Math 220H Notes
Matrices

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I began writing these notes while teaching M220H (Honors Matrices) during the spring 2012 semester at Penn State University. It was a much larger endeavor than I had anticipated, but now that I have these notes, I hope to refine them and produce a better course for 2013.

Please be aware that the examples I use in these notes will be different from the examples presented in lecture. The goal of this is to give you a few more examples to look at.

An unfortunate consequence of having a course taught out of lecture notes designed by the instructor is that you may end up with a narrow perspective of the subject. I will tend to teach how to solve problems the way I think about them, and this same emphasis will show up in my notes. There are other ways to teach many of these concepts, so when things aren't clear, come talk to me and I'll do my best to help.

When I taught the course in 2012, the official text was Lay 4th edition. In these notes, when I refer to "the text", this is what I'm talking about. You can buy it if you'd like, or borrow it from someone, or go without. I started writing these notes because I didn't think it was a good textbook for an honors course. I'm not going to use it or expect you to have it.

Unlike most textbooks, you'll notice that the exercises are scattered through the notes rather than at the end of a chapter. Generally the exercise is put somewhere to illustrate a point that is being made, or because the method to solve it has just been introduced. If you're having trouble with an exercise, the first step is to look at what appears just before it.

Contents

Chapter 1

Introduction

1.1 Important Concepts

After this chapter you should:

- Be comfortable working in $\mathbb{R}^n$.
- Know what a matrix is and be comfortable with matrix arithmetic.
- Understand what it means for a function to be linear.
- Be able to convert a linear system into the form $\vec{b} = \mathrm{A}\vec{x}$.

If you pass this course and in a later course are unable to multiply and add matrices correctly, then I will have failed in the duty I owe your future professors, and I will be denied tenure. So you will not pass if you can't handle matrix arithmetic.

1.2 Introductory comments

Matrices form a large part of the branch of mathematics known as Linear Algebra. Linear algebra is one of the most fundamental parts of mathematics. Some of the techniques we will use here have been known for nearly (and likely more than) 2000 years. I would argue that linear algebra is as fundamental to science and engineering as arithmetic is to daily life.

Unfortunately there are many tools we need to learn in this course. So the quantity I need to cover can lead to some parts seeming a bit dry. There is a set of tools to learn, and I can't spend as much time on interesting applications as I would like. Most applications will be discussed at the end of the course. That said, particularly with the advent of computers, there are some very important and interesting applications of linear algebra which have only recently emerged (*e.g.*, the PageRank algorithm of Google).

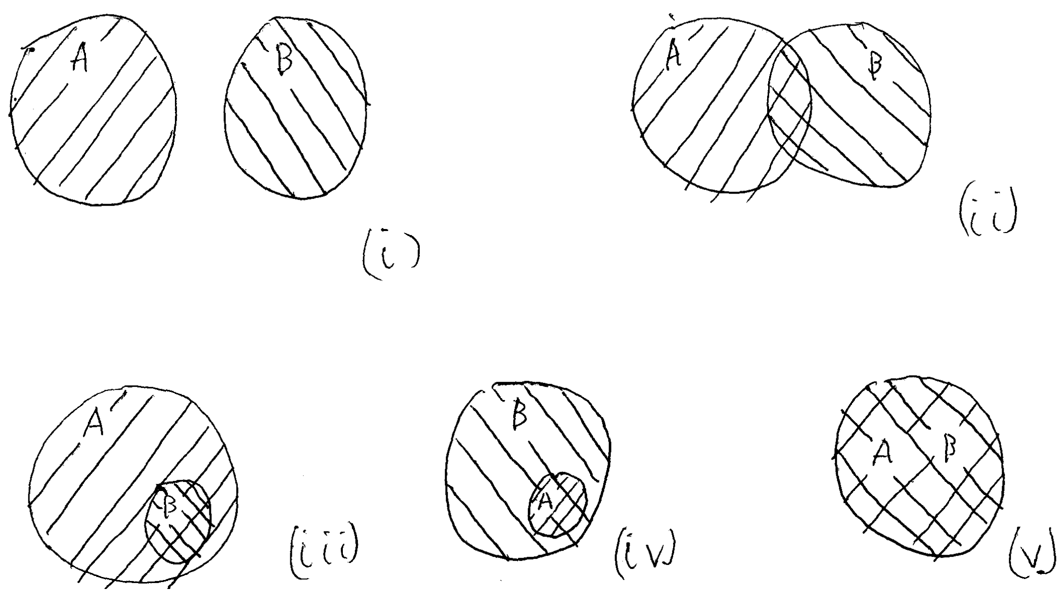
As we explore Linear Algebra, we'll see that there are a number of times where I seem to get overly worked up about introducing some abstract concept. In particular, I make a big deal out of treating functions as if they were very similar to vectors. The reason I do this isn't that I want to confuse people. I promise. I do this because at some point many of you will encounter the same concepts in a different context. If you understand the parallels, then when you learn Fourier Series (as one example) where you approximate a function by a sum of sines and cosines, you'll realize that the instructor is simply repeating things you've already learned in this course. If you don't understand the abstraction, then it's going to be just as hard to learn the second time around as it was the first. By choosing the right level of abstraction, many of these problems will become easy. The way many scientists (particularly mathematicians) operate is to put the effort into finding the right level of abstraction, and then the problems we study become easy. We can alternately put the effort into studying the one specific problem we've encountered, and then we have to put in the same effort when we come across a related problem somewhere else.

1.2.1 Comments on proof technique

Most of you have not been asked to prove things very often or even at all. I will ask you to prove some things in this course. In fact, probably the hardest proofs I might ask you to do are early in this course, exercise 2.19. I will try to give appropriate hints as needed.

Often we are asked to prove that if some mathematical object has property A, then it has property B. To prove this we have to show that any thing with property A has property B. Sometimes we are asked to prove the stronger statement that it has property A if and only if (usually abbreviated 'iff') it has property B. To do this, we must prove not only that anything with property A has property B, but also that anything with property B has property A. In rare cases I will ask whether anything with property A must have property B when this is not true.

Let's put this in Venn Diagrams: Here are the possible ways to have property A and/or property B.



If we are asked to prove that if it has property A then it has property B, what we are being asked is to show that we are either in case (iv) or case (v). If we are aked to prove property A iff property B, we are being asked to show that we are in is case (v). If you are showing that the claim that anything with property A has property B is false, you must prove that it is one of cases (i), (ii), or (iii).

There is a big point I need to make: Proof by example is not proof. By this I mean that if I ask you to prove that anything with property A has property B, and you show me one example, all you've done is proven that A and B have some overlap. In other words, the only case you've ruled out is (i).

If you are proving that a statement is true, you have to show that no matter how hard I look for a counter-example, I will fail. To prove that a general statement is not true, however, it is enough to find a single counter-example, that is, show one case where an object of property A does not have B.

Think of this as a game you play against another student. The rules are as follows: I claim a statement is true. You get the option to "pass" or "play". If you "play", then you have to find a counter-example to my statement. If you "pass", then the other student must play. When do you want to pass? Exactly those times when you can prove the claim is always right.

Let's look at a few cases:

Example 1.1 *An incorrect proof of a true statement: The sum of any two even integers is even.*
The following proof is wrong: "2 is even, 4 is even and 2+4 is even, therefore the sum of any two even numbers is even".

Why is this proof wrong? Well, because it only shows that this works for one pair of numbers. I can find another pair of even integers, in which case the "proof" tells me nothing. This method is commonly referred to (derisively) as "proof by example".

Here is a correct proof of the claim that the sum of two even integers is even:

Example 1.2 *A correct proof of the statement "the sum of two even integers is even"*
Let x be even and y be even. We must show $x+y$ is even. Because x and y are even, we have $x = 2a$ and $y = 2b$ for some integers a and b. Then

$$\begin{aligned} x + y &= 2a + 2b \\ &= 2(a + b) \end{aligned}$$

which is even because $a + b$ is an integer.

The reason we are not allowed to use proof by example is that it can sometimes "prove" a false statement. Let's see an example of where proof by example really fails:

Example 1.3 *The same incorrect proof technique, but now of a false statement: The sum of any two integers is even.*
The following proof is wrong: "2 is an integer, 4 is an integer and 2+4 is even, therefore the sum of any two integers is even".

Here is an example of how to prove that something is not true (for this it is enough to show just one example):

Example 1.4 *Proving that a statement is false: Consider the claim "the sum of any two integers is even". Consider the integers 1 and 2. Their sum is 3, which is not even. Therefore there is at least one counter-example to the claim, so the claim is false.*

Basically, if we're trying to prove that something is always true, and we do it by showing an example, all we've done is shown that it is sometimes true, or at least that it is true in one case. This is not a good enough proof.[1] On the other hand, if the claim is that it is always true, and we show a counter example, then we have proven that it is not always true — the claim was wrong.

Just to be clear: if I ever ask you to show something like "the sum of two even integers is even", I do not mean "show me an example in which the sum of two even integers is even"; I mean, "show me that no matter what two even integers I chose, they have an even sum."

Exercise 1.5 *Prove that if n is an even integer, then n^2 is even.*

1.2.2 Notation

I need to introduce a little notation. We denote the set of real numbers by $\mathbb{R}$, the complex numbers by $\mathbb{C}$, and the rational numbers by $\mathbb{Q}$. The integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ are denoted $\mathbb{Z}$. Notice that $\mathbb{R}$, $\mathbb{C}$, and

[1] An engineer, a physicist and a mathematician (it is said) were holidaying in Scotland. Glancing from a train window, they observed a black sheep in the middle of a field. "How interesting" observed the engineer, "all scottish sheep are black!" To which the physicist responded, "No, no! Some Scottish sheep are black!" The mathematician corrected them, "In Scotland there exists at least one field, containing at least one sheep, at least one side of which is black."

$\mathbb{Q}$ are closed[2] under division (except division by zero), whereas with $\mathbb{Z}$, it is easy to find cases where division of one integer by another results in something other than an integer. I will use $\mathbb{R}$ a lot. The others will generally not be needed and I will remind you of their meaning when they come up.

We write '$\in$' to represent that something is a member of a set. So for example $x \in \mathbb{R}$ means x is a real number. As a few more examples, $1.5 \in \mathbb{Q}$, $1 + i \in \mathbb{C}$, $1.5 \notin \mathbb{Z}$, and $\sqrt{3} \notin \mathbb{Q}$.

We will use an arrow over the top of a variable name to denote that it is a 'vector'. So $\vec{x}$ is a vector. Other conventions exist. In some cases a vector is denoted by using a bold font, while in others a vector is denoted by underlining the variable name.

If I write "$f : X \to Y$," then f is a function, X is a set of things, Y is a set of things, X is the domain of f, and Y contains the range of f. In spoken language, we typically read $f : X \to Y$ as "f maps X to Y". More colloquially what this means is that f eats things in X and spits out things in Y. Note that $f(x)$ must be defined for any $x \in X$, but there is no requirement that for every $y \in Y$ there is some x such that $f(x) = y$; we do not have to be able to reach everything in Y. In our previous experience, X and Y are often just the real numbers [like $f(x) = 2x$ or $f(x) = x^2$]. But we might have $f : \mathbb{R} \to \mathbb{C}$, for example with $f(x) = \sqrt{x}$.

Example 1.6 *For the examples below, $f : X \to Y$ and:*

- *In Example 1.7, $X = \mathbb{R}$ and $Y = \mathbb{R}$.*
- *In Example 1.8, X is the set of integrable functions and $Y = \mathbb{R}$.*
- *In Example 1.9, X is the set of differentiable functions and Y is the set of functions.*

1.3 Linearity (sec 1.8 of text)

This material is covered in section 1.8 of Lay's fourth edition on Linear Algebra[3]

Assume we have a function $f(x)$; often x is simply a number, but x could itself be a function. What is the most common (and generally incorrect) assumption that we might make about $f(y+z)$? Have you ever seen (or done yourself) the following: $(y + z)^2 = y^2 + z^2$? It's wrong, but it's tempting.

We really want $f(y + z)$ to equal $f(y) + f(z)$. In fact, more generally, we want $f(ay + bz) = af(y) + bf(z)$. Let's temporarily call the belief that $f(ay + bz) = af(y) + bf(z)$ the "freshman's dream". Are there any cases where the freshman's dream is right?

Example 1.7 *If $x \in \mathbb{R}$ and $f(x) = 5x$, then*

$$\begin{aligned} f(ay + bz) &= 5(ay + bz) \\ &= a5y + b5z \\ &= af(y) + bf(z), \end{aligned}$$

and the freshman's dream holds for this particular $f(x)$.

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[2] "closed" means if I take two things in the set and perform the operation, I get something else in the set. For example: the positive integers are closed under addition and multiplication, but not under subtraction or division

[3] When I taught the course in 2012, the official text was Lay 4th edition. In these notes, when I refer to "the text", this is what I'm talking about. You can buy it if you'd like, or borrow it from someone, or go without. I started writing these notes because I didn't think it was a good textbook for an honors course. I'm not going to use Lay or expect you to have it.

Example 1.8 *If x is an integrable function of a variable s, then we can define $f(x(s)) = \int_0^1 x(s)\,\mathrm{d}s$. Then*

$$\begin{aligned} f(ay(s)+bz(s)) &= \int_0^1 ay(s)+bz(s)\,\mathrm{d}s \\ &= \int_0^1 ay(s)\,\mathrm{d}s + \int_0^1 bz(s)\,\mathrm{d}s \\ &= a\int_0^1 y(s)\,\mathrm{d}s + b\int_0^1 z(s)\,\mathrm{d}s \\ &= af(y(s))+bf(z(s)) \end{aligned}$$

and again the freshman's dream is correct for this particular f.

Example 1.9 *If x is a differentiable function of s, then we can define $f(x(s)) = \frac{\mathrm{d}}{\mathrm{d}s}x(s)$. We get*

$$\begin{aligned} f(ay(s)+bz(s)) &= \frac{\mathrm{d}}{\mathrm{d}s}(ay(s)+bz(s)) \\ &= a\frac{\mathrm{d}}{\mathrm{d}s}y(s)+b\frac{\mathrm{d}}{\mathrm{d}s}z(s) \\ &= af(y(s))+bf(z(s)) \end{aligned}$$

Once more the freshman's dream holds.

So obviously there are a lot of cases where the freshman's dream is true. To be clear, there are many examples where it's not. For example if $f(x) = x^2$, then $f(y+z) = y^2+2yz+z^2 = 2yz+f(y)+f(z) \neq f(y)+f(z)$. Another example worth noting is $f(x) = x+1$. Even though this doesn't have higher powers, it fails because of the constant term: $f(y+z) = y+z+1 = y+1+z+1-1 = f(y)+f(z)-1 \neq f(y)+f(z)$.

There are many counterexamples to the freshman's dream, but there are some important cases where it holds. So it's worth learning what we can say about these types of functions.

Definition 1.10 *A function $f : X \to Y$ is said to be* ***linear*** *over the reals if for any $y, z \in X$ and any $a, b \in \mathbb{R}$, $f(ay+bz) = af(y)+bf(z)$.*

So a function is linear if this "freshman's dream" holds. These are the sorts of functions we will study in this course. We refer to these functions as linear functions.

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Exercise 1.11 *Many textbooks define the function f to be linear if $f(\vec{u}+\vec{v}) = f(\vec{u})+f(\vec{v})$ and $f(c\vec{u}) = cf(\vec{u})$, but I have simply defined f to be linear if $f(a\vec{u}+b\vec{v}) = af(\vec{u})+bf(\vec{v})$. Show that the two definitions are equivalent by doing the following*

(a). If f is a function that satisfies both conditions of the alternate definition here show it also satisfies definition 1.10.

(b). If f satisfies definition 1.10 show it also satisfies both conditions in the alternate definition here. [hint: if you choose the coefficients carefully in defintion 1.10 you can get something that looks like each part of the alternative definition]

Here the arrows are used to denote that the things may not be numbers. They could themselves be functions, or they could be vectors (which we define later).

You may be wondering why we need to check both directions. To show that the definitions are equivalent, I need to show how to derive each definition from the other. If I weren't careful, I could run into something like the following: suppose I were to define 'even' as meaning a number is divisible by 4 and argue it's equivalent to the usual definition of being even. I could show that any number that satisfies my definition satisfies the usual definition, but that's not enough to show that the definitions are the same because I haven't shown that every number satisfying the usual definition also satisfies my definition. In fact, there are even numbers that don't satisfy my definition, so the definitions are not equivalent.

Example 1.12 *Consider the function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = cx$ for some constant c. Is this linear (over the reals)?*

To check, we must show for any $a, b \in \mathbb{R}$ and $y, z \in \mathbb{R}$ that $f(ay + bz) = af(y) + bf(z)$. We know that

$$\begin{aligned} f(ay + bz) &= c(ay + bz) \\ &= c(ay) + c(bz) \\ &= a(cy) + b(cz) \\ &= af(y) + bf(z) \end{aligned}$$

So this is linear.

Example 1.13 *In this example, we show that all functions $f : \mathbb{R} \to \mathbb{R}$ that are linear take the form $f(x) = cx$.*

Assume that $f : \mathbb{R} \to \mathbb{R}$ is linear. Define $c = f(1)$. Let x be some real number. We note that $x = x \cdot 1 + 0 \cdot y$ for any y. Then

$$\begin{aligned} f(x) &= f(x \cdot 1 + 0 \cdot y) \\ &= xf(1) + 0f(y) \\ &= xf(1) + 0 \\ &= xc \\ &= cx \end{aligned}$$

we used the fact f is linear to get to the second line.

So if f is linear it takes the form $f(x) = cx$. Combined with the previous example, we have proven the following theorem

Theorem 1.14 *The function $f : \mathbb{R} \to \mathbb{R}$ is linear if and only if $f(x) = cx$ for some constant c.*

Exercise 1.15 *Which of the following functions f are linear functions? Show that each either does or does not satisfy the linearity property. To show that it does satisfy the linearity property, you can follow example 1.12 or use any other approach you feel appropriate. To show that it does not satisfy the linearity property you only need one counterexample.*

(a). $f(x) = x^2$.

raight line, not (b). $f(x) = 3x$.

(c). $f(x(s)) = x''(s)$. *[the prime denotes differentiation]*

(d). $f(x(s)) = x'(s)x(s)$.

(e). $f(x(s)) = x''(s) + 2x'(s) + x(s)$.

(f). $f(x(s)) = \int_0^\infty x(s)\,\mathrm{d}s$.

We know that "f is linear" means for the sum of 2 things, $f(ax + by) = af(x) + bf(y)$. What about the sum of three things? Does $f(ax + by + cz) = af(x) + bf(y) + cf(z)$? This works out as you would expect. In general:

Theorem 1.16 *If f is a linear function, then $f(\sum_{i=1}^n a_i x_i) = \sum_{i=1}^n a_i f(x_i)$.*

Proof. The proof of this can be done by a mathematical technique known as "induction".

We know the theorem is true for $n = 2$. I'll prove it's true for $n = 3$, and then we will be able to see that repeating the same proof shows it's true for $n = 4$, $n = 5$, etc.

Consider $f(\sum_{i=1}^3 a_i x_i)$. We can rewrite $\sum_{i=1}^3 a_i x_i$ as $1(\sum_{i=1}^2 a_i x_i) + a_3 x_3$. So

$$\begin{aligned} f(\sum_{i=1}^3 a_i x_i) &= f\left(1(\sum_{i=1}^2 a_i x_i) + a_3 x_3\right) \\ &= 1f\left(\sum_{i=1}^2 a_i x_i\right) + a_3 f(x_3) \\ &= 1\left(\sum_{i=1}^2 a_i f(x_i)\right) + a_3 f(x_3) \\ &= \sum_{i=1}^3 a_i f(x_i) \end{aligned}$$

Going from the first to the second line, we used the fact that f is linear. We used it again to go from the second to the third line, we used the fact that we know the theorem holds for $n = 2$.

We can repeat the argument as many times as we need to prove it is true for any n. □

Theorem 1.16 basically underlies why linear functions and linear combinations are particularly well-behaved. Colloquially speaking, if f is linear, we can add the x-s and then perform f to the entire sum or we can perform f to each x and then add. The result is the same.

1.4 Matrices and Vectors

In Example 1.13, we showed that the set of linear functions from $\mathbb{R}$ to $\mathbb{R}$ is very restricted, it's just functions like $f(x) = cx$. To do anything interesting, we need to look for functions $f : X \to Y$ where X and Y are more general sets.

In the most common examples of linear functions (and our focus in this course), $f : X \to Y$ has $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$ (defined below). We'll see that all such functions f can be represented by matrices. Often these functions are referred to as linear transformations and denoted by the symbol T rather than f. Let me make sure it's clear what I mean by $\mathbb{R}^n$ and $\mathbb{R}^m$ before I go into detail about the transformations.

1.4.1 Vectors (sec 1.3 of text — up to pg 30)

A (real-valued) vector with n elements is an ordered set of n (real) numbers. An example vector with 2 elements is $\begin{bmatrix}1\\2\end{bmatrix}$. An example vector with 3 elements is $\begin{bmatrix}11\\3\\4\end{bmatrix}$. By default the vectors we study will be 'column vectors'. The notation $\mathbb{R}^n$ denotes the set of all vectors having n real numbers. So for example $\mathbb{R}^3$ consists of all vectors of 3 elements. When I say they are ordered, I don't mean the numbers are in numerical order. I mean that I care about the order they are in: $\begin{bmatrix}1\\2\\3\end{bmatrix} \neq \begin{bmatrix}3\\2\\1\end{bmatrix}$.

Definition 1.17 *For a vector*

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$$

the entry x_i is referred to as the i-th ***entry*** *of the vector. It is frequently also referred to as the i-th value, the i-th component, or the i-th element.*

I will most frequently use 'entry'.

It is tempting to use the word "length" to refer to the number of entries in a vector. Don't. Generally when we speak about the "length" of a vector, we are talking about the Euclidean length $\sqrt{x_1^2 + \cdots + x_n^2}$, discussed later.

I will use the notation $\vec{x}$ to represent a column vector, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Any time there is an arrow over something, we will mean that it is a vector. I need to be clear that there will be other contexts in which I am talking about a collection of vectors $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$, in which case I will often refer to a vector $\vec{x}_i$. This is the i-th vector of my set. It is not the i-th entry of the vector $\vec{x}$.

We still need define scalar multiplication and vector addition.

Definition 1.18 *In the context of vectors, a* ***scalar*** *is a real number rather than a vector.*

We can multiply a vector and a scalar. We cannot add a vector and a scalar.

Definition 1.19 ***Scalar multiplication*** *of a vector means multiplying the vector by a real number. The way we do this is to multiply each element of the vector by that real number.*

$$c\vec{x} = c \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$$

So for example $5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$.

Definition 1.20 ***Vector addition*** *is defined by taking two vectors with the same number of entries and adding their corresponding components to get a new vector*

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

So for example

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

If the number of entries in each vector are different, then the sum is not defined — we can only add vectors that have the same number of entries.

Exercise 1.21

(a). *What does the notation $T : \mathbb{R}^4 \to \mathbb{R}^2$ mean?*

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If $T : \mathbb{R}^4 \to \mathbb{R}^2$, then which of the following might be a true statement? (note we do not require that T be linear).

(b). $T(\begin{bmatrix}1\\2\\3\\4\end{bmatrix}) = \begin{bmatrix}6\\1\end{bmatrix}$

(c). $\begin{bmatrix}1\\2\end{bmatrix} = T(4)$

(d). $\begin{bmatrix}1\\2\end{bmatrix} = T(\begin{bmatrix}0\\0\\0\\0\end{bmatrix})$

(e). $\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = T(\begin{bmatrix}1\\2\end{bmatrix})$

Exercise 1.22 *Let*

$$\vec{u} = \begin{bmatrix}1\\2\\3\end{bmatrix} \quad \vec{v} = \begin{bmatrix}2\\4\end{bmatrix} \quad \vec{w} = \begin{bmatrix}4\\8\\6\end{bmatrix} \quad \vec{x} = \begin{bmatrix}1\\-2\end{bmatrix}$$

For each of the following find its value or explain why it is undefined.

(a). $\vec{u} + \vec{v}$

(b). $\vec{u} + 2\vec{w}$

(c). $\vec{v} + 0\vec{w}$

(d). $\vec{w} + 0\vec{u}$

(e). $\vec{x} - 2\vec{v}$

Theorem 1.23 *Scalar multiplication is distributive (in two ways)*

(a). $(a + b)\vec{x} = a\vec{x} + b\vec{x}$

(b). $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$

Proof. I'll prove one result $(a + b)\vec{x} = a\vec{x} + b\vec{x}$ and leave the other as an exercise. The approach is to take one of the expressions (either right or left hand side) and after a series of equalities get the other expression.

$$\begin{aligned}
(a+b)\vec{x} &= (a+b)\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= \begin{bmatrix} (a+b)x_1 \\ (a+b)x_2 \\ \vdots \\ (a+b)x_n \end{bmatrix} \\
&= \begin{bmatrix} ax_1 + bx_1 \\ ax_2 + bx_2 \\ \vdots \\ ax_n + bx_n \end{bmatrix} \\
&= \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix} + \begin{bmatrix} bx_1 \\ bx_2 \\ \vdots \\ bx_n \end{bmatrix} \\
&= a\vec{x} + b\vec{x}
\end{aligned}$$

□

Exercise 1.24 *Prove that $a(\vec{x}+\vec{y}) = a\vec{x} + a\vec{y}$.*

Definition 1.25 *Given a set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n\}$, we call any sum of the form*

$$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + \cdots + a_n\vec{v}_n$$

(where the a_i are real numbers) a ***linear combination*** *of the vectors.*

So Theorem 1.16 shows that if $f : \mathbb{R}^m \to \mathbb{R}^n$ is a linear function, $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is a set of vectors in $\mathbb{R}^m$, and $\vec{b} = \sum_{i=1}^{k} a_i\vec{x}_i$, then $f(\vec{b}) = \sum_{i=1}^{k} a_i f(\vec{x}_i)$.

There are some particularly important vectors that show up repeatedly.

Definition 1.26 *The vector $\vec{e}_i$ is a vector in $\mathbb{R}^n$ whose entries are all 0 except the i-th entry, which is 1. The set $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is known as the* ***canonical basis*** *of $\mathbb{R}^n$.*

We will wait till later to define a "basis". In three dimensions, physicists typically use $\vec{\imath} = \vec{e}_1$, $\vec{\jmath} = \vec{e}_2$, and $\vec{k} = \vec{e}_3$.

Note that when we talk about $\vec{e}_i$, the number of entries it has depends on the context — what is n? Generally it will be obvious what n must be. If not, it must be explicitly stated.

Example 1.27 *Any vector* $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ *can be written as a linear combination of the canonical basis vectors:*

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \sum_{i=1}^{n} x_i \vec{e}_i$$

It is important to note that there is only one way to do this. That is, if $\sum a_i \vec{e}_i = \sum b_i \vec{e}_i$, *then for each* i, $a_i = b_i$.

Exercise 1.28 *Show that*

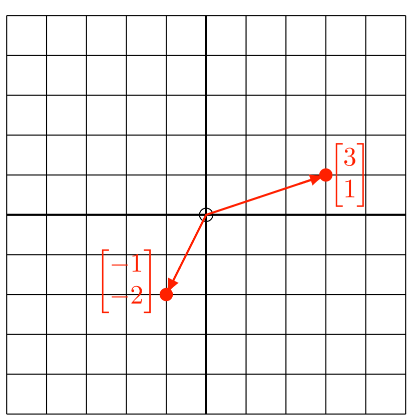
(a). *If* $T : \mathbb{R}^n \to \mathbb{R}$ *takes the form* $T(\vec{x}) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$, *show that* T *is linear. [hint: follow example 1.12.]*

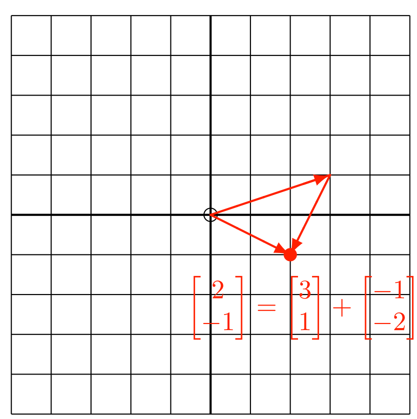
(b). *Show any linear function from* $\mathbb{R}^n$ *to* $\mathbb{R}$ *takes the form* $T(\vec{x}) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$ *where each* $a_i \in \mathbb{R}$. *[hint: look at ~~example 1.13 and~~ example 1.27: let* $a_i = T(\vec{e}_i)$ *and take* $\vec{x} = \sum x_i \vec{e}_i$*]*

Vectors as arrows

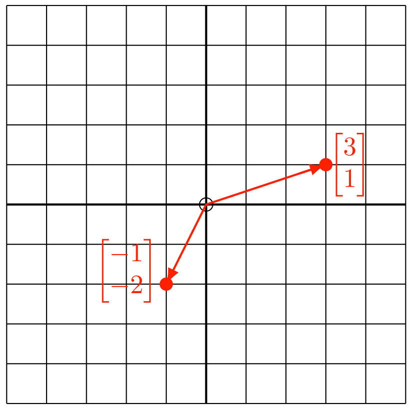
A common geometric interpretation of vectors is to represent them as arrows. Let's look at this in $\mathbb{R}^2$. We think of the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ as an arrow from the origin to the point (x, y).

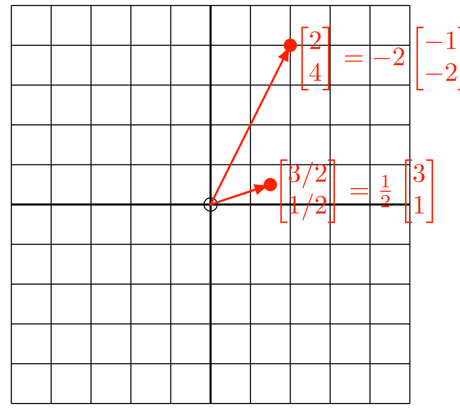
Using this concept, we define addition of the arrow to $u = (u_1, u_2)$ and the arrow to $v = (v_1, v_2)$ by the following process. We take the arrow to u and then at the tip of the arrow, put another arrow that is parallel to and the same length as the arrow to v. The arrow from the origin to the tip of this is defined to be the sum of the two arrows. The sum is thus the arrow from the origin to $(u_1 + v_1, u_2 + v_2)$. Notice, it doesn't matter what order we do them in. This is the same thing we would get if we first added the lists representing the vectors and then translated them into the arrows.





We define scalar multiplication so that if we multiply a point by $c > 0$, we simply take the point farther from the origin by a factor of c. We multiply the arrow length by c. If $c < 0$, we multiply the length by c, meaning it goes in the opposite direction from the origin.





You may have come across vectors in a physics course at some point before. It may have defined vectors simply as having magnitude and direction. This example shows why that definition is equivalent to ours. Because the arrows in space really do behave exactly like column vectors as far as addition and scalar multiplication are concerned, we can switch between these two interpretations freely and think of the column vector as being the arrow. They aren't the same thing: one is a set of numbers, the other is a arrow in space, but there is nothing gained by making that distinction, so we treat them as interchangeable.

Sometimes we drop the arrow from the origin altogether and think about a vector as just the point in space that the arrow would point to.

Exercise 1.29 *Using the arrow representation of vectors, draw*

(a). $\begin{bmatrix}1\\2\end{bmatrix}$,

(b). $\begin{bmatrix}-5\\3\end{bmatrix}$,

(c). $\begin{bmatrix}1\\-4\end{bmatrix}$,

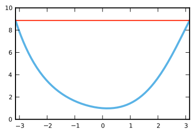
(d). $\frac{5}{3}\begin{bmatrix}3\\6\end{bmatrix}$, *and*

(e). demonstrate how the sum $\begin{bmatrix}-1\\2\end{bmatrix} + \begin{bmatrix}2\\2\end{bmatrix}$ *is calculated.*

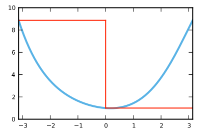
Using vectors to ~~represent~~ approximate functions

Consider an infinitely differentiable function $f(x)$ defined for $x \in [-\pi, \pi]$. There are a number of ways we can represent f using vectors. In our examples, we assume $f(x) = x/\pi + x^2 - x^7/\pi^7 + \cos x - (\sin x)/2$.

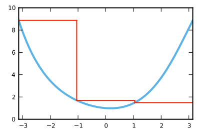
Example 1.30 *Our first method to represent a function with a vector is to subdivide the interval into smaller intervals and approximate the function as constant on each interval. Consider the points* $x_0 = -\pi,\ x_1 = -\pi + \delta x,\ x_2 = -\pi + 2\delta x,\ \ldots,\ x_n = \pi - \delta x$. *Then we represent an approximation to* $f(x)$ *with the vector* $\vec{f} = \begin{bmatrix}f(x_0)\\f(x_1)\\\vdots\\f(x_n)\end{bmatrix}$. *To find this vector, we simply evaluate* f *at each point. We show* $f(x)$ *in blue and the approximation from* $\vec{f}$ *for different* n *in red below.*



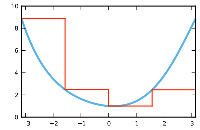
$\vec{f} = \begin{bmatrix} f(-\pi) \end{bmatrix}$



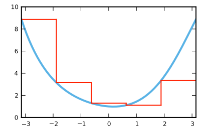
$\vec{f} = \begin{bmatrix} f(-\pi) \\ f(0) \end{bmatrix}$



$\vec{f} = \begin{bmatrix} f(-\pi) \\ f(-\pi/3) \\ f(\pi/3) \end{bmatrix}$

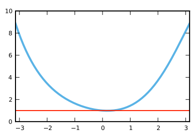


$\vec{f} = \begin{bmatrix} f(-\pi) \\ f(-\pi/2) \\ f(0) \\ f(\pi/2) \end{bmatrix}$

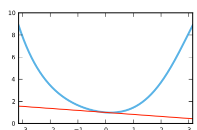


$\vec{f} = \begin{bmatrix} f(-\pi) \\ f(3\pi/5) \\ f(-\pi/5) \\ f(\pi/5) \\ f(3\pi/5) \end{bmatrix}$

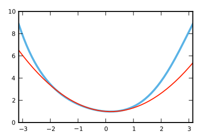
Example 1.31 *If the Taylor Series of f converges, we can write f as $f(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$. We can approximate f by keeping the first n terms of this series. Then we can represent the approximation $f(x) \approx \sum_{k=0}^{n} \frac{a_k}{k!} x^k$ by the vector $\vec{f} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$. To find this vector, we note that $a_k = f^{(k)}(0)$. We show $f(x)$ in blue and the approximation from $\vec{f}$ for different n in red below.*



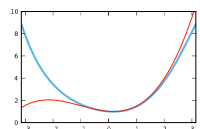
$\vec{f} = \begin{bmatrix} f(0) \end{bmatrix}$



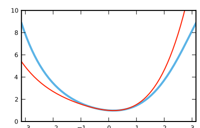
$\vec{f} = \begin{bmatrix} f(0) \\ f'(0) \end{bmatrix}$



$\vec{f} = \begin{bmatrix} f(0) \\ f'(0) \\ f''(0) \end{bmatrix}$

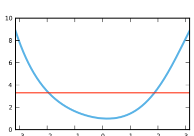


$\vec{f} = \begin{bmatrix} f(0) \\ f'(0) \\ f''(0) \\ f'''(0) \end{bmatrix}$

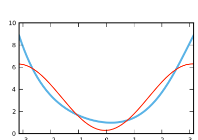


$\vec{f} = \begin{bmatrix} f(0) \\ f'(0) \\ f''(0) \\ f'''(0) \\ f^{(iv)}(0) \end{bmatrix}$

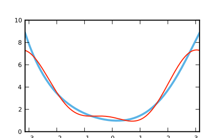
Example 1.32 *Alternately, we may find that rather than writing f as a sum of powers of x, it is better to write it as a sum of* cos *and* sin *(generally this approach is used when f is periodic or only defined on a finite interval). This is known as* <u>*Fourier Series*</u>*. Then $f(x) = a_0/2 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$. We can represent the approximation $f(x) \approx a_0/2 + \sum_{k=1}^{n} a_k(\cos kx + b_k \sin kx)$ by the vector $\vec{f} = \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \\ \vdots \\ a_n \\ b_n \end{bmatrix}$. Later in the course we will learn how to find the entries in this vector. We show $f(x)$ in blue and the approximation from $\vec{f}$ for different n in red below.*



$\vec{f} = \begin{bmatrix} a_0 \end{bmatrix}$



$\vec{f} = \begin{bmatrix} a_0 \\ a_1 \\ b_1 \end{bmatrix}$



$\vec{f} = \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix}$

In each case, as n grows, the vector provides more information about f. Which approach gives the best approximation depends a lot on the context. Generally the first method is more straightforward, but requires a very long vector to get a good approximation. The second and third method both tend to give good approximations, but the error in the Fourier series tends to be more uniformly distributed, while the Taylor series is very accurate close to 0 and less good farther away. Fourier series are particularly useful for functions that "wiggle" a lot.

For the following exercises, it will be useful to look at Example 1.12.

Exercise 1.33 *Assume n is a given ~~number~~ positive integer. Consider the function T_n which takes a function f defined on $[-\pi, \pi]$ and outputs the vector* $T_n(f)\ \cancel{\vec{f}} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}$ *defined in example 1.30.*

Is T_n linear? Prove your answer.

Exercise 1.34 *Assume n is a given ~~number~~ positive integer. Consider the function $\cancel{S_n}\ T_n$ which takes an infinitely differentiable function f defined on $[-\pi, \pi]$ and outputs the vector* $T_n(f)\ \cancel{\vec{f}} = \begin{bmatrix} f(0) \\ f'(0) \\ \vdots \\ f^{(n)}(0)) \end{bmatrix}$ *defined in example 1.31 (note that the notation $f^{(m)}$ means the m-th derivative of f).*

Is $\cancel{S_n}\ T_n$ linear? Prove your answer.

Exercise 1.35 *Consider the set S of all polynomials of degree at most 3. Define $T : S \to \mathbb{R}^4$ such that for the polynomial $P(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$,* $T(P) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$. *Is T linear? Prove your answer.*

trouble under-
T does. Put it
look at exercise

We will revisit exerciese 1.35 in exercise 1.71.

1.4.2 Matrices (sec 1.9 of text)

We frequently call linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ "linear transformations". Due to this term, we often use T to represent them rather than f. Consider any linear transformation T that takes a vector with n numbers and spits out a vector with m numbers. In other words, T is linear and $T : \mathbb{R}^n \to \mathbb{R}^m$. So for any vector $\vec{x} \in \mathbb{R}^n$ there is a $\vec{y} \in \mathbb{R}^m$ such that $\vec{y} = T(\vec{x})$. What we're going to learn is that the function T can be represented by a matrix — in fact every linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be represented by a matrix. The converse holds as well. Every matrix represents a linear function, so we can learn everything there is to know about linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ by studying matrices.

First though, let's go over what a matrix is, begining with an example

Example 1.36 *Let's look at an example. Assume $T : \mathbb{R}^3 \to \mathbb{R}^4$ with*

$$T(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = \begin{bmatrix} x_1 + x_3 \\ x_1 + 0.3x_2 + 5x_3 \\ x_2 \\ 0 \end{bmatrix}$$

We can rewrite this as

$$T(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = \begin{bmatrix} x_1 + 0x_2 + x_3 \\ x_1 + 0.3x_2 + 5x_3 \\ 0x_1 + x_2 + 0x_2 \\ 0x_1 + 0x_2 + 0x_3 \end{bmatrix}$$

But it quickly becomes tedious to write down all the '+' symbols and all the x_i variables. It's much more efficient to just write down the coefficients. We say

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0.3 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or simply $T(\vec{x}) = \mathrm{A}\vec{x}$ where the matrix A *is*

$$\mathrm{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0.3 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the function T is entirely encoded by A, *which is a 4×3 matrix.*

Let's make the definition more precise:

Definition 1.37 *An $m \times n$* ***matrix*** A *is a collection of m rows of numbers, forming n columns.*

Example 1.38 *The following are all examples of matrices, with their sizes below them.*

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 9 \\ 16 & 25 & 36 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

$$2 \times 2 \quad , \quad 1 \times 1 \quad , \quad 2 \times 3 \quad , \quad 5 \times 1 \quad , \quad 1 \times 5$$

1.4.3 Matrix Arithmetic

It is possible to add or subtract matrices, as long as they have the same sizes. We can multiply matrices together, or multiply matrices with vectors, but there is a more complicated size requirement. Division of matrices is not allowed — we will see that in some cases matrices have an "inverse", and we can do something like division by multiplying by that inverse, but this does not always work.

There is one additional new type of operation for matrices, called transposition, which corresponds to switching rows and columns of the matrix.

Matrix times vector

In example 1.36, we saw an example of a matrix representing a transformation of one vector into another. This is done by "matrix multiplication". To make this a bit more rigorous, we define what it means to multiply a vector by a matrix. There is more than one way to think about this multiplication, but the result is always the same. Here are two equivalent definitions. Depending on context one may be more useful than the other.

Definition 1.39 *(first definition)* ***Multiplication of a vector by a matrix*** *proceeds as follows: given an $m \times n$ matrix* $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, *we define* $\vec{a}_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}$ *to be vector giving the i-th column of* A. *Then the product $\mathrm{A}\vec{x}$ is $\vec{b} = \mathrm{A}\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$. That is, the product $\mathrm{A}\vec{x}$ is the linear combination of the columns of* A *with coefficients given by the entries in $\vec{x}$.*

Notice that under this definition if $\vec{e}_i$ is the i-th canonical basis vector, then $\mathrm{A}\vec{e}_i$ is the i-th column of ~~$matA$~~ A. So if we know what a linear transformation ~~$T:\mathbb{R}^n \to \mathbb{R}^n$~~ $T:\mathbb{R}^n \to \mathbb{R}^m$ does to each canonical basis vector, then we know what matrix A represents T.

Definition 1.40 *(second definition)* ***Multiplication of a vector by a matrix*** *can also be calculated in a different, equivalent way. Given an $m \times n$ matrix* $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ *and a vector of n entries* $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$*, the result is a vector of m entries,* $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$*. The values of $\vec{b}$ are*

$$b_i = \sum_{k=1}^{n} a_{ik} x_k$$

This second definition is frequently summarized as "row times column".

These two concepts of multiplication give the same result. In either case, it requires that the vector $\vec{x}$ has the same number of entries as there are entries in a row of A. If the numbers of entries are not the same, the product is not defined.

Example 1.41 *Two examples using the first definition:*

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 1 & -1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 39 \\ 28 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

Example 1.42 *The same examples using the second definition:*

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 1 & -1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2 \cdot 3 + 0 \cdot 5 \\ 2 \cdot 1 + 4 \cdot 3 + 5 \cdot 5 \\ 1 \cdot 1 - 1 \cdot 3 + 6 \cdot 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 39 \\ 28 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 0 \cdot 5 \\ 0 \cdot 2 + 1 \cdot 5 \\ 1 \cdot 2 + 1 \cdot 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

Example 1.43 *A case where the multiplication is undefined:*

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 1 & -1 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

The product is not defined because the number of columns in the matrix does not match the number of entries in the vector.

Exercise 1.44 *Find $\mathrm{A}\vec{x}$ in each case, or say why it is undefined:*

(a). $\mathrm{A} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

(b). $\mathrm{A} = \begin{bmatrix} 0 & 2 \\ 11 & -1 \\ 2 & 2 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$.

(c). $\mathrm{A} = \begin{bmatrix} 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(d). $\mathrm{A} = \begin{bmatrix} 1 & 3 & 5 & 7 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(e). $\mathrm{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 5 \\ 4 \\ 6 \\ 3 \end{bmatrix}$.

(f). $\mathrm{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 5 \\ 4 \\ 6 \\ 3 \end{bmatrix}$.

We will see later that if $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then there is a matrix A which represents it, such that $T(\vec{x}) = \mathrm{A}\vec{x}$. If we want to find this matrix, there are several ways to do it. One is to follow example 1.36. We write out each entry of the output vector as a linear combination of the input vector's entries (in order). Once we've done this, the coefficients become the entries in the matrix.

An alternate way is through definition 1.39. If we assume $\mathrm{A} = [\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_n]$, we know that $\mathrm{A}\vec{e}_i = \vec{a}_i$ for every i (you should check for yourself why this is true). So to find the i-th column of the matrix A, just calculate $T(\vec{e}_i)$.

Exercise 1.45 *Find the matrix* A *so that* $T(\vec{x}) = \mathrm{A}\vec{x}$ *where* $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ *for each of the following*

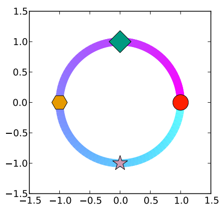
(a). $T : \mathbb{R}^4 \to \mathbb{R}^3$ *by* $T(\vec{x}) = \begin{bmatrix} x_1 - x_2 \\ x_2 + x_3 \\ x_4 \end{bmatrix}$. *Here* $n = 4$, *so* $\vec{x}$ *has 4 entries.*

(b). $T : \mathbb{R}^3 \to \mathbb{R}^3$ *by* $T(\vec{x}) = \begin{bmatrix} x_3 - x_2 - x_1 \\ x_2 - x_1 \\ x_1 \end{bmatrix}$. *Here* $n = 3$, *so* $\vec{x}$ *has 3 entries. (pay attention to the order of* x_1, x_2, *and* x_3*)*

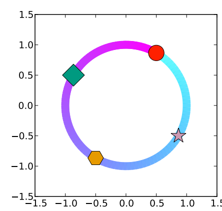
(c). $\cancel{T : \mathbb{R}^4 \to \mathbb{R}^2}$ $T : \mathbb{R}^3 \to \mathbb{R}^2$ *by* $T(\vec{x}) = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$. *Here* $n = 3$ *so* $\vec{x}$ *has 3 entries. (careful you don't make your matrix too small)*

Geometric interpretation Let's think temporarilly about a square $n \times n$ matrix A. When we take a vector $\vec{x} \in \mathbb{R}^n$, and multiply by A, we get a new vector in $\mathbb{R}^n$. It may point in a different direction. It may have a different length. Typically we can think of an $n \times n$ matrix as a possibly reflecting the vector across some line through the origin, then rotating the vector around the origin, and finally rescaling some directions.

This is a very useful conceptualization. In 2 dimensions it converts a circle centered at the origin into an ellipse centered at the origin which may be rotated and reflected from the original. In higher dimensions it converts a "hypersphere" centered at the origin into a "hyperellipse" centered at the origin which again may be rotated and reflected compared to the original.

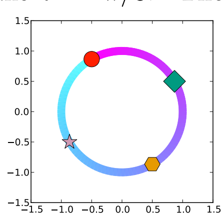
In the following examples, we show what the given matrix does to the unit circle (that is the circle of radius 1), and specifically to a few points on the unit circle: $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$, $\begin{bmatrix}-1\\0\end{bmatrix}$, and $\begin{bmatrix}0\\-1\end{bmatrix}$. The starting points are shown here 

Example 1.46 *A rotation only. The matrix* $\mathrm{R}(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ *rotates all vectors around the origin by the amount* θ*.*

In this case we take $\theta = \pi/3$ *(60°). So we multiply by* $\mathrm{R}(\pi/3) = \begin{bmatrix}1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{bmatrix}$*. The result of multiplying vectors on the unit circle by* $\mathrm{R}(\pi/3)$ *is shown here* 

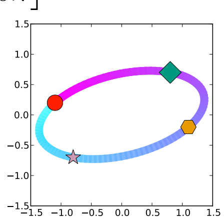
Example 1.47 *A reflection only. To reflect across the line through the origin with angle* θ *from the horizontal, we multiply by* $\mathrm{S}(\theta) = \begin{bmatrix}\cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta\\ 2\cos\theta\sin\theta & \sin^2\theta - \cos^2\theta\end{bmatrix}$.

We take $\theta = \pi/3$*. The result of multiplying vectors on the unit circle by* $S(\pi/3)$ *is shown here*



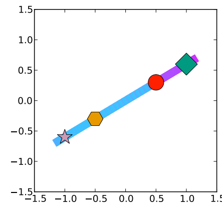
.

Notice the order of points going around the circle has been reversed.

Example 1.48 *We now consider an arbitrary matrix and multiply it times vectors on the unit circle. We take* $\mathrm{M} = \begin{bmatrix}-1.1 & 0.8\\ 0.2 & 0.7\end{bmatrix}$*. This matrix reflects, rotates and then rescales some directions to make the circle an ellipse.*  .

Notice that the order of points going around the ellipse has been reversed. This is the result of the reflection.

Example 1.49 *Some matrices actually take the circle and remap it to an ellipse which has zero width. That is, they turn the circle into a line segment. Each point of the line segment (except the very ends) have two "source points" on the unit circle. More generally each point on the line also has many "source points" not on the unit circle. Here we take* $\mathrm{N} = \begin{bmatrix} 0.5 & 1 \\ 0.3 & 0.6 \end{bmatrix}$.



A note for those who are interested (not vital for the course): Later we'll talk about determinants and invertible matrices. If the matrix contains a reflection in it, then it is referred to as "orientation reversing" and it has a negative determinant. Otherwise it is "orientation preserving" and it has a positive determinant. The value of the determinant actually measures the ratio of the area inside the final ellipse compared with the area inside the initial circle. If there are multiple points sent to the same point, then the matrix is not invertible (we don't know which $\vec{x}$ we might have started with to get to a given $\vec{b} = \mathrm{A}\vec{x}$), and its determinant is zero. In higher dimensions, these concepts still make sense except that we discuss volume or the higher dimensional equivalent.

We've defined what it means to multiply a vector by a matrix. There are also many cases where we will want to add or multiply two matrices together.

Matrix addition and subtraction

The addition and subtraction of (identically-sized) matrices proceeds exactly as we would naturally expect.

Definition 1.50 *Matrix addition of two* $m \times n$ *matrices is performed as*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

If they do not have the same number of rows and columns addition is not defined.

It can be shown that the order of addition does not matter (that is, $\mathrm{A} + \mathrm{B} = \mathrm{B} + \mathrm{A}$).

Example 1.51 *For example, we can add* 4×3 *matrices to get*

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 10 & 20 & 30 \\ 100 & 500 & 10000 \end{bmatrix} + \begin{bmatrix} 1 & 10 & 100 \\ 2 & 20 & 200 \\ 3 & 30 & 300 \\ 4 & 40 & 400 \end{bmatrix} = \begin{bmatrix} 2 & 12 & 103 \\ 6 & 25 & 206 \\ 13 & 50 & 330 \\ 104 & 540 & 10400 \end{bmatrix} \tag{1.1}$$

Example 1.52 *The sum*

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \tag{1.2}$$

is undefined because the matrices do not have the same size.

Scalar multiplication of a matrix

We define scalar multiplication of a matrix A by the scalar c as simply multiplying each entry of $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ by c. Note that order does not matter:

$$c\mathrm{A} = \mathrm{A}c = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

Matrix times matrix

Matrix multiplication is a bit more complicated. It is a generalization of the multiplication of a matrix times a vector. If AB = C, each column of C corresponds to the result of multiplying A by the corresponding column of B

Like multiplication of a matrix with a vector, this has two interpretations.

Here is the first interpretation

Definition 1.53 *An $m \times n$ matrix* A *and an $n \times r$ matrix* B *can be multiplied together so that*

$$\mathrm{AB} = \mathrm{A}\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_r \end{bmatrix} = \begin{bmatrix} \mathrm{A}\vec{b}_1 & \mathrm{A}\vec{b}_2 & \cdots & \mathrm{A}\vec{b}_r \end{bmatrix}$$

We have used $\vec{b}_i$ to denote the i-th column of B. *Each column of the product is simply* A *times the corresponding column of* B.

If we want to use the second interpretation then the definition would be

Definition 1.54 *(equivalent definition of matrix multiplication) An $m \times n$ and an $n \times r$ matrix can be multiplied together so that*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nr} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1r} \\ c_{21} & c_{22} & \cdots & c_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mr} \end{bmatrix}$$

where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

Notice that AB is undefined if the number of columns of A does not match the numer of rows of B.

An important property of matrix multiplication is that the order of multiplication matters. Sometimes AB is defined when BA isn't. Even if both are defined, almost always AB $\neq$ BA. It is incredibly rare that they are equal. This is one of the most common errors made by students when they first start using matrices.

Example 1.55 *To highlight the second definition, consider the product below*

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \\ 3 & 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 & \mathbf{7} & 8 \\ 5 & 6 & 7 & \mathbf{8} & 9 \\ 6 & 7 & 8 & \mathbf{9} & 10 \\ 7 & 8 & 9 & \mathbf{10} & 11 \end{bmatrix} = \begin{bmatrix} 60 & 70 & 80 & 90 & 100 \\ 82 & 96 & 110 & \mathbf{124} & 138 \\ 104 & 122 & 140 & 158 & 176 \end{bmatrix}$$

I have highlighted the terms involved in calculating $c_{2,4}$. The entry $c_{2,4} = 124$ is found by $124 = 2 \times 7 + 3 \times 8 + 4 \times 9 + 5 \times 10$.

Example 1.56 *The product of the matrices in equation (1.1)*

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 10 & 20 & 30 \\ 100 & 500 & 10000 \end{bmatrix} \begin{bmatrix} 1 & 10 & 100 \\ 2 & 20 & 200 \\ 3 & 30 & 300 \\ 4 & 40 & 400 \end{bmatrix}$$

is undefined because the number of elements in each row (3) in the first matrix is not equal to the number of elements in each column (4) of the second matrix. It would also be undefined if we switched the order.

Example 1.57 *The matrices we used in (1.2) can be multiplied (in either order) even though they cannot be added. We find*

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 9 & 12 & 15 \\ 14 & 19 & 24 \\ 19 & 26 & 33 \end{bmatrix} \qquad \textit{and} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 14 & 20 \\ 32 & 47 \end{bmatrix}$$

Definition 1.58 *The $n \times n$ (square) matrix whose diagonal entries are 1 and off-diagonal entries are 0:*

$$\mathrm{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

is referred to as the ***identity*** *matrix. It is sometimes denoted I_n to make clear its size.*

The identity matrix has the property that if we multiply it by another matrix, we get the other matrix back. That is, $\mathrm{AI} = \mathrm{A}$ and also $\mathrm{IA} = \mathrm{A}$. Note that if A is not square, and is in fact $m \times n$, then we need to use a different size I in each case: $\mathrm{AI}_n = \mathrm{A} = \mathrm{I}_m\mathrm{A}$.

Theorem 1.59 *The only matrix S that satisfies $\mathrm{SA} = \mathrm{A}$ for every $m \times n$ matrix A is the identity I_m. The only matrix S that satisfies $\mathrm{AS} = \mathrm{A}$ for every $m \times n$ matrix A is the identity I_n.*

It can be shown that matrix multiplication is associative:

$$\mathrm{A(BC)} = \mathrm{(AB)C}$$

and distributive:

$$\mathrm{A(B + C)} = \mathrm{AB} + \mathrm{AC}\,.$$

It also has these properties with respect to vectors. If A and B are matrices and $\vec{u}$, $\vec{v}$ are vectors then

$$\mathrm{A}(\mathrm{B}\vec{u}) = (\mathrm{AB})\vec{u}$$
$$\mathrm{A}(\vec{u} + \vec{v}) = \mathrm{A}\vec{u} + \mathrm{A}\vec{v}$$

The fact that $\mathrm{A(BC)} = \mathrm{(AB)C}$ and $\mathrm{A}(\mathrm{B}\vec{u}) = (\mathrm{AB})\vec{u}$ means that we don't have to use parentheses when we write out a long product of matrices.

