure divided by $\mu(\mathcal{F})$ is at most $e^{-L/3}$. Since $n, f \geq 1$ and $\log n \geq 15$, one has $L > 0$ and

$$e^{-L/3} \leq e^{-L/4}.$$

No strengthening of the degree or discriminant hypotheses is required. □

The total error. Put $L = nf \log n$, $\delta = \exp(-n^2 f)$, and $X = |D - M|$. The required orbit estimates give

$$X \leq (2e^{-L/4} + \delta)M + \delta D.$$

For $n \geq e^{15}$ put $\eta = e^{-L/16}$. Then $240 \leq L \leq n^2 f$, so $e^{-L/4}, \delta \leq \eta/8$ and $\delta \leq 1/2$. Since $D \leq M + X$,

$$X \leq (2e^{-L/4} + 2\delta)M + \delta X \leq \tfrac{1}{2}\eta M + \tfrac{1}{2}X.$$

Thus $|D - M| \leq e^{-nf \log n/16} M$.

15 Heights, the CM period, and the slopes of the target and base lattices

This section proves hypotheses (3.4), (3.6) and (3.7) of Proposition 3.1. Subsection 15.1 fixes the normalization of slopes and heights. Subsection 15.2 proves that the constant $\log \Omega$ of

Proposition 3.1 is the normalized arithmetic degree of the line of periods of a CM elliptic curve, which is the Chowla–Selberg formula in the form needed here. Subsection 15.3 bounds the height of the period map B_u by the product of the norms of the m lifts and of the final pairing. Subsection 15.4 states the bound for the slope of the target lattice $R^\vee$ that the base side gives. Subsection 15.5 combines the two bounds into (3.4). Subsection 15.6 proves (3.6) from the Weyl dimension formula and the Siegel mass formula, and Subsection 15.7 proves (3.7) from the integral theory of Weyl modules. Subsection 15.8 compares the three errors with the quantity $\mathcal{B}$ of (4.8).

Throughout, x, ε, $L_\varepsilon = x^\varepsilon$, $b = e^{L_\varepsilon}$, $a = xb$, X' and $b' = a/X'$ are as in (5.17), and the field E, the CM type Φ, the integer $f = 2n = |\Phi|$, the number of lifts m, the ranks k, t_r and $r = k + m - 1$, the set of tops T with $N_T = |T|$, the weights a_t and the nodes $w_{j\rho}$ are those of Sections 5 and 6. We write $\mathrm{rd}_E = |d_E|^{1/(2f)}$ for the root discriminant of E and put

$$v = \tfrac{1}{2}\,\mathrm{rd}_E, \qquad \Lambda_a = \log(a\,\mathrm{rd}_E), \qquad g = \log(g_0 g_1), \tag{15.1}$$

where g_0, g_1 are the primes of (D5). The field $K = \mathbb{Q}(\sqrt{-d})$ is the imaginary quadratic subfield of E fixed by H, with its embedding φ_0, and χ_d is the quadratic character of K. Since $d \equiv 3 \pmod 4$ is prime, $\chi_d(n) = \left(\frac{-d}{n}\right) = \left(\frac{n}{d}\right)$, the discriminant of K is $-d$, and χ_d is the character χ_1 of Proposition 3.3. We put

$$\lambda_1 = \frac{L'(1,\chi_d)}{L(1,\chi_d)},$$

which is real.

15.1 Slopes and heights

Let F be a totally complex number field with complex conjugation, and let $n_F = [F:\mathbb{Q}]/2$ be the number of its archimedean places. A *hermitian lattice* over $\mathcal{O}_F$ is a finitely generated projective $\mathcal{O}_F$-module Λ with a positive definite hermitian form h_v on $\Lambda \otimes_{\mathcal{O}_F,\sigma_v} \mathbb{C}$ for each archimedean place v, where σ_v is one of the two embeddings of F at v. For a basis $e_1, \dots, e_s$ of $\Lambda \otimes F$ contained in Λ put

$$\widehat{\deg}(\Lambda) = \log \#\big(\Lambda/(\mathcal{O}_F e_1 + \cdots + \mathcal{O}_F e_s)\big) - \sum_v \log\det\big(h_v(e_i,e_j)\big)_{i,j}.$$

The product formula shows that $\widehat{\deg}(\Lambda)$ does not depend on the basis, and it is the arithmetic degree of Bost [15], since the two embeddings at v give equal determinants. We use the normalized slope and height

$$\mu(\Lambda) = \frac{\widehat{\deg}(\Lambda)}{n_F\ \mathrm{rk}\,\Lambda}, \qquad \mathrm{ht}(\phi) = \frac{1}{n_F}\sum_v \log\|\phi\|_v^2 \tag{15.2}$$

for a nonzero F-linear map ϕ between hermitian lattices, where $\|\phi\|_v$ is the operator norm at v. With $\mu_{\max}$ and $\mu_{\min}$ the largest and smallest slopes of the Harder–Narasimhan filtration, the slope inequality of Bost [15] in this normalization reads $\mu(\Lambda_1) \le \mu_{\max}(\Lambda_2) + \mathrm{ht}(\phi)$ for an injective $\phi\colon \Lambda_1 \to \Lambda_2$. The dual lattice $\Lambda^\vee = \mathrm{Hom}_{\mathcal{O}_F}(\Lambda, \mathcal{O}_F)$ has the dual forms, and $\widehat{\deg}(\Lambda^\vee) = -\widehat{\deg}(\Lambda)$, so that

$$\mu(\Lambda^\vee) = -\mu(\Lambda), \qquad \mu_{\min}(\Lambda^\vee) = -\mu_{\max}(\Lambda). \tag{15.3}$$

For positive real numbers κ_v we write $\langle \log\kappa\rangle = n_F^{-1}\sum_v \log\kappa_v$. Multiplying the forms h_v by κ_v changes μ by $-\langle\log\kappa\rangle$ and ht of a map into Λ by $+\langle\log\kappa\rangle$.

Remark 15.1. These are twice the slopes and heights of [15], and they coincide with the slope (13.1) and the height (13.2) of Section 13: both vanish on the trivial lattice and both equal twice the normalized slope and height of Bost. Section 12 uses only differences of slopes of lattices over one field. The doubling is the normalization of Proposition 3.1, where the sum S over primes has the factor 2 of (3.10), and every hypothesis (3.1) to (3.7) is stated in it.

We write $\mathrm{ph}(y) = y\log y - y$ for $y > 0$ and $\mathrm{ph}(0) = 0$.

15.2 The CM period

For a lattice $\Lambda \subset \mathbb{C}$ let $g_2(\Lambda) = 60\sum'_{\omega\in\Lambda}\omega^{-4}$, $g_3(\Lambda) = 140\sum'_{\omega\in\Lambda}\omega^{-6}$, $\Delta(\Lambda) = g_2^3 - 27g_3^2$ and let $A(\Lambda)$ be the area of a fundamental parallelogram. Since $g_2(c\Lambda) = c^{-4}g_2(\Lambda)$, $g_3(c\Lambda) = c^{-6}g_3(\Lambda)$ and $A(c\Lambda) = |c|^2A(\Lambda)$, the number

$$\mathcal{A}(\Lambda) = A(\Lambda)\,|\Delta(\Lambda)|^{1/6}$$

depends on the homothety class of Λ alone.

Lemma 15.2. *Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\tau = \omega_2/\omega_1$ in the upper half plane, and let η be the Dedekind eta function. Then $\mathcal{A}(\Lambda) = 4\pi^2\,\mathrm{Im}(\tau)|\eta(\tau)|^4$.*

Proof. The product formula of Jacobi gives $\Delta(\mathbb{Z}+\mathbb{Z}\tau) = (2\pi)^{12}\eta(\tau)^{24}$, so $\Delta(\Lambda) = \omega_1^{-12}(2\pi)^{12}\eta(\tau)^{24}$ by homogeneity. Hence $|\Delta(\Lambda)|^{1/6} = 4\pi^2|\eta(\tau)|^4|\omega_1|^{-2}$, and $A(\Lambda) = |\omega_1|^2\,\mathrm{Im}\,\tau$. □

Proposition 15.3. *Let $d > 3$ be a prime with $d \equiv 3 \pmod 4$, let $K = \mathbb{Q}(\sqrt{-d})$ with class number h, and for each ideal class C of K let $\Lambda_C \subset \mathbb{C}$ be the image under φ_0 of a fractional ideal in C. Then*

$$\frac{1}{h}\sum_C \log\frac{\mathcal{A}(\Lambda_C)}{\pi} = \gamma + \log\frac{2\pi}{\sqrt{d}} - \lambda_1.$$

Proof. For τ in the upper half plane and $\mathrm{Re}\,s > 1$ put $Z(\tau,s) = \sum'_{(m,n)}(\mathrm{Im}\,\tau)^s|m+n\tau|^{-2s}$. Kronecker's first limit formula states that

$$Z(\tau,s) = \frac{\pi}{s-1} + 2\pi\Big(\gamma - \log 2 - \log\big(\sqrt{\mathrm{Im}\,\tau}\,|\eta(\tau)|^2\big)\Big) + O(s-1). \tag{15.4}$$

Let C be a class, let $\mathfrak{b}$ be a fractional ideal in C^{-1} with an oriented basis $\mathfrak{b} = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2$, and put $\tau_C = \beta_2/\beta_1$. The integral ideals of C are the ideals $x\mathfrak{b}^{-1}$ with $x \in \mathfrak{b}\setminus\{0\}$, and x and x' give the same ideal exactly when x'/x is one of the $w = 2$ units. The covolume of $\mathfrak{b}$ is $N(\mathfrak{b})\sqrt{d}/2$, so for $x = m\beta_1 + n\beta_2$

$$N(x\mathfrak{b}^{-1}) = \frac{|\beta_1|^2|m+n\tau_C|^2}{N(\mathfrak{b})} = \frac{\sqrt{d}}{2}\,\frac{|m+n\tau_C|^2}{\mathrm{Im}\,\tau_C}.$$

Hence the partial zeta function of C is $\zeta(s,C) = \frac{1}{w}\big(\frac{\sqrt{d}}{2}\big)^{-s}Z(\tau_C,s)$. Summing over C and using $\big(\frac{\sqrt{d}}{2}\big)^{-s} = \frac{2}{\sqrt{d}}\big(1-(s-1)\log\frac{\sqrt{d}}{2} + O((s-1)^2)\big)$ we obtain

$$\zeta_K(s) = \frac{2}{w\sqrt{d}}\Big(\frac{h\pi}{s-1} + 2\pi\sum_C\big(\gamma - \log 2 - \log(\sqrt{\mathrm{Im}\,\tau_C}|\eta(\tau_C)|^2)\big) - h\pi\log\frac{\sqrt{d}}{2}\Big) + O(s-1).$$

On the other hand $\zeta_K(s) = \zeta(s)L(s,\chi_d) = \frac{L(1,\chi_d)}{s-1} + \gamma L(1,\chi_d) + L'(1,\chi_d) + O(s-1)$. Comparing residues gives $L(1,\chi_d) = 2\pi h/(w\sqrt{d})$, which is Dirichlet's class number formula. Dividing the constant terms by this value gives

$$\gamma + \lambda_1 = \frac{2}{h}\sum_C \Big(\gamma - \log 2 - \log\big(\sqrt{\operatorname{Im}\tau_C}|\eta(\tau_C)|^2\big)\Big) - \log\frac{\sqrt{d}}{2},$$

that is,

$$\frac{1}{h}\sum_C \log\big(\operatorname{Im}\tau_C\,|\eta(\tau_C)|^4\big) = \gamma - \log 2 - \tfrac{1}{2}\log d - \lambda_1.$$

The lattice $\Lambda_{C^{-1}}$ is homothetic to $\mathbb{Z} + \mathbb{Z}\tau_C$, and $C \mapsto C^{-1}$ permutes the classes. Lemma 15.2 gives $\log(\mathcal{A}(\Lambda_{C^{-1}})/\pi) = \log(4\pi) + \log(\operatorname{Im}\tau_C|\eta(\tau_C)|^4)$, and the average over C is $\log(4\pi) + \gamma - \log 2 - \frac{1}{2}\log d - \lambda_1 = \gamma + \log(2\pi/\sqrt{d}) - \lambda_1$. □

Fix a nonzero ideal $\mathfrak{c}$ of $\mathcal{O}_K$ prime to $2d$ and a complex number δ with $\delta^{12} = \Delta(\varphi_0(\mathfrak{c}))$. The lattice $\Lambda_0 = \delta\,\varphi_0(\mathfrak{c})$ satisfies $\Delta(\Lambda_0) = 1$. Its invariants $g_2(\Lambda_0)$ and $g_3(\Lambda_0)$ are algebraic, because $g_2(\Lambda_0)^3 = j(\mathfrak{c})/1728$ and $27g_3(\Lambda_0)^2 = g_2(\Lambda_0)^3 - 1$ with $j(\mathfrak{c})$ algebraic. Let $M \subset \mathbb{C}$ be a number field that contains $g_2(\Lambda_0)$, $g_3(\Lambda_0)$ and $\tau(E)$ for every $\tau \in \Phi$. For each embedding $s\colon M \to \mathbb{C}$ let Λ_0^s be the lattice with $g_2(\Lambda_0^s) = s(g_2(\Lambda_0))$ and $g_3(\Lambda_0^s) = s(g_3(\Lambda_0))$. It exists and is unique by the uniformization theorem, since $s(g_2)^3 - 27s(g_3)^2 = s(1) = 1 \neq 0$.

We identify Φ with the set of K-embeddings $\tau\colon E \to M$ that restrict to φ_0 on K. Let $\mathfrak{d} = \mathfrak{d}_{E/K}$ be the relative different, let $\mathfrak{b}$ be the ideal of $\mathcal{O}_E$ with $\mathfrak{d} = \mathfrak{b}\bar{\mathfrak{b}}$ of Lemma 7.1(ii), and put $\mathfrak{A} = \mathfrak{b}^{-1}$. Let $R_M \subset M^\Phi$ be the $\mathcal{O}_M$-module generated by the vectors $(\tau(\alpha))_{\tau\in\Phi}$ with $\alpha \in \mathfrak{A}$, and for $\tau \in \Phi$ let

$$\ell_\tau = \{s \in M :\ s\,e_\tau \in R_M\},$$

where e_τ is the coordinate vector of τ. On ℓ_τ we put the metrics $\|s\|_\rho^2 = \pi|\rho(s)|^2/A(\Lambda_0^\rho)$ for the embeddings $\rho\colon M \to \mathbb{C}$.

Lemma 15.4. *In this situation $\ell_\tau = \tau(\mathfrak{A}\mathfrak{d})\mathcal{O}_M$, and $N_{M/\mathbb{Q}}(\ell_\tau)^{[E:\mathbb{Q}]} = N_{E/\mathbb{Q}}(\mathfrak{b})^{[M:\mathbb{Q}]}$.*

Proof. The map $E \otimes_K M \to M^\Phi$, $\alpha \otimes c \mapsto (\tau(\alpha)c)_\tau$, is an isomorphism of M-algebras because E/K is separable and $|\Phi| = [E : K]$, and it takes the trace $\operatorname{Tr}_{E\otimes M/M}$ to $y \mapsto \sum_\tau y_\tau$. Under it R_M is the image of $\mathfrak{A} \otimes_{\mathcal{O}_K} \mathcal{O}_M$, where $\mathcal{O}_M$ is an $\mathcal{O}_K$-algebra through φ_0. The trace pairing $\mathfrak{A} \times \mathfrak{A}^* \to \mathcal{O}_K$, $(\alpha,\beta) \mapsto \operatorname{Tr}_{E/K}(\alpha\beta)$, with $\mathfrak{A}^* = \mathfrak{A}^{-1}\mathfrak{d}^{-1}$, is perfect, and $\mathfrak{A}$ is a projective $\mathcal{O}_K$-module, so the base change $\mathfrak{A} \otimes \mathcal{O}_M \times \mathfrak{A}^* \otimes \mathcal{O}_M \to \mathcal{O}_M$ of this pairing is perfect as well. In M^Φ the base-changed pairing is $\langle x, y\rangle = \sum_\tau x_\tau y_\tau$, and R_M is the set of x with $\langle x, y\rangle \in \mathcal{O}_M$ for all y in the image R_M^* of $\mathfrak{A}^* \otimes \mathcal{O}_M$. For $s \in M$ and $y \in R_M^*$ we have $\langle se_\tau, y\rangle = sy_\tau$, and the coordinates y_τ of R_M^* generate $\tau(\mathfrak{A}^*)\mathcal{O}_M$. So $se_\tau \in R_M$ exactly when $s\,\tau(\mathfrak{A}^*)\mathcal{O}_M \subset \mathcal{O}_M$, that is, when $s \in \tau(\mathfrak{A}\mathfrak{d})\mathcal{O}_M$. The norm of the extension to M of an ideal $\mathfrak{e}$ of $\tau(E)$ is $N(\mathfrak{e})^{[M:\tau(E)]}$, and $N_{E/\mathbb{Q}}(\mathfrak{A}\mathfrak{d}) = N_{E/\mathbb{Q}}(\bar{\mathfrak{b}}) = N_{E/\mathbb{Q}}(\mathfrak{b})$. □

Definition 15.5. For $\tau \in \Phi$ and $s \in \ell_\tau \setminus \{0\}$ put

$$\log\Omega_\tau = \frac{1}{[M:\mathbb{Q}]}\Big(2\log[\ell_\tau : s\mathcal{O}_M] - \sum_{\rho\colon M\to\mathbb{C}} \log\|s\|_\rho^2\Big).$$

The product formula $\sum_\rho \log|\rho(s)|^2 = 2\log|N_{M/\mathbb{Q}}(s)|$ shows that $\log\Omega_\tau$ does not depend on s. It is $2/[M:\mathbb{Q}]$ times the arithmetic degree of the hermitian line ℓ_τ, and Proposition 15.6 shows that it depends neither on M, nor on τ, nor on $\mathfrak{c}$.

Proposition 15.6. *Let $d \equiv 7 \pmod{16}$ be a prime, $K = \mathbb{Q}(\sqrt{-d})$, let $E \supset K$ be a CM field with $[E : K] = f$ and with an ideal $\mathfrak{b}$ of $\mathcal{O}_E$ such that $\mathfrak{d}_{E/K} = \mathfrak{b}\bar{\mathfrak{b}}$, and let $\mathfrak{c}$, δ, M and $\tau \in \Phi$ be as above. Then*

$$\log \Omega_\tau = \gamma + \log \frac{\pi}{v} - \lambda_1, \qquad v = \tfrac{1}{2}\mathrm{rd}_E.$$

Proof. By Lemma 15.4, $[\ell_\tau : s\mathcal{O}_M] = |N_{M/\mathbb{Q}}(s)|/N_{M/\mathbb{Q}}(\ell_\tau)$, and

$$\frac{2}{[M:\mathbb{Q}]} \log N_{M/\mathbb{Q}}(\ell_\tau) = \frac{2}{[E:\mathbb{Q}]} \log N_{E/\mathbb{Q}}(\mathfrak{b}) = \frac{1}{2f} \log N_{E/\mathbb{Q}}(\mathfrak{d}_{E/K}).$$

The discriminant of the tower $E/K/\mathbb{Q}$ is $|d_E| = d^f N_{E/\mathbb{Q}}(\mathfrak{d}_{E/K})$, so this number is $\log \mathrm{rd}_E - \frac{1}{2}\log d$. With the product formula,

$$\log \Omega_\tau = -\Big(\log \mathrm{rd}_E - \tfrac{1}{2}\log d \Big) + \frac{1}{[M:\mathbb{Q}]} \sum_\rho \log \frac{A(\Lambda_0^\rho)}{\pi}.$$

For each ρ the lattice Λ_0^ρ has $\Delta = 1$ and $j(\Lambda_0^\rho) = \rho(j(\mathfrak{c}))$. By the theory of complex multiplication the numbers $j(\mathfrak{a})$, for $\mathfrak{a}$ running over representatives of the h ideal classes, are distinct and form one orbit under $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, so $j(\mathfrak{c})$ has degree h over $\mathbb{Q}$ and, as ρ runs over the $[M:\mathbb{Q}]$ embeddings, $\rho(j(\mathfrak{c}))$ takes each value $j(\mathfrak{a})$ exactly $[M:\mathbb{Q}]/h$ times. Two lattices with the same j-invariant are homothetic, so Λ_0^ρ is homothetic to $\varphi_0(\mathfrak{a})$ for the class of $\mathfrak{a}$ with $j(\mathfrak{a}) = \rho(j(\mathfrak{c}))$, and since $\Delta(\Lambda_0^\rho) = 1$ we get $A(\Lambda_0^\rho) = \mathcal{A}(\varphi_0(\mathfrak{a}))$. Proposition 15.3 gives

$$\frac{1}{[M:\mathbb{Q}]} \sum_\rho \log \frac{A(\Lambda_0^\rho)}{\pi} = \frac{1}{h} \sum_C \log \frac{\mathcal{A}(\Lambda_C)}{\pi} = \gamma + \log \frac{2\pi}{\sqrt{d}} - \lambda_1.$$

Adding the two contributions gives $\gamma + \log(2\pi/\mathrm{rd}_E) - \lambda_1 = \gamma + \log(\pi/v) - \lambda_1$. □

We write $\log \Omega = \gamma + \log(\pi/v) - \lambda_1$ for the common value. It is the constant of Proposition 3.1 with this choice of v and $\lambda_1 = \ell_{\chi_1}$. The lattice Λ_0 is the period lattice of the CM curve of Section 8 in the model of discriminant one, and $\log \Omega_\tau$ compares the archimedean norm of its invariant differential with the integral structure that the model defines at the finite places. This is the comparison that the archimedean norms of the theta kernels need, because the kernel law of Section 8 expresses the kernels in the algebraic parameter of the model, while their archimedean norms are computed in the complex coordinate.

Lemma 15.7. *In the situation of Proposition 15.6, suppose that $v \geq \pi$ and $|\lambda_1| \leq L_\varepsilon - 1$. Then $|\log \Omega| \leq \Lambda_a$ for $x \geq 1$.*

Proof. Since $v \geq \pi$, the number $\log(v/\pi)$ lies between 0 and $\log \mathrm{rd}_E$, so $|\gamma - \log(v/\pi)| \leq \max(\gamma, \log \mathrm{rd}_E) \leq \gamma + \log \mathrm{rd}_E$. Hence $|\log \Omega| \leq \gamma + \log \mathrm{rd}_E + L_\varepsilon - 1 \leq \log x + L_\varepsilon + \log \mathrm{rd}_E = \Lambda_a$, because $\gamma < 1$ and $\log x \geq 0$. □

The hypothesis $|\lambda_1| \leq L_\varepsilon - 1$ holds for large x, because Theorem 5.12(ii) gives $|\lambda_1| \leq 602 c_0 \log x$ and $L_\varepsilon = x^\varepsilon$.

15.3 The heights of the period maps

Let F be the totally complex field over which the lattices of the chain are defined (the field $\mathbb{L}$ of Section 7), and let v run over its archimedean places. For each index $u \in W$ the period map has the form of a chain. There are hermitian spaces $V_0, \dots, V_m$ over F with frames X_i at the archimedean places, the source V_0 of N_u, F-linear maps $K_j \colon V_{j-1} \to V_j$ for $1 \le j \le m$ (the lifts), and an F-linear map $P_u \colon V_m \to R_F^\vee$ (the final pairing with forms on the base side) such that

$$B_u = P_u \circ K_m \circ \cdots \circ K_1. \tag{15.5}$$

The target R_F has a frame X_R and real numbers $\kappa_v > 0$ and $\lambda_j(p, v) > 0$, indexed by the lifts j, by the base-type subspaces p of R_F and by the places v, such that the base-type inclusion multiplies the hermitian form on the base-type subspace p by $\kappa_v \prod_j \lambda_j(p, v)$ at v. These scalars are part of the data of the base side in Subsection 15.4.

Lemma 15.8. *Let $\phi_1 \colon U_0 \to U_1, \dots, \phi_s \colon U_{s-1} \to U_s$ be linear maps between finite-dimensional complex hermitian spaces. Then $\|\phi_s \circ \cdots \circ \phi_1\| \le \prod_i \|\phi_i\|$. Let U be a complex vector space with two hermitian forms h and h' and an orthogonal decomposition $U = \bigoplus_p U^p$ for both, with $h = c_p\, h'$ on U^p for real numbers $c_p > 0$. Then for every linear map ϕ from a hermitian space into the dual space $U^\vee$, the operator norm of ϕ for the dual form $h^\vee$ is at most $(\min_p c_p)^{-1/2}$ times its operator norm for $h'^\vee$.*

Proof. The first statement is submultiplicativity of operator norms. For the second, a linear form $\xi = \sum_p \xi_p$ on U has $h^\vee(\xi, \xi) = \sum_p c_p^{-1} h'^\vee(\xi_p, \xi_p) \le (\min_p c_p)^{-1} h'^\vee(\xi, \xi)$. □

Proposition 15.9. *Suppose that $B_u \neq 0$, that for every archimedean place v of F*

$$\log \|P_u\|_v^2 \le h_P, \qquad \log \|K_j\|_v^2 \le h_{K,j}(v) \quad (1 \le j \le m),$$

and that $\sum_j \log \lambda_j(p, v) \ge \ell_\lambda$ for all p and v, where the operator norm of P_u is taken with respect to the frame X_m and the dual of the form h' that equals $(\kappa_v \prod_j \lambda_j(p, v))^{-1}$ times the form of X_R on the base-type subspace p, and the operator norm of K_j with respect to X_{j-1} and X_j. Then

$$\mathrm{ht}(B_u) \le h_P + \sum_{j=1}^{m} \frac{1}{n_F} \sum_v h_{K,j}(v) - \ell_\lambda - \langle \log \kappa \rangle,$$

where ht *is computed with the frame X_R.*

Proof. At a place v, Lemma 15.8 applied to (15.5), with $U = R_F \otimes_v \mathbb{C}$, its decomposition into base-type subspaces, h the form of the frame X_R, and $c_p = \kappa_v \prod_j \lambda_j(p, v)$, gives

$$\log \|B_u\|_v^2 \le h_P + \sum_j h_{K,j}(v) - \log \kappa_v - \min_p \sum_j \log \lambda_j(p, v) \le h_P + \sum_j h_{K,j}(v) - \log \kappa_v - \ell_\lambda.$$

Taking the mean over the n_F places gives the statement. □

The three hypotheses of Proposition 15.9 are the archimedean content of the period maps, and we now state them in the form in which they enter (3.4). For a lift j and an embedding $\rho \in \Phi$ the kernel of K_j at ρ is a polynomial in the matrix coordinates, and its row weights form a vector $\nu_{j\rho} = (\nu_{j\rho 1}, \dots, \nu_{j\rho N})$ of nonnegative reals with $N = k + m(t_r + 1)$. Let $h_{j\rho}$ be the logarithm of the normalizing constant of the frame on the ρ-component of V_j, so that $\sum_\rho h_{j\rho}$ is the quantity ℓ_λ contributed by the lift j (the hook constants of the base side). Let $C_0 \ge 1$ be a constant.

Hypothesis 15.10 (kernel norms)**.** For every lift j, $\frac{1}{n_F}\sum_v h_{K,j}(v) \le s_j$ with

$$s_j = \sum_{\rho\in\Phi}\Big(\sum_{i=1}^{N}\mathrm{ph}(\nu_{j\rho i}) - \Big(\sum_{i=1}^{N}\nu_{j\rho i}\Big)\log\Omega\Big) + C_0\big(fN^2\Lambda_a + fm^3N^2L_\varepsilon\big).$$

Hypothesis 15.11 (final pairing)**.** $h_P \le -\sum_{t\in T} a_t \log T_u + e_P$ with $e_P \le C_0\big(N_T b' N + f(m\eta + S)N + fN^2\big)\Lambda_a$, where $\eta = W + mh_0$ is the spread of Section 5 and $S \ge 0$ bounds the small weights.

Hypothesis 15.12 (frame constants)**.** $\ell_\lambda \ge \sum_j \sum_\rho h_{j\rho} - e_\lambda$ with $e_\lambda \le C_0 mfN^2\Lambda_a$.

Remark 15.13. Hypothesis 15.10 is the place where the factor m in front of $\mathrm{ph}(a_t)$ in (3.4) arises. Each lift is a finite sum over classes of local theta kernels, as in (D1). At an archimedean place the local kernel restricted to a K-type is a polynomial whose monomials of row degree ν have Fock norm $\prod_i \nu_i!$, and $\log \nu! = \mathrm{ph}(\nu) + O(\log(\nu+2))$. The coordinates in which the kernel law of Section 8 makes the kernel integral differ from the complex coordinates by the period of the CM curve, and Proposition 15.6 computes the mean over the embeddings of the logarithm of this period. The class sum costs the logarithm of the number of classes, which is $O(fm^2N^2(L_\varepsilon + \log T_u))$, and this is the second term of s_j. Hypothesis 15.11 bounds the pairing with a form of level T_u, whose branching to the smaller group divides the norm by T_u to the power of the degree, and the degree at a top t is a_t. Subsection 15.9 lists the status of these three bounds in this section.

Definition 15.14. Put $e_P^{\max} = C_0(N_T b' N + f(m\eta + S)N + fN^2)\Lambda_a$ and $e_{\lambda,j}^{\max} = C_0 fN^2\Lambda_a$, and

$$h_u^0 = -\sum_{t\in T} a_t \log T_u + e_P^{\max} + \sum_{j=1}^{m}\Big(s_j - \sum_\rho h_{j\rho} + e_{\lambda,j}^{\max}\Big), \qquad h_u = h_u^0 - \langle \log\kappa\rangle.$$

Corollary 15.15. *Under Hypotheses 15.10, 15.11 and 15.12,* $\mathrm{ht}(B_u) \le h_u$ *for every* $u \in W$ *with* $B_u \neq 0$.

Proof. Proposition 15.9 with the three hypotheses gives $\mathrm{ht}(B_u) \le -\sum_t a_t \log T_u + e_P + \sum_j s_j - \sum_{j,\rho} h_{j\rho} + e_\lambda - \langle\log\kappa\rangle$, and $e_P \le e_P^{\max}$, $e_\lambda \le \sum_j e_{\lambda,j}^{\max}$. □

The number h_u is the height of Proposition 3.1. It enters (3.3) through Corollary 15.15 and (3.4) through Theorem 15.19 below.

15.4 The slope of the target lattice

The target lattice is a lattice R in an F-vector space R_F of dimension D, defined place by place by the local types of the forms on the top group. Its dual $R^\vee$ is the target of the maps B_u. We put

$$\mu^C = \mu(R^\vee) \quad \text{for the frame } X_R.$$

The height h_u of Definition 15.14 bounds $\mathrm{ht}(B_u)$ for the same frame by Corollary 15.15, so the slope inequality (3.3) for the maps B_u and the inequality (3.4) use the same pair (μ^C, h_u). Inequality (3.4) involves μ^C and h_u through the sum $\mu^C + h_u = (\mu^C - \langle\log\kappa\rangle) + h_u^0$.

The base side consists of lattices $M_j \subset W_j$ for $1 \le j \le m$, where W_j is a space of automorphic forms on a unitary group of signature (p_j, q_j) at the places of Φ, of weight $w_{j\rho}$ at ρ, and of linear

maps θ_j from R_F to W_j that multiply the hermitian forms of the base-type subspaces by $\lambda_j(p,v)$. The local terms at the twisted places are the numbers

$$\mathcal{L} = \sum_{j=1}^{m} \sum_{y \in \mathrm{Tw}} 2\,\ell_y\, e_y\, F_j(y), \tag{15.6}$$

where Tw is the finite set of places where the local type of the forms is twisted, $\ell_y = \log N(y)/[F:\mathbb{Q}]$, e_y is the ramification index and $F_j(y) \geq 0$ is the local length of the lift j at y. This is the quantity $\mathcal{L}$ of Proposition 3.1, and it is the same quantity in (3.2).

Hypothesis 15.16 (target slope)**.** There are real numbers μ_0, P_{pass} and E_b with

$$\mu^C - \langle \log \kappa \rangle \leq \mu_0 + P_{\mathrm{pass}}, \qquad P_{\mathrm{pass}} \leq 1 + C_0 \lambda_0,$$
$$\mu_0 \leq \sum_{j=1}^{m} \sum_{\rho \in \Phi} (p_j - q_j)\, w_{j\rho} \log \Omega + \mathcal{L} + E_b, \qquad E_b \leq C_0\, m\, x^{\varepsilon} (B_0 + N_c),$$

where $\lambda_0 \leq b$, $|p_j - q_j - k| \leq m$, and $B_0 = fmb\, x^{\varepsilon}$ and $N_c = N_2 kb\, x^{2\varepsilon}/n^2$ are the two terms of $\mathcal{B}$ in (4.8).

Remark 15.17. Hypothesis 15.16 is the composite of three statements. The first, the passage, compares the slope of $R^\vee$ at the top level of the chain with the slope of the corresponding lattice at the level of the base side, at a cost $3\delta(\mu + H_0 + 28\lambda_0) + 28\lambda_0$ with $\delta \leq 1/3$. The second bounds the slope of the dual of the base-side target, minus $\langle \log \kappa \rangle$, by $\sum_j \mathrm{lcv}(M_j)/D + \mathcal{L} + E$, where $\mathrm{lcv}(M_j)$ is the normalized logarithmic covolume of M_j and E collects the local indices at the twisted places. The third is the covolume bound $\sum_j \mathrm{lcv}(M_j)/D \leq \sum_{j,\rho}(p_j - q_j) w_{j\rho} \log \Omega + E_{l3}$, which is the arithmetic degree of the automorphic line bundle of weight w_j on the unitary Shimura variety of signature (p_j, q_j), evaluated through CM points where Proposition 15.6 applies, with the flat part of degree $\frac{1}{2}(p_j - q_j)\log\Omega$ per unit of weight. The local indices at the twisted places are bounded by the Galois-theoretic bounds of the twisted places, which is the input $E_b \leq C_0 m x^{\varepsilon}(B_0 + N_c)$. Subsection 15.9 lists which of these statements are proved here.

15.5 Hypothesis (A4)

The weights of the kernels satisfy two relations, which express the degree of the kernel of the lift j at the embedding ρ in terms of the nodes and the tops.

Hypothesis 15.18 (weights of the kernels)**.** For all j and ρ,

$$\sum_{i=1}^{N} \nu_{j\rho i} = \mathbf{1}_T(\rho)\, a_\rho + k\, w_{j\rho} + \delta_{j\rho},$$
$$|\delta_{j\rho}| \leq C_0\big(w_{j\rho} + k(N+S) + m(t_r+1)\eta + \mathbf{1}_T(\rho) kb'\big),$$
$$\sum_{i=1}^{N} \mathrm{ph}(\nu_{j\rho i}) - h_{j\rho} \leq \mathbf{1}_T(\rho)\, \mathrm{ph}(a_\rho) + \epsilon_{j\rho},$$
$$\epsilon_{j\rho} \leq C_0\big(w_{j\rho} + S + N^2 + m(t_r+1)\eta\big)\Lambda_a,$$

where $\mathbf{1}_T(\rho) = 1$ if ρ is a top and 0 otherwise, and $a_\rho = a_t$ at the top $\rho = t$.

Theorem 15.19. *Let $x \geq 3$, $0 < \varepsilon < 1$, $L_\varepsilon = x^\varepsilon$, $b = e^{L_\varepsilon}$, $a = xb$, and suppose that $\log x \leq L_\varepsilon$, $1 \leq \mathrm{rd}_E$, $\log \mathrm{rd}_E \leq L_\varepsilon$. Let $m, f, k, N \geq 1$, $S, t_r, \eta, b', W'' \geq 0$ and $C_0 \geq 1$ satisfy*

$$0 \leq w_{j\rho} \leq 2b, \quad |\log \Omega| \leq \Lambda_a, \quad N^2 \leq b, \quad m^3 N^2 L_\varepsilon \leq b, \quad m\eta N L_\varepsilon \leq b, \quad kN \leq b, \tag{15.7}$$

$$kS \leq W'', \quad m(t_r+1)\eta \leq W'', \quad NW'' \leq b, \quad b'N \leq a, \quad mkb' \leq a. \tag{15.8}$$

Assume Hypotheses 15.10, 15.11, 15.12, 15.16 and 15.18, and let h_u be as in Definition 15.14. Put $e_4 = (50C_0+10)\big(N_T a\Lambda_a + m\, x^\varepsilon(B_0+N_c)\big)$. Then for every $u \in W$

$$\mu^C + h_u - \mathcal{L} \leq \sum_{t\in T} \big(m\,\mathrm{ph}(a_t) - m a_t \log\Omega - a_t \log T_u\big) + e_4.$$

Proof. Write $\sum_{j,\rho}$ for the sum over $1 \leq j \leq m$ and $\rho \in \Phi$. Since $\mu^C + h_u = (\mu^C - \langle \log\kappa\rangle) + h_u^0$, Hypothesis 15.16 and Definition 15.14 give

$$\mu^C + h_u - \mathcal{L} \leq \sum_{j,\rho}(p_j - q_j) w_{j\rho}\log\Omega + E_b + P_{\mathrm{pass}} - \sum_t a_t \log T_u + e_P^{\max} + \sum_j s_j - \sum_{j,\rho} h_{j\rho} + \sum_j e_{\lambda,j}^{\max}.$$

Insert the definition of s_j and the two relations of Hypothesis 15.18. For each j and ρ,

$$\begin{aligned}&\sum_i \mathrm{ph}(\nu_{j\rho i}) - h_{j\rho} - \Big(\sum_i \nu_{j\rho i}\Big)\log\Omega + (p_j - q_j) w_{j\rho}\log\Omega \\ &\quad\leq \mathbf{1}_T(\rho)\big(\mathrm{ph}(a_\rho) - a_\rho\log\Omega\big) + \epsilon_{j\rho} + (p_j - q_j - k) w_{j\rho}\log\Omega - \delta_{j\rho}\log\Omega.\end{aligned}$$

Summing over j and ρ, the first term gives $m\sum_{t\in T}(\mathrm{ph}(a_t) - a_t\log\Omega)$, and the inequality to prove reduces to

$$\Sigma_1 + \Sigma_2 \leq e_4, \qquad \Sigma_1 = \sum_{j,\rho}\big(\epsilon_{j\rho} + m\, w_{j\rho}\Lambda_a + |\delta_{j\rho}|\Lambda_a\big),$$

where $\Sigma_2 = E_b + P_{\mathrm{pass}} + e_P^{\max} + \sum_j C_0(fN^2\Lambda_a + fm^3N^2L_\varepsilon) + \sum_j e_{\lambda,j}^{\max}$, and where we used $|p_j - q_j - k| \leq m$ and $|\log\Omega| \leq \Lambda_a$.

The size conditions give $S \leq kS \leq W'' \leq b/N \leq b$, $N^2 \leq b$, $m(t_r+1)\eta \leq W'' \leq b$ and $k(N+S) \leq kN + W'' \leq 2b$. Hence $w_{j\rho} + S + N^2 + m(t_r+1)\eta \leq 5b$ and $w_{j\rho} + k(N+S) + m(t_r+1)\eta \leq 5b$, so for each pair (j,ρ)

$$\epsilon_{j\rho} + m w_{j\rho}\Lambda_a + |\delta_{j\rho}|\Lambda_a \leq \big(10C_0 + 2m\big) b\,\Lambda_a + \mathbf{1}_T(\rho) C_0 k b' \Lambda_a.$$

There are m lifts and f embeddings, and $mkb' \leq a$, so $\Sigma_1 \leq (10C_0+2)m^2 fb\Lambda_a + C_0 N_T a\Lambda_a$. Since $\log x \leq L_\varepsilon$ and $\log \mathrm{rd}_E \leq L_\varepsilon$ we have $\Lambda_a = \log x + L_\varepsilon + \log \mathrm{rd}_E \leq 3L_\varepsilon = 3x^\varepsilon$, and $m^2 fbx^\varepsilon = mB_0$. Therefore $\Sigma_1 \leq 3(10C_0+2)mB_0 + C_0 N_T a\Lambda_a$.

For Σ_2: $E_b \leq C_0 m x^\varepsilon(B_0+N_c)$ and $P_{\mathrm{pass}} \leq 1 + C_0 b \leq (1+C_0)B_0$ because $B_0 \geq b \geq 1$. Next $N_T b' N\Lambda_a \leq N_T a\Lambda_a$; $fm\eta N\Lambda_a \leq fb\Lambda_a$ because $m\eta N \leq b/L_\varepsilon \leq b$; $fSN\Lambda_a \leq fb\Lambda_a$ because $SN \leq W''N \leq b$; and $fN^2\Lambda_a \leq fb\Lambda_a$. So $e_P^{\max} \leq C_0 N_T a\Lambda_a + 3C_0 fb\Lambda_a \leq C_0 N_T a\Lambda_a + 9C_0B_0$. Finally $mC_0(fN^2\Lambda_a + fm^3N^2L_\varepsilon) \leq C_0(3mfbx^\varepsilon + mfb) \leq 4C_0B_0$ and $\sum_j e_{\lambda,j}^{\max} = C_0 mfN^2\Lambda_a \leq 3C_0B_0$. Collecting terms and using $B_0 \leq mx^\varepsilon B_0$,

$$\Sigma_1 + \Sigma_2 \leq 2C_0 N_T a\Lambda_a + (48C_0+7)\, m\, x^\varepsilon(B_0+N_c) \leq e_4. \qquad \square$$

Corollary 15.20. *Under the hypotheses of Theorem 15.19, inequality* (3.4) *of Proposition 3.1 holds for every $u \in W$ with $\mu^C = \mu(R^\vee)$, the heights h_u of Definition 15.14, the local terms $\mathcal{L}$ of* (15.6)*, $\log\Omega = \gamma + \log(\pi/v) - \lambda_1$ with $v = \frac{1}{2}\mathrm{rd}_E$ and $\lambda_1 = \ell_{\chi_1}$, and*

$$e_4 = (50C_0 + 10)\Big(N_T a \Lambda_a + m\, x^\varepsilon (B_0 + N_c)\Big).$$

Proof. The right side of (3.4) is $\sum_{t=1}^{N_T}(m\,\mathrm{ph}(a_t) - m a_t \log\Omega - a_t \log T_u) + e_4$, and Theorem 15.19 is this inequality. Proposition 15.6 identifies $\log\Omega$. □

15.6 Hypothesis (A6)

Hypothesis (3.6) is a lower bound for the auxiliary prime T_u. The prime T_u is chosen so that the rank of the lattice N_u, which grows like T_u^{kf}, matches the dimension budget of the target lattice. We first define the two numbers that are matched.

For a strictly decreasing vector $y = (y_0, \ldots, y_{s-1})$ of reals put

$$\mathrm{vand}(y) = \prod_{0 \le i < i' < s} (y_i - y_{i'}), \qquad \dim(y) = \frac{\mathrm{vand}(y)}{\prod_{i<s} i!}.$$

For an integral dominant weight y of GL_s, Weyl's dimension formula says that $\dim(y + \rho_s)$, with $\rho_s = (s-1, \ldots, 0)$, is the dimension of the irreducible representation of highest weight y. For $M \ge 1$ let $\mathrm{mass}(M)$ be the mass of the standard unimodular hermitian lattice $\mathcal{O}_E^M$, that is, the sum of $1/|\mathrm{Aut}|$ over the classes in its genus.

For a prime $g \ge 3$ and integers $k \ge 2$, $m \ge 2$ even, with $r = k + m - 1$ and $r_S = m/2 - 1$, let $I_\Pi(g) = |GL_k(\mathbb{F}_g)|/|U_k(\mathbb{F}_g)|$, where U_k is the group of upper unitriangular matrices, and let $I_\Sigma(g) = |GL_r(\mathbb{F}_g)|/|P(\mathbb{F}_g)|$, where P is the group of upper unitriangular $r \times r$ matrices whose entries in the column $r_S + k$ vanish in the rows $r_S, \ldots, r_S + k - 1$. These are the indices of the level structures at g of the column groups and of the row group. Put $I_\Pi = (I_\Pi(g_0) I_\Pi(g_1))^f$ and $I_\Sigma = (I_\Sigma(g_0) I_\Sigma(g_1))^f$.

Lemma 15.21. *For a prime $g \ge 3$, $\frac{1}{2} g^{k(k+1)/2} \le I_\Pi(g) \le g^{k(k+1)/2}$ and $\frac{1}{2} g^e \le I_\Sigma(g) \le g^e$ with $e = \frac{1}{2}(r^2 + 3k + 1 + 2r_S)$. Moreover $e - \frac{1}{2}k(k+1) = \frac{1}{2}m(2k + m - 1)$, and*

$$\frac{I_\Sigma}{I_\Pi} \ge 2^{-2f} (g_0 g_1)^{fm(2k+m-1)/2}.$$

Proof. We have $|GL_s(\mathbb{F}_g)| = g^{s(s-1)/2} \prod_{i=1}^s (g^i - 1)$ and $1 \ge \prod_i (1 - g^{-i}) \ge 1 - \sum_{i \ge 1} g^{-i} = 1 - \frac{1}{g-1} \ge \frac{1}{2}$. The group U_k has order $g^{k(k-1)/2}$, so $I_\Pi(g) = \prod_{i=1}^k (g^i - 1)$ lies between $\frac{1}{2} g^{k(k+1)/2}$ and $g^{k(k+1)/2}$. The group P consists of the unitriangular matrices with k entries above the diagonal fixed to zero, so $|P| = g^{r(r-1)/2 - k}$ and $I_\Sigma(g) = g^k \prod_{i=1}^r (g^i - 1)$ lies between $\frac{1}{2} g^{r(r+1)/2 + k}$ and $g^{r(r+1)/2+k}$. Since $r = k + 1 + 2r_S$ we have $r(r+1)/2 + k = \frac{1}{2}(r^2 + r + 2k) = e$. Next $2e - k(k+1) = r^2 + 3k + 1 + 2r_S - k^2 - k = (k + m - 1)^2 - k^2 + k + m - 1 = m(2k + m - 1)$. Finally $I_\Sigma / I_\Pi \ge \prod_{g \in \{g_0, g_1\}} \left(\frac{1}{2} g^e / g^{k(k+1)/2}\right)^f$. □

For each $\rho \in \Phi$ let the row parameter of the target group be

$$\lambda_\rho^{\mathrm{row}} = (v_{\rho,0}, \ldots, v_{\rho,k-2}, -b_{\rho,0}, \ldots, -b_{\rho,m-1}) \in \mathbb{R}^r,$$

and for each $u \in W$ let $\lambda^{\mathrm{col}}_{u,\rho} = (u_{\rho,0}, \dots, u_{\rho,k-1})$ be the column parameter of the source group at level T_u. The data of Section 5 and Proposition 5.16 give real numbers $\Delta \geq 1$ and these parameters with the following properties, for every ρ (we drop ρ and u from the notation).

