# Correlation function of the fractional spread-out Ising models at their upper critical dimension and critical temperature

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**Abstract**

We give a renormalisation-group argument for the critical free two-point function of the spread-out Ising model with kinetic operator $(-\Delta_R)^{d/4}$ in dimensions $1 \le d \le 4$. For sufficiently large fixed spread radius $R$, the critical correlation is asymptotic to a positive constant times $|x|^{-d/2}$.

## 1 Model and main result

Fix $d \in \{1,2,3,4\}$ and an integer $R \ge 1$. Write $|x|$ for the Euclidean norm and $|x|_\infty$ for the maximum norm on $\mathbb{Z}^d$. The box kernel, associated Laplacian, and exponents are

$$D_{R,d}(x) = \frac{\mathbf{1}_{\{0<|x|_\infty \le R\}}}{(2R+1)^d - 1}, \qquad L_R = -\Delta_R = I - D_{R,d}, \qquad \alpha = \frac{d}{4}, \qquad s = 2\alpha = \frac{d}{2}.$$

We use the same notation for a convolution kernel $A$ and the operator $(Af)(x) = \sum_y A(x-y)f(y)$. The identity $I$ has kernel $\delta_0$. For a summable kernel $A$, write $\widehat{A}(k) = \sum_x A(x)e^{ik\cdot x}$ for $k \in [-\pi,\pi]^d$, with inverse Fourier measure $(2\pi)^{-d}\,dk$. Define the fractional power and the Ising pair kernel by

$$\widehat{L_R^\alpha}(k) = \left(1 - \widehat{D}_{R,d}(k)\right)^\alpha, \qquad J_R^{\mathrm{raw}} = I - L_R^\alpha, \qquad K_R(x) = \mathbf{1}_{\{x\neq 0\}} J_R^{\mathrm{raw}}(x).$$

The box kernel is symmetric, so $\widehat{D}_{R,d}$ is real and lies in $[-1,1]$. In particular, $K_R(x) = -L_R^\alpha(x)$ for $x \neq 0$, and the calculation below shows that $K_R$ is nonnegative and summable. For $d = 4$, $\alpha = 1$ and $K_R = D_{R,4}$.

**Spin measures.** On a finite set $\Lambda \subset \mathbb{Z}^d$, each spin $\sigma_x$ takes values in $\{-1,1\}$. At inverse temperature $\tau \ge 0$ and zero external field, the free Gibbs probability is

$$\mathbb{P}^f_{\Lambda,\tau}(\sigma) = \frac{1}{Z^f_{\Lambda,\tau}} \exp\left\{\frac{\tau}{2}\sum_{x,y\in\Lambda} K_R(x-y)\sigma_x\sigma_y\right\}. \tag{1}$$

The partition function $Z^f_{\Lambda,\tau}$ normalizes this probability. The equivalent kinetic weight is

$$\exp\{-\tau\langle\sigma, L_R^\alpha\sigma\rangle_\Lambda/2\},$$

where the quadratic form sums over $x, y \in \Lambda$ and uses the infinite-volume kernel. The difference in the two exponents is the constant $-\tau L_R^\alpha(0)|\Lambda|/2$.

Plus boundary conditions add $\tau \sum_{x\in\Lambda} \sigma_x \sum_{y\notin\Lambda} K_R(x-y)$ to the exponent in (1). On $\Lambda_M = (\mathbb{Z}/M\mathbb{Z})^d$ we use the same Gibbs formula with $K_R$ replaced by its periodization

$$(\mathcal{P}_M K_R)(\bar{x}) = \sum_{n\in\mathbb{Z}^d} K_R(x+Mn).$$

Any resulting diagonal term contributes only a constant.

Write $\langle\cdot\rangle_\tau^f$ and $\langle\cdot\rangle_\tau^+$ for the free and plus Gibbs expectations obtained as the corresponding finite-volume limits along increasing boxes. The critical inverse temperature is

$$\tau_c = \tau_c(d,R) = \inf\{\tau \geq 0 : \langle\sigma_0\rangle_\tau^+ > 0\}.$$

**Theorem 1** (Critical fractional two-point function)**.** *For each $d \in \{1,2,3,4\}$ and all sufficiently large fixed $R$, the model* (1) *has critical free two-point asymptotic*

$$\langle\sigma_0\sigma_x\rangle_{\tau_c}^f \sim C_{d,R}|x|^{-s}, \qquad C_{d,R} > 0, \qquad |x| \longrightarrow \infty. \tag{2}$$

**Ferromagnetism and temperature conventions.** For $0 < \alpha < 1$, put

$$c_n = (-1)^{n+1}\binom{\alpha}{n} = \frac{\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)} > 0 \qquad (n \geq 1).$$

Here $\Gamma$ is the gamma function. The binomial series for $1-(1-t)^\alpha$ has these coefficients. Monotone convergence as $t \uparrow 1$ gives $\sum_{n\geq1} c_n = 1$, so the series converges absolutely for $|t| \leq 1$. Since $|\widehat{D}_{R,d}| \leq 1$, Fourier inversion gives

$$J_R^{\mathrm{raw}} = \sum_{n\geq1} c_n D_{R,d}^{*n}, \tag{3}$$

where $*n$ denotes the $n$-fold convolution. Thus $J_R^{\mathrm{raw}}$ is a nonnegative symmetric probability kernel. For $\alpha = 1$ the same conclusion follows from $J_R^{\mathrm{raw}} = D_{R,d}$.

Put $\kappa_R = L_R^\alpha(0) = 1 - J_R^{\mathrm{raw}}(0)$. Because $\|D_{R,d}^{*n}\|_\infty \leq \|D_{R,d}\|_\infty$ for $n \geq 1$, we have $\kappa_R > 0$ and

$$0 \leq 1-\kappa_R \leq C_d R^{-d}, \qquad \|K_R\|_\infty \leq C_d R^{-d}.$$

The zero-diagonal probability kernel is $D_R^{(\alpha)} = K_R/\kappa_R$. Its inverse temperature $\widetilde{\beta}$ is related to the physical inverse temperature by

$$\widetilde{\beta} = \kappa_R\tau, \qquad \widetilde{\beta}_c = \kappa_R\tau_c, \tag{4}$$

where $\widetilde{\beta}_c$ is the critical inverse temperature for $D_R^{(\alpha)}$. The Gaussian representations use a probability kernel with an added diagonal. For the diagonal parameter $\delta_R$ chosen in the respective reference splits, this kernel and its inverse temperature are

$$D^+ = \frac{J_R^{\mathrm{raw}} + \delta_R I}{1+\delta_R}, \qquad \beta = (1+\delta_R)\tau.$$

At these corresponding temperatures all three pair kernels give the same spin laws. In the four-dimensional calculation the box inverse temperature is also denoted by $\beta^{\mathrm{box}} = \tau$.

**Proof outline.** Section 2 collects the common analytic tools. Part I proves the box case and introduces the reference-pressure comparison, connected expansion, and exact RG coordinate changes used in both proofs. The local stiffness is marginal there and is tuned along with the mass. Part II adapts the construction to $d < 4$. An internal space permits a finite-range decomposition of the nonlocal covariance, and the local kinetic coordinate is irrelevant and evolves forward. In both proofs the susceptibility identifies the physical critical temperature, and the source flow determines the two-point asymptotic.

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## 2 Common RG and limiting arguments

Throughout, $\nabla_i f(x) = f(x+e_i) - f(x)$ is the forward lattice difference and $\nabla^\gamma$ is its multi-index extension. We write $U \lesssim V$ for $U \le CV$ with a constant uniform in the parameters of the estimate, and $U \asymp V$ for the two corresponding inequalities. Positivity of a covariance piece means positive semidefiniteness as an operator. It may have signed kernel entries.

In the renormalisation group (RG), the block length at step $j$ is $b_j = bL^j$, where $L > 1$ is the integer scale factor and $b \asymp R$ is the initial block side. A polymer $X$ is a union of blocks, and $|X|_j$ counts its scale-$j$ blocks. The covariance integrated in that step is $\Gamma_{j+1}$, and $w_j = \sum_{i<j} \Gamma_{i+1}$ is the cumulative covariance. We write $(\theta_\zeta F)(\phi) = F(\phi + \zeta)$ for field translation, and $\mathbb{E}_\Gamma \theta F$ for its expectation over a centered Gaussian $\zeta$ of covariance $\Gamma$.

For marks at $a, c$, use the commuting nilpotent source variables $t_a, t_c$, with $t_a^2 = t_c^2 = 0$. The four sectors are the coefficients of $1, t_a, t_c, t_a t_c$. The coalescence index $k$ is chosen at the first scale at which a small polymer neighborhood can reach both marks, up to a bounded shift of the scale index. In particular $b_k \asymp |a - c|$ for distant marks. Before this index, the mixed sector vanishes on small polymers by source support. The subscript $\sigma$ on a norm radius denotes a single source weight, and $\sigma\sigma$ denotes the mixed weight.

### 2.1 Positive polynomial decomposition of an inverse

**Lemma 2.1** (Scalar positive-polynomial inverse decomposition)**.** *Fix an integer $L > 1$. There are nonnegative polynomials $q_j$ on $[0,4]$, of degree at most $L^{j+1}$, such that for $0 < x \le 4$,*

$$\sum_{j \ge 0} q_j(x) = x^{-1}, \qquad q_j(x) \le C_p L^{2j} (1 + L^{2j} x)^{-p} \quad (p \ge 1).$$

*For a fixed $c_* > 0$, $q_j(x) \ge c L^{2j}$ if $0 \le x \le c_* L^{-2j}$. The initial polynomial is bounded below by a positive constant on all of $[0,4]$. After a fixed spectral rescaling the same assertions hold on any fixed bounded interval.*

*Proof.* Choose a nonnegative, even, nonzero smooth function $\rho$ supported in $[-1/2, 1/2]$, and put

$$\widehat{\rho}(u) = \int \rho(v) e^{iuv}\, dv, \qquad f(u) = |\widehat{\rho}(u)|^2, \qquad C_f = \int_0^\infty u f(u)\, du > 0.$$

The function $f$ is even, nonnegative and rapidly decreasing, $f(0) > 0$, and its inverse Fourier transform has support in $[-1, 1]$. For $t > 0$ set

$$P_t(\theta) = \sum_{n \in \mathbb{Z}} f(t(\theta + 2\pi n)).$$

Poisson summation shows that $P_t$ is an even trigonometric polynomial of degree at most $t$. Hence it is a polynomial in $x = 2 - 2\cos\theta$ of that degree, and is nonnegative. For $0 < t < 1$ it is the positive constant $t^{-1} \int \rho(v)^2\, dv$. For $0 < \theta \le \pi$, Tonelli's theorem and the elementary partial-fraction identity for sin give

$$\int_0^\infty t P_t(\theta)\, dt = C_f \sum_{n \in \mathbb{Z}} (\theta + 2\pi n)^{-2} = \frac{C_f}{4 \sin^2(\theta/2)} = \frac{C_f}{x}.$$

Define $q_0 = C_f^{-1} \int_0^L t P_t\, dt$ and $q_j = C_f^{-1} \int_{L^j}^{L^{j+1}} t P_t\, dt$ for $j \ge 1$. These are polynomials of the asserted degrees and sum to $x^{-1}$. Rapid decrease gives, for $t \ge 1$, $P_t(\theta) \le C_p (1 + t^2 x)^{-p}$, since

$\theta^2 \asymp x$ on $[0,\pi]$. Integrating on each fixed-ratio interval proves the upper bounds; $q_0$ is bounded and $x \le 4$. Finally $f(u) \ge c > 0$ for sufficiently small $|u|$, so $P_t(\theta) \ge c$ if $x \le c_* L^{-2j}$, with $c_*$ small enough that $t|\theta|$ is in that neighborhood for all $L^j \le t \le L^{j+1}$. Integration gives the lower bound. The integral over $0 < t < 1$ gives the global lower bound for $q_0$. □

## 2.2 Bounds for the reference quartic recursion

**Lemma 2.2** (Reference quartic and forward Green function)**.** *Let $0 < \lambda \le 1$, $x_0 > 0$, $U \ge 0$, and $1/4 \le d_0 < 1/2$. For $J \in \mathbb{N} \cup \{\infty\}$, let $\chi_j = 2^{-(j-J)_+}$, with $\chi_j = 1$ for $J = \infty$. Suppose*

$$x_{j+1} = x_j - B_j x_j^2, \qquad 0 \le B_j \le C\lambda^2 \chi_j, \qquad B_j \ge c\lambda^2 \quad (j \le J).$$

*Set $f_j = \chi_j^{3/2} x_j^3 + U d_0^j$ and $A_{j,i} = \prod_{i<l<j}(1 - 2B_l x_l)$ for $i < j$. For sufficiently small $\lambda^2 x_0$, uniformly in $J$,*

$$\frac{x_0}{1 + C'\lambda^2 x_0 j} \le x_j \le x_0, \qquad \sum_{i \ge j} \chi_i x_i^p \le C_p \lambda^{-2} x_j^{p-1} \quad (p = 2, 3). \tag{5}$$

*At finite $J$, $x_j \asymp x_{J+1}$ for $j > J$. The envelope is decreasing and $f_{j+1}/f_j \ge 1/4$. Moreover*

$$A_{j,i} \le (x_j/x_{i+1})^2, \qquad \sum_{i<j} A_{j,i} B_i x_i^2 \le C' x_j,$$
$$\sum_{i<j} A_{j,i} f_i \le C' x_j \left( \frac{x_0}{\lambda^2} + \frac{U}{x_0} \right). \tag{6}$$

*Proof.* Take $\lambda^2 x_0$ small enough that $B_j x_j < 1/4$. The increments of $x_j^{-1}$ lie between $B_j$ and $2B_j$. Summing them gives the first bound in (5). After $J$ their total is $O(\lambda^2)$, so $x_j \asymp x_{J+1}$ on that tail. Before the cutoff, telescope $x_i^{p-1}$, whose decrement is at least a fixed multiple of $\lambda^2 x_i^p$. After the cutoff, use $\sum_{i>J} \chi_i = 1$ and $\lambda^2 x_0 \lesssim 1$. This proves the second bound. The adjacent ratio of $\chi_j^{3/2} x_j^3$ is bounded below by $2^{-3/2}(1 - C\lambda^2 x_0)^3$, which is greater than $1/4$ for small $\lambda^2 x_0$. The transient ratio is $d_0$.

The product bound follows from $1 - 2t \le (1-t)^2$. Since $x_i/x_{i+1} \le 4/3$ and $\sum_{i<j} B_i \le x_j^{-1} - x_0^{-1}$, it also gives $\sum_{i<j} A_{j,i} B_i x_i^2 \le C' x_j$. For the cubic forcing at $i \le J$, the product bound gives $C x_j^2 x_i$ per summand. Here

$$\sum_{\substack{i<j \\ i \le J}} x_i \le C\lambda^{-2} \log(x_0/x_j),$$

by $\log(x_i/x_{i+1}) \ge B_i x_i$. Use $x_j \log(x_0/x_j) \lesssim x_0$. The terms with $i > J$ sum to $O(x_j^2 x_0)$ by the geometric decay of $\chi_i$. For the transient, the lower bound in (5) gives the estimate

$$CU(x_j/x_0)^2 \sum_{i \ge 0} d_0^i (1+i)^2 \le C' x_j U/x_0.$$

Together these estimates prove (6). □

## 2.3 Extension to the larger activity ball

Use a fixed positive reference quartic coupling $G$ in the norms at both real and complex potential inputs. Let $\chi \in [0,1]$ be the mass damping factor for the step. Assume the exact maps $\Psi = K_+$ and $R = V_+ - \Phi^{\mathrm{pt}}(V)$ are holomorphic on the joint balls

$$\|V - V_0\| < cG, \qquad \|K\| < cG^{9/4},$$

and satisfy, throughout these balls,

$$\|\Psi(V,K)\| \le CG^{9/4}, \quad \|\Psi(V,0)\| + \|R(V,0)\| \le C\chi^{3/2}G^3, \quad \|D_K R\| \le C. \tag{7}$$

Assume separately that at real potential inputs $\|D_K\Psi(V,0)\| \le \kappa_0 < 1$. The constants below depend only on the constants in these bounds and on the fixed radius losses.

**Lemma 2.3** (Cauchy extension of a uniform contraction)**.** *Under these analytic inputs, on fixed smaller balls and $\|K\| \le \epsilon G^{9/4}$,*

$$\begin{aligned} \|D_K\Psi(V,K)\| &\le \kappa_0 + C\epsilon && \text{for real } V, \\ \|\Psi(V,K)\| + \|R(V,K)\| &\le C(\chi^{3/2}G^3 + \|K\|), \\ \|D_V\Psi(V,K)\| + \|D_V R(V,K)\| &\le CG^{-1}(\chi^{3/2}G^3 + \|K\|). \end{aligned}$$

*Proof.* Cauchy estimates in activity directions give $\|D_K\Psi\| \le C$ and $\|D_K^2\Psi\| \le CG^{-9/4}$. Integrating the second derivative along the segment from zero to $K$ proves the first assertion. Integrating the first derivatives of $\Psi, R$ gives the second. Cauchy's estimate on a potential disc of radius proportional to $G$ gives the third. The constants depend only on the fixed radius losses. Choose $\epsilon$ before making $G$ small, so that $\kappa_0 + C\epsilon < 1$. An estimate only at $K = 0$ would not supply (7) or this conclusion. □

## 2.4 Exact Gaussian identities for the spin law

Both proofs use a normalized decomposition $D^+ = P+Q$ of the form in Section 1. Put $p = \sum_x P(x)$, $q = \sum_x Q(x)$, $A = \beta P$, and $B = I - \beta Q$. The decompositions below have $A > 0$ and $\beta q < 1$, so the inverses of $A$ and $B$ exist. We use the same symbols for the periodized operators on a finite torus $\Lambda$.

**Proposition 2.4** (Susceptibility and reconstruction)**.** *Let $h$ given $\sigma$ be Gaussian of mean $A\sigma$ and covariance $A$, and let $\psi = \sqrt{\lambda R^s}\, B^{-1}h$ for a positive covariance prefactor $\lambda$. Write $G_\sigma$ and $G_\psi$ for the spin and auxiliary covariances, viewed as translation-invariant kernels. Then*

$$G_\sigma = (\lambda R^s)^{-1}P_* * G_\psi * P_* - A^{-1}, \qquad P_* = BA^{-1}, \qquad \widehat{P}_*(0) = \frac{1-\beta q}{\beta p}. \tag{8}$$

*The identity also holds in infinite volume when the relevant Gaussian and inverse kernels are summable.*

*On a finite torus, suppose the $\psi$ law is expressed with Gaussian covariance $C^\Lambda$ of row sum*

$$C_0^\Lambda = \widehat{C}^\Lambda(0) = \frac{\lambda R^s \beta p}{(1-\beta q)\mu}, \qquad \mu > 0.$$

*Let $Z_{\mathrm{tail}}(v)$ be the remaining partition function after all Gaussian integrations, with deterministic shift $v$ in the field and with any shift-independent factors removed. With $\chi_\sigma^\Lambda = \sum_{x\in\Lambda} G_\sigma(x)$,*

$$\chi_\sigma^\Lambda = \frac{1-\beta q}{\beta p\, \mu} - \frac{1}{\beta p} + \frac{\lambda R^s}{\mu^2|\Lambda|} D^2_{\mathbf{1},\mathbf{1}} \log Z_{\mathrm{tail}}(0). \tag{9}$$

*Here $D^2_{\mathbf{1},\mathbf{1}}$ is the second derivative in the constant-field direction $\mathbf{1}$.*

*Proof.* The conditional covariance formula gives

$$G_\psi = \lambda R^s B^{-1}(A + AG_\sigma A)B^{-1}.$$

This proves (8) by inversion. It also yields

$$\chi^\Lambda_\psi = \lambda R^s \frac{\beta p + (\beta p)^2 \chi^\Lambda_\sigma}{(1-\beta q)^2}$$

on taking row sums. In infinite volume the same covariance calculation is valid by absolute convergence, since spin correlations are bounded.

Completing the square against the source $t\sum_{x\in\Lambda}\psi_x$ gives the log moment generating function

$$\tfrac{1}{2}t^2|\Lambda|C_0^\Lambda + \log Z_{\text{tail}}(tC_0^\Lambda \mathbf{1}) - \log Z_{\text{tail}}(0).$$

Its second derivative divided by $|\Lambda|$ is $\chi^\Lambda_\psi$. Substituting $C_0^\Lambda$ and using the row-sum relation proves (9). □

### 2.5 Identification of the physical critical temperature

A summable bound on periodic correlations allows the finite-volume susceptibility calculation to pass to infinite volume. The renewal kernel below gives this bound for both interactions in the manuscript.

**Proposition 2.5** (A summable-interaction endpoint criterion)**.** *Let the pair interaction be $J(x-y)$, where $J:\mathbb{Z}^d\to[0,\infty)$ is symmetric, $J(0)=0$, $\sum_x J(x)<\infty$, and its support generates $\mathbb{Z}^d$. On the torus $\Lambda_D=(\mathbb{Z}/D\mathbb{Z})^d$, use the periodized interaction*

$$J_D(\bar{x}) = \sum_{n\in\mathbb{Z}^d} J(x+Dn) \quad (\bar{x}\neq 0);$$

*the diagonal term is omitted, since it changes only the partition function's normalization. Write $\chi_D(\beta)=\sum_{\bar{x}\in\Lambda_D}\langle\sigma_0\sigma_{\bar{x}}\rangle_{D,\beta}$. Let $I_0\subset(0,\infty)$ be a compact interval. Suppose that, for every sufficiently small auxiliary mass $\mu>0$, there are tuned parameters $\beta_{\mu,D}\in I_0$ along tori with $D\to\infty$, and that*

$$\chi_D(\beta_{\mu,D}) = \frac{a(\beta_{\mu,D})}{\mu} - b(\beta_{\mu,D}) + O_\mu(D^{-s}), \qquad s>0, \tag{10}$$

*where $a,b$ are continuous on $I_0$ and $\inf_{I_0} a>0$. The error constant is uniform in the chosen tuned parameters at fixed $\mu$.*

*For any subsequential limit $\beta_{\mu,D}\to\beta_\mu$, one has*

$$\beta_\mu < \beta_c(J), \qquad \chi^{\mathrm{f}}(\beta_\mu) = \frac{a(\beta_\mu)}{\mu} - b(\beta_\mu),$$

*and the periodic two-point functions converge locally and in the summed sense to the free infinite-volume two-point function at $\beta_\mu$. Every choice of these subsequential limits satisfies*

$$\beta_\mu \longrightarrow \beta_c(J) \qquad (\mu\downarrow 0).$$

*For each fixed pair of sites their free correlations therefore converge to the critical free correlations. No monotonicity of $\beta_\mu$, or uniqueness of the tuning, is assumed.*

*Proof.* We use the Griffiths inequalities, the Simon–Lieb inequality, and the standard sharpness identification of the susceptibility threshold with the spontaneous-magnetization critical point for summable ferromagnetic Ising interactions; see [9]. In particular susceptibility is finite and the Gibbs state is unique at every fixed strictly subcritical inverse temperature.

Fix $\mu > 0$ and a subsequence with $\beta_{\mu,D} \to \beta_\mu$. Equation (10) bounds the periodic susceptibilities uniformly along this subsequence. If $\beta < \beta_\mu$, every fixed finite free box can eventually be embedded in $\Lambda_D$ with $\beta \le \beta_{\mu,D}$. Its pair interactions are no larger than the corresponding periodized interactions. The Griffiths inequalities therefore bound its partial susceptibility by the periodic susceptibility. Increasing the free boxes gives $\chi^{\mathrm{f}}(\beta) \le C_\mu$. Moreover,

$$\chi^{\mathrm{f}}(\beta_\mu) = \sup_{\beta<\beta_\mu} \chi^{\mathrm{f}}(\beta). \tag{11}$$

Indeed free susceptibility is the supremum of the finite free-box partial susceptibilities, each of which is continuous and increasing in $\beta$; the two suprema commute. This proves $\chi^{\mathrm{f}}(\beta_\mu) < \infty$ before any uniqueness or summability of a limiting periodic state is used.

For a finite box $S$ containing the origin put

$$\varphi_\beta(S) = \sum_{u\in S,\, v\notin S} \langle\sigma_0\sigma_u\rangle^{\mathrm{f}}_{S,\beta} \tanh(\beta J(v-u)).$$

There is a box $S$ with $\varphi_{\beta_\mu}(S) < 1$. To see this, bound the free-box correlation by its infinite-volume value at $\beta_\mu$, and use $\tanh(\beta J) \le \beta J$. For each fixed $u$, the sum over $v \notin S$ tends to zero as the boxes increase. It is bounded by $\beta_\mu \|J\|_1$, so dominated convergence with the summable function $u \mapsto \langle\sigma_0\sigma_u\rangle^{\mathrm{f}}_{\beta_\mu}$ proves $\varphi_{\beta_\mu}(S) \to 0$. For fixed $S$, $\varphi_\beta(S)$ is continuous in $\beta$: the correlations are finite-volume continuous, and the remaining series is dominated on compact intervals by a constant times $J$. Choose $\beta_+ > \beta_\mu$ and $a_1 < 1$ with $\varphi_{\beta_+}(S) < a_1$. The finite-set criterion for sharpness gives $\beta_+ \le \beta_c(J)$, and hence $\beta_\mu < \beta_c(J)$.

We next prove the uniform summability needed for the moving-temperature limit. Choose $\varepsilon > 0$ so small that the nonnegative exit kernel

$$T_*(v) = \mathbf{1}_{\{v\notin S\}} \sum_{u\in S} \big(\langle\sigma_0\sigma_u\rangle^{\mathrm{f}}_{S,\beta_+} + \varepsilon\big) \tanh(\beta_+ J(v-u)) \quad \text{satisfies} \quad \|T_*\|_1 = a_* < 1. \tag{12}$$

Such a choice is possible because the added mass is at most $\varepsilon|S|\beta_+\|J\|_1$. Embed $S$ in all sufficiently large tori. The additional image bonds between any two of its sites tend to zero, by summability of $J$. Finite-volume continuity and monotonicity imply, uniformly for $0 \le \beta \le \beta_+$,

$$\langle\sigma_0\sigma_u\rangle^{\mathrm{f}}_{S,J_D,\beta} \le \langle\sigma_0\sigma_u\rangle^{\mathrm{f}}_{S,J,\beta_+} + \varepsilon \qquad (u \in S)$$

when $D$ is sufficiently large. Here the subscript $S, J_D$ means that only the periodized bonds internal to $S$ are kept. Define the torus exit kernel

$$T_{D,\beta}(\bar v) = \mathbf{1}_{\{\bar v\notin S\}} \sum_{u\in S} \langle\sigma_0\sigma_u\rangle^{\mathrm{f}}_{S,J_D,\beta} \tanh(\beta J_D(\bar v - u)).$$

For nonnegative numbers, $\tanh(\sum_i t_i) \le \sum_i \tanh t_i$. Using this inequality and (12) gives

$$T_{D,\beta} \le \mathcal{P}_D T_*, \qquad (\mathcal{P}_D f)(\bar x) = \sum_{n\in\mathbb{Z}^d} f(x + Dn).$$

If $\bar v \notin S$, every representative $v + Dn$ is outside the embedded box $S$, so the indicator in (12) causes no loss in this comparison. For $\bar v \in S$ its left side is zero. In particular $\|T_{D,\beta}\|_1 \le a_* < 1$.

Let $G_{D,\beta}(\bar{x}) = \langle \sigma_0 \sigma_{\bar{x}} \rangle_{D,\beta}$. The Simon–Lieb inequality outside $S$, and the bound $G_{D,\beta} \le 1$ inside $S$, give on the whole torus

$$G_{D,\beta} \le \mathbf{1}_S + T_{D,\beta} * G_{D,\beta}.$$

Iterating this inequality, the remainder after $n$ iterations has supremum at most $a_*^n$. Nonnegative periodization commutes with convolution. Consequently

$$G_{D,\beta} \le \mathcal{P}_D H, \qquad H = \left( \sum_{n \ge 0} T_*^{*n} \right) * \mathbf{1}_S, \qquad \|H\|_1 \le \frac{|S|}{1 - a_*} < \infty, \tag{13}$$

for all sufficiently large $D$ and $\beta \le \beta_+$. For finite-range $J$, the exit kernel $T_*$ has finite support. Choose $\kappa > 0$ small enough that $\sum_v e^{\kappa |v|} T_*(v) < 1$. The renewal series then gives $\sum_z e^{\kappa |z|} H(z) < \infty$. By periodization,

$$G_{D,\beta}(\bar{x}) \le C e^{-\kappa \, \mathrm{dist}_D(\bar{0}, \bar{x})} \qquad (\beta \le \beta_+), \tag{14}$$

where $\mathrm{dist}_D$ is the Euclidean distance between nearest representatives. Indeed, each image of $\bar{x}$ has norm at least this distance from the origin.

Take one shortest representative of each torus point. For every $K > 0$, each image of a representative with $|x| > K$ also has norm greater than $K$. Hence

$$\sum_{\substack{x \text{ shortest representative} \\ |x| > K}} G_{D,\beta}(\bar{x}) \le \sum_{|z| > K} H(z) \longrightarrow 0 \qquad (K \to \infty),$$

uniformly in these volumes and temperatures.

Since $\beta_\mu < \beta_c(J)$, the infinite-volume Gibbs state there is unique. Every local subsequential limit of the periodic laws at $\beta_{\mu,D}$ satisfies the DLR equations at $\beta_\mu$. For a summable interaction this follows by first truncating the interactions entering a fixed conditional law to a finite neighborhood; the discarded contribution is bounded uniformly by a constant times the tail of $J$. The finite-neighborhood equations pass to the local limit, and then the truncation is removed. Uniqueness therefore gives local convergence to the free state. The uniform tail bound in (13) permits summation, proving

$$\chi^{\mathrm{f}}(\beta_\mu) = \lim_{D \to \infty} \chi_D(\beta_{\mu,D}) = \frac{a(\beta_\mu)}{\mu} - b(\beta_\mu).$$

The compact parameter interval keeps $a$ bounded away from zero and $b$ bounded, so these susceptibilities diverge as $\mu \downarrow 0$. Each $\beta_\mu$ is strictly below $\beta_c(J)$. If an accumulation point were smaller than $\beta_c(J)$, choose a fixed strictly subcritical temperature between them. Monotonicity and subcritical susceptibility finiteness would give a common finite upper bound along that subsequence, a contradiction. Thus every accumulation point is $\beta_c(J)$. Finally, the critical free pair correlation is the supremum of its finite-free-volume correlations. As for (11), continuity of those finite correlations and commutation of the two increasing suprema give left continuity at $\beta_c(J)$. Monotonicity then gives convergence along any of the tuned temperatures just considered, including choices that are not monotone. □

### 2.6 Asymptotic amplitude from a two-source sum

Let $b_j = bL^j$ with $L > 1$, and suppose a massless Gaussian covariance has positive finite-range slices $\Gamma_{j+1}$, with range at most $Cb_j$,

$$\|\Gamma_{j+1}\|_\infty \le C\lambda b_j^{-s}, \qquad C(x) = \sum_{j \ge 0} \Gamma_{j+1}(x) \sim c_0 |x|^{-s}, \quad c_0 > 0.$$

Suppose its interacting two-source sum satisfies

$$q(a,c)=\sum_{j\ge 0} v_{a,j}v_{c,j}\Gamma_{j+1}(a-c)+E(a,c),\qquad |E(a,c)|\le b_k^{-s}\eta_k,\quad \eta_k\longrightarrow 0, \tag{15}$$

where $b_k\asymp|a-c|$. Assume $v_{a,j}\to v_{a,\infty}$ uniformly in the marked site, with $0<c\le v_{a,j}\le C$.

**Proposition 2.6** (Observable conclusion under the source estimates)**.** *If $q$ is translation invariant, then there is $v_\infty>0$ independent of the marked site such that*

$$q(a,c)\sim v_\infty^2 c_0|a-c|^{-s}.$$

*Proof.* The range condition makes the covariance sum start at $k-O(1)$. The sum of the absolute slice bounds from that scale onward is at most $C\lambda b_k^{-s}$. Uniform convergence of $v_{a,j}$ therefore allows their replacement by their limiting values at cost $o(b_k^{-s})$. The source error has the same order. Thus

$$q(a,c)=v_{a,\infty}v_{c,\infty}C(a-c)+o(|a-c|^{-s}),$$

uniformly in the two sites. Translation invariance, comparing $(a,a+x)$ with $(0,x)$, gives $v_{a+x,\infty}/v_{x,\infty}\to r(a)=v_{0,\infty}/v_{a,\infty}$. For every fixed positive integer $p$, multiplying this ratio at the shifts $x,x+a,\dots,x+(p-1)a$ gives $r(pa)=r(a)^p$. All these ratios are bounded above and away from zero independently of $p$; hence $r(a)=1$. This proves that the limiting coefficient is independent of the site and proves the assertion. □

## 2.7 Convolution with a summable kernel

**Lemma 2.7** (Preservation of the two-point asymptotic)**.** *If $|H(y)|\le C(1+|y|)^{-d-s}$ and a bounded function satisfies $F(x)\sim A|x|^{-s}$, $0<s<d$, then*

$$(H*F)(x)\sim A\Big(\sum_y H(y)\Big)|x|^{-s} \tag{16}$$

*whenever the displayed coefficient is nonzero, and in all cases the difference from that leading expression is $o(|x|^{-s})$.*

*Proof.* For fixed $M$, the sum over $|y|\le M$ has leading coefficient $A\sum_{|y|\le M}H(y)$. The sum over $M<|y|\le|x|/2$ is bounded by $C|x|^{-s}\sum_{|y|>M}|H(y)|$. On $|y|>|x|/2$, $|x-y|\le|x|/2$, use $|H(y)|\le C|x|^{-d-s}$ and $\sum_{|z|\le|x|/2}|F(z)|\le C|x|^{d-s}$, giving $O(|x|^{-2s})$. The remaining region is bounded by the same order using the tail of $H$ and $|F(x-y)|\le C|x|^{-s}$. Take $|x|\to\infty$, then $M\to\infty$. The argument also preserves the global $O((1+|x|)^{-s})$ bound, so it can be used twice. □

## 2.8 Fourier coefficients of a symbol remainder

Suppose a multiplier is smooth off the origin and, near zero, differs from a smooth function by a symbol $m$ satisfying

$$|\partial^\gamma m(k)|\le C_\gamma|k|^{\rho-|\gamma|},\qquad d+\rho>0.$$

A smooth partition into annuli of radius $2^{-j}$ and integration by parts bound the Fourier coefficient of the $j$-th term by $C_M 2^{-j(d+\rho)}(1+2^{-j}|x|)^{-M}$. For $M>d+\rho$ the sum is $O((1+|x|)^{-d-\rho})$. The smooth part has arbitrarily rapid decay.

# Part I
# The four-dimensional specialization

In this part we prove Theorem 1 for $d = 4$. Use the model of Section 1 with $D_R = D_{R,4}$, $\beta^{\mathrm{box}} = \tau$, and $\beta_c^{\mathrm{box}} = \tau_c(4, R)$. The spread radius $R$ is an integer and is fixed in the large-distance limit. We use $g = R^{-4}$ for the physical small parameter, $\varrho \in (0, 1)$ for the polymer-size weight, and $\mathcal{D} = -\Delta_{\mathrm{nn}}$ for the positive nearest-neighbor Laplacian, with symbol $2\sum_{i=1}^{4}(1 - \cos k_i)$.

The proof first splits the interaction, compares the exact reference pressure with its mean-field pressure, and absorbs the nonlocal Hessian. The connected expansion then constructs the initial activity around a local quartic reference. Next we verify the interior RG map estimates, tune the mass and stiffness on finite tori, and take the positive-mass and critical limits. The RG algebra and norm estimates use the finite-range framework of Bauerschmidt, Brydges and Slade and the regulator of [4, 5]. The supplementary estimates after the proof quantify what the untransformed connected expansion and local subtractions yield.