Change this pr
3 matrices, and
"is this logic co
tiply by A first
C, so $\vec{x} = \mathrm{ABC}$

Exercise 1.60 *Consider the vector $\vec{u}$. Let M be a matrix and define $\vec{v} = \mathrm{M}\vec{u}$. Define $\vec{w} = \mathrm{N}\vec{v}$. Does $\vec{w} = (\mathrm{MN})\vec{u}$ or $\vec{w} = (\mathrm{NM})\vec{u}$? Can you explain?*

Even if it's not assigned, think through exercise 1.60. It highlights a common error.

1.4.4 Matrix Transposition

Definition 1.61 *The* ***transpose*** *of an* $m \times n$ *matrix* $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ *is an* $n \times m$ *matrix denoted* A^T *which we read out loud as "A transpose" and defined by*

$$\mathrm{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

That is, the rows and columns of A are switched to create A^T.

Example 1.62 *Here are some concrete examples*

$$\begin{bmatrix} 1 & 2 & 3 \\ 22 & 5 & 11 \end{bmatrix}^T = \begin{bmatrix} 1 & 22 \\ 2 & 5 \\ 3 & 11 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

I'll list a few properties about how transposition interacts with matrix addition and multiplication. In each statement below, assume that A and B have the appropriate size so that the operations are defined.

- $(\mathrm{A}+\mathrm{B})^T = \mathrm{A}^T + \mathrm{B}^T$
- $(\mathrm{AB})^T = \mathrm{B}^T\mathrm{A}^T$
- $(\mathrm{A}^T)^T = \mathrm{A}$

Exercise 1.63 *Prove that*

(a). $(\mathrm{A}+\mathrm{B})^T = \mathrm{A}^T + \mathrm{B}^T$

(b). $(\mathrm{AB})^T = \mathrm{B}^T\mathrm{A}^T$.

(c). $(c\mathrm{A})^T = c\mathrm{A}^T$ *where* c *is a scalar*

To do this, find the i, j *entry of the matrix on the left hand side and the right hand side in terms of the entries of A and B, and show the results are the same.*

and B playing

Exercise 1.64 *Using the fact that* $(\mathrm{AB})^T = \mathrm{B}^T\mathrm{A}^T$, *prove that* $(\mathrm{ABC})^T = \mathrm{C}^T\mathrm{B}^T\mathrm{A}^T$,

Exercise 1.65 *Let*

$$\mathrm{A} = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix} \qquad \mathrm{B} = \begin{bmatrix} 0 & 0 & 0 \\ 5 & 6 & 7 \end{bmatrix} \qquad \mathrm{C} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \qquad \mathrm{D} = \begin{bmatrix} 5 & 3 \\ 2 & 2 \end{bmatrix} \qquad \mathrm{E} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

For each of the following, either find its value or explain why it does not exist.

(a). *(i)* $\mathrm{A}+\mathrm{B}$
(ii) $\mathrm{B}+\mathrm{A}$

(b). *(i)* AB
(ii) BA

(iii) AB^T

(c). (i) $4AC$

(ii) $4CA$

(d). (i) $A + 3C$

(ii) $3C + A$

(e). (i) $C + D$

(ii) $D + C$

(f). (i) CD

(ii) DC

(g). (i) DE

(ii) ED

(h). (i) $AE + 2C$

(ii) $EA + C$

(i). (i) $A + E^T$

(ii) AE^T

(j). (i) $E^T D$

(ii) ED^T

Exercise 1.66 *For* A, B, C, D, *and* E *as in the previous problem, give the size that the identity matrix* I *must have for each of the following to be defined (recall* I *is the identity matrix).*

(a). AI

(b). IA

(c). BI

(d). IB^T

(e). $C^T I$

(f). IC

(g). DI

(h). ID

(i). EI

(j). IE

(k). $AB^T I$

(l). AIB^T

1.5 Linearity and Matrices

Recall that the definition of a linear function is that $f(ax+by) = af(x) + bf(y)$.

We're going to show that a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear iff it can be written as $T(\vec{x}) = \mathrm{A}\vec{x}$ for some matrix A. In other words, if we want to study linear functions of vectors, then all we have to do is study matrices.

Let's first show that every linear function $T : \mathbb{R}^n \to \mathbb{R}^m$ can be written as $T(\vec{x}) = \mathrm{A}\vec{x}$.

Consider the canonical basis $\{\vec{e}_1, \dots, \vec{e}_n\}$ of $\mathbb{R}^n$. If T is linear, then

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1 + \cdots + x_n\vec{e}_n) \\ &= x_1 T(\vec{e}_1) + \cdots + x_n T(\vec{e}_n) \end{aligned}$$

If we define $\vec{a}_i = T(\vec{e}_i)$, then we have

$$T(\vec{x}) = \sum x_i \vec{a}_i$$

But this is just one of our definitions of $\mathrm{A}\vec{x}$ for $\mathrm{A} = [\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_n]$.

Now we show the opposite direction. If $T(\vec{x}) = \mathrm{A}\vec{x}$, then T is linear. Whenever we want to show that something is linear, we look at what it does to $a\vec{x}+b\vec{y}$. Except in this case since I've already got 'a's floating around, I'm going to look at $c\vec{x}+b\vec{y}$. In this case

$$T(c\vec{x}+b\vec{y}) = \mathrm{A}(c\vec{x}+b\vec{y})$$

This can be rewritten as

$$\begin{aligned} T(c\vec{x}+b\vec{y}) &= \mathrm{A}\begin{bmatrix} cx_1 + by_1 \\ cx_2 + by_2 \\ \vdots \\ cx_n + by_n \end{bmatrix} \\ &= \sum (cx_i + by_i)\vec{a}_i \end{aligned}$$

where $\vec{a}_i$ is the i-th column of A. We used the definition of multiplication by A as being a linear combination of the columns of A.

So we have

$$\begin{aligned} T(c\vec{x}+b\vec{y}) &= \sum cx_i\vec{a}_i + \sum by_i\vec{a}_i \\ &= c\mathrm{A}\vec{x} + b\mathrm{A}\vec{y} \end{aligned}$$

So what we've shown now is that

Theorem 1.67 *The ~~linear~~ function $T : \mathbb{R}^n \to \mathbb{R}^m$ can be represented by $T(\vec{x}) = \mathrm{A}\vec{x}$ for some $m \times n$ matrix A iff T is linear.*

The matrix A for T is unique.

This theorem gives an easy way to check if these functions are linear. If you really want, you can check that something satisfies the original definition of a linear function, but if the function you're asked to look at is from $\mathbb{R}^n$ to $\mathbb{R}^m$, I'd recommend checking if it can be written as $T(\vec{x}) = \mathrm{A}\vec{x}$.

Example 1.68 *Let the function $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by*

$$T(\begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix}) = \begin{bmatrix} x_1 + x_2x_3 \\ x_2 \end{bmatrix}$$

Is T linear?
No. The first entry of the output is not a linear combination of the input vector's entries.
You might argue that this can be written in the form $\mathrm{A}\vec{x}$, for example by

$$T(\begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 1 & x_3 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

But the matrix does not have all constant entries. It is not fixed - so really A depends on $\vec{x}$.

1.6 Applications

Let's get some practice going from "real-world" examples to an appropriate matrix representation.

Frequently we have a set of variables, call them $b_1, b_2, \ldots, b_m$ whose values depend on another set of variables, call them $x_1, x_2, \ldots, x_n$. Under some circumstances we can get an equation of the form $\vec{b} = \mathrm{A}\vec{x}$.

Generically, if the values of the b variables depend on the x variables, then $\vec{b} = T(\vec{x})$ for some function T. We can rewrite this in the form $\vec{b} = \mathrm{A}\vec{x}$ iff T is linear (theorem 1.67). To find A we use the following steps:

Finding A such that $\vec{b} = \mathrm{A}\vec{x}$

(a). Write each b_i in terms of the entries in $\vec{x}$.

(b). Check that each equation is of the form $b_i = \sum_{j=1}^{n} a_{ij}x_j$ (otherwise the system is not linear and this can't be done).

(c). If it's linear, then $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{bmatrix}$.

(d). Then $\vec{b} = \mathrm{A}\vec{x}$ where $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$.

Example 1.69 *A bakery makes bread, muffins, and cookies. Each loaf of bread requires 5 cups of flour, 1 tablespoon of butter, 3 tablespoons of sugar, and 1 tablespoon of yeast. A dozen muffins require 2 cups of flour, 4 tablespoons of butter, 8 tablespoons of sugar and 1 tablespoon of baking powder. A dozen cookies require 3/4 cup of flour, 4 tablespoons of butter, 8 tablespoons of sugar, and 1/6 tablespoon of baking soda.*
We want to find out how much of each ingredient is needed in terms of the products the bakery makes.

We use the following for our variables:

b_1	*the number of cups of flour*
b_2	*the number of tablespoons of butter*
b_3	*the number of tablespoons of sugar*
b_4	*the number of tablespoons of yeast*
b_5	*the number of tablespoons of baking powder*
b_6	*the number of tablespoons of baking soda*
x_1	*loaves of bread*
x_2	*dozen muffins*
x_3	*dozen cookies*

Many people have a tendancy to get the next step reversed. As a self-test, try to find a formula for b_1 in terms of x_1, x_2, and x_3 on your own before looking at the equations below. Bear in mind, we're trying to find the b values in terms of the x values, not vice versa.
We get

$$\begin{aligned}
b_1 &= 5x_1 + 2x_2 + (3/4)x_3 \\
b_2 &= 1x_1 + 4x_2 + 4x_3 \\
b_3 &= 3x_1 + 8x_2 + 8x_3 \\
b_4 &= 1x_1 \\
b_5 &= 1x_2 \\
b_6 &= (1/6)x_3
\end{aligned}$$

So

$$\vec{b} = \begin{bmatrix} 5 & 2 & 3/4 \\ 1 & 4 & 4 \\ 3 & 8 & 8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/6 \end{bmatrix} \vec{x}$$

Exercise 1.70 *Which of the following can be written in the form $\vec{b} = \mathrm{A}\vec{x}$? If they can, do so. Be careful — I'll play around with the order of variables. In all cases, assume $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.*

(a).

$$\begin{aligned}
b_1 &= x_1 - x_2 - x_3 \\
b_2 &= x_2 + x_3 + 2x_1 \\
b_3 &= x_1 - x_3 \\
b_4 &= x_3 + x_1
\end{aligned}$$

(b).

$$\begin{aligned}
b_1 &= x_1 - x_3 \\
b_2 &= x_3 + x_2 + 0.1
\end{aligned}$$

(c).

$$\begin{aligned}
b_1 &= x_1 - x_3 \\
b_2 &= x_3 + x_2
\end{aligned}$$

(d).

$$\begin{aligned}
b_1 &= x_1 - x_2 - x_3 \\
b_2 &= x_2 + x_3 + 2x_1 \\
b_3 &= x_1 - x_3 \\
b_4 &= x_3 + x_1^2
\end{aligned}$$

Exercise 1.71 *In exercise 1.35 we showed how to represent polynomials of degree at most 3 with vectors of 4 numbers. Let $P(t)$ be one of these polynomials and $Q(t) = P'(t)$ [the derivative of $P(t)$],* *(a). If $P(t) = a_0+a_1t+a_2t^2+a_3t^3$, find $Q(t)$ in terms of the a_i and then write down the vector representations $\vec{p} = T(P)$ and $\vec{q} = T(Q)$ following exercise 1.35. The vectors should both have 4 entries.*

(b). Find a square matrix D such that that $\mathrm{D}\vec{p} = \vec{q}$ for any vector $\vec{p}$.

(c). Calculate D^2 and then find $\vec{r} = (\mathrm{D}^2)\vec{p}$. (note that the exponent means what we expect: for a matrix A, $\mathrm{A}^2 = \mathrm{AA}$. It can only be defined for square matrices.)

(d). Here is another way to find $\vec{r}$. Calculate $\vec{q} = \mathrm{D}\vec{p}$ and then calculate $\vec{r} = \mathrm{D}\vec{q}$.

(e). Did you find the same answer for $\vec{r}$ each time? Which method takes fewer additions and multiplications, (c) or (d)?

(f). What polynomial $R(t)$ has $\vec{r} = T(R)$? What is the relation of $R(t)$ to $P(t)$?

A message from this exercise is that at least for polynomials of given maximum degree, the derivative behaves exactly like a matrix multiplication. This holds more generally for the derivative, but we have to allow for "infinite-dimensional" vectors and matrices for the analogy to be exact.

We now look at a few frequently encountered special cases.

1.6.1 Differential Equations

There are many cases where we have a set of variables $x_1(t)$, $x_2(t)$, ..., $x_n(t)$ which change continuously in time. In some cases, we can use matrices to write down the equations in a simple form. We will later learn how to solve these, for now I just want you to be able to know if it is possible to write it down in a simple form and how to do so.

We can often (but not always) write down a system of differential equations of the form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{j=1}^{n} a_{ji}x_j(t)$$

We can write[4]

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}x_1 \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2 \\ \vdots \\ \frac{\mathrm{d}}{\mathrm{d}t}x_n \end{bmatrix}$$

When $\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{j=1}^{n} a_{ji}x_j(t)$, the system can be expressed in the form

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \vec{x}(t)$$

In other words, the function T which takes $\vec{x}$ and outputs the derivative of $\vec{x}$ is linear. In these cases, we can write the system of equations as $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathrm{A}\vec{x}$.

[4] In case uncomfortable just accepting that I can write it this way (it bothered me when I first saw it), let's go through the definition:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \lim_{\Delta t \to 0} \frac{\vec{x}(t+\Delta t) - \vec{x}(t)}{\Delta t} = \lim_{\Delta t \to 0} \begin{bmatrix} [x_1(t+\Delta t) - x_1(t)]/\Delta t \\ [x_2(t+\Delta t) - x_2(t)]/\Delta t \\ \vdots \\ [x_n(t+\Delta t) - x_n(t)]/\Delta t \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}x_1 \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2 \\ \vdots \\ \frac{\mathrm{d}}{\mathrm{d}t}x_n \end{bmatrix}$$

We know that equations like $\frac{\mathrm{d}y}{\mathrm{d}t} = ay$ have a simple solution: $y = Ce^{at}$. Here we have a similar equation form except with matrices and vectors. We will discuss later in the course how to solve equations like this. It will turn out to be very similar to the solution we've just seen.

Example 1.72 *Consider a massless spring with spring constant* $k\,\mathrm{Nm}^{-1}$. *One end is fixed in place and the other is attached to a mass of* $m\,\mathrm{kg}$. *Assume gravity has been turned off (or place it on a frictionless table if you want). Assume that the spring has a damping coefficient of* $c\,\mathrm{Nsm}^{-1}$.

Let the displacement of the mass from equilibrium be represented by $x\,\mathrm{m}$ *and the velocity of the mass be represented by* $v\,\mathrm{ms}^{-1}$.

The force provided by the spring is $-kx\,\mathrm{N}$*: it is proportional to the displacement, but in the opposite direction with constant* k. *The force provided by the damping is* $-cv\,\mathrm{N}$*: it is proportional to the velocity, but in the opposite direction with constant* c. *So the total force acting on the mass is* $(-kx - cv)\,\mathrm{N}$. *Since the mass is* $m\,\mathrm{kg}$*, the acceleration is force over mass:* $-\frac{kx+cv}{m}\,\mathrm{ms}^{-2}$. *So (dropping units) we arrive at*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}x &= v\\ \frac{\mathrm{d}}{\mathrm{d}t}v &= \frac{-kx - cv}{m}\end{aligned}$$

We rewrite this in matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$

Exercise 1.73 *Romeo and Juliet have a dysfunctional ~~disfunctional~~ relationship. Romeo tends to take Juliet for granted when she is being kind to him. So his kindness to Juliet decreases when she is kind. In contrast, as she stops being kind to him, he makes an effort to win her back and is nicer to her. The rate of increase/decrease of Romeo's kindness to Juliet is directly proportional to Juliet's kindness to Romeo with proportionality constant* $-c$ *where* c *is a positive constant.*

Juliet has the opposite response. If Romeo is kind to her, she will start being kinder to him. The rate of increase/decrease of Juliet's kindness to Romeo is directly proportional to Romeo's kindness to Juliet with proportionality constant d *where* d *is a positive constant.*

Let Romeo's kindness to Juliet be measured by R *and Juliet's kindness to Romeo be measured by* J.

(a). Given R *and* J*, find* $\frac{\mathrm{d}}{\mathrm{d}t}R$*, the rate of increase/decrease of* R*, and* $\frac{\mathrm{d}}{\mathrm{d}t}J$*, the rate of increase/decrease of* J*. Explain how you arrived at your equations.*

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(b). Is this system linear? If so, write it in the form $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathrm{A}\vec{x}$.

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Exercise 1.74 *A "soft" spring loses strength when it is stretched. The force from the spring is* $-(k + k_1 x^2)x$ *where* $k_1 < 0$.

(a). Rederive the equations for $\frac{\mathrm{d}}{\mathrm{d}t}x$ *and* $\frac{\mathrm{d}}{\mathrm{d}t}v$ *from example 1.72 assuming a soft spring.*

(b). Is the system linear? If so, write it in the form $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathrm{A}\vec{x}$.

1.6.2 Iterative time stepping

There are many cases where we have a set of variables $x_1(t)$, $x_2(t)$, ..., $x_n(t)$ which change from one time to the next. We can represent these by the vector $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$.

Although the steps are almost identical to those I gave above, I repeat them here because my experience is that that many people find A^T rather than A.

Finding A **such that** $\vec{x}(t+1) = \mathrm{A}\vec{x}(t)$ Our steps to writing $\vec{x}(d+1)$ as a matrix times $\vec{x}(d)$ are to:

(a). Write each $x_i(d+1)$ in terms of the entries of $\vec{x}(d)$.

(b). Check that each equation is of the form $x_i(d+1) = \sum_{j=1}^{n} a_{ij}x_j(d)$ (otherwise the system is not linear, so this can't be done).

(c). If it's linear, then $\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ \vdots \\ x_n(t+1) \end{bmatrix} = \begin{bmatrix} a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) \\ a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) \\ \vdots \\ a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) \end{bmatrix}$.

(d). Then $\vec{x}(d+1) = \mathrm{A}\vec{x}(d)$ where $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{nn} \end{bmatrix}$.

Note that in this case, the matrices are square.

Example 1.75 *Consider an infectious disease with the following properties. A person who becomes infected on one day will be infectious on each of the following two days. On each of those two days, the infectious person will infect 1 person.*

For day d, let $x_1(d)$ be the number of infectious people who are on their final day of infectiousness, and $x_2(d)$ be the number of infectious people who are on their next to last day of infectiousness. Let $\vec{x}(d) = \begin{bmatrix} x_1(d) \\ x_2(d) \end{bmatrix}$.

The first step is to write down equations for $x_1(d+1)$ and $x_2(d+1)$ in terms of $x_1(d)$ and $x_2(d)$.

Let's find $x_1(d+1)$. On day $d+1$, how many people are on their last day of infectiousness? Well, it's the same as the number of people who on the previous day were on their next-to-last day. That is

$$x_1(d+1) = 1x_1(d) + 0x_2(d)$$

Now we find $x_2(d+1)$. How many people on day $d+1$ have just become infected? By the rules of the disease, it's equal to the number of people who were in their first or second day of infection on day d. That is

$$x_2(d+1) = 1x_1(d) + 1x_2(d)$$

Both of these take the form $x_i(d+1) = \sum_{j=1}^{n} a_{ij}x_j(d)$. So we have

$$\vec{x}(d+1) = \mathrm{A}\vec{x}(d)$$

where

$$\mathrm{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

We will return to this disease example a number of times in this course.

Example 1.76 *Consider three cities, Albuquerque, Boston, and Chicago. Each year 5% move to Boston, 10% of the people in Albuquerque move to Chicago, and the remaining 85% stay in Albuquerque. In a similar fashion, 35% of those in Boston move to Albuquerque and 15% move to Chicago, with the remaining 50% remaining in Boston. Finally, 20% of the people in Chicago move to Boston, and 20% move to Albuquerque, with the remaining 60% remaining in Chicago. We ignore birth and death.*

Let $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ where $x_1(t)$ is the number of people in Albuquerque at year t, $x_2(t)$ is the number in Boston, and $x_3(t)$ is the number in Chicago.

Compressing a few steps of the previous example, we have

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{bmatrix} = \begin{bmatrix} 0.85x_1(t) + 0.35x_2(t) + 0.2x_3(t) \\ 0.05x_1(t) + 0.5x_2(t) + 0.2x_3(t) \\ 0.1x_1(t) + 0.15x_2(t) + 0.6x_3(t) \end{bmatrix}$$

gives the new populations in year $t+1$ given their values in year t. This can be represented by

$$\vec{x}(t+1) = \mathrm{A}\vec{x}(t)$$

where

$$\mathrm{A} = \begin{bmatrix} 0.85 & 0.35 & 0.2 \\ 0.05 & 0.5 & 0.2 \\ 0.1 & 0.15 & 0.6 \end{bmatrix}$$

Exercise 1.77 *Consider a disease for which people who get infected on day d become infectious on day $d+1$ and then recover. Assume each child infects two children and one adult before recovering and each adult infects one adult before recovering. Let $x_1(d)$ be the number of infectious adults and $x_2(d)$ be the number of infectious children on day d. Let $\vec{x}(d) = \begin{bmatrix} x_1(d) \\ x_2(d) \end{bmatrix}$*

(a). Find $x_1(d+1)$ and $x_2(d+1)$ in terms of $x_1(d)$ and $x_2(d)$.

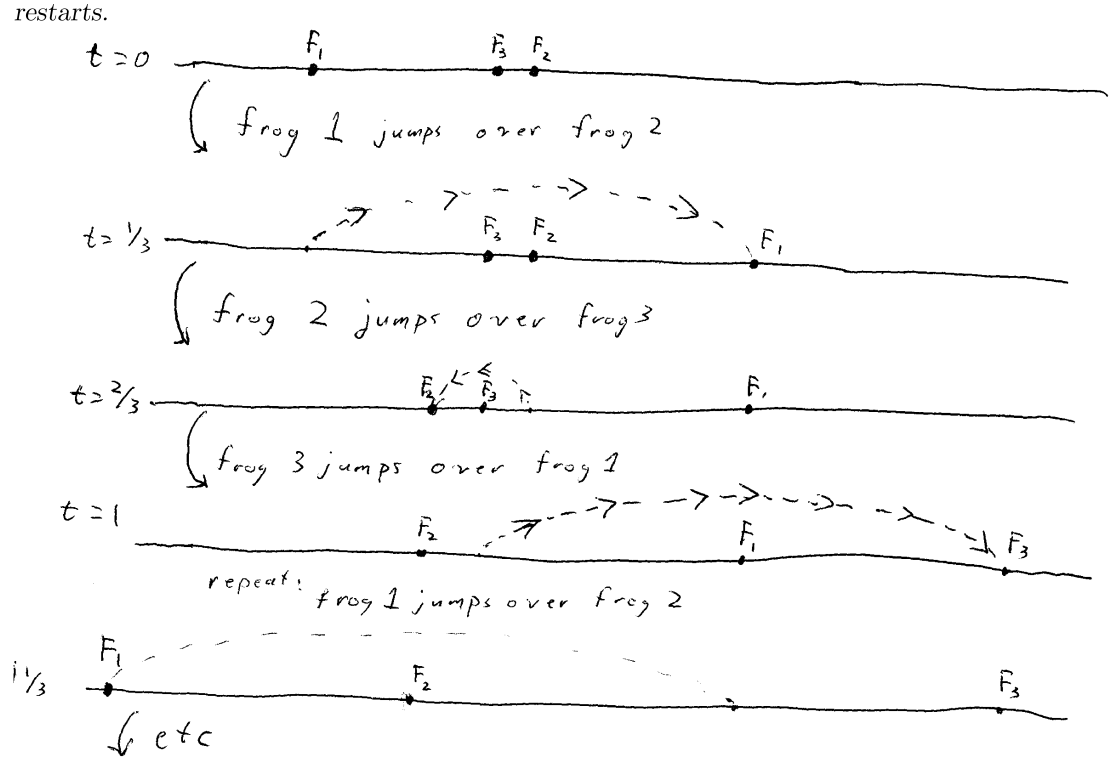
t — simply ask ”? as in previ-

(b). How do you know that this can be written as $\vec{x}(d+1) = \mathrm{A}\vec{x}(d)$?

(c). Find the matrix A such that $\vec{x}(d+1) = \mathrm{A}\vec{x}(d)$.

Exercise 1.78 *Leap Frog:*

Three frogs are in a line. At each full time step, the frogs do three actions: first (at time t) frog 1 jumps over frog 2, so that the distance between frogs 1 and 2 remain the same. Then (at time $t+1/3$) frog 2 jumps over frog 3 maintaining the same distance betweem 2 and 3. Finally (at time $t+2/3$) frog 3 jumps over frog 1 again keeping the same distance between 3 and 1. At time $t+1$ the process restarts.



Let $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the positions at the start of each full time step, so assume in this problem that t is always an integer.

(a). Let $x_1(t+1/3)$, $x_2(t+1/3)$ and $x_3(t+1/3)$ be their positions at the end of the jump by frog 1. Derive equations for $x_1(t+1/3)$, $x_2(t+1/3)$, and $x_3(t+1/3)$ in terms of $x_1(t)$, $x_2(t)$, and $x_3(t)$. Make sure it is clear how you arrived at your equations.

(b). Find a matrix A_1 so that $\vec{x}(t+1/3) = \mathrm{A}_1\vec{x}(t)$. Did you get $\mathrm{A}_1 = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$? If not, revisit the previous part.

(c). Find A_2 so that $\vec{x}(t+2/3) = \mathrm{A}_2\vec{x}(t+1/3)$.

(d). Similarly, find A_3 so that $\vec{x}(t+1) = \mathrm{A}_3\vec{x}(t+2/3)$.

(e). Find the matrix A in terms of A_1, A_2, and A_3 that gives $x(t+1) = \mathrm{A}x(t)$.

1.7 Matlab/Octave/Python comments

Linear Algebra these days is really done computationally. In this course I'll generally expect you to do the calculations by hand, but it's a good idea if you can check your work with a computer and if you know how to do many of the calculations with a computer.

I will try to provide enough guidance that you can use Matlab, Octave or Python.

A comment on the relative merits: Matlab has more infrastructure in place. It's more commonly used. However, you can't see the source code, and if you find a bug and report it, it may (eventually) get fixed, but you'll have to buy the new version. I can go on a long rant about the frustrations to a PhD student of finding that the reason his calculation is getting the wrong answer is because 1) the matlab code doesn't work for the given matrices and 2) the code doesn't bother to check whether it will work so it gives an answer but no warning that the answer might be wrong. Add to that the fact that the people at matlab didn't see this as a bug so were in no hurry to fix it (and would have charged for the new license anyways), and you get a professor who wants his students to know about alternatives.

The most common software used for linear algebra is Matlab. Matlab costs a lot (if you aren't at a university, it can easily cost well over \$1000 for a license per user plus more if you want any "toolboxes"). Matlab is relatively cheap while you are a student. I suspect part of the marketing strategy is to make sure that all engineers exclusively use Matlab while students so that when they go into the "real world" they can't work without it.

Octave is basically a free version of Matlab. It's missing many of the more advanced tools, but if you don't need those tools (and many people don't) it's much cheaper. I haven't needed anything Octave doesn't have for a very long time.

Python is a free, fairly powerful, and relatively simple computer language. It has a package, called SciPy which performs linear algebra calculations. It isn't (yet) as extensive as Matlab. I am less familiar with SciPy than Matlab, but I now use SciPy almost exclusively in my research because I use Python for most of my other work. You have access to the source code, meaning if you find a bug, then at least you can fix it on your own computer. If you convince people your code is better than the existing version, it'll become part of the main distribution everyone uses. If a new version comes out, you can update your code without having to buy a new license. You also get to use the fact that Python can do many things beyond just mathematics.

1.8 A short review of linearity

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Since a fair number of people have struggled with linearity, I want to go through it in a bit more detail.

Let us say we are given a function T and asked to show that it is linear.

Let's review the definition: $T : X \to Y$ is linear iff for any $\vec{u}, \vec{v} \in X$ and $a, b \in \mathbb{R}$, we have

$$T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$$

Note that the use of the vector notation here just means $\vec{u}$ and $\vec{v}$ are in X. They do not have to be vectors in $\mathbb{R}^n$ unless $X = \mathbb{R}^n$. More generally, they could be functions, polynomials, vectors. . . .

The steps are as follows:

(a). First, determine what X is.

(b). Then let $\vec{u}$ and $\vec{v}$ be chosen arbitrarily from X and let a and b be chosen arbitrarily from $\mathbb{R}$.

(c). Look at $T(a\vec{u} + b\vec{v})$.

(d). Expand $T(a\vec{u} + b\vec{v})$, and then collect terms with a and collect terms with b. We should have a times something plus b times something, where those somethings are $T(\vec{u})$ and $T(\vec{v})$

(e). Show that we get to $aT(\vec{u}) + bT(\vec{v})$.

Right now the ONLY exception to following these steps comes from the fact that we know if $T(\vec{x}) = \mathrm{A}\vec{x}$ for some matrix A the T is linear. So if you can show $T(\vec{x}) = \mathrm{A}\vec{x}$, then you're done. Of course, when we proved the theorem that lets you do this, we followed the steps above.

Chapter 2

Vector Spaces

2.1 Important concepts

After this chapter you should:

- Understand what it means to be closed under addition/scalar multiplication
- Be able to show that something is or is not a vector space.
- Understand what a subspace is.
- Be able to identify what the span of a set is.
- Be able to identify what a basis is.

2.2 Vector Spaces

There are many properties of linear transformations that are universal, whether they are linear transformations of vectors, of functions, or of any other objects. I'm going to introduce this chapter with a specific case that plays a significant role across much of mathematics. In each of the following examples, T is a linear transformation from some set X to another set Y. We're going to look at the solutions to $T(\vec{x}) = \vec{y}$ for some given $\vec{y} \in Y$. It's not really important that you understand all the details of the next few examples, but rather that you recognize and appreciate the similarity across these examples despite the diversity of X and Y.

Example 2.1 *Define T such that $T(x(s)) = \mathrm{d}x(s)/\mathrm{d}s$. Consider the set of $x(s)$ for which $T(x(s)) = 1$, that is, the functions of s whose derivative is 1. We know one solution: $x(s) = s$. But there are many others. Every solution can be written in the form $x(s) = s + C$ where C is a constant (and everything of this form is a solution). Note that $T(C) = \mathrm{d}C/\mathrm{d}s = 0$.*

Example 2.2 *Define $T(x(s))$ such that $T(x(s)) = \int_0^1 x(s)\,\mathrm{d}s$. Consider the set of $x(s)$ for which $T(x(s)) = 1$, that is, the functions whose integral from 0 to 1 is 1. One solution is $x(s) = 1$, but the general solution form is $x(s) = 1 + w(s)$ where $w(s)$ is any function for which $T(w(s)) = \int_0^1 w(s)\,\mathrm{d}s = 0$.*

Example 2.3 *Define $T(x(s))$ such that $T(x(s)) = x''(s) + 9x(s)$. Consider the set of $x(s)$ for which $T(x(s)) = 18$. If you've taken a differential equations class, you've probably learned how to solve this. The solutions can be written as $2 + A\cos 3s + B\sin 3s$. Here, 2 is a "particular" solution and $A\cos 3s + B\sin 3s$ is the solution to the "homogeneous problem" $T(x(s)) = 0$.*

Example 2.4 *Set* $\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *and define* $T(\vec{x}) = \mathrm{A}\vec{x}$ *(so* $\vec{x} \in \mathbb{R}^3$*). When we look for solutions to* $T(\vec{x}) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$*, it is fairly easy to see that the first entry of* $\vec{x}$ *doesn't matter, but the second one must be 2 and the third must be 5. So any solution can be written of the form* $\vec{x} = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} + \vec{w}$ *where the second and third entries of* $\vec{w}$ *are 0. Note that if the second and third entries of* $\vec{w}$ *are 0, then* $T(\vec{w}) = \mathrm{A}\vec{w} = \vec{0}$.

In all of these examples, we had a linear transformation T which mapped some set X to to another set[1] Y, and we were looking for solutions $\vec{x}$ to equations of the form $T(\vec{x}) = \vec{y}$ for a given $\vec{y}$. In all cases, we found that any solution could be written like $\vec{x} = \vec{x}_p + \vec{w}$ where $\vec{w}$ was some solution of $T(\vec{w}) = \vec{0}$ and $\vec{x}_p$ was one particular solution to $T(\vec{x}) = \vec{y}$.

The fact that the solutions looked like $\vec{x}_p + \vec{w}$ in all these different cases results from the fact that T is linear. In a later chapter we will show this. Our examples here show that this can work with X and Y being functions or vectors. Is it even more general? There are many other things that are universal to all linear fuctions. How widely can we use them? Well, clearly we need to be able to talk about linear functions, and if we look at the definition of a linear function, we clearly need X and Y to have certain properties. We start this chapter learning about these properties.

So what do we need? Defining linearity requires the ability to take linear combinations of things in X and things in Y. We can't use just any sets X and Y, we have to know that for both X and Y addition and scalar multiplication make sense. The main thing we need to know is that for $\vec{u}, \vec{v} \in X$ and $a, b \in \mathbb{R}$ that $a\vec{u} + b\vec{v} \in X$. We also want to make sure that addition and scalar multiplication really behave correctly. I could have defined vector addition such that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 - y_1 \\ x_2 + 2y_2 \end{bmatrix}$, but that would have been a stupid definition. Lots of things we expect to be true would fail: for example, $\vec{x} \oplus \vec{y} \neq \vec{y} \oplus \vec{x}$ and $\vec{x} \oplus \vec{x} \neq 2\vec{x}$.

So what we need to know is that addition and scalar multiplication make sense for X and that they have the properties we expect. We need similar properties for Y. The official definition we need is that X and Y are "vector spaces".

Before I define a vector space, let me introduce some concepts that will be needed.

Definition 2.5 *A set* X *is* ***closed under addition*** *if for any* $\vec{u}, \vec{v} \in X$ *we have* $\vec{u} + \vec{v} \in X$.

Definition 2.6 *A set* X *is* ***closed under scalar multiplication*** *if for any* $\vec{u} \in X$ *and* $a \in \mathbb{R}$ *we have* $a\vec{u} \in X$.

Definition 2.7 *A set* X *is* ***closed under linear combination*** *if for any* $\vec{u}, \vec{v} \in X$ *and* $a, b \in \mathbb{R}$ *we have* $a\vec{u} + b\vec{v} \in X$.

The method used for exercise 1.11 shows that a a set is closed under linear combination if and only if it is also closed under both addition and scalar multiplication.

Example 2.8 *The set of all polynomials of arbitrary degree is closed under both addition and scalar multiplication.*

This follows because if we add two polynomials $f(x)$ *and* $g(x)$*, we get a polynomial* $f(x) + g(x)$*. If we multiply a polynomial* $f(x)$ *by any constant* c *(including zero), we get another polynomial* $cf(x)$.

This is enough to show that the polynomials are closed under linear combination. We could alternately show this by taking $af(x) + bg(x)$ *and showing that this is a polynomial if* $a, b \in \mathbb{R}$ *and* $f(x), g(x)$ *are polynomials.*

[1] In the first example, X was the set of differentiable functions and Y the set of functions. In the second, X was the set of integrable functions and Y was $\mathbb{R}$. In the third, X was the set of twice differentiable functions and Y was the set of functions. In the final example X was $\mathbb{R}^3$ and Y was $\mathbb{R}^2$.

Example 2.9 *The set of vectors in $\mathbb{R}^2$ with all entries positive is closed under addition, but not closed under scalar multiplication. (Take any vector with all positive entries and multiply by any $c \leq 0$).*

A vector space V has two operations, vector addition '+' and scalar multiplication '$\cdot$' (though usually the dot is omitted in writing it out). Vector addition is denoted as $\vec{u} + \vec{v}$ for $\vec{u}, \vec{v} \in V$, while scalar multiplication is denoted as $a\vec{v}$ for $a \in \mathbb{R}$ and $\vec{v} \in V$. We require a number of properties to hold. Most of these properties are "obvious", but we still need to include them in our definition in order to exclude the possibility of using weird definitions of addition or scalar multiplication as '$\oplus$' above. This makes the definition a bit cumbersome.

Definition 2.10 *A set V of "vectors" is a* ***vector space*** *over $\mathbb{R}$ if the following hold:*

(a). Vector addition properties

- *(i) For any $\vec{u}, \vec{v} \in V$, we have $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutativity of vector addition).*
- *(ii) For any $\vec{u}, \vec{v}, \vec{w} \in V$, we have $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (associativity of vector addition).*
- *(iii) There is a vector $\vec{0}$ in V such that $\vec{u} + \vec{0} = \vec{u}$ for any $\vec{u} \in V$ (existence of additive identity).*
- *(iv) For each $\vec{u} \in V$, there is a vector $\vec{v} \in V$ such that $\vec{u} + \vec{v} = \vec{0}$. We call $\vec{v}$ '$-\vec{u}$' (existence of additive inverse).*

(b). Scalar multiplication properties

- *(i) For all $a \in \mathbb{R}$ and $\vec{u}, \vec{v} \in V$, $a(\vec{u}+\vec{v}) = a\vec{u}+a\vec{v}$ (distributivity of scalar multiplication with respect to vector addition).*
- *(ii) For all $a, b \in \mathbb{R}$ and $\vec{u} \in V$, $(a+b)\vec{u} = a\vec{u}+b\vec{u}$ (distributivity of scalar multiplication with respect to field addition).*
- *(iii) For all $a, b \in \mathbb{R}$ and $\vec{u} \in V$, $(ab)\vec{u} = a(b\vec{u})$ (respect of scalar multiplication over field multiplication).*
- *(iv) For all $\vec{u} \in V$, $1\vec{u} = \vec{u}$ (existence of identity of scalar multiplication).*

(c). Linear combination properties:

- *(i) For every $\vec{u}, \vec{v} \in V$ and $a, b \in \mathbb{R}$ we have $a\vec{u} + b\vec{v} \in V$ (closure under linear combination).*

Let me be clear: I will not ask you to memorize these properties. I will expect you to know that a vector space has two operations, addition and scalar multiplication. You can safely assume that addition and scalar multiplication behave the way you want. The important feature you will need to be able to check is whether there is a linear combination of things in V which is outside of V.

In the definition above we took the scalars to be $\mathbb{R}$. However, the same concepts would work if we allowed the scalars to be the complex numbers $\mathbb{C}$ instead, in which case we would call it a vector space over $\mathbb{C}$. More generally, we can use any "field", but I haven't told you what a field is. Let me give you a few examples before I give a definition.

Example 2.11 *Here are some examples of things that are and aren't fields.*

- *The set of real numbers $\mathbb{R}$ is a field.*
- *The set of complex numbers $\mathbb{C}$ form a field.*
- *The set of rational numbers $\mathbb{Q}$ form a field.*
- *The set of integers $\mathbb{Z}$ does NOT form a field.*

We now give a rough definition of a field.

Definition 2.12 *(This is only a partial definition). Basically a* ***field*** *is a set of "numbers" that are closed under addition, subtraction, multiplication and division (with the exception that division by 0 is not allowed). Both multiplication and addition are commutative.*

That is, if you take any two numbers in a field F, you can add them, subtract them, multiply them, and divide them (so long as the divisor is not zero), and the result is another number in F. Given this definition, can you see why the integers are not a field?

Unless otherwise mentioned, we will always assume that the field we are using is $\mathbb{R}$.

For each of the following examples, you should know how to check that the set is closed under linear combinations.

Example 2.13 *$\mathbb{R}^n$ is a vector space over $\mathbb{R}$.*

Example 2.14 *For any given $n \geq 2$, the set of vectors in $\mathbb{R}^n$ whose second entry is 0 forms a vector space over $\mathbb{R}$.*

Example 2.15 *The set of all functions from $\mathbb{R}$ to $\mathbb{R}$ is a vector space over $\mathbb{R}$.*

Example 2.16 *The set of continuous functions that are 0 at $x = 5$ is a vector space over $\mathbb{R}$*

Example 2.17 *The set of continous functions that are 1 at $x = 0$ is not a vector space.*

Example 2.18 *The set of differentiable functions is a vector space over $\mathbb{R}$.*

Exercise 2.19 *Let's look at some properties of vector spaces. The definition of a vector space listed a lot of seemingly obvious properties of addition and scalar multiplication. But it left other properties that are also "obvious" out. The reason for this is that we can prove these other properties from the ones listed.*

(a). *Show that there can only be one zero vector, $\vec{0}$. That is, show that if $\vec{w}$ and $\vec{x}$ both satisfy the definition of $\vec{0}$ in (a)iii of definition 2.10, then in fact they must equal each other. The starting point is to look at $\vec{w} + \vec{x}$ and use this and the definition of a zero vector to show $\vec{w} = \vec{x}$.*

(b). *Show that given $\vec{u}$, there is only one $-\vec{u}$. That is, show that if $\vec{u} + \vec{w} = \vec{0}$ and $\vec{u} + \vec{x} = \vec{0}$, then $\vec{x} = \vec{w}$. The starting point is to look at $\vec{w} + \vec{u} + \vec{x}$ and find two ways to simplify it using the information we have to show $\vec{x} = \vec{w}$.*

(c). *Show that $0\vec{u} = \vec{0}$. The starting point is to note that $0\vec{u} = (0 + 0)\vec{u}$, and then add $\vec{w} = -(0\vec{u})$ to each side.*

(d). *Show that $-1\vec{u} = -\vec{u}$. That is, show that $\vec{u} + (-1)\vec{u} = \vec{0}$. The starting point is to use $\vec{u} = 1\vec{u}$.*

Exercise 2.20 *For each of the following, you may assume all the properties of a vector space are satisfied except for closure under linear combination. Given this assumption, show that the following are vector spaces. You need only show that they are closed under linear combination.*

(a). *The set of polynomials of maximum degree at most 4 (with real coefficients).*

(b). *The set of vectors of the form $\vec{y} = a\vec{u} + b\vec{v}$ with a, b arbitrary real numbers and* ~~$\vec{x}$ and $\vec{y}$~~ $\vec{u}$ and $\vec{v}$ *two fixed vectors in $\mathbb{R}^n$.*

(c). *The set of continuous functions $V = \{f$ such that $f : \mathbb{R} \to \mathbb{R}$ and f is continuous $\}$.*

(d). *The set of real 2×2 matrices.*

(e). *The set of monomials of the form cx^4.*

(f). *The set of functions $f(x)$ defined for $x \in [-\pi, \pi]$ for which $f(-\pi) = f(\pi)$ (this is what makes Fourier series work).*

of these prob-

Exercise 2.21 *Recall that $\mathbb{Q}$ is the field of rational numbers.*

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(a). *Show that the set of all polynomials with rational coefficients is closed under linear combination if the scalars are in $\mathbb{Q}$. The other properties of a vector space also hold, so these form a vector space (over $\mathbb{Q}$).*

(b). Show that the set of numbers of the form $a + b\sqrt{2}$ where $a, b \in \mathbb{Q}$ is closed under linear combination if the scalars are in $\mathbb{Q}$. Again, these form a vector space (over $\mathbb{Q}$).

Exercise 2.22 *Show that the set of numbers between 0 and 1, that is $\{x : 0 \leq x \leq 1\}$, do not form a vector space over the reals.*

Exercise 2.23 *Consider $\mathbb{R}^2$, but replace the usual definition of addition by $\vec{u} \oplus \vec{v} = \begin{bmatrix} u_1 + v_1 \\ 2u_2 + 2v_2 \end{bmatrix}$. Show that $(\vec{u} \oplus \vec{v}) \oplus \vec{w} \neq \vec{u} \oplus (\vec{v} \oplus \vec{w})$ in general. This proves that $\mathbb{R}^2$ with this definition of addition does not constitute a vector space (it does with the usual definition).*

For one final exercise, I ask you to show all the properties hold:

Exercise 2.24 *Let n be given. Prove that $\mathbb{R}^n$ with the usual addition and scalar multiplication satisfies all the properties of a vector space over the reals.*

Finite dimensional vector spaces over the reals can be transformed into the usual vectors $\mathbb{R}^n$. We will hopefully show this later in the course, but before we do that we'll have to make clear what the definition of dimension is and a number of other details. Note that some vectors spaces such as the set of continuous functions are not finite-dimensional.

2.3 Subspaces

Consider the vectors in $\mathbb{R}^3$. These form a vector space. Focus now on just the subset of vectors whose first entry is zero, and call it Y. We can show Y forms a vector space. We know most of the properties of definition 2.10 hold automatically because Y is a subset of $\mathbb{R}^3$. All we have to check is that Y is closed under linear combination.[2] Obviously if we take two vectors $\begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ y_2 \\ y_3 \end{bmatrix}$ in Y and $a, b \in \mathbb{R}$, then $a \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} + b \begin{bmatrix} 0 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ ax_2 + by_2 \\ ax_3 + by_3 \end{bmatrix}$ is also in Y.

Since Y is a subset of $\mathbb{R}^3$ and it is itself a vector space, we refer to it as a "subspace".

A quick notational comment: given two sets W and V, the notation $W \subseteq V$ means that everything in W is in V. In other words, W is a subset of V.

Definition 2.25 *Given a vector space V, if $W \subseteq V$ and W is itself a vector space, then W is called a* ***subspace*** *of V.*

Note that technically V is a subspace of V, and $\{\vec{0}\}$ is a subspace of V.

A lot of the properties of a vector space are automatically inherited by W from V (for example, addition must commute since it commutes for V). So we don't need to check all the properties of a vector space to show that a subset W is a vector space. An equivalent definition of a subspace (but simpler to test) is

Theorem 2.26 *Given a vector space V, a nonempty subset $W \subseteq V$ is a subspace of V iff W is closed under linear combination.*

Proof. It's clear that if W is a subspace of V, then this holds. We just need to show the converse: if the nonempty subset $W \subseteq V$ is closed under linear combination then it is a subspace of V.

I'll just sketch how the proof would go. We can assume: $W \subseteq V$ so to show that W is a subspace of V, we must show that W is a vector space. We know W is closed under linear combination. We must show that every other property in the definition of a vector space is satisfied.

[2] technically we have to show also that $\vec{0} \in Y$ and for any $\vec{u} \in Y$ that $-\vec{u} \in Y$. This is addressed inour proof of theorem 2.26.

Many of these properties are inherited from V. For example, if $\vec{u}, \vec{v} \in W$, then since $W \subseteq V$, they are also in V. Thus we know $u + v = v + u$ because V is a vector space. So addition in W commutes. An almost identical proof applies for the other properties except for two: We must still show $\vec{0} \in W$ and for any $\vec{u} \in W$ we have $-\vec{u} \in W$.

To show $\vec{0} \in W$, note that W is nonempty, so there is some vector $\vec{u} \in W$. Taking the linear combination $0\vec{u} + 0\vec{u}$ we get $\vec{0}$.

To show $-\vec{u} \in W$, we note that $(-1)\vec{u}+0\vec{u} = -\vec{u}$. So if $\vec{u} \in W$, then $-\vec{u}$ can be found as a linear combination, so $-\vec{u} \in W$ as well. □

In particular, this theorem is useful because it means that in many cases we can show that something is a vector space by showing that it is a subspace of some larger vector space. This saves us the effort of showing that all the addition and scalar multiplication properties hold. In this course we will almost always be working within a larger vector space, so this is why all I insist that you learn is that a vector space is closed under linear combination.

Note: when asked to show that something is not a vector space, the first place to look is generally to see whether $\vec{0}$ is in the set. If not, then it's not a vector space. This won't always work, but it's a good starting point.

Example 2.27 *The set of vectors in $\mathbb{R}^2$ for which the second entry is twice the first is a subspace of $\mathbb{R}^2$.*
To prove this, we note first that it is clearly a subset of $\mathbb{R}^2$. If we take the two vectors $\vec{u} = \begin{bmatrix} u_1 \\ 2u_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ 2v_1 \end{bmatrix}$ and the two scalars $a, b \in \mathbb{R}$, then we get $a\vec{u} + b\vec{v} = \begin{bmatrix} au_1 + bv_1 \\ 2(au_1 + bv_1) \end{bmatrix}$. The second entry of this is twice the first, so this is closed under linear combination. Thus since it is contained within $\mathbb{R}^2$ and is closed under linear combination it is a subspace of $\mathbb{R}^2$.

Example 2.28 *If $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m \in \mathbb{R}^n$, then the set of all linear combinations of the form $a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_m\vec{u}_m$ where $a_i \in \mathbb{R}$ form a subspace of $\mathbb{R}^n$.*
To see this, take two linear combinations of these vectors: $\vec{v} = \sum_{i=1}^m \alpha_i \vec{u}_i$ and $\vec{w} = \sum_{i=1}^m \beta_i \vec{u}_i$ where for each i, $\alpha_i, \beta_i \in \mathbb{R}$. Then take any $b, c \in \mathbb{R}$. The sum $b\vec{v} + c\vec{w} = \sum_{i=1}^m (b\alpha_i + c\beta_i)\vec{u}_i$ which is of the form $a_1\vec{u}_1 + \cdots + a_m\vec{u}_m$.

Very soon we'll define this more specifically to be the "span" of the vectors $\vec{u}_1, \ldots, \vec{u}_m$.

2.3.1 Special Subspaces

If we have an $m \times n$ matrix A, there are two important subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ which are determined from A. These are the Column Space Col(A) and the Null Space Nul(A) of A. They will be used widely when we study systems of linear equations. For now they are simply useful examples of subspaces.

Column Space

Exercise 2.29 *Let* $\mathrm{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. *Let B denote the set of all vectors $\vec{b}$ such that there is a solution $\vec{x}$ to $\mathrm{A}\vec{x} = \vec{b}$.*
Prove B is a subspace of $\mathbb{R}^3$. [Hint: there are (at least) two ways to approach this. One is to look at $\mathrm{A}\vec{x}$ and note its relation to linear combinations of the columns of A. The other is to take two vectors which can be written as $\vec{b}_1 = \mathrm{A}\vec{x}_1$ and $\vec{b}_2 = \mathrm{A}\vec{x}_2$ and consider linear combinations of them. It is easier to leave A as A throughout, rather than using its specific form.]

Theorem 2.30 *Given any $m \times n$ matrix A, let $B \subseteq \mathbb{R}^m$ denote the set of vectors $\vec{b}$ for which a vector $\vec{x}$ exists such that $\mathrm{A}\vec{x} = \vec{b}$. Then B is a vector space.*

The proof of this theorem probably follows immediately from your solution to 2.29.

Definition 2.31 *Given any $m \times n$ matrix A, the subspace $B \subseteq \mathbb{R}^m$ of vectors $\vec{b}$ for which $\mathrm{A}\vec{x} = \vec{b}$ has a solution is called the* ***Column Space*** *of A, and is denoted $\mathrm{Col}(\mathrm{A})$.*

The Column Space always has at least $\vec{0}$.

It is called the column space of A because it is the set of all linear combinations of the columns of A.

Null Space

Exercise 2.32 *Let* $\mathrm{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. *Let W denote the set of all vectors $\vec{w}$ such that $\mathrm{A}\vec{w} = \vec{0}$.*

Prove W is a subspace of $\mathbb{R}^4$. [Hint: use theorem 2.26. It is simpler if you don't use what A looks like, just leave it as A.]

Theorem 2.33 *Given any $m \times n$ matrix A, let $W \subseteq \mathbb{R}^n$ denote the set of vectors $\vec{w}$ for which $\mathrm{A}\vec{w} = \vec{0}$. Then W is a vector space.*

The proof of this theorem follows from the method used in exercise 2.32. This subspace is called the null space of matrix A.

Definition 2.34 *Given any $m \times n$ matrix A, the subspace $W \subseteq \mathbb{R}^n$ of vectors $\vec{w}$ for which $\mathrm{A}\vec{w} = \vec{0}$ is called the* ***Null Space*** *of A and is denoted $\mathrm{Nul}(\mathrm{A})$.*

Definition 2.35 *The system $\mathrm{A}\vec{w} = \vec{0}$ is often called the* ***homogeneous system*** *for the matrix A.*

The Null Space is thus the solution to the homogeneous system. It is worth noting the relation of the Null space and the examples that opened this chapter.

The Null Space always has at least $\vec{0}$. For some matrices that is the only solution to $\mathrm{A}\vec{w} = \vec{0}$. For other matrices there are infinitely many solutions.

Definition 2.36 *The solution $\vec{x} = \vec{0}$ of the system $\mathrm{A}\vec{x} = \vec{0}$ is referred to as the* ***trivial solution****. Any other solution is a* ***nontrivial solution****.*

Note that $\mathrm{Col}(\mathrm{A}) \subseteq \mathbb{R}^m$, but $\mathrm{Nul}(\mathrm{A}) \subseteq \mathbb{R}^n$.

Exercise 2.37 *More generally, for a linear transformation $T : X \to Y$, the nullspace of T is the set of all $\vec{x} \in X$ for which $T(\vec{x}) = \vec{0}$. Find the nullspace of the following. That is, find all functions of s such that $T(x(s)) = 0$.*

(a). ~~*T if*~~ $T(x(s)) = \frac{\mathrm{d}x(s)}{\mathrm{d}s}$

(b). ~~*T if*~~ $T(x(s)) = \frac{\mathrm{d}x(s)}{\mathrm{d}s} - 5x(s)$

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2.4 Span

We've seen already (example 2.28) that if we have a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ each in X, then the set of all linear combinations of the form $\sum a_i\vec{v}_i$ forms a subspace of X. This subspace is in fact the smallest subspace of X which contains $\vec{v}_1, \ldots, \vec{v}_m$. This frequently comes up in applications, so we will study it in more detail.

Definition 2.38 *Given a set of nonzero vectors* $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ *, their* ***span****, denoted* $\mathrm{Span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\})$ *is the set of all linear combinations of the form* $\sum_{i=1}^{n} a_i \vec{v}_i$.

Speaking heuristically, the span of a set of vectors is all places that can be reached by moving some distance from the origin in the direction of each vector in the set.

Like many English words, we can use "span" as a noun (as above), or as a verb.

Definition 2.39 *If* V *is the span of a set of vectors, we say those vectors* ***span*** V.

Example 2.40

(a). If $S = \{\begin{bmatrix}0\\1\end{bmatrix}\}$ *is a set consisting just of the single vector* $\begin{bmatrix}0\\1\end{bmatrix}$*, then* $\mathrm{Span}(S)$ *is the set of vectors in* $\mathbb{R}^2$ *whose first entry is zero.*

(b). If $S = \{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\}$ *then any vector in* $\mathbb{R}^2$*, say* $\begin{bmatrix}c\\d\end{bmatrix}$ *can be written as* $c\begin{bmatrix}1\\0\end{bmatrix} + d\begin{bmatrix}0\\1\end{bmatrix}$*. So every vector in* $\mathbb{R}^2$ *is in the span. So* $\mathbb{R}^2 = \mathrm{Span}(S)$*.*

(c). If $S = \{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\3\end{bmatrix}\}$ *then* $\mathrm{Span}(S)$ *is the set of vectors in* $\mathbb{R}^2$ *whose first entry is zero. In this case including the second vector did not increase the span.*

Theorem 2.41 *If* $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ *are vectors in some vector space* V*, then the span of* $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ *is a subspace of* V*.*

This is just a restatement of example 2.28.