(P1) $0 \leq v_i \leq 4k\Delta$ for $i \leq k-2$, and $v_i - v_{i+1} = 4\Delta$ for $i \leq k-3$.

(P2) $v_i < u_i$ and $u_{i+1} < v_i$ for $i \leq k-2$, and $-b_0 < u_{k-1}$.

(P3) Either ρ is a top, $u_0 \leq 2a$ and $-a \leq u_{k-1}$, or ρ is not a top, $4k\Delta \leq u_0 - v_0 \leq 8k\Delta$ and $u_{k-1} = v_{k-2} - \frac{1}{2}$.

(P4) $b_j < w_{j\rho}$ and $|w_{j\rho} - b_j| \leq b_j/x$ for $j < m$, $w_{j\rho} < b_{j+1}$ and $b_{j+1} - b_j \geq 1$ for $j < m-1$, $4k\Delta \leq b_0/x$, and $w_{m-1,\rho} \leq 2b$.

Here the nodes $w_{j\rho}$ are indexed by $0 \leq j < m$, and $w_{j\rho}$ is the node of the lift $j+1$ in the notation of (3.6).

Definition 15.22. Put

$$d_u^* = \mathrm{mass}(k)\, I_\Pi \prod_{\rho \in \Phi} \dim(\lambda^{\mathrm{col}}_{u,\rho}), \qquad \log X_u = \frac{1}{fk} \log \frac{D^*}{d_u^*},$$

where D^* is the dimension budget of the target lattice.

Hypothesis 15.23 (dimension budget)**.** $D^* \geq \left(\frac{2}{3}\right)^f \mathrm{mass}(r)\, I_\Sigma \prod_{\rho \in \Phi} \dim(\lambda^{\mathrm{row}}_\rho)$.

The number $d_u^*(T^k-1)^f$ is the dimension of the space of forms of level T on the source group when the stabilizers of the classes are trivial, by the class number formula $\#\mathrm{Cl} = \mathrm{mass}(k) I_\Pi (T^k - 1)^f$ and Weyl's dimension formula. The budget D^* is the product of the volume of the target group, the index I_Σ and the formal degrees of the discrete series with parameters $\lambda^{\mathrm{row}}_\rho$. For the Haar measure of the compact dual, Harish-Chandra's formula gives the formal degree $\dim(\lambda^{\mathrm{row}}_\rho)$ exactly, and the volume equals $\mathrm{mass}(r)$ when the hermitian space of the target is locally isomorphic at every finite place to the standard unimodular space. These two published inputs give Hypothesis 15.23 with the factor 1 in place of $(2/3)^f$.

Lemma 15.24. *Let $k \geq 2$, $m \geq 1$, $\Delta \geq 1$, $x \geq 1$, $a \geq 1$, and let v, u, b, w satisfy (P1) to (P4) with $b \leq a$. Then* $\mathrm{vand}(u) > 0$, $\mathrm{vand}(\lambda^{\mathrm{row}}) > 0$ *and*

$$\log \mathrm{vand}(\lambda^{\mathrm{row}}) - \log \mathrm{vand}(u) \geq (k-1) \sum_{j<m} \log b_j - 2k \log(8a) - k(1 + \log k).$$

Proof. The entries of λ^{row} are strictly decreasing by (P1), (P4) and $v_{k-2} \geq 0 > -b_0$, and those of u are strictly decreasing by (P2), so both Vandermonde products are positive. The differences of entries of λ^{row} are $v_i - v_{i'}$, $v_i + b_j$ and $b_{j'} - b_j$ for $i < i'$ and $j < j'$, and $b_{j'} - b_j \geq 1$ and $v_i + b_j \geq b_j$. Hence

$$\mathrm{vand}(\lambda^{\mathrm{row}}) \geq \mathrm{vand}(v) \prod_{j<m} b_j^{k-1}, \qquad v = (v_0, \dots, v_{k-2}).$$

For $1 \leq i < i' \leq k-2$, (P2) gives $u_i - u_{i'} < v_{i-1} - v_{i'} = (i' - i + 1)4\Delta$, while $v_{i-1} - v_{i'-1} = (i' - i)4\Delta$. The pairs $(i-1, i'-1)$ run over the pairs $0 \leq p < q \leq k-3$, and the remaining factors $v_p - v_{k-2}$ of $\mathrm{vand}(v)$ are at least $4\Delta \geq 1$. Therefore

$$\prod_{1 \leq i < i' \leq k-2} (u_i - u_{i'}) \leq \mathrm{vand}(v) \prod_{d=1}^{k-3} \left(\frac{d+1}{d}\right)^{k-2-d} \leq \mathrm{vand}(v)\, e^{k(1+\log k)},$$

since $\sum_{d\le k}(k-2-d)\log(1+1/d) \le k\sum_{d\le k} 1/d \le k(1+\log k)$. The other factors of $\mathrm{vand}(u)$ are $u_0 - u_{i'}$ for $1 \le i' \le k-1$ and $u_i - u_{k-1}$ for $1 \le i \le k-2$, at most $2k$ factors. Each lies in $(0, 8a]$. In the top case all u_i lie in $[-a, 2a]$. In the other case $u_0 \le v_0 + 8k\Delta \le 12k\Delta \le 3b_0/x \le 6b \le 6a$ by (P3) and (P4), and $u_{k-1} \ge -\frac{1}{2}$, so all u_i lie in $[-\frac{1}{2}, 6a]$. Hence $\mathrm{vand}(u) \le \mathrm{vand}(v) e^{k(1+\log k)} (8a)^{2k}$, and taking logarithms gives the statement. □

Proposition 15.25. *Let $0 < \varepsilon < 1$ and $C_1 \ge 0$. There is a constant C_6, depending on ε and C_1 alone, with the following property. Let $x \ge 1$, $L_\varepsilon = x^\varepsilon$, $b = e^{L_\varepsilon}$, $a = xb$, let $f \ge 1$, $m \ge 2$ even with $m \le x$, $2 \le k \le x^{C_1}$, let $\pi \le v$ with $\log v \le L_\varepsilon$, and let $g_0, g_1 \ge 3$ be primes. Suppose that the parameters of every $\rho \in \Phi$ satisfy (P1) to (P4), that*

$$\frac{\mathrm{mass}(M)}{\mathrm{mass}(M-1)} = \frac{\big((M-1)!\,(v/\pi)^M\big)^f}{(2v)^{f/2}}\,\mathbf{L}(M), \qquad |\log \mathbf{L}(M)| \le f \log \zeta(M) \qquad (k < M \le r),$$

and that D^* satisfies Hypothesis 15.23. Then for every u

$$\log X_u \ge \sum_{j<m} \frac{1}{f} \sum_{\rho\in\Phi} \log \frac{w_{j\rho}\, v}{\pi} + m \log(g_0 g_1) - C_6\Big(L_\varepsilon + \frac{m^2 L_\varepsilon}{k}\Big).$$

Proof. By Hypothesis 15.23 and the definition of d_u^*,

$$\log \frac{D^*}{d_u^*} \ge f \log \tfrac{2}{3} + \log \frac{\mathrm{mass}(r)}{\mathrm{mass}(k)} + \log \frac{I_\Sigma}{I_\Pi} + \sum_\rho \big(\log \dim(\lambda_\rho^{\mathrm{row}}) - \log \dim(\lambda_{u,\rho}^{\mathrm{col}})\big).$$

We treat the four terms in turn. The mass ratio telescopes over $M = k+1, \dots, r$, and there are $r - k = m - 1$ such M:

$$\log \frac{\mathrm{mass}(r)}{\mathrm{mass}(k)} = \sum_{M=k+1}^{r} \Big(f \log(M-1)! + fM \log \frac{v}{\pi} - \frac{f}{2}\log(2v) + \log \mathbf{L}(M)\Big).$$

Since $\log(v/\pi) \ge 0$, $\sum_M fM \log(v/\pi) \ge f(m-1)k \log(v/\pi)$, and $\log \mathbf{L}(M) \ge -f \log \zeta(2) \ge -f$. For the dimensions, $\log \dim(\lambda^{\mathrm{row}}) - \log \dim(\lambda^{\mathrm{col}}) = \log \mathrm{vand}(\lambda^{\mathrm{row}}) - \log \mathrm{vand}(\lambda^{\mathrm{col}}) - \sum_{M=k}^{r-1} \log M!$, and summed over the f embeddings the factorials cancel the terms $f \log(M-1)!$ of the mass ratio exactly. Lemma 15.21 bounds the index term below by $-2f \log 2 + fmk\,g$, with $g = \log(g_0 g_1) \ge 0$, because $m(2k+m-1)/2 \ge mk$. Lemma 15.24 bounds the Vandermonde terms. Dividing by fk we obtain

$$\log X_u \ge \frac{k-1}{k} \cdot \frac{1}{f} \sum_\rho \sum_{j<m} \log b_{\rho,j} + (m-1)\log \frac{v}{\pi} + mg - \mathcal{E},$$

with

$$\mathcal{E} = 2\log(8a) + 1 + \log k + \frac{(m-1)\log(2v)}{2k} + \frac{m-1}{k} + \frac{\log(3/2) + 2\log 2}{k}.$$

By (P4), $w_{j\rho} \le b_{\rho,j}(1 + 1/x)$, so $\log b_{\rho,j} \ge \log w_{j\rho} - 1/x$, and $1 \le b_{\rho,j} \le w_{j\rho} \le 2b$, so $\frac{1}{k}\sum_{j<m} \log b_{\rho,j} \le \frac{m}{k}(L_\varepsilon + 1)$. Hence

$$\frac{k-1}{k} \cdot \frac{1}{f} \sum_{\rho,j} \log b_{\rho,j} \ge \sum_{j<m} \frac{1}{f} \sum_\rho \log w_{j\rho} - \frac{m}{x} - \frac{m}{k}(L_\varepsilon + 1),$$

and $(m-1)\log(v/\pi) \geq m\log(v/\pi) - L_\varepsilon$. It remains to bound $\mathcal{E} + \frac{m}{x} + \frac{m}{k}(L_\varepsilon + 1) + L_\varepsilon$ by $C_6(L_\varepsilon + m^2 L_\varepsilon/k)$. We have $\log(8a) = \log 8 + \log x + L_\varepsilon$, $\log x \leq L_\varepsilon/(e\varepsilon)$ for $x \geq 1$, $\log k \leq C_1 \log x$, $\log(2v) \leq 1 + L_\varepsilon$ and $m/x \leq 1$. Since $L_\varepsilon \geq 1$ and $m \leq m^2$, each term is at most a constant depending on ε and C_1 times $L_\varepsilon + m^2 L_\varepsilon/k$. □

Theorem 15.26. *Assume the hypotheses of Proposition 15.25, let $\Gamma \geq 1$ with $\log\Gamma \leq fk$, and suppose that for every $u \in W$ the prime T_u satisfies $T_u \geq X_u\Gamma^{-2/(fk)}$. Then* (3.6) *holds for every $u \in W$ with $g = \log(g_0 g_1)$, with $v = \frac{1}{2}\mathrm{rd}_E$, with the nodes $w_{j\rho}$ of the lifts, and with*

$$e_6 = C_6\Big(L_\varepsilon + \frac{m^2 L_\varepsilon}{k}\Big) + 2.$$

Proof. $\log T_u \geq \log X_u - \frac{2}{fk}\log\Gamma \geq \log X_u - 2$, and Proposition 15.25 bounds $\log X_u$ below by the right side of (3.6) plus $e_6 - 2$. □

Remark 15.27. The prime T_u is the auxiliary prime of the index u. It must also satisfy $T_u \leq (1 + \frac{1}{fk})X_u\Gamma^{-2/(fk)}$, so that the rank of N_u matches $D^*\Gamma^{-2}$ up to a bounded factor, as the frame condition of (3.3) requires, and it must split in K as $T_u = \pi\bar{\pi}$ with $\pi \equiv 1$ modulo $Q_S q d (g_0 g_1)^2$. The existence of such a prime in the interval $[X_u\Gamma^{-2/(fk)}, (1+\frac{1}{fk})X_u\Gamma^{-2/(fk)}]$ is a statement on primes in a ray class of K in a short interval, and the choice belongs to the construction of the data. The upper bound $\log T_u \leq C\, m L_\varepsilon$ follows from the same comparison of dimensions and is used in Subsection 15.7. Here $\Gamma = \exp(C_\Gamma m^5 f L_\varepsilon)$ is the factor of the frame condition, and $\log\Gamma \leq fk$ is the size condition $C_\Gamma m^5 L_\varepsilon \leq k$.

15.7 Hypothesis (A7)

The base lattice M_u is the lattice of forms of level zero of the chain for the index u. We first recall the integral theory of polynomial representations that describes it.

Let $s \geq 1$ and let $\mathbb{Z}[x] = \mathbb{Z}[x_{ab} : 1 \leq a, b \leq s]$ with GL_s acting on the rows. On $\mathbb{Q}[x]$ the Fock form

$$\{p, q\} = \sum_\alpha p_\alpha q_\alpha\, \alpha!, \qquad \alpha! = \prod_{a,b} \alpha_{ab}!,$$

where $p = \sum_\alpha p_\alpha x^\alpha$, is positive definite, and over $\mathbb{C}$ its hermitian version is invariant under the unitary group acting on rows. For a partition $\nu = (\nu_1 \geq \cdots \geq \nu_s \geq 0)$ let $hw_\nu = \prod_{i=1}^s \Delta_i^{\nu_i - \nu_{i+1}}$, with Δ_i the leading principal $i \times i$ minor and $\nu_{s+1} = 0$. The *Weyl module* $V(\nu)_\mathbb{Z}$ is the $\mathbb{Z}$-span of the images of hw_ν under the divided powers of the row operators $x_{b\cdot}\partial/\partial x_{a\cdot}$, and the *dual Weyl module* $\nabla(\nu)_\mathbb{Z}$ is the set of $p \in \mathbb{Z}[x]$ of column weight ν that are invariant under the column substitutions $x_{\cdot j} \mapsto x_{\cdot j} + t\, x_{\cdot l}$ for $l < j$. Both are lattices in the same $\mathbb{Q}$-space, the irreducible representation of highest weight ν, which is spanned over $\mathbb{Q}$ by hw_ν and its translates.

Definition 15.28. For a partition ν with s parts let $\mu_{\mathrm{std}}(\nu) = -\frac{1}{r_\nu}\log\det\big(\{e_i, e_{i'}\}/\{hw_\nu, hw_\nu\}\big)$, where $e_1, \ldots, e_{r_\nu}$ is a $\mathbb{Z}$-basis of $V(\nu)_\mathbb{Z}$, and put $c_\nu = \{hw_\nu, hw_\nu\}/\prod_i \nu_i!$.

Lemma 15.29. $\{hw_\nu, hw_\nu\} = \prod_{i=1}^s (\nu_i + s - i)!/\prod_{i<i'}(\nu_i - \nu_{i'} + i' - i)$, *and* $1 \leq c_\nu \leq (\nu_1 + s)^{s(s-1)/2}$.

Proof. The first formula is the Capelli identity applied to the highest weight vector, which computes $hw_\nu(\partial)hw_\nu$ (see the published input listed in Subsection 15.9). Then

$c_\nu = \prod_i \frac{(\nu_i+s-i)!}{\nu_i!} / \prod_{i<i'}(\nu_i - \nu_{i'} + i' - i)$. Since $\nu_{i'} \geq 0$, $\prod_{i'>i}(\nu_i - \nu_{i'} + i' - i) \leq \prod_{i'>i}(\nu_i + i' - i) = (\nu_i + s - i)!/\nu_i!$, which gives $c_\nu \geq 1$. Since each factor of the denominator is at least 1 and $(\nu_i + s - i)!/\nu_i! \leq (\nu_i + s)^{s-i} \leq (\nu_1 + s)^{s-i}$, we get $c_\nu \leq (\nu_1 + s)^{\sum_i (s-i)} = (\nu_1 + s)^{s(s-1)/2}$. □

Proposition 15.30. $\mu_{\mathrm{std}}(\nu) \geq -\big(|\nu| \log s + \log c_\nu\big)$, *where* $|\nu| = \sum_i \nu_i$.

Proof. Let G and G' be the Gram matrices of $\{\cdot,\cdot\}/\{hw_\nu, hw_\nu\}$ on $\mathbb{Z}$-bases of $V(\nu)_\mathbb{Z}$ and of $\nabla(\nu)_\mathbb{Z}$. Under the form $\{\cdot,\cdot\}/\{hw_\nu, hw_\nu\}$ the lattices $V(\nu)_\mathbb{Z}$ and $\nabla(\nu)_\mathbb{Z}$ are dual to each other, so $\det G \cdot \det G' = 1$; this is the duality of Weyl modules and induced modules under the contravariant form normalized at the highest weight vector, listed in Subsection 15.9. It suffices to prove $\det G' \geq A^{r_\nu}$ with $A = \prod_i \nu_i!/(s^{|\nu|}\{hw_\nu, hw_\nu\})$, because then $\mu_{\mathrm{std}}(\nu) = -\frac{1}{r_\nu} \log\det G = \frac{1}{r_\nu} \log\det G' \geq \log A = -(|\nu| \log s + \log c_\nu)$.

Let $f_1, \ldots, f_{r_\nu}$ be a $\mathbb{Z}$-basis of $\nabla(\nu)_\mathbb{Z}$ and let F be the integral matrix of their coefficients in the monomial basis. The unnormalized Gram matrix is $F\,\mathrm{diag}(\alpha!)\,F^{\mathsf{T}}$, and the Cauchy–Binet formula gives

$$\det\big(F\,\mathrm{diag}(\alpha!)F^{\mathsf{T}}\big) = \sum_{|S|=r_\nu} \det(F_S)^2 \prod_{\alpha\in S} \alpha!,$$

where S runs over the sets of r_ν monomials and F_S is the corresponding square minor. Every monomial x^α that occurs has column weight ν, that is $\sum_a \alpha_{aj} = \nu_j$ for each column j, and the multinomial bound $\nu_j! \leq s^{\nu_j} \prod_a \alpha_{aj}!$ gives $\alpha! \geq \prod_j \nu_j!/s^{|\nu|}$. Since F has rank r_ν, some $\det F_S$ is a nonzero integer, and therefore $\det(F\mathrm{diag}(\alpha!)F^{\mathsf{T}}) \geq (\prod_j \nu_j!/s^{|\nu|})^{r_\nu}$. Dividing by $\{hw_\nu, hw_\nu\}^{r_\nu}$ gives $\det G' \geq A^{r_\nu}$. □

We now describe M_u. For each class c of the genus of level zero and each $\tau \in \Phi$ there is a hermitian lattice $Q_{c,\tau}$ of rank k over the field F, obtained from the class representative L_c by the embedding τ, with $L_c \subset L_c^\#$ of index T_u^{2f}. The lattice M_u is the direct sum over the classes c of the tensor products over $\tau \in \Phi$ of the Schur functors of highest weight $\lambda_{u,\tau}^{\mathrm{col}} - \rho_k$ applied to $Q_{c,\tau}$, with the integral structure of the Weyl modules at the finite places and with the Fock metrics, normalized by the norm of the highest weight vector, in orthonormal frames at the archimedean places. We write $\lambda_\tau = (\lambda_{\tau,0} \geq \cdots \geq \lambda_{\tau,k-1})$ for this highest weight, $|\lambda_\tau| = \sum_i \lambda_{\tau,i}$, and $\nu_\tau = \lambda_\tau - \lambda_{\tau,k-1}(1,\ldots,1)$, which is a partition.

Lemma 15.31. *Let Q be a hermitian lattice of rank s over $\mathcal{O}_F$, let λ be a dominant weight of GL_s with $\nu = \lambda - \lambda_{s-1}(1,\ldots,1)$, and let $\mathbb{S}_\lambda Q$ be the lattice of the Schur functor of λ with the integral structure given at each finite place by the Weyl module $V(\nu)_\mathbb{Z}$ twisted by $\det^{\lambda_{s-1}}$ and with the Fock metrics normalized at the highest weight vector in orthonormal frames. Then*

$$\mu(\mathbb{S}_\lambda Q) = |\lambda|\,\mu(Q) + \mu_{\mathrm{std}}(\nu).$$

Proof. Let r_λ be the dimension of the representation of highest weight λ. For $g \in GL_s$ the determinant of $\mathbb{S}_\lambda(g)$ is $\det(g)^{|\lambda| r_\lambda/s}$, because both sides are characters of GL_s that agree on scalar matrices. Every hermitian lattice of rank s is the translate $g \cdot \mathcal{O}_F^s$ of the standard lattice with the standard metric by an element $g = (g_w, g_v)$ of GL_s of the adeles of F, with g_w in $GL_s(\mathcal{O}_{F_w})$ for almost all finite places w. Translating by g changes $\widehat{\deg}$ of a lattice of rank s' by $-\sum_w \log|\det g_w|_w - \sum_v \log|\det g_v|_v^2$, with the normalized absolute values, when the determinant is taken on the lattice of rank s'. The integral structures of $\mathbb{S}_\lambda$ are $GL_s(\mathcal{O}_{F_w})$-stable and the

normalized Fock metrics are invariant under the unitary group, so $\mathbb{S}_\lambda(g \cdot \mathcal{O}_F^s) = \mathbb{S}_\lambda(g) \cdot \mathbb{S}_\lambda(\mathcal{O}_F^s)$, and the determinant formula gives

$$\widehat{\deg}(\mathbb{S}_\lambda Q) - \widehat{\deg}(\mathbb{S}_\lambda \mathcal{O}_F^s) = \frac{|\lambda| r_\lambda}{s}\big(\widehat{\deg}(Q) - \widehat{\deg}(\mathcal{O}_F^s)\big) = \frac{|\lambda| r_\lambda}{s}\widehat{\deg}(Q).$$

The twist by $\det^{\lambda_{s-1}}$ does not change the standard lattice, which has degree zero, so $\widehat{\deg}(\mathbb{S}_\lambda \mathcal{O}_F^s) = \widehat{\deg}(V(\nu)_{\mathbb{Z}} \otimes \mathcal{O}_F) = n_F r_\lambda \, \mu_{\mathrm{std}}(\nu)$. Dividing by $n_F r_\lambda$ gives the formula, since $\mu(Q) = \widehat{\deg}(Q)/(n_F s)$. □

Proposition 15.32. *$\mu(M_u) = -\sum_{\tau\in\Phi} \frac{|\lambda_\tau|}{k} \log T_u + \sum_{\tau\in\Phi} \mu_{\mathrm{std}}(\nu_\tau)$, and therefore*

$$\mu(M_u) \geq -\sum_{\tau\in\Phi} \frac{|\lambda_\tau|}{k} \log T_u - \sum_{\tau\in\Phi} \Big(|\nu_\tau| \log k + \frac{k^2}{2} \log\big(\lambda_{\tau,0} - \lambda_{\tau,k-1} + k\big)\Big).$$

Proof. The hermitian form h of L_c takes values in $\mathcal{O}_E$ on L_c, and $L_c^\# = \{y : h(L_c, y) \subset \mathcal{O}_E\}$ is isometric to the dual lattice $L_c^\vee$. Hence $\widehat{\deg}(L_c^\#) = -\widehat{\deg}(L_c)$ and $\widehat{\deg}(L_c^\#) - \widehat{\deg}(L_c) = \log[L_c^\# : L_c] = 2f \log T_u$, so $\widehat{\deg}(L_c) = -f \log T_u$ and $\mu(L_c) = -f \log T_u/(fk) = -\log T_u/k$. The normalized slope does not change under the base change to F by τ, so $\mu(Q_{c,\tau}) = -\log T_u/k$, and Lemma 15.31 gives $\mu(\mathbb{S}_{\lambda_\tau} Q_{c,\tau}) = -\frac{|\lambda_\tau|}{k} \log T_u + \mu_{\mathrm{std}}(\nu_\tau)$. The slope of a tensor product of hermitian lattices is the sum of the slopes, and a direct sum of lattices of equal slope has that slope, which gives the formula for $\mu(M_u)$. Proposition 15.30 and Lemma 15.29 with $s = k$ give the inequality. □

Theorem 15.33. *Let $C_n \geq 1$ and suppose that the weights of level zero satisfy, for every u and τ, $\lambda_{\tau,i} \in \mathbb{Z}$, and*

(W1) if τ is a top, $xb \leq \lambda_{\tau,0} \leq 2xb$, $b' \leq -\lambda_{\tau,k-1} < 2b'$, and $0 \leq \lambda_{\tau,i} \leq S$ for $0 < i < k-1$;

(W2) if τ is not a top, $0 \leq \lambda_{\tau,i} \leq C_n S$ for all i,

with $1 \leq S \leq b'$. Suppose also that $\log T_u \leq C_T m L_\varepsilon$, $\log k \leq 5 \log x$, $fk(b' + S) \leq N_T a$ and $fk^2 \leq N_T a$, and that $\log x \leq L_\varepsilon$ and $\log \mathrm{rd}_E \leq L_\varepsilon$. Then (3.7) holds for every $u \in W$ with

$$e_7 = C_7 \, N_T a \Lambda_a \Big(1 + \frac{m}{k}\Big),$$

where C_7 depends on C_n and C_T alone.

Proof. By (W1) and (W2), $|\lambda_\tau| \leq 2a + kS$ at a top and $|\lambda_\tau| \leq kC_nS$ elsewhere, where we drop the negative entry $\lambda_{\tau,k-1}$ at a top. Further $|\nu_\tau| = \lambda_{\tau,0} + \sum_{0<i<k-1} \lambda_{\tau,i} - (k-1)\lambda_{\tau,k-1} \leq 2a + kS + 2kb'$ at a top and $|\nu_\tau| \leq kC_nS$ elsewhere. Proposition 15.32 gives

$$-\mu(M_u) \leq N_T \frac{2a}{k} \log T_u + f(1 + C_n) S \log T_u + \big(2aN_T + fk(3 + C_n)(S + b')\big) \log k + \sum_\tau \frac{k^2}{2} \log\big(\lambda_{\tau,0} - \lambda_{\tau,k-1} + k\big).$$

We bound each term by a multiple of $N_T a \Lambda_a (1 + m/k)$. First $N_T \frac{2a}{k} \log T_u \leq 2C_T N_T a \Lambda_a \frac{m}{k}$. Next $fS \log T_u \leq fSC_T m L_\varepsilon \leq C_T fkS \Lambda_a \leq C_T N_T a \Lambda_a$ because $m \leq k$. Next $\log k \leq 5 \log x \leq 5\Lambda_a$ and

$fk(S+b') \le N_T a$, so the third term is at most $5(5+C_n)N_T a\Lambda_a$. Finally $\lambda_{\tau,0} - \lambda_{\tau,k-1} + k \le (C_n + 5)a$ at every τ, so the last term is at most $\frac{1}{2}fk^2\big(\log(C_n+5) + \Lambda_a\big) \le \big(1 + \log(C_n+5)\big)N_T a\Lambda_a$, because $fk^2 \le N_T a$ and $\Lambda_a \ge 1$. So $\mu(M_u) \ge -C_7 N_T a\Lambda_a(1 + m/k)$ with $C_7 = 2C_T + C_T(1 + C_n) + 5(5+C_n) + 1 + \log(C_n+5)$. □

Remark 15.34. The same description of M_u bounds $\mu_{\max}(M_u)$ above by $C\,N_T a\Lambda_a$, through the duality of Proposition 15.30 and the bound $c_\nu \le (\nu_1 + s)^{s(s-1)/2}$. That bound enters (3.2) and is not needed for (3.7).

15.8 The sizes of the errors

Proposition 15.35. *Let $\varepsilon \in (0, \frac{1}{52})$ and suppose that the size conditions* (4.6) *and* (4.7) *of Proposition 4.3 hold, that $A = \sum_t a_t \le 2N_T a$, and that $\log x \le L_\varepsilon$ and $\log \mathrm{rd}_E \le L_\varepsilon$. Then*

$$e_4 + Ae_6 + e_7 \le C_8\, x^\varepsilon\, \mathcal{B},$$

with C_8 depending on C_0, C_6 and C_7 alone, where $\mathcal{B} = N_T a(1 + m^2/k) + m(B_0 + N_c)$ is the number (4.8).

Proof. Since $\Lambda_a \le 3x^\varepsilon$, $e_4 \le (50C_0 + 10)\cdot 3x^\varepsilon\big(N_T a + m(B_0 + N_c)\big) \le 3(50C_0+10)x^\varepsilon\mathcal{B}$. Next $Ae_6 \le 2N_T a\big(C_6 x^\varepsilon(1 + m^2/k) + 2\big) \le 2(C_6+2)x^\varepsilon N_T a(1 + m^2/k)$. Finally $e_7 \le 3C_7 x^\varepsilon N_T a(1 + m/k) \le 3C_7 x^\varepsilon N_T a(1 + m^2/k)$. □

The constant C_8 depends on ε through C_6, which contains the factor $1/(e\varepsilon)$ of the bound $\log x \le L_\varepsilon/(e\varepsilon)$. For $x \ge x_0(\varepsilon)$ this factor is absorbed in x^ε, so the three errors are at most $x^{2\varepsilon}\mathcal{B}$ with an absolute implied constant once x is large in terms of ε.

15.9 Inputs of this section

This section proves, from the hypotheses named in them, the mean period formula (Proposition 15.6), the height bound (Proposition 15.9 and Corollary 15.15), the deduction of (3.4) (Theorem 15.19), the deduction of (3.6) (Proposition 15.25 and Theorem 15.26), the deduction of (3.7) (Theorem 15.33), and the comparison with $\mathcal{B}$ (Proposition 15.35). Subsections 15.11 to 15.14 reduce Hypotheses 15.10, 15.11, 15.12 and 15.16 to the statements stated there. Hypotheses 15.10, 15.11, 15.12, 15.16, 15.18 and 15.23 are not proved here. The published theorems used are Kronecker's first limit formula, the product formula of Jacobi for Δ, Dirichlet's class number formula, the main theorem of complex multiplication for the conjugates of $j(\mathfrak{c})$, the discriminant formula for a tower of fields, Siegel's mass formula for unimodular hermitian lattices, Harish-Chandra's formula for formal degrees, Weyl's dimension formula, the Capelli identity, and the duality of Weyl modules and dual Weyl modules under the contravariant form.

15.10 The archimedean hypotheses: conventions

We use the conventions of Subsection 15.1. F is a totally complex field with n_F archimedean places, $\widehat{\deg}$ is the arithmetic degree, $\mu = \widehat{\deg}/(n_F \,\mathrm{rk})$ is twice the slope of Bost, and $\mathrm{ht}(\phi) = n_F^{-1}\sum_v \log\|\phi\|_v^2$. For positive reals κ_v we write $\langle \log \kappa\rangle = n_F^{-1}\sum_v \log\kappa_v$. We write $\mathrm{ph}(y) = y\log y - y$ for $y > 0$ and $\mathrm{ph}(0) = 0$. For a hermitian lattice Λ of rank D we put $\mathrm{lcv}(\Lambda) = -\widehat{\deg}(\Lambda)/n_F$, so that $\mu(\Lambda^\vee) = \mathrm{lcv}(\Lambda)/D$.

For a finite set κ we write $\mathbb{C}^\kappa$ with the standard norm $|u|^2 = \sum_x |u_x|^2$, Lebesgue measure du, and for $e \in \mathbb{N}^\kappa$ we put $u^e = \prod_x u_x^{e_x}$, $e! = \prod_x e_x!$ and $|e| = \sum_x e_x$.

15.11 The archimedean norm of each lift

15.11.1 Three elementary estimates

Lemma 15.36. *Let $y \geq 0$, $0 < \epsilon < 1$ and $s \geq 0$. Then $s^y \exp(-(1-\epsilon)s^2/2) \leq (1-\epsilon)^{-y/2} \exp(\mathrm{ph}(y)/2)$. For $n \geq 0$ one has $(1-\frac{1}{n+1})^{-n/2} \leq e^{1/2}$.*

Proof. For $y = 0$ both sides are at most 1. For $y > 0$ the function $g(s) = y \log s - (1-\epsilon)s^2/2$ has its maximum at $s^2 = y/(1-\epsilon)$, where $g = \frac{y}{2}\log y - \frac{y}{2} - \frac{y}{2}\log(1-\epsilon) = \frac{1}{2}\,\mathrm{ph}(y) - \frac{y}{2}\log(1-\epsilon)$. For the second statement, $(1-\frac{1}{n+1})^{-n/2} = (1+\frac{1}{n})^{n/2} \leq e^{1/2}$ for $n \geq 1$, and the case $n = 0$ is trivial. □

Lemma 15.37. *Let ι be a finite set of cardinality q, let $\nu_1, \ldots, \nu_r \geq 0$ with $|\nu| = \sum_i \nu_i$, and let $s_1, \ldots, s_r$ be nonnegative measurable functions on $\mathbb{C}^\iota$ with $\sum_i s_i(u)^2 = |u|^2$. Then*

$$\pi^{-q} \int_{\mathbb{C}^\iota} \prod_i s_i(u)^{\nu_i}\, e^{-|u|^2/2}\, du \leq e^{1/2}\,(2|\nu|+2)^q\, \exp\Big(\frac{1}{2}\sum_i \mathrm{ph}(\nu_i)\Big).$$

Proof. Put $\epsilon = 1/(|\nu|+1)$. Write $e^{-|u|^2/2} = \prod_i e^{-(1-\epsilon)s_i^2/2} \cdot e^{-\epsilon|u|^2/2}$. Lemma 15.36 applied to each factor $s_i^{\nu_i} e^{-(1-\epsilon)s_i^2/2}$ bounds the integrand by $(1-\epsilon)^{-|\nu|/2} e^{\frac{1}{2}\sum \mathrm{ph}(\nu_i)} e^{-\epsilon|u|^2/2}$. The Gaussian integral is $\pi^{-q}\int e^{-\epsilon|u|^2/2}du = (2/\epsilon)^q = (2|\nu|+2)^q$, and $(1-\epsilon)^{-|\nu|/2} \leq e^{1/2}$ by the second part of Lemma 15.36. □

Lemma 15.38 (coefficients in the Fock space)**.** *Let G be an entire function on $\mathbb{C}^\kappa$ with Taylor expansion $G(u) = \sum_e c_e u^e$ and $|G(u)| \leq S e^{|u|^2/2}$. Then for every $e \in \mathbb{N}^\kappa$ the function $G(u)\bar{u}^e e^{-|u|^2}$ is integrable and*

$$\int_{\mathbb{C}^\kappa} G(u)\, \bar{u}^e\, e^{-|u|^2}\, du = \pi^{|\kappa|}\, e!\, c_e.$$

Proof. Integrability follows from $|G(u)\bar{u}^e|e^{-|u|^2} \leq S|u|^{|e|}e^{-|u|^2/2}$. On the polydisc $P_R = \{|u_x| \leq R\}$ the Taylor series converges uniformly, so $\int_{P_R} G\bar{u}^e e^{-|u|^2} = \sum_{e'} c_{e'} \int_{P_R} u^{e'}\bar{u}^e e^{-|u|^2}$. In polar coordinates in each variable the integral vanishes unless $e' = e$, and for $e' = e$ it equals $\prod_x 2\pi \int_0^R r^{2e_x+1}e^{-r^2}dr$. Letting $R \to \infty$ and using dominated convergence on the left gives $\pi^{|\kappa|}e!c_e$. □

15.11.2 The block bound

Let Φ be a finite set of places (the embeddings τ), let ι be a finite set with $q = |\iota|$, and put $\kappa = \Phi \times \iota$. Fix $n_\tau \in \mathbb{N}$ and weights $\nu_\tau = (\nu_{\tau 1}, \ldots, \nu_{\tau r})$ with $\sum_i \nu_{\tau i} = n_\tau$.

Proposition 15.39 (block bound)**.** *Let W be a Banach space and $R(u) = \sum_e u^e R_e$ a W-valued polynomial on $\mathbb{C}^\kappa$ whose monomials satisfy $\sum_{x\in\iota} e(\tau, x) = n_\tau$ for every τ. Suppose there are nonnegative functions $s_{\tau i}$ on $\mathbb{C}^\iota$ with $\sum_i s_{\tau i}(w)^2 = |w|^2$ and*

$$\|R(\bar{u})\| \leq \prod_\tau \prod_i s_{\tau i}(u_\tau)^{\nu_{\tau i}} \qquad (u = (u_\tau)_\tau \in \mathbb{C}^\kappa).$$

Let $G = \sum c_e u^e$ be entire on $\mathbb{C}^\kappa$ with $|G(u)| \leq S e^{|u|^2/2}$, and let $\lambda_\tau \in \mathbb{C}$ with $|\lambda_\tau|^2 = \Omega_\tau^{-1}$, $\Omega_\tau > 0$. Then the vector $J = \sum_e c_e\, e! \prod_{(\tau,x)} \lambda_\tau^{e(\tau,x)} R_e$ satisfies

$$\|J\|^2 \leq S^2 \prod_\tau \Omega_\tau^{-n_\tau}\, e\, (2n_\tau+2)^{2q} \exp\Big(\sum_i \mathrm{ph}(\nu_{\tau i})\Big).$$

Proof. By homogeneity, $\prod_{(\tau,x)} \lambda_\tau^{e(\tau,x)} = \prod_\tau \lambda_\tau^{n_\tau}$ for every e in the support of R. Lemma 15.38 gives $c_e e! = \pi^{-|\kappa|} \int G(u) \bar{u}^e e^{-|u|^2} du$, hence $J = \prod_\tau \lambda_\tau^{n_\tau} \cdot \pi^{-|\kappa|} \int G(u) R(\bar{u}) e^{-|u|^2} du$, the sum being finite. Therefore

$$\|J\| \le \prod_\tau \Omega_\tau^{-n_\tau/2} \, S \prod_\tau \Big(\pi^{-q} \int_{\mathbb{C}^\iota} \prod_i s_{\tau i}(u_\tau)^{\nu_{\tau i}} e^{-|u_\tau|^2/2} du_\tau \Big),$$

and Lemma 15.37 bounds each factor. Squaring gives the statement. □

In the application R is the K-type projection of the matrix coefficient, R_e are the operators that send a highest weight vector of row degree e to its image, and $s_{\tau i}(w)$ are the singular values of the matrix $w \in M_{a\times b}(\mathbb{C})$. The hypothesis $\|R(\bar{u})\| \le \prod s^\nu$ is the statement that the polynomial representation of $U(a) \times U(b)$ on monomials of degree ν_τ in the singular values is contracting, which holds for the standard Fock-space realization because the operator norm of a matrix power representation is the product of the singular values to the power of the weight (Cartan decomposition $w = k_1 \operatorname{diag}(s) \, k_2$ with unitary k_1, k_2, and the highest weight vector of weight ν is multiplied by $\prod s_i^{\nu_i}$). Proposition 15.39 applies with s the singular values and $n_\tau = |\nu_\tau|$.

15.11.3 Class sums

Lemma 15.40. *Let $\mathcal{C}$ and $\mathcal{C}'$ be finite sets partitioned into blocks $\mathcal{C} = \bigsqcup_{a\in A} \mathcal{C}_a$ and $\mathcal{C}' = \bigsqcup_{b\in A'} \mathcal{C}'_b$ with $|\mathcal{C}_a| \le h$ and $|\mathcal{C}'_b| \le h'$. Let $0 < \gamma_L \le 1$ and $0 < \gamma'_{L'} \le 1$, let $\epsilon_{LL'}$ be complex numbers of modulus one, and let $T_{LL'} \colon W \to W'$ be bounded operators with $\|T_{LL'}\|^2 \le t$ which vanish unless L and L' lie in blocks a, b with (a,b) in a set $E \subset A \times A'$. Define $\Theta(G)_{L'} = \sum_L \gamma_L \epsilon_{LL'} T_{LL'} G_L$. Then*

$$\sum_{L'} \gamma'_{L'} \|\Theta(G)_{L'}\|^2 \le |A| \, |A'| \, h \, h' \, t \sum_L \gamma_L \|G_L\|^2.$$

Proof. By Cauchy–Schwarz, $\|\Theta(G)_{L'}\|^2 \le \big(\sum_L \gamma_L\big)\big(\sum_L \gamma_L \|T_{LL'} G_L\|^2\big) \le |A| h \cdot t \sum_L \gamma_L \|G_L\|^2$, using $\gamma_L \le 1$. Summing over L' with $\gamma'_{L'} \le 1$ multiplies by $|\mathcal{C}'| \le |A'| h'$. □

For the coupled last two steps the same argument applied twice, with an intermediate block set Y of cardinality bounded by the coset count, gives the factor $|Y|^2$ times the two single-step factors. We do not repeat it.

15.11.4 The theta kernel

The local step kernel of the lift j at an archimedean place is a theta function on the abelian variety E^n, $n = f N_{j-1} N_j$, where E is the CM elliptic curve of the host and $f = |\Phi|$. We need its sup norm.

Lemma 15.41 (sup norm of a theta function)**.** *Let $\Lambda \subset \mathbb{C}^n$ be a lattice of covolume V for the standard volume, and let F be holomorphic on $\mathbb{C}^n$ with $|F(z+\ell)|^2 e^{-|z+\ell|^2} = |F(z)|^2 e^{-|z|^2}$ for $\ell \in \Lambda$. Put $\|F\|^2 = V^{-1} \int_{\mathbb{C}^n/\Lambda} |F|^2 e^{-|z|^2} dz$. Let $\rho > 0$ and let $N_\Lambda(\rho)$ be the largest number of translates $\ell + P$ of a fixed fundamental parallelotope P that meet a ball of radius ρ. Then*

$$\sup_z |F(z)|^2 e^{-|z|^2} \le \frac{n! \, e^{\rho^2}}{\rho^{2n} \pi^n} \, N_\Lambda(\rho) \, V \, \|F\|^2.$$

Proof. Fix z and put $g(w) = F(z+w)e^{-\langle w,z\rangle - |z|^2/2}$, which is holomorphic in w, with $|g(w)|^2 = |F(z+w)|^2 e^{-|z+w|^2} e^{|w|^2}$. The sub-mean value property of $|g|^2$ on the ball B_ρ of volume $\pi^n \rho^{2n}/n!$ gives

$$|F(z)|^2 e^{-|z|^2} = |g(0)|^2 \le \frac{n!}{\pi^n \rho^{2n}} \int_{B_\rho} |g|^2 \le \frac{n!\, e^{\rho^2}}{\pi^n \rho^{2n}} \int_{B_\rho(z)} |F|^2 e^{-|\zeta|^2} d\zeta.$$

The ball $B_\rho(z)$ is covered by at most $N_\Lambda(\rho)$ translates of P, and on each translate the integral of the periodic function $|F|^2 e^{-|\zeta|^2}$ equals $V\|F\|^2$. □

For $\Lambda = \mathfrak{a}^n$ with $\mathfrak{a} \subset \mathbb{C}$ a lattice of area A whose reduced basis has lengths $\ell_1 \le \ell_2$, a fundamental parallelotope is the product of n reduced parallelograms of diameter at most $\ell_1 + \ell_2$, and with $\rho = 1$ one has $N_\Lambda(1) \le \big((1 + 2(1+\ell_1+\ell_2)/\ell_1)(1 + 2(1+\ell_1+\ell_2)\ell_1/A)\big)^n$. For the CM lattice of the host, after the normalization of Section 8 of the manuscript, $\ell_1 \ge c$ and $\ell_2 \le C\, v$ with $v = \frac{1}{2}\mathrm{rd}_E$, so that the logarithm of the constant of Lemma 15.41 is at most $n \log(C' n v)$. Below we use only the bound $c_5\, n\, \Lambda_a$ for this logarithm.

Hypothesis 15.42 (normalized kernel)**.** For each lift j and each embedding s of the coefficient field M of the kernels, the local kernel of K_j at s is of the form $\Theta_j = \sum_{L,L'} \gamma_L \epsilon_{LL'} J_{LL'}$ as in Lemma 15.40, where each $J_{LL'}$ is the vector J of Proposition 15.39 built from a theta function $G_{LL'}$ which is a combination $\sum_k r_k G_0(z + y_k)$ of translates of one normalized theta function G_0 on $E^{fN_{j-1}N_j}$, with $\sum_k |r_k^s| \le \exp(\mathrm{MN}_{j,s})$, and with $\|G_0\|^2 = \|\theta_0^s\|^{2fN_{j-1}N_j}$, where θ_0 is the genus-one theta function of the base vector and $\|\cdot\|$ is the norm of Lemma 15.41. Moreover $\mathrm{MN}_{j,s} \le C_2 f m^2 N^2 (\log(g_0 g_1) + \log T_u)$ and $|c_{D,\tau} r_\tau|^2 = \Omega_{\tau,s}^{-1}$ with $\Omega_{\tau,s}$ the conjugate periods of Proposition 15.6.

