The critical two-point function of lattice ϕ_2^4

Jiwoon Park

November 10, 2026

Abstract

In this article, we show that the critical two-point correlation function of the lattice ϕ_2^4 model in the strong coupling regime follows that of the Ising universality class.

AI statement

This document is generated using ChatGPT 6.1 Sol.

1 Main results

We consider real spins ϕ_v on the square lattice with single-site probability measure $\rho_g(\mathrm{d}\phi) \propto e^{-g(\phi^2-1)^2}\mathrm{d}\phi$, $g > 0$. The interaction weight in finite volume is $\exp(\beta \sum_{\langle u,v\rangle} \phi_u\phi_v)$, where the sum counts each unoriented nearest-neighbor edge once. Write $\langle\cdot\rangle_\beta^0$ and $\langle\cdot\rangle_\beta^+$ for the infinite-volume free and plus states, respectively, and set

$$G_{\phi,\beta}(x) = \langle\phi_0\phi_x\rangle_\beta^0, \qquad m_+(\beta) = \langle\phi_0\rangle_\beta^+, \qquad \beta_c(g) = \inf\{\beta > 0 : m_+(\beta) > 0\}.$$

Let μ_{I} denote the unique infinite-volume nearest-neighbor Ising state at $\beta_{\mathrm{I}} = \frac{1}{2}\log(1+\sqrt{2})$, and put $G_{\mathrm{I}}(x) = \mu_{\mathrm{I}}(\sigma_0\sigma_x)$. Use $|x|$ for the Euclidean norm and $|x|_\infty$ for the max norm.

Theorem 1. *There exist $g_0 < \infty$ and $\omega > 0$ such that for every fixed $g \geq g_0$ there is $Z_\phi(g) > 0$ with $Z_\phi(g) = 1 + O(g^{-1})$ and, uniformly as $|x| \to \infty$,*

$$G_{\phi,\beta_c(g)}(x) = Z_\phi(g)^2 G_{\mathrm{I}}(x) + O_g(|x|^{-1/4-\omega}). \tag{1}$$

In particular $G_{\phi,\beta_c(g)}(x) \sim C_g|x|^{-1/4}$ for a constant $C_g > 0$.

The pure Ising inputs are the conditional-averaging estimates and calibration stated in Lemma 4.1, from [1, Proposition 4.1 and Lemma 5.3]. Proposition 2.3 gives the starting lists for the background and the physical insertions. Propositions 3.3 and 4.2 give the analytic and linear blocking estimates. The massless tuning in Proposition 4.3 and the periodic comparison in Proposition 5.1 then supply the backgrounds used in Proposition 6.5. The proof of Theorem 1 uses that comparison to identify the tuned temperature with $\beta_c(g)$.

For the finite-volume argument, let $\mathbb{T}_N = (\mathbb{Z}/N\mathbb{Z})^2$, with its nearest-neighbor edges. Use $\langle\cdot\rangle_{N,\beta}$ and $\mu_{\mathrm{I},N}$ for the scalar-field and critical Ising probability measures on this torus, and define

$$G_{\phi,\beta;N}(x) = \langle\phi_0\phi_x\rangle_{N,\beta}, \qquad G_{\mathrm{I};N}(x) = \mu_{\mathrm{I},N}(\sigma_0\sigma_x).$$

A fixed $x \in \mathbb{Z}^2 \setminus \{0\}$ is identified with its torus image for all sufficiently large N. For two commuting labels $t = (t_0, t_x)$, define the partition integral

$$\mathcal{Z}_{N,\beta}(t) = \int e^{\beta \sum_{\langle u,v \rangle} \phi_u \phi_v} (1 + t_0\phi_0)(1 + t_x\phi_x) \prod_{v \in \mathbb{T}_N} \rho_g(\mathrm{d}\phi_v).$$

Write $M_x = \{0, x\}$ and $t^S = \prod_{u \in S} t_u$ for $S \subseteq M_x$. Source identities are in the ring $\mathbb{C}[t_0, t_x]/(t_0^2, t_x^2)$. We write $\varepsilon = g^{-1}$. Positive constants c, C may change from line to line. Constants in the RG bounds are uniform in the volume, scale index, separation and sufficiently large g, unless dependence is indicated. The blocking ratio and starting radius are fixed before choosing g.

2 Microscopic interaction and sources

Use Λ for $\mathbb{Z}^2$ or $\mathbb{T}_N$. In torus statements with two marks assume $N > 4|x|$ and $N \geq 5$. Write d_Λ for graph distance and $E(\Lambda)$ for the nearest-neighbor edges.

Definition 2.1 (Lists and norms)**.** Let $\Lambda = \mathbb{Z}^2$ or $\mathbb{T}_N$, with the lattice max-norm distance d_∞ (the periodic distance on $\mathbb{T}_N$). All carrier diameters use d_∞. Let $B_r = B_r(0)$ be the closed ball of integer radius $r \geq 1$, write $B_r(z)$ for the ball centered at z and $A + B_r$ for the r-neighborhood of A, and set $n_r(A) = |A + B_r|/r^2$. A list $h = (h_{A,z})$ is indexed by finite nearest-neighbor connected carriers $A \subset \Lambda$ and anchors $z \in A$. The function $h_{A,z}$ depends on spins in A, possibly on a proper subset, and $\|h_{A,z}\|_\infty = \sup_{\sigma \in \{-1,1\}^A} |h_{A,z}(\sigma)|$. Set

$$\|h\|_r = \sup_{y \in \Lambda} \sum_{\substack{A \cap B_r(y) \neq \varnothing \\ z \in A}} e^{n_r(A)} \|h_{A,z}\|_\infty.$$

For a marked list U with fixed nonempty label set S, each carrier contains the sites of all labels in S, and its anchor is the site of one prescribed label of S. We use the norm

$$\|U\|_{r,\bullet} = \sum_{A,z} e^{n_r(A)} \|U_{A,z}\|_\infty.$$

In each case the space of lists with finite norm is a Banach space. For a list whose coefficient functions have absolutely summable sup norms, write $\mathrm{ev}(h) = \sum_{A,z} h_{A,z}$. This applies to marked lists on either lattice and to all lists of finite norm on a finite torus. The projection π_A on a carrier A, abbreviated π, subtracts the mean over independent uniform spins on A. A list is canonical if each coefficient has zero such mean. We store the subtracted constants separately. Equivariance under a lattice symmetry means that applying it to the indices and to the spin configuration preserves each coefficient value. A homogeneous list is translation-equivariant. Spin-flip parity is taken coefficientwise.

For $a > 0$, let ρ_g^+ have density proportional to $e^{-g(a^2-1)^2}$. Under the product of the single-site laws ρ_g, the representation $\phi_u = a_u \sigma_u$ gives independent amplitudes of law ρ_g^+ and independent uniform signs, with the two families independent. Denote expectation over the amplitudes by $\mathbb{E}_{\mathrm{iid}}$ and put $\delta_u = a_u - 1$. For $e = \{u, v\}$ write $\sigma_e = \sigma_u \sigma_v$.

Fix a bounded open complex disk $\mathcal{B}$ about β_{I}.