Theorem 2.42 *Every vector space that contains* $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ *contains all of* $\mathrm{Span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\})$*.*

Exercise 2.43 *Prove theorem 2.42. That is, show that if* $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in V$ *for some vector space* V*, show that any linear combination of the vectors* $\vec{v}_i$ *is in* V*. This isn't as hard as it might seem, so if you're putting a lot of work into it, you're overthinking things.*

uses some trou-

Let's return to the Column Space briefly. The column space of A, Col(A) is the set of vectors $\vec{b}$ that can be written as $\mathrm{A}\vec{x}$. Since $\mathrm{A}\vec{x} = \sum x_i \vec{a}_i$ where $\vec{a}_i$ is the i-th column of A. This means that the column space is the set of vectors that can be written as linear combinations of the columns of A. Or more concisely,

Theorem 2.44 $\mathrm{Col}(\mathrm{A})$ *is the span of the columns of* A*.*

Moveover, if we have a set of vectors $\vec{v}_1, \ldots, \vec{v}_m$ each in $\mathbb{R}^n$, then $\mathrm{Span}(\{\vec{v}_1, \ldots, \vec{v}_m\})$ is the column space of the matrix whose columns are the $\vec{v}_i$: $\mathrm{A} = [\vec{v}_1 \; \vec{v}_2 \; \cdots \; \vec{v}_m]$.

Sometimes increasing the size of a set of vectors doesn't increase the span.

Example 2.45 *The span of* $V_1 = \{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\}$ *is the same as the span of* $V_2 = \{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\}$*.*

To show this, we need to show that anything in $\mathrm{Span}(V_1)$ *is in* $\mathrm{Span}(V_2)$ *and vice versa. First note that anything in* $\mathrm{Span}(V_2)$ *is in* $\mathrm{Span}(V_1)$ *since anything that can be written as* $a\begin{bmatrix}1\\0\\2\end{bmatrix} + b\begin{bmatrix}1\\0\\1\end{bmatrix}$ *is equal to* $a\begin{bmatrix}1\\0\\2\end{bmatrix} + b\begin{bmatrix}1\\0\\1\end{bmatrix} + 0\begin{bmatrix}2\\0\\1\end{bmatrix}$.

Now we show anything in $\mathrm{Span}(V_1)$ *is in* $\mathrm{Span}(V_2)$*, note that* $\begin{bmatrix}2\\0\\1\end{bmatrix} = 3\begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}1\\0\\2\end{bmatrix}$*. So if we take any* *linear combination* $a_1 \begin{bmatrix}1\\0\\2\end{bmatrix} + a_2 \begin{bmatrix}1\\0\\1\end{bmatrix} + a_3 \begin{bmatrix}2\\0\\1\end{bmatrix}$, *we can rewrite this as*

$$\begin{aligned} a_1 \begin{bmatrix}1\\0\\2\end{bmatrix} + a_2 \begin{bmatrix}1\\0\\1\end{bmatrix} + a_3 \begin{bmatrix}2\\0\\1\end{bmatrix} &= a_1 \begin{bmatrix}1\\0\\2\end{bmatrix} + a_2 \begin{bmatrix}1\\0\\1\end{bmatrix} + a_3 \left(3 \begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}1\\0\\2\end{bmatrix} \right) \\ &= (a_1 - a_3) \begin{bmatrix}1\\0\\2\end{bmatrix} + (a_2 + 3a_3) \begin{bmatrix}1\\0\\1\end{bmatrix} \end{aligned}$$

and so anything that can be written as a linear combination of the three vectors can be written as a linear combination of just the first two.

Exercise 2.46 *Let* $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^4$. *Let* $\vec{v}_4$ *be a linear combination of these three:* $\vec{v}_4 = a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3$ *for some coefficients* $a_1, a_2, a_3 \in \mathbb{R}$.

(a). Could there be any vectors in $\operatorname{Span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\})$ *which are not in* $\operatorname{Span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$*? Explain.*

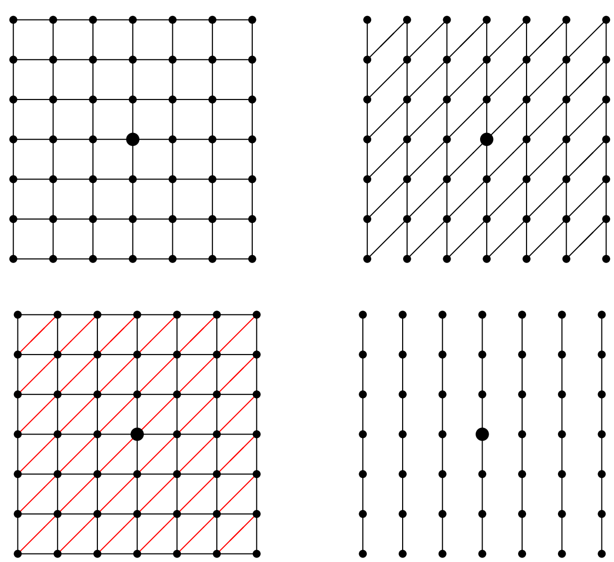
(b). Could there be any vectors in $\operatorname{Span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$ *which are not in* $\operatorname{Span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\})$*? Explain.*

If we have a (finite-dimensional) vector space X, we are often interested in finding a collection of vectors $\{\vec{v}_1, \ldots, \vec{v}_m\}$ which span X. Generally it is most useful if every vector in our collection is really essential. That is, if we remove any vector from our set what is left will no longer span X.

When we have a vector space X and a collection of vectors $\vec{v}_1, \ldots, \vec{v}_m$ which span X, we can represent any vector in X by its "coordinates" in terms of the vectors $\vec{v}_i$. Unfortunately, if we aren't careful we can find ourselves in a situation where one vector can be written in more than one way. Coordinate systems that allow you to denote the same place in more than one way tend to not be useful.

Consider a city whose streets run North-South and East-West on a grid. If we define city hall to be at $\begin{bmatrix}0\\0\end{bmatrix}$, then we can define each location in the city based on how many blocks N/S and how many blocks E/W you must travel from city hall to get to the location.

What if the one set of streets is still North-South, but the cross streets run NorthEast-SouthWest? Would you still give coordinates in terms of N/S and E/W? Or would you choose them in terms of the directions the streets run? You would probably use the directions of the streets.



What would happen if we had all three sets of roads? There would be more than one way to denote the same location. If you go one block North, then one block East, then one block SouthWest, you're back where you started. There are lots of short cuts in this case. For our final case, assume if we only had the North-South streets. You wouldn't be able to get from city hall to every location.

Note that in the first two cases, if we know the coordinates of two places, we can easily determine how to get from one to the other. For example: In the first case if we're at $(1N, 3E)$ and we want to get to $(2N, 1E)$, we just go one block north and two blocks west. In the second case if we're at $(2N, 3SW)$ and we want to get to $(1N, 4SW)$, we go one block south and one block southwest.

The third case is not so simple. We can still get from place to place, but if we are at $(1N, 2W, 1NE)$ and we want to get to $(0N, 3W, 2NE)$, it seems we should go one block south, one west, and then one northeast. This puts us back at our starting point. So this coordinate system leaves something to be desired.

In the final case we don't have ambiguity, but instead we are restricted to just moving north/south. We can never get onto a different street.

You should be able to see a similarity with vector addition here. We can write the same coordinates in terms of different sets of vectors. The locations we can reach along the streets are basically the span of the vectors. The coefficients of the linear combination tell us how far we go along each vector.

The first two cases we saw here, it's possible to get from city hall to every other location following the streets. Further, the streets map out a useful coordinate system. In the third case, it's still possible to get anywhere, but the streets don't give us a good coordinate system. In each of these cases the streets "span" the city, but in the third case we wouldn't want to use all three street directions in a coordinate system. In the final case the streets do not span the city — we cannot get to all locations from city hall.

To get a useful coordinate system, we need a set of roads that span the city, but we need the representation of each location in terms of those roads to be unique. Following our analogy with vector addition, we need a set of vectors that span the space, but without redundancy. To do this we look for a "basis".

2.5 Bases (sections 2.8-2.9 and 4.3-4.4 of text)

When we have the span of a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$, sometimes a few of the vectors $\vec{v}_i$ aren't really needed to "generate" a given space.

When we have a vector space, it's very useful to be able to give "coordinates" for every thing in that space. So we choose a set of vectors that form our coordinate directions. The set needs to span the vector space: otherwise there's no way to reach every point. On the other hand, we want our coordinates to be simple: there should be only one way to represent a point.

Definition 2.47 *A **basis** for a space V is a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ that span V, such that if we remove any of them, they no longer span V.*

The plural of "basis" is "bases", prounounced "base - ease"

Example 2.48 *The canonical basis of $\mathbb{R}^n$:* $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ *forms a basis for $\mathbb{R}^n$.*

Any vector $\vec{x} \in \mathbb{R}^n$ can be written as $\sum x_i \vec{e}_i$, so this certainly spans $\mathbb{R}^n$. If we remove any $\vec{e}_i$, then any vector whose i-th entry is nonzero cannot be writen as a linearc combinations of the remaining. So the full set is a basis.

Definition 2.49 *A set of vectors* $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ *is* ***linearly independent*** *if the only way to write* $\vec{0}$ *as a linear combination* $\vec{0} = a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_n\vec{x}_n$ *is with* $a_1 = a_2 = \cdots = a_n = 0$*.*

Definition 2.50 *A set of vectors that is not linearly independent is* ***linearly dependent****. That is, if* $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ *then* S *is linearly dependent if and only if there is a inear combination* $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$ *which equals* $\vec{0}$ *and at least one* a_i *is nonzero.*

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Exercise 2.51 *Are the following linearly independent? Explain your answer.*

(a). $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}.$

(b). $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}$

(c). $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\3\end{bmatrix}, \begin{bmatrix}3\\6\\10\end{bmatrix}$

Exercise 2.52 *Show that any set of vectors in* $\mathbb{R}^n$ *which includes* $\vec{0}$ *is linearly dependent.*

Exercise 2.53 *Consider a set of vectors* $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$*. Assume that* S *is linearly dependent, so* $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \vec{0}$ *for some coefficients* $a_1, \ldots, a_n$ *which are not all zero.*
Since not all are zero, we can choose one ~~assume that there is some~~ i *such that* $a_i \neq 0$*. Show that the corresponding vector* $\vec{v}_i$ *can be written as a linear combination of the other vectors.*

Theorem 2.54 *The following statements about a set of vectors* B *and a vector space* V *are all equivalent (i.e., if any are true, all are true and if any are false, all are false)*

(a). B *is a basis for the vector space* V*.*

(b). $V = \mathrm{Span}(B)$ *and* B *is linearly independent.*

(c). Every vector in V *can be written as a linear combination of vectors in* B *in exactly one way and* $B \subset V$*.*

The same vector space will generally have infinitely many different possible bases to use. In most cases we want to use the canonical basis. However, if we are studying a physical problem, sometimes certain "directions" are more relevant to the underlying physical process than the coordinate system that we find convenient. In a sense, by using the canonical basis we are forcing our coordinates onto a physical system that doesn't like them.

Example 2.55 *Let* $\phi = (1+\sqrt{5})/2$*. The vectors* $\vec{v}_1 = \begin{bmatrix}1\\\phi\end{bmatrix}$ *and* $\vec{v}_2 = \begin{bmatrix}1\\1-\phi\end{bmatrix}$ *form a basis for* $\mathbb{R}^2$ *(there is nothing special about this value of* ϕ *that makes this true: for any* $\phi \neq 1/2$ *these vectors would form a basis)*
We have a perfectly good basis already, the canonical basis. This basis appears harder to work with. So why on earth would we ever prefer to use $\vec{v}_1$ *and* $\vec{v}_2$ *when we have the canonical basis* $\vec{e}_1$ *and* $\vec{e}_2$ *available? The answer comes from example 1.75. In this example, we had the matrix* $\mathrm{A} = \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}$ *which let us say how many individuals would be infected on a given day. If we start off with an initial vector* $\vec{u}(0) = \begin{bmatrix}0\\1\end{bmatrix}$ *and want to know how many would be infected on day* d*, we need to calculate* $\vec{u}(d) = \mathrm{A}^d\vec{u}(0)$*.*
Would you be upset if your homework assignment asks you to calculate $\vec{u}(1000)$*? I'm going to ask you to do it.*
This is where the special value of ϕ *will become important. It turns out that* $\mathrm{A}\vec{v}_1 = \phi\vec{v}_1$ *and* $\mathrm{A}\vec{v}_2 = (1-\phi)\vec{v}_2$*. Can we use this to our advantage? Well, since* $\vec{v}_1$ *and* $\vec{v}_2$ *form a basis, we know that we can write* $\vec{u}(0) = c_1\vec{v}_1 + c_2\vec{v}_2$*.*

Putting together what we have now gives

$$\begin{aligned}\vec{u}(1000) &= \mathrm{A}^{1000}\vec{u}(0)\\ &= \mathrm{A}^{1000}(c_1\vec{v}_1 + c_2\vec{v}_2)\\ &= c_1\mathrm{A}^{1000}\vec{v}_1 + c_2\mathrm{A}^{1000}\vec{v}_2\end{aligned}$$

But since $\mathrm{A}\vec{v}_1 = \phi\vec{v}_1$*, it is relatively straightforward to show that* $\mathrm{A}^{1000}\vec{v}_1 = \phi^{1000}\vec{v}_1$*. That is, if we are looking at the vector* $\vec{v}_1$*, multiplication by the matrix* A *is the same as multiplication by the scalar* ϕ*. Similarly* $\mathrm{A}^{1000}\vec{v}_2 = (1-\phi)^{1000}\vec{v}_2$*. This means*

$$\vec{u}(1000) = c_1\phi^{1000}\vec{v}_1 + c_2(1-\phi)^{1000}\vec{v}_2$$

So we've reduced doing a thousand multiplications by A *to simply raising two real numbers* ϕ *and* $(1-\phi)$ *to the power* 1000 *and then doing a linear combination.*

This demonstrates the value of using the basis that the physical problem likes. For that basis, multiplication by A becomes an easy process.

The number $\phi = (1+\sqrt{5})/2$ is known as the golden ratio. Its properties come from the fact that $\phi^2 = 1+\phi$.

Later in this course, we will learn how to find vectors like $\vec{v}_1$ and $\vec{v}_2$ as well as different ways to find c_1 and c_2.

Exercise 2.56 *Let* A *be a given* $n \times n$ *matrix. Assume that we have a vector* $\vec{v}$ *and a scalar* λ *for which* $\mathrm{A}\vec{v} = \lambda\vec{v}$*.*

(a). Prove that $\mathrm{A}\lambda^n\vec{v} = \lambda^{n+1}\vec{v}$*.*

(b). Explain why this means that $\mathrm{A}^{n+1}\vec{v} = \lambda^{n+1}\vec{v}$*.*

Exercise 2.57 *Let* $\mathrm{A} = \begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$ *and* $\phi = (1+\sqrt{5})/2$*.*

(a). Show that the solutions of $x^2 - x - 1 = 0$ *are* ϕ *and* $1-\phi$*. Explain why this shows* $\phi^2 = 1+\phi$ *and* $(1-\phi)^2 = 1+1-\phi$*.*

(b). Show that $\mathrm{A}\begin{bmatrix}1\\ \phi\end{bmatrix} = \phi\begin{bmatrix}1\\ \phi\end{bmatrix}$ *and* $\mathrm{A}\begin{bmatrix}1\\ 1-\phi\end{bmatrix} = (1-\phi)\begin{bmatrix}1\\ 1-\phi\end{bmatrix}$*.*

(c). Show that if $c_1 = \phi/(2+\phi)$ *and* $c_2 = -\phi/(2+\phi)$*, then* $c_1\begin{bmatrix}1\\ \phi\end{bmatrix} + c_2\begin{bmatrix}1\\ 1-\phi\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix}$*. [hint: for the second entry in the vectors, expand out the numerators in terms of* ϕ *and use* $\phi^2 = 1+\phi$ *to eliminate any quadratic terms]*

(d). Calculate $\mathrm{A}^{1000}\begin{bmatrix}0\\ 1\end{bmatrix}$ *in terms of* ϕ*.* ~~*Then using a calculator (or by hand) simplify this to a vector of integers.*~~

Exercise 2.58 *Consider the matrix* $\mathrm{A} = \begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix}$*. We define* $\lambda_1 = 1+\sqrt{2}$ *and* $\lambda_2 = 1-\sqrt{2}$*. We set* $\vec{v}_1 = \begin{bmatrix}1\\ \lambda_1\end{bmatrix}$ *and* $\vec{v}_2 = \begin{bmatrix}1\\ \lambda_2\end{bmatrix}$*. The vectors* $\vec{v}_1$ *and* $\vec{v}_2$ *form a basis for* $\mathbb{R}^2$*.*

(a). Show that $\mathrm{A}\vec{v}_1 = \lambda_1\vec{v}_1$ *and* $\mathrm{A}\vec{v}_2 = \lambda_2\vec{v}_2$*.*

(b). Find c so that $\begin{bmatrix}0\\1\end{bmatrix} = c\vec{v}_1 - c\vec{v}_2$*. Because these vectors form a basis, this is the only way to write* $\begin{bmatrix}0\\1\end{bmatrix}$ *as a linear combination of $\vec{v}_1$ and $\vec{v}_2$*

(c). Calculate $\mathrm{A}^{50}\begin{bmatrix}0\\1\end{bmatrix}$*. Leave your answer in terms of λ_1 and λ_2.*

Dimension (section 2.9 and 4.5 of text)

We have a good intuitive idea of what dimension means. It's essentially the number of independent directions available. Thinking of this in terms of a basis, we would expect it's the number of vectors in the basis. That's correct. We just need to tidy up one issue: if one basis has 3 elements, must any other basis for that vector space also have 3 elements?

Theorem 2.59 *If the set of d vectors $B = \{\vec{v}_1, \ldots, \vec{v}_d\}$ is a basis for the vector space V, then any set of more than d vectors in V is linearly dependent.*

We can't prove this theorem yet. The proof requires material from next chapter.

This theorem has a number of important implications. First:

Theorem 2.60 *For a vector space V, all bases have the same number of vectors.*

Proof. Assume some basis, B_1, for V has more vectors than another, B_2. Since B_2 is a basis for V and B_1 has more vectors than B_2, the vectors in B_1 must be linearly dependent by the previous theorem. This contradicts the assumption that it is a basis. □

This allows us to define (finally!) the dimension of a vector space

Definition 2.61 *If the vector space V has a basis with a finite number of vectors, the **dimension** of V is the number of vectors in the basis.*

The theorem we just gave shows that the dimension of V does not depend on what basis we choose.

We also have

Theorem 2.62 *If V is an d-dimensional vector space, and B is any set of d linearly independent vectors in V, then B is a basis for V. and If V is a d-dimensional vector space, and B is a set of d vectors with $V = \mathrm{Span}(B)$, then B is a basis for V.*

Example 2.63

(a). Is $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}$ *a basis for $\mathbb{R}^2$?*

Yes: it is linearly independent and consists of 2 vectors in $\mathbb{R}^2$.

(b). Is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ *a basis for $\mathbb{R}^2$?*

No. The vectors are not in $\mathbb{R}^2$. The span of these vectors will be a two-dimensional space which has the same shape as $\mathbb{R}^2$, but it is not $\mathbb{R}^2$.

Exercise 2.64 *The set of all polynomials that have highest degree at most 2 form a vector space. Call this set P_2.*

(a). Show that x^2, x, and 1 are linearly independent. [hint - if you start writing down a vector, you're taking the wrong path. If the polynomial $c_1x^2 + c_2x + c_3$ is the polynomial 0, what do you know about c_1, c_2, and c_3?]

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and c_3?

(b). *Show that everything in P_2 can be written as a combination of x^2, x, and 1.*

(c). *Why does this prove that x^2, x, and 1 form a basis for P_2?*

(d). *What is the dimension of P_2?*

(e). *If I give you 4 polynomials in P_2 is it possible that they form a basis for P_2?*

Change of coordinates

Given a d-dimensional vector space V with basis $B = \{\vec{v}_1, \ldots, \vec{v}_d\}$ any vector $\vec{w} \in V$ can be written as

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_d\vec{v}_d$$

Addition of such vectors and scalar multiplication of such vectors is really just adding the coefficients and multiplying the coefficients respectively. So if all we are doing is linear operations, we can just as easily represent the vector $\vec{w}$ by the vector $\begin{bmatrix} c_1 \\ \vdots \\ c_d \end{bmatrix}$. If we "translate" our vector in V into this vector in $\mathbb{R}^d$, we can do all of our calculations in $\mathbb{R}^d$. Then we can translate it back to V.

Definition 2.65 *For any basis $B = \{\vec{v}_1, \ldots, \vec{v}_d\}$ of V, the mapping $T : V \to \mathbb{R}^d$ defined by $T(\sum c_i\vec{v}_i) = \begin{bmatrix} c_1 \\ \vdots \\ c_d \end{bmatrix}$ is referred to as a* ***change of coordinates***

The change of coordinates is linear. We can revert back to the original coordinates by the function $T^{-1}(\begin{bmatrix} c_1 \\ \vdots \\ c_d \end{bmatrix}) = \sum c_i\vec{v}_i$.

Why would we ever want to do this? Well, it's because sometimes the operations we are doing are much simpler in the new coordinates. For example, the matrix A from example 2.55 is a little unwieldy to use in the original coordinate system. But in the new coordinate system the translated version of A becomes $\begin{bmatrix} \phi & 0 \\ 0 & 1-\phi \end{bmatrix}$. Multiplication by this matrix is easy. Later, we will study eigenvectors and eigenvalues and learn how to choose the coordinate system that makes a given matrix behave in this easy manner.

Exercise 2.66 *Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Let $\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} 1 \\ 1.00001 \end{bmatrix}$.*

(a). *Show that $\vec{v}_1$ and $\vec{v}_2$ form a basis for $\mathbb{R}^2$, and that $\vec{w}_1$ and $\vec{w}_2$ form a basis for $\mathbb{R}^2$. [hint, this is easy]*

(b). *Write the vector $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, and as a linear combination of $\vec{w}_1$ and $\vec{w}_2$.*

(c). *Assume you will have to store things in a computer in terms of the coefficients of one of these bases, and your computer is only accurate to three significant digits. Which basis would you prefer to use when storing the coefficients of a "random" vector?*

Exercise 2.67 *Let V be a vector space with basis $\vec{v}_1, \ldots, \vec{v}_n$.*
Assume that the vectors $\vec{w}_1, \ldots, \vec{w}_m$ are all in V.

(a). *Prove that if $m > n$, then the $\vec{w}_i$ vectors are not a basis.*

(b). *If $m = n$ and we can write each $\vec{v}_i$ as a linear . combination $\vec{v}_i = \sum a_{ij}\vec{w}_j$, prove that the vectors $\vec{w}_1, \ldots, \vec{w}_m$ form a basis.*

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(c). Given that x^3, x^2, x, and 1 form a basis for P_3, the set of polynomials up to degree 3, prove that the first four Chebyshev Polynomials: 1, x, $2x^2 - 1$ and $4x^3 - 3x$ form a basis for P_3.

(d). In general, for large exponents, odd powers of x look very similar to one another in the interval $[-1, 1]$ — they are close to 0 for a while and then near ± 1 they suddenly grow to ± 1. Similarly even powers also look similar to one another. The different Chebyshev polynomials do not look similar to one another. If you are trying to approximate a "random" function in the interval $[-1, 1]$ by storing it as a vector of coefficients in a computer with roundoff error, which basis would be better? [hint – see exercise 2.66.]

2.6 Showing something is a vector space

Move this into

We are frequently asked to show that something is a vector space. I've told you that in almost all cases I will only ask you to show that the thing is closed under linear combination.

So here is a recipe with the steps we use to show that X is a vector space.

(a). The set X is defined by some property or properties. Let $\vec{u}$ and $\vec{v}$ be two arbitrary things with these properties.

(b). Let a and b be two arbitrary real numbers.

(c). Show that $a\vec{u} + b\vec{v}$ is in X. That is, show that they have the properties that go into the definition of X. Often this is broken into two substeps

  (i) Show that any scalar multiple of something in X is in X.

  (ii) Show that the sum of any two things in X is in X

Example 2.68 *Show that the set of polynomials that are equal to 0 at $x = 0$ is a vector space.*
Here, the set X is the set of polynomials that are 0 at $x = 0$.

(a). Consider two polynomials $u(x)$ and $v(x)$ which are 0 at $x = 0$.

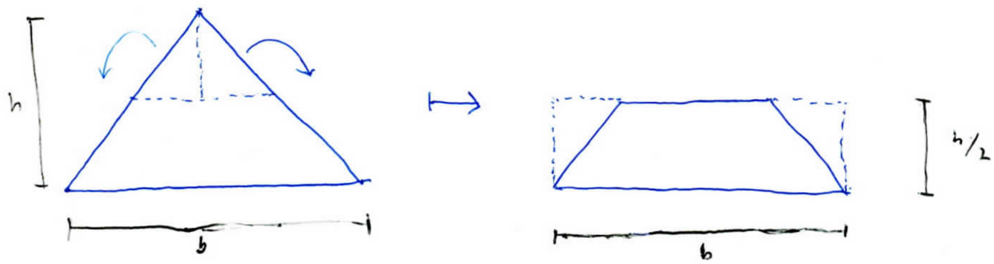
(b). Let a and b be arbitrary real numbers

(c). The sum $au(x) + bv(x)$ is a polynomial (based on things we learned back in Algebra). So we have one of the properties. We need to show that $au(0) + bv(0) = 0$, but $u(0) = v(0) = 0$ because u and v are in X. Thus $au(0) + bv(0) = a0 + b0 = 0$. So we are done.

As an alternate to this, we could also simply note that the polynomials which are 0 at $x = 0$ are the polynomials whose constant term is 0. Following the same steps, when we get to the last step we note that if the constant terms of $u(x)$ and $v(x)$ are 0, then $au(x)$ and $bv(x)$ each have constant term 0, and their sum does as well.

The following exercise is Hilbert's third problem. This was one of a list of twenty-three problems that the prominent mathematician David Hilbert proposed in 1900 as a set of important questions for mathematicians to resolve in the new century. The questions were selected because their answers would give important insights into mathematics. Several of the problems remain unsolved.

Exercise 2.69 *We can derive the formula for the area of a triangle by taking the triangle, cutting it appropriately by straight lines, and reassembling it into a rectangle:*



Because we can do this, there is no need for Calculus or any concept of a limit for us to derive the formula $A = bh/2$.

The formula for the volume of a tetrahedron (or indeed any pyramid) is as simple as that of the triangle: it's $V = Bh/3$ where B is the area of the base and h is the height. However, all known derivations of this formula require some sort of argument relying on limits. If we could find a way to cut a tetrahederon up into a finite number of pieces with straight planes and reassemble it into a rectangular prism, then we'd have a way to derive its volume without relying on limits. Is this possible? In fact, the answer is no.

Before doing the proof, here is some intuition about what is happening. If you were asked to prove that a tetrahedron of one volume cannot be cut up by planar cuts into a finite set of pieces and then reassembled into a rectangular prism of a different volume, it would be straightforward: each cut doesn't change the total volume.[3]

We can't use volume to prove this result because the rectangular prism will have the same volume as the tetrahedron, but we'll find another function to play the same role. We'll define a function, show that it is preserved by the cuts, and then show that it is different for the tetrahedron and the rectangular prism.

Show that there is no way to use planes to cut a regular tetrahedron into a finite number of pieces which can be reassembled into a rectangular prism using the following steps. The one fact we will take as given is that if α is the solid angle of the edge of a regular tetrahedron, then α/π is an irrational number (in fact $\alpha = \arccos(1/3)$).

(a). *Assume that a tetrahedron can be cut into a finite number of pieces that can then be reassembled into into a rectangular prism. Let $\theta_1, \theta_2, \ldots, \theta_n$ be all the angles that occur between faces in all the pieces. These are all real numbers, but we will think of them as vectors. Let V be the span of these vectors over $\mathbb{Q}$, that is, let V be the set of all linear combinations $\sum q_i\theta_i$ where each q_i is a rational number. Given that they can be assembled into a tetrahedron or a rectangular prism, explain why π and α are both in V.*

(b). *We want to find a basis for V over $\mathbb{Q}$. How do we know it is finite dimensional?*

(c). *We can build a basis for V by choosing vectors one after the other in V, and if the chosen vector is not in the span of the previous vectors, we add it to the basis. Explain why this means we can always choose a basis for V whose first two vectors are π and α. Why does this depend on the fact that $\alpha/\pi \notin \mathbb{Q}$?*

(d). *Let $\phi_1,\ \phi_2, \ldots, \phi_k$ be a basis for V over $\mathbb{Q}$ with $\phi_1 = \pi,\ \phi_2 = \alpha$. Show that the function $f : V \to \mathbb{R}$ by $f(\sum c_i\phi_i) = c_2$ is a linear function.*

(e). *Define the function g whose range is the set of polyhedra and whose domain is $\mathbb{R}$ as follows: for a polyhedron P, $g(P)$ is found by first looking at each edge, and calculating its length l_i and its angle*

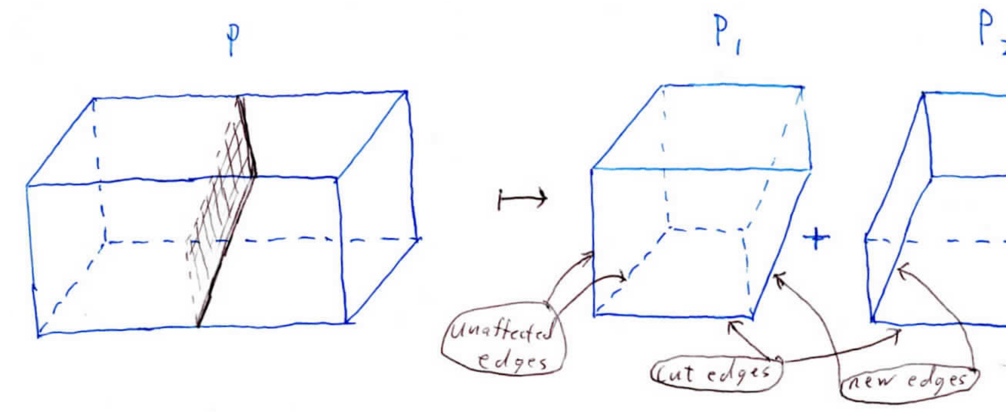
[3] The Banach–Tarski paradox shows that there is a way to take a sphere, cut it into finitely many pieces and reassemble them into two spheres of the same volume. The pieces are very strange-looking objects and nothing at all like what you'd get if you use planar cuts. This proves that there are sets of points for which it doesn't make sense to define "volume".

γ_i. Then

$$g(P) = \sum_{\text{edge } i} l_i f(\gamma_i)$$

Explain why g is zero for any rectangular prism, but nonzero for a regular tetrahedron.

(f). Show that if we use a plane to cut a polyhedron P into two smaller polyhedra, P_1 and P_2, then $g(P) = g(P_1) + g(P_2)$. To do this, you need to consider four types of edges in P_1 and P_2:



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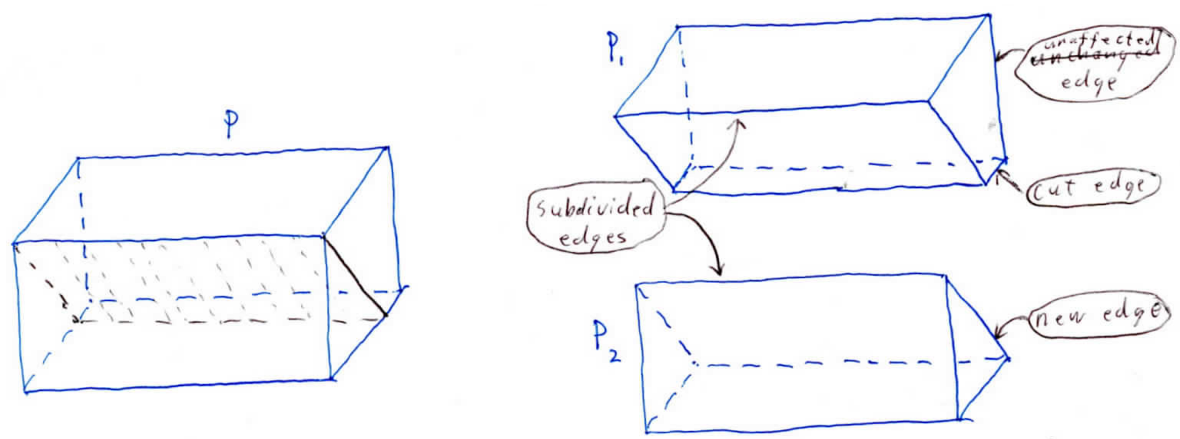

(i) Some edges from P are unaffected and exist entirely in P_1 or entirely in P_2. These are labeled as "unaffected edges". These are the easiest to deal with. You must show that their contribution to the sum $g(P_1) + g(P_2)$ is the same as their contribution to $g(P)$.

(ii) Other edges in P are cut: the angle stays the same, but the length is divided into two parts: one in P_1 and one in P_2. These are labeled as "cut edges". They appear in pairs. You must show that their combined contribution from the two pieces is equal to the contribution of the original edge.

(iii) There are some new edges in P_1 and P_2 resulting from where the planar cut crossed a face of P. These are labeled as "new edges". These appear in pairs (one in P_1 and one in P_2). They have the same length, and their angles sum to π. Show that their contributions to the sum cancel one another.

(iv) *In special cases, one of the existing edges may be part of the plane we cut along. Then the entire length of the edge is in both parts, but the angle is subdivided. These are labeled as "subdivided edges". They appear in pairs with the same length and their angles sum to the original angle. Show that their combined contribution is equal to the contribution of the original edge.*

(g). *Explain why if we cut a tetrahedron into a finite set of pieces using planar cuts, then if we sum over all pieces, we have* $\sum g(P_i) \neq 0$.

(h). *Explain why if we cut a rectangular prism into a finite set of pieces using planar cuts, then if we sum over all pieces, we have* $\sum g(P_i) = 0$. *Why does this contradict the assumption that we can cut a regular tetrahedron up and reassemble it into a rectangular prism?*

Chapter 3

Systems of Linear Equations

3.1 Important concepts

- How to solve linear systems (Gaussian Elimination)
- When is a matrix invertible?
- What is the span of a set of vectors?
- What is linear independence?
- What is the column space or null space of a matrix and how do you find it?

3.2 Introduction

One of the most frequent applications of matrices is in the solution of systems of linear equations.

Definition 3.1 *A **linear equation** is an equation of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ where the a_i coefficients and b are given constants and the x_i variables are unknown. Notice that each term on the left hand side has exactly one x_i variable and it is raised to the first power.*

Definition 3.2 *A **system of linear equations** is a set of m linear equations. The general form is*

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

We assume that all the coefficients a_{ij} are known and that the constant terms b_i are also all known. We have already seen that we can write this in the form $\mathrm{A}\vec{x} = \vec{b}$.

Definition 3.3 *We say that we have a **solution** to the system if we find values of the x_i variables that make all equations in the system true.*

A general rule of thumb that will become clearer later is that if there are more variables than equations, we expect many solutions. If the number of equations and variables are the same, there is a good chance that there is exactly one solution. However, if there are more equations than variables there is a good chance that there is no solution.

Example 3.4 *Probably the simplest possible system of linear equations is the single equation*

$$ax = b \tag{3.1}$$

where a and b are given real numbers and x is an unknown real number.

Exercise 3.5 *For each of the following, find an example of real numbers a and b for which $ax = b$ has:*

(a). Exactly one solution.

(b). No solutions.

(c). Infinitely many solutions.

Hint: try to solve for x in terms of a and b. Under what conditions are the steps you take not allowed?

In general a linear system of equations has either one, zero, or infinitely many solutions.

Example 3.6 *The system of equations*

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ x_2 + 2x_3 &= 4 \\ x_3 &= 1 \end{aligned} \tag{3.2}$$

has only one solution $x_1 = -3$, $x_2 = 2$, and $x_3 = 1$. This is pretty easy to find. The last equation tells us $x_3 = 1$. Plugging this into the middle equation gives us $x_2 = 2$. Plugging these into the first equation gives $x_1 = -3$.
This system can be written in matrix form as

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$$

Example 3.7 *The system of equations*

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_3 &= 1 \end{aligned} \tag{3.3}$$

has infinitely many solutions. They take the form $x_3 = 1$, $x_2 = s$, and $x_1 = 5 - s$ where s can take any value. This system can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

Example 3.8 *The system of equations*

$$\begin{aligned} x_1 + x_2 &= 5 \\ x_1 + x_3 &= 9 \\ x_2 + x_3 &= 0 \\ 3x_2 + x_3 &= 3 \end{aligned}$$

has no solution. To see this, note that the first equation minus the second equation gives $x_2 - x_3 = -4$. If we add twice the third equation to this, we have $3x_2 + x_3 = -4$. This contradicts the third equation. This system can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 0 \\ 3 \end{bmatrix}$$

Exercise 3.9 *Take each of the following systems and find the matrix* A *and vector* $\vec{b}$ *so that* $A\vec{x} = \vec{b}$ *represents the same system. You only need to convert this to a matrix-based equation.*
You do not need to find the solution to the system.

(a).

$$\begin{aligned} x_1 + x_2 &= 3 \\ x_1 - 2x_3 &= 2 \\ x_2 + x_3 &= 3 \end{aligned}$$

(b).

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ 8x_1 + 13x_2 &= 21 \end{aligned}$$

(c).

$$\begin{aligned} 2x_2 + 4x_3 &= 8 \\ x_1 - x_3 &= -2 \end{aligned}$$

(d).

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 + 5x_3 &= 5 \\ x_1 - x_2 - x_3 &= 3 \end{aligned}$$

(e).

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 + 5x_3 &= 5 \\ x_1 - x_2 - x_3 &= -10 \end{aligned}$$

3.3 Gaussian Elimination

Ultimately we want to be able to find all solutions to a system of linear equations. If it's in a nice simple form like in system (3.2) or system (3.3) it's easy to do. The reason those systems were easy to solve is that there was at most one equation with a nonzero coefficient for x_1. Of the remaining equations, at most one had a nonzero coefficient for x_2. Similarly for x_3 etc. In other words, for each x_i, there is at most one equation for which it is the leading term.

So here is our challenge. We want an algorithm to find the solution to a system of linear equations in a finite number of steps.[1] Our approach is to find a sequence of steps that modify the system in such a way that each x_i is the leading term in at most one equation, but ensuring that at each step the solution is unchanged. Let's look at ways we can modify the system of equations without changing the solutions.

- We can change the order of equations. (interchange)

[1] It's not obvious that this must be possible. For example, we can solve a quadratic equation using a simple formula. A cubic or quartic equation has a much more complicated formula for the solution. However, there is no known general algorithm to give a solution to a quintic or higher order equation, and even worse, it has been proven that there is no formula that gives the solution for a general quintic equation.

- We can replace an equation with the sum of itself and a multiple of any other equation. (replacement)
- We can multiply any equation by a nonzero constant. (scaling)

Notice that each of these operations can be undone by another of these operations. This is important because it means that none of our steps "lose information".

We'll do a few examples and then translate this into "elementary row operations" on a matrix. I'll show the method first, and then give a more rigorous explanation.

Example 3.10 *Consider the system*

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ 5x_1 + 5x_2 + x_3 &= -4 \\ x_1 - x_2 + x_3 &= 0 \end{aligned}$$

Our first target is to eliminate all but one of the nonzero coefficients of x_1, and to have the nonzero coefficient in the first equation. We subtract 5 times the first equation from the second equation. We get

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ 0x_1 + 0x_2 - 14x_3 &= -14 \\ x_1 - x_2 + x_3 &= 0 \end{aligned}$$

We now subtract the first equation from the third.

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ 0x_1 + 0x_2 - 14x_3 &= -14 \\ 0x_1 - 2x_2 - 2x_3 &= -2 \end{aligned}$$

We've now made all the coefficients of x_1 be zero, except for the first equation. We now want to do the same thing to the x_2 coefficients, having just the second equation have a nonzero second coefficient. We can now restrict our attention to just the second and third equations. We swap the second and third equations and get

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ 0x_1 - 2x_2 - 2x_3 &= -2 \\ 0x_1 + 0x_2 - 14x_3 &= -14 \end{aligned}$$

At this point, we're done. The last equation tells us $x_3 = 1$, then the second equation tells us $x_2 = 0$, and the first equation tells us $x_1 = -1$. The solution is

$$\begin{aligned} x_1 &= -1 \\ x_2 &= 0 \\ x_3 &= 1 \end{aligned}$$

Example 3.11 *Consider a second example system*

$$\begin{aligned} 0x_1 + 2x_2 + 2x_3 - x_4 &= 1 \\ x_1 - x_2 + 0x_3 + x_4 &= 3 \\ 2x_1 + 4x_2 + 6x_3 + x_4 &= 1 \\ 4x_1 + 0x_2 + 4x_3 + 0x_4 &= 22 \end{aligned}$$

We again try to get just one nonzero coefficient of x_1, and we want to have it in the first equation. To do this, we first, swap the first two equations to put a nonzero coefficient in the first equation.

$$\begin{aligned} x_1 - x_2 + 0x_3 + x_4 &= 3 \\ 0x_1 + 2x_2 + 2x_3 - x_4 &= 1 \\ 2x_1 + 4x_2 + 6x_3 + x_4 &= 1 \\ 4x_1 + 0x_2 + 4x_3 + 0x_4 &= 22 \end{aligned}$$

We now subtract twice the first equation from the third.

$$\begin{aligned} x_1 - x_2 + 0x_3 + x_4 &= 3 \\ 0x_1 + 2x_2 + 2x_3 - x_4 &= 1 \\ 0x_1 + 6x_2 + 6x_3 - x_4 &= -5 \\ 4x_1 + 0x_2 + 4x_3 + 0x_4 &= 22 \end{aligned}$$

and subtract four times the first equation from the last.

$$\begin{aligned} x_1 - x_2 + 0x_3 + x_4 &= 3 \\ 0x_1 + 2x_2 + 2x_3 - x_4 &= 1 \\ 0x_1 + 6x_2 + 6x_3 - x_4 &= -5 \\ 0x_1 + 4x_2 + 4x_3 + -4x_4 &= 10 \end{aligned}$$

We've finished the x_1 part, and we move on to x_2. We're done with the first equation. We subtract three times the second equation from the third equation. We get

$$\begin{aligned} x_1 - x_2 + 0x_3 + x_4 &= 3 \\ 0x_1 + 2x_2 + 2x_3 - x_4 &= 1 \\ 0x_1 + 0x_2 + 0x_3 + 2x_4 &= -8 \\ 0x_1 + 4x_2 + 4x_3 + -4x_4 &= 10 \end{aligned}$$

We subtract twice the second equation from the last

$$\begin{aligned} x_1 - x_2 + 0x_3 + x_4 &= 3 \\ 0x_1 + 2x_2 + 2x_3 - x_4 &= 1 \\ 0x_1 + 0x_2 + 0x_3 + 2x_4 &= -8 \\ 0x_1 + 0x_2 + 0x_3 + -2x_4 &= 8 \end{aligned}$$

We're done with x_2 and with the second equation. We move on to x_3, and oddly, the coefficient of x_3 in all the remaining equations is zero. There is nothing we can do with x_3, so we are done with x_3. We are NOT done with the third equation. We add the third equation to the last equation.

$$\begin{aligned} x_1 - x_2 + 0x_3 + x_4 &= 3 \\ 0x_1 + 2x_2 + 2x_3 - x_4 &= 1 \\ 0x_1 + 0x_2 + 0x_3 + 2x_4 &= -8 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0 \end{aligned}$$

So in our final system, the last equation tells us nothing. The third equation says that $x_4 = -4$. The second equation then gives $2x_2 + 2x_3 = -3$. So there's no restriction on x_3, it can take any value. So we solve for the other variables in terms of x_3. We find $x_2 = -3/2 - x_3$. Plugging this all into the first *equation, we have $x_1 = 17/2 + x_3$. Our final solution is*

$$\begin{aligned} x_1 &= \frac{17}{2} + x_3 \\ x_2 &= -\frac{3}{2} - x_3 \\ x_3 &\textit{ is free} \\ x_4 &= -4 \end{aligned}$$

In the steps I was doing, all I did was manipulate the coefficients of the equations. There's no real need to have the x_i variables be carried along. Let's convert what we just did to a matrix form. First note that each of these equations may be converted to a matrix equation of the form $\mathrm{A}\vec{x} = \vec{b}$ where A is the matrix of coefficients on the right hand side of the systems.

Definition 3.12 *Given an $m \times n$ matrix $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ and the matrix equation $\mathrm{A}\vec{x} = \vec{b}$, the **augmented matrix** M is defined by*

$$\mathrm{M} = [\mathrm{A} \mid \vec{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

That is, we take the matrix A and simply add a new column at the end with the vector $\vec{b}$.

(the vertical line is used to offset the A part and the $\vec{b}$ part of the matrix M — it is a common convention, but not all people use it.)

Let's translate what we just did into operations on the augmented matrix.

Example 3.13 *Consider $\mathrm{A}\vec{x} = \vec{b}$ where*

$$\mathrm{A} = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 5 & 1 \\ 1 & -1 & 1 \end{bmatrix} \qquad \textit{and} \qquad b = \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix}$$

This corresponds to example 3.10. We create the augmented matrix

$$\mathrm{M} = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 5 & 5 & 1 & -4 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

We repeat all the steps we did above, replacing the word "equation" with "row".

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 5 & 5 & 1 & -4 \\ 1 & -1 & 1 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 0 & -14 & -14 \\ 1 & -1 & 1 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 0 & -14 & -14 \\ 0 & -2 & -2 & -2 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & -2 & -2 & -2 \\ 0 & 0 & -14 & -14 \end{array}\right]$$

And then we can read off the solution as before. We find

$$\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Example 3.14 *Consider* $\mathrm{A}\vec{x} = \vec{b}$ *where*

$$\mathrm{A} = \begin{bmatrix} 0 & 2 & 2 & -1 \\ 1 & -1 & 0 & 1 \\ 2 & 4 & 6 & 1 \\ 4 & 0 & 4 & 0 \end{bmatrix} \qquad \text{and} \qquad \vec{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 22 \end{bmatrix}$$

This corresponds to example 3.11. The augmented matrix is

$$\mathrm{M} = \left[\begin{array}{cccc|c} 0 & 2 & 2 & -1 & 1 \\ 1 & -1 & 0 & 1 & 3 \\ 2 & 4 & 6 & 1 & 1 \\ 4 & 0 & 4 & 0 & 22 \end{array}\right]$$

We perform the same operations as before, again replacing "equation" with "row".

$$\left[\begin{array}{cccc|c} 0 & 2 & 2 & -1 & 1 \\ 1 & -1 & 0 & 1 & 3 \\ 2 & 4 & 6 & 1 & 1 \\ 4 & 0 & 4 & 0 & 22 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 2 & 2 & -1 & 1 \\ 2 & 4 & 6 & 1 & 1 \\ 4 & 0 & 4 & 0 & 22 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 2 & 2 & -1 & 1 \\ 0 & 6 & 6 & -1 & -5 \\ 4 & 0 & 4 & 0 & 22 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 2 & 2 & -1 & 1 \\ 0 & 6 & 6 & -1 & -5 \\ 0 & 4 & 4 & -4 & 10 \end{array}\right]$$

$$\mapsto \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 2 & 2 & -1 & 1 \\ 0 & 0 & 0 & 2 & -8 \\ 0 & 4 & 4 & -4 & 10 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 2 & 2 & -1 & 1 \\ 0 & 0 & 0 & 2 & -8 \\ 0 & 0 & 0 & -2 & 8 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 2 & 2 & -1 & 1 \\ 0 & 0 & 0 & 2 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and again, we read off the values as before. We have

$$\vec{x} = \begin{bmatrix} \frac{17}{2} \\ -\frac{3}{2} \\ 0 \\ -4 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} x_3$$

I want to present a final example of this algorithm. I use the same equation as the previous example, except that I change the final value in b by adding just a small amount to it.

Example 3.15 *We take the same* A *as in the previous example, but add* 0.001 *to the final entry of* $\vec{b}$*. We get*

$$\left[\begin{array}{cccc|c} 0 & 2 & 2 & -1 & 1 \\ 1 & -1 & 0 & 1 & 3 \\ 2 & 4 & 6 & 1 & 1 \\ 4 & 0 & 4 & 0 & 22.001 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 2 & 2 & -1 & 1 \\ 2 & 4 & 6 & 1 & 1 \\ 4 & 0 & 4 & 0 & 22.001 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 2 & 2 & -1 & 1 \\ 0 & 6 & 6 & -1 & -5 \\ 4 & 0 & 4 & 0 & 22.001 \end{array}\right]$$

$$\mapsto \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 2 & 2 & -1 & 1 \\ 0 & 6 & 6 & -1 & -5 \\ 0 & 4 & 4 & -4 & 10.001 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 2 & 2 & -1 & 1 \\ 0 & 0 & 0 & 2 & -8 \\ 0 & 4 & 4 & -4 & 10.001 \end{array}\right]$$

$$\mapsto \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 2 & 2 & -1 & 1 \\ 0 & 0 & 0 & 2 & -8 \\ 0 & 0 & 0 & -2 & 8.001 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 0 & 2 & 2 & -1 & 1 \\ 0 & 0 & 0 & 2 & -8 \\ 0 & 0 & 0 & 0 & 0.001 \end{array}\right]$$

So now the final row of the augmented matrix represents the equation $0 = 0.001$*. There is no way to choose the* x_i *variables to make this equation true. So there is no solution to this system of equations. Just a tiny tiny change in* $\vec{b}$ *modified the system from a system with infinitely many solutions to a system with no solutions.*

Now that we've seen examples of the algorithm, let's explain exactly what we've done.

Definition 3.16 *An **elementary row operation** on a matrix is any of the following operations:*

- *Interchange two rows. This is called an **interchange**.*
- *Replace a row with the sum of itself and any multiple of another row. This is called a **row replacement**.*
- *Multiply a row by a nonzero value. This is called a **scaling**.*

Note that it is possible to undo any elementary row operation by using the appropriate elementary row operation. These operations are invertible. Big warning: these are row operations. If you start trying to do "column operations" as well, things will break badly. Do not do it.

Theorem 3.17 *Elementary row operations do not change the solutions to a given system.*

We use elementary row operations to get the system into a form for which the solution is easier to find.

Definition 3.18 *Two matrices* A *and* B *are **row-equivalent** if a sequence of elementary row operations changes* A *into* B*. We use the notation* $\mathrm{A} \sim \mathrm{B}$ *to denote that the matrices are row-equivalent.*

Notice that because these operations are invertible, if we show that A and B can both be changed into another matrix C, then it is possible to change A into B and vice versa. So if $\mathrm{A} \sim \mathrm{C}$ and $\mathrm{B} \sim \mathrm{C}$, then $\mathrm{A} \sim \mathrm{B}$.

Definition 3.19 *A matrix is in **echelon form** if it has the following properties*

- *All nonzero rows are above any rows of all zeros.*
- *Each leading entry (that is, the first nonzero entry) of a row is strictly to the right of the leading entry of the row above it.*

It is perhaps surprising that the echelon form of a matrix is not unique. Depending on the order of steps we take, the echelon form will look different. However, the solution it leads to is always the same. More on this later.

The goal of Gaussian elimination is to reduce a given augmented matrix M into a matrix in echelon form. From here, the solution can be calculated easily.

Exercise 3.20 *For each system in exercise 3.9, find the appropriate augmented matrix* M*. Then use row operations to convert the matrix into a matrix in echelon form. Please make sure it is clear what steps you've taken. Does a solution exist? If so, find it.*

Exercise 3.21 *Consider the* 2×2 *matrix* $\mathrm{A} = \begin{bmatrix} 7 & 3 \\ 7 & 2 \end{bmatrix}$.

(a). Replace the second row of A *with the sum of it and twice the first row of* A.

(b). Perform the same operation on $\mathrm{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *(that is, replace the second row of* I *with the sum of it and twice the first row of* I*). Use* E *to denote the result.*

(c). Then go back to the original matrix A*. Find the product* EA.

(d). What do you notice about EA*?*

(e). Go back to the original matrix A *again. Perform row operations to* A *to convert it to* I.

(f). Perform exactly the same operations you did to A *on* I*. Call the result* B.

(g). What is BA*? What is* AB*?*

3.3.1 Pivot positions

Definition 3.22 *A* ***pivot position*** *of a matrix* C *is a location/position in* C *that corresponds to a leading entry (first nonzero entry of a row) in the echelon form of* C*. A* ***pivot column*** *is a column of* C *that contains a pivot position.*

Pivot positions are not numbers. They are locations in the matrix.

In example 3.14 the pivot columns of M were 1, 2, and 4. In example 3.15 the pivot columns of the matrix were 1, 2, 4, and 5.

Definition 3.23 *The variables x_i that correspond to columns that are not pivot columns are called* ***free variables****.*

Now we have the necessary terms to clearly define the algorithm. Here are the steps

Gaussian Elimination

(a). Begin with the leftmost nonzero column of A. This is a pivot column. The pivot position is at the top.

(b). If there is a zero in the pivot position, interchange the row with a lower row to put a nonzero entry in the top.

(c). Make every entry below the pivot position in that column zero using row replacements.

(d). Now consider the smaller matrix formed by removing the first row. Repeat these steps until all rows have been removed. The matrix is now in echelon form.

(e). For each nonzero row, with a leading entry in the i-th column we can write x_i in terms of x_j for $j > i$.

Successive substitutions allow us to simplify this to writing x_i in terms of the free variables only. If any row has all zeros in the A part, but a nonzero in the $\vec{b}$ part, then there is no solution. The system is "inconsistent".

This Gaussian Elimination algorithm will always work (through roundoff error may be a problem if it is done by computer). Occassionally it may be possible to speed things up using a different order to the steps. Depending on the exact order with which we perform row operations, the echelon form we arrive at may differ. However, the pivot positions are the same.

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if $A\vec{v} = 0$ for
then $A = 0$. (
$A\vec{v} = B\vec{v}$ for
$A = B$

Example 3.24 *Consider the matrix*

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 1 & -1 & 2 \\ 2 & 2 & -2 & 3 \\ 3 & 4 & -1 & 5 \end{bmatrix}$$

Let's look at two different ways to get it into echelon form. For the first, we start by interchanging the first two rows

$$\begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 1 & -1 & 2 \\ 2 & 2 & -2 & 3 \\ 3 & 4 & -1 & 5 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 0 \\ 2 & 2 & -2 & 3 \\ 3 & 4 & -1 & 5 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -1 \\ 3 & 4 & -1 & 5 \end{bmatrix}$$

$$\mapsto \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[the steps were: interchange row 1 and 2; add -2 times row 1 to row 3; add -3 times row 1 to row 4; add -1 times row 2 to row 4; add -1 times row 3 to row 4].