The tensor power identity in Hypothesis 15.42 is the statement that a theta function for the product polarization on E^n whose space is one-dimensional is the product of the genus-one theta functions in each coordinate, and its norm is the n-th power. The coefficient bound is a count of the translates y_k needed to refine the coset indicators of the level structure, of size at most $(g_0 g_1 T_u)^{C_2 f m^2 N^2}$. Both depend on the normalization of θ_0 and of the constant c of Theorem 8.21, which is Hypothesis 11.42. Hypothesis 15.42 is not proved here.

Hypothesis 15.43 (theta-null mean)**.** There is an absolute constant C_θ such that for every admissible host and every number field M containing the data,

$$\frac{1}{[M:\mathbb{Q}]} \sum_{s:\, M \to \mathbf{C}} \log \|\theta_0^s\|^2 \le C_\theta \big(\log v + |\lambda_1| \big), \qquad \lambda_1 = L'/L(1, \chi_{-d}).$$

A sharper bound,

$$\frac{1}{[M:\mathbb{Q}]} \sum_{\rho: M \to \mathbf{C}} \log \|\theta_0^\rho\|^2 \le \frac{1}{2} \big(\log \sqrt{d} - \gamma + \lambda_1 \big),$$

follows from a mean-logarithm argument, with no pointwise estimate on Weber values. Put $n = [M:\mathbb{Q}]$. For each ρ choose compatible period coordinates, and write $y_\rho = \operatorname{Im} \tau_\rho$, Θ_ρ for the norm at zero of the even theta function whose zero is the marked half-period, $E_\rho = |\eta(\tau_\rho)|$, $R_\rho = \|\theta_0^\rho\|^2$, and A_ρ for the area of the corresponding lattice. The theta-null and period-product identities give

$$\log R_\rho = -\tfrac{1}{2}\log 2 - \tfrac{1}{2}\log y_\rho - 2\log \Theta_\rho, \qquad \log A_\rho = \log 4 + 2\log \pi + \log y_\rho + 4 \log E_\rho.$$

The Weber construction gives an element $X \in M$, the same element for all embeddings, such that $X \neq 0$, X is integral over $\mathbb{Z}$, and

$$|\rho X| = (\Theta_\rho / E_\rho)^{12}$$

for every ρ and every compatible choice of the vanishing even characteristic. Indeed, Jacobi's product formula and the half-period identities identify these twelfth powers, up to sign, with the roots of $(T-16)^3 = jT$; integrality of j gives integrality. The algebraic expression in the torsion coordinates makes the formula compatible with ρ (Weber, *Lehrbuch der Algebra* III, §§125–127; Yui and Zagier, *Math. Comp.* 66 (1997), §1). Since X is a nonzero algebraic integer, the norm of X is a nonzero rational integer, and consequently

$$0 \leq \sum_\rho \log|\rho X| = 12(S_\Theta - S_E), \qquad S_\Theta = \sum_\rho \log \Theta_\rho, \quad S_E = \sum_\rho \log E_\rho.$$

Set $S_y = \sum_\rho \log y_\rho$ and $S_R = \sum_\rho \log R_\rho$. Averaging the two displayed identities and using the Chowla–Selberg formula as in Proposition 15.6 give

$$S_R = -\tfrac{n}{2}\log 2 - \tfrac{1}{2}S_y - 2S_\Theta, \qquad S_y + 4S_E = n(\gamma - \log 2 - \log\sqrt{d} - \lambda_1).$$

Since $S_\Theta \geq S_E$, substitution yields

$$\frac{S_R}{n} \leq -\tfrac{1}{2}\log 2 - \tfrac{1}{2n}S_y - \tfrac{2}{n}S_E = \tfrac{1}{2}(\log\sqrt{d} - \gamma + \lambda_1).$$

The transformation/integrality inputs and the compatible period normalizations used above are not reproved in this subsection.

Hypothesis 15.44 (class and index counts)**.** With $U = fm^2N^2\log(g_0g_1T_u)$ there is an absolute C_c with: the number of classes in each block of the class decomposition at level i is at most e^{C_cU}, the number of blocks is at most e^{C_cU}, and the coset count of the coupled step is at most e^{C_cU}.

Classes are orbits of an arithmetic group on decorated lattices of a fixed genus with level structure at $g_0g_1T_u$. Their number is at most the class number of the genus times the index of the principal congruence subgroup, and both are bounded by $N(\mathfrak{n})^{C\,\mathrm{rank}^2}$ with $N(\mathfrak{n}) \leq (g_0g_1T_u)^{Cf}$. The count is not written out here, so Hypothesis 15.44 is not proved here.

15.11.5 The kernel norm theorem

Put $n_j = fN_{j-1}N_j$, $\nu_{j\tau}$ the row weights of the kernel of the lift j at τ, and for an embedding s

$$\begin{aligned}\mathcal{S}_j(s) = \sum_\tau \Big(\sum_i \mathrm{ph}(\nu_{j\tau i}) - |\nu_{j\tau}|\log\Omega_{\tau,s}\Big) + n_j\log\|\theta_0^s\|^2 + n_j\log c_\Lambda \\ + 2\,\mathrm{MN}_{j,s} + \sum_\tau \big(1 + 2N_{j-1}N_j\log(2|\nu_{j\tau}|+2)\big),\end{aligned}$$

where c_Λ is the constant of Lemma 15.41 per coordinate.

Proposition 15.45 (one embedding)**.** *Under Hypotheses 15.42 and 15.44, for every lift j and every embedding s,*

$$\log\|K_j\|_s^2 \leq \mathcal{S}_j(s) + 4C_cU.$$

Proof. By Lemma 15.41 and the triangle inequality, $|G_{LL'}(z)| \le S\, e^{|z|^2/2}$ with $S^2 = c_\Lambda^{n_j}\|\theta_0^s\|^{2n_j} e^{2\mathrm{MN}_{j,s}}$, after the change of variable $z \mapsto r\,z$ that makes the kernel exponent $|z|^2/2$. Proposition 15.39 with $\Omega_\tau = \Omega_{\tau,s}$ and $q = N_{j-1}N_j$ gives $\|J_{LL'}\|^2 \le \exp(\mathcal{S}_j(s))$, the term $\sum_\tau 1$ coming from the factors e. Lemma 15.40 multiplies by $|A||A'|hh' \le e^{4C_cU}$ by Hypothesis 15.44. The frames of the chain are the class decompositions (the isometry $\sum_L \gamma_L\|\Phi(x)_L\|^2 = \|x\|_X^2$ of the chain record), so the operator norm of K_j in the frames equals the norm of Θ_j on the weighted sums. □

Theorem 15.46 (Hypothesis 15.10)**.** *Assume Hypotheses 15.42, 15.43, 15.44, Proposition 15.6, and the size conditions* $\log T_u \le C_T m L_\varepsilon$, $\log(g_0 g_1) \le C_8 L_\varepsilon$, $\log(8 + 32 n_j v) \le c_5 \Lambda_a$, $\log(2|\nu_{j\tau}| + 2) \le c_6 \Lambda_a$, $N_{j-1}N_j \le N^2$, $L_\varepsilon \le \Lambda_a$, $1 \le \Lambda_a$, $m, N \ge 1$. *Then Hypothesis 15.10 holds with*

$$C_0 = C'_\theta + c_5 + 2c_6 + 1 + 2C_2(C_T + C_8) + 2C_c(1 + C_T + C_8) + 2C_8,$$

where C'_θ *is the constant with* $C_\theta(\log v + |\lambda_1|) \le C'_\theta L_\varepsilon$ *under* $\log \mathrm{rd}_E \le L_\varepsilon$ *and* $|\lambda_1| \le L_\varepsilon$.

Proof. Average Proposition 15.45 over the embeddings s, which is the mean over archimedean places of F since each place of F has the same number of embeddings of M above it. Proposition 15.6 gives $\mathrm{mean}_s \log \Omega_{\tau,s} = \log \Omega$ for every τ, so the first term becomes $\sum_\rho(\sum_i \mathrm{ph}(\nu_{j\rho i}) - |\nu_{j\rho}| \log \Omega)$, which is the main term of s_j. The remaining terms are bounded as follows. Hypothesis 15.43 gives $n_j \,\mathrm{mean} \log\|\theta_0\|^2 \le C'_\theta f N^2 L_\varepsilon$. Next $n_j \log c_\Lambda \le c_5 f N^2 \Lambda_a$, $2\,\mathrm{MN} \le 2C_2 f m^2 N^2 (C_8 + C_T) L_\varepsilon$ because $\log T_u \le C_T m L_\varepsilon$ and $\log(g_0 g_1) \le C_8 L_\varepsilon$, $\sum_\tau(1 + 2N_{j-1}N_j \log(2|\nu| + 2)) \le f(1 + 2c_6 N^2 \Lambda_a)$, and $4C_c U \le 4C_c f m^2 N^2 (C_8 + C_T) m L_\varepsilon$. Each error term is at most the corresponding multiple of $f N^2 \Lambda_a$ or of $f m^3 N^2 L_\varepsilon$, and adding the coefficients gives C_0. □

Remark 15.47. Theorem 15.46 has the shape $C_0(f N^2 \Lambda_a + f m^3 N^2 L_\varepsilon)$ of Hypothesis 15.10. It does not use the archimedean decay of the kernel beyond the Gaussian bound. The factor m in front of $\mathrm{ph}(a_t)$ in (3.4) arises because each of the m lifts contributes $\sum_\rho \sum_i \mathrm{ph}(\nu_{j\rho i})$, and Hypothesis 15.18 then extracts $\mathrm{ph}(a_\rho)$ at the tops for each j.

15.12 The final pairing

15.12.1 The branching map

Let $N = n + 1$, let λ be a dominant weight of GL_N and σ a dominant weight of GL_n which interlace, $\lambda_{i+1} \le \sigma_i \le \lambda_i$, with the same determinant shift. Realize V_λ in polynomials on $M_N(\mathbb{C})$ with the Fock form $\langle f, g\rangle = \sum_e f_e \bar{g}_e\, e!$, which is $U(N)$-invariant. Put $n_{\mathrm{Br}} = |\lambda| - |\sigma|$. By the Gelfand–Tsetlin branching rule, V_λ restricted to $GL_n \times GL_1$ contains $V_\sigma \otimes \chi^{n_{\mathrm{Br}}}$ exactly once, where χ is the identity character of GL_1. Let $\mathrm{br}\colon V_\lambda \to V_\sigma$ be the GL_n-equivariant map normalized by $\mathrm{br}(\xi_{\mathrm{Br}}) = c_{\mathrm{Br}} \kappa_{\mathrm{Br}}\, \mathrm{hw}_\sigma$, where ξ_{Br} is the product of leading minors of degree λ that defines the branching map (its explicit form is not written out here), c_{Br} is the integer of Lemma 15.49, and $\kappa_{\mathrm{Br}} = \|\xi_{\mathrm{Br}}\|^2 / \|\mathrm{hw}_\lambda\|^2$.

Let G' be a positive definite hermitian $n \times n$ matrix, $g > 0$, and give V_λ the invariant form for the Gram matrix $\mathrm{diag}(G', g)$ and V_σ the one for G'. Let $t \in \mathbb{C}$ with $\bar{t} g t = T > 0$ and put $\mathrm{br}_t = t^{-n_{\mathrm{Br}}} \mathrm{br}$.

Proposition 15.48 (scaled coisometry)**.** *For all* $y \in V_\sigma$ *and* $x \in V_\lambda$ *one has* $\|\mathrm{br}_t\, x\|^2 \le T^{-n_{\mathrm{Br}}} c_{\mathrm{Br}}^2 \kappa_{\mathrm{Br}} \|x\|^2$, *with equality on the* σ*-isotypic component. In particular the operator norm of* br_t *is* $T^{-n_{\mathrm{Br}}/2} |c_{\mathrm{Br}}| \kappa_{\mathrm{Br}}^{1/2}$.

Proof. First let $G' = 1$ and $g = 1$. The σ-isotypic component $V^\sigma \subset V_\lambda$ is irreducible, and the other components are orthogonal to it by invariance of the form under the compact group $U(n) \times U(1)$. So br vanishes on $(V^\sigma)^\perp$, and on V^σ it is a $U(n)$-equivariant map between irreducible unitary representations, hence a scalar multiple β of an isometry by Schur's lemma. The inclusion $\mathrm{tr}\colon V_\sigma \to V_\lambda$ normalized by $\mathrm{tr}(\mathrm{hw}_\sigma) = c_{\mathrm{Br}}\xi_{\mathrm{Br}}$ is adjoint to br up to the same scalar. Evaluating at ξ_{Br}, which lies in V^σ and is a highest weight vector for $U(n)$ there, gives $|\beta|^2\|\xi_{\mathrm{Br}}\|^2 = c_{\mathrm{Br}}^2\kappa_{\mathrm{Br}}^2\|\mathrm{hw}_\sigma\|^2$. With the extremal normalization $\|\mathrm{hw}_\sigma\| = \|\mathrm{hw}_\lambda\|$ of the Fock model this is $|\beta|^2 = c_{\mathrm{Br}}^2\kappa_{\mathrm{Br}}$.

For general G' and g, write $\mathrm{diag}(G', g) = h^*h$ with $h = \mathrm{diag}(h', g^{1/2})$. The invariant form for $\mathrm{diag}(G', g)$ is the pull back of the standard form by the action of h, and br is GL_n-equivariant, so the computation reduces to $G' = 1$ except for the GL_1-factor. On V^σ the element $\mathrm{diag}(1, u)$ acts by $u^{n_{\mathrm{Br}}}$. Hence the form for (G', g) on V^σ is $g^{n_{\mathrm{Br}}}$ times the form for $(G', 1)$, and replacing br by $t^{-n_{\mathrm{Br}}}\mathrm{br}$ multiplies squared norms by $|t|^{-2n_{\mathrm{Br}}}$. Since $|t|^2 g = T$, the squared norm ratio on V^σ is $T^{-n_{\mathrm{Br}}}c_{\mathrm{Br}}^2\kappa_{\mathrm{Br}}$. □

Lemma 15.49. *Let* $\mathrm{sp}(\lambda) = \lambda_1 - \lambda_N + N$ *and* $d_1 = \sum_{i<N} \min(\lambda_i - \lambda_{i+1}, d_0)$ *with* $d_0 = \sum_i(\sigma_i - \lambda_{i+1})$. *If* $\mathrm{sp}(\lambda) \le A$ *then* $0 < |c_{\mathrm{Br}}| \le A^{n\,d_1}$.

Proof. By definition $c_{\mathrm{Br}} = \prod_{i<N}\prod_{v\in r_i\setminus\{\sigma_i\}}\prod_{j<N}\big((v-i)-(\sigma_j-j)\big)$ with $r_i = [\lambda_{i+1}, \min(\lambda_i, \lambda_{i+1} + d_0)]$. The number of v is $|r_i| - 1 \le \min(\lambda_i - \lambda_{i+1}, d_0)$, so there are at most $n\,d_1$ factors. Each factor is an integer of absolute value at most $\lambda_1 - \lambda_N + n - 1 \le A$. No factor vanishes, because a factor vanishes only if $v - i = \sigma_j - j$, which interlacing excludes for $v \ne \sigma_i$. □

Hypothesis 15.50 (Fock norm of ξ_{Br})**.** $\|\xi_{\mathrm{Br}}\|^2 \le \prod_{i<j\le N}(\lambda_i - \lambda_j + j - i)\,\|\mathrm{hw}_\lambda\|^2$. Consequently $\kappa_{\mathrm{Br}} \le \mathrm{sp}(\lambda)^{N(N-1)/2}$.

The second sentence follows from the first because each factor of the Weyl product is at most $\mathrm{sp}(\lambda)$. The first is a comparison of Fock norms of two products of minors with the same total degree. The Fock norm of a product of leading minors is a ratio of factorials of shifted weights by the Capelli identity. The required comparison includes the normalization of both highest weight vectors. Hypothesis 15.50 is not proved here.

15.12.2 Hypothesis 15.11

Theorem 15.51 (Hypothesis 15.11)**.** *Assume that the final pairing* P_u *is, at each archimedean place, the composite of* br_t *(with* $\bar{t}gt = T_u$*) on each embedding* τ *with an evaluation of operator norm squared at most* e^{ev}, $\mathrm{ev} \le C_{\mathrm{ev}} f N^2 \Lambda_a$, *and assume Hypothesis 15.50 and the weight relations*

$$n_\tau = \lambda_{1,\tau} - D_{0,\tau}, \quad \lambda_{1,t} = a_t - \tfrac{N-k}{2}\ (t \in T), \quad \lambda_{1,\tau} \ge S - \tfrac{N-1}{2}\ (\tau \notin T),$$

$$D_{0,\tau} \le S + m\eta + \mathbf{1}_T(\tau)b_{T,\tau}, \quad D_{1,\tau} \le 3D_{0,\tau} + S + \eta, \quad b_{T,t} \le 2b', \quad \mathrm{sp} \le 2a, \quad \log T_u \le C_T m L_\varepsilon,$$

together with $mL_\varepsilon \le N\Lambda_a$ *and* $\log 2a \le 2\Lambda_a$. *Then* $h_P \le -\sum_{t\in T} a_t \log T_u + e_P$ *with* $e_P \le C_0(N_T b' N + f(m\eta + S)N + fN^2)\Lambda_a$ *for* $C_0 = 2(16 + C_T + C_{\mathrm{ev}})$.

Proof. Proposition 15.48, Lemma 15.49 and Hypothesis 15.50 give

$$h_P \le -\sum_\tau n_\tau \log T_u + \sum_\tau \big(2(N-1)D_{1,\tau} + \tfrac{N(N-1)}{2}\big)\log 2a + \mathrm{ev}.$$

At a top, $-n_t \log T_u = -a_t \log T_u + (\frac{N-k}{2} + D_{0,t}) \log T_u$. Away from the tops $-n_\tau \log T_u \leq (\frac{N-1}{2} + D_{0,\tau}) \log T_u$ since $S \geq 0$. Summing, the coefficient of $\log T_u$ besides $-\sum_t a_t$ is at most $f\frac{N}{2} + f(S + m\eta) + 2N_T b'$, and $\log T_u \leq C_T m L_\varepsilon \leq C_T N \Lambda_a$. The c- and κ-terms are at most $(2N(4S + 4m\eta + 6\mathbf{1}_T b')f + fN^2) \cdot 2\Lambda_a$ after inserting $D_1 \leq 3D_0 + S + \eta$. Collecting coefficients gives the bound with the stated C_0, with room to spare. □

The assumption that P_u factors through br_t is part of the construction of the final pairing, which is not written in this paper.

15.13 The frame constants

The scalars $\lambda_j(p, v)$ are defined by the frame condition of the record. The map $\theta_j \colon R_F \to W_j$ multiplies the hermitian form on the base-type subspace p by $\lambda_j(p, v)$ at v. Equivalently $\lambda_j(p, v)$ is the eigenvalue of $\theta_j^* \theta_j$ on p, where the adjoint is taken for the Petersson forms. Hypothesis 15.12 asks for $\sum_j \log \lambda_j(p, v) \geq \sum_{j,\rho} h_{j\rho} - C_0 m f N^2 \Lambda_a$, where $h_{j\rho}$ is the logarithm of the hook constant of the frame on the ρ-component.

Proposition 15.52 (structure of Hypothesis 15.12)**.** *Suppose that for each j, each base-type p with $W_{j,p} \neq 0$ and each v,*

(i) $\theta_j \theta_j^*$ acts on $W_{j,p}$ by a scalar $a_j z_j$ with $z_j = L^S(\frac{1+t_0}{2}, c\,\chi \mu_{t,j+1}^{-1})$ a partial L-value and a_j a product of local zeta integrals (the Rallis inner product formula);

(ii) $\log(a_j z_j) \geq \sum_\rho \log \mathrm{hook}_{j\rho} - C_{\mathrm{sz}} f N^2 L_p$, where $\log \mathrm{hook}_{j\rho}$ is the archimedean formal degree term written as a hook product, and L_p is the size of the local and L-value corrections.

Then Hypothesis 15.12 holds with $e_\lambda \leq C_{\mathrm{sz}} m f N^2 \Lambda_a$ once $L_p \leq \Lambda_a$.

Proof. If $W_{j,p} = 0$ the scalar is 1 by convention and the inequality is trivial. Otherwise pick $x \neq 0$ in $W_{j,p}$. Then $\lambda_j(p, v)\|x\|^2 = \langle \theta_j^* \theta_j x', x' \rangle$ for the preimage x', and (i) identifies this eigenvalue with $a_j z_j$. Summing (ii) over j gives the statement with $h_{j\rho} = \log \mathrm{hook}_{j\rho}$. □

Item (i) is the Rallis inner product formula in the convergent range, which follows from the Siegel–Weil formula and the doubling method. Item (ii) combines Harish-Chandra's formula for the formal degree of a holomorphic discrete series, written as a hook product, with lower bounds for the partial L-value at $s = \frac{1+t_0}{2}$ in the range of absolute convergence ($\log z_j \geq -\sum_{\mathfrak{p}} \log(1 - N\mathfrak{p}^{-\sigma})^{-1}$). Items (i) and (ii) for the theta lifts of the chain, and the identification of θ_j with the period map of Theorem 14.3, are not proved here.

15.14 The slope of the target lattice

Recall from R4 that Hypothesis 15.16 has three parts. The passage compares two levels of the chain, the base-side bound compares $\mu(R^\vee)$ with the covolumes of the base lattices, and the covolume bound evaluates those covolumes.

15.14.1 The base-side bound

Let $R \subset F^D$ and $M_1, \dots, M_m \subset F^D$ be lattices with hermitian forms X_E on R and X_{W_j} on M_j. Suppose there are invertible matrices ι_t and $\theta_{j,t}$ indexed by local types t, lattices A_t and $T_{j,t}$, and forms $X_{Q,t}$, $X_{T,j,t}$, such that:

(a) at every finite place y, $R_y = \iota_{\mu(y)} A_{\mu(y),y}$;

(b) at every archimedean v and for the base type t_0,

$$\det(\iota_{t_0}^* X_E \iota_{t_0}) / \det X_{Q,t_0} = \kappa_v^D \prod_j \det(\theta_{j,t_0}^* X_{W_j} \theta_{j,t_0}) / \det X_{T,j,t_0}$$

(c) at a twisted type t the same identity holds at the distinguished place v_0 with $\kappa_{v_0}\rho_t$ in place of κ_{v_0}, ρ_t the archimedean value of an element of F which is a unit at all finite places.

Proposition 15.53 (base side)**.** *Under (a)–(c),*

$$\operatorname{lcv}(R) \le D\langle \log \kappa \rangle + \sum_j \operatorname{lcv}(M_j) + \sum_y \ell_y \Big(v_y(G_{\mu(y),y}) + \sum_j \big(i_y(M_j) + i_{\bar{y}}(M_j) + j_y(T_j) \big) \Big),$$

where $\ell_y = \log Ny / [F:\mathbb{Q}]$, $G_{t,y}$ is the Gram determinant of a local basis of $A_{t,y}$ for $X_{Q,t}$, $i_y(M_j)$ is the length of $M_{j,y}$ relative to $\theta_{j,\mu(y)} T_{j,\mu(y),y}$, and j_y is the local discriminant exponent of T_j for the hermitian form. Consequently $\mu(R^\vee) - \langle \log \kappa \rangle \le \sum_j \operatorname{lcv}(M_j)/D + \frac{1}{D}\sum_y \ell_y(\cdots)$.

Proof. Choose a basis of F^D and compute $\widehat{\deg}$ of each lattice as log of the index of the lattice relative to the standard lattice minus the archimedean log-determinants. The archimedean part of $\operatorname{lcv}(R) - \sum_j \operatorname{lcv}(M_j)$ is $\frac{1}{n_F}\sum_v \big(\log\det(\iota^* X_E \iota) - \sum_j \log\det(\theta_j^* X_{W_j} \theta_j) \big)$ plus the archimedean contributions of X_Q and $X_{T,j}$, and (b), (c) turn it into $D\langle \log\kappa\rangle$ plus $\frac{1}{n_F}\sum_v (\log\det X_Q - \sum_j \log\det X_{T,j}) + \log|\rho|$-terms. The product formula applied to $\det \iota_t$, $\det \theta_{j,t}$ and ρ_t (a unit at every finite place) converts the archimedean $\log|\det|$-terms into finite valuations of the same determinants, and (a) together with the definitions of $G_{t,y}$, i_y and j_y identifies the resulting finite part with the sum over y. At untwisted places all local terms vanish by the hypotheses of the record. The second statement uses $\mu(R^\vee) = \operatorname{lcv}(R)/D$. □

The determinant identity (b) follows from the frame condition (the base-type subspaces are orthogonal, ι multiplies the form on p by $\kappa_v \prod_j \lambda_j(p,v)$, θ_j multiplies it by $\lambda_j(p,v)$, and $\dim W_{j,p} = \dim E_p$), since the determinant of a map that is a scalar c_p on orthogonal pieces of dimension d_p is $\prod_p c_p^{d_p}$. The local terms on the right are bounded by Theorem 17.37 and enter $\mathcal{L} + E_b$.

15.14.2 The passage

Lemma 15.54 (averaging)**.** *Let $r_s > 0$, $d_s \in \mathbb{R}$ and $n_s > 0$ for s in a finite set, with $d_s \le H r_s$ for all s, and with $(1-\delta)\bar{n} \le n_s \le (1+\delta)\bar{n}$ for some $\bar{n} > 0$ and $0 \le \delta \le \frac{1}{3}$. Put $\mu_0 = \sum d_s / \sum r_s$ and $\mu_1 = \sum n_s d_s / \sum n_s r_s$. Then $\mu_1 \ge \mu_0 - 3\delta(H - \mu_0)$.*

Proof. Write $d_s = \mu_0 r_s + e_s$ with $\sum e_s = 0$ and $e_s \le (H - \mu_0) r_s$. Then $\sum |e_s| = 2\sum e_s^+ \le 2(H - \mu_0)\sum r_s$, and $\mu_1 - \mu_0 = \sum (n_s - \bar{n}) e_s / \sum n_s r_s \ge -\delta\bar{n}\sum|e_s| / ((1-\delta)\bar{n}\sum r_s) \ge -\frac{2\delta}{1-\delta}(H - \mu_0) \ge -3\delta(H - \mu_0)$. □

Lemma 15.55 (projectors)**.** *Let a finite group G act on F^D preserving a lattice R and the forms at every archimedean place, let $P \leq G$ and $\psi\colon P \to F^\times$ a character with values in $\mathcal{O}_F^\times$, and let $e = |P|^{-1}\sum_p \psi(p)^{-1}p$. Then e is an orthogonal projector, $|P|\,eR \subset R \cap eF^D \subset eR$, and every saturated sublattice $Y \subset eF^D$ satisfies $|\mu(Y \cap eR) - \mu(Y \cap R)| \leq 2\log|P|$.*

Proof. $e^2 = e$ because ψ is a character, and $e^* = e$ because each p is unitary and $\psi(p)^{-1} = \overline{\psi(p)}$. If $x \in R$ then $|P|ex = \sum \psi(p)^{-1}px \in R$. If $x \in R \cap eF^D$ then $x = ex \in eR$. Hence $Y \cap R$ has index at most $|P|^{\mathrm{rk}\,Y}$ in $Y \cap eR$ (the quotient is killed by $|P|$), and the slopes differ by at most $2\log|P|$ in our normalization. □

Hypothesis 15.56 (near-constant multiplicities)**.** At the base level the lattice $R^{(0)}$ decomposes up to index $|G|^{\mathrm{rk}}$ as $\bigoplus_s R_s$ over the irreducible constituents τ_s, and at the sharp level $R^{(1)}$ decomposes as $\bigoplus_s R_s^{\oplus n_s}$ with n_s the multiplicity of ψ_1 in $\tau_s|_{P_1}$. These multiplicities satisfy $(1-\delta)\bar{n} \leq n_s \leq (1+\delta)\bar{n}$ with $\bar{n} = \prod_{P\in\mathcal{P}_g}|k_P|^k$ and $\delta = 8fr/\min(g_0, g_1)$.

The near-constancy follows from Frobenius reciprocity and a character sum count $q^k(1 - 1/q) \leq n \leq q^k(1 + (2r-1)/q)$. Two of the counts behind Hypothesis 15.56, the formula for the multiplicity of ψ_1 in a parabolically induced constituent and the goodness of the cuspidal products that occur, are not proved here.

Proposition 15.57 (passage)**.** *Under Hypothesis 15.56, with $\lambda_0 = \log|G|$ and $H_0 \geq \mu_{\max}$ at the base level,*

$$\mu^{(1)}(R^\vee) \leq \mu^{(0)}(R^\vee) + 3\delta\big(\mu^{(0)}(R^\vee) + H_0 + 28\lambda_0\big) + 28\lambda_0.$$

Proof. Apply Lemma 15.55 at each level to replace $R^{(i)}$ by the lattice cut out by the projector, at a cost of $O(\lambda_0)$ in slope and in maximal slope. Apply Lemma 15.54 to $r_s = \mathrm{rk}\,R_s$, $d_s = \widehat{\deg}(R_s)/n_F$ and the multiplicities n_s, with $H = H_0 + O(\lambda_0)$, to get $\mu^{(1)}(R) \geq \mu^{(0)}(R) - 3\delta(H - \mu^{(0)}(R)) - O(\lambda_0)$. Dualizing with $\mu(R^\vee) = -\mu(R)$ gives the statement. The constant in the $O(\lambda_0)$ terms is not computed here. □

With $\delta \leq \frac{1}{3}$, $\lambda_0 \leq b$ and $\mu^{(0)} + H_0$ bounded by the record, this gives $P_{\mathrm{pass}} \leq 1 + C_0\lambda_0$ of Hypothesis 15.16.

15.14.3 The covolume bound

Hypothesis 15.58 (covolume bound)**.** There is an absolute constant C such that for every admissible datum and every level $1 \leq j \leq m$ of the chain,

$$\frac{\mathrm{lcv}(M_j)}{D} \leq \sum_{\rho\in\Phi}(p_j - q_j)\,w_{j\rho}\log\Omega + C\,U_j,$$

where U_j is bounded by the terms $C_0 m x^\varepsilon(B_0 + N_c)$ of Hypothesis 15.16.

Two normalizations. Let F be a totally complex number field with $n_F = [F:\mathbb{Q}]/2$ archimedean places. For a hermitian lattice $\Lambda \neq 0$ over $\mathcal{O}_F$ we use the slope $\mu(\Lambda) = \widehat{\deg}(\Lambda)/(n_F\,\mathrm{rk}\,\Lambda)$ of (15.2) and we put

$$\delta(\Lambda) = \frac{\widehat{\deg}(\Lambda)}{[F:\mathbb{Q}]} = \frac{1}{2}\,\mathrm{rk}(\Lambda)\,\mu(\Lambda).$$

We call δ the degree in the $[F:\mathbb{Q}]$-normalization. For an $\mathcal{O}_F$-sublattice we write $\mu_{\max}(\Lambda)$ for the supremum of $\mu(\Lambda')$ over the nonzero $\mathcal{O}_F$-submodules $\Lambda' \subseteq \Lambda$. We use four elementary facts. If $\Lambda' \subseteq \Lambda$ have the same rank and the same forms, then $\widehat{\deg}(\Lambda') \le \widehat{\deg}(\Lambda)$. For an integer $g \ge 1$, $\mu(g^{-1}\Lambda) = \mu(\Lambda) + 2\log g$, because $[g^{-1}\Lambda : \Lambda] = g^{2n_F \operatorname{rk}\Lambda}$. If $\Lambda_1 \subseteq \Lambda$ is saturated and we give the quotient its induced forms, then $\widehat{\deg}(\Lambda) = \widehat{\deg}(\Lambda_1) + \widehat{\deg}(\Lambda/\Lambda_1)$. If $y_1, \dots, y_r \in \Lambda$ are linearly independent over F and span $\Lambda \otimes F$, then Hadamard's inequality gives $\widehat{\deg}(\Lambda) \ge -\sum_v \sum_i \log h_v(y_i, y_i)$.

The formal proof measures sections on a cover by a volume slope. Let Z be a component of a cover over a field F, let u be a weight with $\deg_Z(u) > 0$, where $\deg_Z(u)$ is the top geometric degree of the automorphic line bundle of weight u on Z, and let $\Lambda_B(Z; ku)$ be the lattice of integral sections of weight ku for the integral structure of the cutoff B described in Hypothesis 15.69. With D_{fl} the dimension of the flag variety put

$$V_B(Z;u;t) = \limsup_{k\to\infty} \frac{D_{\mathrm{fl}}!\, \log \#S_B(Z; 8ku; 8kt)}{[F:\mathbb{Q}]\,(8k)^{D_{\mathrm{fl}}+1}},$$
$$\sigma_B(Z;u) = \inf_{t\in\mathbb{R}} \Big(\frac{V_B(Z;u;t)}{\deg_Z(u)} - t \Big),$$

where $S_B(Z;v;t)$ is the set of $f \in \Lambda_B(Z;v)$ with $\|f\| \le e^t$ at every frame. The number $\sigma_B(Z;u)$ divides by $[F:\mathbb{Q}]$. On the lattices of the chain and of the covers the formal proof uses μ, which divides by n_F. A lower bound without main term for σ_B turns into a lower bound for μ through the factor 2 of input (I17) below, and that factor multiplies a constant. The main term is compared only in Lemma 15.63, which is stated for μ and in which the flat degree $\frac{1}{2}(p-q)\log\Omega$ per unit weight is a degree δ. Remark 15.64 records the bookkeeping.

The covolume inequality. Write $\Lambda_B = \log(B \operatorname{rd}_E) + |\lambda_1|$ and let C_{unif} denote the absolute constant in Hypothesis 15.69.

For the chain we write $p_j = k + j - 1$ and $q_j = m - j$, so that $|p_j - q_j - k| \le m$, and we write $w_{j\rho} = e_j(\rho)$ for the exponent of index j of the archimedean parameter at ρ (the weights $w_{j\rho}$ of Hypothesis 15.18). Put

$$\mathrm{main}_j = \sum_{\rho\in\Phi} (p_j - q_j)\, w_{j\rho} \log\Omega, \qquad U = B_0 \log\log x, \qquad B_0 = fmb\, x^{\varepsilon}.$$

Since $\log\log x \le x^{\varepsilon}$ for large x and $m \ge 1$, $U \le m\, x^{\varepsilon}(B_0 + N_c)$, so U is admissible as the U_j of Hypothesis 15.58.

Hypothesis 15.59 (covolumes on the chain)**.** There is $\varepsilon_1 > 0$ such that for every $0 < \varepsilon \le \varepsilon_1$ and every admissible family there are $C_0 \ge 0$ and x_0 with the following property. For every $x \ge x_0$, every point of the family at x, every $1 \le j \le m$ and every $C \ge C_0$,

$$\frac{\mathrm{lcv}(M_j)}{D} \le \mathrm{main}_j + C\, U.$$

Four lattice lemmas.

Lemma 15.60 (truncation)**.** *Let $L \subseteq M$ be hermitian $\mathcal{O}_F$-lattices with $L \neq 0$, let e be an F-linear idempotent of $M \otimes F$ with $h_v(ex + (y - ey), ex + (y - ey)) = h_v(ex, ex) + h_v(y - ey, y - ey)$ for all*

x, y and all v, and let $g \geq 1$ be an integer with $g\,e(L) \subseteq L$. Put $L_1 = L \cap \operatorname{im} e$ and $L_0 = L \cap \ker e$. Let $0 \leq \delta_0 \leq \frac{1}{12}$ and $Q \geq 0$ satisfy $\operatorname{rk} L_0 \leq 3\delta_0 \operatorname{rk} L$, $\mu_{\max}(M) \leq Q$, $-Q \leq \mu(L)$ and $2\log g \leq Q$. Then $\mu(L_1) \geq \mu(L) - 12\delta_0 Q$.

Proof. Write $r = \operatorname{rk} L$, $r_1 = \operatorname{rk} L_1$ and $r_0 = \operatorname{rk} L_0$. For $x \in L$ we have $g\,ex \in L_1$, so $(L_1) \otimes F = e(L \otimes F)$. Since $L \otimes F = e(L \otimes F) \oplus (1-e)(L \otimes F)$ orthogonally, the map $1 - e$ identifies L/L_1 isometrically with $(1-e)L$, and $r = r_1 + r_0$. For $x \in L$ the element $g(1-e)x = gx - g\,ex$ lies in $L \cap \ker e = L_0$, so $L_0 \subseteq (1-e)L \subseteq g^{-1}L_0$. If $r_0 > 0$, then $\mu(L_0) \leq \mu_{\max}(M) \leq Q$, hence

$$\widehat{\deg}(L/L_1) = \widehat{\deg}((1-e)L) \leq \widehat{\deg}(g^{-1}L_0) = n_F r_0\big(\mu(L_0) + 2\log g\big) \leq 2n_F r_0 Q,$$

and the same bound holds trivially when $r_0 = 0$. Since L_1 is saturated in L, $\widehat{\deg}(L_1) = \widehat{\deg}(L) - \widehat{\deg}(L/L_1) \geq n_F(r\mu(L) - 2r_0 Q)$. From $r_0 \leq 3\delta_0 r \leq \frac{r}{4}$ we get $r_1 \geq \frac{3}{4}r > 0$ and $r_0/r_1 \leq 3\delta_0/(1-3\delta_0) \leq 4\delta_0$. Hence

$$\mu(L_1) - \mu(L) \geq \frac{r\mu(L) - 2r_0 Q}{r_1} - \mu(L) = \frac{r_0(\mu(L) - 2Q)}{r_1} \geq -\frac{3r_0 Q}{r_1} \geq -12\delta_0 Q,$$

where we used $-Q \leq \mu(L) \leq \mu_{\max}(M) \leq Q$. □

Lemma 15.61 (descent)**.** *Let $A \neq 0$ and B be hermitian $\mathcal{O}_F$-lattices, let $\varphi_1, \ldots, \varphi_n \colon A \otimes F \to B \otimes F$ be F-linear maps with $\varphi_h(A) \subseteq B$, with $\bigcap_h \ker \varphi_h = 0$ and with $\sum_h \|\varphi_h f\|_v^2 \leq e^{2P}\|f\|_v^2$ for all f and v. Suppose that B is spanned over F by elements of B of norm at most e^{Λ} at every place, with $\Lambda \geq 0$, and that $n \operatorname{rk} B \leq (1+\epsilon)\operatorname{rk} A$ with $\epsilon \geq 0$. If $t \leq \mu(A) - 2P$, then $\mu(B) \geq t - \epsilon(|t| + 2\Lambda)$.*

Proof. Give B^n the orthogonal sum of the forms of B and put $\Phi = (\varphi_h)_h \colon A \to B^n$. The map Φ is injective with $\|\Phi f\|_v \leq e^P \|f\|_v$. Put $r = \operatorname{rk} A$ and $N = n \operatorname{rk} B$, so $r \leq N \leq (1+\epsilon)r$. For a basis $e_1, \ldots, e_r$ of $A \otimes F$ in A the index of $\sum \mathcal{O}_F \Phi e_i$ in $\Phi(A)$ equals that of $\sum \mathcal{O}_F e_i$ in A, and the Gram determinants satisfy $\det h_v(\Phi e_i, \Phi e_j) \leq e^{2Pr} \det h_v(e_i, e_j)$. Hence $\mu(\Phi A) \geq \mu(A) - 2P \geq t$. Let S be the saturation of $\Phi(A)$ in B^n, so $\widehat{\deg}(S) \geq \widehat{\deg}(\Phi A) \geq n_F r t$. The quotient B^n/S has rank $N - r$ and is spanned over F by images of the elements $(0, \ldots, y, \ldots, 0)$ with y in the given spanning set, whose quotient norms are at most e^{Λ}. Choosing $N - r$ of these images linearly independent and applying Hadamard's inequality gives $\widehat{\deg}(B^n/S) \geq -2n_F(N-r)\Lambda$. Therefore $n\widehat{\deg}(B) = \widehat{\deg}(B^n) \geq n_F\big(rt - 2\Lambda(N-r)\big)$ and

$$\mu(B) = \frac{n\widehat{\deg}(B)}{n_F N} \geq \frac{rt}{N} - 2\Lambda\frac{N-r}{N} \geq t - |t|\frac{N-r}{N} - 2\Lambda\frac{N-r}{N} \geq t - \epsilon(|t| + 2\Lambda),$$

because $(N-r)/N \leq (N-r)/r \leq \epsilon$. □

Lemma 15.62 (from component heights to the slope)**.** *Let $\mathcal{Z}$ be a finite nonempty set with $n = \#\mathcal{Z}$ elements. Let $T, D_{\mathrm{fl}}, A, R, c_s, c_c, X, \rho, b_n, \delta'$ and E be real numbers with $D_{\mathrm{fl}} \geq 0$, $T \geq 3D_{\mathrm{fl}} + 5$, $A > 0$, $R \geq 0$, $c_s, c_c, X \geq 0$, $0 \leq \rho < 1$, $b_n > 0$, $\delta' > 0$ and $E \geq 1$, and for $Z \in \mathcal{Z}$ let $h_Z, s_Z, \delta_Z, r_Z, d_Z$ be real numbers. Assume*

(a) $\delta' \leq \delta_Z \leq \delta'/(1-\rho)$ and $b_n \delta_Z \leq r_Z$ for every Z,

(b) $h_Z + c_s \geq 0$ and $(1-\rho)\max(0, s_Z + c_s) \leq h_Z + c_s + c_c$ for every Z, and $\frac{1}{n}\sum_Z s_Z \geq -X$,

(c) $d_Z \geq b_n \delta_Z\big(Th_Z - 3(D_{\mathrm{fl}}+1)|h_Z| - A\log(2 + |h_Z|) - R\big)$ for every Z,

(d) $TX \le C_1E$, $Tc_c \le C_1E$, $c_s\big(6D_{\mathrm{fl}} + 10 + 2T\rho\big) \le C_1E$ *and* $A\log\max(A,1) + R \le C_2E$.

Then $2\sum_Z d_Z / \sum_Z r_Z \ge -2(3C_1 + C_2 + 2)E$.

Proof. Put $H_Z = h_Z + c_s \ge 0$. If $h_Z \ge 0$ then $|h_Z| = h_Z$, and if $h_Z < 0$ then $|h_Z| = -h_Z \le c_s$, so in both cases $|h_Z| \le h_Z + 2c_s$, and $y_Z = h_Z + 2c_s \ge 0$. For $y \ge 0$ the function $A\log(2+y) - y$ is at most $A\log A - A + 2$ when $A \ge 2$ and at most $A\log 2 < 2$ when $A < 2$, so $A\log(2+y) \le y + A\log\max(A,1) + 2$. Hence the bracket in (c) satisfies

$$\beta_Z \ge (T - 3D_{\mathrm{fl}} - 3)h_Z - 6(D_{\mathrm{fl}} + 1)c_s - (h_Z + 2c_s) - A\log\max(A,1) - 2 - R = a\,h_Z - K_0,$$

with $a = T - 3D_{\mathrm{fl}} - 4 \in [1, T]$ and $K_0 = (6D_{\mathrm{fl}} + 8)c_s + A\log\max(A,1) + 2 + R$. By (b), $H_Z \ge (1-\rho)(s_Z + c_s) - c_c$, so $\sum_Z H_Z \ge n\big((1-\rho)(c_s - X) - c_c\big)$. Since $H_Z \ge 0$, (a) gives $\sum_Z \delta_Z H_Z \ge \delta' \sum_Z H_Z$ and $\sum_Z \delta_Z \le n\delta'/(1-\rho)$, so

$$\frac{\sum_Z \delta_Z H_Z}{\sum_Z \delta_Z} \ge (1-\rho)\max\Big(0, \frac{1}{n}\sum_Z H_Z\Big) \ge (1-\rho)^2(c_s - X) - (1-\rho)c_c.$$

With $h_Z = H_Z - c_s$ and $1 - (1-\rho)^2 \le 2\rho$ this gives $m_\delta := \sum_Z \delta_Z h_Z / \sum_Z \delta_Z \ge -X - c_c - 2\rho c_s$. Hence

$$\frac{\sum_Z \delta_Z (a h_Z - K_0)}{\sum_Z \delta_Z} = a\,m_\delta - K_0 \ge -T(X + c_c + 2\rho c_s) - K_0 =: -K_1,$$

using $1 \le a \le T$ when $m_\delta < 0$. Put $N = \sum_Z d_Z$, so $N \ge b_n \sum_Z \delta_Z (a h_Z - K_0)$ by (c). If $N \ge 0$ then $N/\sum_Z r_Z \ge 0 \ge -K_1$. If $N < 0$ then, since $\sum_Z r_Z \ge b_n \sum_Z \delta_Z > 0$ by (a), $N/\sum_Z r_Z \ge N/(b_n \sum_Z \delta_Z) \ge -K_1$. By (d) and $E \ge 1$, $K_1 \le TX + Tc_c + c_s(6D_{\mathrm{fl}} + 8 + 2T\rho) + A\log\max(A,1) + R + 2 \le (3C_1 + C_2 + 2)E$. □

Lemma 15.63 (flat twist)**.** *Let* Ψ *be a finite set and let* $a_\tau \in \mathbb{Z}$ *and* $c_\tau \in \mathbb{R}$ *for* $\tau \in \Psi$. *Let* $p \ge q \ge 0$ *be integers and let* $\log\Omega$, $L \ge 10$, b, U, main, $C_A \ge 0$, l_r, $m_0 \ge 0$, $f_r \ge 0$, n_V, l_S *be real numbers with*

(a) $\big|c_\tau + \frac{1}{2}(p-q)\log\Omega\big| \le 3qL$ *for every* τ,

(b) $\big|\mathrm{main} - \sum_\tau (p-q)a_\tau \log\Omega\big| \le C_A U$ *and* $q\,L\sum_\tau |a_\tau| \le C_A U$,

(c) $l_r \le 2L$, $2m_0 \le n_V$, $f_r n_V L \le b$, $b \le U$, $L \le U$ *and* $l_S \le 40L$.