## 3 Exact preprocessing for the standard box

For a function of the fields on a finite set, use the absolute Taylor majorant

$$|f|_{a,\phi} = \sum_{\alpha} \frac{a^{|\alpha|}}{\alpha!} |\partial^{\alpha} f(\phi)|.$$

It is submultiplicative and bounds the field part of Park's $T_a(\phi)$ norm, since the test tensors are bounded by $a^k$ at degree $k$. We use this stronger majorant in the local estimates. At large-field radius $a$, let

$$H_B(\phi) = \exp\{-\kappa \|\phi_B\|_2^2/(|B|a^2)\}.$$

At fluctuation radius $\ell$, the other regulators have the following properties. The growing regulator $G_B$ dominates $\exp(c\|\phi_B\|_\infty^2/\ell^2)$, and $\widetilde{G}_B$ dominates

$$\exp\left(c\ell^{-2} \inf_{p \text{ affine}} \|\phi_B - p\|_\infty^2\right)$$

for fixed $c > 0$. These follow from the block test-function norms in [4, §2.4]. Regulators on a bounded union of blocks are products over the blocks.

Work first on a large periodic torus $\Lambda$, and use periodized convolution kernels there. The zero-field weight for $D_R$ is $\exp(\beta^{\mathrm{box}}\langle\sigma, D_R\sigma\rangle/2)$. Let $f_R$ be a nonnegative, even, hypercubic-symmetric smooth bump sampled on $\mathbb{Z}^4$, supported on $|x|_\infty \le R/3$, and normalized to sum to one. Take

$$P_0 = \theta(f_R * f_R) + e^{-R}\delta_0$$

with a sufficiently small fixed $\theta > 0$. Then $P_0(x) \le D_R(x)$ for $x \neq 0$. This follows from $\|f_R * f_R\|_\infty \le \|f_R\|_\infty \lesssim R^{-4}$. Replace $D_R$ by

$$D = \frac{D_R + P_0(0)\delta_0}{1 + P_0(0)}, \qquad P = \frac{P_0}{1 + P_0(0)}, \qquad Q = D - P.$$

This adds a spin-independent diagonal energy and reparametrizes inverse temperature. Write $\delta_R = P_0(0)$. The two temperature parameters satisfy $\beta = (1 + \delta_R)\beta^{\mathrm{box}}$, and consequently

$$\beta_c^D = (1 + \delta_R)\beta_c^{\mathrm{box}}.$$

Below $\beta$ always denotes the inverse temperature for $D$; the endpoint identified by the tuning argument is $\beta_c^D$. Write $p = \sum P$, $q = \sum Q = 1 - p$. We have $p \asymp 1$, $Q(0) = 0$, $Q \geq 0$, $\|Q\|_\infty \lesssim R^{-4}$, and $\widehat{P}(k) > 0$. For $\beta$ in a fixed sufficiently small neighborhood of one, $J = \beta Q$ obeys $\|J\|_1 \leq q_* < 1$.

Let $A = \beta P$, $B = I - J$, and let

$$F(h) = \log \mathbb{E}_{J,0} e^{\langle h,\sigma\rangle} = \log \frac{Z_J(h)}{Z_J(0)}, \qquad Z_J(h) = \sum_{\sigma \in \{\pm 1\}^\Lambda} e^{\langle \sigma, J\sigma\rangle/2 + \langle h,\sigma\rangle}.$$

Here $\mathbb{E}_{J,0}$ is expectation for the zero-field reference Ising law, with interaction $J = \beta Q$ already including the inverse temperature. The Hubbard–Stratonovich transformation applied to $A$ alone gives the exact density for $h$

$$\frac{1}{Z} \exp\{-\tfrac{1}{2}\langle h, A^{-1}h\rangle + F(h)\}. \tag{17}$$

Thus the reference spins have been integrated out. For a parameter $m^2 > 0$ define

$$S_0 = B^{-1}, \qquad c = \frac{1 - m^2 - \beta q}{\beta p}, \qquad M_m = I - J - cA.$$

For $m^2$ small and $\beta$ close to one, $c > 0$ and is bounded above and below uniformly. The Gaussian covariance and residual interaction in (17) can be taken as

$$C_m = (A^{-1} - cS_0)^{-1} = ABM_m^{-1}, \qquad W(h) = \tfrac{c}{2}\langle h, S_0 h\rangle - F(h). \tag{18}$$

All identities are exact. In particular the mass parameter here specifies a Gaussian decomposition, rather than an approximation to the spin measure.

**Proposition 3.1** (Finite-range slices)**.** *On $\mathbb{Z}^4$, the covariance $C_m$ admits positive finite-range pieces at lengths $O(RL^j)$, $j \geq 0$. For any fixed orders $k, N \geq 0$ their kernels obey*

$$\|\nabla^k C_{m,j}\|_\infty \leq C_{k,N,L} R^{-4-k} L^{-(2+k)j} (1 + m^2 L^{2j})^{-N}. \tag{19}$$

*The bound includes the first piece, with $j = 0$. In particular the massless auxiliary fluctuation has site variance $O(R^{-4})$.*

*Proof.* The multiplier of $M_m$ is comparable to $m^2 + \min(R^2|k|^2, 1)$, uniformly in the indicated parameter range. Indeed, $J + cA$ has mass $1 - m^2$ and its off-diagonal kernel is bounded above and below by constant multiples of the box kernel. For the box kernel the lower bound follows from the product Dirichlet sum for the cube. For $|k| \leq c/R$ it also follows by Taylor expansion and the box second moment. In the annulus $c/R \leq |k| \leq C/R$ the rescaled Dirichlet sums converge uniformly to the product of sinc functions, which stays strictly below one there. For $|k| \geq C/R$, the coordinate of largest absolute value bounds the corresponding normalized Dirichlet sum by $O(1/C)$. The numerator $AB$ is positive semidefinite and has range $O(R)$. Repeated summation by parts for the sampled smooth bump yields

$$|\widehat{AB}(k)| \leq C_s(1 + R|k|)^{-s}$$

for every fixed $s$, including the diagonal term $e^{-R}$.

Apply Lemma 2.1 to $M_m$, with a fixed spectral rescaling, and multiply each positive piece by the commuting positive kernel $AB$. The degree bound gives range $O(RL^j)$. For a difference of order $k$, Fourier inversion inserts a factor bounded by $C|p|^k$ at frequency $p$. On $|p| \leq R^{-1}$, use

the polynomial bound and rescale by $RL^j$. For every fixed $N$, taking the polynomial decay order larger than $N + (4 + k)/2$ yields

$$\int_{|p|\le R^{-1}} |p|^k \widehat{AB}(p) q_j(\widehat{M}_m(p))\, dp \le CR^{-4-k} L^{-(2+k)j} (1 + m^2 L^{2j})^{-N}.$$

On $|p| > R^{-1}$, the spectral multiplier of $M_m$ is bounded below uniformly. The rapid Fourier decay of $AB$ and a sufficiently large polynomial decay order give the same bound. □

# 4 Locality and stability of the exact reference pressure

For a real field $h$, let $m_i^{\mathrm{ex}}(h)$ be the magnetization of the reference system. Let $z$ be the unique real solution to

$$z_i = h_i + \sum_k J_{ik} \tanh z_k, \qquad m_i^{\mathrm{mf}} = \tanh z_i. \tag{20}$$

The row sum bound on $J$ gives uniqueness by contraction.

**Proposition 4.1** (Uniform reference comparison)**.** *There are constants $r, \eta > 0$, independent of $R$, of the torus volume, and of the real background $h$, with the following property. Expand $m_i^{\mathrm{ex}}(h + x) - m_i^{\mathrm{mf}}(h + x)$ in $x$ with coefficients $b_{i,\alpha}$. If $\tau(i, S)$ denotes the shortest tree length for the periodic $\ell^\infty$ metric joining $i$ and $S$, allowing intermediate vertices, then*

$$\sum_\alpha |b_{i,\alpha}| r^{|\alpha|} \exp\{\eta \tau(i, \operatorname{supp} \alpha)/R\} \le Cg. \tag{21}$$

*The two magnetizations separately have uniformly bounded norms of this type. In particular, the pressure defect has uniformly small derivative rows, with an exponential tree weight, at every real background.*

*Proof.* For series at anchor $i$, use the norm on the left of (21). The union of two trees rooted at $i$ gives submultiplicativity. Changing the anchor from $k$ to $i$ costs at most $e^{\eta|i-k|/R}$, since one may add an edge between the two anchors. The norm of $x_i$ is $r$.

We give the recursion used for the exact magnetization. On a finite graph with real fields, the half log ratio of the weights conditioned on $\sigma_i = 1$ and $\sigma_i = -1$ is $h_i$ plus half the log of the ratio of the two partition functions on the graph with $i$ deleted. In this ratio the fields at the neighbors are shifted by $+J_{ik}$ and $-J_{ik}$. Order the neighbors and telescope by changing one shift at a time. For a change at $k$, the ratio on the smaller graph is $\cosh(H_k + J_{ik})/\cosh(H_k - J_{ik})$, where $H_k$ is the cavity half log ratio with the other shifts already chosen. Thus, at a node of the recursion with $d$ ancestors removed,

$$H_i = h_i + x_i + \xi_i + \sum_{k \text{ available}} f_{J_{ik}}(H_k^{\mathrm{child}}), \qquad f_b(z) = \tfrac{1}{2} \log \frac{\cosh(z + b)}{\cosh(z - b)}. \tag{22}$$

Here $|\xi_i| \le Cdg$. The deleted vertices remove row mass at most $C(d + 1)g$. The recursion ends after finitely many deletions. At the root its magnetization is $\tanh H_i$.

For real $z$, $|f_b'(z)| \le \tanh b \le b$. On a fixed sufficiently thin strip, the higher Taylor coefficients around any real point are bounded by $Cb$ times a geometric sequence. This follows from the strip formula for $f_b'$, and Cauchy's estimate on a smaller strip. The same argument applied to $f_b(z) - b \tanh z$ gives a bound $Cb^3$ on its Taylor norm at a small fixed radius. These statements

also hold after substituting a series whose nonconstant part is sufficiently small in an anchored norm, by submultiplicativity.

Choose $\eta > 0$ small enough that $e^{\eta} q_* < 1$. Then choose $r > 0$ small. For a sufficiently large fixed $K$ and all series with nonconstant norm at most $Kr$, the Lipschitz bound for the composition with $f_b$ is $b(1 + CKr)$ in the anchored norm. To see this, integrate $f_b'$ along the segment between the two series. The constant part along that segment is real, and the nonconstant part has norm at most $Kr$. The same statement holds for the composition with $b \tanh$. Choose $r$ so that $q' = e^{\eta} q_*(1 + CKr) < 1$ and $r + q' Kr \le Kr$. Backward induction in (22) proves $\|H_i - H_i(0)\|_i \le Kr$. Iterating (20) about its real solution in the same Banach algebra gives the corresponding analytic series for the mean-field solution.

Compare a node of (22) with the full mean-field solution at its anchor. The shifts and omitted terms cost $C(d+1)g$. The cubic approximation to $f_b$ costs at most $C \sum_k J_{ik}^3 \le Cg^2$. Differences at child nodes enter with weights whose sum is at most $q'$, including the anchor change. Expanding this inequality down the finite recursion yields at the root a bound

$$Cg \sum_{d \ge 0} (d+1)(q')^d \le C' g.$$

The composition with tanh has uniformly bounded Lipschitz constant on these series. This proves (21). □

For clarity, let $I_{\mathrm{bin}}(t)$ be the binary relative entropy with respect to the fair spin, continuously defined for $|t| \le 1$. The mean-field pressure increment is

$$F_{\mathrm{mf}}(h) = \sup_{m \in [-1,1]^\Lambda} \left\{ \langle h, m \rangle + \tfrac{1}{2} \langle m, Jm \rangle - \sum_x I_{\mathrm{bin}}(m_x) \right\}.$$

Its Hessian as a function of the maximizing variable in the open cube is negative definite, since $I_{\mathrm{bin}}'' \ge 1$. The maximizer is the magnetization in (20), and $\nabla F_{\mathrm{mf}} = m^{\mathrm{mf}}$. The exact pressure satisfies $\nabla F = m^{\mathrm{ex}}$. Both pressures vanish at zero and are even. Applying Proposition 4.1 at zero to their gradients and integrating along the field proves that $F - F_{\mathrm{mf}}$ has an exact anchored Taylor expansion on $\|h\|_\infty < r$, with degree-$k$ kernels bounded by $Cgr^{-k}$ in the weighted row norm (after reducing $r$ by a fixed factor if necessary). In particular, if $S = D^2 F(0)$,

$$\sup_i \sum_k |(S - S_0)_{ik}| e^{\eta |i-k|/R} \le Cg. \tag{23}$$

**Corollary 4.2** (Coercivity after Hessian subtraction)**.** *For all real $h$, uniformly in the volume and in $R$ large enough,*

$$\tfrac{1}{2} \langle h, Sh \rangle - F(h) \ge c_1 \sum_i \frac{h_i^4}{1 + h_i^2}, \tag{24}$$

$$W(h) \ge c_1 \sum_i \frac{h_i^4}{1 + h_i^2} - C(g + |\beta - 1| + m^2) \|h\|_2^2. \tag{25}$$

*Proof.* Put $u = S_0 h$ and $V_{\mathrm{ent}}(t) = I_{\mathrm{bin}}(t) - t^2/2$. Completing the square in the variational formula gives

$$\tfrac{1}{2} \langle h, S_0 h \rangle - F_{\mathrm{mf}}(h) = \inf_{m \in [-1,1]^\Lambda} \left\{ \tfrac{1}{2} \langle m - u, (I - J)(m - u) \rangle + \sum_i V_{\mathrm{ent}}(m_i) \right\}.$$

Because $I - J \geq (1 - q_*)I$ and $V_{\mathrm{ent}}(t) \geq t^4/12$, the right-hand side is bounded below by $c \sum_i \min(u_i^4, u_i^2)$. This follows pointwise by considering whether $|m_i| \geq |u_i|/2$. For $\Psi(t) = t^4/(1+t^2)$, convexity and $\Psi(bt) \leq b^4 \Psi(t)$ for $b \geq 1$ imply $\sum_i \Psi(((I-J)u)_i) \leq (1+q_*)^4 \sum_i \Psi(u_i)$, by Jensen's inequality and the absolute row and column bounds. Thus the mean-field deficit is at least $c' \sum_i \Psi(h_i)$.

Let $E_4 = F - F_{\mathrm{mf}} - \frac{1}{2}\langle h, (S - S_0)h\rangle$. It vanishes through degree three at zero. The uniform row bounds from Proposition 4.1 give

$$\sup_h \sup_i \sum_k |\partial_i \partial_k E_4(h)| \leq Cg, \qquad \sup_h \sup_i \sum_{k,l,m} |\partial_i \partial_k \partial_l \partial_m E_4(h)| \leq Cg.$$

Split $h = a + b$ with $a_i = h_i$ on $|h_i| \leq 1$ and zero otherwise. Fourth-order Taylor expansion, symmetry of the derivative tensor, and $|a_i a_k a_l a_m| \leq (|a_i|^4 + |a_k|^4 + |a_l|^4 + |a_m|^4)/4$ give $|E_4(a)| \leq Cg \sum_i |a_i|^4$. The Hessian bound yields $\|\nabla E_4(a)\|_\infty \leq Cg$ and

$$|E_4(a+b) - E_4(a)| \leq Cg(\|b\|_1 + \|b\|_2^2) \leq C' g \|b\|_2^2.$$

Consequently $|E_4(h)| \leq Cg \sum_i \Psi(h_i)$, which can be absorbed for large $R$. Equation (23) and $c - 1 = (1 - m^2 - \beta)/(\beta p)$ give the last claim. □

**Corollary 4.3** (Taylor stability of restricted reference weights)**.** *Let $X \subset \Lambda$, and let $W_X$ be $W$ evaluated at fields supported in $X$, with derivatives taken only in those fields. Set $a = R^{-1}$. If $|\beta - 1| + m^2 \lesssim R^{-4}$, then*

$$|e^{-W_X}|_{a,h} \leq \exp\{C|X|a^4 - c \textstyle\sum_{i \in X} \Psi(h_i)\}.$$

*In particular for an $R$-scale block this gives its Gaussian regulator bound, uniformly in the external volume.*

*Proof.* Write $U_*(h) = \frac{1}{2}\langle h, Sh\rangle - F(h)$. The first three derivatives of $U_*$ at zero vanish. The uniform Taylor bounds on the magnetization in Proposition 4.1 give bounded absolute derivative rows at every real field. Split the field into $h^<$, supported on $|h_i| \leq 1$, and $h^>$, supported on $|h_i| > 1$. For $k = 1, 2, 3$, integrate the Taylor formula from zero to $h^<$ using the fourth derivative rows. By symmetry of the derivative tensor and the arithmetic–geometric mean inequality, the sum of the absolute entries of the $k$-th derivative at $h^<$, over indices in $X$, is at most $C \sum_{i \in X} |h_i^<|^{4-k}$. Moving from $h^<$ to $h^< + h^>$ adds at most $C \sum_{i \in X} |h_i^>|$, by the uniform $(k+1)$-st derivative rows and symmetry. The contribution of orders at least four to the absolute Taylor majorant is $O(|X|a^4)$, by the fixed analytic radius of the magnetization. Consequently the nonconstant Taylor terms of $U_*$, at radius $a$, are bounded by $\varepsilon \sum_X \Psi(h_i) + C_\varepsilon |X| a^4$. Here the small-field terms use Young's inequality, while on the large-field sites $|h_i| \leq |h_i|^2$. The quadratic difference between $W$ and $U_*$ has absolute row sum $O(a^4)$ by (23) and the assumed parameter range. Its value and nonconstant Taylor terms obey the same bound. Combine the nonconstant estimates with the constant term of Corollary 4.2 and exponentiate. □

The leading interactions can be computed explicitly. With $u = S_0 h$, the expansion of the mean-field deficit through degree six is

$$\tfrac{1}{2}\langle h, S_0 h\rangle - F_{\mathrm{mf}}(h) = \tfrac{1}{12} \sum_i u_i^4 + \tfrac{1}{30} \sum_i u_i^6 - \tfrac{1}{18}\langle u^3, S_0 u^3\rangle + O(\text{degree } 8). \tag{26}$$

Here $u^3$ is pointwise. Indeed, the stationarity equation in the preceding infimum gives $m = u - S_0 u^3/3 + O(\text{degree } 5)$, and $V_{\mathrm{ent}}(m) = m^4/12 + m^6/30 + O(m^8)$. The second-order correction

from the quadratic term is $\langle u^3, S_0 u^3\rangle/18$, and the correction from the quartic term is $-\langle u^3, S_0 u^3\rangle/9$. This proves the formula. The actual degree-four Taylor kernel differs from its mean-field value by $O(R^{-4})$ in the weighted row norm. Its constant-field coefficient remains positive for large $R$, and has dimensionless size of order $R^{-4}$ on a block of side $R$ with fluctuation scale $R^{-2}$. The degree-six term for this box preparation already includes a bilocal polynomial. Its bilocal contribution is visible before any localization. The pressure comparison gives the following block size estimates in the fluctuation norm before any localization of the Taylor kernels. For degree $k$, the exact-minus-mean-field term has size at most $C_k R^4 R^{-4}(R^{-2})^k$ per anchored block. For $k = 2, 4, 6$ these are $O(g), O(g^2), O(g^3)$, respectively, with $g = R^{-4}$. The quadratic kernel and the quartic correction thus require treatment before the $O(g^3)$ remainder estimate can be used.

## 5 An exact connected expansion of the reference weight

Partition the torus into blocks of side $b$, where $R < b \le MR$ for a fixed $M$. Join two distinct blocks when some microscopic reference edge joins them. This graph has maximum degree $\Delta$ bounded independently of the volume and range. For a set $X$ of blocks let $\Lambda_X$ be their union. In the following construction we restrict the spin system to $\Lambda_X$ as well as restricting the field. The reference pressure in a proper subvolume is therefore computed with free boundary, even when the ambient volume is a torus.

For a real symmetric array $t = (t_{BC}) \in [0,1]^{\binom{X}{2}}$, let $J_X(t)$ be the restriction of $J$ to $\Lambda_X$, with each cross-block edge between $B$ and $C$ multiplied by $t_{BC}$. Keep intra-block edges unchanged. Set

$$W_X(t,h) = \frac{c}{2}\langle h, (I - J_X(t))^{-1} h\rangle - \log \frac{Z_{J_X(t)}(h)}{Z_{J_X(t)}(0)}.$$

Use the same scalar $c$ as in (18) for every subvolume and every interpolation point. No auxiliary Gaussian integration is performed in this construction.

For block functions $I$ and activities $K$, write

$$(I \circ K)(X) = \sum_{Y \subseteq X} \Big( \prod_{B \in X \setminus Y} I(B) \Big) K(Y), \qquad K(\varnothing) = 1.$$

Throughout this part, connectivity refers to the graph of blocks specified in the relevant expansion or RG scale. On disconnected polymers the activity is the product of the component activities.

**Proposition 5.1** (All-field forest expansion)**.** *Assume $\|J\|_1 \le q_* < 1$, $\|J\|_\infty \lesssim R^{-4}$, $J \ge 0$, and $|c - 1| \lesssim R^{-4}$. Put $a = R^{-1}$. There exist connected activities $A_X$, depending only on the fields on $\Lambda_X$, for which*

$$e^{-W_X(\mathbf{1},h)} = \sum_{\pi \in \mathcal{P}(X)} \prod_{Y \in \pi} A_Y(h). \tag{27}$$

*Here $\mathcal{P}(X)$ denotes the set of partitions of $X$, and $A_Y = 0$ if $Y$ is disconnected in the block graph. For constants independent of the volume and range, the Taylor seminorm satisfies*

$$|A_X|_{a,h} \le C_0^{|X|} \exp\left\{ C|\Lambda_X| a^4 - c_0 \sum_{i \in \Lambda_X} \Psi(h_i) \right\}. \tag{28}$$

*The expansion also holds after multiplying the left side by $\exp(\sum_{z \in \{u,v\} \cap \Lambda_X} s_z h_z)$ in the nilpotent source algebra. The activity on $Y$ then acquires exactly the source factor with sites in $\Lambda_Y$. In a seminorm weighting $s_z$ by $\tau_z$, the right side of (28) is multiplied by at most $\prod_{z \in \{u,v\} \cap \Lambda_X} (1 + \tau_z(|h_z| + a))$.*

*Proof.* We first extend the stability bound to a complex neighborhood of all real interpolation points. In this paragraph the notation $t+v$ means a perturbation in the variables indexing pairs of blocks, not a change in the field. There is a fixed $r>0$ such that the proof of Proposition 4.1 works in the joint Taylor algebra of field increments and block-link increments at radius $r$ in each family. For the unweighted bounds needed here use the coefficient norm $\sum_{\alpha,\gamma}|a_{\alpha,\gamma}|r^{|\alpha|+|\gamma|}$. To check this extension, at a microscopic edge replace the real parameter $d=J_{ij}t_{B_iB_j}$ in $f_d$ by $d+J_{ij}v_{B_iB_j}$. At an intra-block edge there is no increment. The increments in the shifts from deleted neighbors have total norm at most $r\sum_j J_{ij}$. They therefore cost $O(r)$ uniformly even deep in the recursion. The joint Taylor expansion of $f_d$, using its strip representation, bounds its derivative in the message coordinate by $J_{ij}(1+O(r))$ after substitution of any series with nonconstant norm $O(r)$. Indeed the constant coefficient has absolute value at most $\tanh d\le J_{ij}$, while the remaining coefficients cost $O(rJ_{ij})$ by Cauchy's inequality on fixed strips for the message and the rescaled coupling variable. Also $f_d(z)-d\tanh z$ costs $O(J_{ij}^3)$ in the joint norm, since it vanishes to order three in $d$ and is analytic uniformly in real $z$. These bounds give the same backward induction with row contraction $q_*(1+O(r))<1$. For the mean-field messages the argument is the same, with $f_d(z)$ replaced by $d\tanh z$. Comparing a cavity node of depth $k$ to the full mean-field solution costs $O((k+1)R^{-4})$ from missing edges and shifts, uniformly in the joint norm. Summing this error with the geometric row contraction bounds the exact-minus-mean-field magnetization by $O(R^{-4})$ in the joint norm. This argument applies to every finite induced subgraph. It defines the complex pressures by integration of the holomorphic magnetization in the field, with pressure normalized at zero. This agrees with the real log partition function on the real parameters and hence with its analytic continuation.

Here are consequences for the pressure stability. At a real background $h$, the second and fourth field derivative rows of $W_X(t+v,h)-W_X(t,h)$ have absolute sums at most $C\varrho$ when $\|v\|_\infty\le\varrho\le r/2$. This follows by retaining the terms with positive link degree in the joint Taylor expansion. At $h=0$ the second field derivative row improves to $C\varrho R^{-4}$, because the exact and mean-field rows differ by $O(R^{-4})$ in the joint norm, the latter Hessian is $(I-J_X(t+v))^{-1}$, and $|c-1|\lesssim R^{-4}$. The same bounds hold for the corresponding field derivatives of the real pressure on the boundary faces $t_{BC}=0,1$, by the analytic construction. The pressure difference is even and zero at zero field. Subtract its Hessian and repeat the split into coordinates of magnitude at most one and greater than one from Corollary 4.2. The fourth derivative controls the former coordinates and the second derivative controls increments on the latter. Thus

$$|W_X(t+v,h)-W_X(t,h)|\le C\varrho\sum_i\Psi(h_i)+C\varrho R^{-4}\|h\|_2^2.$$

Take $\varrho$ sufficiently small and use the real coercivity bound, uniform in $t$ because the row sum is still at most $q_*$. This proves complex coercivity with a fixed positive fraction of $\sum_i\Psi(h_i)$ and quadratic loss $O(R^{-4})\|h\|_2^2$. The proof of Corollary 4.3 now applies with the same absolute derivative row bounds for complex links. Taking the absolute Taylor majorant in the field of radius $a$, it gives

$$|e^{-W_X(t+v,\cdot)}|_{a,h}\le\exp\left\{C|\Lambda_X|a^4-c_0\sum_i\Psi(h_i)\right\}\quad(\|v\|_\infty\le\varrho).\tag{29}$$

For example, the quadratic loss is absorbed by $R^{-4}s^2\le\varepsilon\Psi(s)+C_\varepsilon a^4$, and the nonconstant field Taylor terms are estimated by the derivative rows in the same way as in the real proof. All constants here are uniform in $X$ and the real interpolation point.

Apply the Brydges–Kennedy–Abdesselam–Rivasseau forest formula [1] on the complete graph with vertex set $X$ to $e^{-W_X(t,h)}$. Given a forest $\mathcal{F}$ and interpolation parameters $s_e\in[0,1]$ for its

edges, its evaluation array has entry $\min_{e \text{ on path}} s_e$ for vertices joined by a forest path and zero for vertices in different forest components. The formula sums the integral of $\partial_{\mathcal{F}} e^{-W_X}$ at those arrays, where $\partial_{\mathcal{F}} = \prod_{e\in\mathcal{F}} \partial_{t_e}$. If an edge of $\mathcal{F}$ is inactive in the microscopic interaction, its derivative vanishes. At each evaluation array, $J_X(t)$ is a direct sum over the forest components. Its inverse $(I - J_X(t))^{-1}$ has the same direct-sum property, and the normalized pressure adds across the components. This factorization continues to hold upon varying the coordinates indexed by $\mathcal{F}$ with all intercomponent coordinates fixed to zero. The mixed derivative and the integral therefore factor over the components. Define $A_Y$ to be the sum of the terms corresponding to spanning trees on $Y$, using $W_Y$, and on a singleton use the weight with no link variables. This proves (27) and spatial connectivity.

For a spanning tree on $Y$, multivariable Cauchy bounds its mixed derivative in the Taylor seminorm by $\varrho^{-(|Y|-1)}$ times (29), reducing $\varrho$ by a fixed factor if necessary. One can apply the coefficient formula before taking the Taylor majorant: integrate each field Taylor coefficient on the same link polytorus and then sum its absolute value. There are at most $\Delta^{|Y|-1}$ spanning trees, by orienting each tree toward a fixed root and recording one of at most $\Delta$ parents at each other vertex. This proves (28). The field-linear source factor is independent of the link variables and already factors over forest components, so the same construction proves the source statement by the product rule for the field Taylor seminorm. □

To put the partition expansion in circle-product notation, set $I(B) = A_{\{B\}}$ and define

$$K(Y) = \sum_{\substack{\pi\in\mathcal{P}(Y)\\ |C|\geq 2 \text{ for every } C\in\pi}} \prod_{C\in\pi} A_C, \qquad K(\varnothing) = 1.$$

Partitions containing a disconnected part have zero weight. Thus $K$ factors over the geometric components of $Y$, and (27) is exactly $(I \circ K)(X)$. The number of partitions into connected parts is at most $2^{\Delta|Y|/2}$: for every part choose a spanning tree by a fixed deterministic rule, and encode the partition by the union of the chosen edge sets. Consequently (28) continues to hold with an enlarged $C_0$ for this $K$. This is an exact all-field connected representation with Gaussian damping. The bound has an exponential constant in polymer size and supplies neither the decaying large-set factor nor the small $g^3$ norm required by the one-step RG theorem. The singleton factor here is the full restricted weight, so matching it to the quartic reference factor is a separate localization problem.

**Anchored link coefficients.** The joint Taylor argument above also works with the following tree weight. For an active link $e$, let $E_e$ be the endpoints of the microscopic edges with link label $e$. For field sites $T$ and marked link labels $S$, let $\tau(i, T, S)$ be the shortest tree length joining $i$, $T$, and at least one point of each $E_e$ with $e \in S$. Give the coefficients the weight $e^{\eta\tau(i,T,S)/R}$. The union of two witness trees gives submultiplicativity. Changing the anchor across a reference edge costs at most $e^{\eta}$, and a link incident on the anchor adds zero tree length. Thus taking $e^{\eta} q_* < 1$ and then a small Taylor radius $r$ gives the uniform joint analytic and comparison estimates in this weighted norm. For pressure coefficients the anchor is the site at which the gradient is integrated.

A forest derivative marks each link at most once. We therefore keep only square-free link monomials, working modulo the squares of the link variables. For their support $S$,

$$|S| \leq C_1(1 + b^{-1}\tau(i, T, S)).$$

Each block contains witnesses for at most $\Delta$ link labels. From the blocks containing witnesses one can select a fixed positive fraction at pairwise periodic distance at least three blocks, by deleting

bounded block neighborhoods. Their witnesses are separated by at least $b$, and any joining tree has length at least a constant times $b$ times one fewer than their number. This proves the estimate also on a torus. For any fixed coefficient radius $D > 1$, choose $b/R$ sufficiently large that

$$(D/r)^{|S|} e^{-\eta\tau(i,T,S)/R} \le C_D e^{-\eta\tau(i,T,S)/(2R)}. \tag{30}$$

Thus one can use the larger radius $D$ while reserving half the tree weight. Once the lower bound on $b/R$ is met, $C_D$ is independent of the block aspect.

# 6 A local quartic coordinate and exact quadratic absorption

With $B$, $S_0$, and $S = D^2F(0)$ from Sections 3–4, put $E_\beta = S - S_0$. Make the exact linear field change $h = Bu$. The Gaussian precision in the new variable and the quadratic coefficient of the residual interaction are

$$\begin{aligned} BC_m^{-1}B &= A^{-1}BM_m, \\ H &= (c-1)B - BE_\beta B. \end{aligned}$$

Indeed the residual interaction is $\frac{c}{2}\langle u, Bu\rangle - F(Bu)$. Write $g = R^{-4}$, and work in $|\beta - 1| + m^2 \le C_0 g$. The cavity estimate and the finite-range bounds on $B$ give

$$\sum_x |H(x)| e^{\eta'|x|/R} \le Cg \tag{31}$$

for a fixed $\eta' > 0$. Set $r = \widehat{H}(0)$ and $K = H - rI$. For a dimensionless kinetic parameter $z$, absorb $K + zR^2\mathcal{D}$ in the Gaussian precision. The new covariance is exactly

$$\mathscr{C} = AT^{-1}, \qquad T = BM_m + A(K + zR^2\mathcal{D}). \tag{32}$$

The remaining quadratic interaction is

$$V^{(2)}_{r,z}(u) = \tfrac{1}{2} r \sum_i u_i^2 - \tfrac{1}{2} zR^2 \sum_i u_i (\mathcal{D}u)_i,$$

so its Hessian is $rI - zR^2\mathcal{D}$. All operators in these formulas commute.

**Proposition 6.1** (Uniform finite-range decomposition after Hessian absorption)**.** *Fix finite derivative and mass-decay orders $k_*, N_*$. For fixed sufficiently small $|z|$, the covariance* (32) *has positive finite-range pieces $\mathscr{C}_j$, $j \ge 0$, with range at most $CRL^j$, such that for $0 \le k \le k_*$,*

$$\|\nabla^k \mathscr{C}_j\|_\infty \le CR^{-4-k} L^{-(2+k)j} (1 + m^2 L^{2j})^{-N_*}.$$

*The constants are independent of $R, \beta, m^2, z$ in the stated parameter domain, and the pieces are continuous in the covariance parameters. There is no logarithmic growth of the initial range.*

*Proof.* Use $k$ for the frequency variable in this proof. From (31) and symmetry, $|\widehat{K}(k)| \le Cg \min(R^2|k|^2, 1)$. Also $R^2 \widehat{A}(k)\widehat{\mathcal{D}}(k) \le C \min(R^2|k|^2, 1)$, by rapid Fourier decay of the sampled smooth kernel $A$. Since $B$ is uniformly positive and $M_m$ has the box multiplier comparison, $\widehat{T}(k) \asymp m^2 + \min(R^2|k|^2, 1)$.

Choose a fixed integer $P$, large compared to $k_* + 2N_*$. There is a hypercubic-symmetric finite-range kernel $K_f$ supported in a fixed multiple of the range $R$, with norm $O_P(g)$ in (31),

whose moments through degree $P$ equal those of $K$. To construct it, choose a smooth compactly supported bump positive on an open box. The Gram matrix of the monomials of degrees at most $P$ against this bump is positive definite. Linear combinations of the monomials times the bump therefore give smooth compactly supported functions dual to the moment functionals. Sample them at scale $R$ with the factor $R^{-4}$. The matrix of the sampled, normalized moments converges to the identity. Solving the moment equations costs bounded coefficients times $g$, since the target moments divided by their powers of $R$ are $O_P(g)$. Average the result over the hypercubic symmetry group. This preserves the target moments and the norm bound.

Let $T_f = BM_m + A(K_f + zR^2\mathcal{D})$. It is finite range and has the same multiplier comparison as $T$, since $K_f$ has zero row sum and uniformly bounded scaled second moments of order $g$. The positive polynomial decomposition of the inverse of $T_f$, multiplied by $A$, gives positive pieces $\Gamma_i^f$ of range $O_P(RL^i)$. The Fourier integrals in Proposition 3.1 give the bounds claimed here for $\Gamma_i^f$. They also give the same upper bound on $\int |k|^r |\widehat{\Gamma}_i^f(k)|\, dk$ at derivative order $r$.