On the other hand, we might start by interchanging the first and the third rows.

$$\begin{bmatrix}0&1&2&0\\1&1&-1&2\\2&2&-2&3\\3&4&-1&5\end{bmatrix} \mapsto \begin{bmatrix}2&2&-2&3\\1&1&-1&2\\0&1&2&0\\3&4&-1&5\end{bmatrix} \mapsto \begin{bmatrix}2&2&-2&3\\0&0&0&1/2\\0&1&2&0\\3&4&-1&5\end{bmatrix} \mapsto \begin{bmatrix}2&2&-2&3\\0&0&0&1/2\\0&1&2&0\\0&1&2&1/2\end{bmatrix}$$

$$\mapsto \begin{bmatrix}2&2&-2&3\\0&1&2&0\\0&0&0&1/2\\0&1&2&1/2\end{bmatrix} \mapsto \begin{bmatrix}2&2&-2&3\\0&1&2&0\\0&0&0&1/2\\0&0&0&1/2\end{bmatrix} \mapsto \begin{bmatrix}2&2&-2&3\\0&1&2&0\\0&0&0&1/2\\0&0&0&0\end{bmatrix}$$

[the steps were: interchange rows 1 and 3; add -1/2 times row 1 to row 2; add -3/2 times row 1 to row 3; interchange rows 2 and 3; add -1 times row 2 to row 4; add -1 times row 3 to row 4].

The result is cosmetically different, but the pivot positions are the same. They are the locations with ■ in the following representation of the echelon form:

$$\begin{bmatrix}\blacksquare&*&*&*\\0&\blacksquare&*&*\\0&0&0&\blacksquare\\0&0&0&0\end{bmatrix}$$

When we talk about the pivot locations of the original matrix A, *we mean the (1,1), (2,2), and (3,4) positions of the matrix. We don't mean the entries of those positions, we just mean "the coordinates which will have a leading nonzero entry if we convert the matrix to echelon form".*

Exercise 3.25 *For each matrix* C, *find the pivot positions*

(a). $\mathrm{C} = \begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$

(b). $\mathrm{C} = \begin{bmatrix}1&2&3\\4&5&6\\5&7&9\\3&2&1\end{bmatrix}$

(c). $\mathrm{C} = \begin{bmatrix}0&0&1&3\\0&0&0&5\\0&0&4&2\end{bmatrix}$

Example 3.26 *Using Gaussian Elimination:*

Consider the matrix equation

$$]\,\vec{x}[\; = \begin{bmatrix}3\\5\\3\\11\end{bmatrix}$$

Find all solutions.

We start with the augmented matrix $M = \left[\begin{array}{cccccc|c} 0 & 0 & 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 & 0 & 1 & 5 \\ 0 & 2 & 1 & -1 & 0 & 1 & 3 \\ 0 & 5 & 4 & -1 & 0 & 3 & 11 \end{array}\right]$ *and do Gaussian Elimination:*

$$\left[\begin{array}{cccccc|c} 0 & 0 & 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 & 0 & 1 & 5 \\ 0 & 2 & 1 & -1 & 0 & 1 & 3 \\ 0 & 5 & 4 & -1 & 0 & 3 & 11 \end{array}\right] \mapsto \left[\begin{array}{cccccc|c} 0 & 1 & 2 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 1 & 1 & 3 \\ 0 & 2 & 1 & -1 & 0 & 1 & 3 \\ 0 & 5 & 4 & -1 & 0 & 3 & 11 \end{array}\right] \mapsto \left[\begin{array}{cccccc|c} 0 & 1 & 2 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 & 0 & -1 & -7 \\ 0 & 5 & 4 & -1 & 0 & 3 & 11 \end{array}\right]$$
$$\mapsto \left[\begin{array}{cccccc|c} 0 & 1 & 2 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 & 0 & -1 & -7 \\ 0 & 0 & -6 & -6 & 0 & -2 & -14 \end{array}\right]$$

At this stage we could speed things up by subtracting twice the third row from the fourth. It's fine to do that, but it's hard to program a computer to check for things like this. Blindly following the recipe for Gaussian Elimination above:

$$\left[\begin{array}{cccccc|c} 0 & 1 & 2 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 & 0 & -1 & -7 \\ 0 & 0 & -6 & -6 & 0 & -2 & -14 \end{array}\right] \mapsto \left[\begin{array}{cccccc|c} 0 & 1 & 2 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 3 & 2 & 2 \\ 0 & 0 & -6 & -6 & 0 & -2 & -14 \end{array}\right] \mapsto \left[\begin{array}{cccccc|c} 0 & 1 & 2 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 0 & 6 & 4 & 4 \end{array}\right]$$
$$\mapsto \left[\begin{array}{cccccc|c} 0 & 1 & 2 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 0 & 6 & 4 & 4 \end{array}\right] \mapsto \left[\begin{array}{cccccc|c} 0 & 1 & 2 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So our pivot columns are 2, 3, *and* 5. *Our last row is simply* $0 = 0$. *The free variables are* x_1, x_4, *and* x_6 *The other rows (from bottom to top) say*

$$\begin{aligned} x_5 &= (-2 - 2x_6)/3 \\ x_3 &= 3 - x_4 - x_5 - x_6 \\ x_2 &= 5 - 2x_3 - x_4 - x_5 \end{aligned}$$

Simplifying gives

$$\begin{aligned} x_5 &= -\frac{2}{3} - \frac{2}{3}x_6 \\ x_3 &= \frac{11}{3} - x_4 - \frac{1}{3}x_6 \\ x_2 &= -\frac{5}{3} + x_4 + \frac{4}{3}x_6 \end{aligned}$$

This isn't the preferred form. Let's write it as a vector:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_1 \\ -\frac{5}{3} + x_4 + \frac{4}{3}x_6 \\ \frac{11}{3} - x_4 - \frac{1}{3}x_6 \\ x_4 \\ -\frac{2}{3} - \frac{2}{3}x_6 \\ x_6 \end{bmatrix}$$

We do one more step to the solution to get "parametric form"

$$\vec{x} = \begin{bmatrix} 0 \\ -\frac{5}{3} \\ \frac{11}{3} \\ 0 \\ -\frac{2}{3} \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ \frac{4}{3} \\ -\frac{1}{3} \\ 0 \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Definition 3.27 *A solution with free variables is in* ***parametric form*** *if it is written as a constant vector plus a sum of free variables times constant vectors:*

$$\vec{x} = \vec{v}_0 + \sum_{\text{free variables } x_i} x_i \vec{v}_i$$

Efficiency of Gaussian Elimination Unfortunately, Gaussian Elimination is a relatively slow process. This gets particularly bad as the matrix size increases. For a square matrix, the number of steps required is approximately the cube of the number of variables. If A is $n \times n$ and B is $10n \times 10n$, then using Gaussian Elimination with B takes approximately 10^3 times as many steps as with A.

This is one of the largest restrictions on our ability to do high resolution numerical calculations. To do a three-dimensional calculation in a foot by foot by foot cube with resolution 1 inch in every direction requires 1728 variables. To double our resolution in each direction requires a total of 8 times as many variables. Gaussian Elimination then takes $8^3 = 512$ times as long. To do ten times the resolution requires 10^3 times as many variables, and then Gaussian Elimination requires approximately 10^6 times as many steps.

Even if we have a fast enough computer to do the calculations, we must be careful that computer rounding error doesn't screw up our calculations. There are ways to improve this, using what is called "partial pivoting" where at step (b) in the algorithm rather than exchanging rows only if the entry is zero, we exchange rows if there is any row below it having a larger entry in the column. In practice this works very well, although in incredibly rare cases roundoff error still matters. It is not fully-understood why the cases where this fails are so rare.

There are some ways to speed up Gaussian elimination so that the time to run isn't cubic in n. But they are complicated, and nothing is known to be very close to being quadratic in n, although some hypothesize it can be done (or at least they hypothesize it can be arbitrarily close to quadratic in n).

Exercise 3.28 *Read "The Smart Money Is on Numerical Analysts" in SIAM (Society for Industrial and Applied Mathematics) News, Nov, 2012. Write a summary of about 150–200 words.*
This article talks about the challenges of making Gaussian Elimination faster and robust to rounding error.
A few comments that help with the terminology in the article:

- *"$O(n^3)$" means that if a coputational problem has an input made up of a vector of n entries, there exists a c such that the number of steps required to solve the computational question is less than cn^3.*
- *"FFT" is the "fast fourier transform", which is an algorithm that is widely used in computations.*
- *"$P = NP$?" is a famous computer science question (the answer, I believe, is no, but that's just a guess — no one knows for sure). It has to do with whether there exists a computer algorithm that can solve certain problems such as the traveling salesman question[2] quickly. If you found a fast algorithm and kept your mouth shut, you could probably become wealthy.*

[2] Consider a salesman who visits a city each week and needs to visit the largest 52 cities in the US. He wants to take the shortest path that goes through all of them. How can we find that path? There are $52! = 52 \times 51 \times 50 \times \cdots \times 2 \times 1$ possible paths which is simply too large to check every possible path, even with a super computer. For N cities, checking every path requires an $O(N!)$ algorithm. Is there an alternative algorithm that finds the shortest path in $O(N^a)$ steps where a is some constant?

3.4 Column space and Null Space

3.4.1 Column Space

I'm going to engage in a bit of hand-waving here. When we look at the results of Gaussian Elimination, it doesn't matter what $\vec{b}$ is, the pivot locations we find in the A part of the augmented matrix will always be the same. The free variables will always be the same.

This has some implications for whether we can find a solution to $A\vec{x} = \vec{b}$ or not. If there is at least one solution, and there are any free variables, then there are infinitely many solutions.

Recall that Col(A) is defined to be the set of vectors $\vec{b}$ that can be written in the form $\vec{b} = A\vec{x}$ for some vector $\vec{x}$. We can now say a bit more about exactly what Col(A) is. We've proven that Col(A) is a vector space. If A has m rows, then it's obvious that Col(A) is a subspace of $\mathbb{R}^m$, but which subspace is it?

The fact that $\vec{b} = A\vec{x}$ means that $\vec{b} = \sum x_i\vec{a}_i$ where $\vec{a}_i$ is the i-th column of A. So clearly everything in Col(A) can be written as a linear combination of columns of A [and also any linear combination of the columns can be found in Col(A)], so

Theorem 3.29 Col(A) *is the span of the columns of* A.

We can use our solution form to find a basis for Col(A). There are quite a different bases, we just need to find one. If $\vec{b} \in \mathrm{Col(A)}$, then we can find a solution to $A\vec{x} = \vec{b}$, and if there are free variables, then there are infinitely many. However, if we set all the free variables to 0, we are left just one solution. That solution can be written as $\sum x_i\vec{a}_i$ where the sum includes only the pivot columns. In other words, if $\vec{b} \in \mathrm{Col(A)}$ then there is one and only one way to write $\vec{b}$ as a linear combination of the pivot columns. Since the pivot columns themselves are in Col(A), this proves that the pivot columns are a basis for Col(A).

Determining if a set of vectors is linearly independent

We can use this to determine if a set of vectors is linearly independent. Recall: a set of vectors is a basis for a vector space V iff they are linearly independent and their span is V. As a consequence of this, we can argue that a set of vectors is a basis for their span iff they are linearly independent.

Our theorem above says that if we take our vectors, and create a matrix which has them as its columns, then the pivot columns of the matrix are a basis for their span. So if we create a matrix A whose columns are those vectors, then the vectors are linearly independent iff all columns of A are pivot columns.

This gives us a simple test for whether vectors are linearly independent. We just need to follow the Gaussian Elimination steps that get A into echelon form.

Theorem 3.30 *The vectors* $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_d$ *are linearly independent iff every column of* $A = [\vec{a}_1\,\vec{a}_2\,\cdots\,\vec{a}_d]$ *is a pivot column*

Notice this proves that any set of more than n vectors in $\mathbb{R}^n$ is linearly dependent: The matrix A will have at most one pivot entry per row, so at most n pivot entries. Thus not all columns can be pivot columns.

Example 3.31 *Is the set of vectors* $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\3\\3\end{bmatrix}, \begin{bmatrix}1\\2\\4\end{bmatrix}$ *linearly independent?*

Take $A = \begin{bmatrix}1&2&1\\1&3&2\\2&3&4\end{bmatrix}$ *and reduce it to echelon form:*

$$\begin{bmatrix}1&2&1\\1&3&2\\2&3&4\end{bmatrix} \mapsto \begin{bmatrix}1&2&1\\0&1&1\\2&3&4\end{bmatrix} \mapsto \begin{bmatrix}1&2&1\\0&1&1\\0&-1&2\end{bmatrix} \mapsto \begin{bmatrix}1&2&1\\0&1&1\\0&0&3\end{bmatrix}$$

So there are three pivots. They are linearly independent.

Exercise 3.32 *Find h such that* $\begin{bmatrix}1\\1\\2\end{bmatrix}$, $\begin{bmatrix}2\\3\\3\end{bmatrix}$, $\begin{bmatrix}1\\h\\4\end{bmatrix}$ *is linearly dependent.*

Finding something not in $\mathrm{Col(A)}$

So we've found a basis for $\mathrm{Col(A)}$, but it's still a bit difficult to check if $\vec{b}$ is in $\mathrm{Col(A)}$. In particular, we might be interested in finding a vector that is not in $\mathrm{Col(A)}$.

To do this, we take A, augment it with $\vec{b} = \begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}$ leaving the b_i as unknown.

When we do Gaussian elimination, if there is a row of all zeros in the A part, there will be some linear combination of the b_i in the $\vec{b}$ part. So this linear combination must be zero. It's easy to find a $\vec{b}$ for which this does not hold.

Example 3.33 *For the matrix* $\mathrm{A} = \begin{bmatrix}1&1&1\\2&3&2\\5&6&5\end{bmatrix}$, *find a vector $\vec{b}$ with $\vec{b} \notin \mathrm{Col(A)}$.*

We do Gaussian Elimination:

$$\left[\begin{array}{ccc|c}1&1&1&b_1\\2&3&2&b_2\\5&6&5&b_3\end{array}\right] \mapsto \left[\begin{array}{ccc|c}1&1&1&b_1\\0&1&0&b_2-2b_1\\5&6&5&b_3\end{array}\right] \mapsto \left[\begin{array}{ccc|c}1&1&1&b_1\\0&1&0&b_2-2b_1\\0&1&0&b_3-5b_1\end{array}\right] \mapsto \left[\begin{array}{ccc|c}1&1&1&b_1\\0&1&0&b_2-2b_1\\0&0&0&b_3-b_2-3b_1\end{array}\right]$$

So if $b_3 \neq b_2 + 3b_1$, then $\vec{b} \notin \mathrm{Col(A)}$. For example $b_1 = 1,\ b_2 = 1,\ b_3 = 1$.

Exercise 3.34 *For each of the following, find an h such that* $\vec{b} = \begin{bmatrix}1\\2\\h\end{bmatrix}$ *is in the span of the vectors:*

(a). $\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\\1\end{bmatrix}$

(b). $\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}-1\\1\\-1\end{bmatrix}$, $\begin{bmatrix}1\\1\\1\end{bmatrix}$

(c). I made a mistake here. Instead, show that for any h that it is in the span $\begin{bmatrix}2\\1\\2\end{bmatrix}$, $\begin{bmatrix}-1\\1\\1\end{bmatrix}$, $\begin{bmatrix}7\\2\\5\end{bmatrix}$, $\begin{bmatrix}-4\\1\\0\end{bmatrix}$

Exercise 3.35 *For each of the following find an h such that* $\vec{b} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ *is <u>not</u> in the span of the vectors:*

(a). $\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}1\\h\\3\end{bmatrix}$

(b). $\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}-1\\2\\-1\end{bmatrix}$, $\begin{bmatrix}1\\1\\h\end{bmatrix}$

(c). $\begin{bmatrix}2\\1\\2\end{bmatrix}$, $\begin{bmatrix}-1\\1\\1\end{bmatrix}$, $\begin{bmatrix}7\\2\\5\end{bmatrix}$, $\begin{bmatrix}8\\4\\h\end{bmatrix}$

Using $\mathrm{Col(A)}$ **to study bases**]

A theorem was given (without proof) that the following are equivalent:

(a). B is a basis for V.

(b). $V = \mathrm{Span}(B)$ and B is linearly independent.

(c). Every vector in V can be written as a linear combination of vectors in B in exactly one way and $B \subset V$.

We can now prove these facts, at least if B is made up of vectors in $\mathbb{R}^n$ and V is a subspace of $\mathbb{R}^n$ (or possibly $V = \mathbb{R}^n$).

Proof. Given the set of vectors $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, the column space of $\mathrm{A} = [\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n]$ is the span of B.

By definition, if B is a basis for V, then V is the column space of A. Further by definition, if B is a basis, if we remove any vector from B, that changes the span. That means that if B is a basis for its span, then if we take A, and perform Gaussian Elimination we will not find any free variables. If there were a free variable x_i, then for any $\vec{b} \in \mathrm{Col(A)}$, we could find the solution formula for $\mathrm{A}\vec{x} = \vec{b}$ and set $x_i = 0$. This would give $\vec{b}$ as a linear combination of the other columns. Thus removing $\vec{v}_i$ does not affect the span, contradicting the definition of a basis.

So if it's a basis every column is a pivot column. Thus $\mathrm{A}\vec{x} = \vec{0}$ has only the trivial solution (since no free variables). So if (a) is true, (b) is true.

We can show that if (b), then (a). If B is linearly independent, and we remove $\vec{v}_i$ from B, then there is no way to write $\vec{v}_i$ in terms of the remaining vectors, so we've changed the span. Thus if (b) holds, then (a) holds.

Now we show if (b) then (c). If B is linearly independent, then the matrix A has no free variables. So if $\mathrm{A}\vec{x} = \vec{b}$ for some $\vec{b} \in V$, there is exactly one solution. Thus every vector in V can be written in exactly one way.

Now we show if (c) then (b). If there is exactly one solution to $\mathrm{A}\vec{x} = \vec{b}$ for any $\vec{b} \in V$, then in particular it holds for $\vec{b} = \vec{0}$. We know $\mathrm{A}\vec{0} = \vec{0}$, and there cannot be any other $\vec{x}$ in the Null Space of A. Thus the only sum of the columns of A that give $\vec{0}$ is with all coefficients 0, so the columns are linearly independent. □

Exercise 3.36 *In this exercise we prove Theorem 2.59. Assume $B = \{\vec{v}_1, \ldots \vec{v}_d\}$ is a basis for the vector space V. Let $S = \{\vec{w}_1, \ldots, \vec{w}_n\}$ where $n > d$. We will show that S is linearly dependent. Let $\vec{w}_i = a_{i1}\vec{v}_1 + \cdots + a_{d1}\vec{v}_d$.*

(a). Show that if $\sum x_i \vec{w}_i = \vec{0}$, then $\mathrm{A}\vec{x} = \vec{0}$ where the entries in A are the a_{ij} above.

(b). If $n > d$ show that A has columns that are not pivot columns.

(c). Show that there must be free variables and explain why this shows that S is linearly dependent.

Exercise 3.37 *For each of the following, determine whether the vector $\vec{b}$ is in the span of the vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$. A yes or no (with explanation) will do, you do not need to find the specific coefficients.*

(a). $\vec{a}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix}3\\4\end{bmatrix}$, $\vec{b} = \begin{bmatrix}11\\12\end{bmatrix}$.

(b). $\vec{a}_1 = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix}$, $\vec{a}_3 = \begin{bmatrix} -1 \\ 2 \\ -4 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

(c). $\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, $\vec{a}_3 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ \pi \\ \pi^2 \end{bmatrix}$.

3.4.2 Null Space

We can also use Gaussian Elimination to give insight into the Null Space of A. Recall that the Null Space is defined to be the set of vectors $\vec{x}$ for which $\mathrm{A}\vec{x} = \vec{0}$.

We know that $\vec{x} = \vec{0}$ is one option. But there may be others.

We can solve for $\vec{x}$ with Gaussian Elimination. We can write it in parametric form as $\vec{x} = \vec{v}_0 + \sum_{\text{free variables}} x_i \vec{v}_i$. Look back at the parametric solution in example 3.26. Notice that the constant vector had entries only in the locations corresponding to pivot columns. Notice also that for the free variables, the i-th coordinate of the vector multiplied by x_i was 1, while in all other vectors the i-th coordinate was 0. This always happens. It's related to the fact that the right hand side is $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and none of the free variables depends on other free variables.

Let's look at an example

Example 3.38 *Find the null space of* $\mathrm{A} = \begin{bmatrix} 1 & 1 & 3 & 4 \\ 2 & 2 & 6 & 8 \end{bmatrix}$. *We augment with* $\vec{0}$ *and do Gaussian Elimination*

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & 4 & 0 \\ 2 & 2 & 6 & 8 & 0 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 1 & 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So $x_1 = -x_2 - 3x_3 - 4x_4$. *In parametric form our solution is*

$$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Notice that there is no constant vector in this parametric form, or rather, the constant vector is $\vec{0}$. *But notice also, that the relation about the entries for the* i*-th coordinate of the vector with* x_i *is correct.*

Why is the constant vector $\vec{0}$ in the example? Let's use the "trivial" solution $\vec{x} = \vec{0}$, we see that the fact that this is a solution means we must get $\vec{0}$ when we set all the free variables to 0. But when we set the free variables to $\vec{0}$ we get the constant vector.

So that means that the null space is the span of the vectors in the parametric form (without the constant vector). Can we say they are a basis? To do this, we must show they are linearly independent. Because the i-th coordinate of the vector for x_i is 1, but the i-th coordinate of any other vector is 0, we know that they must be linearly independent vectors. In any linear combination, the i-th entry in the sum is x_i. So if the linear combination is zero, the coefficient of every vector must be 0.

So they are linearly independent, and they span the Null Space. Therefore they are a basis for the Null Space.

Exercise 3.39 *For each set of vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^m$, determine whether they are linearly independent and find the dimension of their span. If there are vectors in $\mathbb{R}^m$ which are not in their span, find one. If not, say so.*

(a). $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(b). $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$

(c). $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 4 \\ 1 \end{bmatrix}$

(d). $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$

Exercise 3.40 *Let A be an $m \times n$ matrix. Give explanations for your answer to each question below. All of your answers should be based on counting pivots.*

(a). *If $\cancel{m > n}$ $m < n$ and there is a solution to $\mathrm{A}\vec{x} = \vec{b}$, how many solutions are there?*

(b). *If $m = n$ and there is a $\vec{b} \in \mathbb{R}^m$ such that $\mathrm{A}\vec{x} = \vec{b}$ has no solution, how many solutions are there to $\mathrm{A}\vec{x} = \vec{0}$?*

(c). *If $m = n$ and you have a $\vec{b}$ for which there is exactly one solution to $\mathrm{A}\vec{x} = \vec{b}$, is it possible to find another vector $\vec{c} \in \mathbb{R}^m$ for which $\mathrm{A}\vec{x} = \vec{c}$ has no solution?*

(d). *If $\cancel{m < n}$ $m > n$, is there a solution to $\mathrm{A}\vec{x} = \vec{b}$ for every $\vec{b} \in \mathbb{R}^m$.*

should change t
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is a solution to

We should be aware that the Null Space of an $m \times n$ matrix A is in $\mathbb{R}^n$, while the Column Space is in $\mathbb{R}^m$. So these are potentially very different looking vector spaces. Nevertheless, their dimensions are closely related to one another:

Theorem 3.41 *The Rank-nullity Theorem*

The dimension of $\mathrm{Col(A)}$ plus the dimension of $\mathrm{Nul(A)}$ is equal to the number of columns of A.

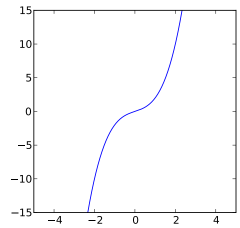
Exercise 3.42 *By counting pivots and free variables, prove the rank-nullity theorem.*

3.5 Onto and one-to-one functions

When we define a function $f : X \to Y$, there are a number of possibilities for how many xs in X can get mapped to a given $y \in Y$.

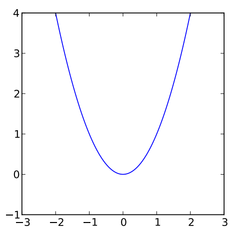
To give some examples, I'll step away from linear functions.

Example 3.43 *The function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^3 + x$ has the property that for any $y \in \mathbb{R}$, there is (at least) one $x \in \mathbb{R}$ such that $f(x) = y$. That is, for any $y \in \mathbb{R}$, there is at least one solution* *to $f(x) = y$. Further it has the property that for any $y \in \mathbb{R}$, there is at most one $x \in \mathbb{R}$ such that*



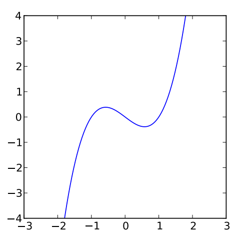
$f(x) = y$. That is, the solution is unique.

Example 3.44 *The function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$ has neither of these properties. We are not guaranteed that there is a solution to $f(x) = y$ for all y, and if there is a solution, we are not*



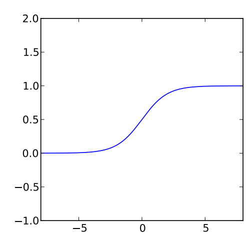
guaranteed it is unique.

Example 3.45 *The function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^3 - x$ has the property that we are guaranteed that there are solutions to $f(x) = y$ for any $y \in \mathbb{R}$, but we are not guaranteed the solution is unique [for*



example, $f(-1) = f(0) = f(1) = 0$].

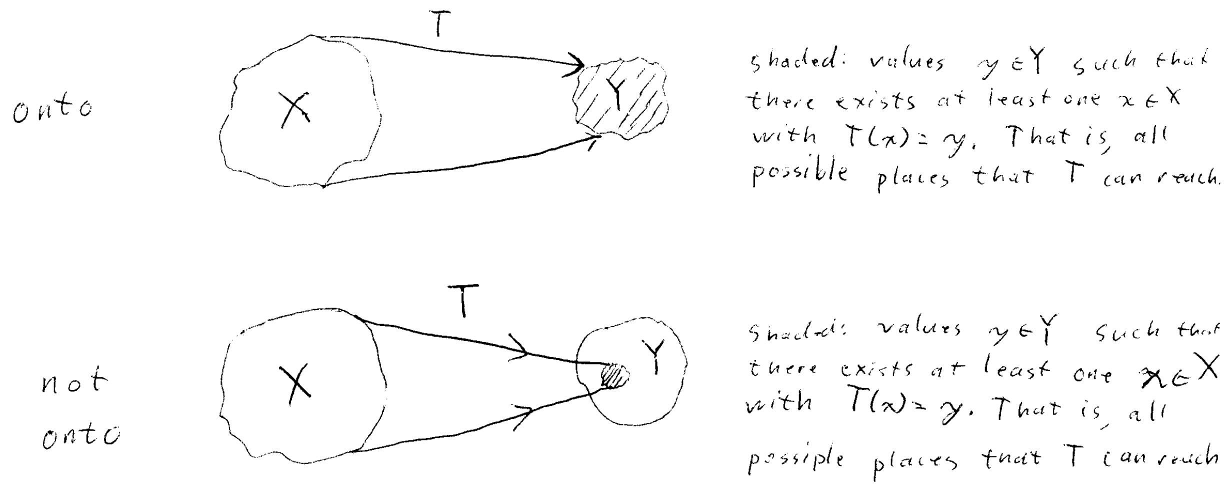
Example 3.46 *The function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = e^x/(e^x+1)$ has the property that if a solution exists to*



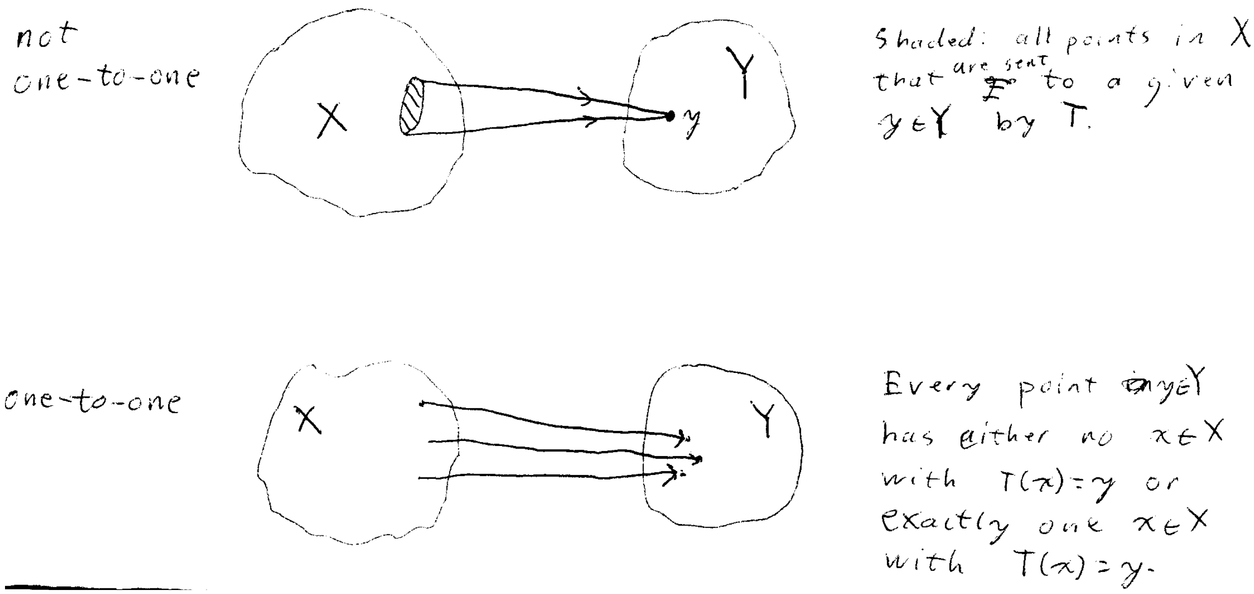
$f(x) = y$, then the solution is unique. However, for many y there is no solution.

So there are really two properties we are interested in. We are interested in whether we can always solve $f(x) = y$ for any given y, and whether if there is a solution that solution is unique.

We have the following idea. f takes X and "maps" it into Y. Think of it as f stretching and deforming X in some way so that it fits into Y. We call the part of Y which is covered by the deformed X the "image" of X. The first question is whether the mapping is "onto". That is, does the image of X cover everything in Y? The second question is whether the mapping is "one-to-one". That is, does everything in the image of X have only one thing sent to it.



A schematic showing how onto functions behave. A function $f : X \to Y$ is onto if every $y \in Y$ has at least one "source" in X.



A schematic showing how one-to-one functions behave. A function $f : X \to Y$ is one-to-one if every $y \in Y$ has at most one "source" in X.

One might argue that a better term for one-to-one would be at-most-one-to-one. That is everything in Y has either one thing sent to it or nothing sent to it by f.

Let's look at some simple examples from linear functions

Example 3.47 *Let* $T : \mathbb{R}^3 \to \mathbb{R}^2$ *by* $T(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$.

Consider any vector $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$*. It is fairly easy to find a vector* $\vec{x}$ *such that* $T(\vec{x}) = \vec{b}$*. All you need is that* $x_1 = b_2$ *and* $x_2 = b_1$*. There is no restriction on* x_3*, so in fact for every* $\vec{b} \in \mathbb{R}^2$ *there are infinitely many* $\vec{x} \in \mathbb{R}^3$ *that get sent to* $\vec{b}$*.*

Thus T is onto, but it is not one-to-one.

Example 3.48 *Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} x_2 \\ x_1 \\ -x_1 - x_2 \end{bmatrix}$.*

Take an arbitrary $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$. We see that if $T(\vec{x}) = \vec{b}$, then $x_1 = b_2$, $x_2 = b_1$, and we also must have $-x_1 - x_2 = b_3$, which means that $b_3 = -b_1 - b_2$.
There are many examples of vectors for which $b_3 \neq -b_1 - b_2$, so T is clearly not onto. However, if we happen to have a vector $\vec{b}$ for which $b_3 = -b_1 - b_2$, then we can find $\vec{x} = \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}$. This is uniquely determined — there is only one such $\vec{x}$. Thus, this function is one-to-one.

Exercise 3.49 *Show that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is both one-to-one and onto if $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 2x_2 \\ -x_1 \end{bmatrix}$.*

Exercise 3.50 *In this exercise we prove Theorem 2.59. Assume $B = \{\vec{v}_1, \dots \vec{v}_d\}$ is a basis for the vector space V. Let $S = \{\vec{w}_1, \dots, \vec{w}_n\}$ where $n > d$. We will show that S is linearly dependent. Let $\vec{w}_i = a_{i1}\vec{v}_1 + \cdots + a_{d1}\vec{v}_d$.*

(a). Show that if $\sum x_i \vec{w}_i = \vec{0}$, then $\mathrm{A}\vec{x} = \vec{0}$ where the entries in A are the a_{ij} above.

(b). If $n > d$ show that A has columns that are not pivot columns.

(c). Show that there must be free variables and explain why this shows that S is linearly dependent.

3.5.1 The "Onto" game

Consider a game between two individuals,'Proposer' and 'Solver'. Take a given $m \times n$ matrix A. Proposer gives Solver some vector $\vec{b}$ of m entries and Solver needs to find <u>any</u> vector $\vec{x}$ such that $\mathrm{A}\vec{x} = \vec{b}$.

Solver wins if he/she can find an $\vec{x}$ such that $\mathrm{A}\vec{x} = \vec{b}$. Proposer wins if Solver does not find such an $\vec{x}$

If the transformation $T(\vec{x}) = \mathrm{A}\vec{x}$ is onto, then no matter what $\vec{b}$ Proposer chooses, Solver can find an $\vec{x}$ using Gaussian elimination. If not, then there is some $\vec{b}$ that Proposer can choose to guarantee a win. Can we figure out whether A represents an onto function? If A is not onto, can we figure out which vectors $\vec{b}$ have a solution?

The algorithm that Solver would use to solve the onto game is to create the augmented matrix $\mathrm{M} = [\mathrm{A} \mid \vec{b}]$ and convert it to echelon form.

Example 3.51 *Try to solve $\mathrm{A}\vec{x} = \vec{b}$ with $\mathrm{A} = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 1 & -4 \\ -2 & -5 & -12 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.*

To do this, we take the augmented matrix $\mathrm{M} = \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 2 & 1 & -4 & 1 \\ -2 & -5 & -12 & 1 \end{array}\right]$ and use Gaussian elimination to reduce it to echelon form.

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 2 & 1 & -4 & 1 \\ -2 & -5 & -13 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -2 & -8 & 0 \\ -2 & -5 & -13 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -2 & -8 & 0 \\ 0 & -2 & -8 & 2 \end{array}\right]$$
$$\mapsto \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -2 & -8 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

and so the final row represents $0 = 2$. There are no values of x_1, x_2, x_3 which make this true. So there is no solution.

What if $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$? Then the same steps lead to

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 2 & 1 & -4 & 1 \\ -2 & -5 & -13 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 0 & -2 & -8 & 1 \\ -2 & -5 & -13 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 0 & -2 & -8 & 1 \\ 0 & -2 & -8 & 1 \end{array}\right]$$
$$\mapsto \left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 0 & -2 & -8 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and there are infinitely many solutions: x_3 is free, $x_2 = -4x_3 - 1/2$, and $x_1 = -3x_2 - 4x_3 = 8x_3 + 3/2$. So the solutions are of the form

$$\vec{x} = \begin{bmatrix} 3/2 + 8x_3 \\ -1/2 - 4x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix} x_3$$

So at least for this matrix A, the choice of $\vec{b}$ by Proposer determines who wins.

How can Proposer choose the vector $\vec{b}$ to guarantee that Solver can't find a solution?

Example 3.52 *Let's put an unknown value for $\vec{b}$ in and try Gaussian elimination. We look for anywhere that we could make something go wrong by the appropriate choice of $\vec{b}$. The augmented matrix is $\mathrm{M} = \left[\begin{array}{ccc|c} 2 & 3 & 4 & b_1 \\ 2 & 1 & -4 & b_2 \\ -2 & -5 & -12 & b_3 \end{array}\right]$. We have*

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & b_1 \\ 2 & 1 & -4 & b_2 \\ -2 & -5 & -12 & b_3 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 2 & 3 & 4 & b_1 \\ 0 & -2 & -8 & b_2 - b_1 \\ -2 & -5 & -12 & b_3 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 2 & 3 & 4 & b_1 \\ 0 & -2 & -8 & b_2 - b_1 \\ 0 & -2 & -8 & b_3 + b_1 \end{array}\right]$$
$$\mapsto \left[\begin{array}{ccc|c} 2 & 3 & 4 & b_1 \\ 0 & -2 & -8 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - b_2 + 2b_1 \end{array}\right]$$

So Proposer can choose any vector $\vec{b}$ for which $b_3 - b_2 + 2b_1 \neq 0$.

What was the feature that allowed Proposer to be able to win? The fact that at the end of Gaussian elimination, when the augmented matrix M is put into echelon form, the A-part of the matrix has a row of just zero entries. All Proposer needs to do is guarantee that the corresponding entry in the $\vec{b}$ part of the echelon form is nonzero.

Let's just restrict our attention to performing Gaussian elimination on A rather than M. If there is a row of all zero entries in the echelon form of A, then there is a $\vec{b}$ for which there is no solution. If there is no such row, then there is always a solution.

Example 3.53 *Consider the matrix $\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 3 & 0 \end{bmatrix}$. Would you want to be Solver or Proposer for the "onto game" with this matrix?*

We can easily get it into echelon form, just by interchanging the two rows. We get $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Both rows of the echelon form have nonzero entries. Therefore, the transformation is onto, and Solver is guaranteed a winning strategy.

Putting the result of our game into more formal language:

Theorem 3.54 *The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{x}) = \mathrm{A}\vec{x}$ is onto iff every row of* A *has a pivot position.*

The "proof" of this theorem was really "proof by example". It does not constitute a complete argument, and would result in red ink on your homeworks. This can be made into a rigorous mathematical argument, but I'm not going to give it here.

Exercise 3.55 *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $T(\vec{x}) = \mathrm{A}\vec{x}$. If* A *has the following echelon form, is T onto (note, this is the echelon form for* A*, not the echelon form for the augmented matrix $[\mathrm{A}|\vec{b}]$)?*

(a). $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(b). $\begin{bmatrix} 2 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

(c). $\begin{bmatrix} 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(d). $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Exercise 3.56 *For the $m \times n$ matrix* A *given, find a $\vec{b}$ with m entries such that $\mathrm{A}\vec{x} = \vec{b}$ has no solution. Follow the method in example 3.52.*

(a). $\mathrm{A} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

(b). $\mathrm{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

(c). $\mathrm{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Exercise 3.57 *For the matrices in the previous exercise, if $\mathrm{A}\vec{x} = \vec{c}$, find a linear combination of the entries of $\vec{c}$ which is zero if and only if there is a solution. To make sure the problem statement is clear: for example 3.52, the answer would be $c_3 - c_2 + 2c_1 = 0$.*

Exercise 3.58 *Is it possible to find a $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{x}) = \mathrm{A}\vec{x}$ that is onto if*

(a). $m < n$

(b). $m = n$

(c). $m > n$

Hint: what is the maximum number of pivot positions A *can have?*

3.5.2 The one-to-one game

Now we consider a different game, which we call the "one-to-one" game. We have the same players: Proposer and Solver. The $m\times n$ matrix A is given. This time Proposer chooses a vector $\vec{p}$ (with n entries) and calculates $\vec{b} = \mathrm{A}\vec{p}$. The only information Solver has is A and $\vec{b}$. Solver wins if he/she can find the $\vec{p}$ that Proposer began with. Otherwise Proposer wins.

The way the game is designed, we know there is at least one solution to $\mathrm{A}\vec{x} = \vec{b}$, but could there be others? In other words, Solver is guaranteed to be able to find some $\vec{x}$ such that $\mathrm{A}\vec{x} = \vec{b}$. But is Solver guaranteed to find the $\vec{p}$ that Proposer started with?

Proposer should win when there are any $\vec{b}$ for which there are multiple solutions to $\mathrm{A}\vec{x} = \vec{b}$. That is if A represents a function that is not one-to-one, proposer should win. The challenge will be whether Proposer can find the right $\vec{b}$. It will turn out in this case that anything Proposer tries for $\vec{p}$ will give an acceptable $\vec{b}$. Conversely, Solver wins whenever the matrix represents a one-to-one function.

Example 3.59 *Who would win the one-to-one game with* $\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 3 & 0 \end{bmatrix}$*?*

It's fairly easy to see that whatever the third entry of $\vec{p}$ is that Proposer chooses, there is no way for Solver to determine what it is — information in $\vec{p}$ is lost when we transform it to $\vec{b}$. So if, for example, Proposer takes $\vec{p} = \begin{bmatrix} 1 \\ 1 \\ \pi \times 10^7 \end{bmatrix}$, *and then gives Solver* $\vec{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ *then Solver would have to make an incredibly lucky guess.*[3]

Let's put this through Gaussian elimination. We have $\mathrm{M} = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 5 \end{array}\right]$. *We get*

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 5 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 2 & 3 & 0 & 5 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

and we've got it in echelon form. The column corresponding to x_3 is <u>not</u> *a pivot column. This means that x_3 is free. It doesn't matter what we put in for $\vec{b}$, x_3 will remain free.*

The last row of the echelon form tells us that $x_2 = 1$. The other row says that $2x_1 = 5 - 3x_2 = 2$, so $x_1 = 1$. The lack of restriction on x_3 means that the solution set is

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_3$$

Example 3.60 *Who would win with* $\mathrm{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$*?*

If Proposer chooses $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$, *then we end up with* $\vec{b} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$. *You can probably see immediately that Solver will win, but just to go through the details, if we do Gaussian elimination we have* $\mathrm{M} = \left[\begin{array}{cc|c} 1 & 0 & p_1 \\ 0 & 1 & p_2 \end{array}\right]$ *and the system is already in echelon form. We find $x_2 = p_2$ and $x_1 = p_1$, so there is only one solution and Solver wins.*

Example 3.61 *Who would win with* $\mathrm{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix}$*?*

Again we take an arbitrary $\vec{p}$. When Solver calculates $\vec{b}$, Solver finds $\vec{b} = \begin{bmatrix} p_1 + 2p_2 + 3p_3 \\ 2p_1 + 3p_2 + 4p_3 \\ p_1 + p_2 + p_3 \end{bmatrix}$. *The*

[3] Did you know that $\pi \times 10^7$ is a very good approximation for the number of seconds in a year? It's actually 363.6... days.

augmented matrix is $\mathrm{M} = \left[\begin{array}{ccc|c} 1 & 2 & 3 & p_1+2p_2+3p_3 \\ 2 & 3 & 4 & 2p_1+3p_2+4p_3 \\ 1 & 1 & 1 & p_1+p_2+p_3 \end{array}\right]$. *Gaussian elimination gives*

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & p_1+2p_2+3p_3 \\ 2 & 3 & 4 & 2p_1+3p_2+4p_3 \\ 1 & 1 & 1 & p_1+p_2+p_3 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 2 & 3 & p_1+2p_2+3p_3 \\ 0 & -1 & -2 & -p_2-2p_3 \\ 1 & 1 & 1 & p_1+p_2+p_3 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 2 & 3 & p_1+2p_2+3p_3 \\ 0 & -1 & -2 & -p_2-2p_3 \\ 0 & -1 & -2 & -p_2-2p_3 \end{array}\right]$$

$$\mapsto \left[\begin{array}{ccc|c} 1 & 2 & 3 & p_1+2p_2+3p_3 \\ 0 & -1 & -2 & -p_2-2p_3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The first thing we might notice is that even though the A*-part has a row of zeros, we can still find a solution since the corresponding entry in the* $\vec{b}$*-part is zero. This is because of the fact that Proposer starts out with* $\vec{p}$ *and finds* $\vec{b} = \mathrm{A}\vec{p}$*. So we are guaranteed at least one solution to* $\mathrm{A}\vec{x} = \vec{b}$ *since we know* $\vec{p}$ *solves it.*
The next thing we notice is that x_3 *is free. We find* $x_2 = p_2 + 2p_3 - 2x_3$ *and*

$$\begin{aligned} x_1 &= p_1 + 2p_2 + 3p_3 - 2x_2 - 3x_3 \\ &= p_1 + 2p_2 + 3p_3 - 2p_2 - 4p_3 + 4x_3 - 3x_3 \\ &= p_1 - p_3 + x_3 \end{aligned}$$

So the full solution is

$$\vec{x} = \begin{bmatrix} p_1 - p_3 + x_3 \\ p_2 + 2p_3 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} p_1 - p_3 \\ p_2 + 2p_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} x_3$$

So there are infinitely many possible solutions. Unless Solver is very lucky, he/she will lose.

We can focus on Gaussian elimination of the matrix A rather than M. If there is a column of A which does not have a pivot position — that is it is not a pivot column — then the corresponding variable is free.

We have

Theorem 3.62 *If the matrix* A *has a column which is* <u>not</u> *a pivot column, and* $\mathrm{A}\vec{x} = \vec{b}$ *has a solution, then the variable corresponding to that column is free and there are infinitely many solutions.*

We also have

Theorem 3.63 *The transformation* $T(\vec{x}) = \mathrm{A}\vec{x}$ *is one-to-one if and only if every column of* A *is a pivot column.*

Let's look a little more at transformations that are not one-to-one. Let's take a matrix A as given and assume that $\mathrm{A}\vec{p} = \vec{b}$ and $\mathrm{A}\vec{y} = \vec{b}$. What can we learn about the relationship between $\vec{p}$ and $\vec{y}$?

$$\begin{aligned} \mathrm{A}\vec{y} &= \mathrm{A}\vec{p} \\ \mathrm{A}\vec{y} - \mathrm{A}\vec{p} &= \vec{0} \\ \mathrm{A}(\vec{y} - \vec{p}) &= \vec{0} \\ \mathrm{A}(\vec{w}) &= \vec{0} \end{aligned}$$

where $\vec{w} = \vec{y} - \vec{p}$. So if $\vec{y} \neq \vec{p}$, then there are nonzero solutions to $\mathrm{A}\vec{w} = \vec{0}$. So $\vec{y} = \vec{p} + \vec{w}$ where $\vec{w}$ is a solution to $\mathrm{A}\vec{w} = \vec{0}$. That is $\vec{w} \in \mathrm{Nul}(\mathrm{A})$. In fact we can make this slightly stronger. Not only does $\vec{y}$ have this form, but everything of this form is a solution. If $\vec{w} \in \mathrm{Nul}(\mathrm{A})$ and $\mathrm{A}\vec{y} = \vec{b}$, then $\mathrm{A}(\vec{y}+\vec{w}) = \mathrm{A}\vec{y} + \mathrm{A}\vec{w} = \mathrm{A}\vec{y} + \vec{0} = \vec{b}$.

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Theorem 3.64 *Let* $\vec{p}$ *be a solution to the equation* $\mathrm{A}\vec{x} = \vec{b}$*. Then* $\vec{x}$ *is a solution of* $\mathrm{A}\vec{x} = \vec{b}$ *iff*

$$\vec{x} = \vec{p} + \vec{w}$$

Where $\vec{w} \in \mathrm{Nul}(\mathrm{A})$ *(the nullspace of* A*).*

So we see it will be important to be able to find the nullspace of the matrix.

Exercise 3.65 *Show that by rewriting x_3 in terms of p_3, any solution of the form at the end of example 3.61 can be written as* $\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \vec{w}$ *where $\vec{w}$ is any vector in* $\mathrm{Nul(A)}$.

Our theorem should look familiar from Calculus and the examples at the begining of Chapter 2. If $p(x)$ is a known solution to $y'(x) = g(x)$, and we are looking for all solutions, they can be written $y(x) = p(x) + C$ where C is a constant. Notice that $C' = 0$. Since $\mathrm{Nul}(d/dx)$ is the set of constants,[4] the analog of the theorem is true for derivatives as well as matrices.

Example 3.66 *Assume we are trying to solve $f'(x) = x^2$. We know that $x^3/3$ is one solution. The set of all solutions is the set of functions of the form $x^3/3 + C$ where C is a constant.*

Exercise 3.67 *We have shown already that the derivative is a linear transformation whose domain is the differentiable functions. For notational convenience, denote the derivative of $f(x)$ by $\mathrm{D}f(x)$. Following the steps above Theorem 3.64, show that if $\mathrm{D}f_1(x) = g(x)$ and $\mathrm{D}f_2(x) = g(x)$, then $f_1(x) - f_2(x)$ is a constant.*

Exercise 3.68 *Let $T : X \to Y$ be a linear transformation from the vector space X to the vector space Y. Let $\vec{b} \in Y$ be given. If $\vec{p} \in X$ is a solution to $T(\vec{x}) = \vec{b}$, show that $\vec{x}$ is a solution of of $T(\vec{x}) = \vec{b}$ iff $\vec{x} = \vec{p} + \vec{w}$ where $T(\vec{w}) = \vec{0}$.*

Does $\mathrm{A}\vec{x} = \vec{0}$ always have at least one solution? Yes, the vector $\vec{x} = \vec{0}$ is a solution (Note that if A is $m \times n$, then when we say $\vec{x} = \vec{0}$ we understand that $\vec{0}$ here has n entries, but when we say $\mathrm{A}\vec{x} = \vec{0}$, we understand that $\vec{0}$ has m entries. The two $\vec{0}$'s mean different things).

So Solver is guaranteed a win if the only solution to $\mathrm{A}\vec{x} = \vec{0}$ is the trivial solution $\vec{x} = \vec{0}$. Otherwise, as long as Proposer chooses something too obscure for Solver to randomly guess, Proposer wins.

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Example 3.69 *Consider the matrix* $\mathrm{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}$. *This matrix has two rows that do not have pivot positions and two columns that do not have pivot positions. Therefore this matrix is neither onto nor one-to-one.*

We summarize what we have learned for $\mathrm{A}\vec{x} = \vec{b}$, where A is $m \times n$. First note that A cannot have more pivot positions than it has either rows or columns. So it has at most $\min\{m, n\}$ pivot positions.

Theorem 3.70 *If every row of* A *has a pivot position, then there is at least one solution for any $\vec{b}$. In other words, if* A *has m pivot positions, there is at least one solution for every $\vec{b}$.*

If $m > n$, then it is not possible for A to have m pivot positions. So if $m > n$, then there are at least some $\vec{b}$ for which no solution exists.

Theorem 3.71 *If every column of* A *has a pivot position, then there is at most one solution for any $\vec{b}$ (there may be no solution if some row of* A *has no pivot position and $\vec{b}$ is chosen appropriately). In other words, if* A *has n pivot positions, there is at most one solution for any $\vec{b}$. Otherwise there are either* 0 *solutions or infinitely many.*

If $n > m$, then it is not possible for A to have n pivot positions. So if $n > m$, and there is a solution to $\mathrm{A}\vec{x} = \vec{b}$, then there are infinitely many solutions.

[4]A bad Null Space joke: e^x was walking down the street and saw 1 running in the opposite direction, with a horrified expression. "What's wrong?" e^x asked. "The derivative is coming, the derivative is coming!!" shouted 1. Of course, for the function 1, being caught by the derivative would be terrible since it would be turned into 0. e^x continued serenely on. Next e^x saw x and x^2 both running past, shouting, "the derivative is coming!". Again, e^x quietly laughed to himself and stopped to idly look in a store window as other functions ran past. Suddenly e^x froze, terrified — reflected behind him in the window was $\mathrm{d}/\mathrm{d}y$.

This leads to the conclusion:

Theorem 3.72 *If* A *is square* ($m = n$) *then* A *has* n *pivot positions iff there is exactly one solution to* $\mathrm{A}\vec{x} = \vec{b}$ *for every* $\vec{b}$.

We can summarize these theorems as follows for $T(\vec{x}) = \mathrm{A}\vec{x}$:

	T is onto	T is not onto
T is one-to-one	Every row and column of A has a pivot position. A must be square.	Every column of A has a pivot position, but some row does not.
T is not one-to-one	Every row of A has a pivot position, but some column does not.	Some row does not have a pivot position and some column does not.

Exercise 3.73 *Let* A *be a given matrix. What do we know about the pivot locations if*

(a). A *is onto but not one-to-one.*

(b). A *is one-to-one but not onto.*

(c). A *is both one-to-one and onto.*

(d). A *is neither one-to-one nor onto.*

Exercise 3.74 *Give an example of an echelon form for a matrix* A *if the transformation* $T(\vec{x}) = \mathrm{A}\vec{x}$ *is*

(a). onto but not one-to-one.

(b). one-to-one but not onto.

(c). both one-to-one and onto.

(d). neither one-to-one nor onto.

3.6 Column Space and Null Space

~~The column space and null space were introduced in chapter 2. For a matrix A, the column space is the set of vectors $\vec{b}$ that can be written in the form $\mathrm{A}\vec{x}$. The null space is the set of vectors $\vec{w}$ for which $\mathrm{A}\vec{w} = \vec{0}$.~~

3.6.1 Column Space

~~By definition $\vec{b} \in \mathrm{Col(A)}$ iff $\vec{b} = \mathrm{A}\vec{x}$ for some $\vec{x}$. If $\mathrm{A} = [\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_n]$ then we know that $\mathrm{A}\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$. This means that the column space of A is in fact the span of the columns of A. This is the source of the name.~~

Chapter 4

Invertible Matrices

4.1 Important Concepts

After this chapter you should:

- Be able to calculate the inverse of A if it exists.
- Know theorem 4.20 which gives many properties of matrices that are equivalent to being invertible.
- Really know theorem 4.21 which gives the most commonly used equivalent properties to being invertible, at least in my experience.
- Be able to calculate the determinant of an $n \times n$ matrix for n up to 3.
- Be able to calculate the determinant of an arbitrary $n \times n$ matrix if some rows or columns have many zeros in them.

4.2 Introduction

Definition 4.1 *The $n \times n$ matrix A is an* ***invertible matrix*** *if there is a matrix B such that $\mathrm{BA} = \mathrm{I}$.*

Definition 4.2 *If A is invertible, the matrix B such that $\mathrm{BA} = \mathrm{I}$ is called the* ***inverse*** *of A. It is usually denoted A^{-1}.*

We haven't yet shown that there is one and only one matrix B such that this works. We also haven't shown that if $\mathrm{BA} = \mathrm{I}$ then $\mathrm{AB} = \mathrm{I}$. Both of these are true, but showing them will come later.

If we we already know the inverse to A, then to solve $\mathrm{A}\vec{x} = \vec{b}$, we simply multiply by A^{-1} on the left hand side of both sides of the equation, getting $\mathrm{I}\vec{x} = \mathrm{A}^{-1}\vec{b}$, which simplifies to $\vec{x} = \mathrm{A}^{-1}\vec{b}$. This is really where the term "inverse" comes from. If we multiply $\vec{x}$ by A to get $\vec{b}$, multiplication by A^{-1} inverts this to get us back to $\vec{x}$.

Multiplication by A^{-1} is a much faster calculation than Gaussian Elimination, but it requires that we know the inverse of A. In general, finding the inverse of A is a slower process than Gaussian Elimination. So there is a tradeoff. If we have a fixed matrix A for some application and we frequently need to solve $\mathrm{A}\vec{x} = \vec{b}$ for different vectors $\vec{b}$, then it's a good idea to calculate A^{-1}. On the other hand, if we are only going to use a given A a few times, just using Gaussian Elimination is faster.