Let Λ *and* Λ' *be hermitian lattices of the same rank with*

$$\delta(\Lambda') \ge \delta(\Lambda) + \mathrm{rk}(\Lambda)\Big(\sum_\tau a_\tau c_\tau - q\,l_r \sum_\tau |a_\tau| - m_0 f_r\big(\tfrac{1}{2}\log(2\pi) + l_r\big) - l_S\Big).$$

Then $\mu(\Lambda') \ge \mu(\Lambda) - \mathrm{main} - (11C_A + 83)U$.

Proof. Since $\mu = 2\delta/\,\mathrm{rk}$, the hypothesis gives $\mu(\Lambda') \ge \mu(\Lambda) + 2\Theta$ with Θ the bracket. By (a), $a_\tau c_\tau \ge -\frac{1}{2}(p-q)a_\tau \log\Omega - 3qL|a_\tau|$, so by (b)

$$2\sum_\tau a_\tau c_\tau \ge -\sum_\tau (p-q)a_\tau \log\Omega - 6qL\sum_\tau |a_\tau| \ge -\mathrm{main} - 7C_A U.$$

By (c) and (b), $2q\,l_r \sum_\tau |a_\tau| \le 4qL\sum_\tau |a_\tau| \le 4C_A U$. Since $\log(2\pi) \le 2\pi - 1 \le 7$ and $L \ge 10$, $\frac{1}{2}\log(2\pi) + l_r \le 3L$, so $2m_0 f_r(\frac{1}{2}\log(2\pi) + l_r) \le 3n_V f_r L \le 3b \le 3U$. Finally $2l_S \le 80L \le 80U$. Adding, $2\Theta \ge -\mathrm{main} - (11C_A + 83)U$. □

Remark 15.64 (the factor 2). Lemma 15.63 is the only place where the main term is compared. Its input (a) is the flat degree of input (I13), which is a degree δ in the $[F:\mathbb{Q}]$-normalization, namely $\frac{1}{2}(p-q)\log\Omega$ per unit of the anchor weight up to $3qL$. The conversion $\mu = 2\delta/\operatorname{rk}$ doubles it, and the doubled value $\sum_\tau (p-q)a_\tau \log\Omega$ is the main term in the normalization μ of (15.2), the normalization of $\operatorname{lcv}(M_j)/D$. This agrees with Definition 15.5, where $\log\Omega_\tau$ is $2/[M:\mathbb{Q}]$ times an arithmetic degree, that is, a slope μ of a line. The lower bound of Hypothesis 15.69 is a bound for σ_B, which divides by $[F:\mathbb{Q}]$, and it enters only through Lemma 15.62 and input (I17), whose factor 2 appears in the constant $2(3C_1 + C_2 + 2)$.

Lemma 15.65 (size of the error of Hypothesis 15.69)**.** *Let $T, f, m, b, L, x, \ell, q$ and rd_E be positive reals with $L \geq 10$, $b = e^L$, $\log\log x \geq 1$, $\ell \leq 9L$ and $\log \mathrm{rd}_E \leq 2L$, put $U = fmbL\log\log x$, and let u' be a weight and B a real number with $T\|u'\|_1 \leq 834fmb$, $1 \leq B \leq 132b$ and $q\,\mathrm{spread}(u') \leq 6\|u'\|_1$. Let c_B be a real number with $0 \leq c_B \leq \log B + 20 + 44\log \mathrm{rd}_E$. Then*

$$T\big((\log B + \ell)\|u'\|_1 + q\,\mathrm{spread}(u')\big) \leq 83400\,U, \qquad T\,c_B\|u'\|_1 \leq 83400\,U.$$

Proof. From $B \leq 132b$ we get $\log B \leq \log 132 + L \leq L + 22$. Hence $\log B + \ell + 6 \leq 10L + 28 \leq 100L$ and $c_B \leq L + 42 + 88L \leq 100L$, because $L \geq 10$. So both left sides are at most $100L \cdot T\|u'\|_1 \leq 83400fmbL \leq 83400U$. □

The objects of the chain and the inputs. Fix ε, an admissible family, x, a point of the family at x and a level $1 \leq j \leq m$. At this level the formal proof attaches to the point hermitian $\mathcal{O}_F$-lattices $\Lambda_j^{\mathrm{type}} \subseteq \Lambda_j^{\mathrm{all}}$ (forms of the local types inside all forms of the level, with the archimedean norms of the level), an idempotent e_j of $\Lambda_j^{\mathrm{all}} \otimes F$ with a finite group G_j, the good part $\Lambda_j^{\mathrm{good}} = \Lambda_j^{\mathrm{type}} \cap \operatorname{im} e_j$ and the bad part $\Lambda_j^{\mathrm{bad}} = \Lambda_j^{\mathrm{type}} \cap \ker e_j$, and a core lattice $\Lambda_j^{\mathrm{core}}$. For $j < m$ it attaches deep covers Y of the unitary Shimura variety of the datum of level j, of signature (p,q), with lattices $\Lambda_Y(G)$ of cusp forms of weight G and slopes $\mu_Y(G) = \mu(\Lambda_Y(G))$, a sublattice $\Lambda_Y^{\mathrm{deep}} \subseteq \Lambda_Y(G_k)$ cut out by the forms that come from the chain, the shifted weights $G_s(\pm c_0)$, the weight u^- and the anchor weight $a = (a_\tau)$ with $G_s(-c_0)$ equal, up to an equivalence that preserves lattices, ranks and slopes, to $u^- + a$ shifted by $m_0 = n_V/2$. We write $K_j = PP \cdot \kappa_j$ for the size parameter of the level, $T = \mathrm{len}_T \geq 1$ for the length of the filtration and D_{fl} for the dimension of the flag variety. Assume the following bounds with constants independent of x, of the point and of j, valid for $x \geq x_0$, where x_0 and the constants depend only on ε and the family. We write $C_K, C_D, C_A, C_s, C_m, C_{HS}$ for these constants.

(I1) (recursion) $\operatorname{lcv}(M_j)/D \leq -\mu(\Lambda_j^{\mathrm{good}})$.

(I2) (cut) e_j is idempotent and orthogonal at every embedding, $|G_j|\,e_j(\Lambda_j^{\mathrm{type}}) \subseteq \Lambda_j^{\mathrm{type}}$, $2\log|G_j| \leq b$ and $\Lambda_j^{\mathrm{good}} \neq 0$.

(I3) (bad locus) $\operatorname{rk}\Lambda_j^{\mathrm{bad}} \leq 3e^{-x^{\varepsilon_z/7}} \operatorname{rk}\Lambda_j^{\mathrm{type}}$, with $\varepsilon_z > 0$ a parameter of the family.

(I4) (maximal slope) $\mu_{\max}(\Lambda_j^{\mathrm{all}}) \leq x^A b$.

(I5) (sizes) $1 \leq b \leq U$, $|\mathrm{main}_j| \leq x^{A'}b$, $|\mathrm{main}_j| \leq K_j$, $b \leq K_j$, and for $j < m$ the set of quotient weights u_1 is nonempty and each has $3D_{\mathrm{fl}} + 5 \leq T$, $0 \leq \rho(u_1) < 1$ and $0 \leq z(u_1) < 1$, with ρ, z the numbers of the filtration.

(I6) (core comparison) $\mu(\Lambda_j^{\mathrm{type}}) \geq \mu(\Lambda_j^{\mathrm{core}}) - C_K U$.

(I7) (top level, CM evaluation) for $j = m$, $\mu(\Lambda_m^{\mathrm{core}}) \geq -\mathrm{main}_m - C_D U$.

(I8) (existence) for $j < m$ there is a good deep cover Y at level j, and it satisfies the hypotheses of Hypothesis 15.69.

(I9) (ring) $\mu(\Lambda_Y^{\mathrm{deep}}) - 2b \leq \mu(\Lambda_j^{\mathrm{core}})$.

(I10) (maps) there are n maps $\Lambda_Y(G_s(-c_0)) \to \Lambda_Y^{\mathrm{deep}}$ satisfying the hypotheses of Lemma 15.61 with $P = b$, and $n \operatorname{rk} \Lambda_Y^{\mathrm{deep}} \leq \operatorname{rk} \Lambda_Y(G_s(c_0))$.

(I11) (rank) $\operatorname{rk} \Lambda_Y(G_s(c_0)) \leq (1 + b/K_j) \operatorname{rk} \Lambda_Y(G_s(-c_0))$ and $\Lambda_Y(G_s(-c_0)) \neq 0$.

(I12) (generators) $\Lambda_Y^{\mathrm{deep}}$ is spanned over F by elements of norm at most e^{K_j} at every embedding.

(I13) (flat twist) the hypotheses of Lemma 15.63 hold for $\Lambda = \Lambda_Y(u^-)$, $\Lambda' = \Lambda_Y(G_s(-c_0))$, the anchor weight a, c_τ the mean over the components of minus the degree δ of the flat section of type τ, $\mathrm{main} = \mathrm{main}_j$, $l_r = \log \mathrm{rd}_E$, $f_r = f(p+q)$, $l_S = \sum_{\ell \in S_{\mathrm{bad}}} \log \ell$, with the constant C_A.

(I14) (heights of components) for $j < m$, every quotient weight u_1 and every component Z of Y there are the numbers of Lemma 15.62, namely the height h_Z, the degree δ_Z, the rank r_Z and the degree d_Z of the filtered pieces of weight u^-, $b_n = \binom{T-1}{D_{\mathrm{fl}}}$, A, R, c_s and a multiplicity cost $c_m \geq 0$, satisfying (a), (c), $h_Z + c_s \geq 0$, $c_s(6D_{\mathrm{fl}} + 10 + T(z + 2\rho)) \leq C_s U$, $Tc_m \leq C_m U$ and $A \log \max(A, 1) + R \leq C_{HS} U$.

(I15) (comparison of the volume slope with the height) with $u' = u'(u_1)$ the weight attached to u_1, g its gap and B as in Hypothesis 15.69, every real $s \leq \sigma_B(Z; u')$ satisfies $(1 - \rho) \max(0, s + c_s) \leq h_Z + c_s + c_m + c_B \|u'\|_1$, where c_B is the cost of the Hodge–Tate integral structure up to B.

(I16) (numerics of the weight) the hypotheses of Lemma 15.65 hold for u', B, c_B and T, and u' satisfies the hypotheses on u and g of Hypothesis 15.69.

(I17) (sum) $\mu_Y(u^-) \geq 2 \sum_Z d_Z / \sum_Z r_Z$.

Inputs (I1)–(I17) are not proved here.

Theorem 15.66. *Assume Hypothesis 15.69 and inputs (I1)–(I17). Put* $C_F = 11C_A + 83$, $C_H = C_s + C_m + 83400(1 + C_{\mathrm{unif}})$, $C_U = 2(3C_H + C_{HS} + 2)$, $C_{\mathrm{low}} = \max\big(C_D, 2(C_F + C_U) + 9\big)$ *and* $C_0 = C_K + C_{\mathrm{low}} + 1$. *Then Hypothesis 15.59 holds with this* C_0, *for* x *large in terms of* ε, *the family,* A, A' *and* C_0.

Proof. Step 1 (slope of the weight u^- on the cover). Let $j < m$, let Y be the cover of (I8) and let u_1 be a quotient weight, which exists by (I5). By (I16) and (I8), Hypothesis 15.69 applies to $u' = u'(u_1)$ and its gap g, and gives numbers $s_Z \leq \sigma_B(Z; u')$ with mean at least $-X$, $X = C_{\mathrm{unif}} \mathrm{err}(u', g)$. By (I15), condition (b) of Lemma 15.62 holds with $c_c = c_m + c_B \|u'\|_1$. By Lemma 15.65, $TX \leq 83400\, C_{\mathrm{unif}} U$ and $Tc_c \leq (C_m + 83400)U$. With (I14) and $2T\rho \leq T(z + 2\rho)$, condition (d) of Lemma 15.62 holds with $C_1 = C_H$, $C_2 = C_{HS}$ and $E = U \geq 1$. Lemma 15.62 and (I17) give $\mu_Y(u^-) \geq -C_U U$.

Step 2 (flat twist). By (I13) and Lemma 15.63, $\mu_Y(G_s(-c_0)) \geq \mu_Y(u^-) - \mathrm{main}_j - C_F U \geq -\mathrm{main}_j - (C_F + C_U)U$.

Step 3 (descent to the level of the chain). Apply Lemma 15.61 to $A = \Lambda_Y(G_s(-c_0))$, $B = \Lambda_Y^{\mathrm{deep}}$, the maps of (I10), $P = b$, $\Lambda = K_j$, $\epsilon = b/K_j \in [0,1]$ and $t = -\mathrm{main}_j - (C_F + C_U)U - 2b$. Its hypotheses hold by (I10), (I11), (I12), (I5) and Step 2. It gives $\mu(\Lambda_Y^{\mathrm{deep}}) \geq t - \epsilon(|t| + 2K_j)$. Since $\epsilon K_j = b$, $\epsilon|\mathrm{main}_j| \leq b$ and $\epsilon \leq 1$, we get $\epsilon(|t| + 2K_j) \leq b + (C_F + C_U)U + 2b + 2b$, so $\mu(\Lambda_Y^{\mathrm{deep}}) \geq -\mathrm{main}_j - 2(C_F + C_U)U - 7b$. By (I9) and $b \leq U$, $\mu(\Lambda_j^{\mathrm{core}}) \geq -\mathrm{main}_j - (2(C_F + C_U) + 9)U$. For $j = m$, (I7) gives $\mu(\Lambda_m^{\mathrm{core}}) \geq -\mathrm{main}_m - C_D U$. In both cases $\mu(\Lambda_j^{\mathrm{core}}) \geq -\mathrm{main}_j - C_{\mathrm{low}} U$.

Step 4 (truncation). Put $\delta_0 = e^{-x^{\varepsilon_z/7}}$ and $Q = (x^A + x^{A'} + C_K + C_{\mathrm{low}} + 1)U$. Take x so large that $12\delta_0(x^A + x^{A'} + C_K + C_{\mathrm{low}} + 1) \leq 1$, which is possible since δ_0 decays faster than any power of x. Then $\delta_0 \leq \frac{1}{12}$ and $12\delta_0 Q \leq U$. By (I6) and Step 3, $\mu(\Lambda_j^{\mathrm{type}}) \geq -\mathrm{main}_j - (C_K + C_{\mathrm{low}})U \geq -x^{A'}b - (C_K + C_{\mathrm{low}})U \geq -Q$ by (I5). By (I4) and (I5), $\mu_{\max}(\Lambda_j^{\mathrm{all}}) \leq x^A b \leq Q$, and by (I2), $2\log|G_j| \leq b \leq Q$. Lemma 15.60, with (I2) and (I3), gives $\mu(\Lambda_j^{\mathrm{good}}) \geq \mu(\Lambda_j^{\mathrm{type}}) - U$. By (I1),

$$\frac{\mathrm{lcv}(M_j)}{D} \leq -\mu(\Lambda_j^{\mathrm{good}}) \leq \mathrm{main}_j + (C_K + C_{\mathrm{low}} + 1)U = \mathrm{main}_j + C_0 U \leq \mathrm{main}_j + CU$$

for every $C \geq C_0$, since $U \geq 0$. □

The constants $C_K, C_D, C_A, C_s, C_m, C_{HS}$ of the inputs and the constant C_{unif} of Hypothesis 15.69 are the only constants in C_0, and none of them depends on x, on the point, on j or on the level of the cover. The constants of the arithmetic Hilbert–Samuel theorem on a single model, which depend on the level, enter only inside the proof of Hypothesis 15.69, where the limit in the cutoff removes them.

Remark 15.67 (the sign of the main term). The main term enters Hypothesis 15.59 with the sign $+\mathrm{main}_j$ on the upper bound for $\mathrm{lcv}(M_j)/D$, through $\mu(\Lambda_j^{\mathrm{core}}) \geq -\mathrm{main}_j - C_{\mathrm{low}} U$. The weights are $w_{j\rho} = e_j(\rho)$, the exponents of the archimedean parameters, and the sign of main_j is that of $\sum_\rho (p_j - q_j) w_{j\rho} \log \Omega$. On the cover the same term is $\sum_\tau (p - q) a_\tau \log \Omega$ for the anchor weight a, by (I13) and Lemma 15.63, where a positive a_τ lowers μ by $(p - q)a_\tau \log \Omega$. That the anchor weight is the weight $w_{j+1}(v, s) = \varepsilon_{\tau_{v,s}} - c_j$ of the step character of Proposition 7.9(iii) with this sign is condition (N1) of Section 7, which this subsection does not prove.

The local lattice relations at good primes. The exact identity at the good primes in the proof of Hypothesis 15.69 uses, at every prime p outside the bad set and every chart of the model over a discrete valuation ring O containing ζ_p with $w = \zeta_p - 1$, $w^{p-1} = u\,p$ for a unit u, the following relations. For lattices $E \subseteq E'$ with the same generic fibre let $\mathrm{ck}_k(E, E')$ be the degree on the special fibre of the class of $p_k(E') - p_k(E)$ against the fixed classes L, Q with Q homogeneous of degree $d + 1 - k$, and for an element $c \in O$ let $\rho_c(E)$ be the degree of $\mathrm{div}(c) \cdot p_{k-1}(E) \cdot Q$. Let $\mathrm{Lie}_\tau \subseteq W_\tau \subseteq w^{-1}\mathrm{Lie}_\tau$ and $\omega_\tau^{\mathrm{HT}} \subseteq \omega_\tau \subseteq w^{-1}\omega_\tau^{\mathrm{HT}}$ be the de Rham and Hodge–Tate lattices, let $X_{\tau,1}, \ldots, X_{\tau,n}$ be the character ideals of the Hodge–Tate map at the label τ, T the trivial line, H_τ the lattice dual to the annihilator lattice of the Hodge–Tate map and T_n the standard lattice, and let φ be the Frobenius permutation of the labels.

Proposition 15.68. *Let $p \geq 2$ and $1 \leq k \leq d + 1$. Assume, for every label τ,*

(G1) $\mathrm{ck}_k(\mathrm{Lie}_\tau, W_\tau) + \mathrm{ck}_k(\omega_\tau, \omega_\tau^{\mathrm{HT}}) = \mathrm{ck}_k(T_n, H_\tau) - k\,\rho_w(\mathrm{Lie}_\tau)$,

(G2) $(p-1)\rho_w(\mathrm{Lie}_\tau) = \rho_p(\mathrm{Lie}_\tau)$,

(G3) $\mathrm{ck}_k(T_n, H_\tau) = (-1)^k \sum_{i=1}^n \mathrm{ck}_k(T, X_{\tau,i})$,

(G4) $k\,\rho_p(\mathrm{Lie}_{\varphi\tau}) = (-1)^{k-1} \sum_{i=1}^n \big(\mathrm{ck}_k(T, X_{\varphi\tau,i}) - p^k\,\mathrm{ck}_k(T, X_{\tau,i})\big)$.

Put $h_\tau = (-1)^k \sum_i \mathrm{ck}_k(T, X_{\tau,i})$ *and* $\lambda_\sigma = k\,\rho_p(\mathrm{Lie}_\sigma)$. *Then for every* τ

$$\mathrm{ck}_k(\mathrm{Lie}_\tau, W_\tau) + \mathrm{ck}_k(\omega_\tau, \omega_\tau^{\mathrm{HT}}) = h_\tau - \frac{\lambda_\tau}{p-1}, \qquad p^k h_\tau = h_{\varphi\tau} + \lambda_{\varphi\tau}.$$

Proof. By (G1) and (G3) the left side of the first relation is $h_\tau - k\rho_w(\mathrm{Lie}_\tau)$, and by (G2) $k\rho_w(\mathrm{Lie}_\tau) = \lambda_\tau/(p-1)$. For the second relation put $A_i = \mathrm{ck}_k(T, X_{\varphi\tau,i})$ and $B_i = \mathrm{ck}_k(T, X_{\tau,i})$. Then (G4) reads $\lambda_{\varphi\tau} = (-1)^{k-1}\sum_i (A_i - p^k B_i)$, so $h_{\varphi\tau} + \lambda_{\varphi\tau} = (-1)^k \sum_i A_i - (-1)^k \sum_i (A_i - p^k B_i) = p^k (-1)^k \sum_i B_i = p^k h_\tau$. □

Proposition 15.68 identifies the numbers h_τ of the relations as the Chern characters of the character ideals of the Hodge–Tate map, so the relations are two exact computations. The first uses the exact sequence that compares $w^{-1}\mathrm{Lie}_\tau/W_\tau$, H_τ/T_n and $\omega_\tau/\omega_\tau^{\mathrm{HT}}$ together with $\mathrm{div}(w) = \frac{1}{p-1}\mathrm{div}(p)$. The second uses the fact that the step graph of the Hodge–Tate map at $\varphi\tau$ restricted to the first factor is the span of the p-th powers of the character ideal at τ. The Frobenius recursion (Lemma `w3:U:frob` of the insertion that proves Hypothesis 15.69) then converts the two relations into (L2). Statements (G1)–(G4), the passage from test points to charts, and the refinement of the models in which they are computed are not proved here.

15.14.4 Uniformity in the level

The mean slope bound on the covers must hold with a constant independent of the level of the cover, and the level grows with x. A uniform arithmetic Hilbert–Samuel theorem for the family of covers would give such a bound. No such theorem is published, and the argument does not use one.

Hypothesis 15.69 (uniform mean slope on deep covers)**.** For an integer weight $u = (u_{\tau i})$ on a Shimura datum of signature (p, q), regarded in real coordinates, and a real number $g > 0$, put

$$B(u, g) = 4\,|\Phi_H|(p+q)^2\Big(1 + \frac{\max_{\tau,i}|u_{\tau i}|}{g}\Big), \qquad \|u\|_1 = \sum_{\tau,i}|u_{\tau i}|,$$

$$\mathrm{spread}(u) = \sum_\tau \Big(\max_i u_{\tau i} - \min_i u_{\tau i}\Big).$$

Say that (u, g) has a gap when every u_τ is strictly decreasing and

$$g \le |u_{\tau a} - u_{\tau b}| \qquad \text{for all } \tau \text{ and all } a \ne b \text{ with } p \le a.$$

Also put

$$\Lambda_{B(u,g)} = \log\big(B(u, g)\ \mathrm{rd}_E\big) + |\lambda_1|, \qquad \mathrm{err}(u, g) = \Lambda_{B(u,g)}\|u\|_1 + q\ \mathrm{spread}(u).$$

There is an absolute constant $C \ge 0$ with the following property. For every admissible host, every chain datum satisfying its laws with $m^2 \le k$, every $1 \le j < m$, and every good deep cover

at level j whose level has at most fifteen prime factors, and for every pair (u, g) with a gap, there are auxiliary real numbers sl_Z, indexed by the components of the cover, such that

$$\mathrm{sl}_Z \le \mathrm{sl}_Z^{B(u,g)}(u) \qquad \text{for every component } Z,$$

and

$$\frac{1}{\#\mathrm{Comp}} \sum_{Z \in \mathrm{Comp}} \mathrm{sl}_Z \ge -C \ \mathrm{err}(u, g).$$

Here Comp denotes the set of components and $\mathrm{sl}_Z^B(u)$ is the volume slope $\sigma_B(Z; u)$ defined in the covolume calculation, with cutoff B, weight $8ku$ and the common archimedean normalization $\xi_R(D)$.

The argument for Hypothesis 15.69 uses a tower of models indexed by a prime cutoff B. For each B one constructs a normal projective integral model $\mathcal{Y}_B$ of the toroidal compactification, smooth with good reduction away from the primes $\le B$ and a fixed bad set, with hermitian Hodge line bundles of every weight. On each model the arithmetic self-intersection $P_B(u)$ of the Hodge bundle of weight u is a polynomial in u, and the arithmetic Hilbert–Samuel theorem of Gillet–Soulé, Zhang and Moriwaki, applied to the single model $\mathcal{Y}_B$, identifies the leading growth of the lattice of sections with $P_B(u)$. Zhang's inequality for successive minima forces P_B to vanish on the Weyl walls up to $O(\log B/B)$, and an antisymmetry under the simple reflections puts it in the form $\mathrm{vol} \cdot (h_{\mathrm{Fl}}(u) - \alpha\,\mathrm{Sh}(u) - \beta\,\mathrm{Tot}(u))$ up to $O(\log B/B)$. An exact identity at the good primes relates P_B and $P_{B'}$ prime by prime. The constants of the asymptotic steps depend on the model, hence on the level, but the exact identity allows $B \to \infty$, and in the limit these constants disappear. What remains is a point-height constant (a slope bound of Bost's type and the formula of Chowla and Selberg for $\mathbb{Q}(\sqrt{-d})$), the constant 4 of a bound for cusp weights, the constant 4 of the first Chern class, the constant 2 of the tail over the primes above B, and the constant 60 of a bound of Mertens type, none of which depends on the level. The published inputs are the arithmetic Hilbert–Samuel theorem for each fixed model, Zhang's theorem on successive minima, Hirzebruch–Mumford proportionality, and the existence of toroidal compactifications with their integral models. The construction of the tower, the exact identity at the good primes, and the limit $B \to \infty$ are not written out here, and Hypothesis 15.69 is not proved here. The uniformity in the level rests on the exactness of the identity at the good primes.

15.14.5 Assembly of Hypothesis 15.16

Theorem 15.70. *Assume Proposition 15.57, Proposition 15.53, Hypothesis 15.58 for each j, and the bounds of Theorems 16.6 and 17.37 for the local terms. Then Hypothesis 15.16 holds with $\mu_0 = \mu^{(0)}(R^\vee)$, $P_{\mathrm{pass}} = \mu^{(1)} - \mu^{(0)}$ and $E_b = E_{l3} + LG + \sum_j (E_{1,j} + E_{2,j}) + E_3$.*

Proof. Proposition 15.57 bounds P_{pass}. Proposition 15.53 bounds $\mu_0 - \langle \log \kappa \rangle$ by $\sum_j \mathrm{lcv}(M_j)/D$ plus the local terms, which Theorem 17.37 and Theorem 16.6 bound by $\mathcal{L} + LG + \sum_j (E_{1,j} + E_{2,j}) + E_3$. Hypothesis 15.58 bounds $\sum_j \mathrm{lcv}(M_j)/D$ by $\sum_{j,\rho} (p_j - q_j) w_{j\rho} \log \Omega + E_{l3}$ with $E_{l3} = C \sum_j U_j$. □

16 The local terms and hypothesis (3.5)

This section defines the numbers E_u of Proposition 3.1 and proves hypothesis (3.5) with

$$g = \log(g_0 g_1), \qquad e_5 = C_E\, m\, B_0, \qquad B_0 = f\, m\, b\, L_\varepsilon = 2\, n\, m\, b\, x^\varepsilon,$$

where g_0, g_1 are the two primes of (D5) and C_E is a constant that depends on ε alone. The number E_u is a sum over the finite places of the field of the lattices, one term for each lift, and each term measures the failure of integrality of one lift at one place. Each lift loses the amount $A \log g_i$ at the places above the level prime g_i, so E_u has the main term $mA \log(g_0 g_1)$, and Proposition 3.1 cancels it against the term mg of (3.6). All other contributions to E_u are bounded by a multiple of mB_0, which is the second term of $\mathcal{B}$ in (4.8).

Section 17 treats the places at which the local type of the lifts is twisted. There a second local quantity, the congruence defect $\eta_j(w)$ of the lattice of integral forms, is bounded by the lengths of Tate–Shafarevich groups of E through a congruence-module estimate, and the lengths are bounded by $\sqrt{b}$. That bound enters the heights of the period maps in (3.4) and the slope inequality (3.3). It does not enter (3.5).

16.1 Places and normalized lengths

Let L be the number field over which the hermitian lattices N_u, M_u and R of Part II are defined. It is a Galois extension of $\mathbb{Q}$ that contains E, and in particular it contains $K = \mathbb{Q}(\sqrt{-d})$. For a finite place w of L write p_w for its residue characteristic, e_w and f_w for its ramification index and residue degree over $\mathbb{Q}$, v_w for the normalized valuation with $v_w(L^\times) = \mathbb{Z}$, and

$$\ell_w = \frac{f_w \log p_w}{[L : \mathbb{Q}]}, \qquad \sigma(w, y) = 2\, \ell_w\, e_w\, y \qquad (y \in \mathbb{R}).$$

The number $\sigma(w, y)$ is the change in the normalized slope of a hermitian lattice over $\mathcal{O}_L$ that is rescaled by ω_w^{-y} at w and its conjugate place, in the normalization of slopes that Part II uses for μ_u^N, μ_u^M and μ^C.

Lemma 16.1. *Let p be a prime number.*

(i) $\sum_{w \mid p} \ell_w e_w = \log p$.

(ii) For $y \in L^\times$, $\sum_{w \mid p} \ell_w\, v_w(y) = \dfrac{v_p\big(N_{L/\mathbb{Q}}(y)\big) \log p}{[L : \mathbb{Q}]}$.

Proof. (i) The places above p satisfy $\sum_{w \mid p} e_w f_w = [L : \mathbb{Q}]$, so $\sum_{w \mid p} \ell_w e_w = \log p \sum_{w \mid p} e_w f_w / [L : \mathbb{Q}] = \log p$.

(ii) The fractional ideal $y\mathcal{O}_L$ factors as $\prod_w \mathfrak{p}_w^{v_w(y)}$, and the absolute norm of $\mathfrak{p}_w$ is $p_w^{f_w}$. Hence $|N_{L/\mathbb{Q}}(y)| = \prod_w p_w^{f_w v_w(y)}$, and the exponent of p in this product is $v_p(N_{L/\mathbb{Q}}(y)) = \sum_{w \mid p} f_w v_w(y)$. Multiplying by $\log p / [L : \mathbb{Q}]$ gives (ii). □

Lemma 16.2. *Let g be a prime number and $\pi \in \mathcal{O}_K$ with $\pi\bar{\pi} = g$. For every automorphism τ of L,*

$$\sum_{w \mid g} \ell_w\, v_w\big(\tau(\bar{\pi})\big) = \frac{1}{2} \log g.$$

Proof. Since $L/\mathbb{Q}$ is Galois, $N_{L/\mathbb{Q}}(\tau(\bar{\pi})) = N_{L/\mathbb{Q}}(\bar{\pi})$, and since $\bar{\pi} \in K$, transitivity of the norm gives $N_{L/\mathbb{Q}}(\bar{\pi}) = N_{K/\mathbb{Q}}(\bar{\pi})^{[L:K]} = (\bar{\pi}\pi)^{[L:K]} = g^{[L:K]}$. Lemma 16.1(ii) with $p = g$ gives $[L : K] \log g / [L : \mathbb{Q}] = \frac{1}{2} \log g$. □

Lemma 16.3. *Let $n \geq 1$ be an integer and P a finite set of primes. Then $\sum_{p \in P} v_p(n) \log p \leq \log n$.*

Proof. The integer $\prod_{p\in P} p^{v_p(n)}$ divides n, so it is at most n, and its logarithm is the sum on the left. □

Lemma 16.4. $\displaystyle\sum_p \frac{\log p}{p^2-1} < 1$, *the sum running over all primes.*

Proof. The terms with $p \le 13$ are 0.2310, 0.1374, 0.0671, 0.0405, 0.0200 and 0.0152, each rounded up in the fourth decimal, with sum less than 0.512. For $t \ge e^{1/2}$ the function $\log t/t^2$ is decreasing, and for an integer $n \ge 17$ we have $n^2 - 1 \ge \frac{255}{256} n^2$. Hence

$$\sum_{p\ge 17} \frac{\log p}{p^2-1} \le \frac{256}{255} \sum_{n\ge 17} \frac{\log n}{n^2} \le \frac{256}{255} \int_{16}^{\infty} \frac{\log t}{t^2}\, dt = \frac{256}{255} \cdot \frac{\log 16 + 1}{16} < 0.237.$$

The total is less than 0.75. □

16.2 The local exponent data

Fix admissible data at the scale x (Definition 6.1) and $u_0 \in A_0$. Part II attaches to them the following objects. Each item names the section in which the object is constructed.

(L1) The finite set W of indices u of Proposition 3.1, and for each $u \in W$ a prime $T_u \notin \{g_0, g_1, d\}$ with $T_u = \varpi_u \bar{\varpi}_u$ for some $\varpi_u \in \mathcal{O}_K$ (Section 18).

(L2) The elements $\pi_0, \pi_1 \in \mathcal{O}_K$ of (D5), with $\pi_i \bar{\pi}_i = g_i$.

(L3) A positive integer D_Γ, a common multiple of the orders of the stabilizers in the class sums that define the lifts $K_1, \dots, K_m$ (Section 11, condition (D1)).

(L4) For $1 \le j \le m$ and each element ρ of the CM type Φ, the node $w_j(\rho) = w_{u_0,j}(\rho) > 0$ of Theorem A (Section 18). For $u \in W$, $1 \le j \le m$ and $\rho \in \Phi$ a real number $n_{u,j}(\rho)$, the exponent of $\bar{\pi}_i$ in the j-th lift at ρ, which is the same for $i = 0$ and $i = 1$. Here $\Phi \subset \mathcal{G} = \mathrm{Gal}(E/\mathbb{Q})$ as in Section 3.2, and we write τ_ρ for an automorphism of L whose restriction to E is ρ. The valuations $v_w(\tau_\rho(y))$ for $y \in E$ below do not depend on the choice of τ_ρ. A real number $\Delta_j \ge 0$, the Fock degree of the polynomial model of the weight of level j, and a real number $\kappa_j \ge 0$ (Section 11).

(L5) Real numbers $\varepsilon_u^T \ge 0$ and ε_j^d, and for each prime p in a finite set P^{in} of primes $p \le X''$ with $X'' = x\, n\, X'$ a real number $\varepsilon_{j,p}^{\mathrm{in}}$ (Sections 12 and 18). The set P^{in} consists of the primes at which Section 9 and Section 12 attach a gain or a loss to the lifts.

For $u \in W$ let S_u be the set of places $w \mid T_u$ of L such that $v_w(\tau_\rho(\varpi_u)) > 0$ for every $\rho \in \Phi$.

Definition 16.5. Let $u \in W$, $1 \le j \le m$, and let w be a finite place of L with residue characteristic $p = p_w$. Define $E_{u,j}(w)$ by the first applicable line.

$$E_{u,j}(w) = \begin{cases} \varepsilon_u^T & \text{if } p = T_u \text{ and } w \in S_u, \\ 0 & \text{if } p = T_u \text{ and } w \notin S_u, \\ v_p(D_\Gamma) - \dfrac{\Delta_j}{2} + \kappa_j + \dfrac{1}{e_w} \displaystyle\sum_{\rho\in\Phi} n_{u,j}(\rho)\, v_w\big(\tau_\rho(\bar{\pi}_i)\big) & \text{if } p = g_i,\ i \in \{0,1\}, \\ v_p(D_\Gamma) + \varepsilon_j^d & \text{if } p = d, \\ v_p(D_\Gamma) + \varepsilon_{j,p}^{\mathrm{in}} & \text{if } p \in P^{\mathrm{in}}, \\ v_p(D_\Gamma) & \text{otherwise.} \end{cases}$$

Put

$$E_{u,j} = \sum_{w} \sigma\big(w, E_{u,j}(w)\big), \qquad E_u = \sum_{j=1}^{m} E_{u,j}.$$

The sum over w is finite, because $E_{u,j}(w) = 0$ unless p_w lies in the finite set $\{T_u, g_0, g_1, d\} \cup P^{\mathrm{in}} \cup \{p : p \mid D_\Gamma\}$. The same numbers $E_{u,j}(w)$ enter the lattice N_u in hypothesis (3.2) and the integrality of the period maps in (3.3). Section 16.5 states the form in which those hypotheses use them.

For the main term put $n_{u,j} = \sum_{\rho \in \Phi} n_{u,j}(\rho)$ and $A = \sum_{t \in T} a_t$. The bound for E_u uses the following six inequalities, in which $C_g, C_T, C_\Gamma, C_d, C_{\mathrm{in}} \geq 0$ are constants and $\delta_j \geq 0$.

(E1) $|n_{u,j} - \Delta_j - A| \leq \delta_j$.

(E2) $(\delta_j + 2\kappa_j) \log(g_0 g_1) \leq C_g \, B_0$.

(E3) $2\, \varepsilon_u^T \log T_u \leq C_T \, B_0$.

(E4) $2 \log D_\Gamma \leq C_\Gamma \, B_0$.

(E5) $2 \log d \, |\varepsilon_j^d| \leq C_d \, B_0$.

(E6) $\sum_{p \in P^{\mathrm{in}}} 2 \log p \, |\varepsilon_{j,p}^{\mathrm{in}}| \leq C_{\mathrm{in}} \, B_0$.

Inequality (E1) is the degree identity of the lifts at the level primes. Subsection 16.4 derives (E2) to (E5) from size bounds for the data and proves (E6) for the part of P^{in} whose loss term has the closed form (16.4).

16.3 The bound for the local terms

Theorem 16.6. *Let $u \in W$ and $1 \leq j \leq m$, let the data (L1) to (L5) be given, and suppose that $g_0 \neq g_1$ and that the inequalities (E1) to (E6) hold for u and j. Then*

$$\big|E_{u,j} - A \log(g_0 g_1)\big| \leq C_E \, B_0, \qquad C_E = C_g + C_T + C_\Gamma + C_d + C_{\mathrm{in}}.$$

Proof. We split the sum $E_{u,j} = \sum_w \sigma(w, E_{u,j}(w))$ into the places above g_0, above g_1, above T_u, and the remaining places. The four sets of places are disjoint because T_u, g_0 and g_1 are distinct primes.

The places above g_i. Fix i and put $c_i = v_{g_i}(D_\Gamma) - \frac{\Delta_j}{2} + \kappa_j$. For $w \mid g_i$ the definition gives

$$\sigma\big(w, E_{u,j}(w)\big) = 2\ell_w e_w c_i + 2\ell_w \sum_{\rho \in \Phi} n_{u,j}(\rho)\, v_w\big(\tau_\rho(\bar{\pi}_i)\big).$$

Summing over $w \mid g_i$, Lemma 16.1(i) turns the first term into $2c_i \log g_i$, and Lemma 16.2 turns the second into $\sum_\rho n_{u,j}(\rho) \log g_i = n_{u,j} \log g_i$. Hence

$$\begin{aligned} \sum_{w \mid g_i} \sigma\big(w, E_{u,j}(w)\big) &= \big(2 v_{g_i}(D_\Gamma) - \Delta_j + 2\kappa_j + n_{u,j}\big) \log g_i \\ &= A \log g_i + \big(n_{u,j} - \Delta_j - A + 2\kappa_j + 2 v_{g_i}(D_\Gamma)\big) \log g_i. \end{aligned}$$

By (E1) the absolute value of the bracket minus $2v_{g_i}(D_\Gamma)$ is at most $\delta_j + 2\kappa_j$, so the sum over the places above g_0 and g_1 differs from $A\log(g_0g_1)$ by at most

$$(\delta_j + 2\kappa_j)\log(g_0g_1) + 2v_{g_0}(D_\Gamma)\log g_0 + 2v_{g_1}(D_\Gamma)\log g_1.$$

The places above T_u. Here $E_{u,j}(w) = \varepsilon_u^T \geq 0$ for $w \in S_u$ and $E_{u,j}(w) = 0$ otherwise. By Lemma 16.1(i),

$$0 \leq \sum_{w|T_u} \sigma\big(w, E_{u,j}(w)\big) = 2\varepsilon_u^T \sum_{w\in S_u} \ell_w e_w \leq 2\varepsilon_u^T \log T_u.$$

The remaining places. Let $p \notin \{T_u, g_0, g_1\}$. For $w \mid p$ the definition gives $E_{u,j}(w) = v_p(D_\Gamma) + \lambda_p$, where $\lambda_d = \varepsilon_j^d$, $\lambda_p = \varepsilon_{j,p}^{\mathrm{in}}$ for $p \in P^{\mathrm{in}} \setminus \{d\}$, and $\lambda_p = 0$ otherwise. The value does not depend on $w \mid p$, so Lemma 16.1(i) gives $\sum_{w|p}\sigma(w, E_{u,j}(w)) = 2(v_p(D_\Gamma) + \lambda_p)\log p$. Summing over these primes,

$$\Big|\sum_{p_w\notin\{T_u,g_0,g_1\}} \sigma\big(w, E_{u,j}(w)\big)\Big| \leq 2\sum_{p\notin\{T_u,g_0,g_1\}} v_p(D_\Gamma)\log p + 2\log d\,|\varepsilon_j^d| + \sum_{p\in P^{\mathrm{in}}} 2\log p\,|\varepsilon_{j,p}^{\mathrm{in}}|.$$

Conclusion. The terms $2v_p(D_\Gamma)\log p$ occur above for $p = g_0$, $p = g_1$ and for the primes outside $\{T_u, g_0, g_1\}$, each prime once, so by Lemma 16.3 their total is at most $2\log D_\Gamma$. Adding the three estimates,

$$\begin{aligned}\big|E_{u,j} - A\log(g_0g_1)\big| \leq (\delta_j + 2\kappa_j)\log(g_0g_1) + 2\varepsilon_u^T\log T_u + 2\log D_\Gamma \\ + 2\log d\,|\varepsilon_j^d| + \sum_{p\in P^{\mathrm{in}}} 2\log p\,|\varepsilon_{j,p}^{\mathrm{in}}|,\end{aligned}$$

and (E2) to (E6) bound the five terms by C_gB_0, C_TB_0, $C_\Gamma B_0$, C_dB_0 and $C_{\mathrm{in}}B_0$. □

Corollary 16.7. *Suppose that the hypotheses of Theorem 16.6 hold for every $u \in W$ and every $1 \leq j \leq m$. Put $g = \log(g_0g_1)$ and $e_5 = C_E\, m\, B_0$. Then for every $u \in W$*

$$mAg - e_5 \leq E_u \leq mAg + e_5,$$

so hypothesis (3.5) *of Proposition 3.1 holds. Moreover $e_5 \leq C_E\,\mathcal{B}$, with $\mathcal{B}$ the number* (4.8).

Proof. Summing Theorem 16.6 over $1 \leq j \leq m$ gives $|E_u - mA\log(g_0g_1)| \leq mC_EB_0 = e_5$. The second term of (4.8) is $m \cdot 2nmbx^\varepsilon = mB_0$, and the other terms of (4.8) are nonnegative, so $mB_0 \leq \mathcal{B}$. □

Remark 16.8. The number g of Proposition 3.1 occurs in (3.5) and in (3.6), and the proof of Proposition 3.1 cancels the two occurrences. Corollary 16.7 fixes $g = \log(g_0g_1)$. Hypothesis (3.6) must be proved with the same number, that is, in the form

$$\log T_u \geq \sum_{j=1}^{m} \frac{1}{f} \sum_{\rho\in\Phi} \log\frac{w_j(\rho)v}{\pi} + m\log(g_0g_1) - e_6.$$

This is consistent with the choice of T_u as a prime of size about $(g_0g_1)^m$ times the product of the nodes, and with the congruence $\omega_u \equiv 1$ modulo $Q_S qd\,(g_0g_1)^2$ of Section 18.

Remark 16.9. The constant C_E depends on ε through C_{in} (Subsection 16.4). Theorem A asks for an absolute constant C in $x^{C\varepsilon}\mathcal{B}$, and Corollary 16.7 gives $e_5 \leq C_E\mathcal{B} \leq x^\varepsilon\mathcal{B}$ for every x with $x^\varepsilon \geq C_E$. The threshold Y_1 of Theorem A may depend on ε, so this contributes 1 to the absolute constant C.

16.4 The inequalities (E2) to (E6) from the sizes

We use the scale (5.17), $L_\varepsilon = x^\varepsilon$, $b = e^{L_\varepsilon}$, $a = xb$, $X' = \exp(x^{\varepsilon/3})$, $b' = a/X'$, the ranks of Proposition 5.15, and

$$N_m = k + m(t_r + 1),$$

the dimension of the polynomial model of the weight at the last level of the chain.

Lemma 16.10. *Let $0 < \varepsilon < \frac{1}{52}$. There is x_0 such that for admissible data at a scale $x \geq x_0$*

(i) $\log g_i \leq 9L_\varepsilon$ *for* $i = 0, 1$, *so* $\log(g_0 g_1) \leq 18L_\varepsilon$;

(ii) $\log d \leq L_\varepsilon$;

(iii) $t_r \leq x^{401}$ *and* $N_m \leq x^{402}$;

(iv) $N_m^2\, X'' \log a \leq b$, *where* $X'' = x\, n\, X'$.

Proof. (i) By (D5), $g_i \leq 2(kb)^8$, so $\log g_i \leq \log 2 + 8\log k + 8L_\varepsilon$. Proposition 5.15 gives $k \leq 2x^{9/2+10\varepsilon} \leq 2x^5$, so $\log g_i \leq 9 + 40\log x + 8L_\varepsilon \leq 9L_\varepsilon$ once $x^\varepsilon \geq 9 + 40\log x$.

(ii) By (D1) the prime d satisfies the conclusion of Theorem 5.12 with $K = \frac{3}{2}$ and $z' = z\log^2 Y$, $z = Y^{\varepsilon/5}$, so $\log d \leq z' = Y^{\varepsilon/5}\log^2 Y$. Since $Y \leq x$, this is at most $x^{\varepsilon/5}\log^2 x \leq x^\varepsilon$ for large x.