Consider the positive multiplier

$$M(k) = \frac{\widehat{T}_f(k)}{\widehat{T}(k)} = 1 - \frac{\widehat{A}(k)(\widehat{K}(k) - \widehat{K}_f(k))}{\widehat{T}(k)}.$$

At $m^2 = 0, k = 0$ use its continuous extension. For $|\alpha| \le P - 2$,

$$\int |\partial_k^\alpha (M(k) - 1)|\, dk \le C_P g R^{|\alpha|-4}. \tag{33}$$

To verify this near the origin, use $\xi = Rk$. The rescaled denominator has uniformly bounded derivatives, its first derivative is $O(|\xi|)$ by symmetry, and it is comparable to $m^2 + |\xi|^2$ on $|\xi| \le 1$. Differentiating its reciprocal therefore costs $(m^2 + |\xi|^2)^{-1-|\alpha|/2}$. Moment matching and the weighted norms give the bound $C_P g |\xi|^{P+1-|\alpha|}$ for the corresponding numerator derivatives. The ratio derivatives are consequently bounded by $C_P g |\xi|^{P-1-|\alpha|}$, also in the zero-mass limit. For $|\xi| \ge 1$, the denominator is bounded below. Its scaled derivatives are uniformly bounded by the moment estimates, and the numerator derivatives have the rapid decay of the sampled kernel $A$ times $O_P(g)$. Integrating in $\xi$ and returning to $k$ proves (33).

Fourier coefficient integration by parts in a coordinate of maximal absolute displacement, using (33), now shows that truncating the real-even Fourier series of $M - 1$ at range $RL^n$ has uniform error at most $\varepsilon_n = C_P g L^{-sn}$, where $s = P - 6$. Denote the truncation plus one by $M^{[n]}$. Set

$$F_n = M^{[n]} - 3\varepsilon_n, \qquad M_0 = F_0, \qquad M_n = F_n - F_{n-1} \quad (n \ge 1).$$

For small $g$, $F_0 \ge 1/2$. Also $F_n \ge F_{n-1}$, since $M^{[n]} - M^{[n-1]} \ge -\varepsilon_n - \varepsilon_{n-1}$ and $3(\varepsilon_{n-1} - \varepsilon_n) \ge \varepsilon_n + \varepsilon_{n-1}$ for $L^{-s} \le 1/2$. Thus the $M_n$ are positive trigonometric polynomials of the required ranges. Their sum is exactly $M$, and for $n \ge 1$, $\|M_n\|_\infty \le C_{P,L} g L^{-sn}$.

Define $\mathscr{C}_j$ by the Fourier multiplier sum of $\widehat{\Gamma}_i^f M_n$ over $\max(i, n) = j$. These products preserve positivity and have range $O_P(RL^j)$. For $i = j$, sum over $n \le j$ and use $\sum_{n \le j} M_n \le M \lesssim 1$ and the weighted Fourier integral bound on $\Gamma_j^f$. For $n = j > i$, use $\|M_j\|_\infty \lesssim g L^{-sj}$ and the sum of the Fourier integral bounds on the $\Gamma_i^f$, which is at most $CR^{-4-k}$ at derivative order $k$. Choose $P$ with $s > 2 + k_* + 2N_*$. Since $m^2 \le 1$, these last terms also obey the asserted mass-decay estimate. Absolute uniform bounds and the fixed truncations give parameter continuity. The sum of all products is $AT_f^{-1}M = AT^{-1}$, as required. □

**Proposition 6.2** (Size of the bubble increments)**.** *Let $A_{\mathrm{sc}} \geq 1$ be a field rescaling factor, put $\lambda = A_{\mathrm{sc}}^{-2}$, and let $C_j = \lambda R^2 \mathscr{C}_j$ be the covariance pieces for $\psi = (R/A_{\mathrm{sc}})u$. Write $\widetilde{w}_j = \sum_{i<j} C_i$ and $\widetilde{w}_{-1} = 0$. For any fixed $a_0 > 0$ define*

$$B_j = a_0 \sum_x (\widetilde{w}_j(x)^2 - \widetilde{w}_{j-1}(x)^2).$$

*Then $B_j \geq 0$, and for every fixed decay order allowed by the decomposition,*

$$B_j \leq C\lambda^2 (1 + m^2 L^{2j})^{-N_*}.$$

*The base positive polynomial decomposition can be chosen so that $B_j \geq c\lambda^2$ whenever $m^2 L^{2j} \leq c_0$ for a fixed $c_0 > 0$. The constants are independent of $R, \lambda$, and the covariance parameters in Proposition 6.1.*

*Proof.* The range and pointwise bounds give $\|C_j\|_1 \leq C\lambda R^2 L^{2j}$ and $\|C_j\|_\infty \leq C\lambda R^{-2} L^{-2j}(1 + m^2 L^{2j})^{-N_*}$. Expanding the difference of squares, the mixed products have nonnegative sums by Plancherel and positive semidefiniteness. Their upper bound is $\|C_j\|_\infty \sum_{i \leq j} \|C_i\|_1$, times a fixed constant.

For the lower bound use the same polynomials from Lemma 2.1 in the construction of $\Gamma_j^f$. The multiplier of $T_f$ is at most $C(m^2 + R^2|k|^2)$ for $|k| \leq R^{-1}$, and $\widehat{A}(k)$ is bounded below near zero on this scale. For sufficiently small fixed $c_0, c_1 > 0$, if $m^2 L^{2j} \leq c_0$ and $|k| \leq c_1 R^{-1} L^{-j}$, the lemma gives $\widehat{\Gamma}_j^f(k) \geq cL^{2j}$, including at $j = 0$. Since $M_0 \geq 1/2$, the positive product decomposition gives $\widehat{C}_j(k) \geq c\lambda R^2 L^{2j}$ on this ball. Its volume is comparable to $R^{-4} L^{-4j}$, so Plancherel gives $\sum_x C_j(x)^2 \geq c\lambda^2$. □

The residual nonlinear interaction in the $u$ variable has the on-site mean-field quartic $\sum_i u_i^4/12$. Its exact quartic defect has anchored weighted row norm $O(g)$ by the cavity comparison and the finite-range field change. At field scale $R^{-2}$ this defect costs $O(g^2)$ per range-scale block. The degree-six term $\sum_i u_i^6/30 - \langle u^3, S_0 u^3 \rangle/18$ also costs $O(g^2)$ there. The exact-minus-mean-field sextic defect costs $O(g^3)$. Thus this change removes the order-$g$ nonlocal quadratic and quartic contributions. The next section constructs the remaining order-$g^2$ activity in a norm retaining large-field decay.

# 7 Periodizing the absorbed covariance

The exact Hessian absorption can be used on every sufficiently large torus. Let $D_N$ be its side and let $E_N = S_N - S_{0,N}$ be the periodic high-temperature Hessian difference. Lift the torus convolution kernel to $\mathbb{Z}^4$ by using shortest coordinate representatives in $[-D_N/2, D_N/2]^4$, splitting the coefficient equally among the representatives when there is a tie. This lift is hypercubic symmetric, has the same row sum, and has exponential weighted $\ell^1$ norm $O(R^{-4})$ by the same torus-distance bound for $E_N$. In particular the moment matching and positive product decomposition of Proposition 6.1 apply to the lift with uniform constants. Periodize the resulting covariance pieces. Their sum is the exact torus covariance for $m^2 > 0$, because periodization samples the Fourier multipliers at the torus frequencies. The lift converges to the infinite-volume $E$ in a possibly weaker exponential weighted $\ell^1$ norm. Indeed, each fixed entry converges by high-temperature locality, while the stronger exponential bound controls the tails uniformly. This argument also gives continuity jointly along converging $\beta$ and $z$ at every fixed scale.

# 8 Connected initialization in the Hessian-subtracted field

For the induced reference graph and its real link array $t$, let $B_t = I - J_X(t)$, $S_t^{\mathrm{ex}} = D^2F_t(0)$, and put

$$\mathcal{U}_X(t,u) = \tfrac{1}{2}\langle B_t u, S_t^{\mathrm{ex}} B_t u\rangle - F_t(B_t u).$$

Here $F_t$ is the normalized reference pressure. At the full volume and all links one, exact Hessian absorption leaves this interaction plus the local mass and nearest-neighbor kinetic terms. For a proper induced block set, $\mathcal{U}_X$ is defined using its own $B_t$ and pressure; it is not obtained by restricting the full-volume transformed interaction. The forest identity below is applied with $X$ equal to the entire ambient torus. Its induced subvolume weights are auxiliary factors in that exact identity. Set $b = MR$, $a \asymp b^{-1}$, $g = R^{-4}$, and $\ell' \asymp A_{\mathrm{sc}} R^{-2}$, where $A_{\mathrm{sc}} \geq 1$ and $A_{\mathrm{sc}}/R \to 0$. The comparability constants are fixed. Use the affine-quotient regulator $\widetilde{G}$ at fluctuation scale $\ell'$, and $H_X = \exp(-\kappa(b^4a^2)^{-1}\sum_{i\in\Lambda_X} u_i^2)$.

**Proposition 8.1** (Exact quartic reference with an order-$g^2$ seed)**.** *For every fixed $\omega \in (0,1)$, one can choose a fixed aspect $M$ sufficiently large that the Hessian-subtracted weight at all real fields has the representation*

$$e^{-\mathcal{U}_X(\mathbf{1},u)} = (I_4 \circ K)(X), \qquad I_4(B) = \exp(-\tfrac{1}{12}\sum_{i\in B} u_i^4).$$

*The polymer activity factors over geometric components. For every nonempty $Y$, uniformly for large $R$,*

$$|K(Y)|_{\ell',u} \leq C_\omega A_{\mathrm{sc}}^6 g^2 \omega^{|Y|} G_Y(u), \qquad |K(Y)|_{a,u} \leq C_\omega g^{1/2}\omega^{|Y|} H_Y(u)\widetilde{G}_Y(u).$$

*The estimates hold with up to two field-linear nilpotent sources at sites, provided the source weights times the corresponding field radius are bounded. If the physical local mass and kinetic terms are included, with normalized coefficients $|r| + |z| = O(g)$ and kinetic term $-zR^2\mathcal{D}$, the same estimates hold in the derivative-sensitive $T$ norms after multiplying the reference and activity by the appropriate block factors. The kinetic term may enlarge their field support to the one-step block neighborhoods.*

*Proof.* We first describe the analytic bounds for the transformed pressure. At any real reference array, $B_t$ and its inverse are bounded in row and column absolute sum, uniformly in the induced volume. Consequently the coercivity estimate in Corollary 4.2 and the convexity and fixed-scale bounds for $\Psi$ give $\mathcal{U}_X(t,u) \geq c\sum_i \Psi(u_i)$. Indeed, $u = B_t^{-1}(B_t u)$ and the weighted averaging inequality for $\Psi$ gives the needed lower bound. Its field derivatives through order three vanish at zero, and its degree-four Taylor polynomial is $\sum_i u_i^4/12$ plus a tensor with absolute weighted rows of order $g$. The leading polynomial is independent of all links, because the mean-field quartic is $\sum_i((I - J_X(t))^{-1}B_t u)_i^4/12$.

Fix a real field $u$. At a link $e = \{B, C\}$, put $r_e^0 = (1 + \sup_{B\cup C}|u|)^{-1}$. When taking Taylor coefficients, regard these numbers as constants. Initially use link increments $r_e^0 v_e$. For any real field $u'$ with $|u_i'| \leq |u_i|$ and field increment $y$, write

$$B_{t+r^0v}(u' + y) = B_t u' + B_t y - (J_{t+r^0v} - J_t)u' - (J_{t+r^0v} - J_t)y.$$

At fixed small Taylor radii in $v, y$, the nonconstant part of this expression has a uniformly small anchored coefficient norm. For the third term this follows from $r_e^0|u_i'| \leq 1$ on the support of link $e$; for the other terms use the absolute row bound and range of $J$. In the weighted tree

norm, a field increment on a neighboring site costs at most $e^{\eta}$ to change its anchor, and a link parameter incident on that site already has an endpoint in its support there. Thus the cavity proof, including comparison with mean field, applies in the same joint coefficient algebra with these external fields. The real backgrounds are arbitrary, but the size of their complex increments is uniformly small. Differentiating the transformed pressure in $u$ applies $B_t$ to the magnetization, and likewise uses the finite-range coefficient matrices for link increments. This gives uniform weighted absolute derivative rows of order at least two for $\mathcal{U}_X$, and analytic sums of all higher orders, uniformly along every field segment with $|u_i'| \le |u_i|$. The covariance in its subtracted quadratic term has the same analytic estimates by differentiating the reference magnetization at zero. The exact-minus-mean-field quartic estimate holds in this joint algebra, by the corresponding magnetization comparison.

We need a small link variation of the exponential in an algebra with a large coefficient radius. The following choice avoids a boundary trace estimate involving the coefficient of $\widetilde{G}$. Choose a large square-free link coefficient radius $D$. The order of choices in this proof is $D$, then $\delta = \delta(D)$, then $\theta = \theta(D, \delta)$, then $M = M(D, \delta, \theta)$, and finally $R$. Constants with a subscript $D$ may grow with $D$; this growth is absorbed by the final lower bound on $R$. Call a block bad if its supremum of $|u|$ exceeds a fixed small $\delta > 0$. Mark a link if one of its endpoint blocks is within two coarse blocks of a bad block, using the periodic block metric. Use increments $r_e v_e$, where $r_e = \theta r_e^0$ on marked links and $r_e = r_e^0$ otherwise, for a fixed $0 < \theta < 1$ chosen below. The square-free link algebra and (30) allow the change from the small Taylor radius to $D$, at the cost of a factor $C_D$ and half the exponential tree weight. Coefficients involving a marked link gain a factor $\theta$ from the scaling. For coefficients with only unmarked links and nonempty link support, every anchor in a bad block is at distance at least $cb$ from every link support involved. The remaining tree weight bounds such rows at bad anchors by $C_D e^{-cM}$. There are at most a fixed number of marked links per bad block, because the block graph has bounded degree.

Let $\Delta\mathcal{U} = \mathcal{U}_X(t + rv, \cdot) - \mathcal{U}_X(t, \cdot)$. The radii are kept fixed under all field and link derivatives at the evaluation field $u$. We claim that in the joint field Taylor seminorm of radius $a$ and square-free link norm of radius $D$,

$$\|\Delta\mathcal{U}\|_{a,u;D} \le \epsilon_0 \sum_i \Psi(u_i) + C|X|, \tag{34}$$

with $\epsilon_0 > 0$ arbitrarily small and $C$ independent of $D$ after choosing $\delta, \theta, M$ in that order and then $R$ sufficiently large. To see this, split $u = s + l$ into coordinates with magnitude at most $\delta$ and greater than $\delta$. Hold the adaptive radii fixed throughout the field interpolation. At $s$, Taylor expansion from zero through order five, with integrated sixth derivative for the rest, bounds the field Taylor terms of orders at most five by

$$C_D \sum_i \big(g(|b_i| + a)^4 + (|b_i| + a)^6\big).$$

This uses the absence of a link-dependent mean-field quartic. For each product of field coordinates, symmetry of the derivative tensor allows the row bound to be anchored at the index carrying the largest power in the arithmetic–geometric mean estimate. This gives at most $C_D(g + \delta^2) \sum_i \Psi(b_i)$ and $C_D|\Lambda_X|(ga^4 + a^6)$. Choose $\delta$ small first. For increments from $s$ to $s + l$, apply the derivative rows anchored at the index of the increment. For field Taylor order zero, first estimate the first derivative at $s$ by integrating the Hessian from zero and then use the second-order formula for the increment. For orders one through five, integrate the next derivative directly. The resulting terms on the large coordinates are bounded by $C_{D,\delta}(\theta + e^{-cM}) \sum_i \Psi(l_i)$, since these coordinates belong to bad blocks. The conversion uses $|l_i| + l_i^2 \le C_\delta \Psi(l_i)$ on $|l_i| > \delta$. The field Taylor orders at least

six contribute $C_D|\Lambda_X|a^6$ at any field by analytic summability. Now choose $\theta$ small and $M$ large. Since $|B|a^4 \asymp 1$, the normalized additive terms are bounded by $|X|$ for large $R$, proving (34).

The real weight has Taylor damping $\exp(C|X| - c\sum \Psi(u_i))$. This follows by the field-split proof of Corollary 4.3, using the transformed derivative rows and real coercivity; here there is no quadratic Taylor term. Applying (34) in the coefficient algebra shows that the same bound, with a smaller damping constant, holds jointly in the links. For the pure quartic $V_X = \sum u_i^4/12$ it holds directly, and hence it holds for exponentials of all convex interpolations between $V_X$ and $\mathcal{U}_X(t + rv, \cdot)$. Constants are uniform for the interpolation parameter in $[0, 1]$.

Write the difference to the local quartic reference by the exact identity

$$e^{-\mathcal{U}_X} - e^{-V_X} = (V_X - \mathcal{U}_X) \int_0^1 e^{-(1-q)V_X - q\mathcal{U}_X}\, dq.$$

At field Taylor radius $a$, the first factor has joint norm at most $C_D \sum_i [g(|u_i| + a)^4 + (|u_i| + a)^6]$. This follows by subtracting its order-$g$ quartic and integrating its sixth derivative for the remaining lower Taylor orders, with the higher orders again bounded by joint analyticity. On each block bound these terms by the corresponding polynomial in $1 + \sup_B |u|/a$ times $|B|(ga^4 + a^6) \lesssim_M g^{1/2}$. The polynomial is absorbed by spare Gaussian damping and the affine-quotient regulator, since $\ell' \le a$ for large $R$. At radius $\ell'$, the exponentials obey the same joint stability by monotonicity, since $\ell' \le a$. The first factor is bounded by

$$C_D \sum_i [g(|u_i| + \ell')^4 + (|u_i| + \ell')^6] \le C_{D,M} A_{\mathrm{sc}}^6 g^2 \sum_{B \in X} (1 + \sup_B |u|/\ell')^6.$$

The growing regulator satisfies $\log G_B(u) \ge c(\sup_B |u|/\ell')^2$, so any fixed fraction of its exponent absorbs these polynomials, leaving the other fractions for the forest extraction and reweighting.

The forest formula for $e^{-\mathcal{U}_X}$ factors across forest components because $B_t$, the reference pressure, and its Hessian all factor there. For each spanning tree extract the square-free coefficient in the normalized link variables $v$, paying $D^{-(|X|-1)}$, and then multiply by $\prod_e r_e^{-1}$ for the original link derivatives. At zero field all these radii equal one. At a real field the extra factors are bounded by

$$\theta^{-C\#\{\text{bad blocks}\}} \prod_{B \in X} (1 + \sup_B |u|)^{\Delta}.$$

The powers of the suprema are absorbed per block at a fixed exponential cost, using spare damping and the affine-quotient regulator at radius $a$, or a further fraction of the growing regulator at radius $\ell'$. For the constant factor on a bad block at radius $a$, use

$$\frac{\sup_B |u|}{a} \le C \left( a\|u_B\|_2 + \frac{\ell'}{a} \sqrt{\log \widetilde{G}_B} \right).$$

A bad block therefore supplies at least $c\delta^2/a^2$ in the sum of the Gaussian and quotient exponents, which absorbs the fixed power of $\theta^{-1}$ for large $R$. There is still a fraction of both exponents left. At radius $\ell'$ the bad block instead gives $\log G_B \ge c\delta^2/(\ell')^2$; the same absorption follows. These estimates also absorb up to two source factors when the radius times their source weight is bounded. The source factors are independent of the links and factor over the forest components.

On a singleton let $\delta A_B$ be the difference of the two weights above. For a larger connected set let $\delta A_X$ be its exact tree activity, which is unchanged by subtracting the link-independent quartic weight under a nonempty spanning-tree derivative. Define $K(Y) = \sum_{\pi \in \mathcal{P}(Y)} \prod_{C \in \pi} \delta A_C$, with zero contribution from disconnected parts. Expanding the exact forest partition by the local

singleton reference and the remaining activities gives $(I_4 \circ K)(X)$. For the bounds, retain the factor $D^{-(|C|-1)}$ for every connected part. At large field the product of the other small factors $(C_D g^{1/2})^k$ is at most $C_D g^{1/2} D^{-(k-1)}$ for large $R$; at radius $\ell'$ use $(C_D A_{\mathrm{sc}}^6 g^2)^k$ in its place. The remaining exponential and the count of connected partitions cost a fixed exponential in $|Y|$, which is absorbed in the original choice of $D$. This proves the two bounds.

Finally, the local mass, nearest-neighbor kinetic term, and field-linear sources at full links are block-multiplicative factors. Use the local kinetic density $u_i(-\Delta_{\mathrm{nn}}u)_i$ at site $i$, so the reference is in Park's $\mathcal{V}^{(0)}$ coordinates. Multiplying by their block product replaces both $I_4(B)$ and $K(Y)$ by the corresponding reweighted factors, with the exact same circle identity. The mass costs $C|r|\sum_i(|u_i|+a)^2$ in the large-field norm and is absorbed because $|r| \ll a^2$. For the kinetic term the two differences in $-\Delta_{\mathrm{nn}}u$ act on the field or on the corresponding test coordinate. The $T$ seminorm at radius $a$ therefore costs at most $C|z|R^2b^2a^2(1+N_{B^\square}(u;a))^2$ per block. Its coefficient tends to zero for $|z| = O(g)$, and the affine bound with the block neighborhood absorbs the field factor using the remaining damping and quotient slack. At radius $\ell'$, the kinetic estimate similarly costs

$$C|z|R^2b^2(\ell')^2(1+N_{B^\square}(u;\ell'))^2 \le C_M A_{\mathrm{sc}}^2 g(1+\log G_B(u)).$$

The mass costs the same order, from $C|r|b^4(\ell')^2(1+\sup_B|u|/\ell')^2$. Since $A_{\mathrm{sc}}^2 g \to 0$, both are absorbed in the remaining growing regulator. The source factors have the previously stated product bound. This proves the reweighted assertion. □

At natural fluctuation scale $A_{\mathrm{sc}} = 1$ the seed has order $g^2$. The rescaling and transient below put this exact seed in the domain of the iterative map.

## 9 Polymer norm and the enlarged analytic ball

At local scale $j$ a block has side $b_j = bL^j$, where $b$ is the initial block side. Write $|X|_j$ for the number of blocks in a polymer and $B^\square$ for the union of small connected polymers containing $B$, with at most 16 blocks each. The scale-$j$ field Taylor seminorm at radius $a$ pairs the derivatives of degree $n$, divided by $n!$, with test tensors $f^{(n)}$ satisfying

$$\max_{|\gamma|\le 12} b_j^{|\gamma|}\|\nabla^\gamma f^{(n)}\|_\infty \le a^n.$$

Here $\gamma$ is a list of differences in the tensor slots, and the seminorm sums the degreewise suprema. For field regulators, $\Phi_j(X)$ takes the infimum of the one-slot test norm over extensions of the field on $X$. Its quotient $\widetilde{\Phi}_j(X)$ also minimizes over subtraction of an affine field on $X$. On each interior block $B$, $G_j(B)$ and $\widetilde{G}_j(B)$ are the exponentials of the corresponding squared seminorms on $B^\square$ at radius $\ell_j$, with a factor $1/2$ in the quotient regulator. The damping $H_j(B)$ is as in Section 3 with radius $h_j$. All three regulators multiply over disjoint blocks. These are the interior-scale definitions of [4, Sections 2.2–2.6].

For a fixed positive reference quartic $G$ at this scale, set $\ell_j = \ell_* b_j^{-1}$ and $h_j = k_0 G^{-1/4} b_j^{-1}$, with fixed $\ell_*, k_0 > 0$. For marks $a, c$ use the commuting nilpotent variables $t_a, t_c$, with $t_a^2 = t_c^2 = 0$. Write $k$ for the first scale at which $48b_k > \mathrm{dist}_\infty(a,c)$, using the torus distance on a finite torus. The full Taylor seminorm $T_j(\phi;\mathfrak{h})$ is the sum of the four field seminorms in the sectors $1, t_a, t_c, s_a s_c$. The weights in the last three sectors are, respectively, $\mathfrak{h}_{\sigma,j}, \mathfrak{h}_{\sigma,j}, \mathfrak{h}_{\sigma\sigma,j}$, where

$$\ell_{\sigma,j} = Gb_{j\wedge k}2^{(j-k)_+}, \qquad \ell_{\sigma\sigma,j} = \ell_{\sigma,j}^2, \qquad h_{\sigma,j} = G^{1/4}b_j, \qquad h_{\sigma\sigma,j} = G^{1/2}b_{j\wedge k}^2 L^{2(1-\varepsilon')(j-k)_+}.$$

The fixed $\varepsilon' > 0$ is chosen as in [4, (2.20)]; in particular $h_{\sigma\sigma,j} \le h_{\sigma,j}^2$. For connected $X$, set $\mathcal{A}_j(X) = \varrho^{(|X|_j - 16)_+}$, and put

$$\mathcal{G}_j(X;\ell) = G_j(X), \qquad \mathcal{G}_j(X;h) = H_j(X)\widetilde{G}_j(X).$$

The subscripted $G_j(X)$ is the growing field regulator. Define

$$\|K\|_{\mathcal{W}_j} = \sum_{\mathfrak{h}\in\{\ell,h\}} \omega(\mathfrak{h}) \sup_{\substack{X\neq\varnothing \text{ connected} \\ \phi}} \frac{\|K(X)\|_{T_j(\phi;\mathfrak{h})}}{\mathcal{A}_j(X)\mathcal{G}_j(X;\mathfrak{h})}, \qquad \omega(\ell) = 1, \quad \omega(h) = G^{9/4}.$$

In intermediate maps a superscript $a$ replaces $\mathcal{A}_j$ by $\mathcal{A}_j^a$, and a regulator parameter $\gamma > 0$ replaces $G_j$ by $G_j^\gamma$ and $H_j\widetilde{G}_j$ by $H_j^{1/\gamma}\widetilde{G}_j^\gamma$. The equivalent norm $\mathcal{Y}_j$ evaluates the fluctuation Taylor seminorm at zero field and keeps the same large-field part [4, Lemma 2.2]. Activities on disconnected sets are the products on their components; their empty-set value is one. Bounds concern nonempty components. Before coalescence the mixed source sector vanishes on small connected polymers, of at most 16 blocks. Large polymers may contain both marks. All analytic activity variations belong to the same linear support, parity and symmetry subspace.

The enlarged activity ball uses the common analytic input (7) with the norm $\mathcal{W}_j$ and the symmetry and support conditions just specified. Proposition 10.1 verifies this input for the box map. Lemma 2.3 then extends its contraction at zero activity to the smaller enlarged ball and bounds the first derivatives in the potential.

For fixed real potential inputs in the stated domains, suppose $G_{j+1}/G_j \ge 1 - CG_j$, $G_{j+1} \le G_j$, and $\chi_j/\chi_{j+1} \le \Omega$. Choose the scale factor first so that $\kappa_0 < \Omega^{-3/2}$, then choose

$$\kappa_0 < \kappa < \vartheta < \Omega^{-3/2}.$$

Choose $\epsilon$ so that $\kappa_0 + C\epsilon < \kappa$, and only afterwards make $G_0$ small. Thus

$$\|K_{j+1}\| \le C\chi_j^{3/2}G_j^3 + \kappa\|K_j\|, \qquad \|K_j\| \le C'\chi_j^{3/2}G_j^3 + \vartheta^j\|K_0\|.$$

Indeed, the ratio of consecutive weights $\chi_j^{3/2}G_j^3$ is at least $\Omega^{-3/2}(1 - CG_0)^3 > \vartheta$, so the inhomogeneous sum is geometric after division by the output weight. If also $G_j \ge cG_0/(1 + G_0 j)$, then

$$\sup_j \frac{\vartheta^j\|K_0\|}{G_j^{9/4}} \le C\frac{\|K_0\|}{G_0^{9/4}} \sup_{j\ge 0} \vartheta^j(1 + G_0 j)^{9/4}.$$

The last supremum is bounded uniformly for small $G_0$. Hence $\|K_0\| = o(G_0^{9/4})$ makes both terms uniformly $o(G_j^{9/4})$, closing induction in the enlarged ball. If $\chi_j \ge \Omega^{-j}$ and $\|K_0\|/G_0^3$ grows at most as a fixed inverse power of $g = R^{-4}$, the transient is bounded by the first term after $O(|\log g|)$ steps: divide by $\chi_j^{3/2}G_j^3$ and use $\vartheta\Omega^{3/2} < 1$. This estimate is uniform in the mass cutoff.

# 10 The interior RG map

## 10.1 Covariance bounds and symmetries

Choose $b = L^{j_0} \in [MR, LMR]$ and write $\Gamma_{j+1} = \lambda R^2 \mathscr{C}_j$ for the piece integrated from local scale $j$ to $j + 1$, periodized on a torus. Write $w_0 = 0$ and $w_j = \sum_{i=1}^{j} \Gamma_i$ for the covariance already

integrated. Earlier physical-scale pieces are zero. The RG reference is $I_j(B) = e^{-V_j(B)}(1 + W_j(B))$, where the canonical potential $V_j$ contains the on-site quadratic and quartic terms, a nearest-neighbor kinetic density, and the linear sources. The polynomial $W_j$ is the part of the quadratic Wick correction left after localization, given by [4, (4.58)–(4.59)] with covariance $w_j$. In particular $W_0 = 0$, and the BBS normal-form change is the identity at this scale. Choose $L > 96$ and then $M$ large enough that each positive piece assigned to the step from $j$ to $j + 1$ has graph range at most $L^{j_0+j+1}/4$. This leaves the fixed collar of fine-block neighborhoods needed for the independence of disconnected coarse components when $L$ is large. The hypercubic and translation symmetries hold for the numerator, both denominator kernels, the symmetrized moment replacement, and the Fourier truncations. They are preserved by every polynomial, positive product, and periodization used in the decomposition.

The derivative estimates in the normalized field are

$$\|\nabla_x^\alpha \nabla_y^\gamma \Gamma_{j+1}\|_\infty \le C_{L,M,p} \lambda b_j^{-(2+|\alpha|+|\gamma|)} (1 + m^2 L^{2j})^{-p} \qquad (|\alpha| + |\gamma| \le 32).$$

The factor $M$ can be chosen fixed. The Park reference mass is comparable to $m^2 R^{-2}$, so the corresponding cutoff indices differ by at most a fixed number. The finite-order version of the interior-scale estimates uses exactly the displayed bound for a fixed sufficiently large $p$. Here is the check of the covariance inputs.

- The algebra in Park Sections 4–5 uses translation and hypercubic symmetries, positive Gaussian integration, finite range for factorization, and Wick contractions of polynomials. The latter involve only differences in the field monomials in $\mathcal{V}$; in four dimensions their order is at most two in each monomial. The same applies to Lemma 4.7 and the moment estimates for the BBS normal-form coefficients, which use two spatial differences at most.

- The bilinear estimates in Park Section 7.2 use the $\Phi_j$ norm of the covariance in (7.37), and at most the mixed derivatives of total order $p_\Phi = 12$ from (2.10). The Gaussian moments in (7.123) and the product estimates in Sections 9–10 use Lemma C.2. More precisely, the Sobolev step (B.2) bounds the field seminorm by square integrals of differences of order at most $p_\Phi + d = 16$. The Gaussian quadratic form bound (C.5) therefore needs at most 16 differences in each covariance variable. Taking powers of these regulator bounds increases the constants in (C.5) but does not increase the derivative order.

- The $H$ expectation uses only the covariance operator norm, bounded by its absolute row sum in (C.14). This follows from the order-zero estimate and finite range. For the last direct Gaussian expectation there are only $L^4$ blocks, and the same Sobolev quadratic form estimate follows from the periodized tail derivative bound; no factorization is used at that step.

This is the finite collection of covariance estimates summarized in Park (1.44) and in the sentence following it. No differentiability in the covariance parameters is used in these map bounds. Continuity suffices for the Gaussian integrals, by the uniform regulator bound and positive Gaussian coupling, and for the finite Wick polynomials. The factor $\lambda \to 0$ makes the fixed powers of $L, M$ in these upper estimates harmless in the fluctuation smallness conditions.

For spatial symmetry write $s = j_0 + j$, and let $\mathsf{G}_s$ be the subgroup of lattice isometries preserving the scale-$s$ partition. Use $\mathsf{G}_s$-equivariance for the bulk activity, and the usual field parity in all source sectors. This is sufficient also on the torus. The only uses of the activity symmetry in the estimates are as follows. In Lemma 4.15, the localization sum uses small polymers and the

equivariance in (4.67) is explicitly restricted to $F(\mathcal{B}) = \mathcal{B}$. Localization assigns the same monomial coefficients to all sites of its domain. Thus the sum on each block has a constant site density, translations by a whole block make its coefficients identical on all blocks, and the signed coordinate permutations and grid-preserving reflections give the hypercubic constraints on those coefficients. This yields the usual translation-invariant effective potential. In Lemma 3.7 the required extra symmetry is exactly the vanishing of the odd-field derivatives in (3.28). Proposition 3.8 invokes it on small polymers only. If a spatial isometry maps a small polymer to another polymer, it preserves the grid: for $L > 2^4$, the projection of a small polymer on each coordinate circle is a proper nonempty union of whole block intervals, so a boundary endpoint in each coordinate fixes the translation residue, taking account of the reversal of endpoints under reflection. There is no use of invariance under a shift that preserves only a large wrapped set. In Section 10 the small-set linearization is Lemma 10.5, invoking Proposition 3.8, and the large-set linearization is Lemma 10.6, which instead uses only $1 - \mathrm{Loc}_Y = 1$ and the geometry of the weights. The rough estimates in Sections 7–9 apply unchanged.

Maps 1–5 are equivariant under the next-scale grid group, which is a subgroup of the input grid group. Indeed the products, component decompositions, small-set neighborhoods, localization, Gaussian expectation, and the covariance redistribution are all equivariant under this group. The forest initial activities are equivariant under the initial grid group for the same reason. The argument applies to connected and disconnected activities, with the latter defined by the component product. Field parity, support and the source grading are preserved throughout. In particular, a small polymer's neighborhood has diameter less than $3 \cdot 2^4 L^s$, so a two-source activity there vanishes before coalescence.

### 10.2 The change to the output coordinates

Write $K_{(5)}$ for the intermediate activity after Map 5 of [4]. The effective output polynomial $U$ and its canonical representative $V_+$ differ by the vacuum and mixed constants and a quadratic density with zero global sum. Denote by $W_{\mathrm{pt}}$ and $W_+$ the Wick factors of the intermediate and canonical block references. Their difference in block weights and the componentwise circle product give the exact output coordinates

$$\Delta I(B) = e^{-V_+(B)}(W_{\mathrm{pt}}(B) - W_+(B)), \qquad K_+ = \Delta I \circ_+ (e^{U-V_+}K_{(5)})$$

on connected polymers, with the corresponding component extension. Pull out the field-independent exponent using $U(\Lambda) - V_+(\Lambda) = \delta u(\Lambda)$. The remaining equality is the circle-product identity of [4, Lemma E.1], since the two block reference factors differ by exactly $\Delta I$. This also keeps the full $\Delta I(B)$ on a singleton.

For this change, let $\tau_K$ be the extended $K$ radius, $G$ the reference coupling, and $\chi$ the mass factor. The redundant stiffness coefficient in $U$ is $O_L(\chi G^2 + \tau_K + \|K\|)$: the input potential is canonical and its Gaussian expectation creates no such coefficient, while $Q$ is linear in $K$ and the perturbative correction is quadratic with covariance factor $\chi$. The local summation-by-parts polynomial therefore has $h$-norm at zero of size $O_L(G^{-1/2}(\chi G^2 + \tau_K + \|K\|))$. Its exponential is a bounded multiplier from the Map 5 regulator with exponent $1/2$ to that with exponent $3/4$. In fact the block Sobolev bound (B.1) bounds its field contribution by this coefficient times $C(1 + \|\varphi/h_\varnothing\|^2_{L^2(B)} + \|\varphi\|^2_{h,\widetilde{\Phi}(B^\square)})$. This is absorbed by the spare $H$ and $\widetilde{G}$ powers. The $\ell$ part can be checked at zero and then in the equivalent mixed norm $\mathcal{Y}$. The mixed constant of $Q$ has bounded $h$-norm $O_L(G^{-9/4}(\|K\| + \tau_K))$ by applying localization directly at $h$. Its exponential is a uniformly bounded field-independent nilpotent factor. The one-source linear coefficient is also

bounded directly at $h$, where $h_\sigma h_\varnothing$ is constant. No factor $(h_\varnothing/\ell_\varnothing)^4$ is used to compare source sectors.