Exercise 4.3 *For each of the following, find $\vec{x}$ such that $\mathrm{A}\vec{x} = \vec{b}$.*

(a). $\mathrm{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. *Note* $\mathrm{A}^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$.

(b). $\mathrm{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. *Note* $\mathrm{A}^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$.

(c). $\mathrm{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$. *Note* $\mathrm{A}^{-1} = \begin{bmatrix} -1/4 & 3/4 & 1/4 \\ 1/4 & 1/4 & -1/4 \\ 3/4 & -1/4 & 1/4 \end{bmatrix}$.

4.3 Finding inverses

We want an algorithm to calculate the inverse of an invertible matrix.

First we go back to Gaussian Elimination and extend the process somewhat.

Definition 4.4 *A matrix is in* ***Reduced Echelon Form*** *if it is in echelon form, all pivot entries are* 1*, and all non-pivot entries in a pivot column are* 0*.*

Unlike echelon form, the reduced echelon form is unique.

Finding reduced echelon form We can transform a matrix into reduced echelon form by first reducing it to echelon form, following the algorithm in section 3.3.1 of these notes. Once the matrix is in echelon form, rescale each row that has a pivot so that the entry in the pivot position is 1. Then subtract appropriate multiples of this row from the rows above it to remove all nonzero entries.

Example 4.5 *Convert the matrix* $\mathrm{A} = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 \\ 6 & 6 & 6 & 4 \end{bmatrix}$ *to reduced echelon form*

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 \\ 6 & 6 & 6 & 4 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & -3 & -6 & -9 \\ 6 & 6 & 6 & 4 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & -3 & -6 & -9 \\ 0 & -3 & -6 & -6 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & -3 & -6 & -9 \\ 0 & -3 & -6 & -6 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & -3 & -6 & -9 \\ 0 & 0 & 0 & 3 \end{bmatrix} \mapsto$$

$$\mapsto \begin{bmatrix} 1 & 3/2 & 2 & 5/2 \\ 0 & -3 & -6 & -9 \\ 0 & 0 & 0 & 3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 3/2 & 2 & 5/2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 3/2 & 2 & 5/2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 3/2 & 2 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto$$

$$\mapsto \begin{bmatrix} 1 & 3/2 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Each pivot entry is 1 *and every other entry in a pivot column is zero. Notice that in columns which are not pivot columns, there may be more than one nonzero number.*

Exercise 4.6 *For each of the following, convert to echelon form and then find the reduced echelon form. For the matrices that are square (same number of rows and columns), perform exactly the same row operations on the square identity matrix* I *with the same number of rows, and call the result* B*. What is* BA*?*

(a). $\mathrm{A} = \begin{bmatrix} 5 & 2 \\ 10 & -2 \end{bmatrix}$

(b). $\mathrm{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c). $\mathrm{A} = \begin{bmatrix} -1 & 2 & -1 \\ 4 & -4 & 2 \\ -1 & 4 & -2 \end{bmatrix}$

(d). $\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 4 & 0 & 2 \\ 0 & 2 & 0 & 4 \end{bmatrix}$

(e). $\mathrm{A} = \begin{bmatrix} 2 & 2 & 0 & 4 & 2 \\ 0 & 0 & 2 & 3 & 1 \\ 1 & 2 & -2 & 2 & 4 \end{bmatrix}$

I will present the algorithm to invert a matrix and then explain why it works.

Earlier, when we were solving linear systems of equations of the form $\mathrm{A}\vec{x} = \vec{b}$, we took the augmented matrix $\mathrm{M} = [\mathrm{A} \mid \vec{b}]$ and performed Gaussian elimination. Now we do much the same with two changes: instead of augmenting the matrix A with just a vector, we augment it with the identity matrix I, and instead of just getting the matrix into echelon form, we go all the way to reduced echelon form.

algorithm to invert a matrix To invert the matrix A:

(a). Start with the augmented matrix $\mathrm{M} = [\mathrm{A} \mid \mathrm{I}]$.

(b). Perform row operations to get M into echelon form.

(c). Scale every row so that the entries in pivot positions are all 1.

(d). Perform row operations to make all entries in the pivot columns 0 (except for the pivot positions which remain 1).

If A is invertible, this converts the matrix $\mathrm{M} = [\mathrm{A} \mid \mathrm{I}]$ to $[\mathrm{I} \mid \mathrm{B}]$ where $\mathrm{BA} = \mathrm{I}$. If A is not invertible, it will not be possible to convert the A part to I.

This algorithm works because for each row operation, there is a matrix which encodes that operation. For example, given a $4 \times n$ matrix M, if we multiply the second row by 3, the result is the same as given by EM where $\mathrm{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. To interchange the second and fourth row, we would multiply by $\mathrm{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. To add twice the first row to the third row, we would multiply by $\mathrm{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Definition 4.7 *A matrix* E *is an* ***elementary matrix*** *if* E *is obtained by performing a row operation to the identity matrix* I.

If we take a matrix A and multiply it by E, then EA is the matrix that would result from performing the row operation associated with E to A.

So if we do r row operations to the matrix $\mathrm{M} = [\mathrm{A} \mid \mathrm{I}]$ to get the matrix $[\mathrm{I}|\mathrm{B}]$, what we are really doing is multiplying M by r elementary matrices E_1, E_2, $\ldots$, E_r. What we have is

$$\begin{aligned}
\mathrm{M} &= [\mathrm{A} \mid \mathrm{I}] \\
\mapsto \mathrm{E}_1\mathrm{M} &= \mathrm{E}_1[\mathrm{A} \mid \mathrm{I}] = [\mathrm{E}_1\mathrm{A} \mid \mathrm{E}_1] \\
\mapsto \mathrm{E}_2\mathrm{E}_1\mathrm{M}\mathrm{E}_2\mathrm{E}_2[\mathrm{A} \mid \mathrm{I}] &= [\mathrm{E}_2\mathrm{E}_1\mathrm{A} \mid \mathrm{E}_2\mathrm{E}_1] \\
&\vdots \\
\mapsto \mathrm{E}_r \cdots \mathrm{E}_2\mathrm{E}_1\mathrm{M} = \mathrm{E}_r \cdots \mathrm{E}_2\mathrm{E}_1[\mathrm{A} \mid \mathrm{I}] &= [\mathrm{I} \mid \mathrm{E}_r \cdots \mathrm{E}_2\mathrm{E}_1] = [\mathrm{I} \mid \mathrm{B}]
\end{aligned}$$

where $\mathrm{B} = \mathrm{E}_r \cdots \mathrm{E}_2\mathrm{E}_1$. So by starting with I in M, our algorithm actually calculates the product $\mathrm{E}_r \cdots \mathrm{E}_2\mathrm{E}_1$. Notice the order of the elementary matrices.

The argument we used to explain why our inverting algorithm works shows that the matrix B that we find is such that $\mathrm{BA} = \mathrm{I}$. Since matrix multiplication isn't commutative, it's not at all obvious that $\mathrm{AB} = \mathrm{I}$ as well. So maybe we've just found an algorithm to find a "right-inverse", but not a "left-inverse".[1]

Here are two arguments that $\mathrm{BA} = \mathrm{I}$, that is, B is a "left-inverse" as well as a "right-inverse".

- The first relies on a simple observation — each row operation can be undone by a row operation. If we can go from $[\mathrm{A} \mid \mathrm{I}]$ to $[\mathrm{I} \mid \mathrm{B}]$ using row operations, then it is possible to go from $[\mathrm{B} \mid \mathrm{I}]$ to $[\mathrm{I} \mid \mathrm{A}]$ by reversing all the operations. This means that if B is a left inverse of A, then we can show that A is a left inverse of B. So $\mathrm{BA} = \mathrm{I} = \mathrm{AB}$, and thus B is a right inverse of A. So the left inverse we found is also a right inverse (and similarly we can conclude that any right inverse is also a left inverse).
- (if you understand the previous argument it's not essential you understand this one, but it may help give some insight) The second argument relies on noting that if we take $\mathrm{A}\vec{x} = \vec{e}_i$ where $\vec{e}_i$ is the i-th canonical basis vector, one way to solve this is by augmenting A with $\vec{e}_i$ and using row operations to change A into I. This isn't possible for all matrices A, but if it is possible, then we've gotten our augmented matrix all the way to $[\mathrm{I}|\vec{x}_i]$, and we can conclude that $\mathrm{A}\vec{x}_i = \vec{e}_i$. Now if instead of augmenting A with just $\vec{e}_i$, we augment it with I, this is like trying to solve $\mathrm{A}\vec{x}_i = \vec{e}_i$ simultaneously. When we've transformed the A part of $[\mathrm{A}|\mathrm{I}]$ all the way to I, each column $\vec{x}_i$ of the part that was originally I is now a solution to $\mathrm{A}\vec{x}_i = \vec{e}_i$. So that means that $\mathrm{AB} = \mathrm{A}[\vec{x}_1\,\vec{x}_2 \cdots \vec{x}_n] = [\mathrm{A}\vec{x}_1\,\mathrm{A}\vec{x}_2 \cdots \mathrm{A}\vec{x}_n] = [\vec{e}_1\,\vec{e}_2 \cdots \vec{e}_n] = \mathrm{I}$.

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Example 4.8 *Find the inverse of* $\mathrm{A} = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 4 & 0 \\ 4 & 0 & 6 \end{bmatrix}$.

We augment A *with* I *to get* $\mathrm{M} = [\mathrm{A}\,|\,\mathrm{I}] = \left[\begin{array}{ccc|ccc} 0 & 2 & 2 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ 4 & 0 & 6 & 0 & 0 & 1 \end{array}\right]$. *We then tranform this into reduced echelon form*

$$\left[\begin{array}{ccc|ccc} 0 & 2 & 2 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ 4 & 0 & 6 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 2 & 4 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 \\ 4 & 0 & 6 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 2 & 4 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & -8 & 6 & 0 & -2 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 2 & 4 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 14 & 4 & -2 & 1 \end{array}\right]$$

$$\mapsto \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 2/7 & -1/7 & 1/14 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 3/14 & 1/7 & -1/14 \\ 0 & 0 & 1 & 2/7 & -1/7 & 1/14 \end{array}\right]$$

$$\mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/7 & 3/14 & 1/7 \\ 0 & 1 & 0 & 3/14 & 1/7 & -1/14 \\ 0 & 0 & 1 & 2/7 & -1/7 & 1/14 \end{array}\right]$$

[1] there are mathematical constructions where left and right inverses can be different things, so this isn't a ridiculous question, but matrices don't do this.

So $\mathrm{B} = \begin{bmatrix} -3/7 & 3/14 & 1/7 \\ 3/14 & 1/7 & -1/14 \\ 2/7 & -1/7 & 1/14 \end{bmatrix}$.

To check

$$\mathrm{BA} = \begin{bmatrix} (-3/7)0+(3/14)2+(1/7)4 & (-3/7)2+(3/14)4+(1/7)0 & (-3/7)2+(3/14)0+(1/7)6 \\ (3/14)0+(1/7)2+(-1/14)4 & (3/14)2+(1/7)4+(-1/14)0 & (3/14)2+(1/7)0+(-1/14)6 \\ (2/7)0+(-1/7)2+(1/14)4 & (2/7)2+(-1/7)4+(1/14)0 & (2/7)2+(-1/7)0+(1/14)6 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So indeed $\mathrm{BA} = \mathrm{I}$.

You should always check this. Odds are good that when you're inverting a matrix by hand an error will appear.

Exercise 4.9 *Find the inverse of the following matrices:*

(a). $\mathrm{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b). $\mathrm{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 0 & 3 \end{bmatrix}$

(c). $\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

Example 4.10 *Here's an example demonstrating that reversing the steps that take* $[\mathrm{A}|\mathrm{I}]$ *to* $[\mathrm{I}|\mathrm{B}]$ *we can take* $[\mathrm{B}|\mathrm{I}]$ *to* $[\mathrm{I}|\mathrm{A}]$.

We invert the steps of example 4.8.

$$[\mathrm{B}\,|\,\mathrm{I}] = \left[\begin{array}{ccc|ccc} -3/7 & 3/14 & 1/7 & 1 & 0 & 0 \\ 3/14 & 1/7 & -1/14 & 0 & 1 & 0 \\ 2/7 & -1/7 & 1/14 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 0 & 1/2 & 0 & 1 & 2 & 0 \\ 3/14 & 1/7 & -1/14 & 0 & 1 & 0 \\ 2/7 & -1/7 & 1/14 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 0 & 1/2 & 0 & 1 & 2 & 0 \\ 1/2 & 0 & 0 & 0 & 1 & 1 \\ 2/7 & -1/7 & 1/14 & 0 & 0 & 1 \end{array}\right]$$
$$\mapsto \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 2 & 4 & 0 \\ 1 & 0 & 0 & 0 & 2 & 2 \\ 4 & -2 & 1 & 0 & 0 & 14 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 2 & 4 & 0 \\ 1 & 0 & 0 & 0 & 2 & 2 \\ 0 & -2 & 1 & 0 & -8 & 6 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 2 & 4 & 0 \\ 1 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 4 & 0 & 6 \end{array}\right]$$
$$\mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 4 & 0 & 6 \end{array}\right] = [\mathrm{I}\,|\,\mathrm{A}]$$

This means that the B *we find by this process satisfies* $\mathrm{BA} = \mathrm{AB} = \mathrm{I}$.

To show there is only one inverse, we imitate steps from homework 2.19. Assume $\mathrm{BA} = \mathrm{I} = \mathrm{AC}$. We will prove that $\mathrm{B} = \mathrm{C}$. Consider BAC. This is equal to both IC and BI. Since these in turn equal C and B, we have $\mathrm{B} = \mathrm{C}$.

Since there is only one inverse, and it doesn't matter whether we multiply A on the right or the left, we don't have to worry about any ambiguity when we talk about <u>the</u> inverse of A. This is what allows us to write A^{-1} without worrying whether our intent is clear. Note, however, that not every matrix is invertible, so we can only write A^{-1} for those matrices which are invertible.

Exercise 4.11 *Consider* $\mathrm{A} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ *and solve* $\mathrm{A}\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$*:*

(a). using Gaussian Elimination

(b). by finding A^{-1} *and calculating* $\vec{x} = \mathrm{A}^{-1}\vec{b}$*. If you don't find the same* $\vec{x}$ *as before there is an error.*

Presumably you found that calculating A^{-1} *was more effort.*

(c). Now solve $\mathrm{A}\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ *by either using Gaussian Elimination or the* A^{-1} *you've already calculated above.*

Exercise 4.12 *For the matrices below, solve* $\mathrm{A}\vec{x} = \vec{b}$ *by using Gaussian Elimination to find echelon form (not all the way to reduced echelon form). Then find* A^{-1} *and solve* $\mathrm{A}\vec{x} = \vec{b}$ *by noting* $\vec{x} = \mathrm{A}^{-1}\vec{b}$*. Does it require more effort to calculate* A^{-1} *or to do Gaussian elimination?*

(a). $\mathrm{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b). $\mathrm{A} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Exercise 4.13 *Assuming that* $ad - bc \neq 0$*, and* $\mathrm{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(a). Find A^{-1} *if* $a \neq 0$

(b). Find A^{-1} *if* $a = 0$*.*

This is a useful "formula" to remember.

I want to emphasize that we need to be careful with how we multiply inverses.

Example 4.14 *Take* $\mathrm{A} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ *and* $\mathrm{B} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$ *(Although it's not important for this problem, it's worth noticing that the columns of* B *are linearly dependent, so* B *is not invertible).*
Does $\mathrm{ABA}^{-1} = \mathrm{B}$*?*
It's natural to think that it does. But the reason it's so appealing is that you're used to multiplication commuting and we've seen that with matrices this isn't necessarilly the case. It happens that A *and* A^{-1} *commute with one another, but that's not good enough to say they commute with* B*. So let's check out what happens.*
First let's find A^{-1}*:*

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

So $\mathrm{A}^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$*. It's always a good idea to test that you haven't made an error.* $\mathrm{A}^{-1}\mathrm{A} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3-2 & 3-3 \\ -2+1 & -2+3 \end{bmatrix} = \mathrm{I}.$

So going back to the original problem, what is ABA^{-1}*? It's*

$$\begin{aligned}\mathrm{ABA}^{-1} &= \begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & -2\end{bmatrix}\begin{bmatrix}3 & -1\\ -2 & 1\end{bmatrix}\\ &= \begin{bmatrix}1+2 & -1-2\\ 2+6 & -2-6\end{bmatrix}\begin{bmatrix}3 & -1\\ -2 & 1\end{bmatrix} = \begin{bmatrix}3 & -3\\ 8 & -8\end{bmatrix}\begin{bmatrix}3 & -1\\ -2 & 1\end{bmatrix}\\ &= \begin{bmatrix}9+6 & -3-3\\ 24+16 & -8-8\end{bmatrix} = \begin{bmatrix}15 & -6\\ 40 & -16\end{bmatrix}\\ &\neq \mathrm{B}\end{aligned}$$

So clearly we need to be careful how we handle this multiplication.

When we take a product of two invertible matrices, its inverse is the product of the inverses of the two matrices, but in the reversed direction:

Exercise 4.15 *Assuming* A *and* B *are invertible, show that* $(\mathrm{AB})^{-1} = \mathrm{B}^{-1}\mathrm{A}^{-1}$. *[hint: this isn't as hard as it looks. If you're doing a lot to do this, look for an easier method]*

Intuitively, this makes sense if we think about each matrix as representing some operation to perform on a vector. Let $\vec{x}$ be an arbitrary vector. If we multiply it by B and then A we've done whatever the B operation is and then we've done the A operation. To undo this, it's like putting on and taking off shoes and socks. You put on socks then shoes. To undo that you don't take off your socks and then your shoes. You deal with them in the opposite order you initially did: shoes off, then socks off.

4.3.1 Matrix calculations with invertible matrices

Example 4.16 *Given that* A *and* B *are invertible: Solve for* C *if we know*

$$\mathrm{ACBA}^{-1} = \mathrm{A} + \mathrm{B}$$

Our previous example shows that we can't just cancel out the A *and* A^{-1} *or the same for* B*. All that we really have to work with is the knowledge that* $\mathrm{AA}^{-1} = \mathrm{A}^{-1}\mathrm{A} = \mathrm{I}$. *and similar for* B*.*
Let's start by getting rid of the A *on the left of the left hand side. We'd like to "divide" by* A*, but really that's multiplication by the inverse of* A*. We get*

$$\mathrm{A}^{-1}\mathrm{ACBA}^{-1} = \mathrm{A}^{-1}(\mathrm{A} + \mathrm{B})$$

and the left hand simplifies to CBA^{-1}*. We could distribute the* A^{-1} *through the sum on the right hand side if we wanted, but for now let's leave it alone.*
We have $\mathrm{CBA}^{-1} = \mathrm{A}^{-1}(\mathrm{A} + \mathrm{B})$*. We get rid of the* A^{-1} *on the left hand side by multiplying by* A *on the right of both sides. We get*

$$\mathrm{CB} = \mathrm{A}^{-1}(\mathrm{A} + \mathrm{B})\mathrm{A}$$

Multiplying both sides by B^{-1} *on the right gives*

$$\mathrm{C} = \mathrm{A}^{-1}(\mathrm{A} + \mathrm{B})\mathrm{AB}^{-1}$$

We can modify this slightly to get

$$\mathrm{C} = (\mathrm{I} + \mathrm{A}^{-1}\mathrm{B})\mathrm{AB}^{-1}$$

or get

$$\mathrm{C} = \mathrm{AB}^{-1} + \mathrm{A}^{-1}\mathrm{BAB}^{-1}$$

Any of these expressions is equally good.

Exercise 4.17 *Given that* A *and* B *are invertible, solve for* C *if*

(a). $\mathrm{ACB}^{-1} = \mathrm{B}^{-1} + \mathrm{I}$

(b). $\mathrm{A}^{-3}\mathrm{BACA}^{3} = \mathrm{A} + \mathrm{A}^{-1} + \mathrm{I}$ *[note* $\mathrm{A}^{-n} = (\mathrm{A}^{-1})^{n}$*]*

4.4 The Invertible Matrix Theorem

Example 4.18 *Try to invert* $\mathrm{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$. *We take* $\mathrm{M} = [\mathrm{A} \mid \mathrm{I}]$ *and begin to find echelon form.*

$$\mathrm{M} = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

When we begin row reductions we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -1 & 1 & -2 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

It's going to be impossible to get the A *part of this to become* I *using row reductions. Why? Because there aren't enough pivots.*

A technical detail: what we've seen basically says that if A does not have enough pivots, then we can't use row reductions to find its inverse. We might hold out hope that if we tried some other method, we could find its inverse.[2] There is in fact no hope. If it doesn't have enough pivots it has no inverse.

Exercise 4.19 *Let* A *be an* $n \times n$ *matrix. Show that if* A *has fewer than* n *pivots, then it has no inverse by using the following steps (without using the invertible matrix theorem):*

(a). Explain why if A *has fewer than* n *pivots, there is a vector* $\vec{v} \neq \vec{0}$ *such that* $\mathrm{A}\vec{v} = \vec{0}$.

(b). Assume that $\mathrm{BA} = \mathrm{I}$: *What can we conclude about* $\vec{v}$ *by multiplying each side of* $\mathrm{A}\vec{v} = \vec{0}$ *by* B *on the left?*

(c). Why does this show that if A *can be inverted then* A *has* n *pivots?*

So we've learned that if an $n \times n$ matrix does not have n pivots it cannot be inverted. Carefully analyzing the algorithm, we could also show that if it does have n pivots, it can be inverted.

So the $n \times n$ matrix A is invertible if and only if it has n pivots. We've already learned that this means that it is both one-to-one and onto.

We've already started to see some properties of invertible matrices. Here is a large collection of them

Theorem 4.20 *If* A *is an* $\underline{n \times n \text{ matrix}}$, *the following are equivalent (that is, all are true or all are false)*

(a). $\mathrm{A}\vec{x} = \vec{b}$ *has exactly one solution* $\vec{x}$ *for every* $\vec{b} \in \mathbb{R}^n$

(b). $T(\vec{x}) = \mathrm{A}\vec{x}$ *is one-to-one and onto*

(c). A *has exactly* n *pivot positions*

(d). The transformation $T(\vec{x}) = \mathrm{A}\vec{x}$ *is onto*

(e). The transformation $T(\vec{x}) = \mathrm{A}\vec{x}$ *is one-to-one*

(f). $\mathrm{Nul}(\mathrm{A}) = \{\vec{0}\}$ *(that is* $\mathrm{A}\vec{x} = \vec{0}$ *has only the trivial solution)*

[2] For example if you were asked to factor $x^2 + 5x + 6$, you would probably look for integers that factor into 6 and add to 5. You'd come up with $(x+3)(x+2)$. If you tried this algorithm on $x^2 - x - 1$, it would fail. That doesn't mean you can't factor $x^2 - x - 1$, but you need another algorithm.

(g). $\dim \mathrm{Nul}(\mathrm{A}) = 0$

(h). The columns of A *are linearly independent.*

(i). There is at most one solution to $\mathrm{A}\vec{x} = \vec{b}$ *for every* $\vec{b} \in \mathbb{R}^n$

(j). There is some vector $\vec{b}$ *such that* $\mathrm{A}\vec{x} = \vec{b}$ *has exactly one solution.*

(k). $\mathrm{Col}(\mathrm{A}) = \mathbb{R}^n$

(l). $\dim \mathrm{Col}(\mathrm{A}) = n$

(m). There is at least one solution to $\mathrm{A}\vec{x} = \vec{b}$ *for every* $\vec{b} \in \mathbb{R}^n$

(n). There is a matrix B *such that* $\mathrm{BA} = \mathrm{I}$ *(*A *is invertible)*

(o). There is a matrix C *such that* $\mathrm{AC} = \mathrm{I}$

(p). A *is row equivalent to* I

(q). A^T *is invertible*

(r). The rows of A *are linearly independent*

(s). $\det A \neq 0$

(t). A *does not have a zero eigenvalue.*

The last two points are things we have not yet talked about yet.

Obviously from the statement of the theorem about invertible matrices, if any of these are false, all are false. However, the following seem to be the most common things that we care about when A is not invertible:

Theorem 4.21 *For an* $n \times n$ *matrix* A *the following are equivalent:*

(a). A *is NOT invertible*

(b). $\det(\mathrm{A}) = 0$

(c). Some row of A *may be written as a linear combination of the other rows.*

(d). Some column of A *may be written as a linear combination of the other columns.*

(e). There is a nontrivial solution to $\mathrm{A}\vec{x} = \vec{0}$

(f). The solution to $\mathrm{A}\vec{x} = \vec{b}$ *may not exist, but if it does, there are infinitely many solutions.*

Example 4.22 *It is easy to show that the columns of* $\mathrm{A} = \begin{bmatrix} 1 & 3 & 13 \\ 4 & 5 & 3 \\ 5 & 8 & 16 \end{bmatrix}$ *are linearly dependent.*

The sum of the first two rows is the third. Therefore the rows are linearly dependent. Therefore A *is not invertible. Therefore the columns are linearly dependent.*

Exercise 4.23 *Consider the product* AB *where* A *and* B *are both* $n \times n$*.*

(a). If B *is not invertible, show that there is a vector* $\vec{v} \neq \vec{0}$ *such that* $\mathrm{B}\vec{v} = \vec{0}$*. Calculate* $\mathrm{AB}\vec{v}$ *and explain why this proves* AB *is not invertible if* B *is not invertible.*

(b). Now assume B *is invertible. If* A *is not invertible, there is a vector* $\vec{w}$ *such that* $A\vec{w} = \vec{0}$*. Show that there is a* $\vec{v}$ *such that* $\vec{w} = B\vec{v}$*. Calculate* $AB\vec{v}$ *and explain why this proves* AB *is not invertible if* A *is not invertible.*

So if A and B are square matrices then AB is invertible iff A and B are both invertible. More generally, the product of many square matrices is invertible iff all of them are invertible.

Exercise 4.24 *In exercise 2.57 we showed that for the matrix* $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$*, the basis* $\vec{v}_1 = \begin{bmatrix} 1 \\ \phi \end{bmatrix}$ *and* $\vec{v}_2 = \begin{bmatrix} 1 \\ 1-\phi \end{bmatrix}$ *has the property that* $A\vec{v}_1 = \phi\vec{v}_1$ *and* $A\vec{v}_2 = (1-\phi)\vec{v}_2$*. We will denote this basis by* B*. In this problem we introduce the concept of* <u>*diagonalization*</u>*.*

(a). Explain why the fact that B *is a basis guarantees us that the matrix* $P = \begin{bmatrix} 1 & 1 \\ \phi & 1-\phi \end{bmatrix}$ *is invertible.*

(b). Find P^{-1}*.*

(c). Let $\vec{w} \in \mathbb{R}^2$*. If* $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$*, find* $\vec{s} = A\vec{w}$ *as a linear combination of* $\vec{v}_1$ *and* $\vec{v}_2$*.*

(d). We use the notation $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_B$ *to represent the coordinates of* $\vec{w}$ *in terms of the basis* B*. Similarly if* $\vec{s} = d_1\vec{v}_1 + d_2\vec{v}_2$*, we use* $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}_B$ *to represent* $\vec{s}$ *in terms of the basis* B*. Explain why* $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_B = P^{-1}\vec{w}$ *and* $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}_B = P^{-1}\vec{s}$*. [hint: multiply both sides of the equations by* P*.] So* P^{-1} *represents a change of coordinates from the canonical basis to the basis* B *and* P *undoes the change of coordinates.*

(e). Find the matrix D *such that* $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}_B = D \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_B$*. The matrix* D *represents* A *in the new coordinate system.*

(f). Without explicitly calculating anything, explain why $PDP^{-1}\vec{w} = A\vec{w}$ *for any vector* $\vec{w}$ *[hint: consider the role we've identified for each of these matrices.] So* $A = PDP^{-1}$*.*

(g). Find A^{50} *by using this expression for* A*. You can leave your answer in terms of* ϕ*.*

The following problem puts together a lot of the pieces we have learned so far.

Exercise 4.25 THIS PROBLEM WILL NEED TO BE CLEANED UP BEFORE IT'S READY TO GO. *Let* A *be an* $n \times n$ *matrix and let* $\vec{v}_1, \dots \vec{v}_n$ *be a set of linearly independent vectors with* $\lambda_1, \dots, \lambda_n$ *a set of scalars with the property that for every* i*,* $A\vec{v}_i = \lambda_i\vec{v}_i$*.*

(a). How do we know that $\vec{v}_1, \dots, \vec{v}_n$ *form a basis for* $\mathbb{R}^n$*?*

It follows that every vector $\vec{w} \in \mathbb{R}^n$ *can be written in the form* $\vec{w} = \sum_{i=1}^n a_i\vec{v}_i$ *for some coefficients* a_i *which depend on* $\vec{w}$*.*

(b). Let $T : \mathbb{R}^n \to \mathbb{R}^n$ *be linear and defined so that* $T(\vec{v}_i) = \vec{e}_i$*. Show that* $T(\vec{w}) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ *if* $\vec{w} = \sum_{i=1}^n a_i\vec{v}_i$*.*

So T *creates a new vector whose entries are the coefficients we get from expressing* $\vec{w}$ *as a linear combination of the basis vectors* $vecv_i$*.* T *gives a change in coordinates from the usual canonical basis to this new basis.*

Every linear function from $\mathbb{R}^n$ *to* $\mathbb{R}^n$ *can be represented by a square matrix, but it's a little tricky to find the one for this* T*. We'll go at it by a "back door" approach. For the remainder of this problem, assume we are in* $\mathbb{R}^2$ *with* $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$*,* $\vec{v}_1 = \begin{bmatrix} 1 \\ \phi \end{bmatrix}$*,* $\vec{v}_2 = \begin{bmatrix} 1 \\ 1-\phi \end{bmatrix}$*,* $\lambda_1 = \phi$*, and* $\lambda_2 = 1-\phi$*.*

(c). *The linear function* $T^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ *inverts* T*. That is, if* $T(\vec{w}) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$*, then* $T^{-1}(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}) = \vec{w}$*. Find the matrix* P *that represents* T^{-1}*. [recall* $\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2$*]*

(d). *How do we know* P *is invertible?*

Let P^{-1} *be the inverse of* P*. The matrix* P *represents the function* T^{-1} *and* P^{-1} *represents* T*. Yes, it's unfortunate notationally, but it's the convention that's used.*
If $\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2$*, what is*

(e). $\mathrm{A}\vec{w}$*?*

(f). $\mathrm{P}^{-1}\mathrm{A}\vec{w}$*?*

Now we're going to take $\mathrm{P}^{-1}\mathrm{A}\vec{w}$ *and "multiply by* I*" carefully. What is*

(g). $\mathrm{P}^{-1}\mathrm{A}\mathrm{P}\mathrm{P}^{-1}\vec{w}$*? [hint, you've already found this above].*

(h). $\mathrm{P}^{-1}\vec{w}$*?*

Given your answer in the previous 2 parts what is

(i). $\mathrm{D} = \mathrm{P}^{-1}\mathrm{A}\mathrm{P}$*?*

Here's what's happening. We know that $\mathrm{A}\vec{w} = \mathrm{P}\mathrm{P}^{-1}\mathrm{A}\mathrm{P}\mathrm{P}^{-1}\vec{w}$*. ..... If you calculate* $\mathrm{P}^{-1}\vec{w}$*, you're* !!!!!!!!!!!!!
changing coordinates to a coordinate system based on the vectors $\vec{v}_i$ *rather than the vectors* $\vec{e}_i$*. Because of the special properties the* $\vec{v}_i$ *have with respect to the matrix* A*, the behavior of the matrix in this new coordinate system (given by* $\mathrm{P}^{-1}\mathrm{A}\mathrm{P}$ *is particularly nice. So you do your calculations in that nice system, and then multiply by ...* !!!!!!!!!!!!!

4.5 Determinants

The determinant is a function that takes any square matrix and outputs a real number. The definition I'll give below is somewhat clunky: we'll need to introduce some concepts to give the general definition.

The determinant has a number of uses, but in particular it is most used in finding out whether a matrix is or is not invertible. It has the property that $\det(\mathrm{A}) = 0$ if and only if A is not invertible. So this may be an easy thing to calculate compared to trying to solve, say $\mathrm{A}\vec{x} = \vec{0}$ by Gaussian elimination when trying to decide if A is invertible.

Let me first give three examples where a simple formula exists for the determinant:

- The determinant of a 1×1 matrix $\mathrm{A} = \begin{bmatrix} a \end{bmatrix}$ is $\det \mathrm{A} = a$.
- The determinant of a 2×2 matrix $\mathrm{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, is $\det \mathrm{A} = ad - bc$. This can be thought of as the product of the main diagonal minus the product of the "other" diagonal. $\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}$
- A special case of the determinant of a 3×3 matrix comes up in taking cross products. More generally, if $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$. Here is a conceptual way to think of this: we write the matrix and then repeat the first two columns. We then add the three diagonal products going down and to the right, and subtract the three diagonal

products going down and to the left: $\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$ Unfortunately this approach doesn't generalize to larger matrices. We'll need to give some definitions in order to sort out the more general case.

Notational comment There are a number of notations available for the determinant. Sometimes it's written $\det \mathrm{A}$. Often it appears as $\det(\mathrm{A})$. It also can be written $|\mathrm{A}|$. Note that although this looks like absolute values, there's nothing wrong with a negative determinant. Finally, if $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, the determinant is often written as $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$. I will generally use either $\det \mathrm{A}$ or $\det(\mathrm{A})$. These are interchangeable.

To define $\det(\mathrm{A})$ when A is larger than 3×3, we need a few definitions.

Definition 4.26 *Given a matrix A, the* ***minor matrix****i, j-minor matrix of A, often denoted $\mathrm{A}_{i,j}$ is the matrix formed by deleting the i-th row and j-th column of A.*

It is unfortunate notation, because sometimes $\mathrm{A}_{i,j}$ is used to denote the entry in the i-th row and j-th column of A. In principle it should be clear from the context, but I will always refer to "the minor matrix $\mathrm{A}_{i,j}$" when I do this.

Example 4.27 *Given* $\mathrm{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,j-1} & a_{2,j} & a_{2,j+1} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i,1} & a_{i,2} & \cdots & a_{i,j-1} & a_{i,j} & a_{i,j+1} & \cdots & a_{i,n} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,j-1} & a_{n,j} & a_{n,j+1} & \cdots & a_{n,n} \end{bmatrix}$, *to find the minor matrix $\mathrm{A}_{i,j}$, we find the i-th row and j-th column and remove them,*

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,j-1} & \cancel{a_{1,j}} & a_{1,j+1} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,j-1} & \cancel{a_{2,j}} & a_{2,j+1} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \cancel{\vdots} & \vdots & \ddots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,j-1} & \cancel{a_{i-1,j}} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ \cancel{a_{i,1}} & \cancel{a_{i,2}} & \cancel{\cdots} & \cancel{a_{i,j-1}} & \cancel{a_{i,j}} & \cancel{a_{i,j+1}} & \cancel{\cdots} & \cancel{a_{i,n}} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,j-1} & \cancel{a_{i+1,j}} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots & \cancel{\vdots} & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,j-1} & \cancel{a_{n,j}} & a_{n,j+1} & \cdots & a_{n,n} \end{bmatrix}$$

which yields

$$\mathrm{A}_{i,j} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{n,n} \end{bmatrix}$$

The formula for the determinant of an $n \times n$ matrix is given in terms of $n-1 \times n-1$ matrices. We first give the formula for 1×1 matrices which form the starting point

Definition 4.28 *The* ***determinant of a*** 1×1 ***matrix*** $\mathrm{A} = \begin{bmatrix} c \end{bmatrix}$ *is* $\det(\mathrm{A}) = c$.

For $n > 1$ the determinant is defined by

Definition 4.29 *Given an* $n \times n$ *matrix* $\mathrm{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}$ *with* $n > 1$*, the* ***determinant***$n \times n$ *determinant is*

$$\det(\mathrm{A}) = \sum_{j=1}^{n} (-1)^{1+j} a_{1,j} \det(\mathrm{A}_{1,j})$$

where $\mathrm{A}_{1,j}$ *is the minor matrix formed by removing row* 1 *and column* j.

Example 4.30 *Find the determinant of* $\mathrm{A} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 3 & 0 \end{bmatrix}$.

To do this according to the definition we take the first row in A *and "expand" the determinant to get:*

$$\begin{aligned} \det \mathrm{A} &= (-1)^{1+1} 1 \det \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix} + (-1)^{1+2} 2 \det \begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix} + (-1)^{1+3} 3 \det \begin{bmatrix} 0 & 3 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} + (-1)^{1+4} 0 \det \begin{bmatrix} 0 & 3 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \\ &= 1 \det \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix} - 2 \det \begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix} + 3 \det \begin{bmatrix} 0 & 3 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{aligned}$$

We are able to avoid calculating the determinant for the last term in the first row because it already has a coefficient of zero. Continuing on we get

$$\det \mathrm{A} = 1 \cdot (-8) - 2 \cdot 1 + 3 \cdot 3 = -1$$

Using the general definition, taking the determinant of a 4×4 matrix requires taking the determinant of four 3×3 matrices. More generally, taking the determinant of an $n \times n$ matrix requires taking the determinant of n $n-1 \times n-1$ matrices, each of which requires $n-1$ determinants of $n-2 \times n-2$ matrices. So it can quickly become a huge number of calculations. However, many of the important large matrices encountered in practice have a lot of zero entries. The following theorem makes it much easier to calculate the determinant of a matrix in this case.

Theorem 4.31 *To take the determinant of an $n \times n$ matrix* $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ *the determinant is*

$$\sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det \mathrm{A}_{i,j}$$

where $\mathrm{A}_{i,j}$ is the i, j minor of A *and i is a* <u>*fixed*</u> *row of* A.

What this theorem says is that to calculate the determinant of A we can choose any <u>one</u> row of A and expand the determinant across that row in much the same manner as in the definition where we just expand across the first row. We just have to be a bit careful of the exponents of -1. This can greatly simplify things.

We reconsider the original example using this theorem:

Example 4.32 *Find the determinant of* $\mathrm{A} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 3 & 0 \end{bmatrix}$.

We choose the row with two zero entries since that will reduce the number of determinants we must take. We get

$$\begin{aligned} \det \mathrm{A} &= (-1)^{2+1} 0 \det \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix} + (-1)^{2+2} 3 \det \begin{bmatrix} 1 & 3 & 0 \\ 2 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix} + (-1)^{2+3} 1 \det \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} + (-1)^{2+4} 0 \det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \\ &= 3 \det \begin{bmatrix} 1 & 3 & 0 \\ 2 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix} - \det \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\ &= 3 \cdot 0 - 1 \\ &= -1 \end{aligned}$$

There are fewer determinants to take, so less chance of an error, and it's quicker. Be sure you understand where the exponents on -1 came from.

We can go a step better still. Another theorem says that almost the same formula applies, but we need to hold our column fixed rather than our row:

Theorem 4.33 *To take the determinant of an $n \times n$ matrix* $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ *the determinant is*

$$\sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det \mathrm{A}_{i,j}$$

where $\mathrm{A}_{i,j}$ is the i, j minor of A *and j is a* <u>*fixed*</u> *column of* A.

Notice that in this theorem the sum is from $i = 1$ to n rather than in the previous theorem where it was $j = 1$ to n.

A simple picture to have in mind when keeping track of which terms get -1 or $+1$ is the following "matrix":

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

When we expand along a row or column, the coefficient of the a_{ij} term is $(-1)^{i+j}$ is either $+1$ or -1 depending on the i, j entry in this matrix.

Now using this theorem, we can take the determinant of the matrix in the example faster

Example 4.34 *Find the determinant of* $\mathrm{A} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 3 & 0 \end{bmatrix}$.

We choose to "expand" across the fourth column since it has a bunch of zeroes:

$$\begin{aligned} \det \mathrm{A} &= (-1)^{3+4} 1 \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \\ &= -1 \cdot [(9+2+0) - (9+1+0)] \\ &= -1 \end{aligned}$$

All three examples found the same result. This is a consequence of the theorems. Notice that by taking advantage of the rows or columns with zeroes, we can significantly reduce the effort we need to put in.

Exercise 4.35 *Calculate the determinant of each matrix (show your work)*

(a). $\begin{bmatrix} -2 \end{bmatrix}$

(b). $\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$

(c). $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & -4 & -2 \end{bmatrix}$

(d). $\begin{bmatrix} 1 & 0 & 2 & 4 \\ 3 & 0 & -4 & -1 \\ 1 & 2 & 2 & 3 \\ 3 & 0 & 0 & 0 \end{bmatrix}$

(e). $\begin{bmatrix} 2 & 0 & -1 & 0 & 3 & 1 \\ 2 & 0 & 4 & 0 & 4 & 2 \\ 1 & 2 & 1 & 3 & 2 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & -4 & 0 & 3 & 1 \\ 1 & 0 & 4 & 0 & 3 & 2 \end{bmatrix}$

The primary value of the determinant comes from the following theorem (which we gave as part of the invertible matrix theorem, theorem 4.20):

Theorem 4.36 *If* A *is a square matrix then* $\det \mathrm{A} = 0$ *iff* A *is* <u>*not*</u> *invertible.*

The determinant has a number of other uses, but because it can be quite difficult to calculate for large matrices, most of these applications have fallen out of use since in modern days the matrices we encounter in applications can be easily have thousands of rows and columns.

Example 4.37 *Show that the vectors* $\begin{bmatrix}1\\2\\4\end{bmatrix}$, $\begin{bmatrix}2\\1\\5\end{bmatrix}$, *and* $\begin{bmatrix}4\\-2\\6\end{bmatrix}$ *are linearly dependent.*

Up to now, the way we have been able to do this is to take the matrix $\mathrm{A} = \begin{bmatrix}1 & 2 & 4\\2 & 1 & -2\\4 & 5 & 6\end{bmatrix}$ *and show that there is a nontrivial solution to* $\mathrm{A}\vec{x} = \vec{0}$.

However, we now have a faster way. We find the same matrix $\mathrm{A} = \begin{bmatrix}1 & 2 & 4\\2 & 1 & -2\\4 & 5 & 6\end{bmatrix}$ *and calculate its determinant. We get* $6 + (-16) + 40 - (-10 + 24 + 16) = 30 - 30 = 0$. *Since* $\det \mathrm{A} = 0$, *we know that* A *is not invertible, and therefore the columns of* A *are linearly dependent.*

Exercise 4.38 *Show that the determinants of the following matrices are zero. You can do this by directly calculating the determinant or using theorem 4.36*

(a). $\mathrm{A} = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$

(b). $\mathrm{A} = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\1 & 2 & 3\end{bmatrix}$

(c). $\mathrm{A} = \begin{bmatrix}1 & 2 & 1\\4 & 5 & 4\\8 & 2 & 8\end{bmatrix}$

(d). $\mathrm{A} = \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\1 & 1 & 1 & 1\\1 & 2 & 3 & 4\end{bmatrix}$

(e). $\mathrm{A} = \begin{bmatrix}1 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\\2 & 3 & 4 & 5 & 6\\6 & 9 & 12 & 15 & 18\end{bmatrix}$

Exercise 4.39 *Consider the matrix* $\mathrm{A} = \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}$, *which we first saw in example 1.75.*

(a). Write down the 2×2 *matrix* $\mathrm{A} - \lambda\mathrm{I}$

(b). Find the determinant of $\mathrm{A} - \lambda\mathrm{I}$. *The resulting polynomial is the "characteristic polynomial" of* A.

(c). Find λ *such that* $\mathrm{A} - \lambda\mathrm{I}$ *is <u>not</u> invertible [hint, use the characteristic polynomial].*

(d). Your result from the determinant of $\mathrm{A} - \lambda\mathrm{I}$ *should look something like* $p(\lambda) = c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_k\lambda^k$ *for some* k. *Calculate the value of* $c_0\mathrm{I} + c_1\mathrm{A} + c_2\mathrm{A}^2 + \cdots + c_k\mathrm{A}^k$.

Exercise 4.40 *Find the determinant of*

(a). $\begin{bmatrix}1 & 0\\2 & 3\end{bmatrix}$

(b). $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 3 & 5 \end{bmatrix}$

(c). $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 23 & 42 & 2 & 0 \\ 567 & 234 & 21 & 5 \end{bmatrix}$

(d). $\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$ *where* $a_{ij} = 0$ *if* $j > i$.

There are two final points to make about the determinant.

Theorem 4.41 *If* A *and* B *are* $n \times n$ *matrices, then* $\det(\mathrm{AB}) = (\det \mathrm{A})(\det \mathrm{B})$.

Note the corresponding result for sums is NOT TRUE: $\det(\mathrm{A} + \mathrm{B}) \neq \det \mathrm{A} + \det \mathrm{B}$.

Note also that although the product of matrices does not commute: $\mathrm{AB} \neq \mathrm{BA}$, we do have $\det(\mathrm{AB}) = \det(\mathrm{BA})$.

Theorem 4.42 *If* A *is an* $n \times n$ *matrix, then* $\det \mathrm{A} = \det \mathrm{A}^T$.

Exercise 4.43 *For* $\mathrm{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ *and* $\mathrm{B} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$*: do the following without using theorems 4.41 and 4.42 (but please think about what your answers have to do with those theorems)*

change matrice det (A+B) = coincidence

(a). Find $\det \mathrm{A}$ *and* $\det \mathrm{B}$.

(b). Calculate AB *and* A + B *and use the result to find* $\det(\mathrm{AB})$ *and* $\det(\mathrm{A} + \mathrm{B})$.

(c). Find A^T *and* B^T *and use the result to find* $\det \mathrm{A}^T$ *and* $\det \mathrm{B}^T$.

Exercise 4.44 *Assume the first three columns of* A *are* $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}$, *and* $\begin{bmatrix} 4 \\ 9 \\ 25 \\ 49 \end{bmatrix}$. *Find a nonzero fourth column of* A *such that* $\det(\mathrm{A}) = 0$ *(there are many possible ways to do this). Explain how you found it. [hint: don't try to calculate out the determinant, instead look at theorem 4.20 to find what having a zero determinant means about columns.]*

Chapter 5

Inner Products and Orthogonality

5.1 Important Concepts

- Know what the dot product is
- Know how to define an inner product for functions.
- Understand what is meant by a "projection" and its relation to inner products.
- Given an orthogonal basis and an arbitrary vector (or function), write the given vector (or function) in terms of the basis.
- Given a basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ and a set of vectors $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ such that $\vec{v}_i \cdot \vec{w}_j = 0$ whenever $i \neq j$ and $\vec{v}_i \cdot \vec{w}_j \neq 0$ whenever $i = j$, write a given vector in terms of the basis.
- Be able to understand how Gram-Schmidt works (don't have to memorize it, but should be able to explain why steps work if they are given).

5.2 Inner Products

So far when we have talked about vectors and vector spaces, we've focused on addition and scalar multiplication. As long as we don't change the rules for these two addition and scalar multiplication we can add other operations. One of the most frequently used operations in vector spaces is something called an "inner product". An inner product takes two vectors and produces a real number (with a few conditions on how it behaves). We will see in this chapter that inner products can be very useful tools, particularly when combined with the concept of "orthogonality".

5.2.1 Vector Inner Products

Definition 5.1 *Given two vectors $\vec{u}$ and $\vec{v}$, the **dot product** of $\vec{u}$ and $\vec{v}$ is denoted $\vec{u} \cdot \vec{v}$ and equals $\sum u_i v_i$.*

This can also be thought of as the numerical entry in the 1×1 matrix $\vec{v}^T \vec{u}$.

Example 5.2

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 4 + 10 + 18 = 32$$

Note that the order doesn't matter: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$. In contrast, for the cross product (which we won't discuss) it does matter.

Geometric interpretation

Definition 5.3 *The **magnitude** of a vector is the geometric length of the vector and is denoted using vertical bars:* $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$.

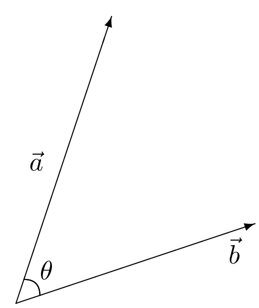
The magnitude is in fact simply $|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$. A simple fact about the magnitude is that if c is a scalar, then $|c\vec{a}| = c|\vec{a}|$ if c is positive, and more generally, $|c\vec{a}| = |c||\vec{a}|$ where the '$|$' around c denotes absolute value.

Definition 5.4 *If the dot product of two vectors is zero, they are called **orthogonal**. For our notation we write* $\vec{u} \perp \vec{v}$ *to denote* $\vec{u}$ *is orthogonal to* $\vec{v}$.

If we draw two orthogonal vectors using the usual geometric interpretation, the vectors are actually at a right angle. More generally, in the geometric interpretation it turns out that

$$\vec{a} \cdot \vec{b} = |\vec{a}|\, |\vec{b}| \cos\theta$$

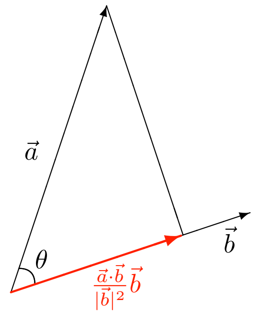
where θ is the angle between $\vec{a}$ and $\vec{b}$.



Notice that this means $\vec{a} \cdot \vec{b}/|\vec{b}| = |\vec{a}| \cos\theta$. If we add a line from the end of vector $\vec{a}$ that is orthogonal to vector $\vec{b}$, we see that $|\vec{a}| \cos\theta$ has an important geometric interpretation. If we draw the vector in the direction of $\vec{b}$ that goes from the origin to the point of intersection of this new line with $\vec{b}$, $|\vec{a}| \cos\theta$ is the length of the vector. This vector is called the "projection" of $\vec{a}$ onto $\vec{b}$. It is

$$\begin{aligned} |\vec{a}| \cos\theta \frac{\vec{b}}{|\vec{b}|} &= \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} \frac{\vec{b}}{|\vec{b}|} \\ &= \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b} \\ &= \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \vec{b} \end{aligned}$$

Its length or magnitude is often called the "component" of $\vec{a}$ in the direction of $\vec{b}$.



In fact, the projection of $\vec{a}$ onto $\vec{b}$ gives the closest vector to $\vec{a}$ that points in the direction of $\vec{b}$. That is, $|\vec{a} - c\vec{b}|$ is minimized for this value of c. We show this to be true in the next exercise:

Exercise 5.5 *Let's set $\vec{a} = c\vec{b} + \vec{x}$ where we want $\vec{x}$ to be as short as possible.*

(a). Explain why $\vec{x}$ should be orthogonal to $\vec{b}$ (Hint: explain why a circle of radius $|\vec{x}|$ centered at the end point of $\vec{a}$ should be tangent to $\vec{b}$.

(b). Given that $\vec{b}$ is orthogonal to $\vec{x}$, explain why $c = \vec{a}\cdot\vec{b}/\vec{b}\cdot\vec{b}$ by taking the dot product of both sides of the assumed equation for $\vec{a}$ with $\vec{b}$.

The fact that $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$ gives the projection of $\vec{a}$ onto $\vec{b}$ is a very important property which we will use to make our lives much easier and will be discussed further below. Among other important mathematical/scientific tools, it is the underlying mechanism by which Fourier Series work.

The fact that it is the closest vector to $\vec{a}$ underlies a lot of image compression. A photograph is stored in a computer as a long vector of numbers, easily millions of entries. So it is a vector in $\mathbb{R}^N$ where N is huge. If we have a subspace of $\mathbb{R}^n$ with a much smaller basis, then we can approximate $\vec{a}$ by finding the closest vector in that subspace. This lets us store a much smaller set of coefficients than N. If we've chosen our subspace well, the error will be small. So one of the challenges people have is finding a good basis that allows us to approximate most images well.

Exercise 5.6 *For each of the following, draw the vectors, find the angle θ between $\vec{a}$ and $\vec{b}$. Calculate $\vec{a}\cdot\vec{b}$ as the sum of the products of corresponding terms, and separately calculate $|\vec{a}|\,|\vec{b}|\cos\theta$ — you should show that these are the same answer. You may want to use the formulas for $\cos(A+B)$ and $\sin(A+B)$ derived in the appendix.* problem a and

For the last few problems, note that the vector $r\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ is a vector of length r whose angle with the positive horizontal axis is θ.

(a). $\vec{a} = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{b} = \begin{bmatrix}0\\1\end{bmatrix}$.

(b). $\vec{a} = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{b} = \begin{bmatrix}0\\1\end{bmatrix}$.

(c). $\vec{a} = 2\begin{bmatrix}\cos\pi/3\\ \sin\pi/3\end{bmatrix}$, $\vec{b} = 4\begin{bmatrix}\cos 2\pi/3\\ \sin 2\pi/3\end{bmatrix}$.

(d). $\vec{a} = \begin{bmatrix}2\cos 6\pi\\ 2\sin 6\pi\end{bmatrix}$, $\vec{b} = 3\begin{bmatrix}\cos 9\pi/2\\ \sin 9\pi/2\end{bmatrix}$.

(e). $\vec{a} = 2\begin{bmatrix}\cos 13\pi/2\\ \sin 13\pi/2\end{bmatrix}$, $\vec{b} = \begin{bmatrix}\cos\pi/2\\ \sin\pi/2\end{bmatrix}$

5.2.2 General Inner Products

Definition 5.7 *For an arbitrary vector space over $\mathbb{R}$, an* ***inner product*** *of two vectors $\vec{u}$ and $\vec{v}$ is denoted by $\langle \vec{u}, \vec{v} \rangle$. It is a generalization of the dot product. It has the properties*

- $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ *(order doesn't matter[1]).*
- $\langle a\vec{u} + b\vec{v}, \vec{w} \rangle = a\langle \vec{u}, \vec{w} \rangle + b\langle \vec{v}, \vec{w} \rangle$ *(so it's linear in the first argument).*
- $\langle \vec{u}, \vec{u} \rangle > 0$ *whenever $\vec{u} \neq 0$ and $\langle \vec{0}, \vec{0} \rangle = 0$.*

Using the first two properties, we can show that it is linear in the second argument as well.

Definition 5.8 *If a vector space also has an inner product, we call it an* ***inner product space****.*

The inner product gives us a definition of distance. The magnitude of a vector $\vec{u}$ is $\sqrt{\langle \vec{u}, \vec{u} \rangle}$. This is valuable because it allows us to talk about how close two vectors are. If we have an infinite dimensional space, or even just a very high-dimensional space, we often want to choose a subset of the basis which allows us to accurately approximate the vector (for example, in image compression). We have lots of tools from calculus that allow us to minimize an "error", but before we can use them, we need to have a clear definition of how large the error is. This is given by the magnitude.

Function Inner Products

The usual generalization of an inner product to functions rather than vectors is fairly simple. If $g(x)$ and $f(x)$ are functions defined on some interval $[a, b]$, we define their inner product to be $\langle g(x), f(x) \rangle = \int_a^b f(x)g(x)\,\mathrm{d}x$.