(iii) Since $\varepsilon < \frac{1}{52}$, Proposition 5.15 and (D1) give $f = 2n \leq 4x^{2+8\varepsilon} \leq 4x^{2.2}$, $1 \leq m \leq 2\sqrt{x}$, $k \leq 2x^{9/2+10\varepsilon} \leq 2x^{4.7}$, $r = k + m - 1 \leq 3x^{4.7}$, and $L_\varepsilon^2 = x^{2\varepsilon} \leq x^{0.04}$. The definition of n_W in Section 5.8 and $m \geq 1$ give

$$n_W \leq 10\, r\big(f + f^2 k^3 L_\varepsilon^2\big) + 8 \leq 10 \cdot 3x^{4.7} \cdot 2 \cdot 16x^{4.4} \cdot 8x^{14.1} \cdot x^{0.04} + 8 \leq x^{23.5}$$

for large x. With $T_O = \lceil x^{377} \rceil \leq 2x^{377}$ and $t_r = 8fr + 4 + T_O n_W - r$ we get $t_r \leq 8fr + 4 + 2x^{400.5} \leq x^{401}$ for large x. Then $N_m = k + m(t_r + 1) \leq 2x^{4.7} + 2\sqrt{x}\,(x^{401} + 1) \leq x^{402}$ for large x.

(iv) We have $n \leq 2x^3$, $\log a = \log x + L_\varepsilon \leq 2L_\varepsilon$, and by (iii)

$$N_m^2 X'' \log a \leq x^{804} \cdot x \cdot 2x^3 \cdot \exp(x^{\varepsilon/3}) \cdot 2x^\varepsilon \leq \exp(x^\varepsilon) = b$$

for large x, because log of the left side is $x^{\varepsilon/3} + O(\log x)$. □

Lemma 16.11. *Let $0 < \varepsilon < \frac{1}{52}$ and let x_0 be as in Lemma 16.10. Let $c_1, \ldots, c_5 \geq 0$ be constants, and suppose that for admissible data at a scale $x \geq x_0$, for $u \in W$ and for $1 \leq j \leq m$,*

$$\delta_j \leq c_1\, f\, b, \qquad \kappa_j \leq c_2\, f\, m\, b, \tag{16.1}$$

$$\varepsilon_u^T \leq c_3\, f\, b, \qquad \log T_u \leq c_4\, m\, L_\varepsilon, \tag{16.2}$$

$$\log D_\Gamma \leq c_5\, f\, N_m \log(f N_m), \qquad |\varepsilon_j^d| \leq 2\, f\, b. \tag{16.3}$$

Then (E2) holds with $C_g = 18(c_1 + 2c_2)$, (E3) with $C_T = 2c_3c_4$, (E4) with $C_\Gamma = 2c_5$, and (E5) with $C_d = 4$, for $x \geq x_0$ enlarged if necessary.

Proof. Recall $B_0 = fmbL_\varepsilon$ and $m \geq 2$.

(E2) By Lemma 16.10(i), $(\delta_j + 2\kappa_j)\log(g_0 g_1) \leq (c_1 fb + 2c_2 fmb) \cdot 18L_\varepsilon \leq 18(c_1 + 2c_2)B_0$.

(E3) $2\varepsilon_u^T \log T_u \leq 2c_3 fb \cdot c_4 mL_\varepsilon = 2c_3c_4B_0$.

(E4) By Lemma 16.10(iii) and $f \leq 4x^3$, $\log(fN_m) \leq \log(4x^{405}) \leq L_\varepsilon$ and $N_m \leq x^{402} \leq b$ for large x. Hence $2\log D_\Gamma \leq 2c_5 fbL_\varepsilon \leq 2c_5 B_0$.

(E5) By Lemma 16.10(ii), $2\log d\,|\varepsilon_j^d| \leq 2L_\varepsilon \cdot 2fb \leq 4B_0$. □

The loss term at a prime $p \in P^{\mathrm{in}}$ has one of two forms. At the primes at which the lifts gain by the class of the element of Section 9 that is trivial, it has the closed form

$$\varepsilon_{j,p}^{\mathrm{in}} = c_{\mathrm{dig}}\, f\, N_m^2\, \log_p a + \frac{1}{p^2-1} \sum_{\rho \in \Phi} w_j(\rho) \tag{16.4}$$

with an absolute constant c_{dig}. At the other primes of P^{in} it is the orbit loss of Section 12, built from the solution of the linear programme of Section 9 along the Frobenius orbits. Write $P^{\mathrm{in}} = P^{\mathrm{C}} \sqcup P^{\mathrm{I}}$ accordingly.

Lemma 16.12. *Suppose that $0 \le w_j(\rho) \le 3b$ for every $\rho \in \Phi$, and that $x \ge x_0$ with x_0 as in Lemma 16.10. If $\varepsilon_{j,p}^{\mathrm{in}}$ has the form* (16.4) *for $p \in P^{\mathrm{C}}$, then*

$$\sum_{p \in P^{\mathrm{C}}} 2 \log p\, |\varepsilon_{j,p}^{\mathrm{in}}| \le (2c_{\mathrm{dig}} + 6)\, f\, b \le (c_{\mathrm{dig}} + 3)\, B_0.$$

Proof. Both terms of (16.4) are nonnegative. Since $\log p \cdot \log_p a = \log a$ and P^{C} consists of primes $p \le X''$, the first terms contribute at most $2c_{\mathrm{dig}} f N_m^2 \log a \cdot X'' \le 2c_{\mathrm{dig}} f b$ by Lemma 16.10(iv). Since $\sum_\rho w_j(\rho) \le 3fb$, Lemma 16.4 bounds the second terms by $2 \cdot 3fb \sum_p \log p/(p^2-1) \le 6fb$. Finally $fb \le \frac{1}{2} fmbL_\varepsilon = \frac{1}{2} B_0$. □

Hypothesis 16.13 (inert orbit losses)**.** Let $0 < \varepsilon < \frac{1}{52}$. There are C'_{in} and x_0 such that for admissible data at a scale $x \ge x_0$ and every $1 \le j \le m$, $\sum_{p \in P^{\mathrm{I}}} 2 \log p\, |\varepsilon_{j,p}^{\mathrm{in}}| \le C'_{\mathrm{in}} B_0$.

The orbit losses $\varepsilon_{j,p}^{\mathrm{in}}$ for $p \in P^{\mathrm{I}}$ come from the construction of Section 12. Hypothesis 16.13 is not proved here.

With Lemma 16.12 and Hypothesis 16.13, (E6) holds with $C_{\mathrm{in}} = c_{\mathrm{dig}} + 3 + C'_{\mathrm{in}}$.

Hypothesis 16.14 (exponents and sizes of the lifts)**.** Let $0 < \varepsilon < \frac{1}{52}$. There are constants $c_1, \dots, c_5$ and x_0 such that for admissible data at a scale $x \ge x_0$, the exponents and sizes of the lifts satisfy (E1) and (16.1) to (16.3) for every $u \in W$ and $1 \le j \le m$, and $0 \le w_j(\rho) \le 3b$ for every $\rho \in \Phi$.

The degree identity (E1) is a computation with the highest weights of the lifts of Section 11, and (16.1) to (16.3) are size bounds for the data of Section 18. Hypothesis 16.14 is not proved here.

16.5 How hypotheses (3.2) and (3.3) use the local terms

The numbers $E_{u,j}(w)$ of Definition 16.5 are bounds for the denominators of the lifts. With the gain $\mathrm{gain}_j(w)$ and the local terms $F_j(w)$ of Sections 11 and 12, the lattice N_u of (3.2) is the rescaling at each place w with $p_w \ne T_u$ of a sublattice of the base lattice by the exponent

$$s_u(w) = \sum_{j=1}^{m} \big(\mathrm{gain}_j(w) + F_j(w) - E_{u,j}(w)\big),$$

so that μ_u^N differs from the slope of the sublattice by $\sum_w \sigma(w, s_u(w))$, which equals $\Gamma + \mathcal{L} - E_u$ up to the terms $\tau(w)$ of the loss and the rounding to integers. This is why E_u enters (3.2) with a minus sign. Hypothesis (3.3) needs, at each place w with $p_w \ne T_u$, that the period of a lifted

form on the rescaled lattice is integral at w. The argument uses this in the following form. For φ in the base lattice and F an integral form of the local type μ_w of the lifts at w, the period $y \in L$ of the lift of φ against F is 0 or satisfies

$$v_w(y) \geq e_w \sum_{j=1}^{m} \big(F_j(w) - E_{u,j}(w)\big).$$

The proof of this integrality belongs to Sections 12 and 13, and they use Definition 16.5 for $E_{u,j}(w)$.

17 The twisted places

At the places where the local type of the lifts is twisted, the lattice of integral lifted forms differs from the lattice that the period pairing sees, and a congruence module measures the difference. This section defines the congruence defects $\eta_j(w)$, states the two bounds that control them, and proves the parts of those bounds that reduce to commutative algebra and Galois cohomology. Theorem 17.37 combines the two bounds. Its conclusion enters the heights of the period maps in (3.4) and the frame parameters in (3.3). It does not enter (3.5), which Corollary 16.7 proves without it.

The first bound, Hypothesis 17.11, bounds the lengths of Tate–Shafarevich groups of the host field on windows of scales. Subsection 17.4 shows that it implies a bound in the direction of Vandiver's conjecture beyond the bounds published by Soulé [68] and by de la Torre Durán [22], and traces the argument for it to an automorphy statement for residually reducible Galois representations that is not a published theorem.

17.1 Congruence defects

For each finite place w of L the lifts have a local type μ_w, which is either the standard type μ_0 or a twisted type μ_w^{tw}, and $\mu_{\bar{w}} = \mu_w$. We call w *twisted* when $\mu_w \neq \mu_0$. At a twisted place the integral structure of the twisted lifts at level j differs from the integral structure of the eigen-components that the period pairing sees, and the difference is measured by a congruence module.

Definition 17.1. Let w be a twisted place, $l = p_w$, and $1 \leq j \leq m$. Let $\mathcal{O}$ be the ring of integers of the completion of L at w, let Λ be the $\mathcal{O}$-lattice spanned by the integral forms of level j and local type μ_w^{tw}, and let $\mathbb{T}$ be the $\mathcal{O}$-algebra of Hecke operators acting on Λ. For a maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ split $\Lambda_{\mathfrak{m}} \otimes \mathbb{Q}$ into the sum $V_{\mathrm{good}} \oplus V_{\mathrm{bad}}$ of the eigen-components whose eigensystems occur in the lifts of level j and of the others. Put

$$\eta_j(w) = \sum_{\mathfrak{m}} \frac{\mathrm{length}_{\mathcal{O}} \Big(\big(V_{\mathrm{good}} \cap (\Lambda_{\mathfrak{m}} + V_{\mathrm{bad}})\big) / \big(\Lambda_{\mathfrak{m}} \cap V_{\mathrm{good}}\big) \Big)}{\mathrm{length}_{\mathcal{O}}(\mathcal{O}/l)},$$

the normalized length of the congruence module of V_{good} in Λ. For a non-twisted place put $\eta_j(w) = 0$.

The module $(V_{\mathrm{good}} \cap (\Lambda + V_{\mathrm{bad}}))/(\Lambda \cap V_{\mathrm{good}})$ is the projection of Λ to V_{good} modulo the forms of Λ that lie in V_{good}. It vanishes when no good eigensystem is congruent to a bad one modulo $\mathfrak{m}$. For $1 \leq j \leq m$ we write d_j^0 for the rank of the space of lifted forms of level j and standard type μ_0, and $N_c = |T_2|\, k\, b\, x^{2\varepsilon}/n^2$, so that $\mathcal{B} = N_T a(1 + m^2/k) + m(B_0 + N_c)$.

17.2 Tate–Shafarevich groups

Let F be a number field, $\overline{F}$ an algebraic closure, $G_F = \mathrm{Gal}(\overline{F}/F)$ with the Krull topology, and l a prime. Let $\chi\colon G_F \to \mathbb{Z}_l^\times$ be the l-adic cyclotomic character, so that $\sigma(\zeta) = \zeta^{\chi(\sigma)}$ for every l-power root of unity ζ. For an integer e let M_e be the discrete group $\mathbb{Q}_l/\mathbb{Z}_l$ on which σ acts by multiplication by $\chi(\sigma)^{-e}$. For a finite place v of F fix a place of $\overline{F}$ above it, with decomposition group D_v and inertia group I_v. Cohomology is continuous cohomology with discrete coefficients.

Definition 17.2. $Ш_l(F,e)$ is the subgroup of $H^1(G_F, M_e)$ of classes c with $c|_{D_v} = 0$ for every $v \mid l$ and $c|_{I_v} = 0$ for every finite v. For a finite free $\mathbb{Z}_l$-algebra O put $M_{e,O} = O \otimes_{\mathbb{Z}_l} M_e$, with G_F acting on the second factor, and define $Ш_l(F,e;O) \subset H^1(G_F, M_{e,O})$ by the same conditions. For a finite abelian l-group X write $\lambda(X) = \log|X| = \mathrm{len}_{\mathbb{Z}_l}(X)\log l$.

The conditions do not depend on the chosen places above v, because restriction to conjugate subgroups differs by an automorphism of cohomology. There is no condition at archimedean places. For odd l none is needed, because H^1 of a group of order 2 with coefficients in an l-primary group vanishes.

Lemma 17.3. *Let $v \nmid l$ be a finite place of F and $e \geq 1$. An element of $H^1(D_v, M_e)$ that vanishes on I_v is zero.*

Proof. The group I_v acts trivially on M_e, because χ is unramified at v. Inflation–restriction gives an exact sequence $0 \to H^1(D_v/I_v, M_e) \to H^1(D_v, M_e) \to H^1(I_v, M_e)$, so a class that vanishes on I_v comes from $H^1(D_v/I_v, M_e)$. The group D_v/I_v is procyclic, topologically generated by a Frobenius element φ, and M_e is a discrete torsion module, so $H^1(D_v/I_v, M_e) \cong M_e/(\varphi - 1)M_e$. The element φ acts on M_e as multiplication by $a = Nv^{-e}$. Since $Nv > 1$ and $e \geq 1$, $a - 1$ is a nonzero element of $\mathbb{Q}_l$, say $a - 1 = l^k u$ with $u \in \mathbb{Z}_l^\times$. Multiplication by $l^k u$ is surjective on the divisible group $\mathbb{Q}_l/\mathbb{Z}_l$, so $M_e/(\varphi - 1)M_e = 0$. □

Lemma 17.4. *Let $e \geq 1$ and let S be a finite set of finite places of F containing the places above l. Let $F_S \subset \overline{F}$ be the maximal extension of F unramified outside S and the archimedean places, and $G_{F,S} = \mathrm{Gal}(F_S/F)$. Inflation identifies $Ш_l(F,e)$ with*

$$\ker\Big(H^1(G_{F,S}, M_e) \longrightarrow \prod_{v \in S} H^1(D_v, M_e)\Big).$$

In particular $Ш_l(F,e)$ does not depend on S.

Proof. The module M_e is unramified outside l, so $H = \mathrm{Gal}(\overline{F}/F_S)$ acts trivially on it. Inflation–restriction gives $0 \to H^1(G_{F,S}, M_e) \to H^1(G_F, M_e) \to \mathrm{Hom}(H, M_e)$. The group H is the closed normal subgroup of G_F generated by the inertia groups at places outside S. A class $c \in H^1(G_F, M_e)$ that vanishes on every I_v restricts to a homomorphism on H that is trivial on these generators and is invariant under conjugation by G_F (as an element of $\mathrm{Hom}(H, M_e)^{G_F}$), hence trivial on H. So c is inflated from $G_{F,S}$. Conversely an inflated class vanishes on I_v for $v \notin S$. At $v \notin S$ an unramified class is zero by Lemma 17.3. At $v \in S$ with $v \nmid l$, vanishing on D_v implies vanishing on I_v, and vanishing on I_v implies vanishing on D_v by Lemma 17.3. At $v \mid l$ both definitions impose vanishing on D_v, which implies vanishing on I_v. This proves the identification, and it shows that the group does not depend on S. □

Lemma 17.5. *Let O be a finite free $\mathbb{Z}_l$-algebra which is a discrete valuation ring with ramification index $e(O)$ over $\mathbb{Z}_l$. Then $Ш_l(F,e;O) \cong O \otimes_{\mathbb{Z}_l} Ш_l(F,e)$ as O-modules, and if $Ш_l(F,e)$ is finite,*

$$\operatorname{len}_O Ш_l(F,e;O) = e(O)\,\operatorname{len}_{\mathbb{Z}_l} Ш_l(F,e), \qquad \operatorname{len}_O(O/lO) = e(O).$$

Proof. Choose a $\mathbb{Z}_l$-basis $o_1,\dots,o_r$ of O. Then $M_{e,O} = \bigoplus_i o_i \otimes M_e$ as G_F-modules, so $H^1(G_F, M_{e,O}) = \bigoplus_i H^1(G_F, M_e)$, compatibly with restriction to every closed subgroup. Hence $Ш_l(F,e;O) = \bigoplus_i o_i \otimes Ш_l(F,e) = O \otimes_{\mathbb{Z}_l} Ш_l(F,e)$, and the O-module structures agree because O acts on the first tensor factor in both. The functor $X \mapsto O \otimes_{\mathbb{Z}_l} X$ is exact, and it sends $\mathbb{Z}/l$ to O/lO, whose O-length is $e(O)$. By additivity of length along a composition series of X, $\operatorname{len}_O(O \otimes X) = e(O)\operatorname{len}_{\mathbb{Z}_l} X$ for every $\mathbb{Z}_l$-module X of finite length. □

Lemma 17.6. *Let l be odd, E a CM field, $c \in \mathrm{Gal}(E/\mathbb{Q})$ complex conjugation, $e \geq 1$. The natural action of c on $H^1(G_E, M_e)$ preserves $Ш_l(E,e)$ and squares to the identity. If $Ш_l(E,e)$ has finite length then* $\operatorname{len} Ш_l(E,e) = \operatorname{len} Ш_l(E,e)^{c=(-1)^e} + \operatorname{len} Ш_l(E,e)^{c=(-1)^{e+1}}$.

Proof. The natural action of $\sigma \in G_{\mathbb{Q}}$ on $H^1(G_E, M_e)$ is $(\sigma \cdot f)(g) = \sigma f(\sigma^{-1} g \sigma)$ on cocycles. It factors through $\mathrm{Gal}(E/\mathbb{Q})$, permutes the decomposition and inertia groups, and so preserves $Ш_l(E,e)$. The action of c is an involution. Since 2 is a unit in $\mathbb{Z}_l$, $x = \frac{1}{2}(x+cx) + \frac{1}{2}(x-cx)$ splits $Ш_l(E,e)$ as the direct sum of the c-eigenspaces for ± 1, and length is additive. □

Proposition 17.7 (Restriction to abelian fields)**.** *Let $N \geq 1$, let $E \subset \mathbb{Q}(\mu_N)$ be a subfield, let $l \nmid N$ be an odd prime and let $e \geq 1$ with $(l-1) \nmid e$. Restriction $H^1(G_{\mathbb{Q}}, M_e) \to H^1(G_E, M_e)$ maps $Ш_l(\mathbb{Q},e)$ injectively into $Ш_l(E,e)$, onto classes fixed by $\mathrm{Gal}(E/\mathbb{Q})$. If E is a CM field and e is even, the image lies in $Ш_l(E,e)^{c=(-1)^e}$. In particular* $\operatorname{len} Ш_l(\mathbb{Q},e) \leq \operatorname{len} Ш_l(E,e)$.

Proof. Restriction of a class that vanishes on D_v (respectively I_v) vanishes on $D_w = D_v \cap G_E$ (respectively I_w) for every place w of E above v, so the image lies in $Ш_l(E,e)$. The kernel of restriction is $H^1(\mathrm{Gal}(E/\mathbb{Q}), M_e^{G_E})$. Since $l \nmid N$, the fields $\mathbb{Q}(\mu_N)$ and $\mathbb{Q}(\mu_{l^\infty})$ are linearly disjoint over $\mathbb{Q}$, because a subfield of both is unramified at every prime. Hence $\chi(G_E) \supseteq \chi(G_{\mathbb{Q}(\mu_N)}) = \mathbb{Z}_l^\times$. An element $x \in M_e[l]$ fixed by G_E satisfies $u^{-e}x = x$ for all $u \in \mathbb{Z}_l^\times$, and since $(l-1) \nmid e$ there is u with $u^{-e} \not\equiv 1 \pmod{l}$, so $x = 0$. A nonzero G_E-fixed element of M_e of order l^k would give the nonzero fixed element $l^{k-1}x$ of $M_e[l]$. So $M_e^{G_E} = 0$ and restriction is injective. A restricted class $f = c|_{G_E}$ satisfies $(\sigma \cdot f)(g) = \sigma c(\sigma^{-1} g \sigma) = c(g)$ modulo coboundaries, by the cocycle relation for c, so it is fixed by $\mathrm{Gal}(E/\mathbb{Q})$. For E CM and e even, c acts trivially on the image, and $1 = (-1)^e$. □

Proposition 17.8 (Class groups)**.** *Let l be an odd prime, $e \geq 1$ with $(l-1) \nmid e$, $F_l = \mathbb{Q}(\mu_l)$, $\Delta = \mathrm{Gal}(F_l/\mathbb{Q})$, $\omega\colon \Delta \to \mathbb{F}_l^\times$ the Teichmüller character, and $A = \mathrm{Cl}(F_l) \otimes \mathbb{Z}_l$. For a character θ of Δ let $A^{(\theta)} = \{a \in A : \delta a = \theta(\delta) a \text{ for all } \delta\}$. Then*

$$\dim_{\mathbb{F}_l} Ш_l(\mathbb{Q},e)[l] = \dim_{\mathbb{F}_l} \left(A/lA\right)^{(\omega^{-e})}.$$

In particular $Ш_l(\mathbb{Q},e) = 0$ if and only if $A^{(\omega^{-e})} = 0$.

Proof. Step 1. $H^0(G_{\mathbb{Q}}, M_e) = 0$ and $H^0(D_l, M_e) = 0$. The group D_l maps onto Δ (the prime l is totally ramified in F_l) and acts on $M_e[l]$ through ω^{-e}, which is nontrivial because $(l-1) \nmid e$. So $M_e[l]^{D_l} = 0$, and as in the proof of Proposition 17.7 this gives $M_e^{D_l} = 0$, and a fortiori $M_e^{G_{\mathbb{Q}}} = 0$.

Step 2. From $0 \to M_e[l] \to M_e \xrightarrow{l} M_e \to 0$ and Step 1, the map $H^1(G_{\mathbb{Q}}, M_e[l]) \to H^1(G_{\mathbb{Q}}, M_e)[l]$ is an isomorphism. For $v \neq l$ the group I_v acts trivially on M_e, so the kernel of $H^1(I_v, M_e[l]) \to H^1(I_v, M_e)$ is $M_e/lM_e = 0$. At l the kernel of $H^1(D_l, M_e[l]) \to H^1(D_l, M_e)$ is $H^0(D_l, M_e)/l = 0$. Hence $Ш_l(\mathbb{Q}, e)[l]$ is identified with the group $\mathcal{S}$ of $c \in H^1(G_{\mathbb{Q}}, \mathbb{F}_l(\omega^{-e}))$ that vanish on I_v for $v \neq l$ and on D_l.

Step 3. Since $|\Delta| = l - 1$ is prime to l, restriction gives an isomorphism $H^1(G_{\mathbb{Q}}, \mathbb{F}_l(\omega^{-e})) \cong H^1(G_{F_l}, \mathbb{F}_l(\omega^{-e}))^{\Delta} = \mathrm{Hom}(G_{F_l}, \mathbb{F}_l)^{\Delta}$, where G_{F_l} acts trivially on the coefficients and Δ-invariance of f means $f(\tilde{\delta} g \tilde{\delta}^{-1}) = \omega^{-e}(\delta) f(g)$. For $v \neq l$ the extension $F_l/\mathbb{Q}$ is unramified at v, so $I_v \subset G_{F_l}$, and c vanishes on I_v iff f does. Restriction from D_l to $D_l \cap G_{F_l} = D_\lambda$ is injective on $H^1(-, \mathbb{F}_l(\omega^{-e}))$ (index prime to l), so $c|_{D_l} = 0$ iff $f|_{D_\lambda} = 0$, where $\lambda = (1 - \zeta_l)$ is the prime above l.

Step 4. The field F_l is totally imaginary. By class field theory a homomorphism $G_{F_l} \to \mathbb{F}_l$ that vanishes on every inertia group factors through the Artin isomorphism $\mathrm{Art}\colon \mathrm{Cl}(F_l) \to \mathrm{Gal}(H/F_l)$ of the Hilbert class field H. Its restriction to D_λ is determined by its value on $\mathrm{Art}([\lambda])$, which is 0 because λ is principal. The Artin map satisfies $\tilde{\delta}\,\mathrm{Art}(a)\tilde{\delta}^{-1} = \mathrm{Art}(\delta a)$. So $\mathcal{S} \cong \mathrm{Hom}_{\Delta}(A/lA, \mathbb{F}_l(\omega^{-e}))$, homomorphisms with $f(\delta a) = \omega^{-e}(\delta) f(a)$. Since $|\Delta|$ is prime to l, $A/lA = \bigoplus_\theta (A/lA)^{(\theta)}$, and such an f vanishes on $(A/lA)^{(\theta)}$ for $\theta \neq \omega^{-e}$. Hence $\mathcal{S} \cong \mathrm{Hom}_{\mathbb{F}_l}((A/lA)^{(\omega^{-e})}, \mathbb{F}_l)$. Finally $Ш_l(\mathbb{Q}, e)$ is an l-primary torsion group, so it vanishes iff its l-torsion does, and $(A/lA)^{(\theta)} = A^{(\theta)}/lA^{(\theta)}$ vanishes iff $A^{(\theta)}$ does. □

For even e the character ω^{-e} is even. Vandiver's conjecture for l is the vanishing of $A^{(\theta)}$ for all even θ. By Proposition 17.8 it is equivalent to $Ш_l(\mathbb{Q}, e) = 0$ for all even e with $2 \leq e \leq l - 3$, since the trivial character contributes 0.

For comparison we record the following published identification. Let l be odd and $e \geq 1$ with $(l-1) \nmid e$. By Lemma 17.4 with $S = \{l\}$ and Poitou–Tate duality [46, Theorem I.4.10], $Ш_l(\mathbb{Q}, e)$ is Pontryagin dual to $H^2(\mathbb{Z}[1/l], \mathbb{Z}_l(1+e))$, and this group is $K_{2e}(\mathbb{Z}) \otimes \mathbb{Z}_l$ by the theorem of Rost and Voevodsky on the Quillen–Lichtenbaum conjecture for $\mathbb{Z}$ (see the survey of Weibel [77]). Nothing below depends on this identification.

17.3 The Selmer bound on windows

Definition 17.9 (Window pairs)**.** Let $0 < \varepsilon < \frac{1}{52}$ and $Y \geq 2$. A *window pair at scale* Y is a pair of primes (q, d) with $q \equiv 3 \pmod 8$, $\frac{1}{2} Y^{\varepsilon/5} < \log q \leq Y^{\varepsilon/5}$, $d \equiv 7 \pmod{16}$, $\frac{1}{2} Y^{\varepsilon/5} \log^2 Y < \log d \leq Y^{\varepsilon/5} \log^2 Y$, $q \mid d + 1$, and $p \mid d + 1$ for every odd prime $p \leq Y^{\varepsilon/5}$.

Definition 17.10 (χ-fields)**.** Let ψ_0 be a primitive Dirichlet character of conductor N_0, $e \geq 0$, and q, d odd primes. Put $N = \mathrm{lcm}(2^{e+2}, N_0, q, d)$. The field $E(\psi_0, e, q, d)$ is the subfield of $\mathbb{Q}(\mu_N)$ fixed by the automorphisms $\zeta \mapsto \zeta^a$ with $a \equiv \pm 1 \pmod{2^{e+2}}$, $\psi_0(a) = 1$, $\left(\frac{a}{q}\right) = \left(\frac{a}{d}\right) = 1$. For $q \equiv 3 \pmod 4$ it is a CM field.

Hypothesis 17.11 (Selmer bound on windows)**.** For every primitive Dirichlet character ψ_0, every $0 < \varepsilon < \frac{1}{52}$ and every function $e\colon [1, \infty) \to \mathbb{N}$ with $x^{2+8\varepsilon} \leq 2^{e(x)} \leq 2x^{2+8\varepsilon}$, there are $J \geq 1$ and X_0 with the following property. For every $X \geq X_0$ and every window pair (q, d) at scale $X^{1/J}$ there is Y with $X^{1/J} \leq Y \leq X^{1/2}$ such that for every $x \in [Y, Y^2]$, with $t = \lfloor x^{401} \rfloor$ and $E = E(\psi_0, e(x), q, d)$, every $Ш_l(E, e')$ with $0 < e' < t$ is finite and

$$\sum_{l} \sum_{1 \leq e' < t} \lambda\big(Ш_l(E, e')\big) \leq \exp\big(\tfrac{1}{2} x^{\varepsilon}\big),$$

the sum over all primes l (including $l = 2$).

Proposition 18.3 deduces (D6) by rescaling the window.
For a CM field E, $t \geq 1$ and $B \geq 0$ write

- $(\mathrm{P}_2)(E,t,B)$: $\sum_{1\leq e'<t} \mathrm{len}_{\mathbb{Z}_2}\, Ш_2(E,e') \leq B$, all finite;
- $(\mathrm{P}_\pm)(E,t,B)$: for odd l, the $c = \pm(-1)^{e'}$ parts are finite and

$$\sum_{l \text{ odd}} \log l \sum_{e'<t} \mathrm{len}\, Ш_l(E,e')^{c=\pm(-1)^{e'}} \leq B.$$

Theorem 17.12 (decomposition)**.** *Hypothesis 17.11 follows from the following two statements, with the notation of Hypothesis 17.11.*

(I) (Plus part and $l = 2$ on windows.) *For every ψ_0, ε and every pair of functions $e(x), t(x)$ with $2 \leq e(x)$, $2^{e(x)} \leq x^4$, $2 \leq t(x) \leq x^{401}$ for $x \geq 2$, there are $J, C, P, X_1 > 0$ such that for $X \geq X_1$ and primes $q \equiv 3\ (8)$, $d \equiv 7\ (16)$ prime to N_0, with $q, d > \max(P, 2X^{401}+1)$ and $\log(qd) \leq X^{\varepsilon/(4J)}$, there is Y with $X^{1/J} \leq Y \leq X^{1/2}$ such that for all $x \in [Y, Y^2]$ and $E = E(\psi_0, e(x), q, d)$: $(\mathrm{P}_2)(E, t(x), e^{x^{4\varepsilon/5}})$ and $(\mathrm{P}_+)(E, t(x), (\mathrm{cond}(E)t(x))^C + Ce^{x^{4\varepsilon/5}})$.*

(II) (Minus part.) *There is C such that for all such E of degree n and conductor f, and $t \geq 2$: $(\mathrm{P}_-)(E,t,\beta)$ with $\beta = 3nt^2\log(tf) + C\sum_{l\mid o,\ l \text{ odd}} l\, n^2t^2\log^2(tlf)$, where o is the order of ψ_0.*

Proof. Fix ψ_0, ε and $e(x)$ as in Hypothesis 17.11, and put $t(x) = \lfloor x^{401} \rfloor$. For $x \geq 2$, $2^{e(x)} \leq 2x^{2+8\varepsilon} \leq x^4$ and $2^{e(x)} \geq x^2 \geq 4$, so (I) applies; let $J_{\mathrm{I}}, C, P, X_1$ be its constants and put $J = \max(J_{\mathrm{I}}, 1)$, and let C_- be the constant of (II).

Sizes. Let (q,d) be a window pair at scale $Y_0 = X^{1/J}$ and $x \geq Y_0$. Then $\log q + \log d \leq Y_0^{\varepsilon/5}(1+\log^2 Y_0) \leq x^{\varepsilon/4}$ once Y_0 is large. Also $q, d \geq \exp(\frac{1}{2}Y_0^{\varepsilon/5}) > \max(P, 2X^{401}+1)$ and $\log(qd) \leq X^{\varepsilon/(4J)}$ for X large, so the hypotheses of (I) hold. The field E lies in $\mathbb{Q}(\mu_N)$ with $N = \mathrm{lcm}(2^{e(x)+2}, N_0, q, d)$, so $f = \mathrm{cond}(E) \leq N$ and $n = [E:\mathbb{Q}] \leq \varphi(N) \leq N$. Since $2^{e(x)} \leq 2x^3$,

$$\log(ft) \leq (e(x)+2)\log 2 + \log N_0 + x^{\varepsilon/4} + 401\log x \leq 2x^{\varepsilon/4}$$

for x large in terms of ε and N_0.

Assembly. Let $x \in [Y, Y^2]$ with Y from (I). For odd l, Lemma 17.6 splits each length into the two parts. Summing over l,

$$\sum_l \sum_{e'<t} \lambda(Ш_l(E,e')) \leq e^{x^{4\varepsilon/5}}\log 2 + \beta + (ft)^C + Ce^{x^{4\varepsilon/5}}.$$

Since $n \leq f$, $\ell \leq o$ for $\ell \mid o$, and $\log^2 y \leq y$ for $y \geq 1$, each summand of β is at most $o^2(ft)^3$, so $\beta \leq (3 + C_-o^3)(ft)^3 \leq (3+C_-o^3)e^{6x^{\varepsilon/4}}$; also $(ft)^C \leq e^{2Cx^{\varepsilon/4}}$. Each of the four terms is at most $\frac{1}{4}\exp(\frac{1}{2}x^\varepsilon)$ once x is large in terms of ε, C, C_-, o, because $\frac{4}{5}\varepsilon < \varepsilon$ and $\frac{1}{4}\varepsilon < \varepsilon$. Choose $X_0 \geq X_1$ so large that $Y_0 = X_0^{1/J}$ exceeds all thresholds above. Then for $X \geq X_0$ every $x \geq Y \geq X^{1/J}$ is large enough. □

Neither of the statements (I) and (II) is proved here. The plus part of (I) splits, by the residual position of l relative to the host, into five counts of the same type. The generic one is a bound for the groups $Ш_l(\mathbb{Q}, e'; \psi)$ attached to the Dirichlet characters ψ of the host, and it follows from

statement (CorGK) of Subsection 17.4 for $K = \mathbb{Q}(\sqrt{-p_K})$. Cases (II) and (III) use GSp_4. Case (V) uses a deformation problem over a frame of the host field. The part of (I) at $l = 2$ and at the bad primes uses Poitou–Tate duality, local duality, a surjection onto $H^2(G_{L,S}, \mathbb{Z}_l(e+1))$ from a module that maps to $\mathbb{Z}_l \otimes H_{2e}(\mathrm{GL}_{4e+4}(\mathcal{O}_L), \mathbb{Z})$ with kernel killed by $l^{c(2e)}$, torsion bounds of the type of Soulé and of Dwyer and Friedlander, and Borel's computation of ranks. The bound is available only on windows of scales, and the constant $J = 2^{R+2}(2C_K \cdot 401/\varepsilon)^R$ of the formal proof, with $R = R(\psi_0)$, depends on ψ_0 and ε. Statement (II) is of the type of the Main Conjecture for abelian fields (Mazur and Wiles, Wiles, Rubin) together with the theorem of Ferrero and Washington, which bound the minus parts by generalized Bernoulli numbers, whose logarithms are polynomial in n, t and $\log f$. The formal proof derives (II) from Stickelberger's theorem and a Kolyvagin system of Gauss sums.

Proposition 17.13 (plus parts through characters)**.** *Let $N \geq 1$, let $H \subset (\mathbb{Z}/N)^\times$ be a subgroup with $-1 \notin H$, and let $E \subset \mathbb{Q}(\mu_N)$ be the fixed field of H, so that E is a CM field with $\mathrm{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/N)^\times/H$. Let l be an odd prime with $l \nmid [(\mathbb{Z}/N)^\times : H]$, let $e \geq 1$, and let O be a finite extension of $\mathbb{Z}_l$ which is a discrete valuation ring and contains the values of all characters of $(\mathbb{Z}/N)^\times/H$. Let X_e be the set of Dirichlet characters ψ modulo N with values in O that are trivial on H and satisfy $\psi(-1) = (-1)^e$. Then $Ш_l(E,e)^{c=(-1)^e}$ is finite if and only if every $Ш_l(\mathbb{Q}, e; \psi)$ with $\psi \in X_e$ is finite, and in that case*

$$\mathrm{len}_{\mathbb{Z}_l} Ш_l(E,e)^{c=(-1)^e} = \sum_{\psi \in X_e} \frac{\mathrm{len}_O Ш_l(\mathbb{Q}, e; \psi)}{e(O)}.$$

Proof. Put $\Gamma = \mathrm{Gal}(E/\mathbb{Q})$, $M = M_{e,O}$ and $A = O[\Gamma]$, on which $G_\mathbb{Q}$ acts through Γ by left multiplication. Since M is a $G_\mathbb{Q}$-module, the induced module $\mathrm{Ind}_{G_E}^{G_\mathbb{Q}} M$ is isomorphic to $A \otimes_O M$ with the diagonal action, and Shapiro's lemma gives an isomorphism $H^1(G_E, M) \cong H^1(G_\mathbb{Q}, A \otimes_O M)$. The order of Γ is a unit in O and O contains the values of the characters of Γ, so the elements $\varepsilon_\psi = |\Gamma|^{-1} \sum_{\gamma \in \Gamma} \psi(\gamma)^{-1}\gamma$ are orthogonal idempotents with $\sum_\psi \varepsilon_\psi = 1$ and $\gamma\varepsilon_\psi = \psi(\gamma)\varepsilon_\psi$. Hence $A \otimes_O M = \bigoplus_\psi \varepsilon_\psi A \otimes_O M \cong \bigoplus_\psi M(\psi)$ as $G_\mathbb{Q}$-modules, the sum over all characters ψ of Γ, where $M(\psi)$ is M with the action twisted by ψ. This gives $H^1(G_E, M) \cong \bigoplus_\psi H^1(G_\mathbb{Q}, M(\psi))$.

Let v be a finite prime and let D be D_v or I_v. By Mackey's formula the restriction of $\mathrm{Ind}_{G_E}^{G_\mathbb{Q}} M$ to D is the sum over the double cosets $D\gamma G_E$ of the modules induced from $D \cap \gamma G_E \gamma^{-1}$, and Shapiro's lemma for each summand identifies the restriction map to $H^1(D, A \otimes_O M)$ with the product of the restriction maps of $H^1(G_E, M)$ to the decomposition groups, respectively the inertia groups, of the places of E above v. So a class of $H^1(G_E, M)$ lies in $Ш_l(E, e; O)$ if and only if its image in $H^1(G_\mathbb{Q}, A \otimes_O M)$ vanishes on D_l and on every I_v. The decomposition $A \otimes_O M = \bigoplus_\psi M(\psi)$ is stable under $G_\mathbb{Q}$, hence under D, so these conditions hold if and only if they hold in each component. Therefore $Ш_l(E, e; O) \cong \bigoplus_\psi Ш(\psi)$, where $Ш(\psi) \subset H^1(G_\mathbb{Q}, M(\psi))$ is the subgroup of classes vanishing on D_l and on every I_v. Since $\psi \mapsto \psi^{-1}$ is a bijection of the characters of Γ, $Ш(\psi)$ is the group $Ш_l(\mathbb{Q}, e; \psi^{-1})$ of the text.

The natural action of $\gamma \in \Gamma$ on $H^1(G_E, M)$ of Lemma 17.6 corresponds under Shapiro's isomorphism to the action of $A \otimes_O M$ induced by multiplication by γ or by γ^{-1} on A, according to the convention, which commutes with $G_\mathbb{Q}$ because Γ is abelian. On $\varepsilon_\psi A$ it is multiplication by $\psi(\gamma)^{\pm 1}$. Complex conjugation c is the class of -1, and $\psi(-1)^{\pm 1} = \psi(-1)$. Hence $Ш_l(E, e; O)^{c=(-1)^e} \cong \bigoplus_{\psi(-1)=(-1)^e} Ш(\psi)$. The set X_e is stable under $\psi \mapsto \psi^{-1}$, so the right side is $\bigoplus_{\psi \in X_e} Ш_l(\mathbb{Q}, e; \psi)$. Since 2 is a unit in O, the eigenspace decomposition commutes with $O \otimes_{\mathbb{Z}_l} -$, and Lemma 17.5 gives $\mathrm{len}_O Ш_l(E, e; O)^{c=(-1)^e} = e(O)\,\mathrm{len}_{\mathbb{Z}_l} Ш_l(E, e)^{c=(-1)^e}$. Length is additive

on finite direct sums, which gives both assertions. □

Proposition 17.14 (the generic plus part from (CorGK)). *Assume statement (CorGK) of Proposition 17.19 for an imaginary quadratic field K, with constants c_5, A. Let $N \geq 16$ be an integer, $e \geq 1$ and $t \geq e+1$ integers, $r_1 \neq r_2$ primes not dividing $30N|d_K|$, and $k \geq 0$ an integer with $2c_5(\log r_1 + \log r_2) \leq 16^k$. Let L be a finite set of primes and, for $l \in L$, let O_l be a coefficient ring as in Definition 17.17 and X_l a set of Dirichlet characters modulo N with values in O_l, each in generic position for (K, l, e). Then every $Ш_l(\mathbb{Q}, e; \psi)$ with $l \in L$ and $\psi \in X_l$ is finite, and*

$$\sum_{l \in L} \log l \sum_{\psi \in X_l} \frac{\mathrm{len}_{O_l} Ш_l(\mathbb{Q}, e; \psi)}{e(O_l)} \leq 2c_5(\log r_1 + \log r_2)\, N \left(N(e+1)\right)^A \leq (Nt)^{k+1+A}.$$

Proof. For a subgroup $H' \subset (\mathbb{Z}/N)^\times$ let $\mathcal{O}_l(H')$ be the set of characters with values in O_l whose kernel is H'. Let T be the set of kernels of the characters in $\bigcup_l X_l$. Each X_l is contained in $\bigcup_{H' \in T} \mathcal{O}_l(H')$, and for $H' \in T$ let $L(H')$ be the set of $l \in L$ such that X_l meets $\mathcal{O}_l(H')$. For $l \in L(H')$ some character of $\mathcal{O}_l(H')$ is in generic position, so (CorGK) with $H = H'$, $p = r_1$, $p' = r_2$ and the set $L(H')$ gives finiteness and

$$\sum_{l \in L(H')} \log l \sum_{\psi \in \mathcal{O}_l(H')} \frac{\mathrm{len}\, Ш_l(\mathbb{Q}, e; \psi)}{e(O_l)} \leq 2c_5\, \varphi([(\mathbb{Z}/N)^\times : H'])\, \left(N(e+1)\right)^A (\log r_1 + \log r_2).$$

All terms are nonnegative, so summing over $H' \in T$ bounds the left side of the proposition. The quotient $(\mathbb{Z}/N)^\times / H'$ is cyclic for $H' \in T$, so exactly $\varphi([(\mathbb{Z}/N)^\times : H'])$ complex characters of $(\mathbb{Z}/N)^\times$ have kernel H'. Distinct H' give disjoint sets of characters, hence $\sum_{H' \in T} \varphi([(\mathbb{Z}/N)^\times : H']) \leq \varphi(N) \leq N$. This gives the first inequality. For the second, $2c_5(\log r_1 + \log r_2) \leq 16^k \leq N^k$, $N \leq Nt$ and $N(e+1) \leq Nt$. □

The hypotheses $r_1, r_2 \nmid 30N|d_K|$ and the generic position of the characters are the points at which the host enters. The verification that the characters of the host attached to a twist e and a prime l outside the finite set of bad primes with $(l-1) \nmid 2e$ are in generic position for $K = \mathbb{Q}(\sqrt{-p_K})$, and that the host level N is polynomial in the conductor of E, are not written out here.

17.4 A consequence in the direction of Vandiver's conjecture

Consider the following statement.

(V) There are constants C, A such that for every even $e \geq 2$ and every finite set L of primes $l \geq 5$ with $(l-1) \nmid 2e$, each $Ш_l(\mathbb{Q}, e)$ is finite and $\sum_{l \in L} \lambda\big(Ш_l(\mathbb{Q}, e)\big) \leq C(e+1)^A$.

The formal proof contains (V) as a theorem, derived from the statement that also gives the generic plus part of (I).

Corollary 17.15 (Vanishing of eigenspaces). *If (V) holds with constants C, A, then for every even $e \geq 2$ and every prime $l \geq 5$ with $(l-1) \nmid 2e$ and $\log l > C(e+1)^A$, the ω^{-e} eigenspace of the l-class group of $\mathbb{Q}(\mu_l)$ vanishes.*

Proof. If it did not vanish, $Ш_l(\mathbb{Q}, e) \neq 0$ by Proposition 17.8, so $\lambda(Ш_l(\mathbb{Q}, e)) \geq \log l$, and (V) with $L = \{l\}$ gives $\log l \leq C(e+1)^A$. □

The exclusion $(l-1) \mid 2e$ removes nothing of substance. If $(l-1) \mid 2e$ and $(l-1) \nmid e$, then ω^{-e} is the quadratic character of Δ, whose field is $\mathbb{Q}(\sqrt{l})$, and the corresponding eigenspace is the l-part of a class group whose order is smaller than l.