Since $W$ ignores field-independent terms and is bilinear, Lemmas 7.3–7.4 bound $\Delta I(B)$, after polynomial absorption, by $O_L(\chi G(\chi G^2 + \tau_K + \|K\|))$ at $\ell$ and $O_L(\chi G^{-1/2}(\chi G^2 + \tau_K + \|K\|))$ at $h$. These sizes are admissible in the weighted output norm. To keep the precise large-set margin, use the following count on a connected $X$. With the fixed margin $\xi > 0$ of [4, Section 9], set $a = 1 + \xi/10$, $a' = 1 + \xi/16$, and $N_s = 2^4$. After multiplication by $e^{U-V_+}$ the activity on a component $C$ has unweighted size at most $\delta\mathcal{A}^a(C)$ times its regulator. The cost of the multiplier is at most $2^{|C|}$, absorbed by the margin from $1+\xi/8$ to $a$ and a dimension-dependent constant. Write $d_1$ for the unweighted singleton bound on $\Delta I$. Both $\delta \le \varrho^{aN_s}$ and $d_1 \le \varrho^a$ hold by the choices of the extended radius and small coupling. For $Y \subseteq X$ with $m \ge 1$ components and $q = |X \setminus Y|$, the unweighted bound on its convolution term is

$$\delta^m d_1^q \mathcal{A}^a(Y) \le \delta\, \varrho^{a(|X|-N_s)_+},$$

since $\sum_{C\in\mathrm{Comp}(Y)}(|C| - N_s)_+ + N_s(m-1) + q \ge (|X| - N_s)_+$. For $Y = \varnothing$ the same bound uses $d_1$ in place of $\delta$. The at most $2^{|X|}$ terms are absorbed by $a - a'$, with a dimension-dependent constant. In the derivative in $K_{(5)}$, one of the $\delta$'s is replaced by the variation norm and the number of choices is at most $|X|$; the same margin absorbs this factor as well. This proves the rough output estimate. For the derivative in the intermediate activity at original input $K = 0$, separate $Y = X$. There the derivative is just multiplication by $e^{(U-V_+)(X)}$. At $K = 0$ its block norm is at most $(1 + \epsilon_L(G))\mathcal{G}^{(3/4)}/\mathcal{G}^{(1/2)}$ with $\epsilon_L(G) \to 0$. Indeed the vacuum and mixed coefficients tend to zero in both norms, and the field-dependent quadratic coefficient uses the spare regulator power by the same block Sobolev bound. For a connected polymer of size $n$, the multiplier in the output norm therefore costs at most

$$(1 + \epsilon_L(G))^n \varrho^{(1+\xi/8-a')(n-N_s)_+} \le (1 + \epsilon_L(G))^{N_s}.$$

Every other term of the intermediate derivative has at least one $\Delta I$ block. Extract its size $d_1 = o_G(1)$ and use the preceding count on the remaining factors. This loses at most one factor $\varrho^{-a}$, independent of $X$ and $G$, since the exponent in the count decreases by at most one. The sum of all these terms is $o_G(1)$ in operator norm for fixed $L, \varrho$. Thus the intermediate derivative has norm at most $1 + o_G(1)$ in the stronger output norm. With $K_{(5)}$ held fixed at its $K = 0$ value, its partial derivative in the original $K$ is $O_L(G^{3/4})$ by Cauchy on $\tau_K = c_L G^{9/4}$, because the full weighted output on that disc is $O_L(\chi^{3/2}G^3 + \chi G\tau_K)$. Use Park (10.74) through $i = 5$, before the last change, then the chain rule. This preserves the arbitrarily small contraction at $K = 0$, with $L$ chosen large and $G$ small. The enlarged-domain Cauchy argument then transfers it to $o(G^{9/4})$ activities.

### 10.3 Analytic and perturbative estimates

**Proposition 10.1** (Interior map at large spread radius)**.** *Use the covariance of Proposition 6.1 on the finite tori and the initial activity of Proposition 8.1. Choose $1/12 < a_* < 1/10$, set $A_{\mathrm{sc}} = g^{-a_*}$, $\lambda = A_{\mathrm{sc}}^{-2}$, and let $G_0 \asymp gA_{\mathrm{sc}}^4$ denote the initial quartic coefficient in the field $\psi = Ru/A_{\mathrm{sc}}$. There are constants $\epsilon > 0$, $\kappa < 1/8$, and $C$, independent of the spread radius, volume and mass cutoff, for which the exact graded RG map obeys, on $\|K\|_{\mathcal{W}_j} \le \epsilon G^{9/4}$,*

$$\|D_K\Phi_{j+1}^K\| \le \kappa, \qquad \|\Phi_{j+1}^K(V,K)\| + |R_j(V,K)| \le C(\chi_j^{3/2}G^3 + \|K\|_{\mathcal{W}_j}).$$

*Here $R_j$ is the bulk remainder in the normal-form coordinates; the source estimates use the four-sector norm specified above. This interior-scale map satisfies the bounds in Proposition 11.1, with constants uniform in the spread radius, torus size and mass cutoff.*

*Proof.* First check the initialization in the precise Park norms. In physical $u$ units the Park radii on the initial block are $\ell_\varnothing = \ell_* A_{\mathrm{sc}}/(RL^{j_0})$ and $h_\varnothing \asymp k_0 L^{-j_0}$. Choose $M$ large compared to $\ell_*$ and $k_0$ small. The radii in Proposition 8.1 dominate these. Its affine regulator uses a physical small-field radius no smaller than that of Park, so is bounded by the corresponding Park regulator. Its $H$ coefficient can be any sufficiently large fixed constant: by $\Psi(u) \geq c_\kappa L^{-2j_0} u^2 - C_\kappa L^{-4j_0}$, one may reserve the necessary $H$ power in the real damping and absorb the fixed loss per block into the choice of the forest coefficient radius. This matches $H_{\mathrm{Park}}$ also after the change of field amplitude. The small-field source weight times its field radius is $O(G_0)$, and the large-field product is $O(1)$. The two-source weight is no larger than the product of the one-source weights. Thus the source part of the same initial bound holds for one-source and two-source flows. The absolute Taylor norm used in the initialization majorizes the $T$ norm by its derivative test bounds. The kinetic factor is treated by the derivative-sensitive estimate at the end of that proposition. Choosing the forest rate smaller than the Park large-set rate and using $\mathcal{Y}$-to-$\mathcal{W}$ equivalence yields

$$U := C\left(g^2 A_{\mathrm{sc}}^6 + G_0^{9/4} g^{1/2}\right), \qquad \|K_0\|_{\mathcal{W}} \leq U.$$

The smallness ratios for $U$ are recorded in Section 12.

We give a source-norm detail for the application of the rough map bounds on the enlarged ball. Use the extended Taylor radii $\tau_K = c_L G^{9/4}$ in $K$ and $c_L G$ in the bulk potential. The space of extended directions uses the same grid symmetry, parity and mixed support conditions as its real centers. In (9.12)–(9.14) of Park, localize $K/I$ separately in each radius using Proposition 3.4. This gives $O_L(\tau_K)$ at $\ell$ and $O_L(\omega(h)^{-1}\tau_K)$ at $h$. The latter is uniformly as small as required by E.7 on choosing $c_L$ small. The stable product $I^X J_B(X)$ obeys the same estimates, using the polynomial bound and absorbing its field polynomial in the quartic damping. This suffices for Map 1. In Map 2,

$$e^{-V} - e^{-(V-Q)} = -\int_0^1 e^{-(V-rQ)} Q\, dr.$$

The sign plays no role in the bound. At both radii, the integrand is bounded by the corresponding stable factor times the localized coefficient estimate just given. The difference of the two $W$'s is bilinear and has at least one $Q$. The scalar source coefficient in $Q$ has absolute size $O_L(\omega(h)^{-1}\tau_K)$, since $h_\sigma h_\varnothing$ is constant. The mixed constant is field-independent and does not enter $W$. This gives the inputs for the circle estimate in Lemma 9.5.

In Map 3 the field-independent part of $\widehat{V}$ drops out of $\theta\widehat{V} - \mathbb{E}_+\theta\widehat{V}$ and may be factored exactly from the exponentials. The other source terms are linear and their fluctuation derivatives gain $\ell_\varnothing/h_\varnothing$ at $h$. The full mixed covariance contribution at $h$ is at most

$$h_\sigma^2 |\lambda_a \lambda_b|\, |\Gamma_+(a,b)| \leq C_L \chi G^{1/2}.$$

The corresponding $\ell$ bound is $C_L \chi G^2$. The vacuum and mixed constants of the perturbative correction thus satisfy the same bounds as its other sectors. This gives the bounds of Lemmas 9.10–9.13 on the one-block difference, of size $O_L(\chi^{1/2} G)$ at $\ell$ and $O_L(\chi^{1/2} G^{1/4})$ at $h$. For the Gaussian estimate on products of segment bounds, each block factor uses its own scalar multiple of the same full fluctuation field. Pointwise,

$$H_B(\varphi + r\zeta) \leq H_B(\varphi)^{1/2} e^{\kappa L^{-4s} h_\varnothing^{-2} \sum_B \zeta_x^2}, \qquad \widetilde{G}_B(\varphi + r\zeta) \leq \widetilde{G}_B(\varphi)^2 \widetilde{G}_B(\zeta)^2.$$

These bounds hold uniformly in $r \in [0,1]$, as does the identical growing-regulator bound for $G_B$. The extra Gaussian factor and fixed fluctuation polynomials are bounded by a fixed power of $G_B(\zeta)$ times a constant. Lemma C.2 gives the product estimate of Lemma 9.7 with the suprema inside the expectation. The covariance reapportionment in Map 4 cancels the one- and two-block second-order terms as in Lemma 5.5. Using the printed covariance of $V$, instead of $\widehat{V}$, changes this cancellation by at most $C_L\chi G\tau_K$ at $\ell$ and $C_L\chi G^{1/2}\omega(h)^{-1}\tau_K$ at $h$, by bilinearity, centering, and the direct bound on $Q$. Both fit the input size of Lemma 9.17. Map 5 and the corrected final change use the same circle estimate as above. At $K=0$, the rough output is $O_L(\chi^{3/2}G^3)$; on the enlarged complex ball it is $O_L(\chi^{3/2}G^3+\tau_K)$. These are bounds on the full joint complex neighborhood, with constants depending only on the fixed scale and regulator choices and the stated finite collection of covariance upper bounds. In particular no lower bound on the covariance amplitude, and no negative power of $\lambda$, is used in these analytic estimates. Here is the holomorphy check, including the order of constants. On the extended ball the extracted $Q$ has $\ell$-norm $O_L(c_LG^{9/4})$ and $h$-norm $O_L(c_L)$. The reference quartic has fixed positive normalized $h$-size proportional to $k_0^4$. Choose $c_L$ small relative to that size and to the spare regulator margins. The quartic part of $\mathrm{Re}(V-tQ)$ then retains a fixed positive coefficient for every $t\in[0,1]$; lower-degree terms are absorbed by the same stability and regulator estimates. The source algebra is finite and nilpotent, so bounded source coefficients add only fixed-degree field polynomials. The body of every zero-field factor inverted by localization is bounded away from zero, and its nilpotent inverse is a finite sum. Finite-volume circle products are finite sums. Gaussian expectation is a bounded linear map between the indicated regulator spaces, with a common integrable majorant on smaller complex balls. These facts prove Fréchet holomorphy of the composed maps and justify differentiation under the Gaussian integrals there. This is the whole-neighborhood input to Lemma 2.3; it is logically separate from the derivative contraction at $K=0$ verified above. Choose the scale so that this last contraction has the desired strict margin, then choose the enlarged radius $\epsilon$, and only then make $R$ large. Small $G$ alone would not absorb the term $C\epsilon$ in that lemma. The bulk polynomial remainder in the original chart has the explicit formula (8.7), with $Q$ linear in $K$; its norm is bounded by $C_L\|K\|$, analytically in the bulk potential. The argument of Lemma 2.3 and the small derivative through Park Map 5 now give the derivatives and contraction assumed in Proposition 11.1.

For the perturbative bulk part use the scalar Wick formulas and normal-form change in [7, Section 3.2, especially (3.16)–(3.26) and (3.33) onwards], together with the covariance-moment estimates of [8, Sections 3–4]. These are Wick-polynomial identities for the finite-range covariance, with the scalar combinatorial coefficients, and make no use of the nearest-neighbor resolvent in the computation. For clarity, the estimates on their coefficients can be read directly from the covariance. At physical scale $s$, with $w$ the cumulative covariance before the step, finite range and the slice estimates give

$$\|w\|_1 \le C\lambda L^{2s}, \qquad \sum_x |x|^2|w(x)| \le C\lambda L^{4s}, \qquad \sum_x |x|^2 w(x)^2 \le C\lambda^2 L^{2s}.$$

For the last bound sum the pointwise inequality $|w(x)| \le C\lambda(1\vee|x|)^{-2}$ over its support, with the origin-scale bound on the first block. The unscaled mass coefficient $\eta_j'$ is a constant times $\Gamma_+(0)$; its dimensionless size is $O(\lambda\chi)$. The $\beta,\omega,\pi$ coefficients use differences of $w^2$ or $w\Delta w$, hence have size $O(\lambda^2\chi)$. For $w\Delta w$ move $\Delta$ to $\Gamma_+$ in the mixed term by summation by parts, then use the $\ell^1$ bound on $w$. The $\theta,\xi$ coefficients have three covariance factors and have size $O(\lambda^3\chi)$. For the potentially logarithmic part of $\xi$, combine $\delta\sum w^3$ with $-3\Gamma_+(0)\sum w^2$. The mixed term is then $3\sum w(x)^2[\Gamma_+(x)-\Gamma_+(0)]$. Symmetry and the second differences bound the

bracket by $C\lambda\chi L^{-4s}|x|^2$; the last displayed moment bound applies. The other terms use the same three moments and the pointwise $\Gamma_+$ bound. Positivity and the lower bubble bound come from Proposition 6.2; its cutoff may differ from Park's by a bounded number of scales, which changes only fixed constants.

The coordinate transformations in BBS have quadratic coefficients bounded by the normalized moments $L^{-2s}\|w\|_1$ and $L^{-4s}\sum|x|^2|w(x)|$, hence $O(\lambda)$. Their inverse is analytic on a fixed small neighborhood by uniform smallness of the quadratic derivative. The normal-form error is analytic and starts at degree three. All its coefficients vanish when the covariance moments entering the perturbative map and the normal-form changes are zero. The fresh covariance coefficients decay faster than $\chi^{3/2}$ past the mass scale by the slice bounds. The normalized cumulative moments also do: summing the slices gives

$$L^{-2s}\|w\|_1 \lesssim \lambda L^{-2(j-J)_+}, \qquad L^{-4s}\sum_x |x|^2|w(x)| \lesssim \lambda L^{-4(j-J)_+}.$$

Here one chooses the mass-decay power larger than two. The coefficients of the perturbative map, the quadratic changes, and their inverses are analytic functions of these finitely many normalized moments and fresh covariance coefficients, on a uniform small potential disc. Their degree-three-and-higher part vanishes at zero covariance data, so it is bounded on that disc by a constant times the maximum size of these data. This supplies the $\chi^{3/2}$ factor on a fixed disc by taking $L$ large. Taylor and Cauchy bounds on the smaller potential discs give the claimed cubic and first derivative estimates. This establishes the full normal-form bounds. □

## 11 A weighted recursion with a small bubble coefficient

The following lemma concerns the recursion once the required one-step bounds have been proved. It separates those bounds from the issue of uniformity as the bubble coefficient tends to zero.

**Proposition 11.1** (Uniform critical tuning for the normal-form recursion)**.** *Let $0 < \lambda \le 1$, $x_0 > 0$, and $\chi_j = 2^{-(j-J)_+}$ with $J \in \mathbb{N}\cup\{\infty\}$. Suppose*

$$x_{j+1} = x_j - B_j x_j^2, \qquad 0 \le B_j \le C\lambda^2\chi_j, \qquad B_j \ge c\lambda^2 \quad (j \le J).$$

*Fix $d_0 = 1/4$, and put $f_j = \chi_j^{3/2}x_j^3 + Ud_0^j$. Suppose the controlled recursion, in normal-form coordinates, is*

$$\begin{aligned} g_{j+1} &= g_j - B_j g_j^2 + e_j^g(v_j, K_j),\\ z_{j+1} &= z_j - \theta_j g_j^2 + e_j^z(v_j, K_j),\\ \mu_{j+1} &= L^2\mu_j + \eta_j g_j - \xi_j g_j^2 - \omega_j g_j\mu_j - \pi_j g_j z_j + e_j^\mu(v_j, K_j),\\ K_{j+1} &= P_j(v_j, K_j), \qquad v_j = (g_j, z_j, \mu_j), \end{aligned}$$

*with*

$$|\eta_j| \le C\lambda\chi_j, \qquad |\theta_j| + |\xi_j| \le C\lambda^3\chi_j, \qquad |\omega_j| + |\pi_j| \le C\lambda^2\chi_j.$$

*Assume the following estimates hold on $|g_j - x_j| \le \varepsilon x_j$, $|z_j| + |\mu_j| \le M\lambda x_j$, $\|K_j\| \le M_K f_j$, for arbitrary fixed $M, M_K$ and all sufficiently small parameters. The constant in the last two estimates*

*may depend on these fixed constants:*

$$\begin{aligned}
|e_j(v,K)| &\le C(\chi_j^{3/2}x_j^3 + \|K\|),\\
\|P_j(v,0)\| &\le C\chi_j^{3/2}x_j^3, \qquad \|P_j(v,K)-P_j(v,K')\| \le \kappa\|K-K'\|, \quad \kappa < 1/8,\\
|e_j(v,K)-e_j(v',K')| &\le C(f_j|v-v'|/x_j + \|K-K'\|),\\
\|P_j(v,K)-P_j(v',K)\| &\le Cf_j|v-v'|/x_j.
\end{aligned}$$

*Here the constants in the first two lines are uniform over the indicated coordinate boxes. If*

$$\lambda \longrightarrow 0, \qquad \frac{x_0}{\lambda^3} \longrightarrow 0, \qquad \frac{U}{\lambda x_0} \longrightarrow 0,$$

*then for every $K_0$ of norm at most $U$ the recursion has a solution with $g_0 = x_0$, $z_j, \mu_j \to 0$, and*

$$|g_j - x_j| \le \varepsilon x_j, \qquad |z_j| + |\mu_j| \le C\lambda\chi_j x_j, \qquad \|K_j\| \le Cf_j.$$

*The same statement, with both terminal coordinates set to zero at the last local scale, holds on a finite horizon with the same constants. If the inputs are continuous in external parameters in a fixed Banach-space chart, so are the two initial tuned couplings.*

*Proof.* Apply Lemma 2.2 for the reference sequence, the adjacent envelope ratios, and the forward Green function. In particular (5) gives

$$\sum_{i\ge j} f_i \le C(\lambda^{-2}x_j^2 + Ud_0^j), \qquad \frac{Ud_0^j}{x_j} \le C\frac{U}{x_0}. \tag{35}$$

Monotonicity of the envelope also gives

$$\sum_{i\ge j} L^{-2(i-j+1)} f_i \le C(x_j^3 + Ud_0^j), \qquad f_j/f_{j+1} \le 4. \tag{36}$$

Write $t_j = g_j - x_j$. On sequences with $t_0 = 0$ and the prescribed $K_0$, define a fixed-point map by the following Green formulas. The output $t$ solves the forward linear equation with coefficient $1 - 2B_jx_j$ and forcing $-B_jt_j^2 + e_j^g$. The output $z$ is

$$z_j = \sum_{i\ge j}(\theta_i g_i^2 - e_i^z).$$

Writing $F_i^\mu = \eta_i g_i - \xi_i g_i^2 - \omega_i g_i\mu_i - \pi_i g_i z_i + e_i^\mu$, the output mass is

$$\mu_j = -\sum_{i\ge j} L^{-2(i-j+1)} F_i^\mu.$$

All right-hand sides use the input sequence. The output activity is $K_{j+1} = P_j(v_j, K_j)$. For a finite horizon replace both backward sums by sums over the local update indices; then the two terminal coordinates are exactly zero. Every estimate above remains valid for these truncated sums.

Use the maximum of the four weighted suprema with denominators

$$\varepsilon x_j, \quad M\lambda x_j, \quad M\lambda x_j, \quad M_K f_j.$$

The closed product box is complete; using separate bounds for $z$ and $\mu$ is legitimate by replacing $M$ in the hypotheses by $2M$. Choose $M_K$ large enough that the activity box is preserved: its

normalized output is at most $4C/M_K + 4\kappa < 1$. Choose $M$ large compared with the uniform coefficients in the leading $\theta$ and $\eta$ forcings. These forcings have size at most $C\lambda x_j$. All other backward forcings have size $o(\lambda x_j)$ for these fixed boxes. In fact, set

$$a = \frac{x_0}{\lambda^2} + \frac{U}{x_0}, \qquad b = \frac{x_0}{\lambda^3} + \frac{U}{\lambda x_0}.$$

Then $a = \lambda b \to 0$ and $b \to 0$. The activity and cubic errors in the $z$ sum, divided by $\lambda x_j$, are $O(b)$ by (35); those in the mass sum are $O(x_0^2/\lambda + U/(\lambda x_0)) = o(1)$. The remaining perturbative mass terms are smaller as well. The normalized forward output is at most $C\varepsilon + Ca/\varepsilon$. Thus a sufficiently small fixed $\varepsilon$, followed by small parameters, preserves all four boxes.

Here are the difference estimates that give uniform contraction. Let $D_t, D_z, D_\mu, D_K$ be the four normalized input distances. Constants in an $o(1)$ below may depend on the now fixed box sizes and on $\varepsilon$, but are independent of $J$ and the terminal scale. The forward Green estimate and the Lipschitz hypothesis on $e$ give

$$D_t^{\mathrm{out}} \le (C\varepsilon + o(1))D_t + o(1)(D_z + D_\mu + D_K).$$

Indeed the error contributions are bounded by fixed multiples of $a$, $M\lambda a/\varepsilon$, and $M_K a/\varepsilon$; all tend to zero. For the backward coordinates the leading perturbative dependence on $t$ is $C\varepsilon/M$, whereas the error dependence uses $b \to 0$. Hence

$$\max(D_z^{\mathrm{out}}, D_\mu^{\mathrm{out}}) \le (C\varepsilon/M + o(1))D_t + o(1)(D_z + D_\mu + D_K).$$

For example the $K$-to-$z$ coefficient is at most $C(M_K/M)b$, which is the potentially delicate backward sum when the bubble coefficient is small. Finally, the activity estimates and $f_j/f_{j+1} \le 4$ give

$$D_K^{\mathrm{out}} \le 4\kappa D_K + \frac{4C_{M,M_K}}{M_K}\big(\varepsilon D_t + M\lambda(D_z + D_\mu)\big).$$

Choose $\varepsilon$ after $M, M_K$ so that the fixed terms in all these row sums are strictly less than one; then make $\lambda, b$ small. The maximum row sum is bounded by some $q < 1$ independent of the mass scale and terminal scale. This proves the claimed uniform contraction, rather than relying on a contraction in any unweighted coordinate norm. Its unique fixed point in the box solves the recursion.

For $j > J$, the backward forcing bounds improve by $\chi_j$. Here $x_i \asymp x_j$ for $i \ge j$, $\sum_{i\ge j} \chi_i x_i^p \le C\chi_j x_j^p$, and

$$U d_0^j \le C(U/x_0)\chi_j x_j.$$

For the last estimate use $\chi_j \ge 2^{-j}$, $2d_0 < 1$, and the lower bound on $x_j$. The same statements for $\chi^{3/2}$ and the geometrically weighted mass sums follow immediately. Inserting these bounds in the Green formulas gives $|z_j| + |\mu_j| \le C\lambda\chi_j x_j$. For $j \le J$ the previously proved bound suffices. Consequently both terminal coordinates tend to zero, including when $J = \infty$, since then $x_j \to 0$.

For continuity, iterate this uniformly contracting map starting from the reference bulk sequence and an admissible activity sequence with the prescribed initial value. At each fixed index, finite-coordinate operations are continuous by hypothesis. The infinite backward mass sum has a geometric majorant. The backward $z$ sum has a geometric tail near a fixed finite mass scale. As the mass scale tends to infinity, its tail starting at $i$ is bounded by $C(\lambda x_i + U d_0^i)$, and for $J \ge i$ $x_i \le (x_0^{-1} + c\lambda^2 i)^{-1}$. This tends to zero locally uniformly in the external parameters with $\lambda, x_0 > 0$. Thus fixed coordinates of every finite iterate are continuous. The common contraction constant makes their convergence uniform at each fixed index, proving continuity of the two initial tuned couplings. The same reasoning is immediate on a finite horizon. □

## 12 Matching the tuned trajectory to the Ising parameters

Use the notation of the exact field change and write $A_{\mathrm{sc}} = g^{-a_*}$, with $1/12 < a_* < 1/10$, for the amplitude factor. It is distinct from the kernel $A = \beta P$. In the variable $\psi = (R/A_{\mathrm{sc}})u$, the initial quartic coefficient, in the convention $g_0\psi^4/4$, and the covariance prefactor are

$$G_0 = \frac{A_{\mathrm{sc}}^4 g}{3}, \qquad \lambda = A_{\mathrm{sc}}^{-2}.$$

Choose the initial block side $b$ to be a power of $L$ with $MR \le b < MLR$, using the fixed sufficiently large aspect from the connected initialization. The physical fluctuation scale in $u$ is a fixed multiple of $A_{\mathrm{sc}}R^{-2}$, and its large-field scale is a fixed multiple of $b^{-1}$. Indeed the large-field scale in $\psi$ is $G_0^{-1/4}b^{-1}$, and $A_{\mathrm{sc}}G_0^{-1/4}/R$ is fixed. Thus Proposition 8.1, with the large-set factor chosen for the Park norm, gives the initial bound

$$U = C(A_{\mathrm{sc}}^6 g^2 + G_0^{9/4} g^{1/2})$$

on the full polymer norm. Thus the smallness ratios required by Propositions 10.1 and 11.1 are

$$\begin{aligned} U/G_0^{9/4} &\le C(g^{3a_*-1/4} + g^{1/2}) = o(1), \ G_0/\lambda^3 &&= \tfrac{1}{3} g^{1-10a_*} = o(1), \\ U/(\lambda G_0) &\le C(g^{1-4a_*} + g^{7/4-7a_*}) = o(1). \end{aligned}$$

The norm estimate includes the source sectors by the initialization check in Proposition 10.1.

**Proposition 12.1** (Physical matching at positive auxiliary mass)**.** *For sufficiently large $R$, fix $0 < m^2 \le g$ and a sufficiently large torus of side $bL^N$. There are parameters $\beta_{m,N}, z_{m,N}$ in the exact Ising representation such that*

$$|\beta_{m,N} - (1 - m^2)| + |z_{m,N}| \le Cg$$

*and the bulk flow through scale $N-1$ satisfies the controlled bounds of Proposition 11.1, with terminal normal-form $z, \mu$ equal to zero. The constants are uniform in $m^2$ and $N$.*

*Proof.* Lift the exact torus Hessian as in Section 7 and construct the covariance slices for this lift. Use the prescribed torus periodization for the local steps. Choose $b/R$ large enough that the support of each such slice is less than $bL^{j+1}/4$ at the step from $j$ to $j+1$ and its support does not wrap before taking periodization. Proposition 10.1 verifies the exact map and normal-form hypotheses on these local steps, uniformly in the lift. In particular, Proposition 6.2 gives a bubble coefficient comparable to $\lambda^2$ up to the mass scale. The fixed multiplicative constant in the mass cutoff can be chosen to match the lower bound in that proposition. The finite-range bounds give the required decay afterwards.

For each covariance parameter pair in a physical box, fix $g_0 = G_0$ and the exact initial activity and solve the finite horizon recursion of Proposition 11.1 through scale $N-1$. In that proposition truncate both backward Green sums at the last local update, so $z_{N-1} = \mu_{N-1} = 0$. The sum estimates and the contraction are unchanged. The two returned initial couplings $(\mu_0^*, z_0^*)$ obey

$$|\mu_0^*| + |z_0^*| \le C\lambda G_0$$

by the smallness ratios above. They are continuous in the external parameters at fixed $N$. For this, iterate the uniform contraction starting with the reference sequence. At each finite iteration,

the coordinates and the relevant field jets are continuous by the local finite sums and Gaussian integrals in the step, using the regulator bounds to dominate the integrals. The initial activities have this continuity from their finite exact formulas. At a fixed scale the contraction makes these iterates converge uniformly in the parameters for every coordinate and field-jet evaluation under consideration. This also proves continuity when the reference couplings, and hence the norms of the activity spaces, vary with the covariance parameters.

Let $\widehat{E}_N(0)$ denote the row sum of the reference Hessian difference on the torus. The physical residual mass is

$$r_N(\beta, m^2) = \frac{(1-\beta q)(1-m^2-\beta)}{\beta p} - (1-\beta q)^2 \widehat{E}_N(0).$$

The cavity estimate is uniform for $\beta$ in a fixed small complex neighborhood of the real reference interval. This follows by treating its increment as a single coupling variable in the same recursion, with coupling change $(\delta\beta)Q$. Cauchy's estimate therefore gives $\partial_\beta \widehat{E}_N(0) = O(g)$ uniformly. Consequently $\partial_\beta r_N = -1 + O(g)$ when $|\beta - 1| + m^2 = O(g)$. For fixed $m^2$, one can use $r$ in place of $\beta$ on an interval $|r| \le C_1 g$, taking a large enough constant in the allowed $\beta$ interval about $1 - m^2$.

Before the first Gaussian integration the cumulative covariance is zero, so the quadratic normal-form change of coordinates is the identity. The bare local mass and kinetic couplings in that chart are

$$\mu_0^{\text{bare}} = \frac{b^2}{R^2\lambda} r, \qquad z_0^{\text{bare}} = -\frac{z}{\lambda}.$$

Here $z$ is the coefficient used in the covariance absorption of $zR^2\mathcal{D}$, so the residual interaction has kinetic coefficient $-zR^2$ in $u$. On the physical box $|r|, |z| \le C_1 g$ consider the map

$$(r, z) \longmapsto \left( \frac{R^2}{b^2} \lambda \mu_0^*,\ -\lambda z_0^* \right),$$

where the starred quantities are computed at the covariance and initial activity of that external pair. Its image has size $O(\lambda^2 G_0) = O(g)$. All constants in this estimate can be chosen before $C_1$: the covariance estimates need only a fixed small upper bound on the weighted Hessian norm and on $|z|$, and the large-field initialization absorbs the local quadratic factors when $C_1 g^{1/2}$ is small. Choose $C_1$ large, then take $R$ sufficiently large. The map is a continuous self-map of the closed box and has a fixed point by Brouwer's theorem. At that point both bare local couplings agree with the tuned initial values, while the quartic and activity were already matched. This proves the proposition. □

## 13 The last covariance and the physical endpoint

The last Gaussian integral can be handled directly when the torus side is much larger than the mass length. Let its side be $D_N = bL^N$ and stop the local RG at scale $N-1$, with block side $D_N/L$. There are then exactly $L^4$ blocks. Denote the remaining periodized covariance by $C_{\text{tail}}$. For a piece with spatial scale $RL^j$, each periodic point receives at most $C(1 + RL^j/D_N)^4$ translates. Thus the derivative bounds for the piece, summed over the tail, give the same bounds at the last block scale, with arbitrary postmass decay after decreasing the exponent in the original bound. In particular all the rescaled derivative bounds of $C_{\text{tail}}$ divided by the square of the fluctuation field scale tend to zero for each fixed $m^2 > 0$ as $N \to \infty$. The Gaussian regulator-moment argument of Park's Lemma C.2 applies to this fixed number of blocks. More explicitly, the block Sobolev inequality bounds the sum of their regulator exponents by a positive quadratic form $Q(\zeta)$. The

normalized derivative bound gives $\mathbb{E}Q(\zeta) = o(1)$. The largest eigenvalue of the corresponding Gaussian quadratic form is at most its trace, so its determinant formula bounds every fixed exponential moment uniformly for large $N$. It uses the derivative bounds rather than finite range at this last integration. Consequently the shifted Gaussian expectation of any last-scale regulated activity has its zero-field Taylor seminorm bounded by a fixed multiple of its regulated input norm.

More precisely, assume the local controlled RG flow through scale $N-1$ is constructed with the uniform envelopes of Proposition 11.1, including any indicated observable sector. Write $\mathcal{Z}_{N-1} = I_{N-1} \circ K_{N-1}$ for the weight after removal of its field-independent factor. Since there are only $L^4$ blocks, the local stability estimate $\|I_{N-1} - 1\|_{\ell,G} = O_L(x_{N-1})$, the polymer bound, and the last Gaussian bound give

$$Z_{\mathrm{tail}}(0) = 1 + O_L(x_{N-1} + f_{N-1}), \qquad \|Z_{\mathrm{tail}}\|_{T_\ell(0)} \le C_L, \qquad Z_{\mathrm{tail}} = \mathbb{E}_{C_{\mathrm{tail}}} \theta \mathcal{Z}_{N-1}.$$

The first estimate here refers to the bulk component. In particular that component is bounded below by $1/2$ for a small initial coupling. The estimates follow directly by expanding the finite circle product around the product of unit singleton factors, using its product bound and the Gaussian moment estimate for the product of regulators. With two fixed source sites, the last contribution to the source-pair coefficient of $\log Z_{\mathrm{tail}}(0)$ is $O_L(\ell_{N-1,\sigma\sigma}^{-1})$, which tends to zero at fixed $m^2 > 0$ because $x_{N-1}$ stays bounded below and the source-pair scale increases exponentially after coalescence.

The bulk Taylor bound and the lower bound at zero give

$$|\Lambda_N|^{-1} |D^2_{\mathbf{1},\mathbf{1}} \log Z_{\mathrm{tail}}(0)| \le C_L D_N^{-2}.$$

In (32), $K + zR^2\mathcal{D}$ has zero row sum and $\widehat{M}_m(0) = m^2$, so the rescaled covariance has row sum

$$\widehat{C}^{\Lambda_N}(0) = \frac{\lambda R^2 \beta p}{(1 - \beta q) m^2}.$$

Thus (9), with $\mu = m^2$, gives at the finite-volume tuned parameters

$$\chi_\sigma^{\Lambda_N}(\beta_{m,N}) = \frac{1 - \beta_{m,N} q}{\beta_{m,N} p \, m^2} - \frac{1}{\beta_{m,N} p} + O_{R,m}(D_N^{-2}). \tag{37}$$

The error is bounded by $C_L \lambda R^2 (m^2)^{-2} D_N^{-2}$ uniformly over the tuned parameters in the physical box.

Apply Proposition 2.5 to the off-diagonal interaction of $D$, with $\mu = m^2$ and the error exponent $s = 2$. In its notation

$$a(\beta) = \frac{1 - \beta q}{\beta p}, \qquad b(\beta) = \frac{1}{\beta p}.$$

The physical parameter box keeps these functions continuous and $a(\beta)$ bounded away from zero. Thus for each positive mass and each tuned subsequence converging to $\beta_m$, its periodic spin laws converge locally to the subcritical free state and their susceptibilities converge to

$$\chi_\sigma^{\mathrm{f}}(\beta_m) = \frac{1 - \beta_m q}{\beta_m p \, m^2} - \frac{1}{\beta_m p}. \tag{38}$$

The same criterion gives

$$\beta_m \longrightarrow \beta_c^D \qquad (m^2 \downarrow 0) \tag{39}$$

and convergence of the free correlations at each fixed pair to their critical values. The conversion to the original box temperature uses $\beta_c^D = (1 + \delta_R)\beta_c^{\mathrm{box}}$.