It may not be obvious why this is a generalization of the vector inner product. But let's assume that we try to approximate f and g by piecewise constant functions, as was done in example 1.30. That is, we divide the interval $[a, b]$ into n smaller intervals and treat f and g as constant in those intervals. As the resolution increases, the approximation improves. Let $\vec{f}_n$ denote the vector used to approximate f with n intervals: we approximate f as constant in each interval $[a, a+\delta x)$, $[a+\delta x, a+2\delta x)$, $\ldots$, $[a+(n-1)\delta x, b]$. We similarly define $\vec{g}_n$. Then $\vec{f}_n \cdot \vec{g}_n \delta x$ is approximately $\int_a^b f(x)g(x)\,\mathrm{d}x$, with the approximation improving as $\delta x \to 0$ (this is just the definition of Riemann integration). So the function inner product looks just like the limit of the vector inner product, except that there is an extra factor of $\delta x = (b-a)/n$ put in to prevent it from blowing up as the number of terms increases.

if use approxi- y to define an- limiting angle

Definition 5.9 *We say two functions are* ***orthogonal*** *if their inner product is zero.*

We still define $|f| = \sqrt{\int_a^b [f(x)]^2\,\mathrm{d}x}$. It still makes sense to say the projection of g onto f is given by $\frac{\int_a^b f(x)g(x)\,\mathrm{d}x}{|f|^2} f(x)$.

Weighted Inner Products

The following is worth being aware of. I will not emphasize it in this course.

Sometimes we modify our inner product forms. This is relatively rare with vectors, but more common with functions. When we use a "weighted" inner product with vectors, what we do is we define the weighted inner product of $\langle \vec{a}, \vec{b} \rangle = \vec{a} \cdot (\mathrm{W}\vec{b})$ for some matrix W which must satisfy certain properties (e.g., it must be symmetric and positive definite — I haven't defined this: basically we need the properties of an inner product to be true). This is equivalent to the numerical entry of $\vec{a}^T \mathrm{W} \vec{b}$.

[1] the rules change if we are using vector spaces over the complex numbers $\mathbb{C}$

With functions, we define the weighted inner product to be $\langle f(x), g(x)\rangle = \int_a^b f(x)g(x)w(x)\,\mathrm{d}x$ for some function $w(x)$. Again w must satisfy certain properties (it must be non-negative and can only be zero at discrete points).

Exercise 5.10 *Show that if there exists i and j such that the ij entry of A and the ji entry are not equal, then if we define $\langle \vec{u}, \vec{v}\rangle = \vec{u}^T \mathrm{A}\vec{v}$, this is not an inner product. To do this, consider $\vec{e}_i$ and $\vec{e}_j$ and show that for these two vectors, one of the properties of an inner product is not satisfied.*

Inner products with complex numbers

I will not focus on this in this course, but inner products can also be defined for vector spaces over complex numbers. The definition changes slightly.

Definition 5.11 *Let V be a vector space over the complex numbers $\mathbb{C}$. An **inner product** on V is a function of two vectors in V such that for any $\vec{u}$, $\vec{v}$, $\vec{w} \in V$ and $a, b \in \mathbb{C}$*

- $\langle \vec{u}, \vec{v}\rangle = \overline{\langle \vec{v}, \vec{u}\rangle}$.
- $\langle a\vec{u} + b\vec{v}, \vec{w}\rangle = a\langle \vec{u}, \vec{w}\rangle + b\langle \vec{v}, \vec{w}\rangle$
- $\langle \vec{u}, \vec{u}\rangle > 0$ *whenever* $\vec{u} \neq \vec{0}$ *and* $\langle \vec{0}, \vec{0}\rangle = 0$.

Here the overlining means taking complex conjugate (and is not to be confused with the arrow denoting a vector).

The change in the first property means that it is no longer linear over the second argument. It is something close however: $\langle \vec{w}, a\vec{u} + b\vec{v}\rangle = \overline{a}\langle \vec{w}, \vec{u}\rangle + \overline{b}\langle \vec{w}\vec{v}\rangle$. It remains linear in the first argument.

5.3 Orthogonality

Let's assume we are working with the canonical basis in $\mathbb{R}^n$: $\vec{e}_1$, ..., $\vec{e}_n$. I have a vector $\vec{u}$, and I want you to write $\vec{u}$ as $c_1\vec{e}_1 + c_2\vec{e}_2 + \cdots + c_n\vec{e}_n$. If I explicitly tell you $\vec{u}$, then it's easy to find the c_i — each is just the i-th component of $\vec{u}$. Unfortunately, I won't tell you what $\vec{u}$ is. I will, however, tell you the dot product of $\vec{u}$ with any vector you give me.

By asking me for the dot product of $\vec{u}$ with well-chosen vectors, can you find the coefficients c_1, c_2, ...? After some thought (and please do think a bit before you read on), you may come to the following observation. My vector is $\vec{u} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$. Even though I won't tell you what $\vec{u}$ is directly, you can get each c_i by just asking me for the dot product $\vec{u}\cdot\vec{e}_i$.

Let me put this another way which will be more useful in generalizing the process. We know that $\vec{u} = c_1\vec{e}_1 + \cdots + c_n\vec{e}_n$. If we take $\vec{e}_i \cdot \vec{e}_j$ for any $j \neq i$, we get 0. If we take $\vec{e}_i \cdot \vec{e}_i$, we get 1. So $\vec{e}_i \cdot \vec{u} = c_1\vec{e}_1 \cdot \vec{e}_i + \cdots + c_n\vec{e}_n \cdot \vec{e}_i$. A bunch of the terms are 0. The only one left is $c_i\vec{e}_i \cdot \vec{e}_i = c_i$. So $c_i = \vec{e}_i \cdot \vec{u}$. The fact that $\vec{e}_i \cdot \vec{e}_j = 0$ whenever $i \neq j$ allows us to use dot products to pick out just the coefficient of a particular vector. We'll start doing this with more interesting bases, and some of the steps will look like black magic if you don't recognize that this sort of feature is underlying everything.

5.3.1 Change of Basis

One of the big advantages ofo rthogonality is that it gives us a quick way to change from one basis to another. Let's assume we have a basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for some n-dimensional space. Let's further assume that we have a second set of vectors $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ that have the property that whenever $i \neq j$,

$$\langle \vec{v}_i, \vec{w}_j \rangle = 0$$

while $\langle \vec{v}_i, \vec{w}_i \rangle \neq 0$. So $\vec{w}_i$ is orthogonal to every $\vec{v}_j$ except the case $j = i$.

Take a vector $\vec{b}$ which we would like to write in terms of the vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$. That is, we want to write $\vec{b} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$, but we don't know what the coefficients are.

So far, the only way we know how to do this is to try to solve the system

$$\mathrm{A}\vec{x} = \vec{b}$$

where the columns of A are the vectors $\vec{v}_i$. However, the orthogonality relation we have gives us a shortcut. We have

$$\begin{aligned}
\vec{w}_i \cdot \vec{b} &= \vec{w}_i \cdot \sum_{j=1}^{n} x_j \vec{v}_j \\
&= \sum_{j=1}^{n} x_j \vec{w}_i \cdot \vec{v}_j \\
&= x_1 \vec{w}_i \cdot \vec{v}_1 + \cdots + x_i \vec{w}_i \cdot \vec{v}_i + \cdots + x_n \vec{w}_i \cdot \vec{v}_n \\
&= 0 + \cdots + 0 + x_i \vec{w}_i \cdot \vec{v}_i + 0 + \cdots + 0 \\
&= x_i \vec{w}_i \cdot \vec{v}_i
\end{aligned}$$

Double check that you understand what has happened here. This is important. By taking the inner product with $\vec{w}_i$, we manage to pick out just the term with $x_i\vec{v}_i$. All the other terms go to zero. Rearranging terms a bit we have

$$x_i = \frac{\vec{w}_i \cdot \vec{b}}{\vec{w}_i \cdot \vec{v}_i}$$

However, every term on the right hand side is something we can calculate directly. So we can calculate each x_i with just a couple of dot products. There's nothing special about $\mathbb{R}^n$ that makes this work, so the same equations hold for other spaces with the dot product replaced by the appropriate inner product.

One of the advantages of this approach is that it works even when the vector space is not $\mathbb{R}^n$, in particular for infinite dimensional vector spaces such as function spaces. Gaussian Elimination assumes that at some point we are done with our row operations and can then read off the answers. If we tried it in some sort of infinite dimensional space, we'd never finish, because we'd always have infinitely many row operations left. In contrast, if we know the vectors $\vec{w}_i$ and $\vec{v}_i$, then at least we can find as many of the coefficients as we care to find.

Exercise 5.12 *Consider the vectors* $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

(a). Show that the vectors $\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$ *and* $\vec{w}_3 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ *have the property that* $\vec{w}_j \cdot \vec{v}_i = 0$ *iff* $i \neq j$.

For each of the below, find c_1, c_2, *and* c_3 *such that* $\vec{b} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ *(without using matrices).*

(b). $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(c). $\vec{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

(d). $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

Theorem 5.13 *If we have two sets of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ such that $\langle \vec{v}_i, \vec{w}_j \rangle$ is nonzero iff $i = j$ then the vectors $\vec{v}_i$ are all linearly independent.*

Exercise 5.14 *Prove theorem 5.13 by writing $\vec{0} = \sum c_i \vec{v}_i$. Show that $c_j = 0$ for all j.*

Example 5.15 *Consider the space spanned by the basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ -1 \end{bmatrix} \right\}$. The vector $\vec{b} = \begin{bmatrix} 4 \\ 9 \\ 0 \\ -7 \end{bmatrix}$ is in this space. How can we write $\vec{b}$ as a linear combination of the basis vectors?*

The vectors $\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \}$ each have the property of not being orthogonal to the corresponding vector of the basis, but being orthogonal to all others in the basis.

So if we write $\vec{b} = x_1 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \\ 2 \\ -1 \end{bmatrix}$, we know that

$$x_1 = \frac{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 9 \\ 0 \\ -7 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}} = \frac{9 - 7}{1 + 1 - 1} = 2$$

$$x_2 = \frac{\begin{bmatrix} 5 \\ -1 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 9 \\ 0 \\ -7 \end{bmatrix}}{\begin{bmatrix} 5 \\ -1 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix}} = \frac{20 - 9 - 35}{2 + 10} = -2$$

$$x_3 = \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 9 \\ 0 \\ -7 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 2 \\ -1 \end{bmatrix}} = \frac{4 + 9 - 7}{2 + 3 + 2 - 1} = 1$$

$$\text{So } \begin{bmatrix} 4 \\ 9 \\ 0 \\ -7 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 3 \\ 2 \\ -1 \end{bmatrix}. \textit{ Check it, it works.}$$

Warning — the method I used here would give coefficients x_1, x_2, and x_3 even if $\vec{b}$ were not in the span of the vectors. So in general, you need to somehow know in advance that $\vec{b}$ is in the vector space spanned by these vectors (perhaps we have n basis vectors in $\mathbb{R}^n$) or check that the sum you get really is $\vec{b}$.

Note that if we have a basis of n vectors for $\mathbb{R}^n$ there is a close relation between this process and the inverse of the matrix whose columns are made up of the basis.

In the special case where $\langle \vec{w}_i, \vec{v}_i \rangle = 1$ this has a special name:

Definition 5.16 *Given a basis $\vec{v}_1, \ldots, \vec{v}_n$ for some vector space, the* ***dual basis*** *is the set of vectors $\vec{w}_1, \ldots, \vec{w}_n$ for which $\langle \vec{v}_i, \vec{w}_j \rangle = 0$ if $i \neq j$ and $= 1$ if $i = j$.*

There is a large amount of mathematical tools built up around the "dual space": I'm just defining this so that you have some recognition if you hear the term in the future.

Definition 5.17 *The* ***dual space*** *of a vector space V is the vector space of linear functions $T : V \to \mathbb{R}$.*

In the case of $V = \mathbb{R}^n$, we know that to find any linear function $T(\vec{x})$ from $\mathbb{R}^n$ to $\mathbb{R}$, we must have $T(\vec{x}) = \mathrm{A}\vec{x}$ where A is a $1 \times n$ matrix — so $\mathrm{A} = \vec{a}^T$ for some vector $\vec{a}$. So given the technical definition of the dual space, it becomes clear that if we're talking about the dual space of $\mathbb{R}^n$, the dual space is actually $\mathbb{R}^n$.

Exercise 5.18 *Assume an invertible matrix A is given to you.*
Show that the rows of A^{-1} form the dual basis for the basis made up of the columns of A.

Exercise 5.19 *Consider the invertible matrix* $\mathrm{A} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 4 & -1 \end{bmatrix}$. *Let* $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(a). Explain how you know the i-th row of A^{-1} is orthogonal to all columns of A except the i-th column.

(b). We can write $\vec{b} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ where $\vec{v}_i$ is the i-th column of A. For some obscure reason, I am only interested in c_2. Given that the second row of A^{-1} is $[-1\ 1\ 0]$, find c_2.

Often, if we need to find these vectors $\vec{w}$, it takes the same amount of effort as to invert a matrix. Of course, if we invert the matrix A, then we can find $\vec{x}$ directly using what we already knew. So have we actually gained anything? We'll see that later in the course there are times when we actually don't care about finding all of the coefficients of the linear combination, but rather just one or two. In such a case, it can be fast to find the corresponding vectors $\vec{w}_i$ for just the coefficients we need. Then rather than inverting a matrix, we can get just the coefficients we want. This is especially useful when we have an infinite dimensional space, in which case we don't have a hope of inverting the "matrix".

5.3.2 Orthogonal basis

In the previous section, we saw that if we have a set of basis vectors $\vec{v}_i$ and a set of additional vectors $\vec{w}_j$ such that $\langle \vec{w}_j, \vec{w}_i \rangle = 0$ whenever $i \neq j$, then we can use this to help write vectors in terms of the basis vectors. It is a bit awkward that we rely on the additional vectors $\vec{w}_j$. Sometimes these auxiliary vectors are actually the same as the basis vectors.

Definition 5.20 *A basis is called an* ***orthogonal basis*** *if every basis vector is orthogonal to every other basis vector. If every basis vector also has magnitude 1, it is called an* ***orthonormal basis****.*

When a basis is orthogonal, it will be much easier to work with conceptually, but also it will behave better in other ways. For example, the round-off issues that appeared in 2.66 will be much lest significant.

Exercise 5.21 *Show that the canonical basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is an orthonormal basis for $\mathbb{R}^n$ (you may assume it is a basis).*

For the most part, I will not emphasize orthonormal bases. If you have an orthogonal basis and you want to create an orthonormal basis, simply divide each vector by its magnitude.

When we have an orthogonal basis, our calculations become a little cleaner. If $\vec{v}_1, \ldots, \vec{v}_n$ are an orthogonal basis for a space containing $\vec{b}$, then $\vec{b} = \sum c_i \vec{v}_i$ where

$$c_i = \langle \vec{v}_i, \vec{b} \rangle / |\vec{v}_i|^2$$

This is the same formula as before, using the fact that the $\vec{v}$ vectors are orthogonal to one another. The advantage of an orthonormal basis is that the denominator in the formula above is 1. So if a basis $\vec{v}_1, \ldots, \vec{v}_n$ is orthonormal and $\vec{b} = \sum c_i \vec{v}_i$, then $c_i = \langle \vec{v}_i, \vec{b} \rangle$.

Example 5.22 *The vectors* $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ *form a basis for the same vector space considered in example 5.15. However, this basis has a nicer property that each basis vector is orthogonal to the other basis vectors. We can write* $\vec{b} = \begin{bmatrix} 4 \\ 9 \\ 0 \\ -7 \end{bmatrix}$ *as a linear combination of the basis vectors*

$$x_1 = \frac{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 9 \\ 0 \\ -7 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}} = \frac{4+9+7}{1+1+1+1} = 5$$

$$x_2 = \frac{\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 9 \\ 0 \\ -7 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}} = \frac{-4+9+7}{1+1+1+1} = 3$$

$$x_3 = \frac{\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 9 \\ 0 \\ -7 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}} = \frac{8+9-7}{4+1+4+1} = 1$$

and $\begin{bmatrix} 4 \\ 9 \\ 0 \\ -7 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$.

Exercise 5.23 *Let* $\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{a}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, *and* $\vec{a}_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

(a). *Show that these vectors are orthogonal and explain how you know they are a basis for* $\mathbb{R}^4$.

For each $\vec{b}$ *below, find the coefficients* c_1, c_2, c_3, c_4 *so that* $\vec{b} = \sum c_i \vec{a}_i$.

(b). $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}$.

(c). $\vec{b} = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$.

(d). $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ -2 \end{bmatrix}$.

Exercise 5.24 *Repeat all parts of exercise 5.23, but using the basis* $\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{a}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$,

$\vec{a}_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$.

Exercise 5.25 *In exercise 2.57 I had you find coefficients such that* $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ \phi \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1-\phi \end{bmatrix}$. *In 2.58 I had you do the same with* $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c \begin{bmatrix} 1 \\ 1+\sqrt{2} \end{bmatrix} - c \begin{bmatrix} 1 \\ 1-\sqrt{2} \end{bmatrix}$.

(a). *Show that the basis* $\begin{bmatrix} 1 \\ \phi \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1-\phi \end{bmatrix}$ *is orthogonal. (recall* $\phi^2 = 1 + \phi$ *)*

(b). *Use this to find* c_1 *and* c_2.

(c). *Show that the basis* $\begin{bmatrix} 1 \\ 1+\sqrt{2} \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1-\sqrt{2} \end{bmatrix}$ *is orthogonal.*

(d). *Use this to find* c.

Later we will learn that because the matrices that led to the creation of these bases were symmetric, we know that they are orthogonal.

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Exercise 5.26 *This question considers Haar Wavelets, which are functions defined on the interval* $[0,1]$ *of the form:*

$$\psi_{m,n}(x) = \begin{cases} 1 & \text{if } \frac{m}{2^n} \le x < \frac{m+1/2}{2^n} \\ -1 & \text{if } \frac{m+1/2}{2^n} \le x < \frac{m+1}{2^n} \\ 0 & \text{otherwise} \end{cases}$$

for $n \ge 0$ *and* $m = 0, \ldots, 2^n - 1$, *and the constant function*

$$\phi(x) = 1$$

(a). *Sketch the functions* $\phi(x)$, $\psi_{0,0}(x)$, $\psi_{0,1}(t)$, *and* $\psi_{1,1}(t)$ *in the interval* $[0,1]$.

(b). *Taking the inner product of* $g_1(x)$, $g_2(x)$ *to be* $\int_0^1 g_1(x) g_2(x)\,\mathrm{d}x$, *show that* $\psi_{0,0}(x)$ *and* $\psi_{0,1}(x)$ *are orthogonal.*

(c). *Show that* $\psi_{0,1}(x)$ *and* $\psi_{1,1}(x)$ *are orthogonal.*

(d). *Consider a function in the span of these functions:*

$$f(x) = c_0\phi(x) + \sum_{n=0}^{\infty} \sum_{m=0}^{2^n-1} c_{m,n}\psi_{m,n}(x)$$

The Haar basis functions are all orthogonal to one another. Given this, explain why $c_0 = \int_0^1 f(x)\,\mathrm{d}x$ *and* $c_{p,q} = \int_0^1 f(x)\psi_{p,q}(x) / \int_0^1 [\psi_{p,q}(x)]^2\,\mathrm{d}x$

(e). *Write the function:*

$$f(x) = \begin{cases} 1 & 0 \le t < 1/4 \\ 3 & 1/4 \le t < 1/2 \\ 2 & 1/2 \le t < 3/4 \\ 5 & 3/4 \le t < 1 \end{cases}$$

as a sum of Haar wavelet functions.

5.3.3 Magnitude of a vector

We've seen that if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ form an orthogonal basis for a vector space V, then for any $\vec{x} \in V$ we can write

$$\vec{x} = \sum c_i \vec{v}_i$$

where $c_i = \frac{\langle \vec{v}_i, \vec{x} \rangle}{\langle \vec{v}_i, \vec{v}_i \rangle}$.

We can now find $|\vec{x}|$ by calculating

$$\langle \vec{x}, \vec{x} \rangle = \left\langle \left(\sum c_i \vec{v}_i\right), \left(\sum c_i \vec{v}_i\right) \right\rangle$$

Most terms in the product give zero. The product of these sums collapses down to just

$$\langle \vec{x}, \vec{x} \rangle = \sum c_i^2 \langle \vec{v}_i, \vec{v}_i \rangle$$

Make sure you understand how I got this collapse to happen.

The formula in terms of c_i^2 is often the most useful expression, however, sometimes it is worth "simplifying" further. Since $c_i = \langle \vec{x}, \vec{v}_i \rangle / \langle \vec{v}_i, \vec{v}_i \rangle$, this becomes

Theorem 5.27 *If $\vec{x} = \sum c_i \vec{v}_i$ then*
$$\langle \vec{x}, \vec{x} \rangle = \sum c_i^2 \langle \vec{v}_i \cdot \vec{v}_i \rangle$$

We know that $c_i = \langle \vec{v}_i, \vec{x} \rangle / \langle \vec{v}_i, \vec{v}_i \rangle$ so we can substitute this in for c_i to "simplify" further. . In the case where the vectors $\vec{v}_i$ form an orthonormal basis (each has magnitude 1), we have Parseval's identity:

Theorem 5.28 *If $\vec{v}_1, \ldots,$ are an orthonormal basis for the (possibly infinite-dimensional) vector space V, then*
$$\langle \vec{x}, \vec{x} \rangle = \sum \langle \vec{x}, \vec{v}_i \rangle^2$$

This is in fact a higher-dimensional form of the Pythagorean theorem.

Exercise 5.29 *Consider the function $y(x) = x$ on the interval $(-\pi, \pi)$. This function is odd [that is $y(-x) = -y(x)$].*
If we use the inner product $\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,\mathrm{d}x$ then the functions $\sin nx$ for $n = 1, 2, \ldots$ are orthogonal to one another (see later exercise 6.41). It can be shown that in the interval $(-\pi, \pi)$, $y(x) = x$ is in their span.

(a). Find $\langle y(x), y(x) \rangle$ by directly integrating x^2.

(b). Calculate $\langle y(x), \sin(nx) \rangle$. This requires an integration by parts (sorry).

(c). Show that $\langle \sin nx, \sin nx \rangle = \pi$ (there is a way to do this without doing the integral, but you can do the integral if you like).

(d). Write $y(x) = \sum c_n \sin nx$. What is c_n?

(e). By writing $\langle y(x), y(x) \rangle$ in terms of c_n, and noting your result must match your earlier answer, find $\sum_{n=1}^{\infty} 1/n^2$.

5.3.4 Approximation/Projection

Sometimes want to reduce dimension (throw out information). For example, a typical map of State College doesn't really need elevation information. So rather than create a 3-dimensional map, we create a 2-dimensional approximation to the 3-dimensional space.

In general, we have a point in an n-dimensional vector space which we want to approximate by a point in a lower-dimensional vector space. If the approximation is good enough, this means we can represent the point with fewer coefficients. This saves space on a computer hard drive.

This sort of calculation frequently occurs when performing image compression. An image can be thought of as a long vector, with each entry giving the color (or other property) of each pixel. With a good choice of the subspace, we can often reduce the dimension by 90% without any significant loss of quality.

Exercise 5.30 *read article on wavelets*

How do we find the best approximation?

exercise 2.66

Let's assume we have an orthogonal basis $\vec{v}_1, \ldots, \vec{v}_n$ for our subspace W. Let $\vec{x}$ be the vector we are trying to approximate. Let $\vec{w}$ be the vector in W which is closest to $\vec{x}$. In other words, we want $\vec{x} - \vec{w}$ to be as small as possible. How do we find $\vec{w}$?

There are several ways to approach this, but all lead to the same result (of course). I will give a geometric interpretation.

For illustration purposes, assume X is a 2-dimensional space, and W is a 1-dimensional subspace, a line (which must go through the origin). Let $\{\vec{v}_1\}$ be a basis for W. We might as well assume that $\vec{x} \notin W$, otherwise we are already done. If we draw a very small small circle around the end point of $\vec{x}$, it won't intersect W. As we draw progressively larger circles, eventually the circle does intersect W, and it does so in exactly one place. This is the closest point in W to $\vec{x}$. Let r be the radius of this circle.

Geometrically, this means that the line corresponding to W is tangent to the circle of radius r about $\vec{x}$. The vector $\vec{w}$ is the vector from the origin to that point of tangency. What have we learned from high school geometry about this? The length of $\vec{x} - \vec{w}$ is r. The vector $\vec{y} = \vec{x} - \vec{w}$ is orthogonal to $\vec{c}_1$.

That means

$$\vec{v}_1 \cdot \vec{y} = 0$$

Because $\vec{w}$ is in the span of $\{\vec{v}_1\}$, we have $\vec{w} = c\vec{v}_1$. So $\vec{x} = c\vec{v}_1 + \vec{y}$. Dotting this with $\vec{v}_1$ we have

$$\begin{aligned} \vec{v}_1 \cdot \vec{x} &= c\vec{v}_1 \cdot \vec{x} + \vec{y} \cdot \vec{v}_1 \\ &= c\vec{v}_1 \cdot \vec{x} \\ c &= \frac{\vec{v}_1 \cdot \vec{x}}{\vec{v}_1 \cdot \vec{v}_1} \end{aligned}$$

Although we used the dot product which allows us to think in terms of geometric concepts like spheres and cubes, in fact this will hold for any inner product.

So this means if we start with our target vector $\vec{x}$ and a basis vector for W, $\vec{v}_1$, then we can easily use the dot product to find the closest approximation to $\vec{x}$ in W. Please note, although I used the concept of drawing circles around $\vec{x}$, that was just to derive the formula. Since we have the formula, you don't need to think about these circles every time.

What happens in higher dimensions? Rather than drawing circles, we draw spheres (or hyper-spheres) around the point $\vec{x}$. Let W be the vector from the origin to the point of tangency. The same argument shows that the vector $\vec{y} = \vec{x} - \vec{w}$ must be orthogonal to every vector within W. If we have a basis $\{\vec{v}_1, \ldots, \vec{v}_m\}$ for W, then we can write $\vec{w} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$. From this it follows that

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m + \vec{y}$$

where $\vec{y}$ is orthogonal to every $\vec{v}_i$. Thus, <u>if the basis of W made up of the vectors $\vec{v}_i$ is an orthogonal basis</u>, taking the dot product with $\vec{v}_i$ gives

$$\vec{v}_i \cdot \vec{x} = c_i\vec{v}_i \cdot \vec{v}_i$$

and so

$$c_i = \frac{\vec{v}_i \cdot \vec{x}}{\vec{v}_i \cdot \vec{v}_i}$$

Exercise 5.31 *Why does the formula developed above fail if the basis for W is <u>not</u> orthogonal?*

How good is the approximation?

We've shown how to find the best approximation. But, is it good enough? If the vector $\vec{y} = \vec{x} - \vec{w}$ is large, then our approximation isn't very good. Typically in this case we add some more basis vectors to our set until we have enough that the approximation is good.

To do this well, we need a way to measure $|\vec{y}|$. It's actually very easy to calculate if the basis is orthogonal.

Let's start with a two observations: $\vec{w} = \sum_{i=1}^{m} c_i\vec{v}_i$ and for any $\vec{v}_i$, we have $\vec{v}_i \cdot \vec{y} = 0$.

We know the target vector $\vec{x}$, so we can easily calculate $\vec{x} \cdot \vec{x}$ just from the definition of the inner product.

But we can also express it in terms of the c_i. We have

$$\begin{aligned}\vec{x}\cdot\vec{x} &= \left(\vec{y}+\sum_{i=1}^{m} c_i\vec{v}_i\right)\cdot\left(\vec{y}+\sum_{i=1}^{m} c_i\vec{v}_i\right)\\ &= \vec{y}\cdot\vec{y}+\sum_{i=1}^{m} c_i^2\vec{v}_i\cdot\vec{v}_i\end{aligned}$$

This looks like magic, so make sure you understand exactly why I was able to do this step. I haven't put it here because I want you to have to think about it. But realize that if you don't understand why orthogonality played a massive role in this expression, then you need to look over it until you do. Why aren't there terms that look like $c_i c_j \vec{v}_i \cdot \vec{v}_j$?

From this, simple rearrangement gives

$$\vec{y}\cdot\vec{y} = \vec{x}\cdot\vec{x} - \sum_{i=1}^{m} c_i^2\vec{v}_i\cdot\vec{v}_i$$

where $c_i = \vec{v}_i \cdot \vec{x}/\vec{v}_i \cdot \vec{v}_i$. This gives us the ability to calculate $|\vec{y}| = \sqrt{\vec{y}\cdot\vec{y}}$ in a relatively easy way.

Theorem 5.32 *If $\vec{v}_1, \ldots, \vec{v}_n$ are orthogonal and $\vec{x} = \sum c_i\vec{v}_i$ is in their span, then $|\vec{x}|^2 = \sum c_i^2|\vec{v}_i|^2$ (Parseval's identity). If $\vec{x}$ is not in their span, then the closest approximation to $\vec{x}$ has an error of magnitude $|\vec{x}|^2 - \sum_{i=1}^{n} c_i^2|\vec{v}_i|^2$ (which is positive).*

The statement that $|\vec{x}|^2 - \sum_{i=1}^{n} c_i^2|\vec{v}_i|^2 \geq 0$ is frequently known as Bessel's inequality.

5.3.5 Creating an orthogonal basis

We've seen that if we have an orthogonal basis for a subspace, we can do a lot with it.

This leaves the question, how do we get an orthogonal basis. There is a straightforward algorithm to create an orthogonal basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ from any known basis $\{\vec{u}_1, \ldots, \vec{u}_n\}$.

The first step of the method is to set $\vec{v}_1 = \vec{u}_1$.

To find $\vec{v}_2$, we know that $\vec{u}_2$ can be written as a vector in the direction of $\vec{v}_1$ plus a vector orthogonal to $\vec{v}_1$. So we write

$$\vec{u}_2 = c_{21}\vec{v}_1 + \vec{v}_2$$

where we assume that $\vec{v}_1 \perp \vec{v}_2$. Either using our results from earlier, or simply taking the dot product of this equation with $\vec{v}_1$ we arrive at $c_{21} = \vec{u}_2 \cdot \vec{v}_1/|\vec{v}_1|^2$. So we have $\vec{v}_2 = \vec{u}_2 - c_{21}\vec{v}_1$ where c_{21} is straightforward to calculate.

To find $\vec{v}_3$, we write

$$\vec{u}_3 = c_{31}\vec{v}_1 + c_{32}\vec{v}_2 + \vec{v}_3$$

where we assume all the $\vec{v}$ vectors are orthogonal to one another. Taking the dot product of this equation with $\vec{v}_1$ we get $c_{31} = \vec{u}_3 \cdot \vec{v}_1/|\vec{v}_1|^2$. Taking the dot product with $\vec{v}_2$ we get $c_{32} = \vec{u}_3 \cdot \vec{v}_2/|\vec{v}_2|^2$. Then we have $\vec{v}_3 = \vec{u}_3 - c_{31}\vec{v}_1 - c_{32}\vec{v}_2$.

We repeat this process until we are done. This is effectively the "Gram-Schmidt" algorithm.

Gram-Schmidt

The Gram-Schmidt method is usually used as a method for creating an orthonormal basis from any existing basis. I'll focus on just using it to create an orthogonal basis.

Let's assume we have a basis $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ and we want a new basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ which has the same span, but is orthogonal.

We define

$$\vec{v}_1 = \vec{u}_1$$

We iteratively find $\vec{v}_i$ for $i > 1$ using the following formula

$$\vec{v}_i = \vec{u}_i - \sum_{j=1}^{i-1} \frac{\vec{u}_i \cdot \vec{v}_j}{\vec{v}_j \cdot \vec{v}_j}$$

If you would like to then make this an orthonormal basis, you can then divide each $\vec{v}$ by its magnitude.

Example 5.33 *In example 5.15 we saw a basis which was not orthogonal. In example 5.22 I showed another basis for the same space which I claimed was orthogonal. In fact this orthogonal basis can be constructed from the original basis using Gram-Schmidt.*

We begin with $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ -1 \end{bmatrix} \right\}$.

We first set

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

Then we set

$$\vec{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix} - \frac{-4}{4} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Finally we set

$$\vec{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 2 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 2 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 2 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ -1 \end{bmatrix} - \frac{4}{4} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} - \frac{-4}{4} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Exercise 5.34 GRAM-SCHMIDT TO A SIMPLE SET OF VECTORS. THEN TO A SET WHICH IS NOT A BASIS. WHERE DOES IT FAIL? !!!!!!!!!!!!!

Exercise 5.35 *Consider the set of functions whose Taylor series converge to the function in the interval* $[-1, 1]$. *Each of these functions can be written as an infinite sum of* $1, x, x^2, x^3, \ldots$. *This is not an orthogonal basis.*

We want to find a more useful basis. Let's use Gram-Schmidt to construct an orthogonal basis where our inner product is the integral from -1 *to* 1 *weighted by* $w(x) = 1/\sqrt{1-x^2}$. *We will call the result* $T_0(x)$, $T_1(x)$, ... *We will find* $T_0(x)$, $T_1(x)$, $T_2(x)$, *and* $T_3(x)$.

Give definition
function. Po
$1/\sqrt{1-x^2}$ is e

I'll give you a few identities to save you from doing some trig substitutions[2]*:* $\int_{-1}^{1} 1/\sqrt{1-x^2}\,\mathrm{d}x = \pi$, $\int_{-1}^{1} x^2/\sqrt{1-x^2}\,\mathrm{d}x = \pi/2$, *and* $\int_{-1}^{1} x^4/\sqrt{1-x^2}\,\mathrm{d}x = 3\pi/8$.
Show your work.

(a). *First show that if* $f(x)$ *is an even function and* $g(x)$ *an odd function then* $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)w(x)\,\mathrm{d}x = 0$. *This will prevent you from needing to do any integrals other than the ones I gave above.*

(b). *Start with* $T_0(x) = 1$. *Using the previous part of the problem and taking* x *as the next function from our starting basis, use Gram-Schmidt to find* $T_1(x)$. *[hint: I expect a few people in office hours to be wondering if they are doing this wrong, keep going to the next part once you've got an answer.]*

(c). *Using* x^2 *as the next function, find* $T_2(x)$.

(d). *Using* x^3 *as the next function, find* $T_3(x)$.

(e). *Compare your answers to the Chebyshev polynomials (find them on wikipedia, or earlier in this text). [there should be a mismatch in normalization]*

[2] Take a look at this link for an example where online calculators get things wrong. When last checked it had $\int_{-1}^{1} 1/\sqrt{1-x^2}\,\mathrm{d}x = -3.1416$ — can you see why there's no way this integral could be negative?

Chapter 6

Eigenvalues and Eigenvectors

6.1 Important Concepts

- Know the definition of an eigenvalue and eigenvector.
- Be able to calculate eigenvalues and eigenvectors of small matrices.
- Understand the power method for calculating the dominant eigenvalue, eigenvector of a large matrix.
- Understand left eigenvectors and their orthogonality relations with right eigenvectors.
- Understand diagonalization.

Look at ho and the effor into calculati ues/eigenvector their lives easie add exercise sh λ is an evec of evec of A^n. In example that i trices share eve uct of matrices and eval is pro

6.2 Other resource

A good resource for this chapter is the wikipedia page: http://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors.

6.3 Introduction

Probably the most important result about matrices is the existence of eigenvalues and eigenvectors.

Definition 6.1 *Let* A *be an* $n \times n$ *matrix. If there is a nonzero vector* $\vec{v}$ *and a scalar (a number)* λ *such that* $\mathrm{A}\vec{v} = \lambda\vec{v}$*, then* $\vec{v}$ *is an* ***eigenvector*** *and* λ *is an* ***eigenvalue****.*

Exercise 6.2 *Let* A *be an* $n \times n$ *matrix. Show that if all rows of* A *have the same sum,* c*, then* c *is an eigenvalue of* A *with eigenvector*

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Exercise 6.3

(a). Let A *be an* $n \times n$ *matrix. By using the definition of an eigenvector, show that if* $\vec{v}$ *is an eigenvector of* A *with eigenvalue* λ*, then* $\vec{v}$ *is an eigenvector of* A^k *with eigenvalue* λ^k*.*

(b). We now generalize the previous result: Let $\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_k$ *be* k *matrices. Assume there is a vector* $\vec{v}$ *which is an eigenvector of each matrix. Assume* $\lambda_1, \ldots, \lambda_k$ *are the corresponding eigenvalues (that is,*

$A_i\vec{v} = \lambda_i\vec{v}$ *for all* i. *Show that* $\vec{v}$ *is an eigenvector of the product* $A_1A_2\cdots A_k$ *and that the eigenvalue is* $\lambda_1\lambda_2\cdots\lambda_k$.

The word *eigen* is a German word having to do with "self". Perhaps a good translation would be "inherent vector/value".

A commonly used property of eigenvectors is that if v is an eigenvector, then $A^k\vec{v} = (A^{k-1})(A\vec{v}) = (A^{k-1})\lambda\vec{v} = \lambda A^{k-1}\vec{v} = \lambda(A^{k-2})(A\vec{v}) = \cdots = \lambda^k\vec{v}$.

We've already seen hints about why these are useful: if the eigenvectors of A form a basis for $\mathbb{R}^n$, then we can write any vector $\vec{w}$ in $\mathbb{R}^n$ in a unique way as $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ where $\vec{v}_i$ is an eigenvector of A. Once we've done this it is relatively easy to calculate $A^k\vec{w}$. It is simply

$$\begin{aligned}
A^k\vec{w} &= A^k\sum_{i=1}^{n} c_i\vec{v}_i \\
&= \sum_{i=1}^{n} A^k c_i\vec{v}_i \\
&= \sum_{i=1}^{n} c_i A^k\vec{v}_i \\
&= \sum_{i=1}^{n} c_i\lambda_i^k\vec{v}_i
\end{aligned}$$

So if the eigenvectors form a basis for $\mathbb{R}^n$, then multiplying a vector by A k times is as simple as figuring out how to write the vector as a linear combination of the eigenvectors and then raise the eigenvalues the power k.

6.4 Finding eigenvalues

How do we find eigenvalues and eigenvectors? Let's start from the definition of the eigenvalues/eigenvectors:

$$\begin{aligned}
A\vec{v} &= \lambda\vec{v} \\
A\vec{v} - \lambda\vec{v} &= \vec{0}
\end{aligned}$$

We would like to factor out $\vec{v}$ on the left hand side. But we can't really. That would leave us $A - \lambda$ where A is a matrix and λ a number. This isn't defined.[1] So let's look at $\lambda\vec{v}$ a bit.

It turns out that $I\vec{v} = \vec{v}$. So $\lambda\vec{v} = \lambda I\vec{v}$. Using this, we have

$$\begin{aligned}
A\vec{v} - \lambda\vec{v} &= A\vec{v} - \lambda I\vec{v} \\
&= (A - \lambda I)\vec{v}
\end{aligned}$$

where we've written $\lambda\vec{v}$ in terms of a matrix times $\vec{v}$ which allows us to factor out the $\vec{v}$.

Thus if $A\vec{v} - \lambda\vec{v} = \vec{0}$, then we must have

$$(A - \lambda I)\vec{v} = \vec{0}$$

That means $\vec{v}$ is in the null space of $A - \lambda I$. Going back to our theorem on invertible matrices, there is a vector $\vec{v} \neq \vec{0}$ that solves this equation iff $A - \lambda I$ is <u>not</u> invertible. So, such a $\vec{v}$ exists iff $\det(A - \lambda I) = 0$.

[1] Although matlab allows you to do this, the result it gets in this case would be wrong — matlab interprets $A - \lambda$ as A minus the $n \times n$ matrix all of whose entries are λ. This is not going to be right. Many hours of my life as a PhD student were spent trying to find places where I had accidentally written c where I meant cI in some matlab code.

So if we calculate $\det(\mathrm{A} - \lambda \mathrm{I})$ and set it to zero, we can solve for λ to find the eigenvalues. This is what we did in exercise 4.39. Using the determinant in this way is the main use of the determinant in modern science applications.

Exercise 6.4 *In this problem, you get to make up your own numbers. If you're working with a friend, please choose different values from your friend.*
Choose a (nonzero) 2×2 matrix, and call it A (any one you want). Choose any nonzero vector you want with 2 entries, call it $\vec{v}$. Take a nonzero number and call it c.

(a). Find $\mathrm{A}\vec{v} - c\vec{v}$ for your values.

(b). Find $\mathrm{M} = \mathrm{A} - c\mathrm{I}$, and calculate $\mathrm{M}\vec{v}$.

(c). Find the matrix N found by subtracting c from every entry of A, and calculate $\mathrm{N}\vec{v}$.

(hint, if this is hard, you're thinking too much. Just do the calculation - I'm just trying to show you why the factoring of $\mathrm{A}\vec{v} - c\vec{v}$ works the way it does.)

Algorithm to find eigenvalues To find eigenvalues of an $n \times n$ matrix A,

(a). Find $\mathrm{A} - \lambda \mathrm{I}$.

(b). Take the determinant and set it equal to 0.

(c). Solve for the roots. Each λ is an eigenvalue.

Example 6.5 *Find the eigenvalues of* $\mathrm{A} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$. *We take* $\mathrm{A} - \lambda \mathrm{I} = \begin{bmatrix} -\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$ *and calculate its determinant:*

$$p(\lambda) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (-\lambda)(2-\lambda) - 1 = \lambda^2 - 2\lambda - 1$$

So the eigenvalues are solutions to $p(\lambda) = 0$ where $p(\lambda) = \lambda^2 - 2\lambda - 1$. The solutions are

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$$

Example 6.6 *Find the eigenvalues of* $\mathrm{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. *We take* $\mathrm{A} - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}$ *and find its characteristic polynomial is λ^2+1. The eigenvalues are $\pm i$. There is nothing wrong with this being a complex number. The eigenvectors can also be complex. However, if the matrix is real and the eigenvalue is real, then so are the corresponding eigenvectors. In differential equations and physics applications, complex numbers will occur frequently — basically whenever a system (such as a spring) oscillates there is a complex eigenvector behind it.*

Notice that we had a 2×2 matrix and we found a degree 2 polynomial. For an $n \times n$ matrix, we get a degree n polynomial. In general, for an $n \times n$ matrix, we anticipate that there should be n eigenvalues. Sometimes however, there may be repeated roots:

Example 6.7 *Find the eigenvalues of* $\mathrm{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$. *We take* $\mathrm{A} - \lambda \mathrm{I} = \begin{bmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & 0 \\ 1 & 2 & -\lambda \end{bmatrix}$. *The determinant of this is $-\lambda^3$. So we find that 0 is a repeated eigenvalue.*

Notice that when we took the determinant of the 2×2 matrix we got a $+1$ in front of the λ^2. When we took the determinant of the 3×3 matrix, we got a -1 in front of λ^3. We don't really like these $-$ signs, so we just multiply by -1 when they occur, or equivalently we multiply by $(-1)^n$.

Definition 6.8 *The* ***characteristic polynomial*** $p_{\mathrm{A}}(\lambda)$ *of the* $n \times n$ *matrix* A *is* $(-1)^n \det(\mathrm{A} - \lambda\mathrm{I})$. *Equivalently, this is the determinant of* $\lambda\mathrm{I} - \mathrm{A}$.

Definition 6.9 *The* ***characteristic equation*** *of the* $n \times n$ *matrix* A *is* $p_{\mathrm{A}}(\lambda) = 0$.

Theorem 6.10 *The characteristic polynomial of an* $n \times n$ *matrix* A *is always a degree* n *polynomial, whose leading coefficient is* 1.

Definition 6.11 *If* λ_0 *is a repeated root of* $p_{\mathrm{A}}(\lambda) = 0$, *then we say that* λ_0 *is a* ***repeated eigenvalue*** *of* A.

Definition 6.12 *The* ***algebraic multiplicity*** *of* λ *is the number of times it appears as a root of* $p_{\mathrm{A}}(\lambda) = 0$.

Note that the sum of the algebraic multiplicities is n.

Even if A has repeated eigenvalues, we will usually write $\lambda_1, \lambda_2, \ldots, \lambda_n$ to denote its eigenvalues. This just means that for some i, j we have $\lambda_i = \lambda_j$.

Example 6.13 *Consider the matrix* $\mathrm{A} = \begin{bmatrix} 7 & -2 & -1 \\ -2 & 10 & 2 \\ -1 & 2 & 7 \end{bmatrix}$. *Find the eigenvalues and their algebraic multiplicities.*
Note that this matrix is (real and) symmetric. Soon we will learn that this means something about the eigenvalues and eigenvectors.
We find $\mathrm{A} - \lambda I = \begin{bmatrix} 7-\lambda & -2 & -1 \\ -2 & 10-\lambda & 2 \\ -1 & 2 & 7-\lambda \end{bmatrix}$. *The determinant is* $[(7-\lambda)(10-\lambda)(7-\lambda) + 4 + 4] - [4(7-\lambda) + 4(7-\lambda) + 10 - \lambda] = -\lambda^3 + 24\lambda^2 - 180\lambda + 432$. *The characteristic polynomial is* $p_A(\lambda) = \lambda^3 - 24\lambda^2 + 180\lambda - 432$.
Solving $p_A(\lambda) = 0$, *we find* $\lambda_1 = \lambda_2 = 6$ *and* $\lambda_3 = 12$ *(I don't expect it to be obvious that this is how you factor it). So we have a repeated eigenvalue,* $\lambda_1 = \lambda_2 = 6$ *with algebraic multiplicity* 2.

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Exercise 6.14 *For each matrix, find the eigenvalues. What are their algebraic multiplicities?*

(a). $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

(b). $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 6 & 6 & 2 \end{bmatrix}$

(c). $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d). $\begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}$

(e). $\begin{bmatrix} 2 & 3 & 0 \\ 2 & 3 & 0 \\ 1 & 1 & 5 \end{bmatrix}$

Exercise 6.15 *For the* 2×2 *matrix*

$$\mathrm{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(a). Find the characteristic polynomial and express it in terms of $\det \mathrm{A}$ *and* $\operatorname{tr} \mathrm{A}$ *(where* $\operatorname{tr} \mathrm{A}$ *is the "trace" of* A, *the sum of the diagonal elements).*

(b). Under what condition on $\det \mathrm{A}$ *and* $\operatorname{tr} \mathrm{A}$ *are the eigenvalues real? [hint, what's the quadratic formula? — I've added an appendix deriving the quadratic formula.]*

(c). If the eigenvalues are real, under what condition are they both negative?

clarify how w
trace less than,

6.5 Finding eigenvectors

We now know how to find the eigenvalues which are involved in $A\vec{v} = \lambda\vec{v}$. But can we find the eigenvectors? We know that

$$(A - \lambda I)\vec{v} = \vec{0}$$

So once we find λ, we just plug it into this equation and then solve for $\vec{v}$. We're just finding the null space of $A - \lambda I$.

Algorithm to find eigenvectors

(a). Find an eigenvalue λ.

(b). Take the matrix $A - \lambda I$

(c). Find all solutions to $(A - \lambda I)\vec{x} = \vec{0}$ in parametric form.

(d). The resulting vectors in parametric form are a basis for the eigenspace of λ.

Example 6.16 *Find the eigenvectors of* $A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$. *We've already found that the eigenvalues are* $\lambda_{1,2} = 1 \pm \sqrt{2}$. *That is, they are* $\lambda_1 = 1 + \sqrt{2}$ *and* $\lambda_2 = 1 - \sqrt{2}$. *Then to find the first eigenvector, we take* $(A - \lambda_1 I)\vec{x} = 0$*:*

$$\begin{aligned} \vec{0} &= (A - \lambda_1 I)\vec{x} \\ &= \begin{bmatrix} -\lambda_1 & 1 \\ 1 & 2 - \lambda_1 \end{bmatrix} \vec{x} \\ &= \begin{bmatrix} -1 - \sqrt{2} & 1 \\ 1 & 1 - \sqrt{2} \end{bmatrix} \vec{x} \end{aligned}$$

This is something of the form $M\vec{x} = \vec{0}$ *where* M *is a matrix. We do gaussian elimination to solve.*

$$\left[\begin{array}{cc|c} -1 - \sqrt{2} & 1 & 0 \\ 1 & 1 - \sqrt{2} & 0 \end{array}\right] \mapsto \left[\begin{array}{cc|c} -1 - \sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

We see that x_2 *is a free variable and* $x_1 = x_2/(1 - \sqrt{2})$. *Rationalizing the denominator we have* $x_1 = (1 + \sqrt{2})x_2/3$. *So* $x = x_2 \begin{bmatrix} \frac{1+\sqrt{2}}{3} \\ 1 \end{bmatrix}$. *Thus our eigenvector is*

$$\vec{v}_1 = \begin{bmatrix} \frac{1+\sqrt{2}}{3} \\ 1 \end{bmatrix}$$

The eigenvector of λ_2 *is found similarly.*

Notice that if $\vec{v}$ is an eigenvector of A then any nonzero multiple $\vec{w} = c\vec{v}$ is also an eigenvector of A corresponding to the same eigenvalue: $A\vec{w} = Ac\vec{v} = cA\vec{v} = c\lambda\vec{v} = \lambda\vec{w}$. This makes grading this course a headache. It is most common that either the first or last (non-zero) entry of an eigenvector is set to 1. If we use the "recipe" above, this will result in the last entry being 1.

Example 6.17 *Consider the matrix* $\mathrm{A} = \begin{bmatrix} 7 & -2 & -1 \\ -2 & 10 & 2 \\ -1 & 2 & 7 \end{bmatrix}$ *from example 6.13. Find the eigenvectors.*

The eigenvalues are 6, 6, *and* 12. *We start with* 6.

$$\mathrm{A} - 6\mathrm{I} = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

Solving $(\mathrm{A} - 6\mathrm{I})\vec{v} = \vec{0}$ *using Gaussian Elimination we have*

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ -2 & 4 & 2 & 0 \\ -1 & 2 & 1 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the solution has v_2 *and* v_3 *free, with* $v_1 = 2v_2 + v_3$. *We write this in parametric form:*

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2v_2 + v3 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The basis for the eigenspace of the eigenvalue 6 *is thus* $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

We now look at $\lambda = 12$. *We have* $\mathrm{A} - 12\mathrm{I} = \begin{bmatrix} -5 & -2 & -1 \\ -2 & -2 & 2 \\ -1 & 2 & -5 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} -5 & -2 & -1 & 0 \\ -2 & -2 & 2 & 0 \\ -1 & 2 & -5 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} -1 & 2 & -5 & 0 \\ -2 & -2 & 2 & 0 \\ -5 & -2 & -1 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} -1 & 2 & -5 & 0 \\ 0 & -6 & 12 & 0 \\ 0 & -12 & 24 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} -1 & 2 & -5 & 0 \\ 0 & -6 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So v_3 *is free and* $v_2 = 2v_3$. *We have* $v_1 = 2v_2 - 5v_3 = -v_3$. *So in parametric form we have*

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -v_3 \\ 2v_3 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$$

Thus we have $\vec{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

Finding eigenvectors of 2×2 matrices A comment is in order. For 2×2 matrices, we can avoid Gaussian Elimination. For almost any 2×2 matrix, once we find λ, we can take $\mathrm{A} - \lambda\mathrm{I}$ and guess that the first entry of $\vec{v}$ is 1. Effectively, we are guessing that the first entry is nonzero, and so an appropriate multiple will get the first entry to be 1.

So $\vec{v} = \begin{bmatrix} 1 \\ c \end{bmatrix}$, and then looking at the first row of $\mathrm{A} - \lambda\mathrm{I}$ multiplied with $\vec{v}$ the result must give 0, so we get an equation for c. We can solve for c. We then test it with the second row of $\mathrm{A} - \lambda\mathrm{I}$. If we get zero from multiplying with the second row, it's an eigenvector. If something has gone wrong, then either the assumption that the first entry is nonzero was wrong so $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector or you've made a mistake.

Example 6.18 *In example 6.16, when we get to* $\begin{bmatrix} -1-\sqrt{2} & 1 \\ 1 & 1-\sqrt{2} \end{bmatrix} \vec{v}_1 = 0$*, we can guess* $\vec{v}_1 = \begin{bmatrix} 1 \\ c \end{bmatrix}$*. Then the first row times* $\vec{v}_1$ *gives* $-1-\sqrt{2}+c=0$*. From that we conclude* $c = 1+\sqrt{2}$*. We just need to check that this works with the second row. That we need to check that* $1(1)+(1-\sqrt{2})(1+\sqrt{2})$ *is zero. Since* $(1-\sqrt{2})(1+\sqrt{2}) = 1-\sqrt{2}^2 = -1$*, we have* $1(1)-1=0$*, and it is done.*

So we have found that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1+\sqrt{2} \end{bmatrix}$*. This appears different from the result in the example, but this is just multiple of* $1+\sqrt{2}$ *times the previously found eigenvector.*

Example 6.19 *Consider the matrix* A *from example 1.75:* $\mathrm{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$*. What are the eigenvalues and eigenvectors of* A*?*

We take $\mathrm{A} - \lambda\mathrm{I} = \begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$ *and take the determinant and set it equal to* 0*. We get* $\lambda^2-\lambda-1=0$*.*

The solutions are $(1 \pm \sqrt{1+4})/2 = (1 \pm \sqrt{5})/2$*.*

So we have the eigenvalues: they are ϕ *and* $1-\phi$ *where* $\phi = (1+\sqrt{5})/2$ *is the golden ratio.*

To find the eigenvector of ϕ*, we take* $\mathrm{A} - \phi\mathrm{I} = \begin{bmatrix} -\phi & 1 \\ 1 & 1-\phi \end{bmatrix}$*. This is a* 2×2 *matrix, so we assume* $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ *where* $v_1 = 1$*. The first row tells us that* $-\phi + v_2 = 0$*, so* $v_2 = \phi$*. Checking with the second row we get* $1+\phi-\phi^2=0$*, but because we know* ϕ *solves* $\lambda^2-\lambda-1=0$*, this holds automatically.*

So $\vec{v} = \begin{bmatrix} 1 \\ \phi \end{bmatrix}$*.*

Similar steps with $1-\phi$ *yield an eigenvector of* $\begin{bmatrix} 1 \\ 1-\phi \end{bmatrix}$*.*

Exercise 6.20 *For the matrix* A *in example 6.16, find an eigenvector corresponding to* λ_2*.*

Example 6.21 *Find the eigenvectors of* $\mathrm{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$*. We've already found that the eigenvalues are all* 0*. So any eigenvector of* A *satisfies* $(\mathrm{A} - 0\mathrm{I})\vec{v} = \vec{0}$*. So* $\mathrm{A}\vec{v} = \vec{0}$*.*

Setting $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$*, setting it up to perform Gaussian Elimination, we have it almost in the right form to begin with, we just need a few steps*

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So we must have $a = b = 0$*, but* c *is free. Thus the null space is* $c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$*, and we can choose as our eigenvector* $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$*.*

Notice that we can only choose one linearly independent eigenvector in this example. So while the eigenvalue was repeated, there is only one eigenvector.

Definition 6.22 *Given the eigenvalue* λ*, the set of all vectors* $\vec{v}$ *such that* $\mathrm{A}\vec{v} = \lambda\vec{v}$ *is the* ***eigenspace*** *of* λ*.*

Note that $\vec{0}$ is in the eigenspace of λ even though it is not an eigenvector. Every other vector in the eigenspace of λ is an eigenvector.