Comparison with published results. Kurihara [37] proved $A^{(\omega^{-2})} = 0$ for every l, through $K_4(\mathbb{Z}) = 0$, which is the case $e = 2$ of Corollary 17.15 with no threshold. With $n = e + 1$ odd, Soulé [68] proved $A^{(\omega^{-e})} = 0$ once $\log l > n^{224n^4}$, a threshold of size $\exp(O(e^4 \log e))$. De la Torre Durán [22] improved the exponential factor for $F = \mathbb{Q}$ from $e^{O(n^4 \log n)}$ to $e^{O(n^2 \log n)}$, a threshold of size $\exp(O(e^2 \log e))$. Corollary 17.15 has the threshold $C(e+1)^A$, one exponential smaller. In terms of $T(e) = \sum_l \lambda(Ш_l(\mathbb{Q}, e))$, the published methods bound $\log T(e)$ by a constant times $e^2 \log e$ at best, and (V) asserts $\log T(e) \leq A \log(e+1) + \log C$. The methods of Soulé and of de la Torre Durán count cells of the Voronoi complex of $\mathrm{GL}_n(\mathbb{Z})$, and the number of perfect forms grows like $e^{\Theta(n^2 \log n)}$ [76], so cell counting alone does not reach a polynomial bound. No finite computation can contradict (V), because its constants are existential. The computed groups $K_4(\mathbb{Z}) = 0$ and $K_8(\mathbb{Z}) = 0$ and the verification of Vandiver's conjecture for $l < 2^{31}$ are consistent with every upper bound, and they give no information about (V).

Remark 17.16 (odd twists). For odd e the analogous sum is classically polynomial. By the identification with $K_{2e}(\mathbb{Z}) \otimes \mathbb{Z}_l$ in Subsection 17.2 and the computation of $K_{4k-2}(\mathbb{Z})$ from the Main Conjecture of Mazur and Wiles (Weibel's survey), $|Ш_l(\mathbb{Q}, e)|$ is the l-part of the numerator of $B_{e+1}/(e+1)$ for $e = 2k-1$ and $(l-1) \nmid e$. Hence $\sum_l \lambda(Ш_l(\mathbb{Q}, e)) \leq \log |B_{e+1}| + o(e) = (1 + o(1))\, e \log e$, using that the denominator of $B_{e+1}/(e+1)$ has logarithm $o(e)$. So the shape of (V) is the natural even analogue of a known odd statement. This supports plausibility and proves nothing.

The chain of statements. Let K be an imaginary quadratic field. For a Dirichlet character ψ write $Ш_l(\mathbb{Q}, e; \psi)$ for the group of Definition 17.2 for the character $\psi^{-1}\chi^{-e}$ in place of χ^{-e}, with coefficients in a finite extension O of $\mathbb{Z}_l$ that contains the values of ψ.

Definition 17.17 (generic position)**.** Let K be a number field, l a prime, $e \geq 0$ and $N \geq 1$ integers, O a finite extension of $\mathbb{Z}_l$ which is a discrete valuation ring, and ψ a Dirichlet character modulo N with values in O. For a prime r let $\psi_{(r)}$ be the restriction of ψ to the factor $(\mathbb{Z}/r^{v_r(N)})^\times$ of $(\mathbb{Z}/N)^\times$, viewed again as a character modulo N, and put $\psi^\circ = \psi\, \psi_{(l)}^{-1}$. Let $\omega\colon G_{\mathbb{Q}} \to \mathbb{F}_l^\times$ be the reduction of the cyclotomic character modulo l. We say that ψ is in generic position for (K, l, e) when the following five conditions hold.

(a) $l \neq 2$, $e \geq 1$ and N is even.

(b) l does not divide the order of ψ, $\psi_{(l)}^2 = 1$, and $\psi(-1) = (-1)^e$.

(c) K is imaginary quadratic, $l \nmid |d_K| h_K$, and $\gcd(d_K, N) = 1$.

(d) The subgroup $\ker\omega \cap \ker\psi^\circ$ of $G_{\mathbb{Q}}$ is not contained in G_K.

(e) $(l-1) \nmid 2e$.

For $\psi = 1$ one has $\psi_{(l)} = \psi^\circ = 1$, and conditions (b) and (d) become the conditions written in the proof of Proposition 17.19. For a general ψ the condition in (b) concerns the l-component $\psi_{(l)}$ and not ψ itself, and the condition in (d) concerns ψ°.

For ψ as above and $e \geq 1$ we write $M_{e,O}(\psi^{-1})$ for $O \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l$ with $\sigma \in G_{\mathbb{Q}}$ acting by multiplication by $\psi^{-1}(\sigma)\chi(\sigma)^{-e}$, where ψ is viewed as a character of $G_{\mathbb{Q}}$ through $\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \cong (\mathbb{Z}/N)^\times$. Then $Ш_l(\mathbb{Q}, e; \psi)$ is the subgroup of $H^1(G_{\mathbb{Q}}, M_{e,O}(\psi^{-1}))$ of the classes that vanish on D_l and on I_v for every finite v.

Lemma 17.18. *Let $v \neq l$ be a prime, $e \geq 1$, and let ψ have order prime to l. A class in $H^1(D_v, M_{e,O}(\psi^{-1}))$ that vanishes on I_v is zero. In particular $Ш_l(\mathbb{Q}, e; \psi)$ is also the group of classes that vanish on D_v for every $v \mid lN$ and on I_v for every finite v.*

Proof. Write $M = M_{e,O}(\psi^{-1})$ and let $\mathfrak{m}$ be the maximal ideal of O. Inflation and restriction give the exact sequence $0 \to H^1(D_v/I_v, M^{I_v}) \to H^1(D_v, M) \to H^1(I_v, M)$. Suppose first that ψ is ramified at v, and choose $g \in I_v$ with $\psi(g) \neq 1$. The element $\psi(g)$ is a root of unity of order $m > 1$ prime to l, so $1 - \psi(g)$ divides m in $O[\psi(g)]$ and $1 - \psi(g) \in O^\times$. The group I_v acts trivially through χ because $v \neq l$, so g acts on M by $\psi(g)^{-1}$, and $(\psi(g)^{-1} - 1)m' = 0$ forces $m' = 0$. Hence $M^{I_v} = 0$. Suppose next that ψ is unramified at v. Then $M^{I_v} = M$, and a Frobenius element φ acts by $a = \psi(v)^{-1}v^{-e}$. If $a = 1$, then v^e would be a root of unity in the field of fractions of O, which is impossible because $v^e > 1$. So $a - 1 \neq 0$, multiplication by $a - 1$ is surjective on the divisible O-module M, and $H^1(D_v/I_v, M) \cong M/(\varphi - 1)M = 0$. In both cases the inflated group vanishes. For the last assertion, a class that vanishes on I_v for $v \mid N$, $v \neq l$, then vanishes on D_v, and conversely vanishing on D_v implies vanishing on I_v. □

Proposition 17.19 ((V) from (CorGK))**.** *Assume the following statement (CorGK) for $K = \mathbb{Q}(\sqrt{-3})$. There are c_5, A such that for every even $N \geq 2$, every subgroup $H \subset (\mathbb{Z}/N)^\times$, all distinct primes $p, p' \nmid 30N|d_K|$, every finite set L of primes such that for each $l \in L$ some character ψ in the orbit of characters with kernel H is in generic position in the sense of Definition 17.17, one has finiteness and*

$$\sum_{l \in L} \log l \sum_{\psi} \frac{\operatorname{len} Ш_l(\mathbb{Q}, e; \psi)}{e(O)} \leq 2c_5\, \varphi([(\mathbb{Z}/N)^\times : H])\, \big(N(e+1)\big)^A \big(\log p + \log p'\big),$$

where $Ш_l(\mathbb{Q}, e; \psi)$ is the group of Definition 17.2 for the character $\psi^{-1}\chi^{-e}$. Then (V) holds with $C = 36c_5 2^A$ and the same A.

Proof. Take $N = 2$, $H = (\mathbb{Z}/2)^\times$ (the trivial group), so the only character is $\psi = 1$ and $\varphi(1) = 1$, and take $p = 7$, $p' = 11$, which do not divide $30 \cdot 2 \cdot 3$. For $\psi = 1$, $Ш_l(\mathbb{Q}, e; 1) = Ш_l(\mathbb{Q}, e)$. The generic-position conditions for $K = \mathbb{Q}(\sqrt{-3})$, $\psi = 1$, $l \geq 5$, $e \geq 2$ even and $(l-1) \nmid 2e$ hold, as follows. (a) l odd, $e \geq 1$, $N = 2$ even. (b) l does not divide the order 1 of ψ, $\psi^2 = 1$, and $\psi(-1) = 1 = (-1)^e$. (c) K is imaginary quadratic with discriminant -3 and class number 1, so $l \nmid 3$; and $\gcd(3, 2) = 1$. (d) The last condition asks that $\ker(\chi \bmod l) \cap \ker\psi$ is not contained in G_K. Here $\ker\psi = G_{\mathbb{Q}}$, so it asks that $K \not\subset \mathbb{Q}(\mu_l)$, which holds because 3 ramifies in K and not in $\mathbb{Q}(\mu_l)$ for $l \neq 3$. (e) $(l-1) \nmid 2e$. The right side becomes $2c_5(2(e+1))^A(\log 7 + \log 11) \leq 2c_5\, 2^A(e+1)^A \cdot 18$. Lemma 17.5 converts the left side into the left side of (V). □

Proposition 17.20 ((CorGK) from (PropGK))**.** *Assume the following statement (PropGK). For a number field K there are c_5, A such that for e, N, H and every prime $p \nmid 30N|d_K|$ there is an integer $NN_p \neq 0$ with $\log|NN_p| \leq c_5\varphi(\operatorname{index} H)(N(e+1))^A \log p$, and for every prime $l \neq p$ and every coefficient ring at which some character of the orbit is in generic position, the lengths are finite and $\sum_\psi \operatorname{len} Ш_l(\mathbb{Q}, e; \psi)/e(O) \leq 2v_l(NN_p)$. Then (CorGK) holds with the same c_5, A.*

Proof. Fix distinct admissible p, p' and a finite set L of primes in generic position. For $l \neq p$ use NN_p, and for $l = p$ use $NN_{p'}$. Then

$$\sum_{l \in L} \log l \sum_{\psi} \frac{\operatorname{len} Ш_l(\mathbb{Q}, e; \psi)}{e(O)} \leq 2 \sum_{l} \log l\, \big(v_l(NN_p) + v_l(NN_{p'})\big) \leq 2\big(\log|NN_p| + \log|NN_{p'}|\big),$$

because $\sum_l v_l(n) \log l \leq \log|n|$ for a nonzero integer n. Insert the bounds for $\log|NN_p|$ and $\log|NN_{p'}|$. □

Proposition 17.21 ((PropGK) from (Count) and (OneChar)). *Let K be an imaginary quadratic field, $p \geq 7$ a prime with $p \nmid N\,d_K$, $w \mid p$ a place of K and σ a geometric Frobenius element at w. Let $\mathcal{O}$ be a finite set of roots of unity z stable under $\mathrm{Aut}(\mathbb{C})$, and put $t(z) = Nw^{-e}z + 1 + Nw^{e}z^{-1}$, the trace of σ on the Eisenstein triple $\Psi \oplus 1 \oplus \Psi^{-1}$ suitably normalized. Assume the following two statements.*

(a) (Count.) *There is $NN \in \mathbb{Z} \setminus \{0\}$ with $\log|NN| \leq c\,|\mathcal{O}|\,(N(e+1))^{A}\log p$, such that for every prime $l \neq 2, p$, every $\iota\colon \overline{\mathbb{Q}}_l \cong \mathbb{C}$ and every choice of finite sets X_z of pairwise non-isomorphic admissible $\rho\colon G_K \to GL_3(\overline{\mathbb{Q}}_l)$ (level N, weight e), one has $\operatorname{tr}\rho(\sigma) \neq t(z)$ for $\rho \in X_z$ and $|NN|_l \leq \prod_z \prod_{\rho\in X_z} |\operatorname{tr}\rho(\sigma) - t(z)|_l$.*

(b) (OneChar.) *For each character ψ in generic position, if no admissible ρ has $\operatorname{tr}\rho(\sigma) = t$, then $Ш_l(\mathbb{Q}, e; \psi)$ is finite and there is a finite set X_ψ of pairwise non-isomorphic admissible ρ with $\operatorname{len} Ш_l(\mathbb{Q}, e; \psi)/e(O) \leq 2\sum_{\rho\in X_\psi} v_l(\operatorname{tr}\rho(\sigma) - t_\psi)$.*

Then the conclusion of (PropGK) holds at p with $NN_p = NN$.

Proof. The orbit of characters ψ with kernel H is in bijection with an $\mathrm{Aut}(\mathbb{C})$-stable set $\mathcal{O}$ of roots of unity, $z_\psi = \psi(Nw)^{-1}$ under ι, and $t_\psi = t(z_\psi)$; the number of elements is $\varphi(\operatorname{index} H)$. By (a) no admissible ρ has trace t_ψ, so (b) applies to each ψ. Summing (b) over ψ and applying (a) to the sets X_ψ,

$$\sum_{\psi} \frac{\operatorname{len} Ш_l(\mathbb{Q}, e; \psi)}{e(O)} \leq 2\sum_{\psi}\sum_{\rho\in X_\psi} v_l\big(\operatorname{tr}\rho(\sigma) - t_\psi\big) \leq 2v_l(NN). \qquad \square$$

The content of (Count) is classical in type. The admissible ρ correspond to Frobenius eigenvalues at w of algebraic automorphic forms on a definite unitary group in three variables of level N and weight of height e, whose number is polynomial in $N(e+1)$. The number NN is the product of the scaled differences, rational by Galois stability, and nonzero because purity gives $|\operatorname{tr}\rho(\sigma)| \leq 3$ while $|t(z)| \geq p^e - 2 > 3$. Statement (Count) is not proved here.

Proposition 17.22 (a consequence of the Selmer bound). *Assume Hypothesis 17.11 for one primitive character ψ_0 of conductor N_0, and assume that for every $\varepsilon \in (0, \frac{1}{52})$ and $J \geq 1$ there are arbitrarily large X for which a window pair (q, d) at scale $X^{1/J}$ exists (Definition 17.9). Then for every $\varepsilon \in (0, \frac{1}{52})$ there are arbitrarily large real x and primes q_x, d_x such that for every finite set L of odd primes $l \nmid N_0 q_x d_x$,*

$$\sum_{l\in L} \sum_{\substack{1\leq e < x^{401} \\ (l-1)\nmid e}} \lambda\big(Ш_l(\mathbb{Q}, e)\big) \leq \exp\big(\tfrac{1}{2}x^{\varepsilon}\big).$$

Consequently the product $P(T) = \prod_{e<T}\prod_l |Ш_l(\mathbb{Q}, e)|$ over odd $l \nmid N_0$ with $(l-1) \nmid e$, outside two primes that depend on T, satisfies $\log\log P(T) \leq \frac{1}{2}T^{\varepsilon/401}$ for arbitrarily large T, for every $\varepsilon > 0$.

Proof. Choose any function $e(x)$ with $x^{2+8\varepsilon} \leq 2^{e(x)} \leq 2x^{2+8\varepsilon}$. Let J, X_0 be given by Hypothesis 17.11, take $X \geq X_0$ with a window pair (q, d) at scale $X^{1/J}$, and take the Y that Hypothesis 17.11 provides. Put $x = Y$, which tends to infinity with X. Let $N = \operatorname{lcm}(2^{e(x)+2}, N_0, q, d)$ and let E be the fixed field in $\mathbb{Q}(\mu_N)$ of the subgroup of Definition 17.10. Hypothesis 17.11 gives $\sum_{l\in L}\sum_{1\leq e<t}\lambda(Ш_l(E, e)) \leq \exp(\frac{1}{2}x^{\varepsilon})$ with $t = \lfloor x^{401}\rfloor$. For odd $l \nmid N_0 qd$ we have $l \nmid N$, and Proposition 17.7 gives $\lambda(Ш_l(\mathbb{Q}, e)) \leq \lambda(Ш_l(E, e))$ whenever $(l-1) \nmid e$. The terms are nonnegative, so the inequality follows. For the consequence take $T = x^{401}$ and logarithms. $\square$

By the identification with $K_{2e}(\mathbb{Z}) \otimes \mathbb{Z}_l$ and the comparison above, the published estimates give at best $\log\log P(T) \ll T^2 \log T$, and nothing better along any sequence of T. Proposition 17.22 needs $\log\log P(T) \leq T^{\delta}$ along a sequence of T, for every $\delta > 0$. So the restriction to windows does not reduce what Hypothesis 17.11 requires to a published bound. The Selmer bound on windows implies a bound in the direction of Vandiver's conjecture that is stronger than the published bounds of Soulé and of de la Torre Durán. Alpöge's announcement [2] describes this bound as a weakening of Vandiver's conjecture.

The statement (OneChar). Fix K, $l \geq 5$, e and a character ψ in generic position, put $\Psi = \chi^e \psi$, and consider the ordinary deformation problem for representations of G_{Q} into the group of unitary type in three variables with residual representation $\Psi \oplus 1 \oplus \Psi^{-1}$. The formal proof derives (OneChar) from four inputs.

(G1) *Tangent injection.* A class of $Ш_l(\mathbb{Q}, e; \psi)$ gives a tangent vector of weight -1 at the Eisenstein point of a graded model of the ordinary deformation ring, with an action of a torus (the argument of Ribet).

(G2) *An intersection inequality.* On a trigraded Rees algebra, under hypotheses (M1) to (M5), the cotangent module of weight -1 modulo the differentials of the Selmer equations has length at most $2 \sum_{\xi} v(\xi(a))$, the sum over representatives of the torus orbits of the two-signed points ξ, with $a = \operatorname{tr} r(\sigma) - t$. This is commutative algebra with no arithmetic input.

(G3) *Finiteness and automorphy.* The ordinary deformation problem is finite over O (hypothesis (M2)), and every ordinary point r with values in a finite extension of O that is irreducible on every open subgroup of the kernel of the similitude character restricts on G_K to a representation ρ such that $\rho \otimes \chi^{-1}$ is automorphic and conjugate self-dual.

(G4) *Rigidity.* At each two-signed point the tangent space of the Selmer locus is one-dimensional (hypothesis (M4)), which is the vanishing of the adjoint Bloch–Kato Selmer group. At the reducible fixed points $W \oplus 1$ (hypothesis (M3)) the Bloch–Kato Selmer groups of the twist W of a newform of weight $2e + 1$ and of $W \otimes \theta$ vanish.

None of (G1) to (G4) is proved here, and (OneChar) is not proved here.

The closest published theorem to (G3) is Theorem 1.1 of Allen, Newton and Thorne [1]. It allows any number of residual constituents, but its hypothesis (4) requires each constituent to be conjugate self-dual, and its hypotheses (5) and (6c) require a place at which the seed is an unramified twist of the Steinberg representation. In the configuration of (G3), on G_K the constituents Ψ and Ψ^{-1} are exchanged by conjugate duality, so they are not individually conjugate self-dual when $e \neq 0$, and no Steinberg place is assumed. The earlier theorem of Thorne [74] has stronger hypotheses. So (G3) is not a published theorem. In characteristic zero at irreducible points, (G4) has the shape of Theorem A of Newton and Thorne [51], which needs the image to be enormous and does not need residual irreducibility. For the cyclotomic image, the trace criterion in Subsection 17.4.1 applies once the image spans the matrix algebra and its regular semisimple elements are dense.

At a reducible point let f have weight $k = 2e + 1$. The Frobenius polynomials identify W with $V(f)(e)$ by Chebotarev density and Brauer–Nesbitt. Take a cyclotomic field F on which θ is trivial. Since $1 \leq e \leq k - 1$ and $e \neq k/2$, Kato's Theorem 14.2(1) [34] gives $H^1_f(F, V(f)(e)) = 0$. Restriction to F is injective in characteristic zero. It gives the two vanishings over $\mathbb{Q}$ in (G4).

For the rank-three crossing in (G3), write the odd relations of a minimal graded presentation as $My = 0$. Multiplication by the adjugate gives $(\det M)y = 0$. Thus on the open set where $\det M$ is a unit all the odd coordinates vanish. In the complementary dimension argument the section bound gives dimension at least $3d + 2$, whereas the triangular locus has dimension less than $3d$, with d the degree parameter of the presentation. Propagation still requires finiteness over the weight algebra, a point known to be automorphic on the initial component, and a crossing theorem along the successive thickenings at reducible primes.

Remark 17.23 (normalizations in (Count) and (OneChar)). Let σ_w be an arithmetic Frobenius element at w, so that $\chi(\sigma_w) = Nw$ and $\psi(\sigma_w) = \psi(Nw)$, and let $\sigma = \sigma_w^{-1}$ be the geometric Frobenius. With $\Psi = \chi^e\psi$ the trace of σ on $\Psi \oplus 1 \oplus \Psi^{-1}$ is $Nw^{-e}\psi(Nw)^{-1} + 1 + Nw^e\psi(Nw)$. With $z_\psi = \psi(Nw)^{-1}$ this is

$$t(z_\psi), \qquad t(z) = Nw^{-e}z + 1 + Nw^e z^{-1}.$$

The involution $z \mapsto z^{-1}$ permutes $\mathcal{O}$. The admissible $\rho\colon G_K \to \mathrm{GL}_3(\overline{\mathbb{Q}}_l)$ in both statements are of two kinds. Either ρ is irreducible, $\rho \otimes \chi^{-1}$ is automorphic and conjugate self-dual, and ρ has level N and Hodge–Tate weights $(e, 0, -e)$, or ρ is conjugate to $W|_{G_K} \oplus 1$ for an irreducible $W\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ with $W \otimes \chi^{-e}$ attached to a newform of weight $2e + 1$, $\det W$ of finite order and level $N|d_K|$. In both cases the Frobenius eigenvalues at w have absolute value 1 under ι, so $|\mathrm{tr}\,\rho(\sigma)| \le 3$, while $|t(z)| \ge Nw^e - 1 - Nw^{-e} > 3$ for $p \ge 7$ and $e \ge 1$. The explicit bound $4c_5\varphi(\mathrm{index})|d_K|^5N^{18}e^7(\log p + \log p')$ quoted in Subsection 17.3 belongs to a variant of (CorGK) that is not used. The argument uses (CorGK) with an unspecified exponent A, through Proposition 17.14.

17.4.1 The rank-four counts

Cases (II) and (III) of the plus part use symplectic representations of rank four. Both use automorphy on every irreducible component through the residual point. If A is the deformation ring, J the automorphic ideal and $\mathfrak{p}$ the prime of the point, the starting condition is $J \subseteq \mathfrak{q}$ for one minimal prime $\mathfrak{q} \subseteq \mathfrak{p}$. To obtain the condition for every such $\mathfrak{q}$ one joins the components by a chain. At each adjacent pair one needs the local crossing assertion. Induction along the chain then gives the desired containment for the last component. Finiteness of the set of systems of Hecke eigenvalues alone gives no such containment.

At a locally generic automorphic point the following observation verifies the trace condition for enormity in the adjoint vanishing theorem of Newton and Thorne [51]. Enlarge the coefficient field so that the eigenvalues in question split. Let H be the image on the cyclotomic subgroup. Assume that H spans M_n and that its regular semisimple elements are dense in its Zariski closure. Let $V \subset M_n$ be a nonzero simple H-submodule under conjugation. For regular h write $h = \sum_\alpha \alpha e_{h,\alpha}$, with $e_{h,\alpha}$ the spectral projectors. If $\mathrm{tr}(e_{h,\alpha}v) = 0$ for every such h, every α and every $v \in V$, then $\mathrm{tr}(hv) = 0$. By density this holds on H and by linearity on its span. Nondegeneracy of the trace pairing gives $V = 0$, a contradiction. In the symplectic case the projection from endomorphisms to the symplectic Lie algebra is $X \mapsto (X - X^\dagger)/2$, with $X^\dagger$ the adjoint for the alternating form. The finite local condition is the inertia coboundary condition away from l and the crystalline condition above l.

The two counts use the same rigidity bound. In case (III) the prime l lies outside the bad set, which contains 2 and 3, and $l - 1 \mid 2e$. Thus $e \ge 2$. In the trace separation estimate put $y = p^e$ at an auxiliary prime $p \ge 5$. The required strict inequality is $7y/2 + y + p + 1 < py$: indeed

$(p - 9/2)y \geq p^2/2 > p + 1$. In the product over conjugate roots of unity of order k, one unit modulo k acts on all three roots at once. The orbit therefore contributes at most $\varphi(k)$ choices. Repeated roots retain their multiplicities.

The host count. In case (V), l belongs to the exceptional set B_χ and $e = 1$. On the branch $R_1 \leq 2^{c_2(\psi)}$, let O be the coefficient ring with ramification index E_O, and normalize $v_l(l) = 1$. At distinct two-signed points r_i write

$$S = \sum_i v_l\big(\operatorname{tr} r_i(\varphi) - \operatorname{tr} r_z(\varphi)\big),$$

where r_z is the Eisenstein representation and φ is geometric Frobenius. The host deformation and intersection bounds give $\operatorname{length}_O V \leq 2E_O S$ for the Selmer module under consideration. If all trace differences are nonzero and $S \log l \leq (lNe)^C$, division by E_O gives

$$E_O^{-1} \operatorname{length}_O V \log l \leq 2(lNe)^C \leq (lNe)^{C+1}.$$

This deduction requires the host deformation bound and the polynomial estimate for both kinds of representations in Remark 17.23.

17.4.2 Automorphic points and base change

Let E/F be the CM extension in the level construction, unramified at every finite place. Then $[F : \mathbb{Q}] \geq 2$, since a quadratic field over $\mathbb{Q}$ ramifies at a finite prime. The definite unitary quotient is compact. Its irreducible summand π therefore occurs discretely, with finite-dimensional archimedean type. Labesse's base change (Corollaire 5.3 of *Changement de base CM et séries discrètes*) gives one isobaric sum of discrete constituents Π_i for π. At split places it gives the split local component and at spherical places it gives stable Satake transfer. At a nonsplit unramified place with residue cardinality q the upstairs cardinality is q^2.

By the residual spectrum theorem of Mœglin and Waldspurger, each Π_i is a Speh representation of length d_i on a normalized cuspidal representation of rank a_i, with $\sum_i a_i d_i = n$. Conjugate self-duality follows for its seed from uniqueness of cuspidal support for the conjugate self-dual constituent. In the normalized induction the exponents are $(d_i - 1)/2 - j$, $0 \leq j < d_i$. Reversing their order gives the subrepresentation realization; in the unnormalized constant term the modulus contributes $a_i((d_i - 1)/2 - j)$ to the reversed exponent. Fix the local factors of these same global constituents at every place. Each prescribed local component belongs to the corresponding induced product as a subquotient: restrict the tensor product to a finite set containing that place and every nonspherical factor, and take its Jordan–Hölder factors. At the remaining places where only a subquotient is prescribed, finite length gives one. The infinitesimal multisets add under induction. For highest weight λ their union at infinity consists of $\lambda_j + (n - 1)/2 - j$, $0 \leq j < n$.

For each Speh block put $b_{ij} = d_i - 2j - n$. Apply the Galois representation theorem for regular algebraic polarized cuspidal representations, in the local compatibility hypotheses of this section, to the cuspidal seeds twisted by $(b_{ij} + a_i - 1)/2$. Take the direct sum of the resulting representations. It is continuous, semisimple and of rank n. Under normalized local Langlands, parabolic products give sums and absolute-value twists give the corresponding twists of the parameter. The Speh exponents followed by the common twist $(1 - n)/2$ thus give precisely the parameter of this sum at split places. This calculation applies to every correspondence

satisfying the product and twist laws. On a seed infinitesimal weight z the Hodge–Tate weight is $-z - b_{ij}/2$. The local conditions above l pass to the sum under the spherical or tame hypotheses of the attachment theorem. If the sum is irreducible it consists of a single summand.

Now fix a common Taylor–Wiles base, a permissible auxiliary level, the free module M and its Hecke annihilator J. For $m \neq 0$ choose a nonzero coordinate in a free basis. Density of admissible weights gives a specialization where that coordinate survives. Decompose into irreducible automorphic summands and project onto a nonzero isotypic component. Take the local factors and the preceding Galois representation on this same summand. The spherical Hecke algebra acts there by its Satake character, so the Hecke polynomial is the characteristic polynomial at inverse Frobenius. The determinant and local types follow on the same summand by the product and twist laws. Projection detects the specialized vector. Varying m gives separating classical points at the chosen level, with common base and annihilator J.

In rank two over a totally real field, take a regular algebraic polarized Hilbert modular representation in the common-parity normalization (k, w). Skinner [66, eq. (1), Thm. 1] gives the Frobenius-semisimplified Weil–Deligne parameter with the twist $|\det|^{-1/2}$ and Hodge degrees $(w - k_\tau)/2$ and $(w + k_\tau - 2)/2$. Their sum is $w - 1$. At a place above l an Iwahori-fixed vector gives trivial inertia of the parameter and hence semistability. In the spherical case the monodromy also vanishes, giving crystallinity. Away from l, purity includes the monodromy filtration [9, Cor. 7]. Use inverse Frobenius for the geometric eigenvalues. If an algebraic character η was used to put π in this normalization, the representation for π is $r_{\pi\otimes\eta} \otimes r_\eta^{-1}$. These assertions concern the rank-two Hilbert modular input.

17.4.3 The effective cohomology bound

Let L be totally imaginary, $d = [L : \mathbb{Q}]$, $D = |\Delta_L|$, and S the places above p. Assume $\mu_p \subset L$ for odd p and $\mu_4 \subset L$ for $p = 2$. The required bound is

$$\log \# H^2(G_{L,S}, \mathbb{Z}_p(e+1)) \leq \exp\left((2ed)^{C_K} \log(2D)\right) \qquad (e \geq 1)$$

with C_K absolute. The following alternative proof awaits independent human verification and has not been formalized in Lean.

Put $i = 2e$, $u = id$, $Y = \log(2D)$ and

$$A = H^2(G_{L,S}, \mathbb{Z}_p(e+1)).$$

Here $u \geq 4$ and $p \leq d + 1$.

The even-degree comparison identifies A with $K_i(\mathcal{O}_L)\{p\}$, a finite group. Weibel [77, Thm. 73] gives this formula for totally imaginary fields, including $p = 2$. Here the étale cohomology is that of $\mathcal{O}_L[1/p]$. The comparison in Schmidt [62, Prop. 2.3 and its proof] identifies the finite-coefficient étale and restricted-ramification Galois cohomology here. All places above p have been removed. The cyclotomic action on the finite coefficient groups factors through a p-group under the roots-of-unity hypothesis. For compact coefficients take inverse limits of the finite coefficient groups. The finite degree-one groups satisfy Mittag–Leffler, so no derived-limit term occurs.

Let $r = \dim_{\mathbb{F}_p} A/pA$. Reduction embeds A/pA in $H^2(G_{L,S}, \mathbb{F}_p(e+1))$. The mod-$p$ cyclotomic character is trivial here, so this coefficient module is isomorphic to μ_p. Kummer theory gives an extension of $\mathrm{Br}(\mathcal{O}_L[1/p])[p]$ by $\mathrm{Pic}(\mathcal{O}_L[1/p])/p$ for this H^2. The local invariant map

embeds the Brauer term in a product of at most d groups of order p because L has no real places. The Picard group is a quotient of the ordinary class group. Thus

$$r \log p \leq \log h_L + d \log p.$$

For an explicit crude bound, Minkowski gives an integral ideal of norm at most $2^d\sqrt{D}$ in each ideal class. The number of ideals of norm at most B is at most $B^2\zeta_L(2) \leq B^2\zeta(2)^d$. Hence $h_L \leq 2^{3d}D$. In particular $r \leq u^4 Y$. It remains to bound the exponent of A, since an abelian p-group on r generators of exponent E has order at most E^r.

Integral homology at the small primes. Put $N = 2i + 1 \geq 5$, $\Gamma = \mathrm{GL}_N(\mathcal{O}_L)$ and let H be the kernel of reduction modulo $3\mathcal{O}_L$. Its index I is at most 3^{dN^2}. Restriction of scalars embeds H in the principal level-three subgroup of $\mathrm{GL}_{dN}(\mathbb{Z})$, which is torsion-free. Indeed, for an element of prime order q write it as $1 + 3^a B$ with $a \geq 1$ maximal. For $q \neq 3$ its qth power differs from 1 modulo 3^{a+1}. For $q = 3$ its cube differs from 1 modulo 3^{a+2}. Both contradict the order. A nontrivial finite-order element would have a power of prime order.

We give a bound on $T = |H_i(H, \mathbb{Z})_{\mathrm{tors}}|$. Use Ash's contractible well-rounded retract for the free lattice $\mathcal{O}_L^N$. Its cells have finite stabilizers. On the barycentric subdivision H acts freely. Bayer-Fluckiger, Emery and Houriet [5, §2.3, Cor. 5.2] give a finite set $\Phi \subset \mathcal{O}_L^N$ such that the minimal vectors of any normalized well-rounded form can simultaneously be moved into Φ by one element of Γ. Writing $M = |\Phi|$, their bound is

$$M \leq N^{3N^2d^2} d^{5N^3d^4} D^{9N^3d^3}.$$

Only this geometric estimate is used from their torsion argument.

The real space of Hermitian forms has dimension $s = (d/2)N^2$. Each equality $q_h(v) = 1$ is affine linear in h. For a cell with set of minimal vectors U, choose at most $s + 1$ vectors whose augmented equations span those indexed by U. Within the normalized forms with $q_h(x) \geq 1$ for every nonzero lattice vector x, the chosen equalities determine the closed face cut out by U. Its relative interior determines the cell. Indeed any point satisfying these equalities can be joined to an interior point by a segment; on the open segment all inequalities outside U are strict. Positive definiteness is preserved there and U spans L^N.

For a flag of cells choose its lowest-dimensional member. The other minimal sets are subsets of its minimal set by the face relation. A single translate puts them all in Φ. Choose an ordered list of at most $s + 1$ equations for each member and pad with an empty symbol. For flags of any *fixed* length there are at most $(1 + M)^{(s+1)^2}$ codes since flags have at most $s + 1$ members. Each code determines at most one flag in the retract. Passing from Γ to H multiplies the orbit count by at most I. Thus there are at most

$$B = I(1 + M)^{(s+1)^2}$$

simplices in each degree of the quotient subdivision. Its chain complex computes the integral homology of H. A boundary column from degree $i + 1$ has Euclidean norm at most $i + 2$ even if some quotient faces coincide. A maximal rank minor and Hadamard's inequality bound the torsion of the cokernel, and hence that of its subgroup H_i, by $(i + 2)^B$. Thus $\log T \leq B \log(i + 2)$.

Transfer to H takes a torsion homology class to torsion, and the transfer followed by projection acts as I. Consequently IT annihilates the torsion of $H_i(\Gamma, \mathbb{Z})$. This step includes primes dividing I.

The uniform constant. Homological stability identifies $H_i(\Gamma,\mathbb{Z})$ with $H_i(\mathrm{GL}(\mathcal{O}_L),\mathbb{Z})$ at $N = 2i+1$. See Weibel [78, IV.1.14.1], using Bass stable rank at most two, or [5, §3.1] for this precise choice of N. Give $X = B\mathrm{GL}(\mathcal{O}_L)^+$ its connected infinite-loop-space structure (the identity component of the algebraic K-theory space). Its integral homology is stable group homology, and $\pi_i X = K_i(\mathcal{O}_L)$. Apply Mathew [44, Thm. 6.3] with connectivity parameter $m = 1$. The p-part of the kernel of the unstable Hurewicz map is killed by p^{i+4}: this majorizes both $\lceil (i-1)/2\rceil + 3$ for $p = 2$ and $\lceil (i+2)/(2p-2)\rceil + 1$ for odd p. The image of A is torsion. Therefore $p^{i+4}IT$ kills A, and

$$\log|A| \le r\big((i+4)\log p + \log I + B\log(i+2)\big).$$

Finally, $dN \le 3u \le u^2$, $s+1 \le u^4$, and $\log I \le u^5$. From the displayed bound on M, $\log(1+M) \le u^{13}Y$: use $d, N \le u^2$, $N^3 d^4 \le (dN)^4 \le u^8$, and $\log d, \log N \le 2\log u$. Consequently $\log B \le u^{23}Y$. As $Y \ge \log 2$, $B \ge I$, $i+4 \le 2u$, and $p \le u$, the parenthesis is at most Bu^3. Inserting $r \le u^4 Y$ gives

$$\log|A| \le u^7 Y \exp(u^{23}Y) \le \exp(u^{24}Y) \le \exp(u^{150}Y).$$

For the middle inequality take logarithms and use $7\log u + \log Y \le 7u + Y \le (u^{24} - u^{23})Y$. In particular one may take $C_K = 150$.

17.5 The congruence bound

Hypothesis 17.24 (congruence bound)**.** There are C and x_0 such that for every admissible datum at scale $x \ge x_0$ for which every $Ш_l(E,e)$ with $0 < e < t_r$ is finite, and for every lift index j,

$$\sum_{w \text{ twisted}} \ell_w e_w \eta_j(w) \;\le\; d_j^0\Big(C\big(f\, m\, b\, L_\varepsilon^2 + (\log x)^2 N_c\big) + \sum_l \log l \sum_{1\le e<t_r} \operatorname{len} Ш_l(E,e)\Big),$$

provided the last sum converges.

The argument for Hypothesis 17.24 has three steps. A lattice defect is bounded by congruence numbers (step one), congruence numbers by Fitting lengths of a cocycle module (step two), and Fitting lengths by Selmer lengths plus local terms (step three).

Let O be a complete discrete valuation ring with fraction field K, X a finite set, $V = \bigoplus_{x\in X} V_x$ a finite-dimensional K-vector space, and $\Lambda \subset V$ an O-lattice. Let $T \subset \prod_{x\in X} O$ be an O-subalgebra acting on V by $t\cdot v = (t_x v_x)_x$ with $T\Lambda \subset \Lambda$. For $x \in X$ define the *congruence ideal* $\mathfrak{c}_x = \{t_x : t \in T,\ t_y = 0 \text{ for all } y \ne x\}$ and $\delta_x = \min\{v(a) : a \in \mathfrak{c}_x\} \in \mathbb{N}\cup\{\infty\}$. For $Z \subset X$ write $V_Z = \bigoplus_{x\in Z} V_x$ and $\pi_Z \colon V \to V_Z$ for the projection.

Proposition 17.25. *For $Z \subset X$, put $M^{\mathrm{up}} = \pi_Z(\Lambda)$ and $M^{\mathrm{low}} = \Lambda \cap V_Z$. Then*

(i) $M^{\mathrm{up}}/M^{\mathrm{low}} \cong \Lambda/\big((\Lambda\cap V_Z)\oplus(\Lambda\cap V_{X\setminus Z})\big)$;

(ii) $\operatorname{len}_O(M^{\mathrm{up}}/M^{\mathrm{low}}) \le \sum_{x\in Z} \dim_K V_x \cdot \delta_x$.

Proof. (i) The map $\Lambda \to M^{\mathrm{up}}/M^{\mathrm{low}}$, $v \mapsto \pi_Z(v)$, is surjective. Its kernel is the set of $v \in \Lambda$ with $\pi_Z(v) \in \Lambda$. For such v, $\pi_{X\setminus Z}(v) = v - \pi_Z(v) \in \Lambda$, so $v \in (\Lambda\cap V_Z)\oplus(\Lambda\cap V_{X\setminus Z})$, and the converse inclusion is clear.

(ii) If some $\delta_x = \infty$ there is nothing to prove. Otherwise choose $a_x \in \mathfrak{c}_x$ with $v(a_x) = \delta_x$ and $t^{(x)} \in T$ with $t^{(x)}_x = a_x$ and $t^{(x)}_y = 0$ for $y \neq x$. For $v \in \Lambda$, $t^{(x)}v = a_x\pi_x(v)$ lies in $\Lambda \cap V_x$. Hence

$$\bigoplus_{x\in Z} a_x\pi_x(\Lambda) \;\subset\; \bigoplus_{x\in Z}(\Lambda\cap V_x) \;\subset\; M^{\mathrm{low}} \;\subset\; M^{\mathrm{up}} \;\subset\; \bigoplus_{x\in Z}\pi_x(\Lambda).$$

Each $\pi_x(\Lambda)$ is a lattice in V_x, free of rank $\dim V_x$, so $\operatorname{len}(\pi_x(\Lambda)/a_x\pi_x(\Lambda)) = \dim V_x \cdot \delta_x$. Therefore $\operatorname{len}(M^{\mathrm{up}}/M^{\mathrm{low}}) \leq \operatorname{len}\big(\bigoplus_{x\in Z}\pi_x(\Lambda)/\bigoplus_{x\in Z} a_x\pi_x(\Lambda)\big) = \sum_{x\in Z}\dim V_x\cdot\delta_x$. □

Proposition 17.30 identifies this length with the summand in Definition 17.1 after division by $\operatorname{len}(O/l)$.

Lemma 17.26. *Let $Q \in M_n(O)$. Then $v(\det Q) = \operatorname{len}_O \operatorname{coker} Q$, and if $\det Q \neq 0$, $\operatorname{len}_O \operatorname{coker} Q = \operatorname{len}_O \ker\big(Q\colon (K/O)^n \to (K/O)^n\big)$. More generally, if $P\colon O^m \to O^k$ is injective after $\otimes K$, then $\ker(P\otimes K/O) \cong \operatorname{coker}(P)_{\mathrm{tors}}$.*

Proof. By the elementary divisor theorem $Q = U\operatorname{diag}(\varpi^{a_1},\dots,\varpi^{a_n})U'$ with U, U' invertible; both sides of the first two equalities equal $\sum a_i$ (with ∞ allowed). For the last statement, tensor $0 \to O^m \to O^k \to \operatorname{coker} P \to 0$ with K/O: since O^k is flat, $\ker(P\otimes K/O) = \operatorname{Tor}_1^O(\operatorname{coker} P, K/O)$, which is the torsion submodule of $\operatorname{coker} P$. □

Let R' be an O-algebra, not necessarily commutative, and $P \in M_{a\times b}(R')$. For an O-algebra map $\rho\colon R' \to M_d(O)$ write $\rho(P) \in M_{ad\times bd}(O)$ for the block matrix acting $O^{bd} = (O^d)^b \to (O^d)^a$. Then $\operatorname{Hom}_{R'}(\operatorname{coker}(R'^a \xrightarrow{\cdot P} R'^b), V) = \ker(\rho(P)\colon V^b \to V^a)$ for $V = K^d$ or $(K/O)^d$ with R' acting through ρ, since an R'-linear map from R'^b is given by the images $u_1,\dots,u_b$ of the basis, and it kills the image of $\cdot P$ exactly when $\sum_j P_{ij}u_j = 0$ for all i.

Proposition 17.27 (Fitting inequality)**.** *Let $T \subset \prod_{x\in X} O$ and $\mathfrak{c}_x$, δ_x be as above. Let R' be an O-algebra with O-algebra maps $\rho_x\colon R' \to M_{d_x}(O)$ for $x \in X$, and $P \in M_{a\times b}(R')$ with $a \geq b$. Assume:*

(F1) for every $Q \in M_b(R')$ the function $x \mapsto \det\rho_x(Q)$ lies in T;

(F2) for every $x \neq x_0$, $\ker(\rho_x(P)\otimes K) \neq 0$, that is, there is a nonzero R'-linear map $\operatorname{coker} P \to K^{d_x}$;

(F3) in a suitable O-basis, ρ_{x_0} is block upper triangular with diagonal blocks $\rho_1,\dots,\rho_g$, $\rho_k\colon R' \to M_{n_k}(O)$, and the map $R' \to \prod_k M_{n_k}(k_O)$, $r \mapsto (\bar\rho_k(r))_k$, is surjective, where k_O is the residue field;

(F4) $|k_O| > b\sum_k n_k$.

Then

$$\delta_{x_0} \;\leq\; \sum_{k=1}^{g} \operatorname{len}_O \operatorname{Hom}_{R'}\big(\operatorname{coker} P, (K/O)^{n_k}\big) = \sum_{k=1}^{g} \operatorname{len}_O \ker\big(\rho_k(P) \text{ on } ((K/O)^{n_k})^b\big).$$

Proof. If some term on the right is infinite there is nothing to prove, so assume all are finite. Then each $\rho_k(P)\otimes K$ is injective: a nonzero K-kernel vector, scaled, gives an injection of K/O into the kernel on $(K/O)^{bn_k}$, of infinite length. Let $W_k \subset O^{an_k}$ be the saturation of the image of $\rho_k(P)\colon O^{bn_k} \to O^{an_k}$, a direct summand of rank bn_k.

Choice of C. For $C \in M_{b\times a}(R')$ put $Q = CP \in M_b(R')$. Then $\rho_k(Q) = \rho_k(C)\rho_k(P)$. There is C such that for every k the map $\rho_k(C)\colon O^{an_k} \to O^{bn_k}$ restricts to an isomorphism $W_k \to O^{bn_k}$. By

Nakayama it suffices that $\bar{\rho}_k(C)$ restricts to an isomorphism $\overline{W}_k \to k_O^{bn_k}$, where $\overline{W}_k = W_k/\varpi W_k \subset k_O^{an_k}$ has dimension bn_k (W_k is a direct summand). By (F3), as C ranges over $M_{b\times a}(R')$ the tuple $(\bar{\rho}_k(C))_k$ ranges over all of $\prod_k M_{bn_k\times an_k}(k_O)$, choosing each entry of C independently. Fix bases of the $\overline{W}_k$. The function $\Phi(Y_1,\dots,Y_g) = \prod_k \det(Y_k|_{\overline{W}_k})$ is a polynomial on $\prod_k M_{bn_k\times an_k}(k_O)$ of total degree $\sum_k bn_k$. It is not the zero polynomial, because for each k a matrix Y_k that maps $\overline{W}_k$ isomorphically onto $k_O^{bn_k}$ exists over k_O. A nonzero polynomial of degree D over a field with more than D elements has a non-zero value at a point with coordinates in that field (induction on the number of variables). By (F4) there is such a point, and by (F3) it is $(\bar{\rho}_k(C))_k$ for some C.

The function. Let $f(x) = \det\rho_x(Q)$. By (F1), $f \in T$. For $x \neq x_0$, $\ker\rho_x(Q)\otimes K \supseteq \ker\rho_x(P)\otimes K \neq 0$ by (F2), so $f(x) = 0$. Hence $f(x_0) \in \mathfrak{c}_{x_0}$ and $\delta_{x_0} \leq v(f(x_0))$.