## 14 The two-source flow and the massive thermodynamic limit

Let the block length at the $j$-th integration be $b_j = bL^j$. Use the reference quartic $x_j$ of Proposition 11.1, the covariance prefactor $\lambda = A_{\mathrm{sc}}^{-2}$ from the field rescaling, and the activity envelope

$$f_j = \chi_j^{3/2} x_j^3 + U d_0^j, \qquad x_j \asymp \frac{x_0}{1 + \lambda^2 x_0 \min(j, J)}.$$

Assume the controlled bulk trajectory and the extended one-step estimate with the source norms before coalescence. We use the extended estimate during the transient. Afterwards we use the ordinary-size activity bound for the same exact map. We use pairs whose coalescence scale exceeds the transient length $J_* + 2$. To obtain a uniform $J_* = O(|\log g|)$, iterate the contraction inequality for $K$. At $K = 0$ the forcing is at most $(C_{\mathrm{RG}}/2)\chi_{j+1}^{3/2} x_{j+1}^3$, taking $C_{\mathrm{RG}}$ large enough for the actual chart change, as in the proof of Park Proposition 9.1. For small $x_0$ the consecutive ratios of $\chi_j^{3/2} x_j^3$ exceed $q = 1/3$. Taking $\kappa < 1/8$ gives the bound

$$\|K_j\| \le \tfrac{4}{5} C_{\mathrm{RG}} \chi_j^{3/2} x_j^3 + U d_0^j.$$

The ratio of the last term to $\chi_j^{3/2} x_j^3$ is at most $(U/x_0^3)(d_0/q)^j$, and $U/x_0^3$ is at most a fixed power of $1/g$. Thus the trajectory is in the ordinary Park activity ball after $J_*$, uniformly in the mass cutoff. This argument applies in the full source norm as long as the source couplings stay in a fixed bounded domain, which is verified below.

Write $\nu_j, g_j^{\mathrm{loc}}$ for the mass and quartic in the original polynomial coordinates. They obey

$$b_j^2 |\nu_j| \le C \chi_j x_j, \qquad b_j^2 |\nu_{j+1} - \nu_j - \eta_j' g_j^{\mathrm{loc}}| \le C(\chi_j x_j^2 + f_j), \qquad \eta_j' = 3\Gamma_{j+1}(0).$$

The first bound follows from Proposition 11.1 and the inverse normal-form mass change $b_j^2 \nu_j = \mu_j + O(\mu_j^2)$. For the second, subtract the linear Wick mass increment in the original perturbative formula. The remaining quadratic terms have size $O(\chi_j x_j^2)$ by the covariance-moment estimates and cancellation in the proof of Proposition 10.1. The exact remainder contributes $O(f_j)$ in the normalized mass coordinate. The covariance bounds also give

$$\|\Gamma_{j+1}\|_\infty \le C\lambda \chi_j b_j^{-2}, \qquad \left| \sum_y \sum_{i<j} \Gamma_{i+1}(y) \right| \le C\lambda b_j^2.$$

The second bound uses the range of each slice and a geometric sum over the slice lengths. The smallness hypotheses in Proposition 11.1 imply $x_0/\lambda^2 + U/x_0 = o(1)$.

**Proposition 14.1** (Observable consequence of the controlled trajectory)**.** *Let $m > 0$ and let the source pair coalesce after $J_* + 2$. On tori with side tending to infinity, the two-point function of the rescaled field tends to the infinite-volume source sum $q_\infty(a, c)$ if the finite-scale maps converge locally to their infinite-volume versions. This sum tends to zero as $|a - c|$ tends to infinity. If the trajectory and covariance data tend locally along a massive sequence to massless data, then the limits $q_\infty(a, c)$ converge at each fixed pair to the massless source sum. When the latter is a translation-invariant two-point function and the massless Gaussian covariance $C$ satisfies $C(x) \sim C_0 |x|^{-2}$ for $C_0 > 0$, it has the asymptotic form $q_\infty(a, c) \sim v_\infty^2 C_0 |a - c|^{-2}$ for a constant $v_\infty > 0$.*

*Proof.* Use two sources linear in $\psi$, both initially of coefficient one. In the physical $u$ coordinate their insertion is $\exp[t_a(R/A_{\mathrm{sc}})u_a + t_c(R/A_{\mathrm{sc}})u_c]$. Write $v_{a,j}$ for the one-source coefficient, and

$\widehat{v}_{a,j}$ for its value after the small-set extraction $Q_j$. The graded structure of the exact RG maps means the projection to $\mathbb{R}[t_a]/(t_a^2)$ commutes with the maps. To estimate that projection at any finite scale $j$, place the second source at distance more than $48b_j$ and set its coefficient zero. Thus the fluctuation source weight in that sector is $x_j b_j$ for every step under consideration. Because a linear field has fluctuation radius comparable to $b_j^{-1}$, the localization bound at zero gives

$$|\widehat{v}_{a,j} - v_{a,j}| \le C f_j / x_j.$$

The same bound holds for the actual pair of sources, since the one-source map is independent of the second source. This use of an unused source avoids the decreased fluctuation weight for a pair after its coalescence.

Put $w_j^{(1)} = \sum_y \sum_{i<j} \Gamma_{i+1}(y)$ and define

$$d_j = (\nu_j + \eta'_j g_j^{\mathrm{loc}}) w_{j+1}^{(1)} - \nu_j w_j^{(1)}.$$

The one-source perturbative formula [5, (A.7)–(A.10)], applied to $V_j - Q_j$, gives

$$v_{a,j+1} = (1 - d_j) v_{a,j} + e_{a,j}, \qquad |e_{a,j}| \le C f_j / x_j$$

as long as $|v_{a,j}|$ stays in a fixed bounded interval. In fact the bulk extraction changes $\nu_j, g_j$ by at most $C f_j / b_j^2, C f_j$, respectively, so it changes $d_j$ by at most $C f_j$. One can also obtain the one-source perturbative formula by contracting the linear source with the quadratic bulk term after the quartic tadpole has been added.

The definition of $d_j$ telescopes:

$$\sum_{j=k}^{n-1} d_j = \nu_n w_n^{(1)} - \nu_k w_k^{(1)} + \sum_{j=k}^{n-1} (\nu_j + \eta'_j g_j^{\mathrm{loc}} - \nu_{j+1}) w_{j+1}^{(1)}.$$

The right-hand side has a limit as $n$ tends to infinity. Indeed the boundary term at $n$ vanishes, and the remaining sum has absolute bound $C \sum_{j \ge k} (\chi_j x_j^2 + f_j)$. The support and slice bounds, together with the mass estimate, also give $|d_j| \le C \chi_j x_j$, hence $\sum_{j \ge k} d_j^2 \le C \sum_{j \ge k} \chi_j x_j^2$. Finally $\sum_{j \ge k} |e_{a,j}| \le C \sum_{j \ge k} f_j / x_j$. The reciprocal comparison for $x_j$ bounds the sums at $k = 0$ by $C(x_0/\lambda^2 + U/x_0)$. They vanish as $k$ tends to infinity, uniformly in $a$, and uniformly as the mass cutoff tends to infinity. For example, $\sum_{j \ge k} \chi_j x_j^2 \le C \chi_k x_k / \lambda^2$, by summing $x_j^2$ before the cutoff and the geometric tail after it. The factors $\prod_{i<j} (1 - d_i)$ are uniformly close to one and converge, by the expansion of $\log(1 - d_i)$ and the telescoping estimate. Variation of constants gives the same conclusions for $v_{a,j}$, with a limit $v_{a,\infty}$ uniformly bounded above and bounded away from zero. This also closes the bounded-interval assumption by induction. In particular $\widehat{v}_{a,j}$ converges to the same limit, uniformly in $a$.

Let $k$ be the coalescence scale of $a, c$. The mixed constant in the effective potential is extracted from small polymers. Its localized density is supported at the marked sites, so summing it over the volume introduces no volume factor. The part from $Q_j$ is zero for $j < k$ and has absolute value at most $C f_j / (x_j^2 b_k^2 4^{j-k})$ for $j \ge k$, by the two-source fluctuation weight in (2.19) of [4]. The remaining part is exactly the Gaussian contraction of the two linear sources. The mixed part of the interior $W_j$ is zero because this contraction is constant and is removed by localization. These claims also follow directly from the mixed specialization in [5, Lemma 8.4]. With the sign convention that $q_\infty$ is the coefficient in the log partition function, we obtain

$$q_\infty(a, c) = \sum_{j \ge 0} \widehat{v}_{a,j} \widehat{v}_{c,j} \Gamma_{j+1}(a - c) + E(a, c), \qquad |E(a, c)| \le \frac{C}{b_k^2} \left( \chi_k^{3/2} x_k + \frac{U d_0^k}{x_k^2} \right).$$

For the last inequality use the monotonicity of $\chi_j, x_j$ and $x_k/x_j \le C(1+\lambda^2 x_0(j-k))$ in the sum. Covariance support makes the first sum start no earlier than $k - O(1)$. Together with $\|\Gamma_{j+1}\|_\infty \le C\lambda\chi_j b_j^{-2}$, this proves decay at positive mass. At zero mass the error is $o(b_k^{-2})$ as $k$ tends to infinity. Replacing the source coefficients in the covariance sum by their limits also costs $o(b_k^{-2})$, uniformly in the sites.

For the thermodynamic limit, use the last Gaussian integral estimate of Section 13. Spin-flip symmetry sets the one-source coefficients at zero field to zero. The last mixed contribution to the log partition function is therefore $O_L(\ell_{N-1,\sigma\sigma}^{-1})$, which tends to zero at fixed positive mass and fixed pair. The finite-scale increments converge by locality. At each fixed pair their tails are uniformly summable, including as the mass goes to zero. For instance the mixed remainder at step $j \ge k$ is dominated by a constant times $4^{-(j-k)}(1+\lambda^2 x_0 j)^2$. This proves both the thermodynamic limit and the claimed fixed-pair continuity of the source sum. The near pairs are not needed in these arguments. The susceptibility comparison uses the bulk uniform-source derivative of the same last Gaussian integral, independently of the two-source flow.

At zero mass the representation and the bounds just proved give the hypotheses of Proposition 2.6, with $s = 2$. It gives the stated asymptotic when the two-point function is translation invariant. □

## 15 Critical two-point function for the box interaction

*Proof of Theorem 1 for $d = 4$.* At fixed $m^2 > 0$ use Proposition 12.1 on tori tending to infinity. The lifted reference Hessians converge locally and in the weaker weighted kernel norm of Section 7. Choose a subsequence on which the tuned $\beta_{m,N}, z_{m,N}$ converge. The initial local activities converge on each fixed polymer by the finite exact forest formulas. Since each fixed-scale step depends only on finitely many local polymer inputs, its coordinates converge by the continuity of the Gaussian step in the covariance and the activities. Thus the controlled coordinate bounds pass to the corresponding infinite-volume local trajectory. The direct final Gaussian estimate and the endpoint argument of Section 13 identify the limiting spin law as the infinite-volume subcritical law at $\beta_m$ and give (38). In particular $\beta_m \to \beta_c^D$ as $m^2 \downarrow 0$.

Along a sequence of such positive masses tending to zero choose a subsequence with $z_m \to z_*$. The cavity Hessian, the finite-range pieces at each fixed scale, and the local initial activities converge again. Hence every fixed-scale coordinate, including the sources, converges. Proposition 14.1 allows this limit in the two-point function at every fixed pair to which the pair flow applies. This suffices for all sufficiently distant pairs at the fixed $R$. Independently, the critical free spin correlations are the limits of the subcritical ones at $\beta_m$. The auxiliary Gaussian conditional law then identifies the limiting source sum with the translation invariant two-point function of $\psi$ at those pairs.

At the massless endpoint the absorbed denominator has the expansion

$$\widehat{T}(k) = t_2 R^2 |k|^2 + O_R(|k|^4), \qquad t_2 > 0.$$

The constant is positive by the uniform quadratic multiplier comparison. The scalar form of the quadratic part follows from hypercubic symmetry. The kernels in $T$ have exponential moments at fixed $R$, so the expansion and its differentiated bounds follow from Taylor's theorem. Since $\widehat{A}(0) = \beta_c^D p$, the massless covariance in the $\psi$ normalization satisfies

$$\widehat{C}(k) = \lambda R^2 \widehat{A}(k)/\widehat{T}(k) = \frac{\lambda \beta_c^D p}{t_2 |k|^2} + O_R(1)$$

near zero. Subtract the first term with a smooth cutoff equal to one near zero. The remainder has symbol bounds $C_{R,r}|k|^{-r}$ at each fixed derivative order $r$. The estimate of Section 2.8 with $\rho = 0$ therefore bounds its Fourier coefficient by $O_R(|x|^{-4})$. The inverse transform of the cutoff singular term has leading coefficient $1/(4\pi^2)$ times its prefactor, as follows from the fundamental solution of the Laplacian in dimension four and convolution with the Schwartz kernel of the cutoff. Therefore

$$C(x) \sim \frac{\lambda \beta_c^D p}{4\pi^2 t_2} |x|^{-2}.$$

Proposition 14.1 now gives the same leading term for the interacting auxiliary covariance, multiplied by the site-independent source amplitude $v_\infty^2 > 0$.

For the return to spins use (8) with $s = 2$ and $P_* = BA^{-1}$, evaluated at the critical parameter. The auxiliary covariance is bounded at all pairs by the conditional Gaussian representation and the bounded spin correlations. Here $A^{-1}$ and $P_*$ have exponentially summable kernels at fixed $R$. Indeed $\widehat{A}$ is a trigonometric polynomial strictly positive on the real frequency torus, so its reciprocal is analytic in a complex neighborhood. Also $\sum_x P_*(x) = (1 - \beta_c^D q)/(\beta_c^D p)$. Apply (16) twice, with $s = 2$; the exponential kernels satisfy its decay assumption. The additive kernel $A^{-1}$ is $o(|x|^{-2})$. It follows that

$$C_R = \frac{v_\infty^2 (1 - \beta_c^D q)^2}{4\pi^2 t_2 \beta_c^D p R^2} > 0.$$

The diagonal shift and temperature conversion of Section 3 identify the critical free correlation for $D$ at $\beta_c^D$ with that for $D_R$ at $\tau_c(4, R)$, in finite free volumes and in their limit. Thus $C_{4,R} = C_R$ in Theorem 1. □

# Part II
# Fractional construction and flow

## 16 The Gaussian asymptotic and the long-range obstruction

Set

$$a_{R,d} = \frac{1}{2}\sum_x x_1^2 D_{R,d}(x) = \frac{(2R+1)^d}{(2R+1)^d - 1}\frac{R(R+1)}{6}.$$

Hypercubic symmetry and finite range give

$$1 - \widehat{D}_{R,d}(k) = a_{R,d}|k|^2 + O_R(|k|^4), \qquad \widehat{L_R^{\alpha}}(k) = a_{R,d}^{\alpha}|k|^s(1 + O_R(|k|^2)). \tag{40}$$

Here $a_{R,d} \asymp R^2$. Since $0 < s < d$, the massless Gaussian covariance $G_{R,0} = L_R^{-\alpha}$ is well-defined as a Fourier integral.

**Proposition 16.1** (Fractional Gaussian two-point function)**.** *For each fixed $R$ and $d \in \{1, 2, 3, 4\}$,*

$$G_{R,0}(x) \sim \frac{\Gamma((d-s)/2)}{2^s \pi^{d/2}\Gamma(s/2)} a_{R,d}^{-\alpha}|x|^{-(d-s)} = \frac{a_{R,d}^{-d/4}}{(2\pi)^{d/2}}|x|^{-d/2}. \tag{41}$$

*For $d < 4$, the pair interaction has the asymptotic*

$$K_R(x) \sim c_{d,s} a_{R,d}^{\alpha}|x|^{-d-s}, \qquad c_{d,s} = \frac{s2^{s-1}\Gamma((d+s)/2)}{\pi^{d/2}\Gamma(1-s/2)} > 0. \tag{42}$$

*Consequently $\sum_x |x|^t K_R(x) < \infty$ precisely for $0 \le t < s$. In particular the interaction has neither an exponential moment nor a finite second moment when $d < 4$.*

*Proof.* Let $p_t(x) = (e^{-tL_R})(0, x)$, the continuous-time walk kernel. The spectral integral identities are

$$L_R^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1} e^{-tL_R}\, dt,$$

and, for $0 < \alpha < 1$,

$$K_R(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty p_t(x) t^{-1-\alpha}\, dt \qquad (x \neq 0).$$

The finite-range walk has diffusion coefficient $a_{R,d}$. Fourier inversion of $e^{-t(1-\widehat{D}_{R,d})}$, splitting into a neighborhood of zero and its complement, gives the local central limit kernel

$$(4\pi a_{R,d} t)^{-d/2} \exp[-|x|^2/(4a_{R,d}t)],$$

in the regime $t \asymp |x|^2$, with the usual Gaussian and bounded-jump off-diagonal bounds outside that regime. These bounds give integrable majorants in the two displayed integrals after the change $t = |x|^2 u$. Integration of the Gaussian kernel gives respectively

$$\frac{\Gamma(d/2-\alpha)}{4^\alpha \pi^{d/2}\Gamma(\alpha)} a_{R,d}^{-\alpha}, \qquad \frac{\alpha 4^\alpha \Gamma(d/2+\alpha)}{\pi^{d/2}\Gamma(1-\alpha)} a_{R,d}^{\alpha}.$$

These are the constants in (41) and (42). The moment assertion follows by summing $|x|^{t-d-s}$ over lattice shells. □

### 16.1 The positive-mass tail

For $d < 4$, fixed $R$, and $\mu > 0$, the positive-mass Gaussian covariance has an algebraic tail.

$$(\mu + L_R^\alpha)^{-1}(x) \sim \mu^{-2} K_R(x) \asymp_{R,\mu} |x|^{-d-s}. \tag{43}$$

To see this, subtract $\mu^{-2}\widehat{J}_R^{\mathrm{raw}}$ from the resolvent multiplier. With $z = (1 - \widehat{D}_{R,d})^\alpha$, the difference is the constant $\mu^{-1} - \mu^{-2}$ plus $z^2/[\mu^2(\mu + z)]$. Apply the estimate of Section 2.8 with $\rho = 2s$ to the latter term and then use (42). Thus a finite-range covariance estimate with scaling $L^{-(d-s)j}$ cannot have arbitrarily high postmass powers $(1 + \mu L^{sj})^{-p}$: summing pieces whose range reaches $|x|$ would otherwise contradict (43). The maximal tail-compatible power is $p = 2$.

**Subordination.** For $0 < \alpha < 1$, $\mu > 0$, and $t > 0$, the Stieltjes identity is

$$\frac{1}{\mu + t^\alpha} = \int_0^\infty \frac{\rho_\mu(r)}{t + r}\, dr, \qquad \rho_\mu(r) = \frac{\sin(\pi\alpha)}{\pi} \frac{r^\alpha}{r^{2\alpha} + 2\mu r^\alpha \cos(\pi\alpha) + \mu^2}. \tag{44}$$

It follows either by the Cauchy integral formula along the negative real axis or by the Stieltjes inversion formula. The limiting $\mu = 0$ density is $(\sin(\pi\alpha)/\pi) r^{-\alpha}$. Consequently

$$(\mu + L_R^\alpha)^{-1} = \int_0^\infty \rho_\mu(r)(L_R + r)^{-1}\, dr.$$

Positive finite-range pieces of the local resolvent on the right can be integrated separately; their ranges are unchanged and positive semidefiniteness is preserved. The same observation applies to (45), whose extra numerator is finite-range and positive semidefinite. Spatial smoothing and initial-scale bounds still have to be verified for the actual covariance used in the Ising representation.

## 17 A finite-range reference split for the fractional model

Using the full long-range kernel as the high-temperature reference would lose both its exponential tree norm and the bounded-degree block graph. There is an exact split that avoids that particular problem. For $0 < \alpha < 1$, choose an even integer $N \geq 2$ and define

$$Q_N = \sum_{1 \leq n < N} c_n D_{R,d}^{*n}, \qquad P_N = \sum_{n \geq N} c_n D_{R,d}^{*n}.$$

Then $J_R^{\mathrm{raw}} = Q_N + P_N$, $Q_N \geq 0$ has range at most $(N-1)R$, and its row sum is the fixed number $q_N = \sum_{n<N} c_n < 1$. The diagonal of $Q_N$ is $O(R^{-d})$. It may be left in the reference kernel, since it only adds a constant to the Ising energy; a mean-field comparison that retains this diagonal incurs an additional $O(R^{-d})$ self-edge error. Transferring it to the Gaussian part as a bare $O(R^{-d})\delta_0$ term would require a separate check of the initial spatial derivative bounds.

**Proposition 17.1** (Positive Gaussian remainder)**.** *For even $N$, $P_N$ is positive semidefinite as a convolution operator, as well as pointwise nonnegative. If $\varepsilon_R > 0$, then $P_N + \varepsilon_R I$ is strictly positive definite. Adding $\varepsilon_R I$ to the total pair interaction is spin-independent. Moreover,*

$$P_N = \frac{\sin(\pi\alpha)}{\pi} \int_0^1 u^{N-\alpha-1}(1-u)^\alpha D_{R,d}^{*N}(I - uD_{R,d})^{-1}\, du. \tag{45}$$

*The integral is an operator-norm integral.*

*Proof.* The beta integral for $c_n$ is

$$c_n = \frac{\sin(\pi\alpha)}{\pi} \int_0^1 u^{n-\alpha-1}(1-u)^\alpha \, du.$$

Summing its geometric series gives (45). The factor $D_{R,d}^{*N}$ is positive semidefinite for even $N$; $(I - uD_{R,d})^{-1}$ is positive definite for $0 < u < 1$, and the commuting product is positive. The product has operator norm at most $(1-u)^{-1}$, and the weight in (45) is integrable against this bound because $\alpha > 0$. Pointwise nonnegativity follows from the defining sum. Adding $\varepsilon_R I$ bounds the operator below by $\varepsilon_R I$ and shifts the spin energy by a constant. □

One may choose $\varepsilon_R = e^{-R}$. The same integral gives $|\widehat{P}_N(k)| \le C_{\alpha,N}|\widehat{D}_{R,d}(k)|^N$. Thus arbitrarily high fixed orders of high-frequency smoothing can be imposed by choosing $N$ before $R$.

The normalized representation used below is

$$D^+ = \frac{J_R^{\mathrm{raw}} + \varepsilon_R I}{1+\varepsilon_R}, \qquad P = \frac{P_N + \varepsilon_R I}{1+\varepsilon_R}, \qquad Q = \frac{Q_N}{1+\varepsilon_R}.$$

Thus $D^+ = P + Q$, its row sum is one, and its inverse temperature is $\beta = (1+\varepsilon_R)\tau$. This diagonal addition and temperature change leave the spin law exactly unchanged. The reference row sum is below one in a fixed neighborhood of $\beta = 1$. Its diagonal can be kept with the self-edge correction noted above; all reference field changes remain finite-range.

### 17.1 Exact Gaussian representation and Hessian subtraction

Work first on a torus, where all convolution operators in this subsection are periodized. Set $A = \beta P$ and $B = I - \beta Q$. Let $F_Q(h) = \log\langle e^{\langle h,\sigma\rangle}\rangle_{\beta Q}$ be the field-dependent reference pressure, with zero-field expectation for the interaction $\beta Q$, and set

$$S = D_h^2 F_Q(0), \qquad S_0 = B^{-1}, \qquad E = S - S_0.$$

The Hubbard–Stratonovich identity gives the joint spin and Gaussian field law proportional to

$$\exp\Big(-\tfrac{1}{2}\langle h, A^{-1}h\rangle + \langle h,\sigma\rangle + \tfrac{\beta}{2}\langle\sigma, Q\sigma\rangle\Big)\, dh.$$

Thus $h \mid \sigma$ has mean $A\sigma$ and covariance $A$. The invertible finite-range field change $h = Bu$ gives the Hessian $B^2A^{-1} - BSB$ for the negative logarithm of the field density at zero.

Write $p = \sum_x P(x)$, $q = \sum_x Q(x)$, and $c = (1-\mu-\beta q)/(\beta p)$, where $\mu \ge 0$ is the dimensionless auxiliary mass. Then $B - cA$ has row sum $\mu$. The residual Hessian relative to $B(B-cA)A^{-1}$ is

$$\mathscr{R} = (c-1)B - BEB = rI + K, qquadr = \widehat{\mathscr{R}}(0).$$

Here $K$ has zero row sum, and hats at zero denote row sums. The cavity bound in Section 22 makes $K$ exponentially local and small. Absorb $K$ into the Gaussian precision and leave $rI$ in the local potential. This gives the covariance $A/[B(B-cA) + AK]$ for $u$. Its denominator is spatially nonlocal even if $K$ is replaced by a finite-range kernel. Section 18 constructs its finite-range decomposition by introducing an internal space before taking the inverse. The local kinetic coordinate subsequently produced by the flow starts at zero.

# 18 An auxiliary-space construction of the absorbed fractional covariance

Use the operators $A, B, K$ and scalar $c$ from Section 17.1. For $d \in \{1, 2, 3\}$, abbreviate $D = D_{R,d}$, $\ell(k) = 1 - \widehat{D}(k)$, and $\zeta = \beta/(1+\varepsilon_R)$, so $A = \zeta(P_N + \varepsilon_R I)$ and $B = I - \zeta Q_N$. We use $\varepsilon_R = e^{-R}$ and take $N$ to be a sufficiently large fixed even integer. The kernel $K$ is real, hypercubic symmetric, and has zero row sum. Assume the bound

$$\sum_x |K(x)| e^{\eta|x|/R} \le Cg, \qquad g = R^{-d}. \tag{46}$$

Work in $|\beta - 1| + \mu \le C_0 g$. All constants may depend on the fixed even integer $N$, the displayed constants, and the fixed scale factor, but not on $R, \mu, \beta$.

**Proposition 18.1** (Positive fractional pieces after Hessian absorption)**.** *For any prescribed finite spatial derivative order $k_*$, choose $N$ sufficiently large and then $R$ sufficiently large. The exact covariance*

$$\mathscr{C} = \frac{A}{B(B - cA) + AK} \tag{47}$$

*has positive semidefinite finite-range pieces $\mathscr{C}_j$, $j \ge 0$, of range at most $CRL^j$, with*

$$\|\nabla^m \mathscr{C}_j\|_\infty \le CR^{-d-m} L^{-(d-s+m)j} (1 + \mu L^{sj})^{-2}, \qquad 0 \le m \le k_*. \tag{48}$$

*The same bound holds for the Fourier integral with weight $|k|^m$. The pieces are continuous in parameters whenever $K$ is continuous in a fixed weaker exponential kernel norm. For any $\lambda > 0$ write $\Gamma_{j+1} = \lambda R^s \mathscr{C}_j$ for the normalized piece used at the step from $j$ to $j + 1$. It satisfies $\sum_x \Gamma_{j+1}(x)^2 \ge c\lambda^2$ whenever $\mu L^{sj} \le c_0$, for a fixed $c_0 > 0$.*

*Proof.* We first assume $K$ has finite range $C_P R$, and later remove that restriction by moment matching. Its zero row sum, symmetry and second moment give $|\widehat{K}(k)| \le Cg \min(R^2|k|^2, 1)$. The box multiplier comparison makes the latter minimum comparable to $\ell(k)$. Let

$$F = cB - K, \qquad \mathscr{T} = B^2 - AF = B(B - cA) + AK.$$

For large $R$, $F$ and $B$ are bounded above and below by fixed positive constants. Moreover

$$\widehat{\mathscr{T}}(k) \asymp \mu + \ell(k)^\alpha. \tag{49}$$

Indeed, with $t = \widehat{D}(k)$,

$$\widehat{B - cA}(k) = \mu + c\zeta \ell(k)^\alpha + (1 - c)\zeta(Q_N(1) - Q_N(t)).$$

Here $c = 1 + O(g)$, and the last bracket is nonnegative and at most $C_N \ell(k)$; the Hessian term is another $O(g\ell(k))$ perturbation. Since $0 < \alpha < 1$, these perturbations are bounded by a small multiple of $\ell(k)^\alpha$.

The crucial point is to represent $A$ on an internal Hilbert space without introducing spatially nonlocal matrix entries. Put

$$w(u) = \frac{\sin(\pi\alpha)}{\pi} u^{N-\alpha-1}(1 - u)^\alpha, \qquad \mathcal{H}_0 = L^2((0, 1), du) \oplus \mathbb{R}.$$

On $\ell^2(\mathbb{Z}^d;\mathcal{H}_0)$, let $\mathscr{D} = \operatorname{diag}(I - uD, I)$. Define the finite-range map $\mathcal{L} : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d;\mathcal{H}_0)$ by

$$(\mathcal{L}f)(u) = \sqrt{\zeta w(u)}D^{N/2}f, \qquad (\mathcal{L}f)_0 = \sqrt{\zeta\varepsilon_R}\, f.$$

Then $A = \mathcal{L}^*\mathscr{D}^{-1}\mathcal{L}$ as quadratic forms. This is exactly (45). The inverse may be unbounded on the internal space, but its displayed compression is bounded, since $w(u)/(1-u)$ is integrable.

Consider the bounded, finite-range, self-adjoint operator

$$\mathcal{Q} = \begin{pmatrix} \mathscr{D} & -\mathcal{L}F \\ -F\mathcal{L}^* & FB^2 \end{pmatrix} \tag{50}$$

on $\ell^2(\mathbb{Z}^d;\mathcal{H}_0 \oplus \mathbb{R})$. Its range is a fixed multiple of $R$, its norm is uniformly bounded, and it is positive. To see positivity, complete the square in the first component. The Schur complement is $FB^2 - FAF = F\mathscr{T} \geq 0$. This form calculation is valid by approximation with $\mathscr{D} + \delta I$ and passage to $\delta \downarrow 0$, or directly because $\mathscr{D}^{-1/2}\mathcal{L}$ is bounded. The compressed inverse is

$$V_0 = \mathcal{L}^*\big[\mathcal{Q}^{-1}\big]_{00}\mathcal{L} = \frac{AB^2}{B^2 - AF}, \qquad \mathscr{C} = B^{-2}V_0. \tag{51}$$

All inverses in this assertion can equivalently be taken as positive quadratic-form limits of resolvents. In particular no finite internal dimension or uniform gap at zero frequency is being assumed.

We record quantitative estimates for those resolvents. Set $A_r = \mathcal{L}^*(\mathscr{D} + r)^{-1}\mathcal{L}$, $r > 0$. At each frequency let $h_r = B^2 + r/F$. The Schur formula gives

$$V_r = \mathcal{L}^*\big[(\mathcal{Q} + r)^{-1}\big]_{00}\mathcal{L} = \frac{A_r h_r}{h_r - FA_r}, \qquad -V_r' = \frac{h_r^2(-A_r') + A_r^2}{(h_r - FA_r)^2}. \tag{52}$$

The second identity follows by differentiation and $Fh_r' = 1$. It is important to use this derivative: $V_r$ alone would give only one postmass power.

Write $\rho = r + \ell(k)$ and restrict first to small $r, \ell(k)$. The integral defining $A_r$ gives, uniformly,

$$\begin{aligned} A_r &\asymp 1, & -A_r' &\leq C\rho^{\alpha-1}, \\ A - A_r &\asymp r\rho^{\alpha-1}, & h_r - FA_r &\asymp \mu + \rho^\alpha. \end{aligned} \tag{53}$$

For example, near $u = 1$, $w(u) \asymp (1-u)^\alpha$, while $1 + r - u\widehat{D}(k) \asymp r + \ell(k) + 1 - u$. Thus the derivative is bounded by a constant times $\int_0^1 v^\alpha(v+\rho)^{-2}\,dv \leq C\rho^{\alpha-1}$. The difference $A - A_r$ is the positive integral with integrand $rv^\alpha/[(v+\ell)(v+\ell+r)]$; it is comparable to $r\rho^{\alpha-1}$. Combining this with (49) proves the last estimate, since $\ell^\alpha + r\rho^{\alpha-1} \asymp \rho^\alpha$. The bounded white component does not affect these estimates. Consequently

$$-V_r' \leq C\frac{(r+\ell)^{\alpha-1}}{[\mu + (r+\ell)^\alpha]^2}, \qquad V_r \asymp \frac{1}{\mu + (r+\ell)^\alpha} \tag{54}$$

in this low-frequency region.

Away from $\ell = 0$, the denominators are uniformly bounded below. Every derivative of $A_r$ in $r$, for $0 < r \leq 1$, is bounded by $C_m(|\widehat{D}(k)|^N + \varepsilon_R)$. The same is true of $-V_r'$. Indeed $A_r^2 \leq C(-A_r')$, as follows directly from the integral and Cauchy–Schwarz, or from the uniform upper bound on $\mathscr{D} + r$ and on $A_r$.

At each frequency the operator in (50) also satisfies

$$\mathcal{Q}(k) \geq c\ell(k)I. \tag{55}$$

For small positive $\ell$, completion of the square gives a lower bound $c\ell\|z - Jv\|^2 + c(\mu + \ell^\alpha)|v|^2$, where $J = \mathscr{D}^{-1}\mathcal{L}F$. The integral estimate above gives $\|J\|^2 \le C\ell^{\alpha-1}$. Hence $\ell(\|z\|^2 + |v|^2)$ is bounded by a constant times that completed square. Away from zero the same argument has uniformly bounded coefficients. This proves (55).

Apply Lemma 2.1 to the bounded positive operator $\mathcal{Q}$. After a fixed spectral rescaling its polynomials $q_j$ obey

$$\deg q_j \le CL^j, \quad q_j(t) \ge 0, \quad \sum_{j\ge 0} q_j(t) = t^{-1}, \quad q_j(t) \le C_p r^{-1}(1 + t/r)^{-p}, \qquad r = L^{-2j}.$$

They also satisfy $q_j(t) \ge cr^{-1}$ for $0 \le t \le c_* r$. These are scalar statements; the spectral theorem proves them on any Hilbert space, and polynomial degree proves the finite-range property, independently of the internal dimension. The upper bound and (55) imply

$$\mathcal{L}^*[q_j(\mathcal{Q})]_{00}\mathcal{L} \le C_p r(1 + \ell/r)^{-(p-2)}(-V_r').$$

All terms are positive multipliers. Using (54), their low-frequency bound is

$$C_p r^{-\alpha}(1 + \mu r^{-\alpha})^{-2}(1 + \ell/r)^{-(p-1-\alpha)}. \tag{56}$$

Their high-frequency bound is $C_p r^{p-1}(|\widehat{D}|^N + \varepsilon_R)$.

The box multiplier comparison and a change of variables yield

$$\int_{|k|\le C/R} |k|^m (1 + \ell(k)/r)^{-q}\, dk \le CR^{-d-m} r^{(d+m)/2}$$

when $q > (d + m)/2$. Moreover, for $N > m + 1$,

$$\int |k|^m (|\widehat{D}(k)|^N + e^{-R})\, dk \le C_{m,N} R^{-d-m}.$$

For the latter use the product Dirichlet bound $|\widehat{D}(k)| \le C \prod_i \min(1, (R|k_i|)^{-1}) + CR^{-d}$. Choosing $p$ so that $p - 1 > 3\alpha + k_*/2$, and enlarging it further for the low-frequency integral, proves (48) and its weighted Fourier version for these compressed pieces of $V_0$.