Lemma 2.2 (Amplitude and edge bounds)**.** *For fixed $p, C \geq 0$ and all sufficiently large g, one has for $\delta = a - 1$*

$$\mathbb{E}_{\mathrm{iid}}(|\delta|^p e^{Ca^2}) \leq C_{p,C} g^{-p/2}, \qquad |\mathbb{E}_{\mathrm{iid}} \delta| \leq C_0 g^{-1}. \tag{2}$$

For $X_e = a_u a_v - 1$ and $d_e = e^{\beta X_e \sigma_e} - 1$, uniformly over $\beta \in \mathcal{B}$, edges and signs, one has for all sufficiently large g

$$\|d_e\|_{L^p(\mathbb{E}_{\mathrm{iid}})} \le C_{p,\mathcal{B}} g^{-1/2} \quad (p \ge 1), \qquad |\mathbb{E}_{\mathrm{iid}} d_e| \le C_{\mathcal{B}} g^{-1}. \tag{3}$$

Proof. The normalizing integral for ρ_g^+ is at least $cg^{-1/2}$ by restriction to $[1, 1+g^{-1/2}]$. Since $(a^2-1)^2 = (a+1)^2\delta^2 \ge \delta^2$ and $e^{Ca^2} \le e^{2C+2C\delta^2}$, a Gaussian integral bounds the numerator of the first estimate. Integration by parts on $(0,\infty)$, using $ae^{-g(a^2-1)^2} = 0$ at both endpoints, gives $4g\,\mathbb{E}_{\mathrm{iid}}(a^4 - a^2) = 1$. The identity $a^4 - a^2 = 2\delta + \delta^2(a^2+2a+2)$ and the first estimate imply the signed bound.

Now $X_e = \delta_u + \delta_v + \delta_u\delta_v$ and $|X_e| \le 1 + (a_u^2 + a_v^2)/2$. By independence and (2),

$$\mathbb{E}_{\mathrm{iid}}(|X_e|^p e^{C|X_e|}) \le C_{p,C} g^{-p/2}, \qquad \mathbb{E}_{\mathrm{iid}} X_e = 2\mathbb{E}_{\mathrm{iid}}\delta + (\mathbb{E}_{\mathrm{iid}}\delta)^2.$$

The bounds

$$|d_e| \le C_{\mathcal{B}} |X_e| e^{C_{\mathcal{B}}|X_e|}, \qquad |d_e - \beta\sigma_e X_e| \le C_{\mathcal{B}} X_e^2 e^{C_{\mathcal{B}}|X_e|}$$

then give (3). □

In the square-free source ring, $1 + t_u a_u \sigma_u = (1 + t_u\sigma_u)(1 + t_u\delta_u\sigma_u)$. The first factor is independent of the amplitudes and equals $e^{t_u\sigma_u}$ in this ring. Define the torus amplitude polynomial

$$W_N(\sigma; t) = \mathbb{E}_{\mathrm{iid}}\left[\prod_{e \in E(\mathbb{T}_N)} (1 + d_e) \prod_{u \in M_x} (1 + t_u \delta_u \sigma_u)\right]. \tag{4}$$

For estimates we view t as ordinary complex variables on $|t_u| \le 2$ and extract the square-free coefficients from the analytic logarithm. For $S \subseteq M_x$, write $[t^S]$ for extraction of the coefficient of $\prod_{u \in S} t_u$ with all other label exponents zero.

Proposition 2.3 (Microscopic interaction and sources)**.** *For each fixed $b \ge 0$, all sufficiently large g, and each $S \subseteq M_x$, there are coefficient functions $f_{A,z}^S(\beta)$, analytic in $\beta \in \mathcal{B}$, indexed by finite nearest-neighbor connected carriers $A \subseteq \Lambda$ and anchors $z \in A$. For $S \ne \varnothing$, all contributing carriers contain S and are anchored at its prescribed site (u for $S = \{u\}$ and 0 for $S = M_x$). On the torus $[t^S] \log W_N = \sum_{A,z} f_{A,z}^S$, and the plane coefficients are given by the same local cluster formula. The bounds*

$$\begin{aligned} &\sup_{y \in \Lambda} \sum_{\substack{A \ni y \\ z \in A}} e^{b|A|} \|f_{A,z}^{\varnothing}\|_\infty \le C_b \varepsilon, \\ &\sum_{A,z} e^{b|A|} \|f_{A,z}^S\|_\infty \le C_b \varepsilon e^{-d_\Lambda(0,x)} \mathbf{1}_{\{S = M_x\}} \qquad (S \ne \varnothing) \end{aligned} \tag{5}$$

hold uniformly in β, N and the separation. The background list is homogeneous, equivariant under quarter-turns, and even under spin flip. The parity of $f_{A,z}^S$ under spin flip is $(-1)^{|S|}$.

Use π_A to separate the constant of each coefficient function from its canonical list term. Denote the canonical background by h^{mic}, adding the temperature interaction $(\beta - \beta_{\mathrm{I}}) \sum_e \sigma_e$, with half of each edge term assigned to each endpoint. Its site-rooted norm with weight $e^{b|A|}$ is at most $C_b(\varepsilon + |\beta - \beta_{\mathrm{I}}|)$. Denote the canonical marked lists by U_{mic}^S, including the bare spin term σ_u for $S = \{u\}$, and the extracted scalars by c_{mic}^S. Then $c_{\mathrm{mic}}^{\{u\}} = 0$. In the marked norms with weight $e^{b|A|}$, the singleton correction has size $O(\varepsilon)$, and the mixed list and scalar together have size $O(\varepsilon e^{-d_\Lambda(0,x)})$. For the

torus let $c^{\varnothing}_{\mathrm{mic}}$ be the scalar from the unmarked list and $\mathcal{Z}_{\mathrm{I},N}$ the pure Ising partition function with product uniform signs as base measure. The exact partition identity is

$$\frac{\mathcal{Z}_{N,\beta}(t)}{\mathcal{Z}_{\mathrm{I},N}} = e^{c^{\varnothing}_{\mathrm{mic}}+t^{M_x}c^{M_x}_{\mathrm{mic}}}\,\mu_{\mathrm{I},N}\left(\exp\left[\mathrm{ev}(h^{\mathrm{mic}})+\sum_{\varnothing\neq S\subseteq M_x} t^S\,\mathrm{ev}(U^S_{\mathrm{mic}})\right]\right). \tag{6}$$

Proof. Make a graph on the factors in (4), with an edge between factors when their site supports intersect. Use $d_i = d_e$ for an edge factor and $d_i = t_u\delta_u\sigma_u$ for a marked factor. A polymer P is a nonempty connected set of factor vertices, $A(P)$ is the union of its site supports, and $\zeta(P) = \mathbb{E}_{\mathrm{iid}}\prod_{i\in P} d_i$. The components of any subset of factor vertices use disjoint families of amplitudes. Expansion of (4) therefore gives the polymer partition function with activities $\zeta(P)$ and with incompatibility $A(P)\cap A(Q)\neq\varnothing$.

The degree of the factor graph is uniformly bounded, so it admits a proper coloring with a fixed number q of colors. Apply Hölder with exponent q to the products on the color classes. Independence of the amplitudes in each class gives

$$|\zeta(P)| \le \prod_{i\in P}\|d_i\|_{L^q} \le (C_q g^{-1/2})^{|P|}.$$

For $|P|=1$ the signed mean bounds improve this to $C\varepsilon$, including for a marked factor. The number of connected m-factor sets with $y\in A(P)$ is at most C_1^m, by traversing a spanning tree from a factor incident to y. Also $|A(P)|\le 2m$. Let $a(P)$ be the supremum of $|\zeta(P)|$ over the signs, $\beta\in\mathcal{B}$ and $|t_u|\le 2$. The preceding bounds give

$$\sup_y \sum_{P:\,y\in A(P)} a(P)e^{(b+1)|A(P)|} \le C_b\varepsilon.$$

Thus for each polymer Q the same weighted sum over polymers incompatible with Q is bounded by $|A(Q)|C_b\varepsilon$. For $C_b\varepsilon\le 1$, this verifies the cluster criterion for activity majorants $a(P)e^{b|A(P)|}$ and charge $|A(P)|$.

By the pinned cluster estimate [2, Theorem 1], the absolute sum of connected logarithm terms whose generating union contains y, weighted by $e^{b|\bigcup_i A(P_i)|}$, is at most $\sum_{P:\,y\in A(P)} a(P)e^{(b+1)|A(P)|}$. Indeed one pins a polymer containing y and uses $e^{b|\bigcup_i A(P_i)|}\le\prod_i e^{b|A(P_i)|}$. Connectedness of the incompatibility graph makes the full generating union $A=\bigcup_i A(P_i)$ a connected site set. We keep A as carrier even when some of its spins do not enter the resulting coefficient. The pinned estimate on the plane follows by exhaustion, since it is uniform for finite polymer families. It defines each plane coefficient by an absolutely convergent local sum. The corresponding torus series is the analytic logarithm of W_N obtained from the cluster expansion.