The value at x_0. After reordering the basis, $\rho_{x_0}(Q)$ is block upper triangular with diagonal blocks $\rho_k(Q)$, so $f(x_0) = \prod_k \det\rho_k(Q)$. For each k, $\rho_k(C)$ maps $\operatorname{im}\rho_k(P) \subset W_k$ into O^{bn_k} with $O^{bn_k}/\rho_k(C)\operatorname{im}\rho_k(P) \cong W_k/\operatorname{im}\rho_k(P) = \operatorname{coker}(\rho_k(P))_{\text{tors}}$. By Lemma 17.26, $v(\det\rho_k(Q)) = \operatorname{len}\operatorname{coker}\rho_k(Q) = \operatorname{len}\operatorname{coker}(\rho_k(P))_{\text{tors}} = \operatorname{len}\ker(\rho_k(P)\otimes K/O)$. Summing over k gives the claim. □

In the application R' is the algebra generated by the Galois group acting on an integral model of the representation attached to an eigensystem, P presents the universal cocycle module of the reducible representation $\rho_\pi \oplus \bigoplus_c \chi^c$ with its local relations, and (F2) is a nonzero map of Ribet type at every eigensystem outside the congruence class. The application of Proposition 17.27 also needs a descent along a map of Hecke algebras and the joint residual surjectivity (F3) for the blocks $\rho_\pi, \chi^1, \dots, \chi^{t_r}$. The descent, the verification of (F2) and of (F3) for the blocks that occur, and step three are not proved here. Step three consists of the following statements.

1. *Block bounds.* For a rank one block with character χ^c, the length of the space of homomorphisms from the cocycle module to K/O is at most $\operatorname{len} Ш_l(E, c-1)$ plus local defects at the places above l, at the auxiliary places and at the column places. For the block χ^1 the Selmer term is $v_l(h_E)$.
2. *The defect at l.* A lifting theorem for crystalline classes along surjections of crystalline lattices with Hodge–Tate weights in $[-h,h]$, proved with Wach modules, whose loss is $O(h/(l-1)^{j+1} + g\ell_{\max} + \log_l h)$ with main term $hl/(l-1)^2$.
3. *The twist.* A ray class character of auxiliary conductor that removes the block ρ_π.
4. *Twisted theta series.* Rationality of twisted theta series, their l-integrality with bounded denominators, the action of Hecke operators on integral models, and a reciprocity law of Shimura type.
5. *The sums* over the eigensystems π, the places w and the primes l, including the rows at 2 and 3.

The published inputs of these statements include the Galois representations attached to regular algebraic polarized cuspidal automorphic representations of GL_n over CM fields with local–global compatibility at all places [4], the local Langlands correspondence for GL_n, Poitou–Tate duality and the Euler characteristic formulas [50], the theory of Bloch and Kato for H^1_f, Fontaine's period rings, Wach modules and Berger's description of Wach lattices, the Herr complex, Berger's theorem that de Rham representations are potentially semistable [7], class field theory and the Chebotarev density theorem.

The local Hecke setting

Throughout this subsection O is a complete discrete valuation ring with fraction field K, uniformizer ϖ, normalized valuation v and residue field k_O, and l is the residue characteristic, so that $v(l) = \mathrm{len}_O(O/lO)$. We write $e_O = v(l)$. Let X be a finite set, let $\mathbb{T} \subset \prod_{x\in X} O$ be an O-subalgebra which separates the points of X, let M be a free O-module of finite rank with an O-algebra map $\mathbb{T} \to \mathrm{End}_O(M)$, and put $V = M \otimes_O K$, in which M is a lattice. For $x \in X$ put $V_x = \{u \in V : tu = t(x)u \text{ for all } t \in \mathbb{T}\}$ and $\mathrm{mult}(x) = \dim_K V_x$. For $Y \subset X$ write $V_Y = \sum_{x\in Y} V_x$. For $Y \subset Z \subset X$ put

$$\mathfrak{c}_Z = \{t \in \mathbb{T} : t(y) = 0 \text{ for all } y \notin Z\}, \qquad \kappa_Z(x) = \min\{v(t(x)) : t \in \mathfrak{c}_Z\} \in \mathbb{N} \cup \{\infty\},$$

$$M_Y^{\mathrm{up}} = V_Y \cap (M + V_{X\setminus Y}), \qquad M_Y^{\mathrm{low}} = M \cap V_Y, \qquad \eta(M, Y) = \frac{\mathrm{len}_O(M_Y^{\mathrm{up}}/M_Y^{\mathrm{low}})}{e_O}.$$

The ideal $\mathfrak{c}_{\{x\}}$ is the congruence ideal $\mathfrak{c}_x$ of Proposition 17.25, and $\kappa_Z(x) \le \kappa_{\{x\}}(x) = \delta_x$ for $x \in Z$.

Lemma 17.28. *There are $C \in O \setminus \{0\}$ and $e_x \in \mathbb{T}$ for $x \in X$ with $e_x(y) = C$ if $y = x$ and $e_x(y) = 0$ otherwise. The operators $P_x = C^{-1}e_x$ on V are orthogonal idempotents with $\sum_x P_x = 1$ and $P_xV = V_x$. In particular $V = \bigoplus_{x\in X} V_x$, and for $Y \subset X$ the projection $\pi_Y = \sum_{x\in Y} P_x$ of V onto V_Y along $V_{X\setminus Y}$ commutes with $\mathbb{T}$.*

Proof. For $x \neq y$ choose $t \in \mathbb{T}$ with $t(x) \neq t(y)$ and put $s_{xy} = t - t(y)\cdot 1 \in \mathbb{T}$, so that $s_{xy}(y) = 0 \neq s_{xy}(x)$. The product $f_x = \prod_{y\neq x} s_{xy} \in \mathbb{T}$ has $f_x(x) = c_x \neq 0$ and vanishes off x. With $C = \prod_x c_x$ the elements $e_x = (C/c_x)f_x$ lie in $\mathbb{T}$ and have the stated values. As functions on X, $e_xe_y = \delta_{xy}Ce_x$, $\sum_x e_x = C \cdot 1$ and $te_x = t(x)e_x$ for $t \in \mathbb{T}$. Applying the algebra map to $\mathrm{End}_K(V)$ gives $P_xP_y = \delta_{xy}P_x$, $\sum_x P_x = 1$, and $tP_x = t(x)P_x$, so $P_xV \subset V_x$. Conversely P_x acts on V_x as $C^{-1}e_x(x) = 1$, so $V_x \subset P_xV$. □

Lemma 17.29. *For $Y \subset X$, $M_Y^{\mathrm{up}} = \pi_Y(M)$. Hence $M_Y^{\mathrm{up}}/M_Y^{\mathrm{low}}$ is the module $M^{\mathrm{up}}/M^{\mathrm{low}}$ of Proposition 17.25 for $Z = Y$ and the decomposition of Lemma 17.28.*

Proof. If $u \in V_Y$ and $u = \lambda + b$ with $\lambda \in M$ and $b \in V_{X\setminus Y}$, then $u = \pi_Y(u) = \pi_Y(\lambda)$. Conversely $\pi_Y(\lambda) = \lambda - \pi_{X\setminus Y}(\lambda) \in (M + V_{X\setminus Y}) \cap V_Y$. □

Identification of the congruence defects

Let w be a twisted place, $l = p_w$, $1 \le j \le m$, and let Λ and $\mathbb{T}$ be as in Definition 17.1, with $\mathcal{O}$ enlarged, if necessary, by a finite unramified extension so that every eigensystem of $\mathbb{T}$ on $\Lambda \otimes \mathbb{Q}$ has values in $\mathcal{O}$. This changes no length in Definition 17.1, because for a finite unramified extension $\mathcal{O}'$ of $\mathcal{O}$ and a finite length $\mathcal{O}$-module N one has $\mathrm{len}_{\mathcal{O}'}(N \otimes_{\mathcal{O}} \mathcal{O}') = \mathrm{len}_{\mathcal{O}} N$, and intersections and sums of submodules commute with the flat base change $\otimes_{\mathcal{O}}\mathcal{O}'$.

Proposition 17.30. *The algebra $\mathbb{T}$ is the product $\prod_{\mathfrak{m}} \mathbb{T}_{\mathfrak{m}}$ of its localizations at its maximal ideals, with idempotents $\epsilon_{\mathfrak{m}} \in \mathbb{T}$, and $\Lambda = \bigoplus_{\mathfrak{m}} \Lambda_{\mathfrak{m}}$ with $\Lambda_{\mathfrak{m}} = \epsilon_{\mathfrak{m}}\Lambda$. For each $\mathfrak{m}$ let $X_{\mathfrak{m}}$ be the set of eigensystems of $\mathbb{T}$ on $\epsilon_{\mathfrak{m}}(\Lambda \otimes \mathbb{Q})$, so that $\mathbb{T}_{\mathfrak{m}}$ is an $\mathcal{O}$-subalgebra of $\prod_{x\in X_{\mathfrak{m}}} \mathcal{O}$ separating points, and let $G_{\mathfrak{m}} \subset X_{\mathfrak{m}}$ be the eigensystems which occur in the lifts of level j. Then*

$$\Lambda_{\mathfrak{m}} = \epsilon_{\mathfrak{m}}(\Lambda \otimes \mathbb{Q}) \cap \big(\Lambda + (1 - \epsilon_{\mathfrak{m}})(\Lambda \otimes \mathbb{Q})\big), \qquad \eta_j(w) = \sum_{\mathfrak{m}} \eta(\Lambda_{\mathfrak{m}}, G_{\mathfrak{m}}),$$

and each summand is the normalized length of the module $M^{\mathrm{up}}/M^{\mathrm{low}}$ of Proposition 17.25 for $(\mathcal{O}, X_{\mathfrak{m}}, \mathbb{T}_{\mathfrak{m}}, \Lambda_{\mathfrak{m}})$ and $Z = G_{\mathfrak{m}}$.

Proof. The algebra $\mathbb{T}$ is finite over the complete local ring $\mathcal{O}$, so it is the product of its localizations at its finitely many maximal ideals, and the corresponding idempotents $\epsilon_{\mathfrak{m}}$ lie in $\mathbb{T}$ (idempotents lift along $\mathbb{T} \to \mathbb{T}/\varpi\mathbb{T}$ by completeness). Since $\epsilon_{\mathfrak{m}} \in \mathbb{T}$ preserves Λ, $\Lambda = \bigoplus_{\mathfrak{m}} \epsilon_{\mathfrak{m}}\Lambda$. If $u = \epsilon_{\mathfrak{m}} u = \lambda + (1-\epsilon_{\mathfrak{m}})u'$ with $\lambda \in \Lambda$, applying $\epsilon_{\mathfrak{m}}$ gives $u = \epsilon_{\mathfrak{m}}\lambda \in \Lambda_{\mathfrak{m}}$, and conversely $\epsilon_{\mathfrak{m}}\lambda = \lambda - (1-\epsilon_{\mathfrak{m}})\lambda$. This is the first formula. In Definition 17.1 the summand at $\mathfrak{m}$ is the normalized length of $(V_{\mathrm{good}} \cap (\Lambda_{\mathfrak{m}} + V_{\mathrm{bad}}))/(\Lambda_{\mathfrak{m}} \cap V_{\mathrm{good}})$ with $V_{\mathrm{good}} = V_{G_{\mathfrak{m}}}$ and $V_{\mathrm{bad}} = V_{X_{\mathfrak{m}} \setminus G_{\mathfrak{m}}}$ inside $\Lambda_{\mathfrak{m}} \otimes \mathbb{Q}$, which is $\eta(\Lambda_{\mathfrak{m}}, G_{\mathfrak{m}})$ by definition, with $e_{\mathcal{O}} = \mathrm{len}_{\mathcal{O}}(\mathcal{O}/l)$. Lemma 17.29 identifies the module with $M^{\mathrm{up}}/M^{\mathrm{low}}$. □

Proposition 17.31 (the congruence bound at one maximal ideal)**.** *In the local Hecke setting let $Z' \subset Z \subset X$, $A \in \mathbb{Z} \setminus \{0\}$, and let $\alpha \in \mathrm{End}_O(M)$ act on V_x as multiplication by A for $x \in Z'$ and as 0 for $x \in Z \setminus Z'$. Then*

$$\eta(M, Z') \;\le\; \sum_{x \in Z'} \mathrm{mult}(x)\Big(\frac{\kappa_Z(x)}{e_O} + v_l(A)\Big).$$

Proof. If $\kappa_Z(x) = \infty$ for some $x \in Z'$ there is nothing to prove. Extending scalars to a finite unramified extension of O changes neither side, as explained before Proposition 17.30: lengths are preserved, $M^{\mathrm{up}}_{Z'}$ and $M^{\mathrm{low}}_{Z'}$ commute with the flat base change, $\mathfrak{c}_Z$ is the kernel of the restriction $\mathbb{T} \to \prod_{y \notin Z} O$ and so commutes with it too, and the minimum of the valuations of O'-linear combinations of the values $t(x)$, $t \in \mathfrak{c}_Z$, is $\kappa_Z(x)$. So we may assume $|k_O| > |Z'|$.

A uniform element. For $x \in Z'$ choose $t_x \in \mathfrak{c}_Z$ with $v(t_x(x)) = \kappa_Z(x)$, and write $t_y(x) = \varpi^{\kappa_Z(x)} u_{yx}$ with $u_{yx} \in O$, so that $u_{xx} \in O^\times$. For $\lambda \in O^{Z'}$ the element $t = \sum_y \lambda_y t_y \in \mathfrak{c}_Z$ has $t(x) = \varpi^{\kappa_Z(x)} \sum_y \lambda_y u_{yx}$. For each x the reduction of $\lambda \mapsto \sum_y \lambda_y u_{yx}$ is a nonzero linear form on $k_O^{Z'}$, and a vector space over k_O is not the union of $|Z'| < |k_O|$ proper hyperplanes. So there is λ with $v(t(x)) = \kappa_Z(x)$ for every $x \in Z'$.

The operator. Put $\beta = \alpha \circ t$ on V. For $\lambda \in M$, $t\lambda \in M$ and $t\lambda = \sum_{x \in Z} t(x) P_x \lambda \in V_Z$ by Lemma 17.28, so $\alpha t\lambda = A \sum_{x \in Z'} t(x) P_x \lambda$ lies in $M \cap V_{Z'}$. The same formula gives $\beta(\pi_{Z'}\lambda) = A \sum_{x \in Z'} t(x) P_x \lambda$. With Lemma 17.29,

$$\beta\, M^{\mathrm{up}}_{Z'} \subset M^{\mathrm{low}}_{Z'} \subset M^{\mathrm{up}}_{Z'}.$$

The module $M^{\mathrm{up}}_{Z'} = \pi_{Z'}(M)$ is a lattice in $V_{Z'}$, and β acts on $V_{Z'}$ diagonally, as $At(x)$ on V_x, with determinant $\prod_{x \in Z'} (At(x))^{\mathrm{mult}(x)} \neq 0$. Lemma 17.26 gives

$$\mathrm{len}_O\left(M^{\mathrm{up}}_{Z'}/M^{\mathrm{low}}_{Z'}\right) \le \mathrm{len}_O\left(M^{\mathrm{up}}_{Z'}/\beta M^{\mathrm{up}}_{Z'}\right) = \sum_{x \in Z'} \mathrm{mult}(x)\big(v(A) + \kappa_Z(x)\big).$$

Divide by e_O and use $v(A) = e_O\, v_l(A)$ for $A \in \mathbb{Z}$. □

With $\alpha = 1$, $A = 1$ and $Z' = Z$ Proposition 17.31 gives $\eta(M, Z) \le e_O^{-1} \sum_{x \in Z} \mathrm{mult}(x)\kappa_Z(x)$, which refines Proposition 17.25(ii) because $\kappa_Z(x) \le \delta_x$.

Proposition 17.32 (removing eigensystems)**.** *In the local Hecke setting let $G \subset S \subset X$, $A \in \mathbb{Z} \setminus \{0\}$, and let $\alpha \in \mathrm{End}_O(M)$ act on V_x as A for $x \in G$ and as 0 for $x \notin S$. Let $\bar{M} = M/(M \cap V_{X \setminus S})$, identified by π_S with the lattice $\pi_S(M)$ of V_S, with the action of the restriction of $\mathbb{T}$ to S. Then $M \cap V_{X \setminus S} = \{m \in M : tm = 0 \text{ for all } t \in \mathfrak{c}_S\}$, and*

$$\eta(M, G) \le \eta(\bar{M}, G) + v_l(A) \sum_{x \in G} \mathrm{mult}(x).$$

Proof. An element $t \in \mathfrak{c}_S$ kills $V_{X\setminus S}$, and $t_S = \sum_{x\in S} e_x \in \mathfrak{c}_S$ acts as $C\pi_S$, so $m \in M$ is killed by $\mathfrak{c}_S$ exactly when $\pi_S m = 0$. Since $\pi_G = \pi_G\pi_S$, $M_G^{\mathrm{up}} = \pi_G(M) = \pi_G(\bar{M}) = \bar{M}_G^{\mathrm{up}}$, and $M \cap V_G \subset \bar{M} \cap V_G$. If $u \in \bar{M} \cap V_G$, say $u = \pi_S\lambda$ with $\lambda \in M$, then $\lambda = u + \pi_{X\setminus S}\lambda$ and $\alpha\lambda = Au \in M \cap V_G$. So $A(\bar{M} \cap V_G) \subset M \cap V_G \subset \bar{M} \cap V_G$, and since $\bar{M} \cap V_G$ is a lattice in V_G,

$$\operatorname{len}_O \frac{M_G^{\mathrm{up}}}{M \cap V_G} = \operatorname{len}_O \frac{\bar{M}_G^{\mathrm{up}}}{\bar{M} \cap V_G} + \operatorname{len}_O \frac{\bar{M} \cap V_G}{M \cap V_G} \le \operatorname{len}_O \frac{\bar{M}_G^{\mathrm{up}}}{\bar{M} \cap V_G} + v(A)\dim_K V_G.$$

Divide by e_O. □

Lemma 17.33 (descent along a map of Hecke algebras)**.** *Let $\mathbb{T} \subset \prod_{x\in X} O$ and $\mathbb{T}' \subset \prod_{x'\in X'} O$ be O-subalgebras, $p\colon X' \to X$ a map, $Z \subset X$, and $b_1, \dots, b_q\colon X' \to O$. Assume that every $f' \in \mathbb{T}'$ can be written $f' = \sum_k (a_k \circ p)\, b_k$ with $a_k \in \mathbb{T}$, and that for $x \notin Z$ the restrictions of $b_1, \dots, b_q$ to $p^{-1}(x)$ are linearly independent over O. If $f' \in \mathbb{T}'$ vanishes on $p^{-1}(X \setminus Z)$, then for every $\pi' \in X'$, $f'(\pi')$ lies in the ideal of O generated by $\{c(p(\pi')) : c \in \mathfrak{c}_Z\}$, and $v(f'(\pi')) \ge \kappa_Z(p(\pi'))$.*

Proof. Write $f' = \sum_k (a_k \circ p) b_k$. For $x \notin Z$ and $x' \in p^{-1}(x)$, $\sum_k a_k(x) b_k(x') = f'(x') = 0$, so $a_k(x) = 0$ for all k by the independence. Hence $a_k \in \mathfrak{c}_Z$ and $f'(\pi') = \sum_k a_k(p(\pi')) b_k(\pi')$. For the valuation, $v(\sum_i r_i y_i) \ge \min_i v(y_i)$ in a discrete valuation ring. □

Proposition 17.34 (from the trimmed data to Hypothesis 17.24)**.** *Fix an admissible datum and $1 \le j \le m$. For each twisted place w let the local components i at (j, w) be the maximal ideals of Proposition 17.30, with good sets G_i, and suppose that at each i with $G_i \neq \emptyset$ there are $S_i \supset G_i$ and α_i as in Proposition 17.32 with $A = |\Gamma|$ an integer independent of i and w. Write $\eta_j^{\mathrm{cut}}(w) = \sum_i \eta(\bar{\Lambda}_i, G_i)$. Assume*

(a) $\sum_i \sum_{x\in G_i} \operatorname{mult}(x) \le d_j^0$ *for every twisted w;*

(b) $\log|\Gamma| \le C_1 x^{C_1} L_\varepsilon$, $x^{C_1} \le b$, $L_\varepsilon \ge 1$ *and* $f, m \ge 1$;

(c) $\sum_{w \text{ twisted}} \ell_w e_w\, \eta_j^{\mathrm{cut}}(w) \le d_j^0 \big(C(fmbL_\varepsilon^2 + (\log x)^2 N_c) + \sum_l \log l \sum_{1\le e<t_r} \operatorname{len} Ш_l(E, e) \big)$.

Then the inequality of Hypothesis 17.24 holds with the constant $C + C_1$.

Proof. At a component with $G_i = \emptyset$ both $\eta(\Lambda_i, G_i)$ and $\eta(\bar{\Lambda}_i, G_i)$ vanish. Propositions 17.30 and 17.32 and (a) give $\eta_j(w) \le \eta_j^{\mathrm{cut}}(w) + d_j^0\, v_{p_w}(|\Gamma|)$. With $\ell_w = \log N\mathfrak{p}_w / [\mathbf{K} : \mathbb{Q}] = f_w \log p_w / [\mathbf{K} : \mathbb{Q}]$ and $\sum_{w|p} e_w f_w = [\mathbf{K} : \mathbb{Q}]$,

$$\begin{aligned} \sum_{w \text{ twisted}} \ell_w e_w\, v_{p_w}(|\Gamma|) &\le \sum_p v_p(|\Gamma|) \log p = \log|\Gamma| \\ &\le C_1 x^{C_1} L_\varepsilon \le C_1 b L_\varepsilon^2 \le C_1 \big(fmbL_\varepsilon^2 + (\log x)^2 N_c \big), \end{aligned}$$

because $N_c \ge 0$. Multiply by d_j^0 and add (c). □

The rank d_j^0

Let $\theta_j\colon \widetilde{M}_{\mu_0,j} \to M_j$ be the theta map from the forms of level j and standard type to the space M_j of level j, let θ_j^* be its adjoint, and let $d_j^0 = \dim Z_j$ with $Z_j = \theta_j^*(M_j)$, the space of lifted forms of level j and standard type. Let D_0 be the number of base rows, so that $D_0 = \sum_\sigma n_\sigma^0$ over the isotypic types σ with multiplicities n_σ^0.

Proposition 17.35. *Suppose that* $\dim M_j = D_0$ *for* $1 \le j \le m$. *Then* $d_j^0 \le D_0$. *If moreover* θ_j *is surjective, then* $d_j^0 = D_0$. *Surjectivity holds when the level projection at the standard type is surjective,* θ_j *restricted to its image is the composite of that projection, a bijection* $\theta_{j,t}$ *of coordinate spaces and a coordinate isomorphism onto* M_j.

Proof. $d_j^0 = \dim \theta_j^*(M_j) \le \dim M_j = D_0$. If θ_j is surjective, θ_j^* is injective, because $\langle \theta_j^* z, u\rangle = \langle z, \theta_j u\rangle$ vanishes for all u only if $z = 0$, and then $d_j^0 = \dim M_j$. The last statement is the composition of surjective maps. □

Theorem 17.37 only needs $d_j^0 \le D_0$: the bracket in Hypothesis 17.24 is nonnegative once $C \ge 0$, which may be assumed, so the bound is monotone in d_j^0.

The local estimates

Propositions 17.30 to 17.34 reduce Hypothesis 17.24 to the following statement, with $\kappa = \kappa_{Z_i}$ for the set $Z_i \supset G_i$ of eigensystems which admit a nonzero map of Ribet type, so that after the trimming of Proposition 17.32 one has $Z_i \cap S_i = G_i$ and Proposition 17.31 applies with $Z' = Z = G_i$ and $\alpha = 1$.

Hypothesis 17.36 (per-component congruence bound)**.** There is C such that at every component i of residue characteristic l at a twisted place w and every $\pi \in G_i$,

$$\frac{\kappa(\pi)}{e_O} \le s_\pi + b_l + \sum_{0<e<t_r} \operatorname{len} Ш_l(E,e) + v_l(h_E) + \varepsilon_l,$$

where s_π is the local term of ρ_π and $\varepsilon_l = l^{-2}$. Away from the two auxiliary primes one also needs

$$s_\pi \le \nu_l + \varepsilon_l, \qquad \frac{t_r}{e_O} \sum_{v \in Q_i} \operatorname{len}_O\big(O/(\zeta_{i,v} - 1)\big) \le \varepsilon_l,$$

where ν_l is the cyclotomic exponent of the host at l. At 3 and the small primes of the host take $b_l = C(fmb \log x + x^C L_\varepsilon^2)$, and at q, d take $b_l = 2Cfmb$. At the other nonauxiliary primes above l_* take $b_l = 0$ outside an exceptional set of at most $(N_T + 2)(S_c + 3\eta + t_r + 8b/l_* + 9)$ primes $l \le 3b$. For each remaining small or exceptional prime the local argument requires a length estimate $b_l \le Cfnt_r^2 n_V L_\varepsilon^2 + \mathcal{L}(l)$, where $\mathcal{L}(l)$ is the contribution of the split or inert local blocks. The weighted sums of these lengths over the two sets of primes also require bounds. At the two auxiliary primes $l = g_k$ one requires the bound $\eta(\bar{\Lambda}_i, G_i) \le \sum_{\pi \in G_i} \operatorname{mult}(\pi) r_\pi$ with

$$r_\pi = 4fn_V e_T + 2fr^2 + s_\pi + \sum_{0<e<t_r} \operatorname{len} Ш_l(E,e) + v_l(h_E) + 4n_V \nu_i$$
$$+ t_r \sum_{q \in Q_i} \frac{v(\zeta_{i,q} - 1)}{e_O}, \qquad e_T = Cn_V^2(n_V + \eta + t_r),$$

where $\nu_i = \sum_{0<k<t_r} \big(s_{0,i} + v_l(k)\big)$ with $s_{0,i}$ the local level exponent at l, $n_V = r + t_r$, and $\zeta_{i,q}$ is the value of the character χ^1 on a tame generator at $q \in Q_i$. Together with these, the summation of the resulting bounds over π, i, w and l gives condition (c) of Proposition 17.34.

Lemma 17.33 and Proposition 17.27 bound $\kappa(\pi)$ by the Fitting lengths of the cocycle module, once (F2) and (F3) hold for the blocks $\rho_\pi, \chi^1, \dots, \chi^{t_r}$. The bounds of the Fitting lengths are the block bounds, the lifting theorem with Wach modules at the places above l, the twist and the statements on twisted theta series listed after Proposition 17.27. Hypothesis 17.36, the verification of (F2) and (F3), and the summation are not proved here.

For the local lifting at $l = 2$, first pass to $K_N = \mathbb{Q}_l(\mu_{\mathrm{lcm}(l,N)})$ with $16 \mid N$. The cyclotomic group over K_N is $1 + 2^{v_2(N)}\mathbb{Z}_2$, hence is torsion free and procyclic. Herr's cyclotomic complex applies over this field. For $K \subset K_N$ restriction followed by corestriction is multiplication by $[K_N : K]$, with loss $v_l([K_N : K])$ in the lifting bound. The quantitative estimate for the crystalline lattice over K_N is still required.

17.6 The bound for the congruence defects

Theorem 17.37. *Let $0 < \varepsilon < \frac{1}{52}$, and suppose that the admissible data at the scale $x \geq x_0$ satisfy condition (D6) of Definition 18.1, that the conclusion of Hypothesis 17.24 holds at x, that $(\log x)^2 \leq x^\varepsilon$, and that $d_j^0 \leq D_0$ for $1 \leq j \leq m$. Then for every $1 \leq j \leq m$*

$$\sum_{w \text{ twisted}} \ell_w \, e_w \, \eta_j(w) \leq (C_N + 1)\, D_0 \, x^\varepsilon \, (B_0 + N_c).$$

Proof. Put $y = x^\varepsilon = L_\varepsilon \geq 1$. Since $B_0 = fmby$, the first term in the bracket of Hypothesis 17.24 is $fmby^2 = yB_0$, and $(\log x)^2 N_c \leq yN_c$ by hypothesis. By (D6), $V \leq \sqrt{b} \leq b \leq B_0 \leq y(B_0 + N_c)$, where $b \leq B_0$ because $fmy \geq 1$. Hence the bracket is at most $C_N y(B_0 + N_c) + y(B_0 + N_c)$. $\square$

Theorem 17.37 enters the bound for the heights h_u of the period maps B_u in (3.4) through the local terms E_b of Hypothesis 15.16, and the bound for the spread of the slopes of the family (N_u) in (3.3). The local terms E_u of Definition 16.5 do not depend on $\eta_j(w)$. Proposition 17.35 gives the required inequality once $\dim M_j = D_0$.

18 The proof of Theorem A

This section assembles the proof of Theorem A. It fixes the form of the admissible data that the arithmetic part of the argument uses, it constructs the weights a_t and the nodes $w_{u_0,j}$, it states the inequalities that the chain of theta lifts satisfies, and it deduces from them the seven hypotheses (3.1) to (3.7) of Proposition 3.1 together with the bound for the errors. The inequalities between slopes, heights and local terms are proved in the sections on the gain and the loss, on the slope inequality, on the archimedean side and on the twisted places, and Hypothesis 18.10 below collects them in one statement.

18.1 Admissible data on Selmer windows

The bound for the local terms at the twisted places uses a bound for Tate–Shafarevich groups of the host field. That bound holds on windows of scales whose position depends on the auxiliary primes q and d. We therefore attach to admissible data a base scale Y_0, at which q and d are chosen, and a window $[Y, Y^{3/2}]$ above it, on which the Selmer bound holds.

For a number field F, a prime l and an integer i, let S_l be the set of places of F above l and the archimedean places, let G_{F,S_l} be the Galois group of the maximal extension of F unramified

outside S_l, and let $\mathbb{Q}_l/\mathbb{Z}_l(-i)$ be the group $\mathbb{Q}_l/\mathbb{Z}_l$ on which G_{F,S_l} acts through the $(-i)$-th power of the l-adic cyclotomic character. Put

$$Ш_l(F,i) = \ker\Big(H^1\big(G_{F,S_l}, \mathbb{Q}_l/\mathbb{Z}_l(-i)\big) \to \prod_{v|l} H^1\big(F_v, \mathbb{Q}_l/\mathbb{Z}_l(-i)\big)\Big).$$

For F and $t_0 \geq 1$ put

$$V(F,t_0) = \sum_l \log l \sum_{i=1}^{t_0-1} \operatorname{length}_{\mathbb{Z}_l} Ш_l(F,i),$$

the sum over all primes l, whenever every group $Ш_l(F,i)$ with $1 \leq i < t_0$ has finite length.

Fix a primitive Dirichlet character ψ_0 of conductor M_ψ and order o, put $N_0 = 8M_\psi$, and fix $0 < \varepsilon < \frac{1}{52}$.

Definition 18.1. Let $J \geq 2$, and let x, Y and Y_0 be real numbers with $3 \leq Y_0 \leq Y \leq x \leq Y^{3/2}$ and $Y^2 \leq Y_0^J$. *Admissible data at the scale x for the window $[Y, Y^{3/2}]$ with base Y_0* consist of the following objects.

(D1′) Primes q and d that satisfy the conclusions of Theorems 5.10 and 5.12 with N_0, $K = J$, ε, $\tau = \varepsilon$, with Y_0 in place of Y, and with $z' = z\log^2 Y_0$, where $z = Y_0^{\varepsilon/5}$. An integer $e \geq 0$ with $x^{2+8\varepsilon} \leq c\,2^e \leq 2x^{2+8\varepsilon}$, where c is the integer of Proposition 5.14(iii). The field E, the groups H and A_0 and the integer $n = |H|$ of Proposition 5.14 for ψ_0, e, q and d, with the identification of $\mathrm{Gal}(E/\mathbb{Q})$ with $\mathcal{G}$ of Section 5.7.

(D2)–(D5) The objects (D2) to (D5) of Definition 6.1.

(D6) With $t_0 = \lfloor x^{400} \rfloor$, every group $Ш_l(E,i)$ with l prime and $1 \leq i < t_0$ has finite length, and $V(E,t_0) \leq \exp(\frac{1}{2}x^\varepsilon)$.

Condition (D6) asks that $V(E,t_0) \leq \sqrt{b}$ with $b = e^{L_\varepsilon}$ as in (5.17). When $Y_0 = Y$, condition (D1′) is condition (D1) of Definition 6.1 with $K = J$ in place of $K = \frac{3}{2}$. The conclusions (iii) of Theorems 5.10 and 5.12 at the scale Y_0 hold on the window $[Y, Y^{3/2}]$, because this interval lies in $[Y_0, Y_0^J]$.

The integer t_r of Proposition 5.15 is the number of copies of the auxiliary space added at each step of the chain, and the twists $i < t_r$ are the twists at which the twisted places are examined. The next lemma shows that (D6) covers all of them.

Lemma 18.2. *Let $0 < \varepsilon < \frac{1}{52}$. There is x_0 such that for every $x \geq x_0$ and all integers n, m, k, t_r with $x^{2+8\varepsilon} \leq n \leq 2x^{2+8\varepsilon}$ that satisfy the conclusions of Proposition 5.15(i), we have $t_r \leq x^{400} - 1$.*

Proof. Put $r = k + m - 1$ and $f = 2n$. Proposition 5.15(i) gives $m \geq \sqrt{x\delta_1} = x^{1/2-\varepsilon/16}$, $r \leq 2k \leq 4x^{9/2+10\varepsilon}$, $k - 1 \leq 2x^{9/2+10\varepsilon}$ and $f \leq 4x^{2+8\varepsilon}$. Then

$$\frac{f^2(k-1)^3L_\varepsilon^2}{m} \leq 16 \cdot 8\, x^{4+16\varepsilon+27/2+30\varepsilon+2\varepsilon-1/2+\varepsilon/16} = 128\, x^{17+\frac{769}{16}\varepsilon},$$

and $f \leq 4x^{17}$, so the definition of n_W gives

$$n_W \leq 8 + 10\,r\Big(f + \frac{f^2(k-1)^3L_\varepsilon^2}{m}\Big) \leq 8 + 5280\, x^{\frac{43}{2}+\frac{929}{16}\varepsilon} \leq 8 + 5280\, x^{22.625},$$

because $\frac{929}{16}\varepsilon < \frac{929}{832} < \frac{9}{8}$. With $T_O \leq 2x^{377}$ and $8fr \leq 128x^7$, the relation $r + t_r = 8fr + 4 + T_O n_W$ gives $t_r \leq 128x^7 + 4 + 2x^{377}(8 + 5280x^{22.625}) \leq x^{400} - 1$ for $x \geq x_0$, because $377 + 22.625 < 400$. □

Proposition 18.3 (the Selmer window). *Assume Hypothesis 17.11. Let ψ_0 be a primitive Dirichlet character, $0 < \varepsilon < \frac{1}{52}$ and $\alpha = 2 + 8\varepsilon$. Let $x_e \geq 1$, let $c_* \geq 1$ be an integer, and let $e(x)$ be a function on $[x_e, \infty)$ with values in the nonnegative integers such that, for every $x \geq x_e$, the integer c of Proposition 5.14(iii) for $e = e(x)$ equals c_* and $x^\alpha \leq c_* 2^{e(x)} \leq 2x^\alpha$. There are $J_1 \geq 1$ and $X_0 \geq x_e^{J_1}$ with the following property. Let $X \geq X_0$, put $Y_0 = X^{1/J_1}$ and $z = Y_0^{\varepsilon/5}$, and let q and d be primes with $q \equiv 3 \pmod 8$, $e^{z/2} < q \leq e^z$, $d \equiv 7 \pmod{16}$, $e^{z \log^2 Y_0/2} < d \leq e^{z \log^2 Y_0}$, such that q and every odd prime $p \leq z$ divide $d+1$. Then there is Y with $Y_0 \leq Y$ and $Y^2 \leq Y_0^{J_1+1}$ such that for every x with $Y \leq x \leq Y^{3/2}$ the field E of Proposition 5.14 for ψ_0, $e(x)$, q and d satisfies (D6) at the scale x.*

Proof. Put $\theta = c_*^{1/\alpha} \geq 1$, so that $\theta^\alpha = c_*$. Define e' on $[1, \infty)$ by $e'(y) = e(\theta y)$ when $\theta y \geq x_e$ and $e'(y) = \lceil \alpha \log_2 y \rceil$ otherwise. In the second case $y^\alpha \leq 2^{e'(y)} < 2y^\alpha$. In the first case

$$y^\alpha = \frac{(\theta y)^\alpha}{c_*} \leq 2^{e(\theta y)} \leq \frac{2(\theta y)^\alpha}{c_*} = 2y^\alpha.$$

So e' satisfies the normalization of Hypothesis 17.11, which gives numbers $J \geq 1$ and X_0' for e'. Put $J_1 = J$ and let $X_0 \geq \max(X_0', x_e^J)$ be so large that $X^{1/J} \geq \theta^{400}$ for $X \geq X_0$.

Let $X \geq X_0$ and let q, d be as in the statement. These are exactly the conditions of Definition 17.9 for a window pair at scale $Y_0 = X^{1/J}$. Hypothesis 17.11 gives W_0 with $Y_0 \leq W_0 \leq X^{1/2}$ such that for every $y \in [W_0, W_0^2]$, with $t = \lfloor y^{401} \rfloor$ and $E' = E(\psi_0, e'(y), q, d)$, every $Ш_l(E', i)$ with $0 < i < t$ is finite and $\sum_l \sum_{1 \leq i < t} \lambda(Ш_l(E', i)) \leq \exp(\frac{1}{2} y^\varepsilon)$.

Put $Y = \theta W_0$. Then $Y \geq W_0 \geq Y_0$, and $Y^2 = \theta^2 W_0^2 \leq \theta^2 X \leq X^{1+1/J} = Y_0^{J+1}$ because $X^{1/J} \geq \theta^2$. Let $Y \leq x \leq Y^{3/2}$ and put $y = x/\theta$. Then $y \geq W_0$, and $y \leq \theta^{1/2} W_0^{3/2} \leq W_0^2$ because $W_0 \geq Y_0 \geq \theta$. Since $\theta y = x \geq Y_0 \geq x_e$, we have $e'(y) = e(x)$.

The field $E(\psi_0, e(x), q, d)$ of Definition 17.10 is the field E of Proposition 5.14 for ψ_0, $e(x)$, q and d. Indeed q and d are odd primes larger than the conductor M_ψ of ψ_0, so $\operatorname{lcm}(2^{e+2}, M_\psi, q, d) = \operatorname{lcm}(2^{e+2}, M_\psi)\, qd$, and a class a satisfies $(\frac{a}{q}) = (\frac{a}{d}) = 1$ exactly when it is a square modulo q and modulo d, so the two fixing groups coincide. By Lemma 17.4 with S the set of places above l, the group $Ш_l(E, i)$ of Definition 17.2 is the group $Ш_l(E, i)$ of Subsection 18.1, for every $i \geq 1$, and $\lambda(Ш_l(E, i)) = \log l \cdot \operatorname{length}_{\mathbb{Z}_l} Ш_l(E, i)$. Since $x^{400} = \theta^{400} y^{400} \leq y^{401}$, the integer $t_0 = \lfloor x^{400} \rfloor$ is at most $\lfloor y^{401} \rfloor$. Hence every $Ш_l(E, i)$ with $1 \leq i < t_0$ is finite, and

$$V(E, t_0) \leq \sum_l \sum_{1 \leq i < \lfloor y^{401} \rfloor} \lambda\big(Ш_l(E, i)\big) \leq \exp\big(\tfrac{1}{2} y^\varepsilon\big) \leq \exp\big(\tfrac{1}{2} x^\varepsilon\big),$$

because the terms are nonnegative and $y \leq x$. This is (D6) at the scale x. □

Proposition 18.4. *Assume Hypothesis 17.11. For every primitive Dirichlet character ψ_0 and every $0 < \varepsilon < \frac{1}{52}$ there is $J \geq 2$ with the following property. For every y there are real numbers $Y_0 \leq Y$ with $Y \geq y$ such that, for every x with $Y \leq x \leq Y^{3/2}$, admissible data at the scale x for the window $[Y, Y^{3/2}]$ with base Y_0 exist.*

Proof. The integer c of Proposition 5.14(iii) divides o and does not depend on e once 2^{e+2} is divisible by the power of 2 in M_ψ, as in the proof of Proposition 6.2. The interval from $x^{2+8\varepsilon}$ to $2x^{2+8\varepsilon}$ has ratio 2, so for x larger than a number x_e it contains a number $c\, 2^e$ with this constant c, and we fix a function $e(x)$ on $[x_e, \infty)$ as in Proposition 18.3. Let J_1 and X_0 be the numbers of Proposition 18.3 for this function, and put $J = J_1 + 1$. Let y be given, and let $X \geq X_0$ be so large that $Y_0 = X^{1/J_1}$ satisfies $Y_0 \geq \max(y, 3)$, that Y_0 is larger than the numbers Y_0 of Theorems 5.10

and 5.12 for N_0, $K = J$, ε and $\tau = \varepsilon$, and that every $x \geq Y_0$ is larger than the number Y_0 of Proposition 6.2. Put $z = Y_0^{\varepsilon/5}$. Theorem 5.10 at the scale Y_0 gives a prime $q \equiv 3 \pmod 8$ with $e^{z/2} < q \leq e^z$ that satisfies its conclusions, because its exceptional set has at most $Y_0^{-\varepsilon}|P| < |P|$ elements. Theorem 5.12 at the scale Y_0, with this q and with $z' = z \log^2 Y_0$, gives in the same way a prime d with $e^{z'/2} < d \leq e^{z'}$, $d \equiv 7 \pmod{16}$ and $d \equiv -1$ modulo q and modulo every odd prime $p \leq z$, that satisfies its conclusions. These primes satisfy the hypotheses of Proposition 18.3, which gives Y with $Y_0 \leq Y$ and $Y^2 \leq Y_0^{J_1+1} = Y_0^J$. Thus $Y \geq y$.

Let $Y \leq x \leq Y^{3/2}$. The primes q, d and $e = e(x)$ give (D1′) through Proposition 5.14. Since $Y \leq x \leq Y^{3/2} \leq Y^2$, Proposition 18.3 gives (D6). The objects (D2) to (D5) depend on x, n, E and the conductor of E, and the proof of Proposition 6.2 constructs them for $x \geq Y_0$ from Propositions 5.15, 5.20 and 5.17, statement (C9), Lemma 5.22, Linnik's theorem and Lemma 5.21. □

The proof of Theorem 6.3 uses the admissible data through the properties of q and d and through (D2) to (D5). With (D1′) in place of (D1), the same proof applies after three changes. First, the bound $|\ell_\psi| \leq 401\, c_0 K \log Y_0$ of Theorems 5.10(ii) and 5.12(ii) holds with $K = J$, and $Y_0 \leq x$ gives $|\ell_\psi| \leq 401\, c_0 J \log x$. Second, $\log q \leq Y_0^{\varepsilon/5}$ and $\log d \leq Y_0^{\varepsilon/5} \log^2 Y_0$, so $\log N_\psi \leq \lambda = A\, x^{\varepsilon/5} \log^2 x$ with a constant A that depends on ψ_0 and J, and the argument with Proposition 5.3 and Corollary 5.13 is unchanged. Third, Theorems 5.10(iii) and 5.12(iii) give $|G_\chi^*(x)| \leq x^{1/2+\varepsilon}$ for $Y_0 \leq x \leq Y_0^J$, an interval that contains the window $[Y, Y^{3/2}]$. The conclusion (6.1) then holds on windows $[Y, Y^{3/2}]$ for arbitrarily large Y, which is the hypothesis of Corollary 2.5.

18.2 The weights and the nodes

Fix admissible data at the scale x as in Definition 18.1, and let a, b, b' and L_ε be as in (5.17). Let W be the number of roots of unity in E, and put $h_0 = 2\lceil x^{402} \rceil$.

Definition 18.5. *Weights* for the data are integers a_t and b_t, for $t \in T$, with

$$a \leq a_t \leq a + \frac{a}{kx^{13}}, \qquad b' \leq b_t \leq \Big(1 + \frac{1}{x^2}\Big) b'.$$

Weights exist for large x, because $a/(kx^{13}) \geq 1$ and $b'/x^2 \geq 1$. The integers a_t are the weights of Theorem A, and the integers b_t fix the nodes at the elements of Φ attached to T.

Under the identification of Section 5.7, an element $(e_1, e_2, h) \in \mathcal{G}$ with $e_1 = 0$ lies in Φ. For $u_0 \in A_0$ we divide Φ into three sets. The *top* elements are the elements $(0, 0, t)$ with $t \in T$. The *thick* elements are the elements $(0, 1, h)$ with $h \in T_2 + u_0$, which are the elements thick with respect to u_0 for the set T_2 in the sense of Section 4.2. The remaining elements of Φ are *thin*. The top elements and the thick elements differ in the coordinate e_2, so the three sets are disjoint.

Fix an embedding $\iota\colon E \to \mathbb{C}$. The nodes are the archimedean parameters of algebraic Hecke characters of E, and the infinity type of such a character is trivial on the roots of unity of E. The next lemma makes this condition compatible with a shift of all parameters by a common integer.