Exercise 6.23 *Show that an eigenspace is in fact a vector space. Because it is a subset of $\mathbb{R}^n$, all we just need to show is that the eigenspace is closed under linear combination.*

Definition 6.24 *Given an eigenvalue λ, the dimension of the corresponding eigenspace is the* ***geometric multiplicity*** *of λ.*

To be clear: this is not the number of entries in an eigenvector (or even the number of nonzero entries), but rather the number of free variables when solving $(\mathrm{A}-\lambda\mathrm{I})\vec{v}=\vec{0}$.

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f free variables
$(\mathrm{A}-\lambda\mathrm{I})\vec{v}=\vec{0}$.

Theorem 6.25 *The geometric multiplicity of an eigenvalue is always at least one and at most the algebraic multiplicity.*

We've seen an example where the geometric multiplicity is less than the algebraic multiplicity. Here's an example where they are equal.

Example 6.26 *The matrix I has a repeated eigenvalue 1 with algebraic multiplicity 2. Any vector in $\mathbb{R}^2$ satisfies $\mathrm{I}\vec{v}=\vec{v}$. So the eigenspace is $\mathbb{R}^2$, which is two-dimensional.*

Definition 6.27 *If the geometric multiplicity of λ equals its algebraic multiplicity, then we say that λ has a* ***full eigenspace****. Otherwise it has a* ***deficient eigenspace****.*

Deficient eigenspaces are nasty, evil things that make our lives difficult. We avoid them whenever possible, but may discuss them in more detail later in the course. For now we are interested in cases where we have a full eigenspace.

Exercise 6.28 *For the matrices in exercise 6.14, find the eigenvectors and give the geometric multiplicity of each eigenvalue.*

Theorem 6.29 *If every eigenvalue of A is real and has a full eigenspace, then by taking a basis for each eigenspace and combining all of these bases, we arrive at a basis for $\mathbb{R}^n$. If any eigenvalue of A has a deficient eigenspace, then there is no basis for $\mathbb{R}^n$ made up of eigenvectors.*

If you find yourself in the situation of having a deficient eigenspace, look up "generalized eigenvectors"[2] and "Jordan normal form".

Exercise 6.30 *The result of this exercise seems to show up frequently on Math GREs. A matrix satisfies its characteristic equation (when converted to a matrix equation).*
Assume the characteristic polynomial of A is $p_{\mathrm{A}}(\lambda)=\sum c_i\lambda^i$. We can define $p_{\mathrm{A}}(\mathrm{B})$ where B is an $n\times n$ matrix. The result is an $n\times n$ matrix $p_{\mathrm{A}}(\mathrm{B})=\sum c_i\mathrm{B}^i$.

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(a). Let $\vec{w}=\sum a_i\vec{v}_i$ where $\vec{v}_i$ are the eigenvectors. Show that $p_{\mathrm{A}}(\mathrm{A})\vec{w}=\vec{0}$.

(b). Explain why this means $p_{\mathrm{A}}(\mathrm{A})=0$ (that is, the $n\times n$ zero matrix) if every eigenvalue of A has a full eigenspace. [hint: if it has a nonzero entry in the i-th column, what is $\mathrm{A}\vec{e}_i$ where $\vec{e}_i$ is the i-the canonical basis vector?]

[In fact $p_{\mathrm{A}}(\mathrm{A})=0$ even if the eigenspace is not full, but the proof requires a few concepts I don't intend to cover.]

Example 6.31 *The matrix $\mathrm{A}=\begin{bmatrix}1&1&0\\1&1&0\\0&0&2\end{bmatrix}$ has eigenvalues 2 and 0. The eigenvalue 0 has algebraic multiplicity 1 and the eigenvalue 2 has algebraic multiplicity 2.*

To see this: $\mathrm{A}-\lambda\mathrm{I}=\begin{bmatrix}1-\lambda&1&0\\1&1-\lambda&0\\0&0&2-\lambda\end{bmatrix}$. The determinant is $-\lambda(2-\lambda)^2$.

[2] be careful — there is more than one meaning to "generalized eigenvector". You want the version that has $(\mathrm{A}-\lambda\mathrm{I})\vec{v}=\vec{u}$ where $\vec{u}$ is an eigenvector of A. You do not want the generalization that involves solving $\mathrm{A}\vec{v}=\lambda\mathrm{B}\vec{v}$.

We can choose $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ *as the eigenvector of* 0. *We can find two linearly independent eigenvectors of* 2*:* $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. *These are the vectors that come out of finding the null space of* $\mathrm{A} - \lambda\mathrm{I}$.

The vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, *and* $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ *are three linearly independent vectors in* $\mathbb{R}^3$. *So they provide a basis.*

Example 6.32 *The matrix* $\mathrm{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ *also has eigenvalues* 2 *and* 0 *with* 0 *have algebraic multiplicity* 1 *while* 2 *has algebraic multiplicity* 2.

In this case, the eigenvector of 0 *is* $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. *When we try to find the eigenvectors of* 2, *we look for the null space of* $\mathrm{A} - 2\mathrm{I} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. *If* $\vec{v}$ *is in the null space of this, there is only one free variable: we must have* $v_1 = 0$ *and* $v_2 = 0$. *So we can only find a single linearly independent eigenvector,* $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

The vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ *clearly do not span* $\mathbb{R}^3$. *This is a case of a deficient eigenspace.*

This result is important. For most cases we do not have repeated eigenvalues. So if there is no repeated eigenvalue, we are guaranteed that we can choose a basis made up of eigenvectors, which allows us to do our calculations much more easily.

Theorem 6.33 *The product of the eigenvalues of the matrix* A *(accounting for multiplicity) is* $\det \mathrm{A}$. *That is if* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *are eigenvalues of* A *with algebraic multiplicity* $m_1, m_2, \ldots, m_k$, *then* $\lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_k^{m_k} = \det \mathrm{A}$.

Theorem 6.34 *The sum of the eigenvalues of the matrix* A *is the trace of* A, *that is, the sum of the diagonal entries of* A. *That is* $m_1\lambda_1 + \cdots + m_k\lambda_k = a_{11} + a_{22} + \cdots + a_{nn}$.

6.6 Left Eigenvectors

Theorem 6.35 *If* λ *is an eigenvalue of* A, *then it is also an eigenvalue of* A^T.

Exercise 6.36 *Show that Theorem 6.35 is true. The main step is to explain why if* $\det(\mathrm{A} - \lambda\mathrm{I}) = 0$, *then so does* $\det(\mathrm{A}^T - \lambda\mathrm{I})$.

Definition 6.37 *A nonzero row vector* $\vec{w}^T$ *is a* ***left eigenvector*** *if* $\vec{w}^T\mathrm{A} = \lambda\vec{w}^T$ *for some number* λ.

I am likely to sometimes be inconsistent and use $\vec{w}$ rather than $\vec{w}^T$ for a left eigenvector.

apparently this to suggest $\vec{w}^T$ row of A^T

In fact, if $\vec{w}^T$ is a left eigenvector of A with eigenvalue λ, then $\vec{w}$ is an eigenvector of A^T again with eigenvalue λ. Thus, the most obvious way to find left eigenvectors of a matrix is to find eigenvectors of its transpose (recalling that the eigenvalues are the same).

Exercise 6.38 *For the matrices in exercise 6.14, find the left eigenvectors by finding eigenvectors of* A^T.

This brings us to one of my favorite theorems of mathematics.

Theorem 6.39 *Given an $n \times n$ matrix A, if $\vec{w}^T$ is a left eigenvector for eigenvalue λ and $\vec{v}$ is a right eigenvector for eigenvalue $\mu \neq \lambda$, then $\vec{v} \cdot \vec{w} = 0$.*

Assuming n different eigenvalues, this means that each eigenvalue has just a single eigenvector. Thus we have a set of vectors $\vec{v}_1, \ldots, \vec{v}_n$ which are a basis for $\mathbb{R}^n$ and a set of vectors $\vec{w}_1, \ldots, \vec{w}_n$ which have the property that $\vec{w}_i$ is orthogonal to $\vec{v}_j$ whenever $i \neq j$. We spent time talking about things like this in the chapter on orthogonality.

Exercise 6.40 *Prove theorem 6.39 by noting that $(\vec{w}_i^T \mathrm{A})\vec{v}_j - \vec{w}_i^T(\mathrm{A}\vec{v}_j) = 0$.*
(technically this 0 on the right hand side is a 1×1 matrix, but that might as well be just a real number)

Recall that the derivative behaves similarly to a matrix, except that it acts on functions, not vectors. The concept of "eigenfunctions" makes sense for linear operations with functions.

Notice that $\vec{w}^T\mathrm{A}\vec{v}$ is essentially $\vec{w} \cdot (\mathrm{A}\vec{v})$, but it's also $(\mathrm{A}^T\vec{w}) \cdot \vec{v}$. If we use the more general inner product notation, we have $\langle \vec{w}, \mathrm{A}\vec{v} \rangle = \langle \mathrm{A}^T\vec{w}, \vec{v} \rangle$. If we are dealing with functions and linear operations acting on functions, this is the way we define the "adjoint" of the operation — it's the equivalent of the transpose: If L is a linear function of functions, then $\langle f(x), \mathrm{L}g(x) \rangle = \int_a^b f(x)\mathrm{L}g(x)\,\mathrm{d}x$. Usually through some sort of integration by parts we can convert this into $\int_a^b g(x)\mathrm{L}^T f(x)\,\mathrm{d}x$. Here L^T is called the "adjoint" and behaves like the transpose. So L has "adjoint eigenfunctions" as well as "eigenfunctions".

Exercise 6.41

(a). Show that $\cos mx$ and $\sin mx$ are eigenfunctions of the differential operator $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$. What is the eigenvalue?

Someday (if not already) you will take a course that uses Fourier Series and you will be asked to prove the following relations (or the professor will do it in lecture). Most likely the method the professor uses will involve some trigonometric identities, and some steps will get a bit messy. Impress him/her by suggesting this alternate approach (that even most profs haven't seen):
Assume m and n are integers with $m \neq n$ following these steps:

0

(b). By considering $\int_0^{2\pi} \cos mx \frac{\mathrm{d}^2}{\mathrm{d}x^2} \cos nx\,\mathrm{d}x$ show that $\int_0^{2\pi} \cos mx \cos nx\,\mathrm{d}x = 0$.

(i) By differentiating $\cos nx$ twice, find the constant c such that $\int_0^{2\pi} \cos mx \frac{\mathrm{d}^2}{\mathrm{d}x^2} \cos nx\,\mathrm{d}x = c\int_0^{2\pi} \cos mx \cos nx\,\mathrm{d}x$

(ii) By integrating by parts twice find another constant d such that $\int_0^{2\pi} \cos mx \frac{\mathrm{d}^2}{\mathrm{d}x^2} \cos nx\,\mathrm{d}x = d\int_0^{2\pi} \cos mx \cos nx\,\mathrm{d}x$.

(iii) Explain why this means $\int_0^{2\pi} \cos mx \cos nx\,\mathrm{d}x = 0$.

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(c). Repeat these steps for $\int_0^{2\pi} \sin mx \sin nx\,\mathrm{d}x = 0$.

(d). Repeat these steps for $\int_0^{2\pi} \cos mx \sin nx\,\mathrm{d}x = 0$.

This method doesn't prove that for for $\sin$ and $\cos$ with the same m: $\int_0^{2\pi} \sin mx \cos mx\,\mathrm{d}x = 0$, but this is straightforward to do through a symmetry argument (the contributions from 0 to π and π to 2π cancel out). We won't deal with it here.

One way to find the left eigenvectors of a matrix is to find the right eigenvectors of its transpose. There is an alternate method. If the eigenvectors of A form a basis for $\mathbb{R}^n$, we can create the matrix P whose columns are the eigenvectors of A. The matrix P will be invertible. The i-th row of P^{-1} is orthogonal to every column of P except the i-th column. We can use this to show that the i-th row of P^{-1} is a left eigenvector of A corresponding to the same eigenvalue as the i-th column of P. So we can find the left eigenvectors by finding the inverse of P (or, if we know the left eigenvectors, we can find the inverse of P from using the left eigenvectors).

In practice, if I only needed a few left eigenvectors of a large matrix, I would calculate them individually. If I needed all of them, and I knew the right eigenvectors (and the right eigenvectors formed a basis for Re^n), then it would be faster to invert the matrix than to individually calculate each left eigenvector.

In practice though, often we just want one or two eigenvectors. We will see examples of why this would be. In this case it is faster to find just the left eigenvector of interest by looking at A^T.

6.7 Matrices with Full Eigenspaces

In this section we will make frequent use of the orthogonality of left and right eigenvectors. If $\vec{w}$ is a left eigenvector of A and $\vec{v}$ is a right eigenvector for a different eigenvalue then $\vec{w} \cdot \vec{v} = 0$.

The usefulness of this comes from the orthogonality results we had earlier. When we have a vector $\vec{x}$ we frequently want to write it as a linear combination of the eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$.

If no eigenvalues are repeated, then we have a set of vectors $\vec{v}_1, \ldots, \vec{v}_n$ (the right eigenvectors) and another set of vectors $\vec{w}_1, \ldots, \vec{w}_n$ (the left eigenvectors) such that $\vec{w}_i \cdot \vec{v}_j = 0$ whenever $i \neq j$. We saw a bit about why such properties are useful in the orthogonality chapter.

Exercise 6.42 *Show that if every eigenvalue of* A *is real and has algebraic multiplicity* 1*, then it is possible to find a basis for* $\mathbb{R}^n$ *made up of eigenvectors of* A.

If an eigenvalue λ is repeated and its eigenspace is full, then the left eigenvectors of that eigenvalue will be orthogonal to all the right eigenvectors of other eigenvalues, but they are not automatically orthogonal to different right eigenvectors of λ. We can choose our left eigenvectors so that each is orthogonal to all but one of the right eigenvectors (discussed later).

Example 6.43 *A matrix with no repeated eigenvalues. Consider the matrix* $\mathrm{A} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$. *Find the eigenvalues, left and right eigenvectors, and write* $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *as a linear combination of the coefficients.*

We take $\det(\mathrm{A} - \lambda \mathrm{I}) = (1-\lambda)(-1-\lambda) - 3 = \lambda^2 - 4$. *We set this equal to* 0. *The eigenvalues are* 2 *and* -2.

Let's first look for right eigenvectors For $\lambda_1 = 2$, *we solve* $(\mathrm{A} - 2\mathrm{I})\vec{v_1} = \vec{0}$. *We get* $\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \vec{s} = \vec{0}$.

It's fairly straightforward to see that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *is the eigenvector.*

Now $\lambda_2 = -2$: *We have* $(\mathrm{A} + 2\mathrm{I})\vec{v}_2 = \vec{0}$. *We get* $\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \vec{v}_2 = \vec{0}$. *So we can see that* $\vec{v}_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$.

We now look at left eigenvectors. We take $\mathrm{A}^T - \lambda \mathrm{I}$: *For* $\lambda_1 = 2$ *we have* $\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \vec{w}_1 = \vec{0}$, *so* $\vec{w}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Similarly For $\lambda_2 = -2$, *we have* $\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \vec{w}_2 = \vec{0}$. *So* $\vec{w}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

It's easy to check that $\vec{w}_2 \cdot \vec{v}_1 = 0$ *and* $\vec{w}_1 \cdot \vec{v}_2 = 0$.

So in cases where there eigenvalues are not repeated, we quite easily get the orthogonality we expect.

To write $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *as a linear combination* $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$, *we take the dot product of the equation with* $\vec{w}_1$ *to find* $\vec{x} \cdot \vec{w}_1 = c_1 \vec{v}_1 \cdot \vec{w}_1 + 0$ *and find* $c_1 = \vec{x} \cdot \vec{w}_1 / \vec{v}_1 \cdot \vec{w}_1 = 5/4$. *We similarly have* $c_2 = \vec{x} \cdot \vec{w}_2 / \vec{v}_2 \cdot \vec{w}_2 = 1/(4/3) = 3/4$.

So we have $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5/4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3/4 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$, *and a simple check shows this is correct.*

Example 6.44 *A matrix with repeated eigenvalue, and full eigenspaces. Consider the matrix* $\mathrm{A} = \begin{bmatrix} 3 & 0 & -2 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$.

Write $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ *as a linear combination of the eigenvectors.*

I'll expand the determinant across the second column. We find that $\det(\mathrm{A}-\lambda\mathrm{I}) = (1-\lambda)\det\begin{bmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{bmatrix}$. *This is* $(1-\lambda)(\lambda^2 - 3\lambda + 2)$, *so when we set this equal to* 0, *we find* $\lambda = 1$ *or* $\lambda = 2$, *with* $\lambda = 1$ *having algebraic multiplicity* 2.

Let's look at the right eigenvectors of 1. *We solve* $(\mathrm{A} - 1\mathrm{I})\vec{s} = 0$ *(I use* $\vec{s}$ *rather than* $\vec{v}$ *just so that we don't have confusion of using 'v' for too many things). We get*

$$\left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So we find that s_2 *and* s_3 *are free variables and* $s_1 = s_3$. *So the solution is*

$$\vec{s} = \begin{bmatrix} s_3 \\ s_2 \\ s_3 \end{bmatrix} = s_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Where we get it into parametric form. Our vectors are thus $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ *and* $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = 2$, *we again look for solutions to* $(\mathrm{A} - 2\mathrm{I})\vec{s} = \vec{0}$. *We have*

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 1 & -1 & -1 & \\ 1 & 0 & -2 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We find that s_3 *is free,* $s_2 = s_3$, *and* $s_1 = 2s_3$. *So our solution is* $\vec{s} = \begin{bmatrix} 2s_3 \\ s_3 \\ s_3 \end{bmatrix} = s_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. *So we take* $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

Let's look at the left eigenvectors. We start with $(\mathrm{A}^T - \lambda\mathrm{I})\vec{s} = \vec{0}$. *We get*

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & -1 & -1 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So we have s_2 *and* s_3 *both free, with* $s_1 = -(s_2 + s_3)/2$. *Getting into parametric form:*

$$\vec{s} = \begin{bmatrix} -(s_2 + s_3)/2 \\ s_2 \\ s_3 \end{bmatrix} = s_2 \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$$

So $\vec{w}_1 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$ *and* $\vec{w}_2 = \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$.

For the eigenvalue of 2, *we follow these steps to get* $\vec{w}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

So you can check that the left eigenvector we find for $\lambda = 2$ is orthogonal to both right eigenvectors for $\lambda = 1$, and the right eigenvector for $\lambda = 2$ is orthogonal to both left eigenvectors for $\lambda = 1$. However, we can see that $\vec{w}_1$ is not orthogonal to either of $\vec{v}_1$ or $\vec{v}_2$.
It is possible to choose a different pair of vectors in the left eigenspace that will have the orthogonality we want. We could take $\vec{w}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$. These are left eigenvectors of 1 and have the desired orthogonality condition. We'll see soon how I found these.
To write $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ as a linear combination of the eigenvectors, we write $\vec{x} = \sum c_i\vec{v}_i$ where $c_i = \vec{x} \cdot \vec{w}_i / \vec{v}_i \cdot \vec{w}_i$. For this formula to work, we must use the left eigenvectors with the orthogonality relations. We have $c_1 = -1/1 = -1$, $c_2 = -3/1 = -3$, and $c_3 = -2/-1 = 2$. So we have $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

Symmetric matrices are a very important special case. In fact

Theorem 6.45 *If A is an $n \times n$ symmetric real matrix, then all eigenvalues of A are real and have full eigenspace.*

Further, because $A = A^T$, we see that the left and right eigenvectors are the same. So

Theorem 6.46 *If A is an $n \times n$ symmetric real matrix, then we can choose an orthogonal basis $\vec{v}_1, \ldots, \vec{v}_n$ made up of eigenvectors of A.*

If the eigenvalues all have multiplicity 1, the eigenvectors we find are automatically orthogonal. If an eigenvalue is repeated, then we may have to carefully choose our vectors to be orthogonal within each eigenspace.

Example 6.47 *The matrix $A = \begin{bmatrix} 5 & -1 & -1 \\ -1 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix}$ is symmetric. Write $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ as a linear combination of the eigenvectors. Taking $\det(A - \lambda I)$ we have*

$$\begin{aligned}
(5-\lambda)(1-\lambda)(1-\lambda) + (-1)(3)(-1) + (-1)(-1)(3) - [(-1)(1-\lambda)(-1) + (5-\lambda)(3)(3) + (-1)(-1)(1-\lambda)] \\
= (5-\lambda)(1-2\lambda+\lambda^2) + 6 - [1-\lambda+45-9\lambda+1-\lambda] \\
= -\lambda^3 + 7\lambda^2 - 11\lambda + 11 - [45 - 11\lambda] \\
= -\lambda^3 - 2\lambda^2 + 36
\end{aligned}$$

So $p_A(\lambda) = \lambda^3 + 2\lambda^2 - 36$. It's hardly obvious how this factors, though you can guess and check and you'll figure it out.
We find $p_A(\lambda) = (\lambda+2)(\lambda-3)(\lambda-6)$. So $\lambda_1 = -2$, $\lambda_2 = 3$, and $\lambda_3 = 6$. We can use the same methods to calculate the eigenvalues (if you want to test whether you've understood the methods, check that you get these). We have $\vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. You can check easily that these are orthogonal to one another.
We have $\vec{x} = \sum c_i\vec{v}_i$ where $c_i = \vec{x} \cdot \vec{v}_i / \vec{v}_i \cdot \vec{v}_i$. We find $c_1 = 0$, $c_2 = 1/3$, and $c_2 = -2/6 = -1/3$. So $\vec{x} = (1/3)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - (1/3)\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

The method we've used to find left and right eigenvectors will work. However, in the non-symmetric cases there's an awful lot of calculation going on. You may have noticed that once we had the eigenvectors, we could have found $\mathrm{P} = [\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n]$ and looked at $\mathrm{P}^{-1}\vec{x}$ to get the coefficients. In fact finding P^{-1} is the same as finding left eigenvectors of A.

We set $\mathrm{P} = [\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n]$. We can calculate P^{-1}. We note that $\mathrm{P}^{-1}\mathrm{P} = \mathrm{I}$ and so the i-th row of P^{-1} dotted with the j-th colum of P is 0 iff $i \neq j$ and 1 otherwise. A little thought will convince you that this means the i-th row of P^{-1} is a left eigenvector of A corresponding to λ_i. Further, if an eigenvalue has multiplicity greater than 1, this approach will automatically give us left eigenvectors which satisfy the orthogonality requirements we want (in example 6.44 the left eigenvectors of a repeated eigenvalue were not necessarily orthogonal to other right eigenvectors of the same eigenvalue). As one last bonus, we get that $\vec{v}_i \cdot \vec{w}_i = 1$ when we choose $\vec{w}_i$ to be the i-th row of P^{-1}.

Note that this means that in the particular case where each eigenvalue has multiplicity 1, we have

$$\mathrm{P}^{-1} = \begin{bmatrix} \vec{u}_1^T \\ \vec{u}_2^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix}$$

where $\vec{u}_i = \vec{w}_i/(\vec{w}_i \cdot \vec{v}_i$.

So what's going on here is that once we have all of the right eigenvectors, we actually know quite a bit about any given left eigenvector. In particular, we know $n-1$ vectors that it is orthogonal to (the other $n-1$ right eigenvectors). If we start from the beginning and look at $\mathrm{A}^T - \lambda\mathrm{I}$, we're throwing out that information. If we work to invert P instead, we're using that information.

Which of these approaches is better? That depends on the question we're studying. If we need all the right and left eigenvectors, it's best to find the right eigenvectors, write out P and calculate P^{-1}. On the other hand, there are may scenarios where we just care about the eigenvectors of the largest one or two eigenvalues. In this case it's better to find the right and left eigenvectors of just those eigenvalues. We'll see some examples of this, and it will save you a lot of effort in the "leap frog problem" coming up, exercise 6.49.

Example 6.48 *Early in this course, we discussed how to write down a system of linear differential equations in the form $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathrm{A}\vec{x}$ where A is an $n \times n$ matrix. We have not yet discussed how to solve them, so for now I'll just tell you what the solution looks like when we get to that stage. Assume $x_1(0) = 1$, $x_2(0) = -1$.*
Consider

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1 = x_1 - x_2$$
$$\frac{\mathrm{d}}{\mathrm{d}t}x_2 = 2x_1 - 2x_2$$

Assume we want to know what the solution to this behaves like at large time.
Let $\mathrm{A} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathrm{A}\vec{x}$$

The eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = -1$. If $\vec{v}_1$ and $\vec{v}_2$ are their eigenvectors, it can be shown that the solution is (for now, just trust me that it looks like this)

$$c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 = c_1\vec{v}_1 + c_2 e^{-t}\vec{v}_2$$

If we are only interested in what happens at large values of t, the fact that $\lambda_2 < 0$ means no matter what c_2 is, it is unimportant at large enough time. So we ignore it. We can avoid calculating c_2 and $\vec{v}_2$, but we need to know c_1 and $\vec{v}_1$.

We can find that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

But we still need c_1. *Using the orthogonality relations:* $\vec{w}_1 \cdot \vec{x}(0) = \vec{w}_1 \cdot (c_1\vec{v}_1 + c_2\vec{v}_2) = c_1\vec{w}_1 \cdot \vec{v}_1$ *So* $c_1 = \vec{w}_1 \cdot \vec{x}(0)/\vec{w}_1 \cdot \vec{v}_1$.

So we find $\vec{w}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, *and thus*

$$\vec{c}_1 = \frac{\vec{w}_1 \cdot \vec{x}(0)}{\vec{w}_1 \cdot \vec{v}_1} = \frac{\begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \frac{2+1}{2-1} = 3$$

So at large time the solution from the given initial condition converges to $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

Exercise 6.49 *Consider the "leap frog" exercise 1.78. For an integer* n, *the matrix that takes us from* $t = n$ *to* $t = n+1$ *is* $\mathrm{A} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 2 \\ -2 & 4 & -1 \end{bmatrix}$ *(that is if* $\vec{x}(n)$ *is the position at the start of the sequence, then* $\mathrm{A}\vec{x}(n)$ *gives the positions after all three have finished their next jump).*

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(a). Using exercise 6.2, show that 1 is an eigenvalue of A, and find ~~a right eigenvector~~ its left and right eigenvectors. Call these $\vec{w}_2^T$ *and* $\vec{v}_2$.

(b). Find the eigenvalues of A using the standard method. Since you know 1 is an eigenvalue, this should help to factor the characteristic polynomial. You should have one larger than 1 (in magnitude), the one equal to 1, and one less than 1 (in magnitude). Call them λ_1, λ_2, *and* λ_3 *with* $\lambda_2 = 1$ *and* $|\lambda_1| > 1 > |\lambda_3|$.

(c). For λ_1, *find a left eigenvector* $\vec{w}_1^T$. *[note, if it's not orthogonal to the* right *eigenvector of* $\lambda_2 = 1$, *you've made a mistake]*

(d). Find $\vec{w}_2^T$, *the left eigenvector of 1.*

(e). Let the right eigenvectors of A be denoted $\vec{v}_1$, $\vec{v}_2$, *and* $\vec{v}_3$ *(you don't need to calculate them). If* $\vec{x}(0) = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$, *find* $\vec{x}(n) = \mathrm{A}^n\vec{x}(0)$ *in terms of the eigenvalues and the (as yet uncalculated eigenvectors).*

(f). Assume the frogs begin with $\vec{x}(0) = \begin{bmatrix} -1 \\ 0 \\ s \end{bmatrix}$. *If you know that as* $n \to \infty$, *the distance of the frogs from one another does not grow, what do you know about* c_1? *Using this, find* s.

(g). Given s, *find where each frog converges as* $n \to \infty$.

(h). Assume that you enter this value of s *into a computer, but the computer has round-off error, so it has an error of size* $1.1|\lambda_1|^{-10}$ *in the initial value of* c_1. *How many steps will it take before the error for* $\vec{x}(n)$ *in the coefficient of* $\vec{v}_1$ *is as large as 1?*

Notice that in this exercise, we never have to calculate the eigenvectors of λ_3 *to find out what happens at large* n. *This happens in many applications: Frequently we are only interested in the eigenvectors of the largest 1 or 2 eigenvalues.*

6.7.1 Diagonalization

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Directly taking powers of a matrix is generally not an easy process.

In example 1.75, we introduced the matrix $\mathrm{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. In exercise 4.24, we saw a method that allowed us to transform this matrix into a diagonal matrix. In this section we will explore this more closely.

If $\phi = (1+\sqrt{5})/2$, then the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ \phi \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1-\phi \end{bmatrix}$ satisfy $\mathrm{A}\vec{v}_1 = \phi\vec{v}_1$, $\mathrm{A}\vec{v}_2 = (1-\phi)\vec{v}_2$.

We now know that this means ϕ is an eigenvalue with eigenvector $\vec{v}_1$ and $1-\phi$ is an eigenvalue with eigenvector $\vec{v}_2$.

The vectors $\vec{v}_1$ and $\vec{v}_2$ form a basis for $\mathbb{R}^2$. If we assume that $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$ then we can multiply by A very easily. We found that

$$\mathrm{A}^k\vec{x} = c_1\phi^k\vec{v}_1 + c_2(1-\phi)^k\vec{v}_2$$

The key observation is that this works because we found a basis for $\mathbb{R}^2$ that is made up of eigenvectors of A.

How do we find c_1 and c_2? We set $\mathrm{P} = [\vec{v}_1\ \vec{v}_2]$ and note that $\mathrm{P}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathrm{P}^{-1}\vec{x}$$

So multiplying by P^{-1} changes coordinates to the coordinates that the matrix A really prefers.

We want to see how multiplication by A behaves in this coordinate system. If we look at $\mathrm{A}(c_1\vec{v}_1 + c_2\vec{v}_2)$, it is $c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2$. Doing a similar operation to this, we see that

$$\mathrm{P}^{-1}\mathrm{A}\vec{x} = \begin{bmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \end{bmatrix}$$

But if we set $\mathrm{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, we see that

$$\mathrm{P}^{-1}\mathrm{A}\vec{x} = \mathrm{D}\mathrm{P}^{-1}\vec{x}$$

And this must hold for all $\vec{x}$. In other words, we can show that

$$\mathrm{P}^{-1}\mathrm{A} = \mathrm{D}\mathrm{P}^{-1}$$

Which converts to

$$\mathrm{A} = \mathrm{P}\mathrm{D}\mathrm{P}^{-1}$$

There was nothing special about this matrix A that made this work, except the fact that no eigenvectors have deficient eigenspaces.

I want to go through the change of basis a bit better. The eigenvectors of this matrix are a bit awkward to draw. So let's look at another example:

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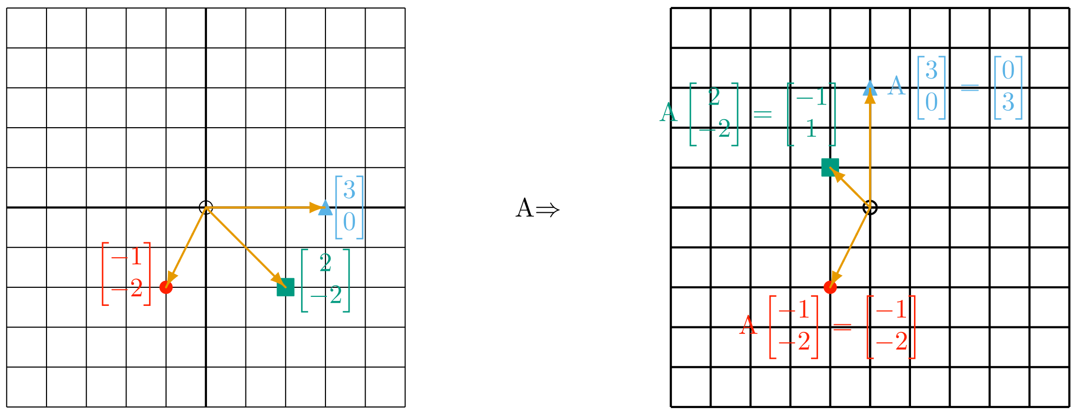
Example 6.50 *Consider the matrix* $\mathrm{A} = \begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix}$.

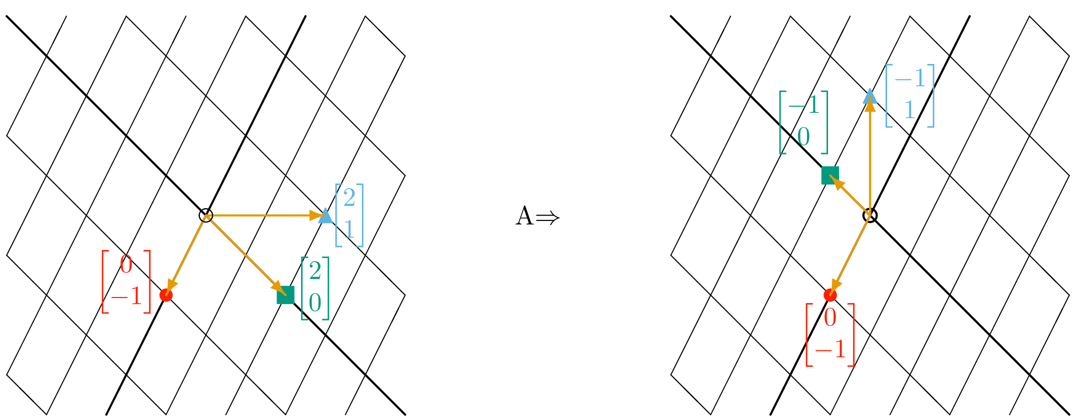
The characteristic polynomial is

$$\begin{aligned} p(\lambda) &= -\lambda\left(\frac{1}{2} - \lambda\right) - \frac{1}{2} \\ &= \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} \\ &= (\lambda - 1)\left(\lambda + \frac{1}{2}\right) \end{aligned}$$

So the eigenvalues are $-1/2$ and 1. Some work shows the eigenvectors are $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$ *respectively.*
Let's look at a few examples. What is $A\begin{bmatrix}3\\0\end{bmatrix}$*? It's* $\begin{bmatrix}0\\3\end{bmatrix}$. *What about* $A\begin{bmatrix}-1\\-2\end{bmatrix}$*? It's* $\begin{bmatrix}-1\\-2\end{bmatrix}$. *And* $A\begin{bmatrix}2\\-2\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$. *Let's look at the picture*



The arrows on the left represent the vectors before multiplying by A. *The points on the right represent the vectors after multiplying by* A *There's no clear pattern for where a vector is and where* A *puts it. But if instead of using the canonical basis, we use the eigenvectors for our coordinates: For a given vector* $\vec{w}$, *we know that* $\vec{w}=c_1\vec{v}_1+c_2\vec{v}_2$. *So we can represent the vector* $\vec{w}$ *with another vector* $\vec{c}=\begin{bmatrix}c_1\\c_2\end{bmatrix}$.
Again, to find $\vec{c}$, *we note that* $\vec{w}=P\vec{c}$, *so* $\vec{c}=P^{-1}\vec{w}$.
For $\begin{bmatrix}3\\0\end{bmatrix}$, *we have* $c_1=2,\ \ c_2=1$. *For* $\begin{bmatrix}-1\\-2\end{bmatrix}$, *we have* $c_1=0,\ \ c_2=-1$. *For* $\begin{bmatrix}2\\-1\end{bmatrix}$ *we have* $c_1=2$, $c_2=0$. *We can do the same transformation to* $A\vec{w}$. *We draw exactly the same points, but label them in the eigenvector coordinates:*



The thick curves are the eigenvectors. In this coordinate system, the role that A *plays is to multiply the first coordinate by* $-1/2$ *and to multiply the second coordinate by* 1. *That is, in the direction of each eigenvector, the matrix multiplies by the eigenvalue. So* <u>*in this coordinate system*</u> A *behaves like the matrix* $D=\begin{bmatrix}-1/2&0\\0&1\end{bmatrix}$.
To be more specific, if we change coordinates and then multiply by D *we get the same result as if we multiply by* A *and then change coordinates. Since the coordinate change is given by multiplication by* P^{-1}, *this means* $DP^{-1}=P^{-1}A$ *We can solve to get* $A=PDP^{-1}$.

In this case $\mathrm{P} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ *and* $\mathrm{D} = \begin{bmatrix} -1/2 & 0 \\ 0 & 1 \end{bmatrix}$. *We can invert* P *to get* $\mathrm{P}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$.
The intuition underlying what we've done is that when we're using A*, the eigenvectors form a better basis for* $\mathbb{R}^n$ *than the canonical basis. The matrix* P^{-1} *changes coordinates to this basis. We then do the equivalent calculations in this basis (multiplication by* D*). Finally we change back into the original coordinates (the canonical basis) by multiplying by* P*. See the diagram below*

$$\begin{tikzcd}
\vec{x} = \sum c_i\vec{v}_i \arrow[r, "\text{Change Coordinates}"] \arrow[d, "\text{Multiply by A}"] & \mathrm{P}^{-1}\vec{x} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \arrow[d, "\text{Multiply by D}"] \\
\mathrm{A}\vec{x} = \mathrm{PDP}^{-1}\vec{x} = \sum \lambda_i c_i \vec{v}_i & \mathrm{DP}^{-1}\vec{x} = \begin{bmatrix} \lambda_1 c_1 \\ \vdots \\ \lambda_n c_n \end{bmatrix} \arrow[l, "\text{Change Coordinates back}"]
\end{tikzcd}$$

In practice, finding P, D, and P^{-1} may take more effort than just calculating $\mathrm{A}\vec{w}$. Whether it's worth the effort depends on whether we multiply by A a lot. For example, a big savings can come from observing $\mathrm{A}^k = \mathrm{PDP}^{-1}\mathrm{PDP}^{-1}\mathrm{P}\cdots\mathrm{P}^{-1}\mathrm{PDP}^{-1} = \mathrm{PD}^k\mathrm{P}^{-1}$

Exercise 6.51 *For each matrix, find the eigenvalues and a basis for each eigenvalue's eigenspace. Diagonalize the matrix and calculate* A^{10}*. Show your work.*

(a). $\mathrm{A} = \begin{bmatrix} 0 & 0 & -2 \\ -2 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix}$

(b). $\mathrm{A} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Exercise 6.52 *Find a matrix with eigenvalues* 1, 2*, and* 3 *with corresponding eigenvectors* $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$*, and* $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$*. (hint: for this matrix, what would* D *and* P *be?)*

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Exercise 6.53 *Using the fact that* $\det(\mathrm{BC}) = \det \mathrm{B} \det \mathrm{C}$ *show that if* $\mathrm{A} = \mathrm{PDP}^{-1}$ *then theorem 6.33 is true.*

Exercise 6.54 *Show that the matrix* A *has the given left and right eigenvectors (just show* $\mathrm{A}\vec{v} = \lambda\vec{v}$ *and* $\vec{w}^T\mathrm{A} = \lambda\vec{w}^T$*). What are the corresponding eigenvalues? Find* P*,* D*, and using the relation between left eigenvectors and* P^{-1} *find* P^{-1}*. Check that* $\mathrm{A} = \mathrm{PDP}^{-1}$*. (note that the left and right eigenvectors*

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(a). $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. *Right eigenvectors:* $\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. *Left eigenvectors:* $\begin{bmatrix} 3 \\ -1 \end{bmatrix}^T, \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T$

(b). $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$ *Right:* $\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. *Left:* $\begin{bmatrix} 1 \\ 2 \end{bmatrix}^T, \begin{bmatrix} 1 \\ -2 \end{bmatrix}^T$

(c). $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. *Right* $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ *Left: the transposes of the right eigenvectors.*

Exercise 6.55 *In this exercise, we look back at example 6.50*

Consider the matrix $A = \begin{bmatrix} 1/2 & 0 \\ -5/2 & 3 \end{bmatrix}$. *The eigenvalues are* $\lambda_1 = 1/2$ *and* $\lambda_2 = 3$, *with eigenvectors* $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Take the vector $\vec{x} = -4\vec{v}_1 + 1\vec{v}_2 = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$.

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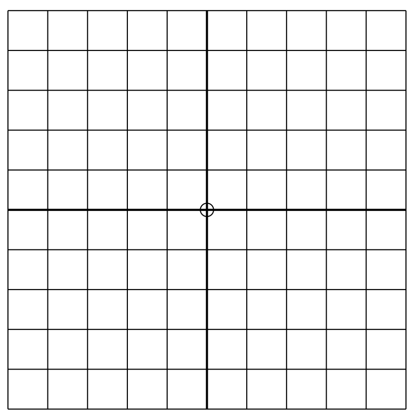
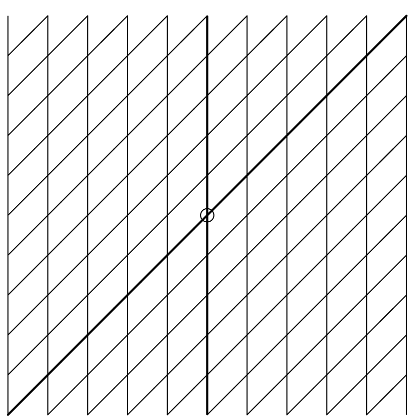
(a). *By directly multiplying by A, find* $A\vec{x}$ *and* $A^2\vec{x}$. *Sketch a grid like the below on your paper and draw* $\vec{x}$, $A\vec{x}$ *and* $A^2\vec{x}$.



(b). *By using the fact that we've written* $\vec{x}$ *as a linear combination of eigenvectors, find* $A\vec{x}$ *and* $A^2\vec{x}$ *as a linear combination of eigenvectors. Sketch a grid like the below on your paper and draw them. Make sure you are comfortable with why this coordinate system is easier to use.*



Exercise 6.56 *In this exercise, we look back at example 6.50*

Consider the matrix $A = \begin{bmatrix} 5/3 & 2/3 \\ 1/3 & 4/3 \end{bmatrix}$. *The eigenvalues are* 2 *and* 1 *The corresponding eigenvectors are* $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(a). Consider the vector $\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. *Find* c_1 *and* c_2 *such that* $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$ *by whatever means you want.*

(b). Draw a coordinate system based on the eigenvectors (as opposed to the canonical basis), and show where $\vec{w}$ *lies. What do* c_1 *and* c_2 *have to do with this?*

(c). Show where $A\vec{w}$ *and* $A^2\vec{w}$ *lie on the coordinate system. Think about how this relates to the original coordinate system.*

Alternate derivation of $A = PDP^{-1}$ Not everyone is happy with a derivation that relies on coordinate transformations. Here is another way to show that $A = PDP^{-1}$.

We start by looking at the columns of $AP = [A\vec{v}_1\ A\vec{v}_2\ \cdots\ A\vec{v}_n] = [\lambda_1\vec{v}_1\ \lambda_2\vec{v}_2\ \cdots\ \lambda_n\vec{v}_n]$.

We then consider the columns of $PD = [P\vec{d}_1\ P\vec{d}_2\ \cdots\ P\vec{d}_n]$ where $\vec{d}_i$ is the i-th column of D. If we look at the i-th column of D, its only nonzero entry is the i-th entry which is λ_i. So the multiplication $P\vec{d}_i$ will pick out the i-th column of P (that is $\vec{v}_i$) and multiply by λ_i. So $PD = [\lambda_1\vec{v}_1\ \lambda_2\vec{v}_2\ \cdots\ \lambda_n\vec{v}_n]$.

Putting this together gives $PD = AP$. Solving for A, we have $A = PDP^{-1}$.

Quite a bit of most introductory differential equations courses is spent on the following exercises (in one form or another). Basically if we have a linear system of differential equations, the way to solve it is to use the coordinate system that it wants. The matrix has a strongly preferred coordinate system — the eigenvectors. Don't try to impose a different coordinate system on the matrix. Use the one it likes. Switch to the eigenvector coordinates and the system changes from n coupled linear differential equations to n independent simple linear differential equations of the form $y'(t) = ay(t)$.

Consider the system of linear differential equations $\frac{d}{dt}\vec{x} = A\vec{x}$ where A is a diagonalizeable matrix that does not depend on t. We'll transform the equations into the more natural coordinate system for A.

Exercise 6.57 *For the below, note that the product rule of differentiation works for vectors and matrices.*

(a). Using the product rule, show that for any constant matrix B *and non-constant vector* $\vec{z}$, $\frac{d}{dt}(B\vec{z}) = B\frac{d}{dt}\vec{z}$.

(b). Consider the vector $\vec{y}(t) = P^{-1}\vec{x}(t)$. *Show that* $\frac{d}{dt}\vec{y} = D\vec{y}$. *So in the "natural" coordinate system for* A, *the system reduces to a diagonal matrix.*

(c). Show that for each i, $\frac{d}{dt}y_i = \lambda_i y_i$. *The solution to this equation is* $y_i(t) = ce^{\lambda_i t}$.

(d). Show that the solution to $\frac{d}{dt}\vec{x} = A\vec{x}$ *is* $\sum_i c_i e^{\lambda_i t}\vec{v}_i$.

(e). If we know the vector $\vec{x}(0)$, *show that* $P\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{x}(0)$, *so* $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = P^{-1}\vec{x}(0)$.

Here is a more direct way to approach the same concepts as exercise 6.57:

Exercise 6.58 *Let* A *be an* $n \times n$ *matrix and assume that all eigenvalues have full eigenspaces. Assume we need to solve* $\frac{d}{dt}\vec{x} = A\vec{x}$.

(a). Explain why we can assume that $\vec{x} = \sum y_i(t)\vec{v}_i$ *where* $\vec{v}_i$ *are the eigenvectors.*

(b). By substituting this into the differential equation, show that $\frac{\mathrm{d}}{\mathrm{d}t} y_i = \lambda_i y_i$. The solution to this equation is of the form $y_i = c_i e^{\lambda_i t}$.

(c). If we know the vector $\vec{x}(0)$, show that $\mathrm{P}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{x}(0)$, *so* $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \mathrm{P}^{-1}\vec{x}(0)$.

Exercise 6.59 *We've shown that if we find n linearly independent eigenvectors for A, we can write A as PDP^{-1} using the eigenvalues and eigenvectors. If we change the order of our eigenvalues and eigenvectors, we'll get a different D and P, but the equation will hold.*
Could we find another P and D that do not involve eigenvalues and eigenvectors such that $\mathrm{A} = \mathrm{PDP}^{-1}$? No.
Show that if A can be written in the form PDP^{-1} where D is a diagonal matrix, then if $\vec{v}_i$ is the ith column of P. Then $\mathrm{A}\vec{v}_i = d_{ii}\vec{v}_i$.

Exercise 6.60 *Given a diagonalizeable matrix $\mathrm{A} = \mathrm{PDP}^{-1}$, show that* $\mathrm{A}^k = \mathrm{P}\begin{bmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}\mathrm{P}^{-1}$.

Exercise 6.61 *In this exercise we revisit exercise 6.30, and use a different approach. Let A have full eigenspaces, so A is diagonalizable.*

(a). By writing $\mathrm{A} = \mathrm{PDP}^{-1}$, show that $p_{\mathrm{A}}(\mathrm{A}) = \mathrm{P}p_{\mathrm{A}}(\mathrm{D})\mathrm{P}^{-1}$.

(b). Explain why $p_{\mathrm{A}}(\mathrm{D}) = \begin{bmatrix} p_{\mathrm{A}}(\lambda_1) & 0 & 0 & \cdots & 0 \\ 0 & p_{\mathrm{A}}(\lambda_2) & 0 & \cdots & 0 \\ 0 & 0 & p_{\mathrm{A}}(\lambda_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p_{\mathrm{A}}(\lambda_n) \end{bmatrix}$

(c). Use this to prove that $p_{\mathrm{A}}(\mathrm{A}) = 0$.

Exercise 6.62 *Assume that A and B are $n \times n$ matrices, and we can find a set of (linearly independent) vectors $\vec{v}_1, \ldots, \vec{v}_n$ which are eigenvectors of both. So $\mathrm{A} = \mathrm{PD_AP}^{-1}$ and $\mathrm{B} = \mathrm{PD_BP}^{-1}$ with the same P in each case. Show that A and B commute: $\mathrm{AB} = \mathrm{BA}$.*

6.7.2 Diagonalizing symmetric matrices

If A is symmetric, we know that we can diagonalize it because every eigenvector has a full eigenspace. However, we can go a bit further. If there are no repeated eigenvalues, then the eigenvectors are automatically orthogonal. If sn eigenvalue is repeated, the basis we find for its eigenspace may not be orthogonal, but we can make it orthogonal using Gram-Schmidt (this method is in the notes, but we haven't discussed it). So we can guarantee an orthogonal basis.

If we go one step further, we can normalize each eigenvector, replacing $\vec{v}_i$ with $\vec{v}_i/|\vec{v}|$. So the new $\vec{v}_i$ satisfies $\vec{v}_i \cdot \vec{v}_i = 1$.

Definition 6.63 *A matrix Q is called an **orthogonal matrix** if $\mathrm{Q}^T = \mathrm{Q}^{-1}$.*

Exercise 6.64 *Show that if A is real and symmetric it is possible to find an orthogonal matrix Q such that $\mathrm{A} = \mathrm{QDQ}^T$.*

So every symmetric matrix can be diagonalized in such a way that the matrix of eigenvectors is an orthogonal matrix.

Example 6.65 *Consider the matrix* $\mathrm{A} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

We can find a matrix Q *such that* $\mathrm{Q}^{-1} = \mathrm{Q}^T$ *and* $\mathrm{A} = \mathrm{QDQ}^T$ *for which* D *is a diagonal matrix.*
To do this: we first find the eigenvalues and the eigenvectors. The eigenvalues are $\lambda_1 = 4$ *and* $\lambda_2 = -2$*, with eigenvectors* $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$*. Note that the fact that* A *is symmetric means that left and right eigenvectors are the same. So* $\vec{v}_1$ *is orthogonal to* $\vec{v}_2$*.*
Any nonzero multiples of $\vec{v}_1$ *and* $\vec{v}_2$ *give a* P *that works for diagonalizing. Let's look for one that gets* $\mathrm{P}^{-1} = \mathrm{P}^T$*.*
$\mathrm{P} = \begin{bmatrix} \frac{\vec{v}_1}{c_1} & \frac{\vec{v}_2}{c_2} \end{bmatrix}$*. Then* $\mathrm{P}^T = \begin{bmatrix} \vec{v}_1^T/c_1 \\ \vec{v}_2^T/c_2 \end{bmatrix}$*. Thus*

$$\mathrm{P}^T\mathrm{P} = \begin{bmatrix} \frac{\vec{v}_1\cdot\vec{v}_1}{c_1^2} & 0 \\ 0 & \frac{\vec{v}_2\cdot\vec{v}_2}{c_2^2} \end{bmatrix}$$

But we want this to equal I*, so we take* $c_1 = |\vec{v}_1|$ *and* $c_2 = |\vec{v}_2|$*.*

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In general we can do this for symmetric matrices and we get: $\mathrm{Q} = \begin{bmatrix} \frac{\vec{v}_1}{|\vec{v}_1|} & \frac{\vec{v}_2}{|\vec{v}_2|} & \cdots & \frac{\vec{v}_n}{|\vec{v}_n|} \end{bmatrix}$.

Exercise 6.66 *For the following matrices* A *find* Q *and* D *such that* $\mathrm{A} = \mathrm{QDQ}^T$, $\mathrm{Q}^{-1} = \mathrm{Q}^T$ *and* D *is diagonal.*

(a). $\mathrm{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

(b). $\mathrm{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

(c). $\mathrm{A} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$

6.7.3 Non-diagonalizeable matrices

Some matrices cannot be diagonalized. This happens when an eigenvalue does not have a full eigenspace. We won't go into detail on these, but it is possible to get something close to diagonal with a "Jordan Normal Form". Look it up on Wikipedia if needed. Basically instead of having a matrix whose only nonzero entries are on the diagonal, we have a matrix whose nonzero entries are on the diagonal and a few entries of 1 just above the main diagonal. The matrix P is no longer a matrix of just eigenvectors: it also has "generalized eigenvectors" to make up for the eigenspaces that are not full.

Exercise 6.67 *Show that it is not possible to find a basis for* $\mathbb{R}^n$ *made up of eigenvectors of the matrices below*

(a). $\mathrm{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

(b). $\mathrm{A} = \begin{bmatrix} 7 & 0 & 0 \\ 1 & 7 & 0 \\ 0 & 2 & 7 \end{bmatrix}$.

6.8 The Power method

Often we just want to know the largest eigenvalue of a matrix A and possibly its corresponding eigenvector. If the matrix is very large, it is simply too difficult to find the characteristic polynomial. So we need another method. For now assume that λ_1 is the largest eigenvalue, and no other eigenvector of equal magnitude exists. We say that A has a unique "dominant" eigenvalue.

If we take a random vector $\vec{w}$, we know that it is of the form $\vec{w} = \sum c_i \vec{v}_i$ where the $\vec{v}_i$ are eigenvectors of A (there is an exception: when an eigenvalue has a deficient eigenspace — the derivation I give needs a tiny bit of modification in that case, but the method still works).

Definition 6.68 *If* A *is an $n \times n$ matrix and λ_1 is an eigenvalue such that $|\lambda_1| > |\lambda_i|$ for $i \geq 2$ then λ_1 is the* ***dominant eigenvalue*** *of* A.

The power method is a method for calculating the dominant eigenvalue and its eigenvector. It does not give any information about other eigenvectors.

The power method for finding the dominant eigenvalue/eigenvector Given a matrix A with a unique dominant eigenvalue, the following method finds the dominant eigenvalue and its eigenvector:

(a). Choose a random initial vector $\vec{w}_0$. Divide $\vec{w}_0$ so that its entry of highest magnitude is 1. (other "normalization" rules are possible — this tends to behave better than others I know).

(b). Set $\vec{x}_k = \mathrm{A}\vec{w}_{k-1}$. Divide $\vec{x}_k$ by its entry of largest magnitude and set the result to be $\vec{w}_k$.

(c). Repeat until the result has converged.

The largest (in magnitude) eigenvalue is the number by which you are dividing through at each step. Its eigenvector is the limit of the $\vec{w}_k$.

Note that if the dominant eigenvalue has more than one eigenvector, this algorithm only finds one of them. If there is more than eigenvalue whose magnitude is a maximum, then the algorithm doesn't converge to a vector, instead it oscillates between different values

Why does it work? Let X_1, X_2, ... be the values by which we divide $\vec{x}_1$, $\vec{x}_2$, ... Then $\vec{w}_k = \frac{\mathrm{A}^k \vec{w}_0}{X_1 X_2 \cdots X_k} = \frac{1}{X_1 X_2 \cdots X_k}(\sum c_i \lambda_i^k \vec{v}_i)$. The $c_1 \lambda_1^k$ term is eventually much larger than all other terms (If $c_1 = 0$, then we can rely on round-off error from the computer to make this statement true anyways).