Divide each unmarked cluster contribution equally among the sites of A. This preserves the bound and the lattice symmetries. For $S\neq\varnothing$, Cauchy extraction of $[t^S]$ costs at most one on the fixed polydisk. Any cluster with a nonzero extracted coefficient has a marked factor at each $u\in S$, so it can be pinned at the prescribed marked site. When $S=M_x$ its generating union satisfies $|A|\ge d_\Lambda(0,x)+1$. Using the pinned estimate with $b+1$ in place of b gives the additional exponential in (5). Simultaneous reversal of the signs and the source labels leaves W_N invariant and gives the asserted parity, also termwise in the local formula.

Subtracting the independent uniform-spin mean on each carrier costs at most a factor two in the list norm. The absolute sum of the removed marked constants is bounded by the same generating-carrier estimate. Odd singleton terms have zero uniform-spin mean. Finally, each original edge weight factors into $e^{\beta_{\mathrm{I}}\sigma_e}e^{(\beta-\beta_{\mathrm{I}})\sigma_e}(1+d_e)$. Combining these identities with $1+t_u\sigma_u = e^{t_u\sigma_u}$ and the amplitude expectation gives (6). □

Since $n_1(A) \le 9|A|$, a bound on the site-rooted sum with weight $e^{b|A|}$ as in (5), with $b > 9$, bounds the unmarked $r = 1$ norm and the $r = 1$ total norm of each marked list. For the unmarked norm, a root in $B_1(y)$ costs at most nine site-rooted sums.

3 The analytic blocking map

We use the following spaces and local transformation from [1, Sections 5.1–5.2]. We give the estimates with the full carrier weight because the microscopic lists need not have finite range.

Definition 3.1 (One blocking step)**.** Let $R = Lr$ with L an integer, and let $M = \lceil (\log L)^2 \rceil$. A carrier is short if $n_r(A) \le M$. For sufficiently large L, a short carrier is contained in the interior of the cutting square $B_R(z)$ about any $z \in A$. Denote by $P_{R,z}$ the pure critical Ising conditional expectation given the spins at sites y with $d_\infty(y, z) \ge R$. For tests supported in $B_R(z)$, use the finite-volume kernel in $\{y : d_\infty(y, z) < R\}$ with the prescribed spins on $\{y : d_\infty(y, z) = R\}$. This specifies the kernel at every boundary configuration and agrees in the plane and on a torus whenever the square lifts isometrically.

For finite input lists, expand $\exp(\mathrm{ev}(h))$ in ordered occurrences of input list terms, with repetitions allowed. For an occurrence of carrier A, put $V_R(A) = A + B_{2R}$. Join two occurrences if their V_R neighborhoods intersect. In each monomial apply $P_{R,z}$ to every isolated short occurrence, leaving all other factors as they are. Take the logarithm of the transformed series and use π term by term. A connected coefficient with at least two occurrences is assigned to the generating carrier $Q = \bigcup_i V_R(A_i)$. Divide an unmarked coefficient equally among its input anchors, counting multiplicity. For a single occurrence use carrier $B_R(z)$ if it was averaged, and A otherwise, with the original anchor in either case. For its estimates retain the auxiliary generating carrier $Q = V_R(A)$. The resulting unmarked list is $\mathcal{R}_{r,R}(h)$; on the torus the sum of removed constants is $\mathcal{C}_{r,R}(h)$.

In the marked version use the square-free variables t_ℓ, so $t_\ell^2 = 0$ and $t^S = \prod_{\ell \in S} t_\ell$. Root each output of type S at its prescribed anchor. For $p = 1, 2$ and input lists U_i with nonempty pairwise disjoint label sets S_i, let $\mathcal{D}^{(p)}_{r,R}(h)[U_1, \dots, U_p]$ be the coefficient of $t^{\bigcup_i S_i}$ obtained with input $h + \sum_i t^{S_i} U_i$. This is the mixed p-input derivative of the transformation and takes values in pairs (output list, new scalar), with projections $\mathrm{pr}_{\mathrm{list}}$ and $\mathrm{pr}_{\mathrm{sc}}$. On pairs use the norm $\|(U, c)\|_{R,\bullet} = \|U\|_{R,\bullet} + |c|$. An input scalar is independent of spins and is carried outside the map.

Lemma 3.2 (Geometry of the carrier weights)**.** *There are absolute $c, C > 0$ such that, for every connected carrier A,*

$$n_r(A) \ge c(1 + \mathrm{diam}(A)/r), \qquad n_R(V_R(A)) \le C(1 + n_r(A)/L).$$

Moreover $A + B_{4R}$ is covered by at most $CL^2 n_r(A)$ balls of radius r. The same bounds hold with periodic distances for $R \le \vartheta N$, where $\vartheta > 0$ is an absolute constant.

Proof. Along a path joining two sites at distance $\mathrm{diam}(A)$, select successive sites at distances from the starting point spaced by $3r$. The disjoint r-balls centered there lie in $A + B_r$, which proves the first bound. In particular every short carrier lies in $B_{CMr}(z)$ for each $z \in A$, and $CMr < R$ for L sufficiently large.

For the second bound, choose a maximal $10R$-separated subset of A. Its $13R$-balls cover $A + B_{3R}$. If there is more than one center, a path in A starting at each center and heading toward another leaves the R-ball about the first center. Its initial part up to that radius has an r-neighborhood of size at least cRr, by the same disjoint-ball argument. These neighborhoods lie in disjoint $2R$-balls and are contained in $A + B_r$. The number of centers is thus at most $C(1 + |A + B_r|/(Rr))$, proving

the bound. For the last assertion choose instead a maximal r-separated subset of A. The disjoint balls of radius $\lfloor r/2 \rfloor$ bound the number of its centers by $Cn_r(A)$. The B_{4R+r} ball about each center costs at most CL^2 balls of radius r. All constructions use the local cardinality bounds for balls, which hold on the stated periodic scales. □

Proposition 3.3 (Blocking with a diameter reserve)**.** *Fix L sufficiently large. There are $\eta_L, C_L, c > 0$, uniform in r and in tori with $R \le \vartheta N$, such that the blocking map is analytic for $\|h\|_r < \eta_L$. Writing $T_{r,R} = \mathrm{D}\mathcal{R}_{r,R}(0)$, one has*

$$\|\mathcal{R}_{r,R}(h) - T_{r,R}h\|_R \le C_L\|h\|_r^2, \qquad \|\mathrm{D}\mathcal{R}_{r,R}(h) - T_{r,R}\| \le C_L\|h\|_r. \tag{7}$$

The linear map $T_{r,R}$ averages short carriers, keeps long carriers and projects by π. The same linear rule applies to $T^{\bullet}_{r,R} = \mathrm{pr}_{\mathrm{list}}\,\mathcal{D}^{(1)}_{r,R}(0)$, with marked anchors.

Superscript $\ge D$ means the sum of the generating terms with $\mathrm{diam}(Q) \ge D$, retained before scalar extraction. For $D \ge 0$, $p \in \{1,2\}$ and $\mathbf{U} = (U_1, \dots, U_p)$, the bounds are

$$\begin{aligned} \|\mathcal{R}^{\ge D}_{r,R}(h)\|_R &\le C_L e^{-cD/R}\|h\|_r, \\ \|\mathcal{D}^{(p),\ge D}_{r,R}(h)[\mathbf{U}]\|_{R,\bullet} &\le C_L e^{-cD/R}\prod_i \|U_i\|_{r,\bullet}, \\ \|(\mathcal{D}^{(p),\ge D}_{r,R}(h) - \mathcal{D}^{(p),\ge D}_{r,R}(0))[\mathbf{U}]\|_{R,\bullet} &\le C_L e^{-cD/R}\|h\|_r\prod_i \|U_i\|_{r,\bullet}. \end{aligned} \tag{8}$$

The right side of the first bound also bounds the rooted sum before projection. The marked estimates include the new scalar.