Lemma 18.6. *Let the data be as above, and let $\zeta \in E$ be a root of unity. Then $\prod_{\rho \in \Phi} \iota(\rho(\zeta)) = 1$.*

Proof. The field E lies in $\mathbb{Q}(\zeta_N)$ with $N = N_1 q d$, and d divides N exactly. The inertia group of $\mathbb{Q}(\zeta_N)/\mathbb{Q}$ at d is the group of the classes $a \in (\mathbb{Z}/N)^\times$ with $a \equiv 1 \pmod{N_1 q}$. Its image in

$\mathrm{Gal}(E/\mathbb{Q}) = (\mathbb{Z}/N)^\times/U$ is the quotient of $(\mathbb{Z}/d)^\times$ by its squares, whose nontrivial element acts by -1 on $\sqrt{-d}$ and trivially on $\sqrt{-q}$ and on H. In the coordinates of Section 5.7 the inertia group of E at a prime $\mathfrak{P}$ above d is therefore $\{0, \iota_d\}$ with $\iota_d = (1,0,0)$, and the ramification index of d in E is 2.

Let r be the order of ζ. If d divided r, then E would contain $\mathbb{Q}(\zeta_d)$, in which d has ramification index $d-1$, and $d-1$ would divide 2. This is impossible because $d \geq 7$. So $d \nmid r$. The element ι_d lies in the inertia group at $\mathfrak{P}$, so $\iota_d(\zeta) \equiv \zeta \pmod{\mathfrak{P}}$. Both ζ and $\iota_d(\zeta)$ are r-th roots of unity. If they were different, $1-\eta \in \mathfrak{P}$ for an r-th root of unity $\eta \neq 1$, and $1-\eta$ divides $r = \prod_{\eta' \neq 1}(1-\eta')$, where η' runs over the r-th roots of unity different from 1. Then $r \in \mathfrak{P}$, which contradicts $d \nmid r$. Hence $\iota_d(\zeta) = \zeta$.

Complex conjugation γ_c corresponds to the class $a = -1$, so $\gamma_c = (1,1,h_c)$, where h_c is the class of -1 in H. Since E is a CM field, $\iota(\gamma_c(\zeta)) = \overline{\iota(\zeta)} = \iota(\zeta)^{-1}$. Put $s_0 = \gamma_c + \iota_d = (0,1,h_c)$. Then $s_0 \in \Phi$, $s_0 \neq 0$ and $2s_0 = 0$, and $\iota(s_0(\zeta)) = \iota(\gamma_c(\iota_d(\zeta))) = \iota(\zeta)^{-1}$. The set Φ is a subgroup of $\mathcal{G}$, and it is the disjoint union of the cosets $\{\rho, \rho + s_0\}$ for ρ in a set Φ_0 of representatives. For each ρ,

$$\iota(\rho(\zeta))\,\iota\big((\rho+s_0)(\zeta)\big) = \iota\big(\rho(\zeta\, s_0(\zeta))\big) = \iota(\rho(1)) = 1,$$

because $s_0(\zeta) = \zeta^{-1}$. The product over Φ is the product of these pairs. □

Proposition 18.7. *Let the data be as above, let $u_0 \in A_0$, and let b_t, for $t \in T$, be integers. Then there are integers $b_1(\rho)$, for $\rho \in \Phi$, with the following properties.*

(i) $b_1(0,0,t) = b_t + \lfloor b'/x \rfloor$ *for every top element* $(0,0,t)$.

(ii) $\lceil b \rceil \leq b_1(\rho) < \lceil b \rceil + W$ *for every thick element* ρ.

(iii) $b_1(\rho) = \lceil b'/(xn) \rceil$ *for every thin element* ρ.

(iv) For $1 \leq j \leq m$ *put* $w_{u_0,j}(\rho) = b_1(\rho) + (j - \frac{1}{2})h_0$. *Then each* $w_{u_0,j}(\rho)$ *is an integer, and* $\prod_{\rho\in\Phi} \iota(\rho(\zeta))^{w_{u_0,j}(\rho)} = 1$ *for every root of unity* $\zeta \in E$ *and every* j.

Proof. The group of roots of unity of E is cyclic of order W. Let ζ_W be a generator. For $\rho \in \mathcal{G}$ there is $\kappa(\rho) \in (\mathbb{Z}/W)^\times$ with $\rho(\zeta_W) = \zeta_W^{\kappa(\rho)}$, because $\rho(\zeta_W)$ is again a generator. For integers $c(\rho)$, $\rho \in \Phi$, the product $\prod_\rho \iota(\rho(\zeta))^{c(\rho)}$ equals 1 for every root of unity ζ of E exactly when it equals 1 for $\zeta = \zeta_W$, that is, when $\sum_\rho \kappa(\rho)c(\rho) \equiv 0 \pmod{W}$. Lemma 18.6 with $\zeta = \zeta_W$ gives $\sum_{\rho\in\Phi} \kappa(\rho) \equiv 0 \pmod{W}$.

The set T_2 is nonempty by (D4), so there is a thick element ρ_*. Define $b_1^0(\rho)$ by (i) and (iii) on the top and thin elements, and put $b_1^0(\rho) = \lceil b \rceil$ on the thick elements. The integer h_0 is even, so $(j-\frac{1}{2})h_0$ is an integer for every j. Put $R_0 = \sum_\rho \kappa(\rho)\big(b_1^0(\rho) + \frac{1}{2}h_0\big)$, a class modulo W. Since $\kappa(\rho_*)$ is a unit modulo W, there is an integer $0 \leq s < W$ with $\kappa(\rho_*)s \equiv -R_0 \pmod{W}$. Put $b_1(\rho_*) = \lceil b \rceil + s$ and $b_1(\rho) = b_1^0(\rho)$ for $\rho \neq \rho_*$. Then (i) to (iii) hold, and for $1 \leq j \leq m$

$$\sum_\rho \kappa(\rho)\, w_{u_0,j}(\rho) = R_0 + \kappa(\rho_*)s + (j-1)h_0 \sum_\rho \kappa(\rho) \equiv 0 \pmod{W},$$

which is (iv). □

Corollary 18.8. *Let $c_h = 402$. There is x_0 such that for every $x \geq x_0$, every choice of admissible data at the scale x, every $u_0 \in A_0$ and all weights of Definition 18.5, the nodes $w_{u_0,j}$ of Proposition 18.7 satisfy the bounds of Proposition 5.16(i) at the thick elements and the bounds of Proposition 5.16(ii) at the other*

elements of Φ. *In particular* $1 \le w_{u_0,j}(\rho) \le 2b'$ *when* ρ *is not thick, and* $b \le w_{u_0,j}(\rho) \le b + W + mh_0$ *when* ρ *is thick.*

Proof. The nodes have the form $b_1(\rho) + (j' + \frac{1}{2})h_0$ with $j' = j - 1$ and $0 \le j' < m$, and $h_0 = 2\lceil x^{c_h} \rceil$. By (D1′), $n = c\,2^e \le 2x^{2+8\varepsilon} \le 2x^{9/4}$, Proposition 5.15 gives $m \le 2\sqrt{x\delta_1} \le 2\sqrt{x}$, and $W \le 32n^2$ by the remark after Proposition 5.16. At a thick element, Proposition 18.7(ii) is the hypothesis of Proposition 5.16(i). At a thin element, (iii) is the first hypothesis of Proposition 5.16(ii). At a top element, (i) gives $b_1 - \lfloor b'/x \rfloor = b_t$, and $b' \le b_t \le (1 + x^{-2})b' \le \frac{5}{4}b'$ is the second hypothesis of Proposition 5.16(ii). The node $w_{u_0,j}(\rho) \ge \lceil b'/(xn) \rceil \ge 1$ in all three cases. □

The archimedean parameters of the m lifts at an element $\rho \in \Phi$ are the integers $b_1(\rho) + j'h_0$ and $b_1(\rho) + (j' + \frac{1}{2})h_0$ for $0 \le j' < m$, and the nodes are the second family. Consecutive parameters differ by $\frac{1}{2}h_0$, and the choice $c_h = 402$ together with Lemma 18.2 gives $\frac{1}{2}h_0 \ge x^{402} \ge \frac{1}{2}(t_r + 3)$. This is the gap between the archimedean parameters that Hypothesis 18.10 uses.

18.3 Chain records

Fix admissible data at the scale x, weights, an element $u_0 \in A_0$ and the nodes $w_{u_0,j}$ of Proposition 18.7. We use the following quantities. Let $N_T = |T|$, $N_2 = |T_2|$, $A = \sum_{t \in T} a_t$ and $f = 2n$. Put

$$B_0 = 2\,n\,m\,b\,x^{\varepsilon}, \qquad N_c = \frac{N_2\,k\,b\,x^{2\varepsilon}}{n^2}, \qquad X'' = x\,n\,X',$$
$$\mathrm{rd}_E = |\mathrm{disc}\,E|^{1/[E:\mathbb{Q}]}, \qquad \Lambda_a = \log\left(a\,\mathrm{rd}_E\right), \qquad v = \tfrac{1}{2}\,\mathrm{rd}_E, \qquad g = \log(g_0 g_1),$$

with X' as in (5.17) and g_0, g_1 the primes of (D5). Then the number $\mathcal{B}$ of (4.8) is

$$\mathcal{B} = N_T\,a\Big(1 + \frac{m^2}{k}\Big) + m\,(B_0 + N_c). \tag{18.1}$$

Let $\lambda_1 = \ell_{\chi_1} = L'(1,\chi_{-d})/L(1,\chi_{-d})$, which is real because $\chi_{-d} = (\frac{-d}{\cdot})$ is real, and put $\log\Omega = \gamma + \log(\pi/v) - \lambda_1$. Let S be the sum over $t \in T$ and $1 \le j \le m$ of the left side of (3.10) with $a = a_t$ and $w = w_{u_0,j}$.

For a hermitian lattice N over $\mathcal{O}_E$ let $\mathrm{rk}\,N$, $\mu(N)$ and $\mu_{\max}(N)$ denote its rank, its slope and its maximal slope, and let $\mu_{\min}(N)$ be its minimal slope. For real numbers $u \ge 0$ and $H > 0$ we read $u\log(1 + H/u)$ as 0 when $u = 0$.

Definition 18.9. Let $C_* \ge 1$, $C_H \ge 0$, $e_D \ge 0$ and $e_L \ge 0$ be real numbers. A *chain record* for the data, the weights and u_0, with constants C_*, C_H and exponents e_D, e_L, consists of

(a) a finite nonempty set $\mathcal{W}$ with $|\mathcal{W}| \le 2^{k-3}$,

(b) for every $u \in \mathcal{W}$ a prime T_u, hermitian $\mathcal{O}_E$-lattices N_u with $\mathrm{rk}\,N_u \ge 1$ and M_u, and a real number h_u,

(c) a hermitian $\mathcal{O}_E$-lattice R with dual lattice $R^\vee$,

(d) real numbers $E_{u,j}$ for $u \in \mathcal{W}$ and $1 \le j \le m$, and real numbers $\mathcal{L}$, Γ, $\mathcal{I}$, δ, η and ϵ,

such that the conditions (K1) to (K8) below hold. In them u and u' run over $\mathcal{W}$ and j over $1 \le j \le m$, and

$$E_u = \sum_{j=1}^{m} E_{u,j}, \qquad s = \frac{\sum_u d_u \big(\mu(N_u) - h_u\big)}{\sum_u d_u}, \qquad u_F = \max_u \big(\mu_{\max}(N_u) - h_u\big) - s, \qquad d_u = \operatorname{rk} N_u.$$

(K1) $S - \Gamma \le C_* N_T\, m\, b\, x^{e_L \varepsilon}$ and $0 \le \mathcal{I} \le C_* N_T\, m\, b\, x^{e_L \varepsilon}$.

(K2) $\mu_{\max}(N_u) \le \mu_{\max}(M_u) + \Gamma + \mathcal{L} - E_u + C_* m B_0$ and $\mu(M_u) + \Gamma + \mathcal{L} - E_u - \mathrm{Lo}_u \le \mu(N_u)$, where

$$\mathrm{Lo}_u = C_* \mathcal{I} + C_* N_T\, m \Big(\frac{a}{k} + b'\Big) \log T_u + C_* X'' + 1.$$

(K3) The numbers δ, η and ϵ satisfy

$$0 \le \delta \le \tfrac{1}{2}, \qquad \eta \ge 0, \qquad 0 \le \epsilon \le \tfrac{1}{4}, \qquad (\eta + 4\sqrt{\epsilon})\, x^{C_H} \le 6,$$

and for every $H > 0$ with $H \ge s - \mu_{\min}(R^\vee)$

$$s - \log|\mathcal{W}| - \frac{1}{1-\delta}\Big(4 u_F \log\Big(1 + \frac{H}{u_F}\Big) + (\eta + 4\sqrt{\epsilon}) H\Big) \le \mu(R^\vee).$$

(K4) $\mu_{\min}(R^\vee) = -\mu_{\max}(R)$, $s + \mu_{\max}(R) \le x^{C_H} b$, and

$$\mu(R^\vee) + h_u - \mathcal{L} \le \sum_{t \in T} \big(m \operatorname{ph}(a_t) - m a_t \log \Omega - a_t \log T_u\big) + C_* N_T\, a\, \Lambda_a + C_* m\, (B_0 + N_c)\, x^{e_D \varepsilon}.$$

(K5) $|E_{u,j} - A\, g| \le C_* B_0$.

(K6) $\big|(h_u + A \log T_u) - (h_{u'} + A \log T_{u'})\big| \le C_* \big(N_T\, a\, \Lambda_a + m B_0\big)$.

(K7) The auxiliary primes satisfy

$$\log T_u \ge \sum_{j=1}^{m} \frac{1}{f} \sum_{\rho \in \Phi} \log \frac{w_{u_0,j}(\rho)\, v}{\pi} + mg - C_* \Big(\Lambda_a + \frac{m^2 \Lambda_a}{k}\Big),$$
$$\log T_u \le C_* m L_\varepsilon, \qquad |\log T_u - \log T_{u'}| \le C_* \log k.$$

(K8) $-C_* N_T\, a\, \Lambda_a \big(1 + \frac{m}{k}\big) \le \mu(M_u)$, $\mu_{\max}(M_u) \le C_* N_T\, a\, \Lambda_a$, and $\mu(N_u) \le \mu_{\max}(N_u)$.

In the application, $\mathcal{W}$ is a set of words that index the auxiliary primes T_u, N_u is the rescaled lattice of images of the Hecke operators at T_u, M_u is the corresponding base lattice, R is the lattice of automorphic forms at the top of the chain, and h_u bounds the height of the period map $B_u \colon N_u \to R^\vee$. The number Γ is the total gain of the m lifts at the inert primes, $\mathcal{L}$ is the sum of the local terms of the lifts, $E_{u,j}$ is the local term of the j-th lift at the twisted places, and $\mathcal{I}$ is the index loss at the inert primes.

Hypothesis 18.10 (chain records)**.** There are absolute constants $e_D, e_L \geq 0$ with the following property. For every primitive Dirichlet character ψ_0 there is $0 < \varepsilon_1 \leq \frac{1}{52}$, and for every $0 < \varepsilon < \varepsilon_1$ and every $J \geq 2$ there are $C_* \geq 1$, $C_H \geq 0$ and x_1 such that the following holds. Let $x \geq x_1$, and let $Y_0 \leq Y$ be such that admissible data at the scale x for the window $[Y, Y^{3/2}]$ with base Y_0 exist, for this J. Then there are admissible data at the scale x for this window with this base, and weights for them, such that for every $u_0 \in A_0$ and every choice of the nodes $w_{u_0,j}$ of Proposition 18.7 there is a chain record with constants C_*, C_H and exponents e_D, e_L. One can take $e_D = 2$ and $e_L = 0$.

Fix admissible data at the scale x, weights, $u_0 \in A_0$ and the nodes $w_{u_0,j}$. Let $\mathcal{W}$, T_u, N_u, M_u, h_u, R, $E_{u,j}$, $\mathcal{L}$, Γ and $\mathcal{I}$ be the objects of the construction, as described after Definition 18.9, with $\mathcal{I} = \mathrm{IO}$ the index loss of Section 12.

Proposition 18.11. *Let $C_H \geq 0$. Suppose that x is so large that Lemma 18.15 applies, and that the following hold for every $u \in \mathcal{W}$.*

(i) The conclusions of Theorems 12.27 and 12.21, with the constant C_T of (L7), and $\mathrm{IO} \geq 0$.

(ii) The hypotheses (a), (b), (d), (e) and (f) of Theorem 13.15, with $\sigma = x^{-C_H}$, with numbers $\eta_0, \eta_1, e_0, \delta'$ that satisfy the conclusions of Lemma 13.19, and with $\delta \leq \frac{1}{2}$.

(iii) $\mu_{\min}(R^\vee) = -\mu_{\max}(R)$ and $s + \mu_{\max}(R) \leq x^{C_H} b$.

(iv) The conclusion of Theorem 15.19 with a constant $C_0 \geq 1$.

(v) The conclusion of Theorem 16.6 with the constant C_E, for every $1 \leq j \leq m$.

(vi) h_u is the number of Definition 15.14, and the numbers $e_P^{\max}$, s_j, $h_{j\rho}$, $e_{\lambda,j}^{\max}$ and $\langle \log \kappa \rangle$ in that definition do not depend on u.

(vii) The conclusion of Theorem 15.26 with the constant C_6, and $\log T_u \leq C_1 m L_\varepsilon$ and $|\log T_u - \log T_{u'}| \leq C_2 \log k$ for $u' \in \mathcal{W}$.

(viii) The conclusion of Theorem 15.33 with the constant C_7, and $\mu_{\max}(M_u) \leq C_3 N_T a \Lambda_a$.

(ix) $1 \leq |\mathcal{W}| \leq 2^{k-3}$.

Then the objects form a chain record with the constants C_H and

$$C_* = \max\big(6,\ C_T + 2,\ 50C_0 + 10,\ C_E,\ C_6 + 2,\ C_1,\ C_2,\ C_3,\ C_7\big),$$

and the exponents $e_D = 1$ and $e_L = 0$. Since $x^\varepsilon \leq x^{2\varepsilon}$, they also form a chain record with $e_D = 2$ and $e_L = 0$.

Proof. We use $N_T \geq 1$, $m \geq 1$, $k \leq a$, $\Lambda_a \geq L_\varepsilon \geq 1$ and $B_0 \geq 0$ from Lemma 18.15, and $T_u \geq 2$.

(a) is (ix).

(K1) Theorem 12.27 gives $S - \Gamma \leq e_1 \leq (2\log 4 + 1) N_T m b$ and $0 \leq \mathrm{IO} \leq N_T m b$, and $2\log 4 + 1 < 6 \leq C_*$.

(K2) Theorem 12.21(ii) gives $\mu_{\max}(N_u) \leq \mu_{\max}(M_u) + \Gamma + \mathcal{L} - E_u$, and $C_* m B_0 \geq 0$. Theorem 12.21(i) gives $\mu(M_u) + \Gamma + \mathcal{L} - E_u - e_2 \leq \mu(N_u)$ with $e_2 = 4\,\mathrm{IO} + \mathrm{LT} + \mathrm{RX} + 1$, where

$\mathrm{LT} = 4\lambda_T \log T_u = C_T N_T m(\frac{a}{k} + b') \log T_u$ and $\mathrm{RX} \le 6X'' + 2\log T_u$. Since $N_T m(\frac{a}{k} + b') \ge 1$ and $\log T_u \ge 0$,

$$e_2 \le 4\,\mathrm{IO} + (C_T + 2) N_T m\Big(\frac{a}{k} + b'\Big) \log T_u + 6X'' + 1 \le \mathrm{Lo}_u.$$

(K3) Put $\eta = 5\sigma + 3\sqrt{\delta'}$ and $\epsilon = 2e_0$. Then $\eta \ge 0$ and $0 \le \epsilon \le \frac{1}{4}$ because $e_0 \le \frac{1}{8}$, and by Lemma 13.19

$$(\eta + 4\sqrt{\epsilon})\, x^{C_H} = 5\sigma x^{C_H} + \big(3\sqrt{\delta'} + 4\sqrt{2e_0}\big) x^{C_H} \le 5 + 1 = 6.$$

Let $H > 0$ with $H \ge s - \mu_{\min}(R^\vee)$. By (iii), $s + \mu_{\max}(R) \le H$, which is hypothesis (c) of Theorem 13.15, and the other hypotheses do not involve H. Theorem 13.15 gives $s - e_3 \le \mu(R^\vee)$ with $e_3 = (1-\delta)^{-1}\big(4u_F \log(1 + H/u_F) + (\eta + 4\sqrt{\epsilon})H\big)$, where $\Lambda(u, H) = u\log(1 + H/u)$ as in Section 13. Since $\log|\mathcal{W}| \ge 0$, this gives (K3).

(K4) The first two statements are (iii). Theorem 15.19 gives the inequality with $e_4 = (50C_0 + 10)\big(N_T a \Lambda_a + m\, x^{\epsilon}(B_0 + N_c)\big)$, which is at most $C_* N_T a \Lambda_a + C_* m(B_0 + N_c) x^{e_D \epsilon}$ for $e_D = 1$.

(K5) is (v), since $C_E \le C_*$.

(K6) By (vi),

$$h_u + A\log T_u = e_P^{\max} + \sum_{j=1}^{m} \Big(s_j - \sum_{\rho} h_{j\rho} + e_{\lambda,j}^{\max}\Big) - \langle \log \kappa \rangle,$$

with $A = \sum_{t \in T} a_t$, and the right side does not depend on u. So the left side of (K6) is 0.

(K7) Theorem 15.26 gives the first inequality with $e_6 = C_6(L_\varepsilon + m^2 L_\varepsilon/k) + 2 \le (C_6 + 2)(\Lambda_a + m^2\Lambda_a/k)$, because $L_\varepsilon \le \Lambda_a$ and $\Lambda_a \ge 1$. The other two inequalities are (vii).

(K8) The first inequality is Theorem 15.33, with $C_7 \le C_*$. The second is (viii). The third holds because N_u is one of the sublattices over which $\mu_{\max}(N_u)$ is the maximum. □

The exponents $e_D = 1$ and $e_L = 0$ do not depend on ψ_0, ε or J, so they are absolute, and Proposition 18.16 gives $\kappa = \max(2, e_D, e_L + 1) = 2$ and the constant $C = 3$ of Theorem A, whether one takes $e_D = 1$ or $e_D = 2$. The constant C_* depends on ε through C_E, which Hypothesis 18.10 allows.

Condition (ix) is compatible with the lower bound $|\mathcal{W}| \ge C_0 \Gamma_0^2 x^{C_H}$ of Lemma 13.19, with $\Gamma_0 = \exp(C_\Gamma m^5 f L_\varepsilon)$. Indeed $m \le 2x^{1/2-\varepsilon/16}$ and $f \le 4x^{2+8\varepsilon}$ give $\log\big(C_0\Gamma_0^2 x^{C_H}\big) \le \log C_0 + 256\, C_\Gamma\, x^{9/2+139\varepsilon/16} + C_H \log x$, while $(k-3)\log 2 \ge \frac{1}{2} x^{9/2+10\varepsilon}$ for large x, and $\frac{139}{16} < 10$.

Hypothesis 18.10 collects in one statement the inequalities proved in Sections 11 to 17. Condition (a) is the choice of the words in Lemma 13.19. Conditions (K1) and (K2) are Theorems 12.27 and 12.21 with the divisibility of Theorem 11.45. Condition (K3) is Theorem 13.15 with Theorems 14.1, 14.2 and 14.3. Condition (K4) is Theorem 15.19 with Theorem 17.37. Condition (K5) is Theorem 16.6. Conditions (K6) and (K7) are proved in Section 15, and (K8) is Theorem 15.33 with $\mu(N_u) \le \mu_{\max}(N_u)$. Each of these results is proved here from hypotheses stated in its section. The conditions (K1) to (K8) are stated with the exponents and constants of the formal proof, and some of the cited results give an inequality of a slightly different shape. Theorem 13.15 gives (K3) without the term $\log|\mathcal{W}|$, which only strengthens it, and Theorem 12.21 gives the two halves of (K2), the upper half without the term $C_* m B_0$. We have not checked line by line that the constants and exponents of Sections 11 to 17 give (K1) to (K8) with absolute e_D and e_L, so Hypothesis 18.10 is not proved here even modulo the hypotheses of those sections.

The proof of Theorem A below uses Hypothesis 18.10 for one admissible datum at each scale. The conditions (K1) to (K8) depend on the window and the base through the field E, the ranks and the sets T and T_2.

18.4 The seven hypotheses of the accounting

We fix a chain record with constants C_*, C_H and exponents e_D, e_L, and we write $y = x^{\varepsilon} = L_{\varepsilon}$. The first lemma bounds the logarithmic term of (K3).

Lemma 18.12. *Let $u \ge 0$, $U \ge u$, $b > 0$, $X > 0$ and $0 < H \le Xb$. Then $u \log(1 + H/u) \le \log(1 + X)\,(U + b)$.*

Proof. For $u = 0$ the left side is 0 and the right side is nonnegative. Let $u > 0$. The inequality $\log z \le z - 1$ for $z > 0$, with $z = (1 + H/u)/(1 + X)$, gives

$$u \log\Big(1 + \frac{H}{u}\Big) \le u \log(1 + X) + \frac{u + H}{1 + X} - u = u \log(1 + X) + \frac{H}{1 + X} - \frac{uX}{1 + X}.$$

The last term is nonpositive, $u \log(1 + X) \le U \log(1 + X)$, and

$$\frac{H}{1 + X} \le \frac{Xb}{1 + X} = b\Big(1 - \frac{1}{1 + X}\Big) \le b \log(1 + X),$$

by the inequality $1 - 1/z \le \log z$ for $z = 1 + X$. □

The second lemma bounds the spread u_F of the slopes of the lattices N_u. Put

$$\begin{aligned} K_F &= C_* N_T\, a\, \Lambda_a \Big(3 + \frac{m}{k}\Big) + 4 C_* m B_0 + C_* A \log k, \\ e_2 &= C_*^2 N_T\, m\, b\, x^{e_L \varepsilon} + C_*^2 N_T\, m^2 \Big(\frac{a}{k} + b'\Big) L_{\varepsilon} + C_* X'' + 1, \\ U &= K_F + e_2. \end{aligned}$$

Lemma 18.13. *For every $u \in \mathcal{W}$ we have $\mathrm{Lo}_u \le e_2$, and $0 \le u_F \le U$.*

Proof. By (K1), $\mathcal{I} \le C_* N_T\, m\, b\, x^{e_L \varepsilon}$, and by (K7), $\log T_u \le C_* m L_{\varepsilon}$. The coefficient $C_* N_T\, m(a/k + b')$ is nonnegative, so $\mathrm{Lo}_u \le e_2$.

By (K8), $\mu(N_u) - h_u \le \mu_{\max}(N_u) - h_u$ for every u, so the weighted mean s is at most the maximum in the definition of u_F, and $u_F \ge 0$.

Let $u, u' \in \mathcal{W}$. The first inequality of (K2) for u and the second for u' give

$$\begin{aligned} \big(\mu_{\max}(N_u) - h_u\big) - \big(\mu(N_{u'}) - h_{u'}\big) \\ \le \mu_{\max}(M_u) - \mu(M_{u'}) + C_* m B_0 + \mathrm{Lo}_{u'} + (E_{u'} - E_u) + (h_{u'} - h_u). \end{aligned}$$

By (K8), $\mu_{\max}(M_u) - \mu(M_{u'}) \le C_* N_T\, a\, \Lambda_a (2 + m/k)$. By (K5), each $E_{u,j}$ lies within $C_* B_0$ of Ag, so $E_{u'} - E_u \le 2 C_* m B_0$. By (K6) and the third inequality of (K7),

$$h_{u'} - h_u \le C_*\big(N_T\, a\, \Lambda_a + m B_0\big) + A\big(\log T_u - \log T_{u'}\big) \le C_*\big(N_T\, a\, \Lambda_a + m B_0\big) + C_* A \log k,$$

where we used $A \ge 0$. Adding these bounds, and using $\mathrm{Lo}_{u'} \le e_2$,

$$\big(\mu_{\max}(N_u) - h_u\big) - \big(\mu(N_{u'}) - h_{u'}\big) \le K_F + e_2 = U.$$

The weighted mean over u' with the weights $d_{u'}$ gives $\mu_{\max}(N_u) - h_u - s \le U$, and the maximum over u gives $u_F \le U$. □

Define

$$e_1 = C_* N_T\, m\, b\, x^{e_L \varepsilon}, \qquad e_3 = \log|\mathcal{W}| + 8\log\left(1 + x^{C_H}\right)(U + b) + 12\, b,$$
$$e_4 = C_* N_T\, a\, \Lambda_a + C_* m\, (B_0 + N_c)\, x^{e_D \varepsilon}, \qquad e_5 = C_* m B_0,$$
$$e_6 = C_* \Lambda_a \Big(1 + \frac{m^2}{k}\Big), \qquad e_7 = C_* N_T\, a\, \Lambda_a \Big(1 + \frac{m}{k}\Big),$$

and let e_2 be as above. All seven numbers are nonnegative, because $C_* \geq 1$, $\log|\mathcal{W}| \geq 0$ and $\Lambda_a \geq \log a \geq 0$.

Proposition 18.14. *Let a chain record be given for admissible data at the scale x, weights and $u_0 \in A_0$. Put $d_u = \operatorname{rk} N_u$, $\mu_u^N = \mu(N_u)$, $\mu_u^M = \mu(M_u)$, $\mu^C = \mu(R^\vee)$, $w_{j\rho} = w_{u_0,j}(\rho)$ for $\rho \in \Phi$, $v = \frac{1}{2}\operatorname{rd}_E$, γ Euler's constant, $\lambda_1 = \ell_{\chi_1}$ and $g = \log(g_0 g_1)$. Then the hypotheses* (3.1) *to* (3.7) *of Proposition 3.1 hold for every $u \in \mathcal{W}$, with the index set $\mathcal{W}$ in place of W, with S, Γ, $\mathcal{L}$, E_u, h_u, T_u and the numbers $e_1, \ldots, e_7$ defined above.*

Proof. The weights $a_t \geq a > 0$, the nodes are positive by Corollary 18.8, $v > 0$, $d_u \geq 1$ and $T_u \geq 2$, so the data are of the kind that Proposition 3.1 admits.

(3.1) is the first inequality of (K1). (3.2) is the second inequality of (K2) with $\mathrm{Lo}_u \leq e_2$ from Lemma 18.13.

For (3.3), put $H = x^{C_H} b$. By (K4), $s - \mu_{\min}(R^\vee) = s + \mu_{\max}(R) \leq H$, and $H > 0$, so (K3) applies with this H. Lemma 18.12 with $u = u_F$, U from Lemma 18.13 and $X = x^{C_H}$ gives $4u_F \log(1 + H/u_F) \leq 4\log(1 + x^{C_H})(U + b)$. By (K3), $(\eta + 4\sqrt{\epsilon})H \leq 6b$ and $(1 - \delta)^{-1} \leq 2$. Both terms in the bracket of (K3) are nonnegative, so (K3) gives

$$s - \log|\mathcal{W}| - 8\log\left(1 + x^{C_H}\right)(U + b) - 12b \leq \mu(R^\vee),$$

which is (3.3) with e_3, because s is the weighted mean of $\mu_u^N - h_u$.

(3.4) is the inequality of (K4) with e_4. For (3.5), summing (K5) over j gives $E_u \leq mAg + C_* m B_0 = mAg + e_5$. (3.6) is the first inequality of (K7) with e_6, and (3.7) is the first inequality of (K8) with e_7. □

18.5 The size of the errors

Lemma 18.15. *Let $0 < \varepsilon < \frac{1}{52}$, $J \geq 2$ and $C_H \geq 0$. There is x_0, depending on ψ_0, ε, J and C_H, such that for every $x \geq x_0$, all admissible data at the scale x with this J, all weights and every chain record with $|\mathcal{W}| \leq 2^{k-3}$ the following hold, with $y = x^\varepsilon$.*

(i) $1 \leq b \leq a$, $1 \leq m \leq x$, $mb \leq a$, $xm \leq k \leq x^5$ *and* $k \leq a$.

(ii) $kb' \leq a$ *and* $X'' \leq a$.

(iii) $0 \leq \Lambda_a \leq 2y$.

(iv) $\log x \leq y$, $\log k \leq 5y$ *and* $\log(1 + x^{C_H}) \leq (1 + C_H)\, y$.

(v) $N_T \geq 1$ *and* $0 \leq A \leq 2N_T a$.

(vi) $\log|\mathcal{W}| \leq k$, *and* $\max(a,\, k,\, b,\, N_T a,\, mB_0,\, mN_c) \leq \mathcal{B}$.

Proof. (i) We have $b = e^y \geq 1$ and $a = xb \geq b$. Proposition 5.15 gives $2 \leq m \leq 2\sqrt{x\delta_1} \leq x$, so $mb \leq xb = a$, and $xm \leq k \leq 2x^{9/2+10\varepsilon} \leq x^5$ because $10\varepsilon < \frac{1}{5}$. For large x, $x^5 \leq xe^y = a$.

(ii) Since $b' = a/X'$, the inequality $kb' \leq a$ is $k \leq X' = \exp(x^{\varepsilon/3})$, which holds for large x by (i). The inequality $X'' \leq a$ is $nX' \leq b$, that is $n \leq \exp(x^{\varepsilon} - x^{\varepsilon/3})$, which holds for large x because $n \leq 2x^{2+8\varepsilon}$.

(iii) The integer $|\operatorname{disc} E|$ is at least 1, so $\Lambda_a \geq \log a \geq 0$. The field E lies in $\mathbb{Q}(\zeta_N)$ with $N = N_1 qd$ and $N_1 = \operatorname{lcm}(2^{e+2}, M_\psi)$. By the conductor–discriminant formula, $|\operatorname{disc} E|$ is the product of the conductors of the $[E:\mathbb{Q}]$ Dirichlet characters of E, and each of them divides N. Hence $\operatorname{rd}_E \leq N$ and

$$\log \operatorname{rd}_E \leq \log\left(2^{e+2} M_\psi\right) + \log q + \log d \leq \log(8M_\psi n) + Y_0^{\varepsilon/5}\left(1 + \log^2 Y_0\right),$$

by (D1′) and $2^e \leq n$. Since $Y_0 \leq x$ and $n \leq 2x^{2+8\varepsilon}$, the right side is at most $x^{\varepsilon/5}(1+\log^2 x) + 3\log x + \log(16M_\psi)$. Then $\Lambda_a = \log x + y + \log \operatorname{rd}_E \leq 2y$ for large x.

(iv) The first inequality holds for large x. The second is $\log k \leq 5 \log x$, from $k \leq x^5$. For the third, $\log(1 + x^{C_H}) \leq \log 2 + C_H \log x \leq (1 + C_H)y$ because $y \geq 1$.

(v) Proposition 5.20 gives $N_T \geq \frac{1}{2}\delta_1 n \geq 1$, and $0 < a_t \leq a + a/(kx^{13}) \leq 2a$.

(vi) $\log|\mathcal{W}| \leq (k-3)\log 2 \leq k$. The first term of (18.1) is at least $N_T a \geq a$, and (i) gives $b \leq a$ and $k \leq a$. The second term of (18.1) is $mB_0 + mN_c$. □

Proposition 18.16. *Let $\kappa = \max(2, e_D, e_L + 1)$. Let $0 < \varepsilon < \frac{1}{52}$, $J \geq 2$, $C_* \geq 1$ and $C_H \geq 0$. There is x_0 such that for every $x \geq x_0$, all admissible data at the scale x with this J, all weights and every chain record with constants C_*, C_H and exponents e_D, e_L, the numbers $e_1, \ldots, e_7$ of Section 18.4 satisfy*

$$e_1 + e_2 + e_3 + e_4 + e_5 + Ae_6 + e_7 \leq x^{(\kappa+1)\varepsilon}\,\mathcal{B}.$$

Proof. Let x_0 be as in Lemma 18.15, enlarged so that $y = x^\varepsilon \geq 260(1 + C_H)C_*^2$ for $x \geq x_0$. We bound each term by a multiple of $\mathcal{B}$, using Lemma 18.15, $y \geq 1$, $C_* \geq 1$ and $y^\alpha \leq y^\kappa$ for $0 \leq \alpha \leq \kappa$.

Since $mb \leq a$, $e_1 \leq C_* y^{e_L} N_T a \leq C_* y^{e_L}\mathcal{B}$. In e_2, the first term is at most $C_*^2 y^{e_L}\mathcal{B}$. Since $b' \leq a/k$, the second term is at most $2C_*^2 y\, N_T a\, m^2/k \leq 2C_*^2 y\,\mathcal{B}$. The third term is at most $C_* a \leq C_*\mathcal{B}$, and $1 \leq \mathcal{B}$. Hence

$$e_2 \leq 5C_*^2 y^{\max(1,e_L)}\mathcal{B}.$$

Since $\Lambda_a \leq 2y$, $A \leq 2N_T a$ and $m \leq k$,

$$e_4 \leq 2C_* y\, N_T a + C_* y^{e_D} m(B_0 + N_c) \leq 3C_* y^{\max(1,e_D)}\mathcal{B}, \qquad e_5 \leq C_*\mathcal{B},$$

$$Ae_6 \leq 2N_T a \cdot C_* \cdot 2y\left(1 + \frac{m^2}{k}\right) \leq 4C_* y\,\mathcal{B}, \qquad e_7 \leq C_* N_T a \cdot 2y \cdot 2 \leq 4C_* y\,\mathcal{B}.$$

For e_3, $\log k \leq 5y$ and $A \leq 2N_T a$ give

$$K_F \leq 8C_* y\, N_T a + 4C_*\mathcal{B} + 10C_* y\, N_T a \leq 22C_* y\,\mathcal{B},$$

so $U + b \leq 22C_* y\,\mathcal{B} + 5C_*^2 y^{\max(1,e_L)}\mathcal{B} + \mathcal{B} \leq 28C_*^2 y^{\max(1,e_L)}\mathcal{B}$. With $\log(1 + x^{C_H}) \leq (1 + C_H)y$, $\log|\mathcal{W}| \leq \mathcal{B}$ and $b \leq \mathcal{B}$,

$$e_3 \leq \mathcal{B} + 8(1 + C_H)y \cdot 28C_*^2 y^{\max(1,e_L)}\mathcal{B} + 12\mathcal{B} \leq 237(1 + C_H)C_*^2 y^\kappa \mathcal{B},$$

because $1 + \max(1, e_L) \leq \kappa$. Adding the seven bounds,

$$e_1 + e_2 + e_3 + e_4 + e_5 + Ae_6 + e_7 \leq \left(18C_*^2 + 237(1 + C_H)C_*^2\right)y^\kappa\mathcal{B} \leq 260(1 + C_H)C_*^2 y^\kappa \mathcal{B} \leq y^{\kappa+1}\mathcal{B}.$$

□

With the exponents $e_D = 2$ and $e_L = 0$ of Hypothesis 18.10, $\kappa = 2$ and the bound is $x^{3\varepsilon}\mathcal{B}$.

18.6 Proof of Theorem A

Assume Hypotheses 17.11 and 18.10. We prove Theorem A in the form stated in Section 6, which asks for one choice of admissible data at each scale of a window, for windows $[Y, Y^{3/2}]$ with Y arbitrarily large.

Proof. Let e_D and e_L be the absolute constants of Hypothesis 18.10, and put $C = \max(2, e_D, e_L + 1) + 1$, which is 3 for $e_D = 2$ and $e_L = 0$. Fix ψ_0, let ε_1 be as in Hypothesis 18.10, and fix $0 < \varepsilon < \varepsilon_1$. Let $J \geq 2$ be the number of Proposition 18.4, and let C_*, C_H and x_1 be the numbers of Hypothesis 18.10 for ε and this J. Let x_0 be the maximum of x_1 and of the numbers x_0 of Lemma 18.2, Corollary 18.8, Lemma 18.15 and Proposition 18.16 for these constants.

Let y be given. Proposition 18.4 gives $Y_0 \leq Y$ with $Y \geq \max(y, x_0)$ such that admissible data with base Y_0 exist for every $x \in [Y, Y^{3/2}]$. Fix such an x. Then $x \geq x_1$, and Hypothesis 18.10 gives admissible data at the scale x, for the window $[Y, Y^{3/2}]$ with base Y_0, and weights a_t, b_t, such that for every $u_0 \in A_0$ the nodes $w_{u_0,j}$ of Proposition 18.7 admit a chain record with constants C_*, C_H and exponents e_D, e_L.

The weights satisfy $a \leq a_t \leq a + a/(kx^{13})$ by Definition 18.5, and Corollary 18.8 gives (i).

Fix $u_0 \in A_0$ and a chain record for it. Proposition 18.14 shows that the hypotheses (3.1) to (3.7) of Proposition 3.1 hold with $\lambda_1 = \ell_{\chi_1}$, with $w_{j\rho} = w_{u_0,j}(\rho)$ for $\rho \in \Phi$, and with S the sum over $t \in T$ and $1 \leq j \leq m$ of the left side of (3.10) with $a = a_t$ and $w = w_{u_0,j}$. Proposition 3.1 gives the conclusion (3.8). The number $\ell_{\chi_1} = L'(1, \chi_{-d})/L(1, \chi_{-d})$ is real, so Corollary 3.4 applies and gives

$$\sum_{t \in T} \sum_{j=1}^{m} \operatorname{Re} \operatorname{De}(a_t, t, w_{u_0,j}) \leq e_1 + e_2 + e_3 + e_4 + e_5 + Ae_6 + e_7.$$

Proposition 18.16 bounds the right side by $x^{C\varepsilon}\mathcal{B}$, which is (ii). □

The constant C does not depend on ε or on ψ_0, which is the uniformity that the last step of the proof of Theorem 6.3 uses. With $C = 3$ that proof gives $\zeta(s) \neq 0$ and $L(s, \chi^{(i)}) \neq 0$ for $\operatorname{Re} s > \frac{1}{2} + 7\varepsilon$, for every $0 < \varepsilon < \varepsilon_1$, and so for $\operatorname{Re} s > \frac{1}{2}$.

The three primes of (D5) are congruent to 1 modulo $15\,Q_S q d$. The construction of the level structure of the lifts uses two primes congruent to 1 modulo $\operatorname{lcm}(15, Q_S q d)$ and a third prime, different from them, congruent to 1 modulo $Q_S q d$, all at most $(Q_S q d)^{C_8}$ for a constant C_8. Since $\operatorname{lcm}(15, Q_S q d)$ divides $15\,Q_S q d$, the primes of (D5) have these properties with $C_8 = 2A_L(1 + A_L)^2$, because $15 \leq Q_S q d$ gives $(15\,Q_S q d)^{A_L(1+A_L)^2} \leq (Q_S q d)^{2A_L(1+A_L)^2}$.

18.7 Hypotheses in the final deduction

The chain and the Hecke bounds. Proposition 11.27 gives the frame valuation under the step-character and determinant normalizations of Definition 11.23. Hypotheses 11.38 and 11.42 concern the odd-position lattices and the normalized kernel. In the Hecke bound, Theorem 12.54 controls the isotypic projector, Proposition 12.40 bounds the window loss, and Lemma 12.63 combines the final share estimates. Hypothesis 12.12 still requires the individual share estimates, the comparisons between levels and the construction of the test systems. Corollary 12.65 uses the Weyl lattice model. Proposition 12.67 requires a polynomial bound for the constant of the base weights. The colength at the auxiliary prime in (L7) is a further input.

Periods and multiplicities. Theorem 14.15 uses the seesaw relation, its nonzero pairing and the period identifications. Theorem 14.29 proves the double-step identity from the general and local Weil representation laws. Lemma 14.24 specifies the three period identities needed for rationality. The multiplicity calculation requires the label identifications and the pre-trace hypotheses of Subsections 14.3.7 and 14.3.8. The estimates on individual rows in Theorem 14.79 also require the kernel expansion, the main row, the loop rows and the degenerate terms. Lemma 14.78 proves the two exponential size inequalities with $r_S = m/2 - 1$.

Heights and local terms. The kernel norm uses the common integral Weber value and the period identities in Section 15. The final pairing requires its factorization through the branching map. For the covolume bound, Theorem 15.66 specifies the descent, flat twist and lattice comparisons under its hypotheses (I1)–(I17). Hypothesis 15.69 gives a lower bound for the average of component slopes with the term q spread(u) in the error. The flat twist gives the main term in the covolume inequality. The exact change of integral structure at good primes still enters the uniformity in the level. At the auxiliary and inert places the degree and orbit bounds enter Corollary 16.7.

Twisted places and assembly. Theorem 17.12 separates the plus and minus parts, including the prime 2. Proposition 17.13 separates the character summands when the group order is a unit. The generic estimate uses (OneChar) and (Count). The two rank-four cases also require propagation of automorphy and rigidity as in Subsection 17.4.1. The rank-two local comparison in Subsection 17.4.2 applies to Hilbert modular forms in the stated normalization. For the twisted lattice, Propositions 17.30 and 17.34 identify the congruence module and reduce its estimate to per-component bounds. The Wach lifting estimate, the bounds for the other local rows, and the integral theta-series comparisons still enter that argument. Proposition 17.35 shows why $\dim M_j = D_0$ suffices at the last step. Theorem 17.37 uses only $d_j^0 \le D_0$.

Published inputs. Section 5.7 uses Gallagher's Theorem 7 and the exceptional-zero bound (C8) for (C7), and the ray class case of Thorner–Zaman with Stark's bound for (C9). For the local arguments Berger's de Rham theorem is Théorème 0.6, equivalently Théorème 5.19, of [7]. For algebraic Hecke characters the Hodge jumps at τ have sign $-n_\tau$ in the local Artin convention. In the CM functional equation the archimedean phase at infinity type $2m$ is $\mathrm{i}^{\sum_i |2m_i|}$: substituting $2m$ in the local phase gives this equality since the exponent is even. The Rallis inner product identity still requires the comparison of the Petersson norm with the operators of Proposition 15.52. The mass and formal-degree formulas must use the measures of the dimension estimate. The final implication to Theorem 1.1 remains conditional on the hypotheses above.

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