Multiplication by $B^{-2}$ preserves a positive finite-range decomposition with these bounds. Indeed the same scalar construction for the finite-range, uniformly gapped operator $B$ gives positive pieces for $B^{-1}$ with arbitrarily rapid decay in their range index. Convolve two such decompositions to obtain one for $B^{-2}$, and group products with the pieces of $V_0$ by the maximum range index. In the terms whose maximum comes from $V_0$, sum the bounded positive multiplier of the other factor. In the remaining terms use the arbitrarily rapid decay and the finite weighted Fourier integral of the full compressed covariance. These are precisely the two estimates used in the product construction of Proposition 6.1.

For completeness the low-frequency lower bound follows from the compressed spectral measure, not from an operator gap of fractional order. Write $V_r(k) = \int_0^S (t + r)^{-1}\, d\nu_k(t)$. For $\mu \le c_0 r^\alpha$ and $\ell(k) \le c_1 r$, (54) implies $V_{\delta r} \ge a\delta^{-\alpha} r^{-\alpha}$ and $V_{c_* r} \le A c_*^{-\alpha} r^{-\alpha}$, if $c_0, c_1$ are taken small relative to $\delta$. On $t > c_* r$, $(t + \delta r)^{-1} \le 2(t + c_* r)^{-1}$. Choose fixed $\delta > 0$ so small that the first lower bound exceeds four times the second upper bound. It follows that $\nu_k([0, c_* r]) \ge cr^{1-\alpha}$. The scalar lower bound on $q_j$ therefore gives a compressed slice at least $cr^{-\alpha}$ on that ball. The initial scale uses the constant positive polynomial in the initial time interval. The zero-scale positive piece of $B^{-2}$ is bounded below, so the same bound holds after the product construction. After multiplying

by $\lambda R^s$, Plancherel on a ball of volume $cR^{-d}L^{-dj}$ proves the stated bubble lower bound, because $d = 2s$.

Finally remove the finite-range assumption on $K$. Match its moments through an arbitrarily high fixed order $P$ by a symmetric finite-range kernel $K_f$ of range $C_P R$, zero row sum, and weighted norm $O_P(g)$, exactly as in Proposition 6.1. Let $T_f = B(B - cA) + AK_f$ and $T = B(B - cA) + AK$. The positive ratio

$$M(k) = \widehat{T}_f(k)/\widehat{T}(k) = 1 - \widehat{A}(k)(\widehat{K}(k) - \widehat{K}_f(k))/\widehat{T}(k)$$

is $1 + O(g)$. In rescaled coordinates $\xi = Rk$, its derivatives through any prescribed order $m$ are integrable, with

$$\int |\partial_k^m (M - 1)|\, dk \le C_{P,m} g R^{m-d},$$

provided $P$ is sufficiently large. Near zero, moment matching makes the derivatives of $K - K_f$ bounded by $Cg|\xi|^{P+1-m}$; differentiation of the fractional denominator costs at most $|\xi|^{-s-m}$. Derivatives of the fractional numerator obey the same symbol bounds, so the resulting powers are integrable for large $P$, uniformly in $\mu \ge 0$. Away from zero the denominator is bounded below and the filter smoothing, with $N$ chosen large after $P$, gives the displayed integral bound. Truncate the Fourier series at range $RL^n$, subtract decreasing error constants, and take consecutive differences as in the proof of Proposition 6.1. This decomposes $M$ into positive trigonometric polynomials $M_n$, with $M_0 \ge 1/2$ and arbitrarily rapid fixed decay in $n$. Multiply them by the already constructed covariance pieces and group by maximum index. The weighted Fourier estimates give (48); choose the decay exponent larger than $d + s + k_*$ to preserve its postmass factor. The factor $M_0 \ge 1/2$ preserves the lower bound. All operations are continuous in the indicated parameter norms. This proves the proposition. □

**Corollary 18.2** (Critical Gaussian asymptotic and bubble envelopes)**.** *At $\mu = 0$, the covariance in* (47) *obeys*

$$(\lambda R^s \mathscr{C})(x) \sim \frac{\lambda R^s \widehat{A}(0)}{\widehat{B}(0) c\zeta a^{\alpha}_{R,d}} \frac{1}{(2\pi)^{d/2}} |x|^{-s}.$$

*For the bubble bounds at any $\mu \ge 0$, fix the Wick contraction constant $a_0 > 0$ and define the cumulative covariance $w_j = \sum_{i<j} \Gamma_{i+1}$, with $w_0 = 0$, and the bubble increment*

$$B_j = a_0 \sum_x (w_{j+1}(x)^2 - w_j(x)^2).$$

*Then $0 \le B_j \le C\lambda^2 (1 + \mu L^{sj})^{-2}$, and $B_j \ge c\lambda^2$ when $\mu L^{sj} \le c_0$.*

*Proof.* The analytic kernels $B, K$ have expansions with remainder $O_R(|k|^2)$, while the only leading nonsmooth contribution to $B - cA$ is $c\zeta a^{\alpha}_{R,d}|k|^s$. Consequently $\widehat{\mathscr{T}}(k) = \widehat{B}(0) c\zeta a^{\alpha}_{R,d}|k|^s + O_R(|k|^2)$. Division shows that the covariance multiplier is its displayed positive multiple of $|k|^{-s}$, plus a symbol of strictly higher order: its derivatives of order $m$ are bounded by $C_{R,m}|k|^{-s+\delta-m}$, where $\delta = \min(s, 2 - s) > 0$. The Fourier estimate of Section 2.8, with $\rho = -s + \delta$ and $d = 2s$, bounds the remainder by $O_R(|x|^{-s-\delta})$. The Riesz kernel coefficient is $(2\pi)^{-d/2}$, because $d = 2s$. This proves the asymptotic. For the bubble bound, positivity gives nonnegative mixed sums by Plancherel. The range and pointwise estimates give $\sum_{i \le j} \|\Gamma_{i+1}\|_1 \le C\lambda R^s L^{sj}$, and multiplying this by the pointwise bound on $\Gamma_{j+1}$ proves the upper envelope. The square of the last piece and the lower bound in Proposition 18.1 give the lower bound. □

# 19 Scaling and covariance moment estimates

Write $s = d/2$, so that the fractional precision has order $s$ and the Gaussian field has scaling dimension

$$[\phi] = \frac{d-s}{2} = \frac{d}{4} = \frac{s}{2}.$$

For a block of side $b_j = bL^j$, its fluctuation radius in canonical field units is therefore proportional to $b_j^{-d/4}$. The scaling degree of a local monomial with $p$ fields and $r$ spatial derivatives is $pd/4 + r$. In particular,

| monomial | dimension minus $d$ | linear RG factor |
|---|---|---|
| $\phi^2$ | $-s$ | $L^s$ |
| $\phi^4$ | $0$ | $1$ |
| $\phi^6$ | $s$ | $L^{-s}$ |
| $\lvert\nabla\phi\rvert^2$ | $2-s$ | $L^{s-2}$ |

The quartic is marginal in every dimension considered here. The local sextic remains irrelevant, including in dimension one. For $d < 4$, the ordinary local kinetic term is irrelevant; the backward tuning of a marginal local stiffness in the four-dimensional argument must not be copied into these dimensions. The nonlocal fractional kinetic term is part of the Gaussian precision and is not a local monomial produced by polynomial localization.

Let $b \asymp R$ be an initial block side chosen sufficiently large relative to the interaction and covariance ranges, and take $L > 1$ to be an integer. We use field radii

$$\ell_j = \ell_* b_j^{-s/2}, \qquad h_j = k_0 G_j^{-1/4} b_j^{-s/2}$$

for fixed positive constants $\ell_*, k_0$ and reference quartic coupling $G_j$. For two distinct marked sites $a, c$, use the nilpotent algebra generated by $t_a, t_c$, with $t_a^2 = t_c^2 = 0$. The subscript $\sigma$ on a radius denotes a source sector. The two marks have coalescence index $k$, the first scale at which a small polymer can contain both, with $b_k \asymp |a-c|$ for distant pairs. The fluctuation source radii are

$$\ell_{\sigma,j} = G_j b_{j\wedge k}^{s/2} 2^{(j-k)_+}, \qquad \ell_{\sigma\sigma,j} = \ell_{\sigma,j}^2.$$

The large-field source radii are

$$h_{\sigma,j} = G_j^{1/4} b_{j\wedge k}^{s/2} 2^{(j-k)_+}, \qquad h_{\sigma\sigma,j} = h_{\sigma,j}^2.$$

Both source radii stop their spatial rescaling at coalescence. Choose $L^{s/2} \geq 2$, so

$$h_{\sigma,j} h_j = k_0 (2L^{-s/2})^{(j-k)_+} \leq k_0.$$

The single and mixed ratios of fluctuation to large-field radii are $G_j^{3/4}$ and $G_j^{3/2}$. After coalescence the mixed radius grows by four per step, up to the adjacent coupling ratio. A mixed two-field remainder therefore gains $4L^{-s}$. The weighted coefficient norms specify the source domain on the complex balls. The physical coefficients are bounded by the single-source projection in Section 22.5.

## 19.1 Covariance moments

Proposition 18.1 supplies translation-invariant, hypercubic-symmetric positive pieces satisfying

$$\operatorname{range}(\Gamma_{j+1}) \leq C b_j, \qquad \|\nabla^r \Gamma_{j+1}\|_\infty \leq C_r \lambda b_j^{-s-r} (1 + \mathfrak{m} b_j^s)^{-2}$$

for the finite collection of derivative orders required below. Constants may depend on the fixed scale factor, but not on the mass cutoff. Here $\mathfrak{m} = \mu/R^s$ if $\mu$ is the dimensionless auxiliary mass used with $b_j \asymp RL^j$; thus the damping is comparable to $(1+\mu L^{sj})^{-2}$. Set $a_j = (1+\mathfrak{m}b_j^s)^{-2}$. If $\mu > 0$, let $J$ be the largest index with $\mu L^{sJ} \le c_0$, where $c_0$ is the constant in Corollary 18.2. Then $\mathfrak{m}b_J^s \asymp 1$, with constants depending on $L$ and the fixed block aspect. For $\mu = 0$ put $J = \infty$. Let $\chi_j = 2^{-(j-J)_+}$, interpreted as one when $J = \infty$.

**Lemma 19.1** (Moment estimates and cancellation of the marginal logarithm)**.** *At every scale,*

$$|w_j(x)| \le C\lambda(b+|x|)^{-s}, \qquad \|w_j\|_1 \le C\lambda b_j^s,$$
$$\sum_x |x|^2|w_j(x)| \le C\lambda b_j^{s+2}, \qquad \sum_x |x|^2 w_j(x)^2 \le C\lambda^2 b_j^2.$$

*Put* $\Gamma = \Gamma_{j+1}$. *Then*

$$\left|\sum_x ((w_j+\Gamma)^2 - w_j^2)(x)\right| \le C\lambda^2 a_j,$$
$$\left|\sum_x \left((w_j+\Gamma)^3 - w_j^3 - 3\Gamma(0)w_j^2\right)(x)\right| \le C\lambda^3 a_j b_j^{-s},$$
$$\sum_x |x|^2 \left|(w_j+\Gamma)^3(x) - w_j^3(x)\right| \le C\lambda^3 a_j b_j^{2-s}.$$

*After the mass scale the normalized cumulative moments satisfy*

$$b_j^{-s}\|w_j\|_1 \le C\lambda L^{-s(j-J)}, \qquad b_j^{-s-2}\sum_x |x|^2|w_j(x)| \le C\lambda L^{-2s(j-J)} \qquad (j \ge J).$$

*Proof.* The range bound makes only pieces with $b_i \ge c|x|$ contribute at $x$; their pointwise bounds form a geometric sum. Summing its result over the support of $w_j$ gives the two absolute moments. Since $d = 2s$, summing $|x|^2(b+|x|)^{-2s}$ gives $O(b_j^2)$, proving the fourth estimate. The bubble bound follows by expanding the difference of squares and using $\|\Gamma\|_\infty\|w_j\|_1$ and $\|\Gamma\|_\infty\|\Gamma\|_1$. Evenness and the second-difference bound give

$$|\Gamma(x) - \Gamma(0)| \le C\lambda a_j b_j^{-s-2}|x|^2.$$

Thus the potentially logarithmic term in the cubic increment is $3\sum w_j(x)^2(\Gamma(x)-\Gamma(0))$, bounded by the fourth moment estimate. The remaining terms are $3\sum w_j\Gamma^2 + \sum\Gamma^3$, bounded by $C\lambda^3 a_j^2 b_j^{-s}$ and $C\lambda^3 a_j^3 b_j^{-s}$, respectively. Inserting $|x|^2$ in the uncorrected cubic expansion instead uses $\sum|x|^2 w_j^2$, $\sum|x|^2|w_j|$, and the range of $\Gamma$, proving the kinetic-moment estimate. Finally, the postmass row-sum slices are bounded by $C\lambda\mathfrak{m}^{-2}b_i^{-s}$, whose sum is $O(\lambda b_J^s)$. Their second absolute moments are bounded by $C\lambda\mathfrak{m}^{-2}b_i^{2-s}$. Since $s < 2$, the latter sum is dominated by its final term. Division by the displayed normalizing powers gives exactly the two postmass estimates. In particular the second moment does not decay faster than the estimate displayed here by this moment argument. □

The normalized Wick mass increment is therefore $O(\lambda a_j)$, the bubble is $O(\lambda^2 a_j)$, the corrected cubic mass coefficient is $O(\lambda^3 a_j)$, and the local kinetic coefficient has the same size after multiplication by $b_j^{s-2}$. For large fixed $L$, the postmass normalized moments are smaller than any prescribed fixed power of $\chi_j$ by choosing $L$ accordingly. The map uses the bound $C\lambda\chi_j^{3/2}$.

**Lemma 19.2** (The decaying Gaussian regulator)**.** *For a block $B$ of $n = b_j^d$ sites, put*

$$\ell_j = \ell_* b_j^{-s/2}, \qquad h_j = k_0 G^{-1/4} b_j^{-s/2}, \qquad H_B(\phi) = \exp[-\kappa \|\phi_B\|_2^2/(n h_j^2)].$$

*If $\zeta$ has covariance $\Gamma_{j+1}$, then $\mathbb{E} H_B(\phi+\zeta) \le H_B(\phi)^{1/2}$ for sufficiently small $\lambda G^{1/2}$, uniformly in the mass scale.*

*Proof.* The covariance restricted to $B$ has operator norm at most $C\lambda a_j b_j^{d-s} = C\lambda a_j b_j^s$. Writing $q = \kappa/(n h_j^2)$, one obtains $2q\|\Gamma_B\|_{\mathrm{op}} \le C\kappa k_0^{-2} \lambda G^{1/2}$. For this quantity at most one, Gaussian completion of the square gives

$$\mathbb{E} H_B(\phi+\zeta) = \det(I + 2q\Gamma_B)^{-1/2} \exp[-q\langle \phi, (I+2q\Gamma_B)^{-1}\phi\rangle] \le H_B(\phi)^{1/2}.$$

□

**The precise algebraic irrelevance gap.** At zero field consider a scalar even local functional, with spatial support of diameter $O(b_j)$, and its coefficient pairing with coarse-scale test tensors. Let $\Delta = s/2$. Coarse test tensors of field degree $p$ have size $\ell_{j+1}^p$, and their spatial derivatives of order $r$ have size $\ell_{j+1}^p b_{j+1}^{-r}$. Localize the constant, mass and quartic terms. Use a canonical even quadratic Fourier symbol, or centered lattice derivatives modulo total differences, to eliminate the one-derivative quadratic density. Then the remaining pairings have factors

$$L^{-6\Delta} = L^{-3s} \quad (p \ge 6), \qquad L^{-2\Delta-2} = L^{-(s+2)} \quad (p=2), \qquad L^{-4\Delta-1} = L^{-(d+1)} \quad (p=4).$$

For $p = 2$ this is the second-order spatial Taylor remainder; for $p = 4$ a first-order remainder suffices. The higher-field terms follow by the ratio $\ell_{j+1}/\ell_j = L^{-\Delta}$. Reblocking adds at most $CL^d$ fine-block positions. Hence the small-field linear irrelevance factor is

$$CL^{-\delta}, \qquad \delta = \min(s, 2-s, 1) = \begin{cases} 1/2, & d = 1, 3, \\ 1, & d = 2. \end{cases} \tag{57}$$

The same calculation for a single source, anchored at its marked site, has no volume factor. After removing the linear source-field term, the first remaining field term is cubic (gain $L^{-s}$) or contains one spatial derivative (gain $L^{-1}$). In the mixed sector after coalescence the constant is extracted and the next even field term gains $L^{-s}$; the growth of the mixed source radius costs only the fixed factor four. A sufficiently large $L$ absorbs this cost.

This is the small-field localization calculation. A proof of a full activity-norm contraction also needs the regulator estimates and the nonlinear reblocking/circle-product theorem; it is not implied by (57) alone.

In the mixed sector the covariance insertion also obeys

$$h_{\sigma\sigma,j} |\Gamma_{j+1}(a-c)| \le C\lambda \chi_j G_j^{1/2}.$$

This follows from the pointwise bound and $(4L^{-s})^{(j-k)_+} \le 1$.

**Lemma 19.3** (The zero-field/large-field mixed norm)**.** *Let $E$ be the finite-volume test-field space and take field jets through a fixed order $p \ge 9$. Assume the test-tensor norms permit insertion of a field direction $v$ at cost at most $\|v\|_E$, as for the product derivative test spaces. Write*

$$|F|_{T_a(\phi)} = \sum_{q=0}^{p} \frac{a^q}{q!} \|D^q F(\phi)\|_{E^{*q}}, \qquad \ell/h = cG^{1/4},$$

*where the fixed factor $c$ is harmless. Suppose*

$$\mathcal{G}(\phi) = e^{c_0\|\phi\|_E^2/\ell^2}, \qquad H(\phi) \le 1, \qquad \widetilde{G}(t\phi) \le e^{c_1\|\phi\|_E^2/\ell^2} \quad (0 \le t \le 1), \qquad c_1 < c_0.$$

*Define*

$$\|F\|_W = \sup_\phi \frac{|F|_{T_\ell(\phi)}}{\mathcal{G}(\phi)} + G^{9/4} \sup_\phi \frac{|F|_{T_h(\phi)}}{H(\phi)\widetilde{G}(\phi)},$$
$$\|F\|_Y = |F|_{T_\ell(0)} + G^{9/4} \sup_\phi \frac{|F|_{T_h(\phi)}}{H(\phi)\widetilde{G}(\phi)}.$$

*Then $\|F\|_Y \le \|F\|_W \le C\|F\|_Y$, with a constant independent of the volume and $G$. The statement applies sectorwise in the nilpotent source algebra if the small-field source weight is at most the corresponding large-field source weight.*

*Proof.* The first inequality follows by evaluating the first supremum at zero. For the reverse inequality put $u = \|\phi\|_E/\ell$. For each $q < p$, Taylor-expand $D^qF(\phi)$ from zero through order $p-q-1$ and use the integral remainder with $D^pF(t\phi)$. For $q = p$, use its value directly. The tensor insertion assumption and the factorial coefficients yield

$$|F|_{T_\ell(\phi)} \le C_p|F|_{T_\ell(0)} \sum_{r=0}^{p-1} u^r + C_p(\ell/h)^p \left(\sum_{r=0}^{p} u^r\right) \sup_{0\le t\le 1} |F|_{T_h(t\phi)}.$$

The first term divided by $\mathcal{G}(\phi)$ is bounded by a constant times the zero-field norm. In the second term use $H(t\phi) \le 1$ and the regulator bound, leaving a polynomial in $u$ times $e^{-(c_0-c_1)u^2}$. Its supremum is finite. Since $(\ell/h)^p \le C_pG^{9/4}$ for $p \ge 9$, this gives exactly the second term in $\|F\|_Y$. For a source sector the same argument introduces only the ratio of its small-field weight to its large-field weight, which is at most one by hypothesis. Applying the estimate to each of the four sectors and adding proves the last statement. □

For the explicit stopped source weights above the sector ratios are $G_j^{3/4}$ and $G_j^{3/2}$, so they are at most one for small $G_j$. The finite-volume polymer version uses the sum of squared local field seminorms in $E$; the affine-quotient seminorm is bounded by the full field seminorm. Choosing the growing regulator with a fixed spare exponent gives the displayed domination uniformly on field segments.

## 20 A fractional regulated localization estimate

Put $\Delta = s/2$, and fix $p_\Phi = 12$. On a block neighborhood use the concrete seminorms

$$\|v\|_{\Phi_j} = \max_{|\gamma|\le p_\Phi} b_j^{|\gamma|}\|\nabla^\gamma v\|_\infty, \qquad \|v\|_{\widetilde{\Phi}_j} = \inf_{p\ affine} \|v-p\|_{\Phi_j}.$$

Fixed enlargements of block neighborhoods change only constants. The field radii are $\ell_j = \ell_* b_j^{-\Delta}$ and $h_j = k_0 G_j^{-1/4} b_j^{-\Delta}$. Define growing and quotient regulators by exponentiating small fixed positive multiples of the squares of these seminorms divided by $\ell_j^2$, and define $H_j$ by the block mean square divided by $h_j^2$, with a negative coefficient. Use strengthened regulator powers on intermediate maps, leaving fixed positive margins between input and output powers. All estimates below hold for these choices; replacing them by the standard equivalent test-function regulators changes fixed constants.

**Lemma 20.1** (The affine complement estimate)**.** *Let $B$ be a coarse block of side $Lb_j$, and let $X \subset B$ contain at most $2^d$ fine blocks. Put $\phi = p + r$, where $p$ is an affine approximation on the coarse neighborhood, and set*

$$z = \left( |B|^{-1} \sum_B p^2 \right)^{1/2} / h_{j+1}, \qquad q = \|r\|_{\Phi_{j+1}} / \ell_{j+1}.$$

*The approximation may be chosen within any fixed factor of the quotient infimum. For fixed sufficiently large $L$, sufficiently small $G_j$, and stable potentials with quartic coefficient comparable to $G_j$, the reference factor on $B \setminus X$ absorbs the input quotient regulator on $X$, the inverse output $H_{j+1}$, and any fixed polynomial in $z, q$. More precisely, after Gaussian expectation their product divided by a fixed spare power of the coarse quotient regulator is bounded by*

$$C_L e^{-cz^2 - cq^2}$$

*before the polynomial is absorbed. The same estimate holds for all field derivatives through a fixed finite order, with a different fixed polynomial before absorption. Nilpotent linear sources give only such polynomials.*

*Proof.* For an affine function, normalized suprema and normalized $L^2$ norms on a cube are equivalent. Removing at most $2^d$ fine blocks removes a fraction at most $2^d L^{-d}$. The normalized supremum of an affine function is bounded by its normalized $L^2$ norm, so

$$|B|^{-1} \sum_{B \setminus X} p^2 \geq c |B|^{-1} \sum_B p^2$$

for fixed large $L$. Restriction of coarse test fields gives

$$\|\phi\|_{\Phi_j(X)} / h_j \leq C L^{-\Delta} (z + G_j^{1/4} q), \qquad \|\phi\|_{\widetilde{\Phi}_j(X)} / \ell_j \leq C L^{-\Delta} q.$$

Indeed coarse scaled differences dominate the restricted fine ones; affine functions disappear in the second inequality. Consequently the input quotient regulator is at most $e^{CL^{-s} q^2}$.

The identity $|B| G_j h_{j+1}^4 \asymp 1$ permits the quartic Young inequality with an arbitrary fixed quadratic coefficient $A > 0$:

$$\frac{G_j}{4} \sum_{B \setminus X} \psi_x^4 \geq A |B|^{-1} \sum_{B \setminus X} \psi_x^2 / h_{j+1}^2 - C_A.$$

The affine complement inequality and $(p + r)^2 \geq p^2/2 - r^2$ then give, for $\psi = \phi$,

$$V_{B \setminus X}(\phi) \geq cAz^2 - C_A G_j^{1/2} q^2 - C_A.$$

The local mass and kinetic coefficients in their normalized domain contribute at most $C_L G_j^{1/2} (z^2 + G_j^{1/2} q^2)$, and are absorbed for small $G_j$. This quadratic argument avoids any uncontrolled positive $G_j q^4$ term. With $\psi = \phi + \zeta$, one loses only $C_A |B|^{-1} \sum_B \zeta^2 / h_{j+1}^2$. Its covariance quadratic form has operator norm at most $C_L \lambda G_j^{1/2}$. The Gaussian determinant formula therefore bounds its exponential moments uniformly for small $\lambda G_j^{1/2}$. The additional fluctuation quotient regulators are controlled by the same formula applied to the discrete Sobolev quadratic forms through order $p_\Phi + d$. The covariance derivative assumption through twice that order bounds their operator

norm by $C_L\lambda/\ell_*^2$ and their trace by that constant times the number of blocks. The fixed number of blocks here gives a fixed bound. On polymers it gives a fixed exponential per block.

Dividing by $H_{j+1}$ costs at most $e^{C\kappa(z^2+G_j^{1/2}q^2)}$. Choose $A$ large enough to leave a negative coefficient of $z^2$. Next choose $L$ so that $CL^{-s}q^2$ uses less than half the spare quotient exponent, and then choose $G_j$ small so that all $G_j^{1/2}q^2$ terms use less than half the remaining margin. This leaves $e^{-cz^2-cq^2}$. Field derivatives of the reference and nilpotent source factors introduce fixed polynomials, absorbed in the retained Gaussian decay. All choices precede taking the spread radius large. □

**Lemma 20.2** (Small-set contraction in the full mixed norm)**.** *Use field jets through order at least nine and the source radii of Section 19. For the linearized small-set reblocking map, with the local bulk and source counterterms extracted and scalar quadratic symbols put in their even canonical form, the mixed norm satisfies*

$$\|\mathcal{L}_j F\|_{W_{j+1}} \le \left(CL^{-\delta} + C_L\lambda^{1/2} + C_L G_j^{1/4}\right)\|F\|_{W_j}, \qquad \delta = \min(s, 2-s, 1) > 0.$$

*The background reference factors on the complement of a small set are included in $\mathcal{L}_j$. The constant multiplying $L^{-\delta}$ is independent of $L$: the coarse quartic sum is normalized by $|B|G_j h_{j+1}^4 \asymp 1$, and the affine norm comparisons have uniform constants. Gaussian moments may be bounded by two after the fixed $L$ is chosen and then $C_L\lambda$ is made small. The remaining nonuniform constants occur only in the displayed small-parameter terms.*

*Proof.* It is enough to use the equivalent zero-field/large-field norm of Lemma 19.3. The small-field local Taylor pairings have exactly the factors in (57). The number of fine-block positions in a coarse block is at most $CL^d$; source sectors are anchored and do not have this volume factor. Gaussian averaging does not enlarge the support of the external field variables. For the lower field jets, its change of zero-field jets is at most $C_L\lambda^{1/2}\|F\|_{W_j}$, by Taylor expansion in the fluctuation and the Gaussian moments. Top-order jets are bounded directly by their coarse test-tensor restriction gain; no additional field derivative is assumed for a Lipschitz estimate on those top jets. If the counterterms are extracted after expectation, this error may instead be incorporated in the localization itself. Critical terms regenerated by Gaussian contractions must not be counted as geometrically irrelevant without one of these two provisions.

For the large-field bound, treat each input source degree $r = 0, 1, 2$ separately and Taylor-expand its field dependence through order $n_r - 1$, where

$$n_r = 9 - 3r.$$

Low-order coefficient tensors are controlled by the zero-field norm. After conversion to output large-field radii and multiplication by the weight $G_{j+1}^{9/4}$, their coupling factor is at most a fixed constant times

$$G_j^{9/4-3r/4-p/4} \le CG_j^{1/4} \qquad (0 \le p \le n_r - 1).$$

Here the inequality means that the factor is bounded above by $CG_j^{1/4}$ for small $G_j$. All powers of the normalized affine field and quotient field are absorbed by Lemma 20.1.

For the integral remainder, each field direction costs $L^{-\Delta}(z + G_j^{1/4}q)$, and each restricted field test tensor has the same $L^{-\Delta}$ factor. The affine lemma again absorbs the resulting polynomial. The large-field source radius changes by at most $CL^{\Delta}$ for a single source, and by at most four after coalescence for the mixed sector. Thus the remainders have factors

$$L^{d-9\Delta} = L^{-5s/2} \quad (r=0), \qquad L^{-6\Delta+\Delta} = L^{-5s/2} \quad (r=1), \qquad 4L^{-3\Delta} = 4L^{-3s/2} \quad (r=2).$$

Before coalescence the small-set mixed sector vanishes. Extra source factors from the complement reference are bounded multipliers because the mixed source weight is at most the product of the single weights; they add only fixed field polynomials. Derivatives of field order above $n_r$ have at least the same restriction gain and are bounded directly by the input large-field seminorm. Adjacent reference-coupling ratios are $1 + O(G_j)$, which is absorbed by the last error term. Combining the low jets and remainders proves the large-field estimate; the zero-field estimate and norm equivalence prove the result. □

**Theorem 2** (The fractional one-step map)**.** *Let $d = 2s \in \{1, 2, 3\}$, and suppose the positive finite-range covariance pieces have the bounds of Section 19.1 through total derivative order 32, with natural mass power two. Use the concrete field norms above, field jets through order at least nine, the four nilpotent sectors, and the explicit mixed large-field source weight. Define extraction, Gaussian integration, second-order covariance redistribution, reblocking, and final componentwise circle products by the algebraic formulas specified in Part I and [4], with canonical mass dimension $s$ and field dimension $s/2$. The estimates for $d < 4$ are supplied below. The local source domain is defined by bounded weighted source norms; a fixed bound on all unweighted coefficients is not required for arbitrary complex activity directions. The bulk potential domain has $g \asymp G > 0$ and normalized mass and local kinetic coefficients bounded by fixed multiples of $G$. The reference couplings in consecutive norms obey $1 - CG \le G_+/G \le 1$; their values are fixed also at complex inputs.*

*Fix universal admissible $k_0, \kappa$ and regulator powers first. There are then choices of fixed $L$, large-set parameter $\varrho$, and a joint complex ball radius $c > 0$, for which the maps are holomorphic on*

$$\|V - V_0\| < cG, \qquad \|K\|_{W_j} < cG^{9/4}.$$

*On fixed smaller balls, the activity map $\Psi$ and bulk normal-form remainder $R$ satisfy*

$$\begin{aligned} \|\Psi(V,K)\| + |R(V,K)| &\le C(\chi_j^{3/2} G^3 + \|K\|), \\ \|D_K R(V,K)\| &\le C, \\ \|D_K \Psi(V,0)\| &\le CL^{-\delta} + C_L \varrho^{\epsilon_0} + C_L(\lambda^{1/2} + G^{1/4}), \qquad \delta = \min(s, 2-s, 1) > 0, \end{aligned}$$

*where $\epsilon_0 > 0$ is fixed. In particular one can make the last bound strictly smaller than $1/32$, uniformly in the mass cutoff, volume and sufficiently large spread radius. The derivative bounds of Lemma 2.3 hold as well. The constants depend on the fixed norm choices, but not on the mass cutoff or volume.*

*Proof.* The submultiplicative test-tensor estimates, localization at zero field, and the finite nilpotent algebra are the same normed-algebra operations in every dimension considered here. The only dimensional inputs to Gaussian integration are the finite derivative bounds and the regulator comparison already proved. Bulk localization retains the vacuum, mass, on-site quartic, and isotropic local kinetic monomials. The kinetic coordinate is retained even when irrelevant and is evolved with factor $L^{s-2}$. Odd quadratic derivative densities are put in canonical form modulo total differences. Single-source localization retains the linear source-field monomial at every scale. Only the mixed localization is constrained to vanish on small sets before coalescence. Consequently projection to a single nilpotent mark commutes exactly with the maps, independently of the position of the unused second mark; this is the all-scales convention needed for the distant-mark source bound. The small-set derivative is Lemma 20.2. We give the remaining counting and remainder bounds explicitly.

**Regulator comparison for large sets.** The quotient regulator also reblocks with a strict margin. On each fine block choose the affine Taylor polynomial at its center. The discrete second-order Taylor bound and the scaled derivatives through order $p_\Phi$ imply

$$\|\phi\|_{\widetilde{\Phi}_j(B_{\mathrm{fine}})} \le CL^{-2}\|\phi\|_{\widetilde{\Phi}_{j+1}(B)}.$$

For derivative orders zero and one this is the Taylor remainder; for orders at least two it is the factor $L^{-|\gamma|} \le L^{-2}$. After summing over at most $L^d$ fine blocks and dividing by the squared radii, the coefficient of the coarse quotient exponent is

$$CL^{d-4}(\ell_{j+1}/\ell_j)^2 = CL^{s-4},$$

which is strictly smaller than any fixed regulator margin for large $L$. Subsets of fine blocks only decrease the left side. On occupied sites the product of the fine $H$ factors has quadratic coefficient larger than the coarse coefficient by $L^s$. On unoccupied sites the quartic reference, by quadratic Young's inequality, supplies the missing coarse quadratic coefficient at a fixed exponential cost per coarse block. The Gaussian shift uses the same determinant bounds as the affine lemma, with a fixed exponential loss per input block. Hence dense occupied sets cause no growing-regulator loss: one uses the $Y$ norm, the decaying $H$, and this quotient comparison, rather than attempting to compare products of growing regulators.

**Circle-product summability.** The nonlinear estimate uses a purely combinatorial bound. For a connected polymer let $A(X) = \varrho^{(|X|-2^d)_+}$. Connected shapes with $n$ fine blocks and a prescribed anchor number at most $C_d^n$. A depth-first walk through a connected set has length at most $2n$; split it into segments of length at most $L/2$. Each segment meets at most $2^d$ coarse blocks. Therefore

$$|\overline{X}| \le 2^d(1 + 4n/L).$$

If $n$ is bounded by a fixed multiple of $2^d$, a sufficiently large $L$ makes $|\overline{X}| \le 2^d$. For larger $n$, the displayed inequality gives $|\overline{X}| \le n/2$ after increasing $L$. Thus large input sets gain a positive power of $\varrho$ in comparison with their output size weight, and this gain absorbs $C_d^n$ when $\varrho$ is small. For a product of $m$ nonempty component activities the elementary count

$$\sum_{i=1}^{m}(|X_i| - 2^d)_+ + 2^d(m-1) \ge \left(\sum_i |X_i| - 2^d\right)_+$$

shows that component amplitudes at most $\varrho^{2^d}$ supply the missing large-set weight. Fixed exponential losses per block are absorbed by the strengthened input weight margins. At zero field, products of activities of norm at most $\tau$ are bounded by $\tau^m$. At large field, with $\tau = cG^{9/4}$, their weighted sizes are at most

$$G^{9/4}(\tau G^{-9/4})^m = \tau c^{m-1}.$$

Hence every circle product and connected expansion used in the exact step is bounded by $C\tau$, uniformly in the volume, for small fixed $c$. The same counting bounds derivatives, with the extra polynomial choice factors absorbed by a smaller weight margin.

**Second-order $W$ terms and exact redistribution.** The polynomial $W_j$ is counted as degree two in the potential. Its norm estimates always mean the polynomial multiplied by its stable reference factor, or its zero-field polynomial majorant followed by quartic/regulator absorption. The bare polynomial does not have a finite decaying $H$ norm. In this precise sense,

$$\|e^{-V}W_j\|_{\ell,G} \le C\chi_j G^2, \qquad \|e^{-V}W_j\|_{h,H\widetilde{G}} \le C\chi_j G^{1/2}. \tag{58}$$

Here and below stable interpolated references may replace $e^{-V}$.