The map preserves translation invariance, square symmetry and spin parity in the corresponding sectors. On each torus its exact identity is

$$\mu_{\mathrm{I},N}\big(e^{\mathrm{ev}(h)}\big) = e^{\mathcal{C}_{r,R}(h)}\mu_{\mathrm{I},N}\big(e^{\mathrm{ev}(\mathcal{R}_{r,R}(h))}\big), \tag{9}$$

The identity also holds in the square-free algebra with the corresponding marked coefficients and scalars. The proposition applies at $r = 1$, $L = r_0$ for a fixed sufficiently large r_0.

Proof. The interiors of the replaced cutting squares in a monomial are disjoint, and the remaining factors are measurable outside their union. The nearest-neighbor Markov property gives the same integral before and after the replacements. The symmetry statements follow from the local rule, the anchor averaging and the uniform-spin projection. Terms whose generating carrier lifts isometrically have the same local rule in plane and torus.

We estimate the logarithm before projection. For fixed labeled occurrences $1, \dots, m$, let $\mathcal{G}$ be their overlap graph. Use auxiliary variables $s_i^2 = 0$. The transformed coefficient for any subset of occurrences is a product over the components of its induced graph. The transformed series in the s_i is therefore the hard-core polynomial of connected subsets of $\mathcal{G}$. A singleton activity contains its linear image and any larger connected activity is the original product. The logarithm coefficient of $\prod_i s_i$ is a sum over partitions into connected blocks. The coefficient of a partition is the alternating sum over connected edge sets of its incompatibility graph. This follows by expanding the hard-core constraint into edge factors and grouping a graph by its connected components. The sum vanishes for a disconnected graph and is bounded in absolute value by its number of spanning trees. To see the bound, fix an edge ordering and group the connected edge sets by their Kruskal spanning tree. For each tree, an eligible extra edge is one larger than every edge on its tree path. The edge sets in that class consist of the tree and any subset of its eligible edges, so their alternating sum has magnitude at most one.

For each block partition and quotient tree, choose one internal tree in each block and one contact edge for each quotient edge, using fixed deterministic choices. Their union is a spanning tree of $\mathcal{G}$, with edges designated internal or external. These designations recover the partition and quotient tree. Writing $\tau(\mathcal{G})$ for the number of spanning trees, the absolute coefficient is consequently bounded by

$$2^{m-1}\tau(\mathcal{G})\prod_{i=1}^{m}\|f_i\|_\infty,$$

where f_i is the function of occurrence i. In particular the logarithmic coefficient has connected generating carrier Q.

By Lemma 3.2 and subadditivity of neighborhood cardinality,

$$2n_R(Q) \le Cm + (C/L)\sum_i n_r(A_i).$$

Take L large enough that $C/L \le 1/2$. For an unmarked output rooted at a ball $B_R(y)$, one occurrence meets $B_{3R}(y)$, which costs $O(L^2)$ input root balls. For a marked output, root the tree at an input occurrence of one prescribed mark and sum it in its total marked norm. A child of an occurrence of carrier A meets $A + B_{4R}$, so summing over the child costs at most $CL^2 n_r(A)$ input root balls. Additional marked inputs can be summed in their total marked norms. Inducting from leaves toward the root gives k such covering factors at a vertex with k children. The reserve $e^{n_r(A)/2}$ absorbs $n_r(A)^k$ at cost $2^k k!$. Thus the tree bound with weight $e^{2n_R(Q)}$ costs at most $C_L^m \prod_i k_i!$ times the product of the input norms. The factorials count the orderings of children. The number of ordered rooted shapes is at most 4^{m-1} and there are at most $m!$ labelings, canceled by the exponential factorial. This yields a bound C_L^m times the product of input norms for the homogeneous degree-m contribution, including the order-one rule.

Every actual output carrier is contained in Q. Since Q is connected, $n_R(Q) \ge cD/R$ on the diameter restriction, by Lemma 3.2. One factor $e^{n_R(Q)}$ therefore pays for the output weight and the other yields $e^{-cD/R}$. This estimate is made before π, which costs at most two in sup norm, and bounds extracted constants as well. Distinguishing p inputs in degree m costs at most m^p. For a difference of derivatives at h and zero the degree is at least $p+1$, so there is at least one factor $\|h\|_r$. Summing the resulting series on a sufficiently small ball proves (7) and (8).

For finite lists the monomial identity and the logarithmic identity hold first as power-series identities. Absolute convergence gives the analytic identities on the stated ball. Rooted estimates define the plane coefficients locally and permit removal of carrier truncations on each torus, where the total sum is bounded by finitely many rooted sums. This gives (9), including its marked coefficients. Only geometry and the reference Markov property enter this proof, including for the preliminary step. □

4 Linear contraction and thermal tuning

For integers $1 \le r < R$ and an anchor z, the cutting square and conditional kernel are those of Section 3. If F is supported in the square about z of radius $s < R$, define

$$d_{R,z}(F) = \mu^{+}_{4R,z}(F) - \mu_1(F),$$

where $\mu^{+}_{4R,z}$ is the nearest-neighbor critical Ising law in the lattice disk of radius $4R$ about z, with plus boundary condition. Put $e_z = \frac{1}{4}\sum_{w:\, w\sim z}\sigma_z\sigma_w$. Let v_r^{th} denote the homogeneous list with one coefficient $r^{-1}\pi P_{r,z}e_z$ on $B_r(z)$ at each z. Let v_r^u denote the singleton list anchored at u with coefficient $r^{1/8}P_{r,u}\sigma_u$ on $B_r(u)$.

Lemma 4.1 (Pure reference estimates)**.** *For each sufficiently large fixed ratio R/s there is a lower threshold on s such that the following estimates hold uniformly in z and in the bounded function F on $B_s(z)$:*

$$|d_{R,z}(F)| \le C(s/R)^d \|F\|_\infty, \qquad \|P_{R,z}F - \mu_1(F)\|_\infty \le C|d_{R,z}(F)| + C(s/R)^p \|F\|_\infty. \tag{10}$$

The pairs (d,p) are $(1,3)$ for spin-flip-even tests invariant under right-angle rotations about z, $(1/8, 9/8)$ for spin-flip-odd tests, and $(1,2)$ for arbitrary spin-flip-even tests. The constants are uniform over sufficiently large fixed ratios.

For all sufficiently large r, $\|v_r^{\mathrm{th}}\|_r$ and $\|v_r^u\|_{r,\bullet}$ are bounded above and below by positive constants. For sufficiently large R/r,

$$|d_{R,z}(r^{-1}\pi P_{r,z} e_z)| \asymp r^{-2}(r/R), \tag{11}$$

$$|d_{R,u}(r^{1/8} P_{r,u}\sigma_u)| \asymp (r/R)^{1/8}. \tag{12}$$

Proof. Equation (10) is the pure Ising estimate of [1, Proposition 4.1]. It extends to complex F by applying it to real and imaginary parts. The distinguished-field bounds and calibrations are [1, Lemma 5.3]. For the thermal norm, the single coefficient has size comparable to r^{-2}, with $O(r^2)$ anchors in a rooted sum and weights bounded above and below on $B_r(z)$. The same weights in the marked norm give the bound for v_r^u. □

Use $T_{r,R}$ and $T_{r,R}^\bullet$ for the pure linear maps on backgrounds and marked lists of Proposition 3.3.

Proposition 4.2 (Linear estimates)**.** *Fix $0 < \rho < 1$. There is a sufficiently large integer L, and then an integer r_0, such that for $r \ge r_0$, $R = Lr$, the following assertions hold.*

1. *In the homogeneous, right-angle-invariant, even background space, $T_{r,R} v_r^{\mathrm{th}} = L v_R^{\mathrm{th}}$. There are real bounded complements to the thermal lines such that the matrix of $T_{r,R}$ in the line and complement coordinates is $\begin{pmatrix} L & B_{r,R}^{\mathrm{th}} \\ 0 & D_{r,R}^{\mathrm{th}} \end{pmatrix}$, with $\|B_{r,R}^{\mathrm{th}}\| \le C_L$ and $\|D_{r,R}^{\mathrm{th}}\| \le \rho$.*
2. *On singleton odd lists anchored at u, $L^{1/8} T_{r,R}^\bullet v_r^u = v_R^u$. The corresponding matrix of $L^{1/8} T_{r,R}^\bullet$ is $\begin{pmatrix} 1 & B_{r,R}^{\mathrm{sp}} \\ 0 & D_{r,R}^{\mathrm{sp}} \end{pmatrix}$ with $\|B_{r,R}^{\mathrm{sp}}\| \le C_L$ and $\|D_{r,R}^{\mathrm{sp}}\| \le \rho$.*
3. *On singleton even lists, the unscaled $T_{r,R}^\bullet$ has norm at most ρ. This last assertion also holds on $\mathbb{T}_N$ when $R \le \vartheta N$ for a sufficiently small fixed $\vartheta > 0$.*

The line and complement coordinates have norms uniformly equivalent to the corresponding list norms.

Proof. Set $M = \lceil (\log L)^2 \rceil$. By the carrier geometry in Lemma 3.2, all short terms at z are supported on $B_s(z)$ with $s = CMr$ for an absolute integer C. For the unchanged long terms, the comparison $n_R(A) \le C + (C/L) n_r(A)$ gives an extra e^{-cM}. In the background norm a root square at scale R can be covered by $O(L^2)$ root squares at scale r. The long operators therefore cost $CL^2 e^{-cM}$ in the background norm and Ce^{-cM} in the marked norm, before any normalization.

For a background list h, sum the short coefficients anchored at z into F_z. The rooted norm counts all anchors in a radius-r square. By translation invariance their total weighted masses are equal, so $\|F_z\|_\infty \le Cr^{-2}\|h\|_r$. The rotation and spin-flip symmetries pass to F_z. By (10) and (11) one may subtract $\alpha r^{-1}\pi P_{r,z} e_z$ to cancel its disk response, where α is independent of z and $|\alpha| \le CM\|h\|_r$. The adjusted function has norm at most $C(1+M)r^{-2}\|h\|_r$. The projection π kills

constants. Using $p = 3$ in (10), its projected image has sup norm at most $C(M/L)^3(1+M)r^{-2}\|h\|_r$. The $O(R^2)$ output anchors have bounded weights on their squares, giving a background list bound $CM^4L^{-1}\|h\|_r$.

For a singleton odd list U, the short sum at u has norm at most $\|U\|_{r,\bullet}$. The cancellation now subtracts a multiple of v_r^u of size at most $CM^{1/8}\|U\|_{r,\bullet}$, by (12). With $p = 9/8$, multiplication by $L^{1/8}$ gives a marked bound

$$CL^{1/8}(M/L)^{9/8}(1+M^{1/8})\|U\|_{r,\bullet} \le CM^{5/4}L^{-1}\|U\|_{r,\bullet}.$$

Each distinguished carrier is short. The tower property gives the claimed multipliers on the two lines. The preceding cancellation bounds hold for every input representative modulo its line, and therefore bound the quotient operator norms after adding the long terms. Choose a bounded real dual functional equal to one on each distinguished vector and complexify it. The calibration guarantees uniform bounds for the resulting projections. Taking the kernels as complements gives the stated triangular matrices. The shear bounds follow from boundedness of the full operators, and the transverse bounds follow from the quotient estimates upon increasing L.

For the unscaled even mark, $(d, p) = (1, 2)$ in (10) gives a short operator bound CM/L after projecting off constants, without any response subtraction. Adding the long bound proves contraction. A short cutting square at $R \le \vartheta N$ lifts isometrically, and its conditional kernel is the same as in the plane by the nearest-neighbor Markov property. Thus the same estimate holds on the torus. Choose L first so the operator estimates are below ρ, then r_0 to meet the thresholds in Lemma 4.1. □

Fix these integers and choose $\rho < \gamma < 1$. Let $r_j = r_0 L^j$ for $j \ge 0$. The preliminary map is $\mathcal{R}_{1,r_0}$ and each subsequent step uses $\mathcal{R}_{r_j,r_{j+1}}$. The preliminary bounds follow from Proposition 3.3 with ratio r_0.

Proposition 4.3 (Massless tuning)**.** *For all sufficiently large g there is $\beta_*(g) = \beta_{\mathrm{I}} + O(\varepsilon)$ such that the plane lists obtained from the microscopic background at $\beta = \beta_*(g)$ satisfy*

$$\|h_j\|_{r_j} \le C\varepsilon\gamma^j \qquad (j \ge 0). \tag{13}$$

The construction selects a unique root of the decaying-graph equation on an interval $|\beta - \beta_{\mathrm{I}}| \le C_0\varepsilon$ with C_0 fixed.

Proof. Use the complements of Proposition 4.2 and write $h_j = \tau_j v^{\mathrm{th}}_{r_j} + w_j$. The recursion takes the form

$$\tau_{j+1} = L\tau_j + B_j^{\mathrm{th}} w_j + a_j(\tau_j, w_j), \qquad w_{j+1} = D_j^{\mathrm{th}} w_j + b_j(\tau_j, w_j).$$

Here $B_j^{\mathrm{th}} = B^{\mathrm{th}}_{r_j,r_{j+1}}$, and similarly for D_j^{th}. The remainders vanish quadratically, and their derivatives are bounded by $C_L(|\tau_j| + \|w_j\|_{r_j})$, by Proposition 3.3. Write $\Phi_{j,i} = D^{\mathrm{th}}_{j-1} \cdots D^{\mathrm{th}}_i$ for $j > i$ and $\Phi_{i,i} = \mathrm{Id}$. For given remainders a_i, b_i the decaying solution with prescribed w_0 is

$$w_j = \Phi_{j,0} w_0 + \sum_{i<j} \Phi_{j,i+1} b_i,$$
$$\tau_j = -\sum_{i \ge j} L^{j-i-1}(B_i^{\mathrm{th}} w_i + a_i).$$

In the norm $\sup_j \gamma^{-j}(|\tau_j| + \|w_j\|_{r_j})$, the solution operator is bounded by the geometric sums with ratios ρ/γ and γ/L. Substituting the nonlinear remainders defines a map on the sequence space. On a ball of radius $A\|w_0\|_{r_0}$ it is a contraction into the ball for fixed large A and sufficiently small

$\|w_0\|_{r_0}$, by the quadratic and derivative bounds. It yields an analytic graph $\tau_0 = \psi(w_0)$ and a trajectory of norm $O(\|w_0\|_{r_0})$. Analyticity follows from the locally uniform contraction iterates in the complex sequence space. The graph and trajectory are real for real w_0.