For clarity, the scalar quartic–quartic Wick contractions give the following finite list. Write $w = w_j$, suppress fixed combinatorial constants, and use the canonical localization of the quadratic symbol:

| contractions | external fields | normalized covariance moment | large-field size |
|---|---|---|---|
| 1 | 6 | $b_j^{-s}\|w\|_1$ | $O(\chi_j G^{1/2})$ |
| 2 | 4 | $b_j^{-1}\sum_x \|x\|w(x)^2$ | $O(\chi_j G)$ |
| 3 | 2 | $b_j^{s-3}\sum_x \|x\|^3\|w(x)\|^3$ | $O(\chi_j G^{3/2})$ |
| 4 | 0 | removed by localization | 0 |

All three small-field sizes are $O(\chi_j G^2)$. The second row uses the first spatial Taylor remainder after quartic localization; using $\sum w^2$ instead would introduce an inadmissible logarithm. The third row uses the third spatial remainder after mass and kinetic localization. A second remainder is enough for a noncanonical bound, but the third remainder makes the postmass estimate immediate. The pointwise finite-range slice sum gives

$$|\nabla^r w_j(x)| \le C_r\lambda(b+|x|)^{-s-r}(1+\mathfrak{m}|x|^s)^{-2}.$$

If $\xi = b_J$, the second-row moment is at most $C\lambda^2\xi$ and the third-row moment at most $C\lambda^3\xi^{3-s}$ after the mass scale. Their normalized decays are respectively $L^{-(j-J)}$ and $L^{-(3-s)(j-J)}$, while the first row decays as $L^{-s(j-J)}$. These dominate $\chi_j$ for a sufficiently large fixed $L$.

The remaining Wick cases do not require a new dimension-dependent estimate. A mass vertex supplies $\nu b_j^s = O(G)$. A kinetic vertex supplies $z_{\mathrm{phys}} b_j^{s-2} = O(G)$; move its differences by summation by parts to external fields whenever a single covariance is contracted. For two contractions, use the corresponding weighted moments of $w\nabla^2 w$ and $(\nabla w)^2$ after quadratic localization. For example, $\sum|x|^3(|w\nabla^2 w| + |\nabla w|^2) \le C\lambda^2 b_j$, with a bound $C\lambda^2\xi$ after the mass scale. This is a direct shell sum of the displayed pointwise derivative bound. A kinetic–kinetic one-contraction term can be written with four external differences and $w$; its normalized row moment has the first-row bound. All zero-row-sum derivative terms have no hidden mass constant. Vacuum terms are localized away. Single-source terms have only one possible contraction; source–quartic terms use the first row, and source–quadratic terms use the first spatial remainder or the external kinetic differences. The mixed source contraction is a constant and is extracted. Stopped source radii only decrease these bounds. This proves (58) for every sector.

Introduce a formal parameter $t$ multiplying the potential. The exact second-order identity is

$$\mathbb{E}[e^{-tV}(1+t^2W_j)] = 1 - t\mathbb{E}V + t^2\left(\tfrac{1}{2}\mathbb{E}V^2 + \mathbb{E}W_j\right) + O(t^3).$$

Put $R_2 = \frac{1}{2}\operatorname{Cov}(V,V) + \mathbb{E}W_j$. The quadratic correction to the output potential is $-\operatorname{Loc} R_2$, and the unlocalized second-order factor is $(1-\operatorname{Loc})R_2$. Expanding

$$e^{-t\mathbb{E}V + t^2 \operatorname{Loc} R_2}[1 + t^2(1-\operatorname{Loc})R_2]$$

shows exactly the same coefficients through degree two. This is the covariance redistribution used in Maps 1–5; it is an algebraic identity and does not depend on the field dimension.

**Higher-order terms.** On a scale-$j$ block, $|B|\ell_j^4 = \ell_*^4$ and $|B|h_j^4 = k_0^4 G_j^{-1}$. These identities and the Gaussian Sobolev bounds give the sizes of the centered potential increments

$$\ell: \quad C_L \chi_j^{1/2} G_j, \qquad h: \quad C_L \chi_j^{1/2} G_j^{1/4}.$$

There are four kinds of remainders after the exact second-order redistribution:

- Cubic integral remainders in centered potential increments have respective sizes

$$C_L \chi_j^{3/2} G_j^3, \qquad C_L \chi_j^{3/2} G_j^{3/4}.$$

- One $W_j$ times a centered increment has the same two sizes, by (58). Terms with two $W$ factors have sizes at most $C_L \chi_j^2 G_j^4$ and $C_L \chi_j^2 G_j$, which are smaller after the large-field weight.

- Bilinear differences from extraction obey

$$\|W(V-Q) - W(V)\|_{\ell,\mathrm{stable}} \le C\chi_j G\tau, \qquad G^{9/4}\|W(V-Q) - W(V)\|_{h,\mathrm{stable}} \le C\chi_j G^{1/2}\tau.$$

  This follows from bilinearity, $\|Q\|_\ell \le C\tau$, $\|Q\|_h \le C\tau G^{-9/4}$, and the covariance contraction factor $\ell^2/h^2 = O(G^{1/2})$. Field-independent source constants do not enter $W$.

- In the final canonical chart let $a = \chi_j G^2 + \tau + \|K\|$ bound the redundant stiffness difference. The singleton difference has stable bounds $C\chi_j G a$ at $\ell$ and $C\chi_j G^{-1/2} a$ at $h$. Its exponential multiplier is bounded by the spare regulator powers. After multiplication by $G^{9/4}$, these bounds are admissible in the same output norm.

Gaussian expectation of products of the segment remainders is bounded by the determinant/Sobolev argument even when segment parameters differ between blocks. Thus all pure-potential outputs have mixed size at most $C_L \chi_j^{3/2} G_j^3$. This conclusion counts $W$ as degree two; it does not assert that every higher-order term contains three fresh centered increments.

For large output polymers the required size weight is also obtained from the perturbative factors. Each single-block increment has size at most $C_L G^{1/4}$ at $h$, and each quadratic $W$ factor has size at most $C_L G^{1/2}$. Finite covariance range below a fixed fraction of the coarse block implies that each factor covers only a fixed number of coarse blocks. A connected contribution occupying $n$ coarse blocks therefore contains at least $cn - C_0$ such perturbative orders, for fixed positive $c, C_0$. Retain three orders for the $G^3$ bound above; the remaining factor $(C_L G^{1/4})^{cn - C_0}$ supplies any prescribed strengthened large-set weight when $R$ is large after $\varrho$ is fixed. There are at most a fixed exponential number of connected choices. Boundedly many exceptional sizes are absorbed into $C_\varrho$. Terms containing input activities use the preceding circle-product count and are bounded by $C_L\|K\|$.

The exact Maps 1–5 formulas, their second-order covariance redistribution, and the componentwise final chart change in Part I are algebraic identities on these same normed algebras. Localization uses bounded finite zero-field inverses; the quartic remains stable on the complex potential disc. Consequently the preceding bounds give holomorphy and the uniform rough estimate on the whole joint ball, not only at zero activity. Combining the small-set lemma with the large-set circle count gives

$$\|D_K \Psi(V, 0)\| \le C L^{-\delta} + C_L \lambda^{1/2} + C_L G^{1/4} + o_\varrho(1).$$

The normal-form bulk coefficient bounds follow from Lemma 19.1. Fix $k_0, \kappa$ and the regulator powers universally first; then choose $L$, then $\varrho$, then the complex ball radius $c$, and finally the spread radius so that $\lambda, G$ are small. This makes the zero-activity derivative at most $1/32$. Choose $\epsilon$ afterwards so that the enlarged-ball Cauchy correction is less than $1/32$. This gives $\|D_K \Psi(V,K)\| < 1/16$, uniformly on $\|K\| \le \epsilon G^{9/4}$, as required by the recursion.

□

# 21 A uniform recursion with an irrelevant local kinetic coordinate

The following is an algebraic result. Theorem 2 supplies its one-step hypotheses. In the application $a_z = L^{s-2} < 1/8$ for sufficiently large $L$, and we use $d_0 = 1/4$.

**Proposition 21.1** (Uniform fractional normal-form tuning)**.** *Let $\chi_j = 2^{-(j-J)_+}$, where $J \in \mathbb{N} \cup \{\infty\}$, and suppose*

$$x_{j+1} = x_j - B_j x_j^2, \qquad 0 \le B_j \le C\lambda^2 \chi_j, \qquad B_j \ge c\lambda^2 \quad (j \le J).$$

*Let $0 \le a_z < 1/8$, $1/4 \le d_0 < 1/2$, and set $f_j = \chi_j^{3/2} x_j^3 + U d_0^j$. Consider the recursion*

$$\begin{aligned}
g_{j+1} &= g_j - B_j g_j^2 + e_j^g(v_j, K_j),\\
z_{j+1} &= a_z z_j - \theta_j g_j^2 + e_j^z(v_j, K_j),\\
\mu_{j+1} &= L^s \mu_j + \eta_j g_j - \xi_j g_j^2 - \omega_j g_j \mu_j - \pi_j g_j z_j + e_j^\mu(v_j, K_j),\\
K_{j+1} &= P_j(v_j, K_j), \qquad v_j = (g_j, z_j, \mu_j),
\end{aligned}$$

*where*

$$|\eta_j| \le C\lambda\chi_j, \qquad |\theta_j| + |\xi_j| \le C\lambda^3\chi_j, \qquad |\omega_j| + |\pi_j| \le C\lambda^2\chi_j.$$

*Assume that on the boxes*

$$|g_j - x_j| \le \varepsilon x_j, \quad |z_j| + |\mu_j| \le M\lambda x_j, \quad \|K_j\| \le M_K f_j$$

*the maps obey*

$$\begin{aligned}
|e_j(v,K)| &\le C(\chi_j^{3/2} x_j^3 + \|K\|),\\
\|P_j(v,0)\| &\le C\chi_j^{3/2} x_j^3, \qquad \|P_j(v,K) - P_j(v,K')\| \le \kappa\|K - K'\|, \quad \kappa < 1/8,\\
|e_j(v,K) - e_j(v',K')| &\le C_{M,M_K}(f_j|v - v'|/x_j + \|K - K'\|),\\
\|P_j(v,K) - P_j(v',K)\| &\le C_{M,M_K} f_j |v - v'|/x_j.
\end{aligned}$$

*The first two constants are uniform over fixed coordinate boxes; constants in the last two lines may depend on their fixed sizes. Suppose*

$$\lambda \to 0, \qquad x_0/\lambda^3 \to 0, \qquad U/(\lambda x_0) \to 0.$$

*For sufficiently small parameters and every prescribed $\|K_0\| \le U$, there is a controlled solution with $g_0 = x_0$, $z_0 = 0$, and $\mu_j \to 0$, satisfying*

$$|g_j - x_j| \le \varepsilon x_j, \qquad |z_j| + |\mu_j| \le C\lambda\chi_j x_j, \qquad \|K_j\| \le C f_j.$$

*It is the unique solution in a fixed box of these weighted sizes. The contraction factor and all constants are independent of $J$. For a finite horizon the same conclusion holds, with $\mu_T = 0$ in place of the condition at infinity; $z$ retains its initial condition $z_0 = 0$ and is not assigned an additional terminal value. If the maps, reference data and prescribed initial activity are continuous in external parameters in a common Banach-space chart, the tuned initial mass is continuous as well.*

*Proof.* Use Lemma 2.2 for the reference sequence, the adjacent envelope ratios, and the forward Green function.

Put

$$a = \frac{x_0}{\lambda^2} + \frac{U}{x_0}, \qquad c_* = \frac{x_0^2}{\lambda} + \frac{U}{\lambda x_0}.$$

Both tend to zero under the stated hypotheses. Since $f_{j+1}/f_j \geq 1/4$ for small parameters and $a_z < 1/8$,

$$\sum_{i<j} a_z^{j-1-i} f_i \leq C f_j.$$

The adjacent ratios of the weight $\chi_j x_j$ are at least $\frac{1}{2}(1 - C\lambda^2 x_0) > 1/4$, so

$$\sum_{i<j} a_z^{j-1-i} \chi_i x_i \leq C \chi_j x_j, \qquad \sum_{i \geq j} L^{-s(i-j+1)} f_i \leq C f_j.$$

The latter estimate follows from monotonicity of $f_i$ and the geometric mass Green function. In particular,

$$\sup_j \frac{f_j}{\lambda \chi_j x_j} \leq C c_*. \tag{59}$$

For its transient term use $\chi_j \geq 2^{-j}$ and the reference lower bound, which give $d_0^j x_0/(\chi_j x_j) \leq (2d_0)^j (1 + C\lambda^2 x_0 j) \leq C$.

Write $t_j = g_j - x_j$. Define a fixed-point map by solving the linearized $t$ equation forward from $t_0 = 0$, with forcing $-B_j t_j^2 + e_j^g$. Solve the stable kinetic equation forward:

$$z_j = \sum_{i<j} a_z^{j-1-i} (-\theta_i g_i^2 + e_i^z).$$

Writing $F_i^\mu = \eta_i g_i - \xi_i g_i^2 - \omega_i g_i \mu_i - \pi_i g_i z_i + e_i^\mu$, solve the mass backward:

$$\mu_j = -\sum_{i \geq j} L^{-s(i-j+1)} F_i^\mu.$$

The output activity is the original forward update $K_{j+1} = P_j(v_j, K_j)$. All forcings on the right use the input sequence. For a finite horizon truncate the backward mass sum at $T - 1$; the other formulas are unchanged.

Use the maximum of the four sup norms divided respectively by $\varepsilon x_j, M\lambda x_j, M\lambda x_j, M_K f_j$. Choose $M_K$ so large that $4C/M_K + 4\kappa < 1$, and then choose $M$ large enough for the leading mass forcing $\eta_j g_j$. Separate bounds on $z, \mu$ fit the hypotheses with $M$ replaced by $2M$. The kinetic forcing has normalized size $O(\lambda^2 x_0 + c_*)$; the nonleading mass forcings have normalized size $o(1)$, by (59). The normalized forward quartic output is at most $C\varepsilon + Ca/\varepsilon$, by (6). Thus a small fixed $\varepsilon$, followed by small parameters, makes the product box invariant.

For explicit contraction estimates let $D_t, D_z, D_\mu, D_K$ denote the four normalized input distances. The forward Green estimate yields

$$D_t^{\mathrm{out}} \le (C\varepsilon + o(1))D_t + o(1)(D_z + D_\mu + D_K).$$

Here the error coefficients are bounded by fixed multiples of $a, M\lambda a/\varepsilon, M_K a/\varepsilon$. For the forward kinetic coordinate,

$$D_z^{\mathrm{out}} \le o(1)(D_t + D_z + D_\mu + D_K).$$

Indeed its leading $t$-coefficient is $O((\varepsilon/M)\lambda^2 x_0)$; its error coefficients are bounded by fixed multiples of $(\varepsilon/M)c_*, \lambda c_*, (M_K/M)c_*$. For the mass coordinate the only nonvanishing leading row entry is from $\eta_j g_j$:

$$D_\mu^{\mathrm{out}} \le (C\varepsilon/M + o(1))D_t + o(1)(D_z + D_\mu + D_K).$$

The other perturbative mass coefficients contain $\lambda^2 x_0$ or $\lambda^2 x_0^2$, and its error entries are controlled by $c_*$. Finally,

$$D_K^{\mathrm{out}} \le 4\kappa D_K + \frac{4C_{M,M_K}}{M_K}(\varepsilon D_t + M\lambda(D_z + D_\mu)).$$

Choose $\varepsilon$ after the fixed box sizes, then make the parameters small. The maximum row sum is bounded by some $q < 1$ independent of $J$ and of the finite horizon. Banach's theorem therefore gives the fixed point and uniqueness in the box.

At that fixed point, all mass forcings are bounded by $C\lambda\chi_j x_j$, using (59) and the preliminary box bounds. The backward geometric sum preserves this weight. The kinetic forcing has the same bound and its forward geometric sum preserves it as well. This proves the stronger displayed estimates with $\chi_j$. They tend to zero at a finite mass cutoff; when $J = \infty$, $x_j \to 0$ by the lower bubble bound. For continuity, iterate the same uniformly contracting sequence map. At each fixed scale its forward coordinates use finite sums. The only backward sum is the mass sum, which has a locally uniform geometric majorant $CL^{-s(i-j+1)}\lambda x_0$. Dominated convergence therefore makes every fixed coordinate of each finite iterate continuous. The common contraction factor makes convergence of the initial mass uniform on a small parameter neighborhood, and proves its asserted continuity. □

# 22 Initialization, physical tuning, and the critical limit

We describe the consequences of the one-step theorem, keeping its use separate from the covariance construction. In this section the one-step estimates of Theorem 2 are used for the exact four-sector map, with constants independent of the spread radius, mass cutoff and torus volume.

## 22.1 Finite-range reference and the initial activity

For the reference in Section 17.1, $\|\beta Q\|_1 \le q_* < 1$, $\mathrm{range}(Q) \le C_N R$, and $\|Q\|_\infty = O(R^{-d})$. The diagonal contributes an omitted self-edge to the cavity comparison, with error $O(R^{-d})$ as in the ancestor-edge estimate. Proposition 4.1 therefore gives, with $g = R^{-d}$,

$$\sup_i \sum_x |E(i,x)| e^{\eta|i-x|/R} \le Cg.$$

Indeed every branch of the cavity recursion has an error at depth $n$ at most $C(n+1)g$, and its weighted row contraction is $q' < 1$. Summing $Cg\sum(n+1)(q')^n$ proves the bound. The joint field/link estimates and real coercivity use the same recursion and contain no spatial-dimensional

identity. With $B = I - \beta Q$, the exact change $h = Bu$ makes the leading mean-field quartic exactly $\sum u_i^4/12$, independently of all reference interpolation links. The residual Hessian satisfies (46), so Proposition 18.1 applies to its nonconstant part.

Set

$$A_{\mathrm{sc}} = g^{-a_*}, \qquad 1/12 < a_* < 1/10, \qquad \lambda = A_{\mathrm{sc}}^{-2}, \qquad \psi = \sqrt{\lambda} R^{s/2} u.$$

The rescaled covariance has slices $\Gamma_{j+1}$, and the coefficient of $\psi^4/4$ in the on-site reference quartic is $G_0 = A_{\mathrm{sc}}^4 g/3$. In the original field units the initial radii are obtained by dividing the $\psi$-radii by $\sqrt{\lambda} R^{s/2}$. The all-field connected initialization of Proposition 8.1 uses the cavity bounds and

$$|B| h_u^4 \asymp 1, \qquad h_u \asymp R^{-d/4}, \qquad \ell_u' \asymp A_{\mathrm{sc}} R^{-d/2}, \qquad \ell_u'/h_u \to 0.$$

Its Taylor and regulator estimates in these dimensions are explicitly

$$|B| g (\ell_u')^4 + |B| (\ell_u')^6 \le C A_{\mathrm{sc}}^6 g^2, \qquad |B| g h_u^4 + |B| h_u^6 \le C g^{1/2}.$$

The joint link coefficient extraction is unchanged: its graph has bounded degree depending only on $d, N$, the adaptive radii are held fixed during differentiation, and the forest factorization uses the finite-range $B_t$, not the long-range Gaussian numerator. Thus in the fractional norm, including its sources,

$$\|K_0\| \le U = C(A_{\mathrm{sc}}^6 g^2 + G_0^{9/4} g^{1/2}).$$

The three ratios tend to zero as $g \downarrow 0$:

$$\frac{U}{G_0^{9/4}} \le C(g^{3a_* - 1/4} + g^{1/2}), \tag{60}$$

$$\frac{G_0}{\lambda^3} = \frac{1}{3} g^{1 - 10 a_*}, \qquad \frac{U}{\lambda G_0} \le C(g^{1 - 4a_*} + g^{7/4 - 7a_*}). \tag{61}$$

The ratios (60)–(61) put the initial activity inside the enlarged ball and ensure the smallness conditions of Proposition 21.1.

### 22.2 Mass matching at finite volume

Use the Hessian lift and periodization described in Section 7. Work on $\Lambda_N = (\mathbb{Z}/D_N\mathbb{Z})^d$ with $D_N = bL^N$. Only the reference Hessian is lifted; its exponential bound follows from the finite-range reference. The fractional numerator itself is periodized directly. The auxiliary-space construction commutes with periodization, since each polynomial piece is a translation-invariant finite-range kernel.

The potential Wick formulas are algebraic Gaussian identities. The bounds in Lemma 19.1 give the bubble and cubic coefficients required in Proposition 21.1. In physical field units the local kinetic coefficient is unchanged by reblocking; after multiplication by $b_j^{s-2}$ its linear factor is $L^{s-2}$. Thus that proposition applies with $z_0 = 0$, forward evolution of $z$, and only a backward terminal condition for the mass. The initial normal-form change is the identity because the cumulative covariance is zero. For each external covariance parameter it returns a continuous initial mass $\mu_0^*$ with $|\mu_0^*| \le C\lambda G_0$, uniformly in the finite horizon and the auxiliary mass.

The residual row sum in Section 17.1 is

$$r(\beta, \mu) = \frac{(1 - \beta q)(1 - \mu - \beta)}{\beta p} - (1 - \beta q)^2 \widehat{E}(0).$$

The complex cavity comparison gives $\partial_\beta \widehat{E}(0) = O(g)$, so $\partial_\beta r = -1 + O(g)$ on the physical box. The bare normalized mass is $\mu_0^{\mathrm{bare}} = b^s r/(\lambda R^s)$. Therefore the continuous map

$$r \longmapsto (R/b)^s \lambda \mu_0^*(r)$$

has image $O(\lambda^2 G_0) = O(g)$. The constant can be chosen before the interval size: use a fixed comparison domain $|\beta - 1| + \mu + \|K\|_{\exp} \le \varepsilon_0$ for the covariance bounds. The initialization absorbs the mass factor uniformly once $C_1 g^{1/2}$ is small. Thus none of the image constants needs to grow with $C_1$. First choose a sufficiently large fixed interval $[-C_1 g, C_1 g]$, then take $R$ large. It is a self-map and has a fixed point. Continuity here can be checked without assuming a common variable-weight Banach chart: at fixed finite horizon, every finite fixed-point iterate is continuous in the local field jets by Gaussian domination and the finite algebraic maps. The common contraction makes the scalar initial-mass iterates converge uniformly on the physical interval. Their limit is continuous. At the fixed point the bare physical mass agrees with the tuned mass, whereas $g_0 = G_0$ and $z_0 = 0$ already agree. No second physical tuning of a marginal local stiffness is used in $d < 4$. The argument works uniformly on finite tori with terminal normalized mass zero.

### 22.3 The final Gaussian integral

At fixed positive auxiliary mass, let the torus side tend to infinity and stop the local flow with $L^d$ blocks remaining. For a covariance piece of physical range $r_j \asymp RL^j$ beyond the mass scale, Proposition 18.1 implies

$$\|\nabla^k C_j\|_\infty \le C\lambda \mu^{-2} R^{2s} r_j^{-(d+s+k)}.$$

Periodization on a torus of side $D_N$ costs at most $C(1 + r_j/D_N)^d$ images. Summing the pieces with $r_j \gtrsim D_N$ gives

$$\|\nabla^k C_{\mathrm{tail}}^{\Lambda_N}\|_\infty \le C\lambda \mu^{-2} R^{2s} D_N^{-(d+s+k)}.$$

After division by the squared fluctuation radius and its derivative units this is $O_{R,\mu}(D_N^{-d})$. The Sobolev/determinant argument for the fixed number of remaining blocks gives the last-scale regulator moments and

$$Z_{\mathrm{tail}}(0) \ge \tfrac{1}{2}, \qquad \|Z_{\mathrm{tail}}\|_{T_{\ell_{N-1}}(0)} \le C_L.$$

It follows that

$$|\Lambda_N|^{-1} |D^2_{\mathbf{1},\mathbf{1}} \log Z_{\mathrm{tail}}(0)| \le C_L D_N^{-s}, \tag{62}$$

since $d = 2s$ and $\ell_{N-1}^{-2} \asymp D_N^s$.

### 22.4 Susceptibility and the critical endpoint

The covariance in (47) gives

$$\widehat{C}^{\Lambda_N}(0) = \frac{\lambda R^s \beta p}{(1 - \beta q)\mu}$$

for the rescaled field. Indeed $B - cA$ has row sum $\mu$ and the absorbed kernel $K$ has zero row sum. With the tail derivative bound, (9) gives

$$\chi_\sigma^{\Lambda_N}(\beta_{\mu,N}) = \frac{1 - \beta_{\mu,N} q}{\beta_{\mu,N} p\, \mu} - \frac{1}{\beta_{\mu,N} p} + O_{R,\mu}(D_N^{-s}). \tag{63}$$

The coefficients of $\mu^{-1}$ and the constant term are continuous in $\beta$ on the physical interval, and the first is bounded away from zero. Apply Proposition 2.5 to the off-diagonal part of $D^+$. Every

massive limiting temperature is subcritical and the limits approach its critical temperature $\beta_c^+$ as $\mu \downarrow 0$. In the original parameter, $\tau_c = \beta_c^+/(1+\varepsilon_R)$.

At fixed $\mu > 0$, choose a subsequence on which the compact physical parameters converge. The lifted reference Hessians converge in a weaker exponential kernel norm by the cavity locality bound. Proposition 18.1 gives continuity of every fixed covariance piece. The initial activity on a fixed polymer is continuous by its exact finite induced-volume forest formula. Every fixed number of RG steps uses finitely many local polymer inputs and Gaussian integrals with the uniform regulator majorants. Its field jets therefore converge locally. Repeating the local argument along a sequence $\mu \downarrow 0$ gives the massless trajectory at each fixed scale.

### 22.5 Linear and mixed sources

The mass matching uses the zero-source trajectory. On that trajectory, iterate the graded map with its initial sources. Projection to the bulk commutes with the map. The same activity contraction gives $\|K_j\| \le C f_j$ in the full norm while the one-source couplings stay in a fixed bounded interval. We verify this bound on the couplings below, using their initial value one. The localization and Gaussian identities for sources are unchanged in physical field coordinates. Before coalescence, the fluctuation source radius obeys $\ell_{\sigma,j}\ell_j \asymp x_j$. For a single-source projection at an arbitrary fixed scale, place an unused second mark outside that scale's dependence neighborhood. This keeps its fluctuation radius equal to $x_j b_j^{s/2}$ and bounds the extraction error by $C f_j/x_j$, independently of any later coalescence of the actual pair.

Write $\nu_j, g_j^{\mathrm{loc}}$ for the mass and quartic coefficients in the original polynomial coordinates. Let $v_{a,j}$ be the coefficient of the linear source at $a$, initially one, and $\widehat{v}_{a,j}$ its value after extraction. The same single-source projection gives $|\widehat{v}_{a,j} - v_{a,j}| \le C f_j/x_j$. Put

$$w_j^{(1)} = \sum_y w_j(y), \qquad \eta_j' = 3\Gamma_{j+1}(0), \qquad d_j = (\nu_j + \eta_j' g_j^{\mathrm{loc}}) w_{j+1}^{(1)} - \nu_j w_j^{(1)}.$$

The one-source Wick formula and the extraction estimate give

$$v_{a,j+1} = (1-d_j)v_{a,j} + e_{a,j}, \qquad |e_{a,j}| \le C f_j/x_j.$$

The inverse normal-form bounds and Lemma 19.1 give

$$b_j^s|\nu_j| \le C\chi_j x_j, \qquad b_j^s|\nu_{j+1} - \nu_j - \eta_j' g_j^{\mathrm{loc}}| \le C(\chi_j x_j^2 + f_j), \qquad |w_j^{(1)}| \le C\lambda b_j^s.$$

The leading kinetic term has zero row sum and contributes no linear source coefficient under constant-field localization. Its higher terms are covered by the displayed errors. Telescoping yields

$$\sum_{j=k}^{n-1} d_j = \nu_n w_n^{(1)} - \nu_k w_k^{(1)} + \sum_{j=k}^{n-1} (\nu_j + \eta_j' g_j^{\mathrm{loc}} - \nu_{j+1}) w_{j+1}^{(1)}.$$

The boundary term at infinity vanishes. The remaining sum, the sum of $d_j^2$, and $\sum |e_{a,j}|$ are bounded by

$$C\left(x_0/\lambda^2 + U/x_0\right) = o(1).$$

Their tails tend to zero uniformly in the marked site and as the mass cutoff tends to infinity. Expanding $\log(1-d_j)$ and using variation of constants proves that the source coefficients remain in a fixed positive interval and converge uniformly to positive limits. This also closes the assumed bounded source domain in the one-step theorem.

For a pair with coalescence index $k$, the mixed extraction is zero on small polymers before $k$. Afterwards its absolute value is at most

$$\frac{Cf_j}{x_j^2 b_k^s 4^{j-k}}.$$

The Gaussian part is the contraction of the two linear sources. Sum the extraction errors using $\chi_j \le \chi_k$, $x_j \le x_k$, and $x_k/x_j \le 1 + C\lambda^2 x_k(j-k)$, which follows by telescoping the reciprocals of the reference recursion. This gives

$$q_\infty(a,c) = \sum_{j\ge 0} \widehat{v}_{a,j}\widehat{v}_{c,j}\Gamma_{j+1}(a-c) + E(a,c), \qquad |E(a,c)| \le \frac{C}{b_k^s}\left(\chi_k^{3/2}x_k + \frac{Ud_0^k}{x_k^2}\right). \tag{64}$$

At zero mass this is $o(b_k^{-s})$. At fixed positive mass the last torus source integral is $O(\ell^{-1}_{\sigma\sigma,N-1}) \to 0$, since $x_{N-1}$ remains positive and $4^{N-1-k} \to \infty$. For fixed marks the covariance terms are bounded by $C\lambda b_j^{-s}$, and the mixed extraction terms by a constant times $4^{-(j-k)}(1+\lambda^2 x_0 j)^2$. Both majorants are summable uniformly in the mass cutoff. Thus the source sums pass first to infinite volume at positive mass and then to the massless limit.

The conditional Gaussian law in Section 17.1 identifies the limit of the massive interacting two-point functions with the $\psi$ two-point function constructed from the critical free spin state. Indeed $A$ and $B^{-1}A$ have summable kernels continuous in $\beta$ in the row-sum norm, and spin correlations are bounded by one and converge at every fixed pair by Proposition 2.5. The resulting $G_\psi$ is translation invariant and bounded also for near pairs. Applying Proposition 2.6 to the massless source sum gives

$$G_\psi(a,c) = v_\infty^2 c_{d,R}|a-c|^{-s} + o(|a-c|^{-s}), \qquad v_\infty > 0. \tag{65}$$

## 23 Convolution back to spin variables

For $d < 4$, consider the even-tail split with a strictly positive diagonal addition. Its multiplier $\widehat{A}$ is smooth away from zero and has a positive constant term. Near zero its difference from that constant, and the corresponding difference for $1/\widehat{A}$, satisfy symbol derivative bounds $C_{R,m}|k|^{s-m}$ of every fixed order $m$. This follows from (40), the analytic polynomial part of $P_N$, and the positive constant term in the inverse. The Fourier estimate in Section 2.8, with $\rho = s$, gives

$$|A^{-1}(x)| + |(BA^{-1})(x)| \le C_R(1+|x|)^{-d-s}$$

since the finite-range multiplier $\widehat{B}$ is smooth.

## 24 The critical spin asymptotic

*Proof of Theorem 1 for $d < 4$.* By (65), the interacting $\psi$ two-point function has positive leading coefficient $v_\infty^2 c_{d,R}$. Here $c_{d,R}$ is the Gaussian coefficient in Corollary 18.2. Apply the reconstruction formula (8) with $\beta = \beta_c^+$. The kernels $P_*$ and $(\beta P)^{-1}$ satisfy $O_R((1+|x|)^{-d-s})$ bounds for $d < 4$, as proved in Section 23. Use the row sum in (8) and apply (16) twice. Since $A^{-1}(x) = o(|x|^{-s})$, this gives

$$C_{d,R} = \frac{v_\infty^2 c_{d,R}}{\lambda R^s}\left(\frac{1-\beta_c^+ q}{\beta_c^+ p}\right)^2 > 0.$$

□

# A Local sextic subtraction and Gaussian transport

These estimates describe a local log-cosh weight and the effect of transporting a sextic factor. The box initialization in Proposition 8.1 uses the full transformed reference pressure.

## A.1 Local subtraction

Put

$$U(t) = \tfrac{1}{2}t^2 - \log\cosh t, \qquad U_r(t) = \tfrac{r}{2}t^2 + U(t), \qquad p_r(t) = \tfrac{r}{2}t^2 + \tfrac{1}{12}t^4.$$

Then $U''(t) = \tanh^2 t$, and

$$U(t) \geq c\min(t^4, t^2), \qquad U(t) = \tfrac{1}{12}t^4 - \tfrac{1}{45}t^6 + O(t^8). \tag{66}$$

Consider a box $B$ of side $R$ and cardinality $n = R^4$. In microscopic field units let

$$h = R^{-1}, \qquad \ell = R^{-2}, \qquad g = R^{-4}, \qquad |r| \leq C_0 g.$$

This is a local estimate under a mesoscopic small-field normalization. The application to the box interaction requires obtaining such a normalization for the exact weight.

**Proposition A.1** (Local sextic subtraction)**.** *Set*

$$P_B = \sum_{y\in B} p_r(\phi_y), \qquad S_B = \tfrac{1}{45}\sum_{y\in B}\phi_y^6, \qquad E_B = e^{-\sum_{y\in B} U_r(\phi_y)} - e^{-P_B}(1+S_B).$$

*For a fixed sufficiently small $\kappa > 0$, uniformly in $R$ sufficiently large,*

$$\sup_\phi \frac{|E_B|_{\ell,\phi}}{G_B(\phi)} \leq Cg^3, \qquad \sup_\phi \frac{|E_B|_{h,\phi}}{H_B(\phi)\widetilde{G}_B(\phi)} \leq Cg. \tag{67}$$

*The first bound fails at order $g^3$ for the direct subtraction with $S_B = 0$. The latter subtraction also leaves an error of order $g^{1/2}$ at $\phi = h\mathbf{1}$.*

*Proof.* Write $M(t) = \min(t^4, t^2)$. For $|t| \leq 1$, the first three derivatives of $U$ are bounded by $C|t|^3, C|t|^2, C|t|$. The sum of the terms of degree at least four in its absolute Taylor majorant is $O(h^4)$, by uniform analyticity of $U''$ in a fixed strip. For $|t| \geq 1$ the first derivative is $O(|t|)$, and the remaining Taylor terms sum to $O(h^2)$. Young's inequality and $|r| \leq C_0 h^4$ give, for $0 < a \leq h$,

$$|e^{-U_r}|_{a,t} \leq e^{Ch^4 - cM(t)}, \qquad |e^{-p_r}|_{a,t} \leq e^{Ch^4 - ct^4}.$$

The same bound with $M$ holds for the exponential of the negative interpolated potential $(1-s)p_r + sU_r$, $0 \leq s \leq 1$. These bounds follow by taking the exponential of the absolute nonconstant Taylor terms and retaining the exact constant term in the exponent. They give $C\exp(-c\sum_B M(\phi_y))$ on the block.