Put $\Delta\beta = \beta - \beta_{\mathrm{I}}$. The sum of the microscopic temperature coefficients at anchor z is $2\Delta\beta\, e_z$. Each temperature carrier is short. They have the same output carrier at z, so their preliminary linear image there is $2\Delta\beta\, \pi P_{r_0,z} e_z$. The amplitude contribution to the microscopic norm and to its β-derivative is $O(\varepsilon)$ on a smaller fixed disk, by Proposition 2.3 and Cauchy's formula. The preliminary operator estimate gives

$$h_0(\beta, g) = 2r_0\Delta\beta\, v_{r_0}^{\mathrm{th}} + O_{\|\cdot\|_{r_0}}((\Delta\beta)^2 + \varepsilon), \qquad \partial_\beta h_0 = 2r_0 v_{r_0}^{\mathrm{th}} + O_{\|\cdot\|_{r_0}}(|\Delta\beta| + \varepsilon).$$

It follows that $f(\beta) = \tau_0(\beta, g) - \psi(w_0(\beta, g))$ satisfies $f(\beta) = 2r_0\Delta\beta + O((\Delta\beta)^2 + \varepsilon)$ and $f'(\beta) = 2r_0 + O(|\Delta\beta| + \varepsilon)$. For a fixed sufficiently large C_0, f changes sign between $\Delta\beta = \pm C_0\varepsilon$ and has positive derivative on this interval. At its root $\|w_0\|_{r_0} = O(\varepsilon)$, so the decaying graph estimate proves (13). □

5 Periodic backgrounds

Fix a small numerical $\kappa > 0$. The projection Π_N of a plane list keeps carriers of max-norm diameter less than κN and projects their sites to $\mathbb{T}_N$. For a homogeneous list use one representative of each anchor modulo N. For a rooted marked list, the chosen lift of its marked anchor fixes the lifts of the retained carriers. In either case the projection is bounded in the list norms for $r \le \vartheta N$ with ϑ sufficiently small, by lifting all retained carriers meeting a fixed norm root box to the chart around that box.

Proposition 5.1 (Periodic comparison)**.** *Let H_j^N denote the torus background at $\beta_*(g)$ after the preliminary step and j subsequent steps. For a sufficiently small $\vartheta > 0$, for all j with $r_j \le \vartheta N$,*

$$E_j := \|H_j^N - \Pi_N h_j\|_{r_j} \le C\varepsilon\gamma^j e^{-cN/r_j}, \qquad \|H_j^N\|_{r_j} \le C\varepsilon\gamma^j. \tag{14}$$

Proof. At the microscopic scale, clusters whose carriers fit the projected chart lift uniquely after their anchors are fixed. Their activities and combinatorial factors coincide. The other connected carriers have at least cN sites, so the size-weight reserve in Proposition 2.3 bounds the discrepancy by $C\varepsilon e^{-cN}$ in the scale-one norm. The temperature term has identical local coefficients. At the preliminary step, the Lipschitz estimate transports this discrepancy. All new terms that fail chart comparison have generating diameter at least cN, so the tail estimate of Proposition 3.3 gives $E_0 \le C\varepsilon e^{-cN/r_0}$.

At a subsequent step compare the projected plane output with the torus output from the projected plane input. On a generating union of diameter less than cN, for c sufficiently small relative to κ, all input and output carriers are retained, the cutting kernels agree, and the same uniform-spin means are subtracted. The two coefficient constructions therefore agree on these terms. The remaining terms, including winding unions, have generating diameter at least cN. Their summed norm is at most $C\|h_j\|_{r_j} e^{-cN/r_{j+1}}$. Changing the torus input from $\Pi_N h_j$ to H_j^N costs KE_j as long as both lie in the analytic ball. Thus

$$E_{j+1} \le KE_j + C\varepsilon\gamma^j e^{-cN/r_{j+1}}.$$

For $X = N/r_j$ direct iteration gives

$$\frac{E_j}{\varepsilon\gamma^j} \le C(K/\gamma)^j e^{-cXL^j} + \frac{C}{\gamma}\sum_{m=0}^{j-1}(K/\gamma)^m e^{-cXL^m} \le C' e^{-c'X}.$$

The last inequality holds uniformly for $X \geq \vartheta^{-1}$ since L^m grows geometrically. Combining this estimate with (13) ensures that the torus input stays in the analytic ball for sufficiently large g, closing the induction. □

6 The two marked sources

Fix the tuned inverse temperature $\beta_*(g)$ from Proposition 4.3 and two distinct sites $0, x$. For $\varnothing \neq S \subseteq M_x$, let U_j^S and c_j^S be the canonical list and the accumulated scalar in the coefficient of t^S in the plane flow at scale r_j. Normalize both by $r_j^{|S|/8}$. Thus the preliminary step includes multiplication by $r_0^{|S|/8}$, and the step from j to $j+1$ includes multiplication by $L^{|S|/8}$. Write $U_{N,j}^S, c_{N,j}^S$ for the periodic counterparts. For superscripts we abbreviate $\{u\}$ by u and $\{0, x\}$ by $0x$. For marked lists set $\mathcal{D}_j^{(p)} = \mathcal{D}_{r_j, r_{j+1}}^{(p)}$. In the following estimates, the constants may depend on the fixed blocking parameters.

Lemma 6.1 (Periodic comparison of marked coefficients)**.** *For each fixed x and j, as $N \to \infty$,*

$$\sum_{\varnothing \neq S \subseteq M_x} \big(\|U_{N,j}^S - \Pi_N U_j^S\|_{r_j,\bullet} + |c_{N,j}^S - c_j^S| \big) \longrightarrow 0.$$

Here Π_N uses the carrier cutoff of Section 5 and the fixed marked anchor. The plane lists omitted by Π_N have marked norm tending to zero for each such x, j.

Proof. For fixed x and large N, the marked sites have unique lifts in any retained chart containing them and based at a prescribed marked anchor. Microscopic coefficients agree on these charts by lifting each amplitude cluster. The errors in the microscopic marked list and scalar have norm tending to zero by the weighted carrier bound of Proposition 2.3. Indeed the generating support of an exceptional cluster has diameter at least cN, and the weight is applied before extracting its scalar.

Write $\Delta_{j,N}$ for the sum in the lemma. The preliminary map at $r = 1$, $R = r_0$ has bounded operator norms by Proposition 3.3. The microscopic background discrepancy tends to zero, and exceptional preliminary test unions have diameter at least cN. The diameter estimate and the microscopic marked convergence give $\Delta_{0,N} \to 0$. For subsequent steps, first compare the maps on projected lists. The kernels, the projections off constants and the coefficient rules agree on test unions that lift into the chart. Every exceptional connected test union has diameter at least cN, provided $r_{j+1} \leq \vartheta N$. The diameter estimate in Proposition 3.3 therefore bounds the corresponding list and scalar errors. On bounded sets of marked inputs, the one- and two-input operators and their background differences give

$$\Delta_{j+1,N} \leq C_j \big(\Delta_{j,N} + E_j + e^{-cN/r_{j+1}} \big), \qquad E_j = \|H_j^N - \Pi_N h_j\|_{r_j}.$$

The marked norms are bounded for each fixed number of steps by the same operator estimates, including the fixed normalization factors. Proposition 5.1 gives $E_j \to 0$ at fixed j, so the recurrence proves the comparison. The omitted plane tails tend to zero by absolute summability in the marked norm. □

Write $v_j^u = v_{r_j}^u$ for the pure spin list of Section 4. Use $\rho < \gamma < 1$ from Section 4 and fix $\theta \in (\gamma, 1)$.

Proposition 6.2 (One-source flow)**.** *There is a real $Z_\phi(g) = 1 + O(\varepsilon)$ such that*

$$U_j^u = Z_\phi(g) v_j^u + \eta_j^u, \qquad \|\eta_j^u\|_{r_j,\bullet} \leq C\varepsilon\theta^j. \tag{15}$$

The factor $Z_\phi(g)$ is independent of u. The one-source scalars vanish at every scale, on the plane and the torus.

Proof. At zero background the preliminary one-input operator sends the bare spin list to v_0^u after normalization. At the tuned temperature the microscopic background norm is $O(\varepsilon)$. The physical source differs from the bare spin by $O(\varepsilon)$ in the microscopic marked norm by Proposition 2.3. The bounded one-input operator and its $O(\varepsilon)$ difference from the operator at zero background give $\|U_0^u - v_0^u\|_{r_0,\bullet} \le C\varepsilon$.