Let $\Delta = \sum_B U_r - P_B$. The identity

$$E_B = -e^{-P_B}(\Delta + S_B) + \Delta^2 \int_0^1 (1-s)e^{-P_B - s\Delta}\, ds$$

is exact. On the single-site level, the absolute Taylor majorants of $U_r - p_r$ and $U_r - p_r + t^6/45$ are bounded by $C(|t|+a)^6$ and $C(|t|+a)^8$, respectively. Near zero this follows from the convergent

expansion, and away from zero it follows from the uniform strip estimates and polynomial growth. With $M_\phi = \|\phi_B\|_\infty$, we obtain

$$|E_B|_{a,\phi} \le C\big[n(M_\phi + a)^8 + n^2(M_\phi + a)^{12}\big]e^{-c\sum_B M(\phi_y)}. \tag{68}$$

For $a = \ell$, division by $G_B$ absorbs the polynomial and gives $O(n\ell^8 + n^2\ell^{12}) = O(g^3)$. For $a = h$, let $p$ be an affine approximation on the block. On affine fields the supremum is bounded uniformly by the normalized block $L^2$ norm. Consequently

$$\frac{M_\phi}{h} \le C\left(\frac{\|\phi_B\|_2}{\sqrt{n}h} + \frac{\|\phi_B - p\|_\infty}{h}\right).$$

Use $M(t) \ge c'h^2t^2 - C'h^4$, $nh^4 = 1$, and $\ell \le h$. A fraction of the exponential in (68) controls the first term in parentheses after division by $H_B$, and $\widetilde{G}_B$ controls the second. The result is $O(nh^8 + n^2h^{12}) = O(g)$. For the lower bound on direct subtraction, the sixth Taylor coefficient tested along the constant direction has size $n\ell^6/45 = g^2/45$. Evaluating the direct subtraction at $h\mathbf{1}$ gives $(e^{-1/12}/45 + o(1))nh^6$. □

The Gaussian part of the regulator propagates exactly. If $\zeta_B$ has covariance $\Gamma_B$ and $b = \kappa/(nh^2)$, completing the square gives

$$\mathbb{E}e^{-b\|\phi_B+\zeta_B\|_2^2} = \det(I + 2b\Gamma_B)^{-1/2}\exp\{-b\langle\phi_B, (I + 2b\Gamma_B)^{-1}\phi_B\rangle\}. \tag{69}$$

In particular it is at most $H_B(\phi)^{1/2}$ if $2b\|\Gamma_B\|_{\mathrm{op}} \le 1$. This is the mechanism of [4, Appendix C]. In the initial comparison, Park's large-field weight $g^{9/4}$ makes the second bound in (67) at most order $g^3$. The direct comparison instead has a weighted contribution of order $g^{11/4}$.

## A.2 Gaussian transport with quartic and source factors

Here we use the canonical field at scale $j$. Let $X$ contain at most $M$ blocks of side $L^j$, with $M$ fixed. Set

$$n = L^{4j}, \qquad \ell = L^{-j}, \qquad h = g^{-1/4}\ell.$$

For the local statements below assume

$$\max_{x,y\in X}|\Gamma_{xy}| \le C_\Gamma\ell^2, \qquad \mathbb{E}(1 + \|\zeta_X\|_\infty/\ell)^k \le C_{k,M} \quad (k \ge 1). \tag{70}$$

The second condition holds for the usual smooth finite-range slices, by a block Sobolev estimate and Gaussian moments. It is included explicitly among the hypotheses here.

Use commuting nilpotent variables $t_a, t_b$ with $t_a^2 = t_b^2 = 0$. The absolute Taylor majorant includes the sum of the four source sectors, weighted by $1, \tau_a, \tau_b, \tau_a\tau_b$. We assume $\tau_\#\ell \lesssim g$ for the fluctuation norm and $\tau_\#h \lesssim 1$ for the large-field norm. For a bulk sextic $S_X = u\sum_{x\in X}\phi_x^6$ assume

$$n|u|\ell^6 \le Cg^2d_j, \qquad 0 \le d_j \le 1. \tag{71}$$

**Lemma A.2** (Linear-source transport)**.** *Assume $a, b \in X$ and let $J_s(\phi) = \lambda_a t_a\phi_a + \lambda_b t_b\phi_b$, with bounded $\lambda_a, \lambda_b$. Define*

$$\mathcal{H}_{6,v}(t) = t^6 + 15vt^4 + 45v^2t^2 + 15v^3.$$

*With $v_x = \Gamma_{xx}$ and $b_x = \lambda_a t_a\Gamma_{xa} + \lambda_b t_b\Gamma_{xb}$, one has*

$$\mathbb{E}\left[e^{J_s(\phi+\zeta)}S_X(\phi + \zeta)\right] = e^{J_s(\phi)+\lambda_a\lambda_b s_a s_b\Gamma_{ab}}\,u\sum_{x\in X}\mathcal{H}_{6,v_x}(\phi_x + b_x). \tag{72}$$

*After dividing by the displayed exponential, the two single-source coefficients are given by*

$$6u\lambda_{\#}\sum_{x\in X}\Gamma_{x\#}(\phi_x^5+10v_x\phi_x^3+15v_x^2\phi_x),$$

*and the double-source coefficient is*

$$30u\lambda_a\lambda_b\sum_{x\in X}\Gamma_{xa}\Gamma_{xb}(\phi_x^4+6v_x\phi_x^2+3v_x^2).$$

*Their weighted Taylor norms at zero are, respectively,*

| | $\ell$ *scale* | $h$ *scale* |
|---|---|---|
| *one source* | $O_M(g^3d_j)$ | $O_M(gd_j)$ |
| *two sources* | $O_M(g^4d_j)$ | $O_M(g^{3/2}d_j)$. |

*For a finite-range slice of range $r_\Gamma$, the last polynomial vanishes if $|a-b|>2r_\Gamma$.*

*Proof.* Complete the square in the Gaussian expectation with a linear source and then take the coefficient in each sector. Equivalently, apply the identity with real source parameters and take the Taylor polynomial with each source degree at most one. Each source replaces one field by $\lambda_{\#}\Gamma_{x\#}$ and carries an additional factor $\tau_{\#}$. Thus one source costs at most $C\tau_{\#}\ell^2/\mathfrak{h}$ relative to the bulk size at field radius $\mathfrak{h}$. At $\ell$ this is $O(g)$ and at $h$ it is $O(g^{1/2})$. The bulk sizes at the two radii are $O_M(g^2d_j)$ and $O_M(g^{1/2}d_j)$, by (71). The support statement follows from the two covariance factors. □

**Proposition A.3** (Local mixed quartic and source bound)**.** *Assume* (70) *and* (71)*. Let*

$$V_X=\sum_{x\in X}(g\phi_x^4/4+\nu\phi_x^2/2),\qquad n|\nu|\ell^2\le Cg,\qquad I_X=e^{-V_X+J_s}.$$

*Put $\mathcal{E}_X=\mathrm{Cov}_\Gamma(S_X(\phi+\zeta),I_X(\phi+\zeta))$. In the full source-weighted Taylor majorant,*

$$\sup_\phi\frac{|\mathcal{E}_X|_{\ell,\phi}}{G_X(\phi)}\le C_Mg^3d_j,\qquad \sup_\phi\frac{|\mathcal{E}_X|_{h,\phi}}{H_X(\phi)\widetilde{G}_X(\phi)}\le C_Mgd_j.$$

*These estimates bound the double-source sector as well as the two single-source sectors.*

*Proof.* Take two centered Gaussian vectors $(\zeta_t,\zeta_t')$, each with marginal covariance $\Gamma$ and with cross covariance $t\Gamma$. Gaussian differentiation of the cross covariance, followed by integration in $t$, gives

$$\mathcal{E}_X=\int_0^1\sum_{x,y\in X}\Gamma_{xy}\mathbb{E}[\partial_xS_X(\phi+\zeta_t)\partial_yI_X(\phi+\zeta_t')]\,dt. \tag{73}$$

The identity holds coefficient by coefficient in the source algebra. For $\mathfrak{h}\in\{\ell,h\}$ write $p(\psi)=1+\|\psi_X\|_\infty/\mathfrak{h}$. Submultiplicativity, the quartic Taylor estimate in the proof of Proposition A.1, and the source bounds give

$$\sum_{x\in X}|\partial_xS_X|_{\mathfrak{h},\psi}\le C_Mn|u|\mathfrak{h}^5p(\psi)^5,$$
$$\sum_{y\in X}|\partial_yI_X|_{\mathfrak{h},\psi}\le C_M(ng\mathfrak{h}^3+n|\nu|\mathfrak{h}+\tau_a+\tau_b)p(\psi)^5e^{-cg\sum_X\psi_y^4}.$$

Sources outside $X$ are omitted. In the second line the factors from $e^{J_s}$ and its field derivatives have been bounded as polynomials using nilpotence. At $\ell$ the prefactors in the two lines are bounded by $C_M g^2 d_j/\ell$ and $C_M g/\ell$. Using (70) in (73), the expectation leaves only a polynomial in $1 + \|\phi_X\|_\infty/\ell$. This is absorbed by $G_X$.

At $h$ the corresponding prefactors are $C_M g^{1/2} d_j/h$ and $C_M/h$. The covariance factor $C_\Gamma \ell^2$ then gives $C_M g d_j$. To keep the decay in the expectation, for any fixed $A > 0$ bound the quartic factor by

$$C_{A,M} \exp\{-A\|\phi_X + \zeta'_t\|_2^2/(nh^2)\}.$$

This follows by completing a square at each site, since $ngh^4 = 1$. Cauchy–Schwarz separates the polynomial fluctuation factor from this Gaussian factor. Apply (69) to the latter, using $\|\Gamma_X\|_{\mathrm{op}} \lesssim_M n\ell^2$ and $\ell^2/h^2 = g^{1/2}$. By taking $A$ a sufficiently large fixed multiple of $\kappa$, there is enough decay left to divide by $H_X$ and absorb powers of the block $L^2$ averages. The remaining powers of the deviations from affine fields are absorbed by $\widetilde{G}_X$, as in Proposition A.1. □

For the single Gaussian integration there is therefore an exact comparison

$$\mathbb{E}_\Gamma[(1 + S_X)I_X] = (1 + \overline{S}_X)\mathbb{E}_\Gamma I_X + \mathcal{E}_X, \qquad \overline{S}_X = \mathbb{E}_\Gamma S_X(\phi + \zeta). \tag{74}$$

The terms of degrees zero, two and four in $\overline{S}_X - S_X$ are explicit from $\mathcal{H}_{6,v}$. They are the required bulk counterterms at first order in the sextic. If $u$ is held fixed in microscopic units, then $n\ell^6$ decreases by $L^{-2}$ per step. In a stable quartic reference factor, replacing $1 + S_X + \overline{S}_X - S_X$ by $(1 + S_X)e^{\overline{S}_X - S_X}$ produces terms quadratic in the sextic, since $e^A - 1 - A = A^2 \int_0^1 (1-t)e^{tA}\,dt$. The quartic part of $A = \overline{S}_X - S_X$ is smaller than the quartic in $V_X$ by $O_M(g d_j)$. The stable interpolation bound thus applies after multiplication by the quartic reference. For nondegenerate $\Gamma$, restricting $\phi + \zeta$ to a fixed bounded set gives a Gaussian lower bound on the bulk convolution. Thus a positive quartic counterterm cannot be exponentiated directly against the fully averaged weight.

# B Diagnostics for the untransformed connected expansion

We keep the notation of Section 5. The first two estimates locate the mixed quartic in the untransformed chart. The subsequent estimates separate its size from the decay of large sets.

**Proposition B.1** (Size of the raw adjacent-block activity)**.** *Use the standard box split and take the minorant small enough that $Q_{xy} \geq cR^{-4}$ for $0 < |x - y|_\infty \leq R$. Let $B_1, B_2$ share a face, write $X = \{B_1, B_2\}$, and let $\ell \asymp R^{-2}$ be the fluctuation-field scale with fixed block aspect $b/R$. For $R$ sufficiently large, the source-free activity of Proposition 5.1 obeys*

$$|K(X)|_{\ell,0} \geq c' R^{-4}.$$

*Here this activity is compared with the singleton factor prescribed by the forest representation.*

*Proof.* Write $J_t = J_X(t)$, $S_t = (I - J_t)^{-1}$, and $r_t = S_t \mathbf{1}$, where $\mathbf{1}$ is the constant field on $\Lambda_X$. For $t = 0, 1$, the mean-field deficit computed in (26) has degree-four coefficient $\sum_i r_{t,i}^4/12$ at constant field $h = z\mathbf{1}$. The argument for (26) applies without translation invariance, as it only uses $S_t = (I - J_t)^{-1}$. The comparison of exact and mean-field pressure jets shows that the degree-four coefficient of $W_X(t, z\mathbf{1})$ differs from this by $O(R^{-4}|\Lambda_X|)$. Its degree-two coefficient is $O(R^{-4}|\Lambda_X|)$ as well, from the Hessian comparison and $|c - 1| \lesssim R^{-4}$.

Entrywise positivity of the walk expansion gives $S_1 - S_0 \geq J_1 - J_0 \geq 0$ and $r_{t,i} \geq 1$. There are at least $cR^4$ sites in the first block in a slab next to the common face and away from the other faces. For each such site $i$, at least $cR^4$ sites of $B_2$ lie within its interaction box. Thus $((J_1 - J_0)\mathbf{1})_i \geq c'$ on the slab. It follows that

$$\sum_i (r_{1,i}^4 - r_{0,i}^4) \geq 4 \sum_i (r_{1,i} - r_{0,i}) \geq c'' R^4.$$

For two blocks the exact tree integral is $K(X) = e^{-W_X(1,h)} - e^{-W_X(0,h)}$. Because $W_X(t,0) = 0$ and it is even, the degree-four coefficient of $e^{-W_X(t,z\mathbf{1})}$ is minus the quartic coefficient of $W_X$ plus half the square of its quadratic coefficient. After scaling $z$ by $\ell$, the exact-minus-mean-field quartic terms and the quadratic squares are both $O(R^{-8})$. The mean-field quartic difference has magnitude at least $c'' R^4 \ell^4 \asymp R^{-4}$. Evaluating the fourth field Taylor coefficient in the scaled constant direction is bounded in absolute value by the Taylor seminorm. This proves the claimed lower bound. □

**Corollary B.2** (A mixed quartic term cannot be removed by block-local sextics)**.** *Let $X = \{B_1, B_2\}$ be as in Proposition B.1. There are nonnegative smooth lattice functions $f_i$, supported in $B_i$, such that, with $h = \ell(t_1 f_1 + t_2 f_2)$,*

$$[t_1^2 t_2^2] e^{-W_X(1,h)} \leq -cg, \qquad g = R^{-4},$$

*for all sufficiently large $R$. If $I_i(h_{B_i})$ are even block-local factors, normalized by $I_i(0) = 1$, with quadratic Taylor coefficients of size $O(g)$ in these directions, then*

$$\left| [t_1^2 t_2^2] \left( e^{-W_X(1,h)} - I_1 I_2 \right) \right| \geq c' g.$$

*In particular this difference cannot have an $O(g^3)$ Taylor norm.*

*Proof.* Choose patches of side proportional to $R$ on opposite sides of the common face, and smooth bumps $f_i$ bounded below on smaller patches. Choose their supports so that every pair of points in the two smaller patches is at distance at most $R$. Then $\|\nabla^k f_i\|_\infty \leq C_k R^{-k}$. For $S = (I - J_X(1))^{-1}$, the coefficient in the leading quartic is

$$[t_1^2 t_2^2] \frac{1}{12} \sum_x (Sh)_x^4 = \frac{\ell^4}{2} \sum_x (Sf_1)_x^2 (Sf_2)_x^2 \geq cR^4 \ell^4.$$

Indeed $Sf_1 \geq f_1$, and $Sf_2 \geq J_X(1) f_2 \geq c$ on the first smaller patch. The reference comparison bounds the exact quartic error by $O(g^2)$ in these directions. All coefficients of $W_X^{(2)}(1,h)$ are $O(g)$, so its square contributes only $O(g^2)$ when exponentiating. This proves the first assertion. The mixed coefficient of $I_1 I_2$ is the product of their two quadratic coefficients, hence is $O(g^2)$. This proves the second. By rescaling the bumps by a fixed constant depending on the block aspect and the finite test-derivative order, the symmetrized tensor $\ell^4 f_1^{\otimes 2} \otimes f_2^{\otimes 2}$ is an admissible test tensor in the RG Taylor seminorm as well. A degree-six prefactor does not change the degree-four coefficient. □

Thus the raw connected expansion has a two-block term already at the order of the marginal quartic coupling. Improving bounds on large polymers or increasing a regulator coefficient cannot remove this low-degree term. A different reference factor or a localization of the two-block quartic is needed before applying the small-activity contraction.

**Proposition B.3** (Large-set decay at zero field)**.** *For every fixed $\omega \in (0,1)$, one can choose a fixed block aspect $b/R$ sufficiently large that the activities above satisfy, for $|X| \geq 2$,*

$$|K(X)|_{\ell,0} \leq C_\omega R^{-4} \omega^{|X|}, \qquad \ell \asymp R^{-2},$$

*for all sufficiently large $R$. This assertion concerns the zero-field Taylor seminorm, before the large-field part of the RG norm is imposed.*

*Proof.* Use the square-free link algebra and the change of coefficient radius from Section 5. At zero field the degree-two coefficient rows of $W_X$ have joint weighted norm $O(g)$, and all coefficients of field degree at least four have a uniform joint weighted norm. There are no terms of field degree zero or odd degree. The quadratic estimate, including nonconstant link coefficients, follows by joint comparison with the mean-field Hessian. Choose $D > 1$ and then $b/R$ large enough for (30). For each anchor this bounds the unweighted square-free link coefficient norm at radius $D$ by a fixed multiple of the old joint norm. Set $\epsilon = R^{-4}$. In the same algebra, with the field Taylor norm at zero of radius $\ell$, summing over anchors now gives

$$\|W_X(t+v,h)\|_{\ell,D} \leq C_D |\Lambda_X| (\epsilon \ell^2 + \ell^4) \leq C'_D \epsilon |X|.$$

The constant may depend on the chosen fixed aspect. We use a quotient norm only in the link variables; the field Taylor norm is the full absolute coefficient sum. The exponential therefore obeys

$$\|e^{-W_X(t+v,h)} - 1\|_{\ell,D} \leq C'_D \epsilon |X| e^{C'_D \epsilon |X|}.$$

Extract the square-free coefficient indexed by a spanning tree, which costs $D^{-(|X|-1)}$. Sum over at most $\Delta^{|X|-1}$ trees. Since $D$ can be made arbitrarily large, the resulting bound on $A_X$ has any prescribed geometric decay factor, times $C_D\epsilon$, after absorbing $|X|$ and the factor $e^{C'_D \epsilon |X|}$ into the rate. For a partition into $k$ connected parts of size at least two, retain the factors $D^{-(|C|-1)}$ from the estimate for each part. Their product is at most $D^{-|X|/2}$. Also $(C'_D\epsilon)^k \leq C'_D\epsilon$ once $R$ is large enough that $C'_D\epsilon \leq 1$, and the products of the remaining factors $|C| e^{C'_D \epsilon |C|} \Delta^{|C|-1}$ cost only a fixed exponential in $|X|$. This exponential and the connected-partition count are independent of $D$ after increasing $R$ as above. Choose $D$ large to prove the bound for $K$. □

**Proposition B.4** (Large-set decay with the decaying regulator)**.** *Fix $\omega \in (0,1)$. For a sufficiently large fixed block aspect $M = b/R$, set $a = b^{-1}$, take $\ell = R^{-2}$, and put*

$$H_X(h) = \exp\left( -\kappa a^2 \sum_{i \in \Lambda_X} |h_i|^2 \right).$$

*Let $\widetilde{G}_X$ be the product of the affine-quotient regulators at block side $b$ and fluctuation scale $\ell$, with at least four lattice derivatives in the defining test norm. Then for some fixed $\kappa > 0$, all real fields, and $|X| \geq 2$,*

$$|K(X)|_{a,h} \leq C_\omega \omega^{|X|} H_X(h) \widetilde{G}_X(h).$$

*The bound can include up to two field-linear nilpotent sources with $a\tau_z$ uniformly bounded. The same bounds hold for the individual connected tree activities. This is a large-set estimate for the raw forest chart, without the $g^3$ smallness conclusion.*

*Proof.* Use the anchored joint norm of Section 5 at an arbitrary real field. Choose $D > 1$ for the square-free link coefficient radius and then $M$ large enough for (30). If $S \neq \varnothing$, at least one

microscopic link endpoint in its witness tree lies within $R$ of a coarse block boundary. Define $w(i) = \exp(-\eta' \operatorname{dist}(i, \partial B_i)/R)$, where $B_i$ is the block containing $i$, and choose $0 < \eta' \le \eta/2$. The leftover tree factor is bounded by $Cw(i)$. Thus at any real background the absolute sums of the field derivative rows of positive link degree, measured at link radius $D$, have a bound $C_D w(i)$ at anchor $i$ for each fixed derivative order at least two. The first derivative need not be bounded at arbitrary field; below it is used only at a field of supremum norm at most one. When summed in field Taylor degree at radius at most $r/2$, their coefficient bounds are geometric. For the Hessian at zero the same statement has the extra factor $\epsilon = R^{-4}$, by joint comparison with the mean-field Hessian.

Let $\mathcal{Q}(v, h) = W_X(t + v, h) - W_X(t, h)$ in the square-free link algebra. We claim

$$\|\mathcal{Q}(v, \cdot)\|_{a,h;D} \le C_D \sum_{i \in \Lambda_X} w(i)(\Psi(h_i) + \epsilon h_i^2 + a^4). \tag{75}$$

For the quadratic term at zero, use the weighted Hessian bound and symmetry of its field indices, bounding $(|h_i| + a)(|h_j| + a)$ by the average of their squares. For the remaining term subtract this quadratic. Its field Taylor coefficients through degree three vanish at zero. For derivative order $k \le 3$ at the field restricted to $|h_i| \le 1$, integrate the fourth derivative along the segment from zero. Bound each field product times $a^k$ by a constant times the sum of the fourth powers of the participating field coordinates and $a^4$. Because the fourth derivative is symmetric in all field indices, anchor at the index carrying each fourth power before using the row bound. The $a^4$ terms use the sum of the rows. Next move to the full field. For $1 \le k \le 3$, the difference is controlled by the $(k + 1)$-st derivative, anchoring at the index of the large field increment. For $k = 0$, use the first derivative at the restricted field and the integrated Hessian for the increment. The first derivative is bounded rowwise by $C_D w(i)$ since the restricted field has supremum at most one. Thus the cost on the large coordinates is bounded by $C_D \sum_{|h_i|>1} w(i)(|h_i| + h_i^2)$, which is at most a constant times the $\Psi$ contribution. All Taylor orders $k \ge 4$ together are bounded by $C_D a^4 \sum_i w(i)$ using the geometric row bounds. This proves (75).

We record the elementary boundary estimate used to control (75). For $p = 4$ and each block $B$, write $D_{p,B} = \max_e b^p \|\nabla_e^p h\|_{\infty,B}$, where only forward differences whose full stencil lies inside $B$ are included. Then

$$\sum_{i \in B} w(i)\Psi(h_i) \le CM^{-1/2} \sum_{i \in B} \Psi(h_i) + C|B|M^{-1-p} D_{p,B}^2. \tag{76}$$

To prove this, bound $w$ by the sum of the corresponding weights for the eight faces. For one face and a site $i$ within $b/4$ of it, let $e$ point inward and set $q = \lfloor b/\sqrt{M} \rfloor$. For $1 \le s \le q$ the $p$-th finite-difference identity writes $h_i$ as a linear combination of $h_{i+lse}$, $1 \le l \le p$, with coefficients of bounded absolute sum and error at most $s^p \|\nabla_e^p h\|_{\infty,B}$. Take $M$ large enough that all the stencils are inside the block. Convexity and fixed-scale bounds on $\Psi$, together with $\Psi(u) \le u^2$, give a bound on $\Psi(h_i)$ by the corresponding sum of $\Psi(h_{i+lse})$ and $C(q/b)^{2p} D_{p,B}^2$. Average over $s$. For fixed $l$ and a given image site, the weights of its possible preimages are bounded by a geometric sum of order $R$ along the normal line. The average therefore costs $CR/q$. The weighted count of sites in the block is $O(|B|/M)$. Sites more than $b/4$ from the face contribute an exponentially small coefficient times $\sum_B \Psi$. Summing over faces proves (76).

The affine-quotient regulator dominates $\exp(c\ell^{-2} D_{p,B}^2)$. Indeed, subtracting any affine function preserves the $p$-th differences inside $B$, and the field norm on the block neighborhood includes their normalized supremum. Since $|B|\ell^2 \asymp M^4$, the last term in (76) costs at most $CM^{3-p} \log \widetilde{G}_B$.

Also, for large $R$ at fixed $M$, $\epsilon s^2 \le \Psi(s) + C\epsilon^2$ and $\epsilon^2 \le a^4$. First fix $D$, then take $M$ sufficiently large. Equations (75) and (76) imply

$$\|\mathcal{Q}(v,\cdot)\|_{a,h;D} \le \delta \sum_{i\in\Lambda_X} \Psi(h_i) + \tfrac{1}{2}\log \widetilde{G}_X(h) + |X|$$

with any prescribed fixed $\delta > 0$. In the last term we used $a^4|B| = 1$ and $\sum_B w \le C|B|/M$. The constants in the real restricted-weight stability bound are independent of $D, M$, provided $R$ is sufficiently large depending on $M$. In fact its proof applies at any $a \le R^{-1}$, with the quadratic loss contributing at most $O(|\Lambda_X|\epsilon^2)$, which is at most $O(|\Lambda_X|a^4)$ here.

Use submultiplicativity in the square-free link algebra to exponentiate $\mathcal{Q}$, and multiply by the real weight. Taking $\delta$ smaller than half the real damping coefficient bounds the resulting joint norm by

$$e^{C|X|} \exp\left(-c' \sum_i \Psi(h_i)\right) \widetilde{G}_X(h)^{1/2},$$

where $C$ is independent of $D$. By $\Psi(s) \ge ca^2s^2 - Ca^4$, the exponential supplies $H_X$ with slack. For a source on $B$, its extra polynomial is controlled with this slack and the unused fraction of $\widetilde{G}_B$. Explicitly, the affine-quotient inequality in the proof of Proposition A.1 bounds $|h_z|/a$ by a constant times $a\|h_B\|_2$ and the affine deviation divided by $a$, since $|B|a^4 = 1$ and $\ell \le a$ for large $R$. The field-linear source factors can therefore be absorbed uniformly for $a\tau_z$ bounded.

Finally extract the square-free coefficient of each spanning tree, which costs $D^{-(|X|-1)}$. This leaves an exponential constant per block independent of $D$, including the number of spanning trees and the normalized damping loss. In the partition formula for $K$, the product of the factors $D^{-(|C|-1)}$ is at most $D^{-|X|/2}$, since all parts have size at least two. The partition count costs only another fixed exponential. Choose $D$ large to obtain the prescribed geometric factor, then choose $M$ as above. □

# C Localizing the mixed quartic

For a symmetric quartic tensor $C(i_1, i_2, i_3, i_4)$, write

$$Q_4(\phi) = \sum_{\boldsymbol{i}} C(\boldsymbol{i}) \prod_{k=1}^{4} \phi_{i_k}, \qquad q_i = \sum_{i_2,i_3,i_4} C(i, i_2, i_3, i_4).$$

We use ordered indices. The tensor includes whatever factorial is used in the convention for the Taylor polynomial. Assume that its anchored exponential moment is bounded, in the sense that

$$\sup_{i_1} \sum_{i_2,i_3,i_4} |C(\boldsymbol{i})| e^{\eta\tau(\boldsymbol{i})/R} \le C_4, \tag{77}$$

where $\tau(\boldsymbol{i})$ is the length of a shortest lattice tree joining the four sites, measured in the periodic lattice metric when the volume is a torus.

**Proposition C.1** (A two-difference quartic remainder)**.** *The exact localization*

$$\sum_i q_i\phi_i^4 - Q_4(\phi) = \sum_{\boldsymbol{i}} C(\boldsymbol{i}) \left(\frac{1}{4}\sum_{k=1}^{4} \phi_{i_k}^4 - \prod_{k=1}^{4} \phi_{i_k}\right) \tag{78}$$

*has a representation by quartic polynomials $D_4(X)$ supported on connected block sets $X$ of side $b$. Every monomial in the representation contains at least two first differences after path telescoping. If the block test norm has at least two discrete derivatives, then for any fixed $\varrho \in (0,1)$, for $b/R$ sufficiently large,*

$$\|D_4(X)\|_{T_{\mathfrak{h},\phi}} \le C_\varrho b^4 \mathfrak{h}^4 (R/b)^2 \varrho^{|X|} (1 + N_X(\phi;\mathfrak{h}))^4. \tag{79}$$

*Here $N_X$ is the maximum of $|\phi_i|/\mathfrak{h}$ on $\Lambda_X$ and of $b|\nabla_e \phi_i|/\mathfrak{h}$ over unit steps with both endpoints in $\Lambda_X$. The estimate is uniform in the volume. With $\ell_b = (Rb)^{-1}$ and $g = R^{-4}$, its zero-field prefactor is $g(R/b)^2$.*

*Proof.* Symmetry of $C$ gives (78). Set $u = \phi_{i_1}$ and $d_k = \phi_{i_k} - u$ for $k = 2,3,4$. The expression in parentheses is exactly

$$u^2 \left( \tfrac{3}{2} \sum_{k=2}^{4} d_k^2 - \sum_{2 \le k < l \le 4} d_k d_l \right) + u \left( \sum_{k=2}^{4} d_k^3 - d_2 d_3 d_4 \right) + \tfrac{1}{4} \sum_{k=2}^{4} d_k^4.$$

For each tuple, choose coordinate shortest paths from $i_1$ to the other three sites. Average over coordinate orders and periodic shortest-path ties when symmetry is needed. The block set visited by their union is connected, and has at most $C(1+\tau(\boldsymbol{i})/b)$ blocks. In fact each coordinate segment of length $s$ meets at most $2 + s/b$ blocks, and the total path length is at most $C\tau(\boldsymbol{i})$. Assign the tuple's contribution to its visited block set and write each $d_k$ as a sum of unit first differences along its path. This defines $D_4(X)$ and proves the exact identity.

In the Taylor pairing of a degree-four monomial, a difference factor is either evaluated on $\phi$ or acts as a difference in one argument of a test tensor. The former costs at most $\mathfrak{h} N_X/b$ for a unit step. The latter costs $b^{-1}$ in the test-function norm. When there are two or more such factors, they act on distinct slots of the tensor, including when the underlying sites coincide. Use two of these differences for the gain, and if there are more, bound each extra difference by the sum of its two endpoint evaluations. There are only finitely many choices of differentiated factors. The Taylor seminorm contribution of a tuple is therefore bounded by

$$C\mathfrak{h}^4 (1 + N_X)^4 (\tau(\boldsymbol{i})/b)^2.$$

Reserve half the exponential weight in (77) for the second moment, giving $C(R/b)^2$. The other half gives $\exp(-cb|X|/R)$ up to a fixed correction in the exponent for the bounded-size case. Equivalently, by taking $b/R$ large, one can bound the remaining factor by $C_\varrho \varrho^{2|X|}$, with $C_\varrho$ independent of larger $b/R$. There are at most $b^4|X|$ choices of the anchor $i_1$ for a given $X$. Absorb $|X|$ in the geometric factor. This gives (79). □

For the exact box pressure the leading tensor at every real forest interpolation point is

$$C_{\mathrm{mf,t}}(\boldsymbol{i}) = \tfrac{1}{12} \sum_{x} \prod_{k=1}^{4} S_t(x, i_k), \qquad S_t = (I - J_X(t))^{-1}.$$

Each row of $S_t$ has uniformly bounded exponential moment at length $R$ by its geometric series, since the weighted row sum of $J_X(t)$ is below one for small enough $\eta$. Symmetry and a star tree

at $x$ prove (77) for $C_{\mathrm{mf,t}}$. The cavity comparison proves the same bound with right side $O(g)$ for the exact-minus-mean-field quartic tensor. For the leading tensor the localized coefficient is

$$q_{i,t}^{\mathrm{mf}} = \tfrac{1}{12} \sum_x S_t(x,i)(S_t\mathbf{1})_x^3 \geq \tfrac{1}{12}.$$

The exact coefficient differs by $O(g)$, so the sitewise localized quartic is uniformly positive even in an induced subvolume. At full translation invariance it is constant, with leading value $\frac{1}{12}(1-\beta q)^{-4}$, where $q = \sum Q$ is the reference interaction mass. At every real interpolation point the full leading quartic is also positive uniformly in $\sum_i \phi_i^4$, by bounded invertibility of $S_t$ on $\ell^4$. Thus interpolation between the local quartic and the full quartic preserves quartic coercivity for large $R$.

The polynomial estimate can also be used inside a stable weight. For example, multiply $D_4(X)$ by the product of the positive localized quartic factors on $\Lambda_X$. At large-field scale $a = b^{-1}$, the extra factor $(1 + N_X(\phi; a))^4$ is absorbed by unused Gaussian damping and the affine-quotient regulator. Indeed, for each block neighborhood subtract an affine function. The affine function's supremum and scaled slope on that neighborhood are bounded by its normalized block $L^2$ norm, while the deviation and its scaled derivative are bounded by the quotient seminorm. The identity $b^4a^4 = 1$ converts the normalized $L^2$ term into the Gaussian damping factor. At the fluctuation scale the growing regulator absorbs the same polynomial. The submultiplicative Taylor seminorm of the positive quartic weight contributes at most a fixed exponential in $|X|$, which can be absorbed by starting with a stronger geometric factor in (79). This does not by itself give the circle-product estimates for a renormalized chart, where one has to integrate the complete weights.

The gain $(R/b)^2$ is the natural size of this localization. On a translation-invariant torus take the real field $\phi_i = \ell_b \cos(k \cdot i)$, with a nonzero allowed frequency $|k| \asymp b^{-1}$. The normalized smoothing operator $T = (1-\beta q)S_0$ multiplies this field by $(1-\beta q)/(1-\widehat{J}(k))$. The box reference has second moment comparable to $R^2$, so for $b/R$ large, $1 - \widehat{T}(k)^4 \asymp (R/b)^2$. Thus the mean-field quartic difference on this field, per block, is of order $b^4\ell_b^4(R/b)^2$. With fixed $b/R$ the localized activity need not have size $O(g^3)$.

# D An exact four-difference quadratic remainder

For a hypercubic-symmetric translation-invariant kernel $E$ with $\sum_x |E(x)|e^{\eta|x|/R} = O(g)$, the quadratic form with symbol $\widehat{E}$ can be localized to its constant and nearest-neighbor kinetic terms, with a four-difference remainder. Here is an explicit finite-difference representation. Put $d(k) = 1 - \cos k$ and $S_n(k) = \sum_{a=0}^{n-1} e^{iak}$ for $n \geq 1$. Then

$$1 - \cos(nk) = d(k)|S_n(k)|^2, \qquad \cos(nk) - 1 + n^2 d(k) = 2d(k)^2 \sum_{m=1}^{n-1} (n-m)|S_m(k)|^2.$$

The first identity follows by factoring $1 - e^{ink}$. The second uses $|S_n|^2 = n^2 - 2\sum_{m=1}^{n-1}(n-m)(1-\cos(mk))$. The multiplier after $d(k)^2$ is a finite-range convolution kernel of absolute sum $O(n^4)$. Coordinate reflections allow us to replace $\cos(k \cdot x)$ in $\widehat{E}(k)$ by $\prod_i \cos(k_i x_i)$. Expand this product as a product of $1 + (\cos(k_i x_i) - 1)$. Its linear terms produce $-\sum_i x_i^2 d(k_i)$ and the one-dimensional remainders above. Every term with at least two factors $\cos(k_i x_i) - 1$ contains a product of two $d(k_i)$'s by the first identity, with a finite-range kernel of absolute sum $O(|x|^4)$ after choosing any two of its difference factors and bounding the remaining factors by their absolute kernel sums.

Consequently the exact residual multiplier is a sum of terms $d(k_i)d(k_j)\widehat{R}_{ij}(k)$ after subtracting $\widehat{E}(0)$ and its kinetic coefficient. The kernels $R_{ij}$ have absolute sums $O(gR^4)$ and retain an

exponential moment at a smaller rate. At the scale $b$, four discrete derivatives give the zero-field quadratic prefactor

$$b^4 \ell_b^2 \, b^{-4} g R^4 = g(R/b)^2.$$

This is an exact localization of the translation-invariant kernel, not a quadratic localization at arbitrary noninvariant forest interpolation points.

## AI Statement

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