Let $A_j = L^{1/8} \operatorname{pr}_{\mathrm{list}} \mathcal{D}_j^{(1)}(0)$. Write $U_j^u = \alpha_j v_j^u + w_j^u$ in the spin-line splitting from Proposition 4.2. In these coordinates A_j has blocks

$$A_j = \begin{pmatrix} 1 & B_j^{\mathrm{sp}} \\ 0 & D_j^{\mathrm{sp}} \end{pmatrix}, \qquad \|B_j^{\mathrm{sp}}\| \le C, \quad \|D_j^{\mathrm{sp}}\| \le \rho.$$

The longitudinal multiplier is one by nested conditional expectation. The true update is $U_{j+1}^u = (A_j + J_j)U_j^u$, with

$$\|J_j\| \le C\|h_j\|_{r_j} \le C\varepsilon\gamma^j$$

by Propositions 3.3 and 4.3. The pure products have norm at most C_0 for every consecutive interval of scales. The transverse block is bounded by the corresponding power of ρ, and the off-diagonal block by $C\sum_{m\ge 0}\rho^m$. Expanding each perturbed product by the positions of the J_i gives the bound $C_0\prod_{i\ge 0}(1 + C_0\|J_i\|)$. In particular $\|U_j^u\|_{r_j,\bullet} \le C$.

The transverse recurrence and the preliminary estimate give

$$\|w_j^u\|_{r_j,\bullet} \le C\varepsilon\rho^j + C\varepsilon\sum_{i=0}^{j-1}\rho^{j-1-i}\gamma^i \le C_\theta\varepsilon\theta^j.$$

The longitudinal increments satisfy $|\alpha_{j+1} - \alpha_j| \le C(\|w_j^u\|_{r_j,\bullet} + \varepsilon\gamma^j)$. Hence α_j converges and its tail is $O(\varepsilon\theta^j)$. Its limit is $1 + O(\varepsilon)$ because $\alpha_0 = 1 + O(\varepsilon)$. Setting the other label to zero commutes with the microscopic integral and each blocking map. Thus the one-source flow depends only on u. Choose the complements by translation from one site. Translation invariance then gives the same limit for all u. The uniform norm bound on v_j^u gives (15). Finally the background is even, the one-source input is odd, and all kernels and the constant projection commute with spin flip. This makes each one-source scalar increment, including the microscopic one, zero. □

Proposition 6.3 (Mixed coefficient up to separation)**.** *Choose $b_0 > 0$ sufficiently small and suppose $b_0|x| \ge r_0$. Let k be the last index with $r_k \le b_0|x|$, so that*

$$b_0|x|/L < r_k \le b_0|x|. \tag{16}$$

Then

$$\|U_k^{0x}\|_{r_k,\bullet} + |c_k^{0x}| \le C\varepsilon\theta^k \tag{17}$$

with the θ of Proposition 6.2.

Proof. Set $D = |x|$ and $B_j = \|U_j^{0x}\|_{r_j,\bullet} + |c_j^{0x}|$. Take b_0 so small that the protective neighborhoods of the two pure single-spin carriers are disjoint at each step up to k, including the preliminary step. In a step ending at r_{j+1} those neighborhoods have radii at most $r_j + 2r_{j+1}$. Their disjointness follows from $\|x\|_\infty \ge D/\sqrt{2}$ and $r_{j+1} \le b_0 D$.

At the preliminary step the microscopic mixed coefficient and its scalar have norm $O(\varepsilon e^{-cD})$. The zero-background bilinear operator on the two bare spins vanishes by the disjoint neighborhoods.

Every remaining connected term contains a background occurrence or a microscopic one-source correction and has a generating test union containing both sites. Proposition 3.3, used with $r = 1$ and $R = r_0$, bounds its joint list and scalar norm by $C\varepsilon e^{-cD/r_0}$, with normalization absorbed into C. Consequently $B_0 \le C\varepsilon e^{-cD/r_0}$.

For $j < k$ the coefficient rule in the square-free ring gives

$$(U^{0x}_{j+1}, c^{0x}_{j+1}) = L^{1/4}\Big((0, c^{0x}_j) + \mathcal{D}^{(1)}_j(h_j)[U^{0x}_j] + \mathcal{D}^{(2)}_j(h_j)[U^0_j, U^x_j]\Big). \tag{18}$$

The two label orders in the quadratic Taylor term cancel its factor $1/2$. The accumulated scalar is spin independent and is carried along separately. The terms linear in (U^{0x}_j, c^{0x}_j) are bounded in joint norm by KB_j, for a scale-independent K.

Write $Z = Z_\phi(g)$. By disjointness, $\mathcal{D}^{(2)}_j(0)[Zv^0_j, Zv^x_j] = 0$, including its scalar part. Subtracting it from the bilinear term in (18) gives

$$\begin{aligned}&\big(\mathcal{D}^{(2)}_j(h_j) - \mathcal{D}^{(2)}_j(0)\big)[U^0_j, U^x_j]\\ &\quad + \mathcal{D}^{(2)}_j(0)[\eta^0_j, U^x_j] + \mathcal{D}^{(2)}_j(0)[Zv^0_j, \eta^x_j].\end{aligned}$$

Each contributing generating union has max-norm diameter at least $D/\sqrt{2}$ because it contains both marks. The joint diameter bound controls the displayed terms by

$$Ce^{-cD/r_{j+1}}\big(\|h_j\|_{r_j} + \|\eta^0_j\|_{r_j,\bullet} + \|\eta^x_j\|_{r_j,\bullet}\big).$$

Here the remaining one-source factors have uniformly bounded norm. It follows that

$$B_{j+1} \le KB_j + C\varepsilon\theta^j e^{-cD/r_{j+1}}.$$

Iterating and using $r_k/r_{k-m} = L^m$ bounds the inhomogeneous sum by

$$C\varepsilon\theta^k \sum_{m\ge 0} (K/\theta)^m e^{-c(D/r_k)L^m} \le C'\varepsilon\theta^k.$$

The initial term obeys the same bound since $K^k e^{-cD/r_0}/\theta^k = (K/\theta)^k e^{-c(D/r_k)L^k}$ is uniformly bounded. This proves (17). □

Define the residual periodic probability law by

$$\nu_{N,j}(d\sigma) = \frac{\exp\big(\mathrm{ev}(H^N_j)(\sigma)\big)}{\mu_{\mathrm{I},N}\big[\exp(\mathrm{ev}(H^N_j))\big]}\,\mu_{\mathrm{I},N}(d\sigma).$$

The real coefficients of H^N_j and finite volume make this a positive probability law.

Lemma 6.4 (Comparison of an unscaled even test)**.** *Let F be an even test carried by a connected set A with specified anchor $u \in A$ on $\mathbb{T}_N$. Suppose $k \le s$ and $r_s \le \vartheta N$. There is a $q < 1$ independent of F, k, s, N such that*

$$|\nu_{N,k}(F) - \mu_{\mathrm{I},N}(F)| \le Ce^{n_{r_k}(A)}\|F\|_\infty\big(\varepsilon\gamma^k + q^{s-k}\big).$$

Proof. Let c be the uniform-spin mean of F on A. Insert $F - c$ at scale k as one unscaled auxiliary marked list f_k, and carry c as its scalar. The list norm is at most $C_F = 2e^{n_{r_k}(A)}\|F\|_\infty$. At scale j the linear transfer and scalar increment are the two components of $\mathcal{D}^{(1)}_j(H^N_j)$, computed on the

torus. Use f_j^0 for the analogous pure lists, with $f_k^0 = f_k$. The unscaled even pure list map contracts by Proposition 4.2. Both the list map and the scalar functional differ from their pure values in operator norm by at most $C\varepsilon\gamma^j$, by Propositions 3.3 and 5.1. For sufficiently large g, both list maps have norm at most some fixed $q < 1$ at all the chart scales. It follows that

$$\|f_j\|_{r_j,\bullet} + \|f_j^0\|_{r_j,\bullet} \le 2C_F q^{j-k}, \qquad \|f_j - f_j^0\|_{r_j,\bullet} \le CC_F\varepsilon\gamma^k(j-k)q^{j-k-1} \quad (j > k).$$

The second bound is obtained by telescoping the products of the two maps at each position $i \in [k, j-1]$.

Let $\ell_j(H_j^N)$ denote the scalar functional of the one-input operator at scale j. The exact partition identity, with one square-free label, gives

$$\nu_{N,k}(F) = c + \sum_{j=k}^{s-1} \ell_j(H_j^N)[f_j] + \nu_{N,s}(\mathrm{ev}(f_s)).$$

There is the corresponding identity with $H_j^N = 0$, f_j^0 and $\mu_{\mathrm{I},N}$. The initial scalar cancels. Boundedness of the scalar functional and the preceding estimates bound the difference of the scalar sums by

$$C\sum_{j=k}^{s-1} \big(\|f_j - f_j^0\|_{r_j,\bullet} + \varepsilon\gamma^j\|f_j\|_{r_j,\bullet}\big) \le CC_F\varepsilon\gamma^k.$$

The two terminal expectations are bounded in absolute value by their list norms, since the measures are positive. This proves the claimed estimate. □

Proposition 6.5 (Two-point comparison at the tuned temperature)**.** *With k as in Proposition 6.3,*

$$\limsup_{N\to\infty} \Big| r_k^{1/4} G_{\phi,\beta_*;N}(x) - Z_\phi(g)^2 r_k^{1/4} G_{\mathrm{I};N}(x) \Big| \le C\varepsilon\theta^k. \tag{19}$$

Here x and k are held fixed in the periodic limit.

Proof. The partition identity at scale k, after canceling the background scalar and using the vanishing one-source scalars, is

$$\frac{\mathcal{Z}_{N,\beta_*}(r_k^{1/8}t)}{\mathcal{Z}_{N,\beta_*}(0)} = e^{t_0 t_x c_{N,k}^{0x}}\, \nu_{N,k}\big[\exp\big(t_0\,\mathrm{ev}(U_{N,k}^0) + t_x\,\mathrm{ev}(U_{N,k}^x) + t_0 t_x\,\mathrm{ev}(U_{N,k}^{0x})\big)\big]\,.$$

The argument $r_k^{1/8}t$ multiplies both labels. Taking the coefficient of $t_0 t_x$ yields

$$\begin{aligned} r_k^{1/4} G_{\phi,\beta_*;N}(x) = c_{N,k}^{0x} &+ \nu_{N,k}(\mathrm{ev}(U_{N,k}^{0x})) \\ &+ \nu_{N,k}\big(\mathrm{ev}(U_{N,k}^0)\,\mathrm{ev}(U_{N,k}^x)\big). \end{aligned} \tag{20}$$

Set $F_k = \mathrm{ev}(v_k^0)\,\mathrm{ev}(v_k^x)$ on the two lifted cutting squares, viewed on the torus. The marked norm bounds the evaluation norm, and chart projection does not increase it. Lemma 6.1 and Propositions 6.2 and 6.3 therefore give

$$r_k^{1/4} G_{\phi,\beta_*;N}(x) = Z_\phi(g)^2 \nu_{N,k}(F_k) + O(\varepsilon\theta^k) + o_N(1).$$

The test F_k is even with uniformly bounded sup norm. By (16), a square of radius $C_{b_0} r_k$ contains both carriers and has n_{r_k} bounded uniformly in x. Lemma 6.4, with the largest s such that $r_s \le \vartheta N$, now gives

$$\nu_{N,k}(F_k) = \mu_{\mathrm{I},N}(F_k) + O(\varepsilon\gamma^k) + o_N(1).$$

Finally, under $\mu_{\mathrm{I},N}$, condition on the exterior of the two cutting interiors. The disjoint interiors have conditionally independent spins and their conditional means are the defining kernels of v_k^0 and v_k^x. Consequently $\mu_{\mathrm{I},N}(F_k) = r_k^{1/4} G_{\mathrm{I};N}(x)$ for all sufficiently large N. Since $\gamma < \theta$, this proves (19). □

7 Thermodynamic limit and criticality

Lemma 7.1 (Fixed-separation torus limits)**.** *For each fixed $\beta \geq 0$, $g > 0$ and $x \in \mathbb{Z}^2$,*

$$\lim_{N\to\infty} G_{\phi,\beta;N}(x) = G_{\phi,\beta}(x), \qquad \lim_{N\to\infty} G_{\mathrm{I};N}(x) = G_{\mathrm{I}}(x).$$

Proof. Let $m_{p,N} = \langle |\phi_0|^p \rangle_{N,\beta}$. Integration by parts in ϕ_0, using the quartic tails, gives

$$4g(m_{4,N} - m_{2,N}) = 1 + \beta \sum_{z:\, z\sim 0} \langle \phi_0 \phi_z \rangle_{N,\beta} \leq 1 + 4\beta m_{2,N}.$$

Since $m_{2,N} \leq m_{4,N}^{1/2}$, the fourth moments are uniformly bounded for fixed g, β. The periodic extensions to $\mathbb{Z}^2$ are therefore locally tight. Any limit is stationary and spin-flip symmetric.

For each finite set, the finite-volume conditional kernels coincide with the infinite-lattice kernels on fixed cylinder tests when N is sufficiently large. For a bounded continuous test of the interior variables, the kernel is continuous in its boundary values by quartic domination on compact sets. Passing the conditional identity to the limit and then using a monotone class gives the DLR equations. They imply that finite marginals are absolutely continuous. Moreover, the lattice pointwise ergodic theorem applied to ϕ_0^2 gives almost surely bounded centered box averages of ϕ_v^2. Thus the limits meet the temperedness requirement of [3, Appendix D]. By [3, Theorem 1.2], each stationary symmetric limit is the symmetric mixture of the plus and minus states, which is also the free state. The uniform fourth moment bound gives uniform integrability of $\phi_0\phi_x$, so the correlations converge to those of that state.

For the pure Ising model, local subsequential limits are critical nearest-neighbor Gibbs measures. Uniqueness at β_{I} [4] gives the second assertion. □

Proof of Theorem 1. Proposition 6.5 and Lemma 7.1 imply, at $\beta = \beta_*(g)$,

$$G_{\phi,\beta_*}(x) = Z_\phi(g)^2 G_{\mathrm{I}}(x) + O(\varepsilon r_k^{-1/4} \theta^k).$$

Here $r_k \asymp |x|$ by (16), so for $\omega = -\log\theta/\log L > 0$ the error is $O(\varepsilon |x|^{-1/4-\omega})$. The pure reference has $G_{\mathrm{I}}(x) \sim C_{\mathrm{I}} |x|^{-1/4}$ with $C_{\mathrm{I}} > 0$ [5]. Thus the correlation at $\beta_*(g)$ tends to zero and its lattice sum diverges.

The three critical inverse temperatures for magnetization, finite susceptibility and long-range order coincide by [3, Proposition C.1]. For $\beta < \beta_c(g)$, the free two-point function is summable. For $\beta > \beta_c(g)$, it has strictly positive infimum over x, using its monotonicity in β by the Griffiths inequality. The two properties at $\beta_*(g)$ force $\beta_*(g) = \beta_c(g)$. The preceding comparison proves (1) with $C_g = C_{\mathrm{I}} Z_\phi(g)^2 > 0$. □

References

[1] OpenAI. *Conformal universality of bulk Ising correlations under weak interactions.* Preprint, 2026. Manuscript.

[2] D. Ueltschi. *Cluster expansions and correlation functions.* arXiv:math-ph/0304003.

[3] T. S. Gunaratnam, C. Panagiotis, R. Panis and F. Severo. *Random tangled currents for φ^4: translation invariant Gibbs measures and continuity of the phase transition.* arXiv:2211.00319.

[4] H. Duminil-Copin, V. Sidoravicius and V. Tassion. *Continuity of the phase transition for planar random-cluster and Potts models with $1 \leq q \leq 4$.* arXiv:1505.04159.

[5] D. Chelkak, K. Izyurov and R. Mahfouf. *Universality of spin correlations in the Ising model on isoradial graphs.* arXiv:2104.12858.