Example 6.69 *Consider the matrix* $\mathrm{A} = \begin{bmatrix} 5 & -1 \\ 3 & -2 \end{bmatrix}$. *Find the dominant eigenvalue and eigenvector using the power method.*

We take as our starting guess $\vec{w}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. *We'll use decimals*

k	$\vec{w}_k$	$\vec{x}_{k+1} = \mathrm{A}\vec{w}_k$
1	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}4\\1\end{bmatrix}$
2	$\begin{bmatrix}1\\0.25\end{bmatrix}$	$\begin{bmatrix}4.75\\2.5\end{bmatrix}$
3	$\begin{bmatrix}1\\0.526\end{bmatrix}$	$\begin{bmatrix}4.47\\1.95\end{bmatrix}$
4	$\begin{bmatrix}1\\0.435\end{bmatrix}$	$\begin{bmatrix}4.56\\2.12\end{bmatrix}$
5	$\begin{bmatrix}1\\0.466\end{bmatrix}$	$\begin{bmatrix}4.53\\2.06\end{bmatrix}$
6	$\begin{bmatrix}1\\0.456\end{bmatrix}$	$\begin{bmatrix}4.533\\2.09\end{bmatrix}$

And so we see that our eigenvector is approximately $\begin{bmatrix}1\\0.456\end{bmatrix}$ *and the eigenvalue is approximately* 4.533*.*

Example 6.70 *We'll change the matrix by multiplying the first row by* -1*. Consider the matrix* $\mathrm{A} = \begin{bmatrix}-5 & 1\\3 & -2\end{bmatrix}$*. Find the dominant eigenvalue and eigenvector using the power method.*

We take as our starting guess $\vec{w}_0 = \begin{bmatrix}1\\1\end{bmatrix}$*.*

k	$\vec{w}_k$	$\vec{x}_{k+1} = \mathrm{A}\vec{w}_k$
1	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}-4\\1\end{bmatrix}$
2	$\begin{bmatrix}1\\-0.25\end{bmatrix}$	$\begin{bmatrix}-5.25\\3.5\end{bmatrix}$
3	$\begin{bmatrix}1\\-0.667\end{bmatrix}$	$\begin{bmatrix}-5.67\\4.3\end{bmatrix}$
4	$\begin{bmatrix}1\\-0.765\end{bmatrix}$	$\begin{bmatrix}-5.76\\4.53\end{bmatrix}$
5	$\begin{bmatrix}1\\-0.786\end{bmatrix}$	$\begin{bmatrix}-5.78\\4.57\end{bmatrix}$
6	$\begin{bmatrix}1\\-0.791\end{bmatrix}$	$\begin{bmatrix}-5.79\\4.58\end{bmatrix}$

This time our eigenvector is approximately $\begin{bmatrix}1\\-0.791\end{bmatrix}$ *and the eigenvalue is approximately* -5.79*.*

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How quickly the method converges depends on how large the next largest eigenvalue is.

Exercise 6.71 *Consider the matrices* $\mathrm{A} = \begin{bmatrix}7/2 & 1\\3/2 & 1\end{bmatrix}$ *and* $\mathrm{B} = \begin{bmatrix}3 & 2\\3 & -2\end{bmatrix}$*. The eigenvalues of* A *are* 4 *and* $1/2$ *with eigenvectors* $\begin{bmatrix}1\\-2\end{bmatrix}$ *and* $\begin{bmatrix}2\\1\end{bmatrix}$ *respectively. The eigenvalues of* B *are* 4 *and* -3*, with the same eigenvectors as* A*. Please note that I switched the eigenvectors inadvertantly.*

(a). Following the power method with $\vec{w}_0 = \begin{bmatrix}1\\1\end{bmatrix}$*, calculate up to* $\vec{x}_5$ *and* $\vec{w}_5$ *for* A *(just do 2 digits of accuracy).*

(b). Using $\vec{w}_0 = \begin{bmatrix}1\\1\end{bmatrix}$*, calculate up to* $\vec{x}_5$ *and* $\vec{w}_5$ *for* B *(just do 2 digits of accuracy).*

(c). Explain the relative speeds with which $\vec{w}_k$ *is converging in each case.*

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Exercise 6.72 *Read sections 1 and 2 of The \$25,000,000,000 Eigenvector SIAM Rev., 48(3), 569–581.*

Chapter 7

Applications of eigenvalues/eigenvectors

7.1 PageRank

Want to make \$25 billion?

When Google went public, it was worth \$25 billion dollars. The basis of Google's search engine is a simple eigenvector. (As of Apr 5 2012 it was \$168.48 billion)

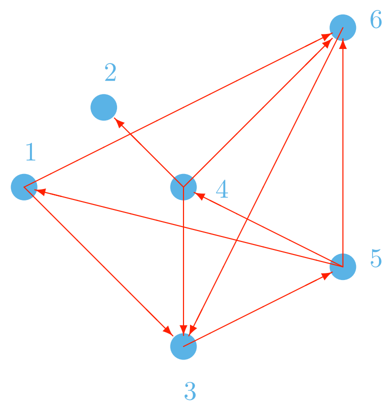
Let's look at the PageRank algorithm, designed by Page and Brin, the founders of Google (and named after Page).

We'll start with a simplification, and then look at the actual algorithm. Consider an individual surfing the web. Assume that when an individual is at a web page, s/he will follow a randomly chosen link from that page. If the webpage has no link, s/he will go to a randomly chosen webpage from the entire web.

After many steps, we might expect that the surfer will be more likely to be at a useful page simply because we assume the people creating links have a tendency to link to good pages rather than bad pages. So let's rank pages by the probability of being at one at large time. Let $p_i(k)$ be the probability of being at page i at step k. We use the vector $\vec{p}(k) = \begin{bmatrix} p_1(k) \\ p_2(k) \\ \vdots \\ p_n(k) \end{bmatrix}$. For now, we'll make the assumption that as k gets large, $\vec{p}(k)$ converges.

It will turn out that $\vec{p}(k)$ is converging to an eigenvector of a matrix.

Consider a network



So here's what our surfer is doing. When at node 1, the surfer goes to node 3 or node 6 each with equal probability. When at node 2 there is no link out. So if at node 2 at the next step the surfer visits a random one of the 6 nodes with equal probability. When at node 3 the surfer goes to node 5. From node 4 the surfer visits node 2 or 6 with equal probability. From node 5 the surfer goes to either node 1, 4, or 6 with equal probability. Finally, from node 6 the surfer goes to node 3.

Let $p_i(k)$ be the probability of being at node i in step k. How can we find $p_i(k+1)$? We need to be careful about this because it's a very natural mistake to come up with the transpose of the correct matrix: To find the probability of being at node 3 at step $k+1$ we must look at the arrows that point to node 3, not the arrows that point away from node 3.

Assume we know the vector $\vec{p}(k) = \begin{bmatrix} p_1(k) \\ p_2(k) \\ \vdots \\ p_n(k) \end{bmatrix}$. We need to find $\vec{p}(k+1)$. Let's look at $p_1(k+1)$.

How can we end up at node 1 at step $k+1$. If the surfer is at node 3, 4, or 6 then it does not go to node 1. It can come from node 5 or node 2. We find that $p_1(k+1) = p_2(k)/6 + p_5(k)/2$. Similarly $p_2(k+1) = p_2(k)/6 + p_4(k)/3$. In general

$$\begin{bmatrix} p_1(k+1) \\ p_2(k+1) \\ p_3(k+1) \\ p_4(k+1) \\ p_5(k+1) \\ p_6(k+1) \end{bmatrix} = \begin{bmatrix} p_2(k)/6 + p_5(k)/3 \\ p_2(k)/6 + p_4(k)/3 \\ p_1(k)/2 + p_2(k)/6 + p_4(k)/3 + p_6(k) \\ p_2(k)/6 + p_5(k)/3 \\ p_2(k)/6 + p_3(k) \\ p_1(k)/2 + p_2(k)/6 + p_4(k)/3 + p_5(k)/3 \end{bmatrix} = \begin{bmatrix} 0 & 1/6 & 0 & 0 & 1/3 & 0 \\ 0 & 1/6 & 0 & 1/3 & 0 & 0 \\ 1/2 & 1/6 & 0 & 1/3 & 0 & 1 \\ 0 & 1/6 & 0 & 0 & 1/3 & 0 \\ 0 & 1/6 & 1 & 0 & 0 & 0 \\ 1/2 & 1/6 & 0 & 1/3 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} p_1(k) \\ p_2(k) \\ p_3(k) \\ p_4(k) \\ p_5(k) \\ p_6(k) \end{bmatrix}$$

We'll call the matrix here A.

Let's make a few observations about A: First, all the columns sum to 1. This means that 1 is an eigenvalue and $[1\ 1\ \cdots\ 1]$ is a left eigenvector.

Let's assume that our surfer starts out with a randomly chosen node, so that $\vec{p}(0) = \begin{bmatrix} 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \end{bmatrix}$. The surfer follows a random link, arriving at $\vec{p}(1) = \mathrm{A}\vec{p}(0)$. Continuing this for a while, we hope that eventually the

$\vec{p}$-s converge. We use the probability that the surfer is at a given page at large time as a ranking mechanism for the pages.

How do we know that the process will converge?

Because each step corresponds to multiplication by A, what we are doing is basically the power method to find the dominant eigenvector. We know that 1 is an eigenvalue, and there is a theorem (below) which guarantees that there is no larger eigenvalue. However, it may happen that 1 is a repeated eigenvalue or that there is another eigenvalue (-1) with equal magnitude.

Exercise 7.1 *Show that if all columns of a matrix sum to* 1*, then the entries of any (right) eigenvector other than the eigenvector of* 1 *must sum to* 0*.*

Theorem 7.2 *(Perron Frobenius) If a matrix has all non-negative entries and all columns sum to* 1*, then*

(a). 1 *is an eigenvalue.*

(b). 1 *has an eigenvector all of whose entries are positive.*

(c). Every other eigenvalue has magnitude at most 1*.*

If all entries are positive then 1 *has algebraic multiplicity* 1*, and no other eigenvalue has magnitude* 1*.*

Exercise 7.3 *If some entries of a matrix are* 0*, the Perron-Frobenius theorem allows for* 1 *to be a repeated eigenvalue or for* -1 *to be an eigenvalue. Find* 2×2 *matrices* A *such that all entries are non-negative, all columns sum to* 1 *and*

(a). 1 *is a repeated eigenvalue of* A*.*

(b). -1 *is an eigenvalue of* A*.*

hint: it may help to start by figuring out what the characteristic polynomial has to be.

End of lecture 24

The full PageRank algorithm does what we've described, but adds one wrinkle. At each step, there is a probability ϵ which is small that the surfer doesn't follow a link, but instead starts over at another random node. Under this rule,

$$\vec{p}(t+1) = (1-\epsilon)\mathrm{A}\vec{p}(t) + \epsilon \begin{bmatrix} 1/N \\ 1/N \\ 1/N \\ \vdots \\ 1/N \end{bmatrix}$$

This is a fairly quick calculation to do because a lot of the entries in A are 0. We can rewrite this as

$$\vec{p}(t+1) = \mathrm{B}\vec{p}(t)$$

where

$$\mathrm{B} = (1-\epsilon)\mathrm{A} + \frac{\epsilon}{N} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

Doing this change makes the matrix B have all positive entries. So we are guaranteed a unique dominant eigenvalue and eigenvector. We wouldn't want the computer to have to calculate $\mathrm{B}\vec{p}$ because even with a computer when a matrix has millions of nonzero entries, multiplication is slow. Doing the calculation by finding $(1-\epsilon)\mathrm{A}\vec{p}$ and then adding $\epsilon \begin{bmatrix} 1/N \\ 1/N \\ 1/N \\ \vdots \\ 1/N \end{bmatrix}$ is fast because for the web, A is mostly made up of zeroes.

So the way we would actually have the computer perform the PageRank algorithm for our example network is to start with a guess for $\vec{p}(0)$ and then solve the following equations for $\vec{p}(1)$, then repeat to find $\vec{p}(2)$ etc

$$\begin{aligned}
p_1(k+1) &= (1-\epsilon)[p_2(k)/6 + p_5(k)/3] + \epsilon/6 \\
p_2(k+1) &= (1-\epsilon)[p_2(k)/6 + p_4(k)/3] + \epsilon/6 \\
p_3(k+1) &= (1-\epsilon)[p_1(k)/2 + p_2(k)/6 + p_4(k)/3 + p_6(k)] + \epsilon/6 \\
p_4(k+1) &= (1-\epsilon)[p_2(k)/6 + p_5(k)/3] + \epsilon/6 \\
p_5(k+1) &= (1-\epsilon)[p_2(k)/6 + p_3(k)] + \epsilon/6 \\
p_6(k+1) &= (1-\epsilon)[p_1(k)/2 + p_2(k)/6 + p_4(k)/3 + p_5(k)/3] + \epsilon/6
\end{aligned}$$

Can you see that this is the same as doing the power method with the matrix B above?

I assume that there is more to the PageRank algorithm these days, but they don't publish their method. One obvious change that could be made is to use a different "reset" vector: when the surfer starts over, s/he could be more likely to go to, say, a `.edu` site which would be less subject to corporations trying to manipulate their rankings.

7.2 Stochastic matrices

Definition 7.4 *A square matrix whose entries are all non-negative (that is, entries are either positive or zero) such that the entries in each column ~~row~~ sum to 1 is called a* ***Stochastic Matrix*** *(or sometimes a transition matrix).*

If a system is in one of N states at any given time, and the state at the next time is random but only depends on its current state, then the transitions can be represented by a stochastic matrix. If q_{ij} is the probability that if the system is in state j at time t then it is in state i at time $t+1$, then the matrix

$$\mathrm{A} = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix}$$

is the stochastic matrix that takes the system from $\vec{p}(t)$ to $\vec{p}(t+1)$. That is $\vec{p}(t+1) = \mathrm{A}\vec{p}(t)$.

Exercise 7.5 *For each of the following, determine whether it is a stochastic matrix*

(a). $\begin{bmatrix} 1 & 1 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.25 \\ 0 & 0 & 0 & 0 & 0.25 \end{bmatrix}$

(b). $\begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$

(c). $\begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.4 & 0.4 & 0.4 \end{bmatrix}$

(d). A^2 *where* $\mathrm{A} = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.4 & 0.4 & 0.4 \end{bmatrix}$

Exercise 7.6 *Define the function* $T : \mathbb{R}^n \to \mathbb{R}$ *by* $T(\vec{p}) = \sum p_i$*. Show that if* A *is a stochastic matrix, then* $T(\vec{p}) = T(\mathrm{A}\vec{p})$*. That is, if* A *is a stochastic matrix, the sum of the entries of* $\mathrm{A}\vec{p}$ *equals the sum of the entries of* $\vec{p}$*.*

Exercise 7.7 *A (newborn) baby has a number of different states. Let* p_1 *be the probability that the baby is eating,* p_2 *the probability it is crying,* p_3 *the probability it is getting its diaper changed,* p_4 *the probability it is soiling its diaper and* p_5 *the probability it is asleep.*

- *If the baby is eating at the current time step, the probability it is still eating at the next time step is* $1/2$*, the probability it is crying at the next time step is* $1/10$*, the probability it is soiling its diaper is* $2/10$*, and the probability it is sleeping at the next time step is* $2/10$*.*
- *If the baby is crying, the probability it is still crying at the next time step is* $1/4$*, the probability it is eating is* $1/2$*, and the probability it is soiling its diaper is* $1/4$
- *If the baby is getting its diaper changed, the probability it is soiling its diaper at the next time step is* $1/2$ *and the probability it is eating is* $3/8$*, and the probability it is asleep is* $1/8$*.*
- *If the baby is soiling its diaper, at the next time step its diaper is changed.*
- *If the baby is asleep, the probability it is still asleep at the next time step is* $3/4$*, the probability it is soiling its diaper is* $1/8$*, and the probability it is crying is* $1/8$*.*

(a). Find the matrix A *that takes* $\vec{p}$ *at one time to* $\vec{p}$ *at the next time step.*

(b). Starting with the vector $\vec{p}(0) = \begin{bmatrix} 1/5 \\ 1/5 \\ 1/5 \\ 1/5 \\ 1/5 \end{bmatrix}$*, do three iterations of the power method.*

(c). Startingwith the vector $\vec{p}(0) = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 0 \\ 0 \end{bmatrix}$ *do three iterations of the power method.*

Exercise 7.8 *A suspect has three hiding places: an apartment (a), a boat (b), or a cave (c). Every hour the suspect has the opportunity to move. Let* $\vec{p}(t) = \begin{bmatrix} p_a \\ p_b \\ p_c \end{bmatrix}$ *denote the probability the suspect is at each position at time* t

If the suspect is in the apartment, he stays with probability $1/2$ *and moves to the boat with probability* $1/4$ *and to the cave with probability* $1/4$*. If he is in the boat, he stays with probability* $2/3$ *and moves to the apartment with probability* $1/12$ *and to the cave with probabillity* $1/4$*. If he is in the cave he stays with probability* $1/3$*, moves to the apartment with probability* $1/4$ *and to the boat with probability* $5/12$*.*

(a). Find the stochastic matrix S *such that* $\vec{p}(t+1) = \mathrm{S}\vec{p}(t)$*.*

(b). If police know that the suspect was in the apartment at time $t = 0$ *(*$\vec{p}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$*), what are the probabilities for the suspect to be in each location at* $t = 2$*?*

(c). Find all eigenvalues. It may make it easier to use the fact that $12\mathrm{A}$ *has eigenvalues that are twelve times the eigenvalues of* A*.*

(d). Find the eigenvector of 1 *and normalize it so that its sum is* 1*. These give the probabilities of being at each location after a long time.*

ADD EXAMPLE OF $\mathrm{A} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$, WITH PLOT OF EVALS — SEE SOLUTIONS FROM LAST YEAR.

Exercise 7.9 *A theorem (which I will not prove or expect you to remember) called the "Gershgorin Circle Theorem" gives us a little bit of knowledge about what the eigenvalues of a matrix can be. The statement of the theorem is as follows: Given a matrix* $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$*, for each diagonal entry* a_{ii}*, sum up the absolute values of the NON-Diagonal entries in the* i*-th column, call the sum* r_i*. Then draw a circle in the complex plane whose center is at* a_{ii} *and whose radius is* r_i*. When we do this for all of the diagonal entries, then every eigenvalue is in one of the circles (there is no guarantee that every circle has an eigenvalue).*

Use the Gershgorin circle theorm to prove that if all diagonal entries of a stochastic matrix are nonzero, then there is no other eigenvalue except 1 *which has magnitude* 1*.*

End of lecture 25

7.3 Application to linear systems of differential equations

When you learn(ed) in a differential equations class how to solve linear systems, most of what you do boils down to doing a change of coordinate system. Once you've got the right coordinate system, the problem becomes easy.

Example 7.10 *Let's start by making sure everyone is familiar with the solution to*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax$$

The solution is $x(0)e^{ax}$ where $x(0)$ is the initial value of x.

Some systems of differential equations are very simple to solve.

Example 7.11 *How would you solve*

$$\frac{\mathrm{d}c_1}{\mathrm{d}t} = \lambda_1 c_1$$
$$\frac{\mathrm{d}c_2}{\mathrm{d}t} = \lambda_2 c_2$$

You'd just look at each part separately and say $c_1 = c_1(0)e^{\lambda_1 t}$, $c_2 = c_2(0)e^{\lambda_2 t}$.

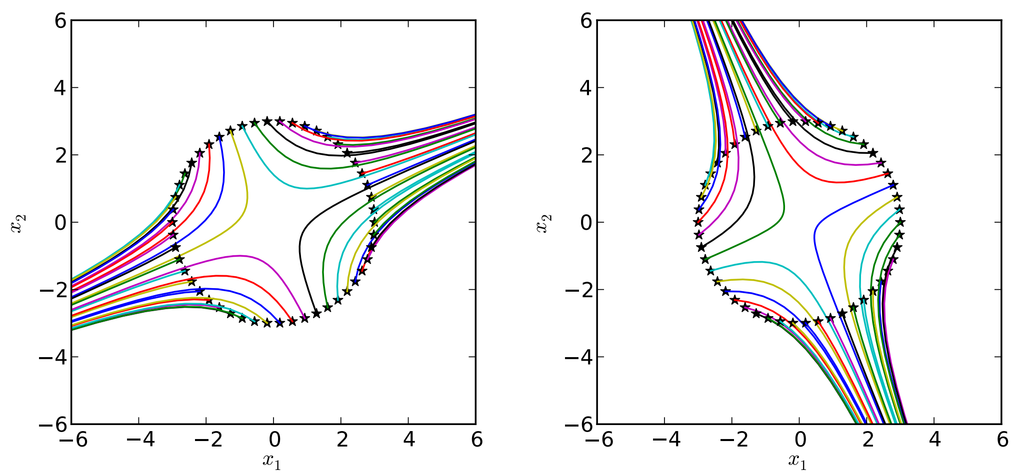
Example 7.12 *Say we have the system*

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1 + x_2$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 - x_2$$

We can rewrite the system as

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \vec{x}$$

First let's show what solutions to this system look like:



The stars denote different values of $\vec{x}(0)$, and the curves denote what happens as time progresses (left forward in time, right backwards). Notice if we go forward in time everything seems to converge to a line. If we were to go backward in time, things would converge to a different line. Can you speculate what these lines are?

How can we solve this system? If you haven't seen this before, it's not obvious what to do.
I claim that the original system can be changed into a much simpler system by changing coordinate systems. If we use the eigenvectors as our basis, the system will become easier. The eigenvalues are $\lambda_1 = \sqrt{2}$ *and* $\lambda_2 = -\sqrt{2}$ *with eigenvectors* $\vec{v}_1 = \begin{bmatrix} 1 \\ -1+\sqrt{2} \end{bmatrix}$ *and* $\vec{v}_2 = \begin{bmatrix} 1 \\ -1-\sqrt{2} \end{bmatrix}$. *So at any time, we can take* $\vec{x}(t)$ *and write it as* $\vec{x}(t) = c_1(t)\vec{v}_1 + c_2(t)\vec{v}_2$. *Plugging this into the system, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}(c_1(t)\vec{v}_1 + c_2(t)\vec{v}_2) = \mathrm{A}(c_1(t)\vec{v}_1 + c_2(t)\vec{v}_2) = c_1(t)\lambda_1\vec{v}_1 + c_2(t)\lambda_2\vec{v}_2$$

But we also have

$$\frac{\mathrm{d}}{\mathrm{d}t}(c_1(t)\vec{v}_1 + c_2(t)\vec{v}_2) = \frac{\mathrm{d}c_1}{\mathrm{d}t}\vec{v}_1 + \frac{\mathrm{d}c_2}{\mathrm{d}t}\vec{v}_2$$

So

$$\frac{\mathrm{d}c_1}{\mathrm{d}t}\vec{v}_1 + \frac{\mathrm{d}c_2}{\mathrm{d}t}\vec{v}_2 = c_1(t)\lambda_1\vec{v}_1 + c_2(t)\lambda_2\vec{v}_2$$

The vectors $\vec{v}_1$ *and* $\vec{v}_2$ *form a basis. From this we conclude that the coefficients of* $\vec{v}_1$ *on each side of the equation are equal and coefficients of* $\vec{v}_2$ *on each side are equal.*
So we arrive at

$$\frac{\mathrm{d}c_1}{\mathrm{d}t} = \lambda_1 c_1$$
$$\frac{\mathrm{d}c_2}{\mathrm{d}t} = \lambda_2 c_2$$

Which we can solve. We get $\vec{x} = c_1(t)\vec{v}_1 + c_2(t)\vec{v}_2$ *where* $c_1(t) = c_1(0)e^{\lambda_1 t}$ *and* $c_2(t) = c_2(0)e^{\lambda_2 t}$. *How do we find* $c_1(0)$ *and* $c_2(0)$*? 4 methods:*

(a). *Turn* $c_1(0)\vec{v}_1 + c_2(0)\vec{v}_2 = \vec{x}(0)$ *into a matrix equation* $\mathrm{P}\vec{c}(0) = \vec{x}(0)$ *where* $\mathrm{P} = [\vec{v}_1 \ \vec{v}_2]$ *and solve using Gaussian elimination.*

(b). *Or take the matrix* $\mathrm{P} = [\vec{v}_1 \ \vec{v}_2]$ *and invert it and solve* $\vec{c}(0) = \mathrm{P}^{-1}\vec{x}(0)$.

(c). *Or take the left eigenvectors and find* $c_i(0) = \vec{w}_i \cdot \vec{x}(0)/\vec{w}_i \cdot \vec{v}_i$. *(because the matrix is symmetric, we can substitute* $\vec{v}_i$ *for* $\vec{w}_i$ *in this formula).*

(d). *If we chose* $\vec{v}_i$ *such that* $|\vec{v}_i| = 1$, *then the formula is even simpler — we would avoid the division. In this case, the* $\vec{v}_i$ *we have chosen does not satisfy this, so we would have to divide it by* $|\vec{v}_i|$ *to normalize it first.*

The general approach I've described might fail if an eigenvalue is repeated. If the eigenvalue still has a full eigenspace, it's still possible to use this approach, but you have to be a bit more careful finding the coefficients of the eigenvectors of the repeated eigenvalue (don't use the dot product short-cut). If the eigenvalue has a deficient eigenspace we need to do a few more steps (the things to look up in wikipedia are "Jordan Canonical Form" and "generalized eigenvectors").

We can summarize our steps to solve $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathrm{A}\vec{x}$ as follows:

(a). Find the eigenvalues and eigenvectors of A.

(b). Write $\vec{x}$ in the form $\sum c_i(t)\vec{v}_i$.

(c). Write the system in the form $\sum \vec{v}_i \frac{\mathrm{d}}{\mathrm{d}t} c_i(t) = \sum c_i(t)\lambda_i\vec{v}_i$.

(d). The coefficient for $\vec{v}_i$ must be the same on both sides of the equation (since they form a basis). So $\frac{\mathrm{d}}{\mathrm{d}t}c_i = \lambda_i c_i$ for each i.

(e). Solve the differential equation for each c_i — the solution will have some arbitrary constants — generally the initial condition. The sum $\sum c_i(0)e^{\lambda_i t}\vec{v}_i\cancel{(t)}$ is the general solution of the system of differential equations.

(f). Find the values of the arbitrary constants $[c_i(0)]$ by solving for $c_i(0)$ from $\vec{x}(0)$. If the matrix happens to be symmetric, $c_i(0) = \vec{v}_i \cdot \vec{x}(0)/\vec{v}_i \cdot \vec{v}_i$, and if not symmetric, but we know the left eigenvectors we can use a similar approach. If not, Gaussian elimination is generally faster than finding the left eigenvectors.

This method will work as long as there is a basis for $\mathbb{R}^n$ made up of eigenvectors of A. In other words, as long as A is diagonalizable. If A is not diagonalizable, a related approach works.

The general solution will be of the form $\vec{x}(t) = \sum c_i(0)e^{\lambda_i t}\vec{v}_i$ where $c_i(0)$ are determined based on the initial conditions $\vec{x}(0)$.

Exercise 7.13 *For each of the following, find the general solution to the system of differential equations $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathrm{A}\vec{x}$. Then find the solution corresponding to the given initial condition. For the 2-D cases I have given a plot which shows a few representative solution curves with $\vec{x}(0)$ on the unit circle. Spend some time looking at the relation between the eigenvectors and the plots. With practice, you should be able to tell which direction the eigenvectors point, what sign the eigenvalues have, and roughly what the relative sizes of the eigenvectors are just by looking at the solutions (and* vice versa *you should be able to sketch the solutions just by looking at the eigenvectors and the eigenvalues).*

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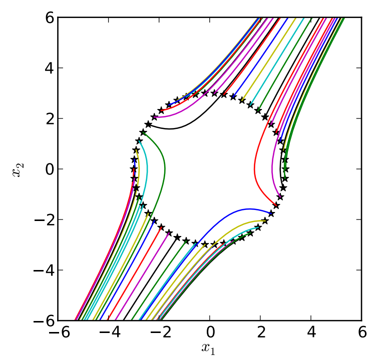
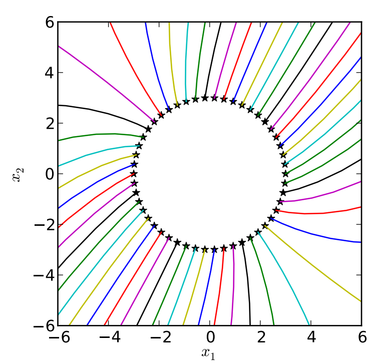
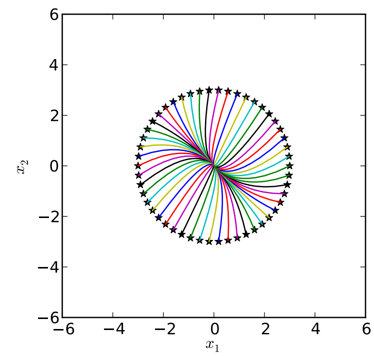


(a). $\mathrm{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\vec{x}(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$.
The eigenvalues are $\lambda_1 = \phi$, $\lambda_2 = 1 - \phi$, and the eigenvectors are $\vec{v}_1 = \begin{bmatrix} 1 \\ \phi \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1-\phi \end{bmatrix}$.

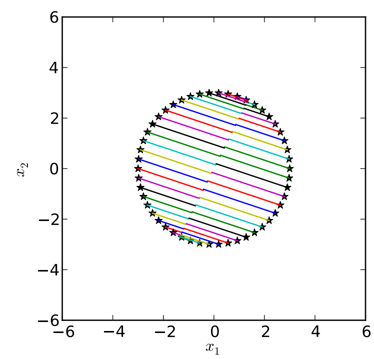


(b). $\mathrm{A} = \begin{bmatrix} 2 & 1/2 \\ 1 & 5/2 \end{bmatrix}$, $\vec{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.
The eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 3/2$, and the eigenvectors are $\vec{v}_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(c). $\mathrm{A} = \begin{bmatrix} -2 & -1/2 \\ -1 & -5/2 \end{bmatrix}$, $\vec{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.
The eigenvaules are $\lambda_1 = -3$, $\lambda_2 = -3/2$, *and the eigenvectors are* $\vec{v}_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ *and* $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.



(d). $\mathrm{A} = \begin{bmatrix} -6 & 3 \\ 2 & -1 \end{bmatrix}$, $\vec{x}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. *The eigenvalues are* $\lambda_1 = -7$ *and* $\lambda_2 = 0$, *and the eigenvectors are* $\vec{v}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ *and* $\vec{v}_2 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$.

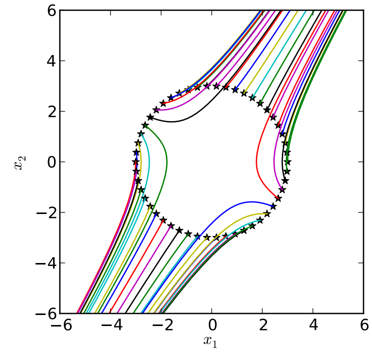


(e). $\mathrm{A} = \begin{bmatrix} -3 & 3 & 8 \\ -5 & 5 & 8 \\ 5 & 3 & 0 \end{bmatrix}$, $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. *The eigenvalues are* $\lambda_1 = 8$, $\lambda_2 = -8$, *and* $\lambda_3 = 2$, *and the eigenvectors are* $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, *and* $\vec{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

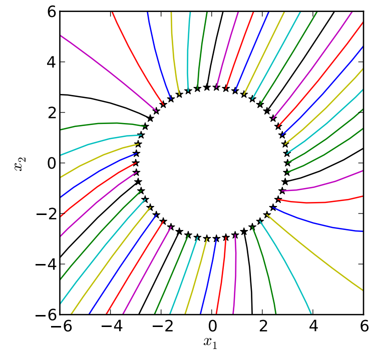
Exercise 7.14 *If all eigenvalues of* A *have negative real part (and* A *is diagonalizable), show that all solutions of* $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathrm{A}\vec{x}$ *approach* $\vec{0}$ *as time increases. This remains true even if it's not diagonalizable, but I haven't taught how to handle this yet.*

Exercise 7.15 *For each of the following, find the general solution to the system of differential equations* $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathrm{A}\vec{x}$. *Then find the solution corresponding to the given initial condition. For the 2-D cases I have given a plot which shows a few representative solution curves with* $\vec{x}(0)$ *on the unit circle. Spend some time looking at the relation between the eigenvectors and the plots. With practice, you should be able to tell which direction the eigenvectors point, what sign the eigenvalues have, and roughly what the relative sizes of the eigenvectors are just by looking at the solutions (and* vice versa *you should be able to sketch the solutions just by looking at the eigenvectors and the eigenvalues).*

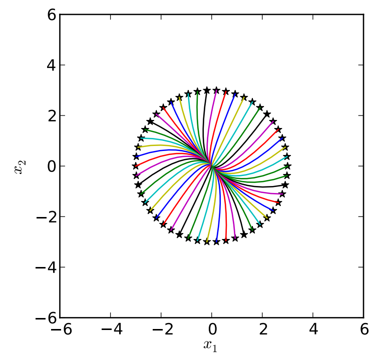
(a). $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\vec{x}(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$.
The eigenvalues are $\lambda_1 = \phi$, $\lambda_2 = 1 - \phi$, *and the eigenvectors are* $\vec{v}_1 = \begin{bmatrix} 1 \\ \phi \end{bmatrix}$ *and* $\vec{v}_2 = \begin{bmatrix} 1 \\ 1-\phi \end{bmatrix}$.



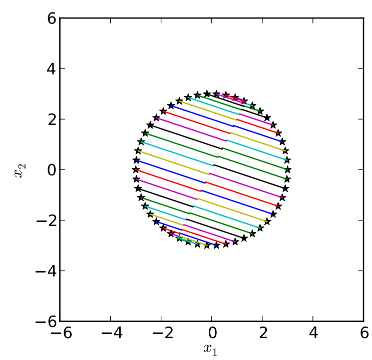
(b). $A = \begin{bmatrix} 2 & 1/2 \\ 1 & 5/2 \end{bmatrix}$, $\vec{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.
The eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 3/2$, *and the eigenvectors are* $\vec{v}_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ *and* $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.



(c). $A = \begin{bmatrix} -2 & -1/2 \\ -1 & -5/2 \end{bmatrix}$, $\vec{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.
The eigenvaules are $\lambda_1 = -3$, $\lambda_2 = -3/2$, *and the eigenvectors are* $\vec{v}_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ *and* $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.



(d). $A = \begin{bmatrix} -6 & 3 \\ 2 & -1 \end{bmatrix}$, $\vec{x}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. *The eigenvalues are* $lambda_1 = -7$ *and* $\lambda_2 = 0$, *and the eigenvectors are* $\vec{v}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ *and* $\vec{v}_2 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$.



(e). $A = \begin{bmatrix} -3 & 3 & 8 \\ -5 & 5 & 8 \\ 5 & 3 & 0 \end{bmatrix}$, $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. *The eigenvalues are* $\lambda_1 = 8$, $\lambda_2 = -8$, *and* $\lambda_3 = 2$, *and the* *eigenvectors are* $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, *and* $\vec{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

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ugates are

Exercise 7.16 *Consider* $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathrm{A}\vec{x}$ *where* A *is* 2×2 *and has all real entries, but complex eigenvalues. Assume the initial condition is* $\vec{x}(0)$ *which has just real entries. If* λ_1 *is the first eigenvalue, then the complex conjugate of it is the other eigenvalue* $\lambda_2 = \overline{\lambda}_1$. *Similarly, if* $\vec{v}_1$ *is the first eigenvector, then its complex conjugate is the second eigenvector.*

(a). *Let* $\lambda_1 = r + is$ *and* $\vec{v}_1 = \vec{\alpha} + i\vec{\beta}$ *where* r *and* s *are real numbers and* $\vec{\alpha}$ *and* $\vec{\beta}$ *are real-valued vectors. The general solution can be written as* $K_1 e^{(r+is)t}(\vec{\alpha} + i\vec{\beta}) + K_2 e^{(r-is)t}(\vec{\alpha} - i\vec{\beta})$ *where* K_1 *and* K_2 *are complex numbers. Explain why the initial condition and differential equation (should) lead you expect that when all the dust settles from choosing* K_1 *and* K_2, *we must have a solution that is entirely real.*

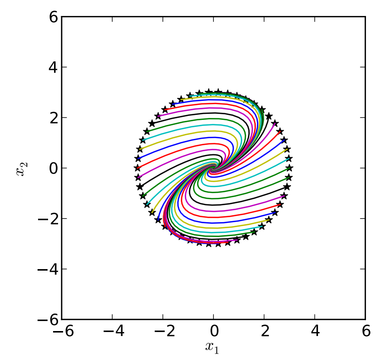
(b). *In order to have it be real at all time, we must have* $K_2 = \overline{K}_1$ *(no need to prove this, but it follows roughly from the fact that for the sum of two complex numbers to be real, the imaginary part of second must be negative that of the first). So write* $K_2 = a - bi$ *and* $K_1 = a + bi$. *Show that the solution looks like* $2ae^{rt}(\vec{\alpha}\cos st - \vec{\beta}\sin st) - 2be^{rt}(\vec{\alpha}\sin st + \vec{\beta}\cos st)$. *Use this to conclude that the general solution when* A *is real and the initial conditions are real is of the form*

$$\vec{x}(t) = C_1 e^{rt}(\vec{\alpha}\cos st - \vec{\beta}\sin st) + C_2 e^{rt}(\vec{\alpha}\sin st + \vec{\beta}\cos st)$$

For the remainder of the problem, let $\mathrm{A} = \begin{bmatrix} -2 & 2 \\ -1 & 0 \end{bmatrix}$, *with initial condition* $\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(c). *Find the eigenvectors and eigenvalues using the usual method. Show that the eigenvalues are complex conjugates of one another and the eigenvectors are complex conjugates.*

(d). *Find the general real-valued solution form for* $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathrm{A}\vec{x}$. *The solutions look like*



(e). *Find the solution for the given initial condition.*

Exercise 7.17 *We can define the matrix exponential by*

$$e^{\mathrm{A}} = \mathrm{I} + \mathrm{A} + \frac{\mathrm{A}^2}{2!} + \frac{\mathrm{A}^3}{3!} + \cdots$$

so $e^{\mathrm{A}t} = \mathrm{I} + t\mathrm{A} + t^2\frac{\mathrm{A}^2}{2!} + t^3\frac{\mathrm{A}^3}{3!} + \cdots$. *If* A *is diagonalizable such that* $\mathrm{A} = \mathrm{P}\Lambda\mathrm{P}^{-1}$ *with* $\mathrm{P} = [\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n]$ *as usual, we know that the problem* $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathrm{A}\vec{x}$ *with a given* $\vec{x}(0)$ *has the solution* $x(t) = c_1(0)e^{\lambda_1 t}\vec{v}_1 + c_2(0)e^{\lambda_2 t}\vec{v}_2 + \cdots + c_n(0)e^{\lambda_n t}\vec{v}_n$ *where* $\begin{bmatrix} c_1(0) \\ c_2(0) \\ \vdots \\ c_n(0) \end{bmatrix} = \mathrm{P}^{-1}\vec{x}(0)$.

Show that this solution $x(t)$ *can be written as* $e^{\mathrm{A}t}\vec{x}(0)$. *[hint, look back at exercise* **??***]*

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Exercise 7.18 *In this exercise we adapt the method for* $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathrm{A}\vec{x}$ *to allow more general derivatives in* t.
Consider $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\vec{x} + \frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathrm{A}\vec{x}$ *where* A *is diagonalizable.*
Write $\vec{x}(t)$ *as* $\sum c_i(t)\vec{v}_i$ *where* $\vec{v}_i$ *are the eigenvectors, and find a differential equation that each* c_i *must solve. (you don't have to solve it)*

End of lecture 26

7.4 Applications to Fourier Series

Let's refresh our memory what the steps were for a linear system of differential equations. To solve a linear system of differential equations $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathrm{A}\vec{x}$:

(a). Find the eigenvalues and eigenvectors of A.

(b). Write $\vec{x}(t)$ in the form $\sum c_i(t)\vec{v}_i$.

(c). Write the system in the form $\sum \vec{v}_i \frac{\mathrm{d}}{\mathrm{d}t} c_i(t) = \sum c_i(t)\lambda_i \vec{v}_i$.

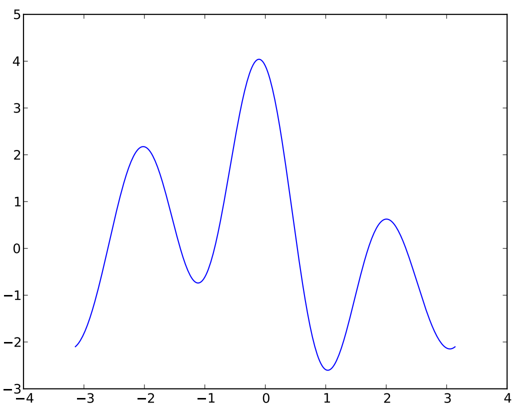
(d). The coefficient for $\vec{v}_i$ must be the same on both sides of the equation (since they form a basis). So $\frac{\mathrm{d}}{\mathrm{d}t} c_i = \lambda_i c_i$ for each i.

(e). Solve the differential equation for each c_i.

(f). Find the initial condition for each c_i, (if A is symmetric and has no repeated eigenvalues, $c_i = \vec{v}_i \cdot \vec{x}(0)/\vec{v}_i \cdot \vec{v}_i$).

In this section, our goal is to solve a linear partial differential equation. The approach we will use is conceptually identical.

Let's look at the heat equation $\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u$. Let's assume we have a wire in the interval $-\pi$ to π and that it has "periodic boundary conditions": $u(-\pi, t) = u(\pi, t)$, $\frac{\partial}{\partial x} u(-\pi, t) = \frac{\partial}{\partial x} u(\pi, t)$ for all time. That is, like in the game PacMan, when you walk off the end of the wire going right, you appear on the leftmost end of the wire. We assume that we know the initial condition $u(x, 0) = f(x)$.



A sample function that satisfies periodic boundary conditions: at $\pm\pi$, both the value and the derivative of the function are the same.

I want to make one observation that I will need for later: the set of functions that solve the heat equation with periodic boundary conditions form a vector space. To show this, I consider have two functions u_1 and u_2 for which each satisfies the boundary conditions and the heat equation. Any linear combination of the two satisfies the boundary conditions and the heat equation. This means that later on, I can add functions that satisfy these without worrying whether the result of the addition satisfies the conditions.

We're going to solve this using basically the same method as we did to solve the linear systems of differential equations. Let's first see how the heat equation compares to the previous equations: we have $\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u$ and $\frac{\partial}{\partial t}\vec{x} = \mathrm{A}\vec{x}$. It's clear that the time derivatives correspond to one another, and that u corresponds to $\vec{x}$.

This means that A corresponds to $\frac{\partial^2}{\partial x^2}$. So we're going to go through the same algorithm as when solving $\frac{\partial}{\partial t}\vec{x} = \mathrm{A}\vec{x}$, but we'll use $\frac{\partial^2}{\partial x^2}$ to play the role of A.

The first step to solving a linear system of differential equations $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \mathrm{A}\vec{x}$ was to find the eigenvalues and eigenvectors of A. So we need to find the eienvalues and eigen<u>functions</u> of $\frac{\partial^2}{\partial x^2}$. What would you guess is meant by an "eigenfunction" of $\frac{\partial^2}{\partial x^2}$? Hopefully it's not a surprise that an eigenfunction is a function $g(x)$ that satisfies $g''(x) = \lambda g(x)$.

We'll find these, then we'll write $u(x,t)$ as a sum of coefficients that depend on time times the eigenfunctions, plug this into the differential equation, match coefficients, and from that solve our system.

What are the eigenfunctions of $\frac{\partial^2}{\partial x^2}$? They are solutions to $g'' - \lambda g = 0$. If $\lambda > 0$, then the solutions are linear combinations of $e^{\pm x\sqrt{\lambda}}$. If $\lambda < 0$, the solutions are linear combinations of $\cos x\sqrt{-\lambda}$ and $\sin x\sqrt{-\lambda}$. This leaves $\lambda = 0$, for which solutions are linear combinations of 1 and x.

In the linear system of differential equations case, we found eigenfunctions that were a basis for $\mathbb{R}^n$. Here we want to find functions that form a basis for the vector space of functions that satisfy periodic boundary conditions at $-\pi$ and π. Of these eigenfunctions, which have the same values and derivatives at $\pm\pi$ — in other words, which are in the vector space we care about?

The only ones with the same values at $\pm\pi$ are the constant 1, and $\cos\sqrt{-\lambda}x$, $\sin\sqrt{-\lambda}x$ where $\sqrt{-\lambda}$ is a positive integer. I've made this claim without proof. Let's prove it. We'll look at $\lambda > 0$, then at $\lambda = 0$, and finally at $\lambda < 0$. In each case we'll look for eigenfunctions in the vector space we want.

If $\lambda > 0$, the solution looks like $v(x) = C_1 e^{x\sqrt{\lambda}} + C_2 e^{-x\sqrt{\lambda}}$. Setting $v(\pi) = v(-\pi)$ and $v'(\pi) = v'(-\pi)$ we get two equations for C_1 and C_2:

$$
\begin{aligned}
C_1 e^{\pi\sqrt{\lambda}} + C_2 e^{-\pi\sqrt{\lambda}} &= C_1 e^{-\pi\sqrt{\lambda}} + C_2 e^{\pi\sqrt{\lambda}} \\
C_1\sqrt{\lambda}e^{\pi\sqrt{\lambda}} - C_2\sqrt{\lambda}e^{-\pi\sqrt{\lambda}} &= C_1\sqrt{\lambda}e^{-\pi\sqrt{\lambda}} - C_2\sqrt{\lambda}e^{\pi\sqrt{\lambda}}
\end{aligned}
$$

Obviously $C_1 = C_2 = 0$ is <u>one</u> solution to this. We can turn it into a matrix equation $\mathrm{A}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \vec{0}$ and show that the determinant of A is nonzero which shows that $C_1 = C_2 = 0$ is the <u>only</u> solution. Here is another way to see it. If we multiply the first equation by $\sqrt{\lambda}$ and add it to the second equation, we have

$$
2C_1\sqrt{\lambda}e^{\pi\sqrt{\lambda}} = 2C_1\sqrt{\lambda}e^{-\pi\sqrt{\lambda}}
$$

We have assumed $\lambda > 0$, the only value of C_1 which makes this true is $C_1 = 0$. Taking $C_1 = 0$, we are left $C_2 e^{-\pi\sqrt{\lambda}} = C_2 e^{\pi\sqrt{\lambda}}$, for which again the only solution is $C_2 = 0$. So there is no eigenfunction that is in the vector space we want if $\lambda > 0$.

If $\lambda = 0$, the solution looks like $v(x) = C_1 x + C_2$. Taking $v(\pi) = v(-\pi)$ we see that $C_1 = 0$. Looking at $v'(\pi) = v'(-\pi)$ doesn't restrict C_2 at all. So any constant is in the vector space we want. We choose 1 as our eigenfunction.

If $\lambda < 0$, we have $v(x) = C_1\cos(x\sqrt{-\lambda}) + C_2\sin(x\sqrt{-\lambda})$. Looking at $x = \pm\pi$ we have two equations

$$
\begin{aligned}
C_1\cos\pi\sqrt{-\lambda} + C_2\sin\pi\sqrt{-\lambda} &= C_1\cos -\pi\sqrt{-\lambda} + C_2\sin -\pi\sqrt{-\lambda} \\
-C_1\sqrt{-\lambda}\sin\pi\sqrt{-\lambda} + C_2\sqrt{-\lambda}\cos\pi\sqrt{-\lambda} &= -C_1\sqrt{-\lambda}\sin -\pi\sqrt{-\lambda} + C_2\sqrt{-\lambda}\cos -\pi\sqrt{-\lambda}
\end{aligned}
$$

Observing that $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, we get some simplifications: the cos terms disappear from both equations:

$$
\begin{aligned}
C_2\sin\pi\sqrt{-\lambda} &= -C_2\sin\pi\sqrt{-\lambda} \\
-C_1\sqrt{-\lambda}\sin\pi\sqrt{-\lambda} &= -C_1\sqrt{-\lambda}\sin -\pi\sqrt{-\lambda}
\end{aligned}
$$

Each equation is decoupled from the other. The first equation says that either $C_2 = 0$ or $\sin \pi\sqrt{-\lambda} = 0$. Similarly the second equation says that either $C_1 = 0$ or $\sin \pi\sqrt{-\lambda} = 0$. So to get a non-zero solution, we have $\sin \pi\sqrt{-\lambda} = 0$. So $\sqrt{-\lambda}$ is a positive integer, which we will call m. So any function of the form $C_1 \cos mx + C_2 \sin mx$ is an eigenfunction, with eigenvalue $-m^2$. We've got two independent functions here. We need two eigenfunctions. There is some freedom how we choose them, but the best (or at least standard) choice is $\cos mx$ and $\sin mx$.

This means that

- 1 is an eigenfunction of the eigenvalue 0.
- for any positive integer, m, $\cos mx$ and $\sin mx$ are eigenfunctions of the eigenvalue $-m^2$.

There are no other functions that are eigenfunctions of the differential equation and satisfy the boundary conditions. So we assume that these form a basis for the vector space of solutions. This completes the first step of the process.

We move on to the second step and write u as

$$u(x,t) = a_0(t) + \sum_{m=1}^{\infty} a_m(t)\cos mx + b_m(t)\sin mx$$

This is where we use the fact that the solutions form a vector space. We know that any linear combination of the eigenfunctions will satisfy the boundary conditions.I have not shown that these eigenfunctions form a basis, but all we really need is that the solution we want is in the span of the eigenfunctions.

We move on to the third step and substitute this into $\frac{\partial}{\partial t}u = \frac{\partial^2}{\partial x^2}u$. We get

$$a_0'(t) + \sum_{m=1}^{\infty} a_m'(t)\cos mx + b_m'(t)\sin mx = 0 + \sum_{m=1}^{\infty} -a_m(t)m^2\cos mx - b_m(t)m^2\sin mx$$

cation for this ependence OR oduct. Maybe near indepen- n exercise us- duct. Or per- r product ear- linear indepen- night help give

The fourth step is to match coefficients.

$$\begin{aligned} a_0'(t) &= 0 \\ a_m'(t) &= -m^2 a_m(t) \\ b_m'(t) &= -m^2 b_m(t) \end{aligned}$$

where $m \geq 1$.

Solving this, the fifth step gives

$$\begin{aligned} a_0(t) &= a_0(0) \\ a_m(t) &= a_m(0)e^{-m^2 t} \\ b_m(t) &= b_m(0)e^{m^2 t} \end{aligned}$$

where $m \geq 1$

We just need to figure out what the initial values of these coefficients are. This comes from the initial condition $u(x,0) = f(x)$.

End of lecture 27

To find the coefficients of this series, we look back at the methods we had for finding the coefficients of solutions to the linear systems of differential equations. The method of Gaussian elimination doesn't work so well here for two reasons: first, we have an infinite set of coefficients and Gaussian elimination basically requires a lot of work until finally we can get all of the coefficients at once at the end. No such method can work when there are infinitely many rows in the matrix. More significantly, Gaussian elimination requires the idea of a pivot entry and a pivot column. We're looking at functions — it doesn't really make sense to talk about the 3-rd entry of $\sin x$ or similar concepts.

Another method we had was to take the left eigenvector and dot it with both sides of $\vec{x} = \sum c_i \vec{v}_i$. On the left hand side we get some numerical value we can calculate. On the right we get a bunch of 0s, and one nonzero entry $c_i \vec{w}_i \cdot \vec{v}_i$. So $c_i = \vec{w}_i \cdot \vec{x} / \vec{w}_i \cdot \vec{v}_i$. We will generalize this approach.

In the case where matrices were symmetric we had $\vec{w}_i = \vec{v}_i$, and so the formula becomes $c_i = \vec{v}_i \cdot \vec{x} / \vec{v}_i \cdot \vec{v}_i$. In this case we have some sort of "symmetric" operator, so something will be very similar. Hopefully this statement will become clear later.

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We're going to note some orthogonality relations. We've shown in our homework that if $m \neq n$, then $\cos mx$ is orthogonal to $\sin nx$, $\cos mx$ is orthogonal to $\cos nx$, and $\sin mx$ is orthogonal to $\cos nx$. There are two remaining things we have to show. Firstly, 1 is orthogonal to $\cos mx$ and $\sin mx$ since we know that their integrals from $-\pi$ to π are zero. Secondly, for the same value of m, $\cos mx$ is orthogonal to $\sin mx$. There are several ways to do this. The easiest is to note that sin is an odd function while cos is an even function. So their product is odd, so the integral of their product from $-\pi$ to π is zero. Alternately, note that their product is $(\sin 2mx)/2$, and integrate this.

So this shows that if I take any two different functions from my set of eigenfunctions, the integral of their product from $-\pi$ to π is 0. We thus immediately have that if

$$f(x) = a_0 + \sum_{m=1}^{\infty} a_m \cos mx + \sum_{m=1}^{\infty} b_m \sin mx$$

Then

$$a_0 = \frac{\langle f(x), 1 \rangle}{\langle 1, 1 \rangle} = \frac{\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x}{\int_{-\pi}^{\pi} 1\,\mathrm{d}x} = \frac{\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x}{2\pi}$$

and

$$a_m = \frac{\langle f(x), \cos mx \rangle}{\langle \cos mx, \cos mx} = \frac{\int_{-\pi}^{\pi} f(x) \cos mx\,\mathrm{d}x}{\pi}$$

and

$$b_m = \frac{\langle f(x), \sin mx \rangle}{\langle \sin mx, \sin mx} = \frac{\int_{-\pi}^{\pi} f(x) \sin mx\,\mathrm{d}x}{\pi}$$

So given whatever initial condition we have, this gives us the coefficients, and

$$u(x,t) = \frac{\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x}{2\pi} + \sum_{m=0}^{\infty} \frac{e^{-m^2 t}}{\pi} \left(\int_{-\pi}^{\pi} f(x) \cos mx\,\mathrm{d}x + \int_{-\pi}^{\pi} f(x) \sin mx\,\mathrm{d}x \right)$$

What does this tell us physically: notice the $e^{-m^2 t}$ terms. These tend to decay quickly, especially for relatively large m. Certainly at large time, this converges to just the first term. If you look at the first term, this says that (given the boundary conditions), the temperature everywhere approaches what was the initial average temperature. If we look at not as large time, we see that the higher oscillations are damped very quickly. The bits that don't oscillate very quickly don't decay as quickly.

give an exercise
conditions are
the coefficients

Although the Fourier series is particularly well suited to cases where we have u_{xx} appear, it is widely used beyond that. We are frequently interested in periodic functions and Fourier series give us a basis for the set of periodic functions, and the basis has the very nice property that the functions are mutually orthogonal: given a function, it is easy to determine what the coefficients must be.

Exercise 7.19 *Consider a string which is stretched between two pegs (at* 0 *and* π *note, not* $-\pi$*) and clamped at both ends. If we pull the string out of its natural rest position (like a violinist plucking the string) and release it, the displacement from the rest position satisfies the wave equation* $\frac{\partial^2}{\partial t^2} u = \frac{\partial^2}{\partial x^2} u$*. Because it is clamped we have* $u(0) = u(\pi) = 0$*. We start it with a position* $u(x, 0) = f(x)$*, and initially when released it is not moving so* $\frac{\partial}{\partial t} u(x, 0) = 0$*.*

(a). *Find the eigenfunctions of* $\frac{\partial^2}{\partial x^2}$ *that satisfy clamped boundary conditions. (you should get* $\sin mx$ *for* $m = 1, 2, \ldots$*, the* $\cos$ *terms disappear).*

(b). *We write* $u(x, t) = a_1(t) \sin x + a_2(t) \sin 2x + \cdots$*. Substitute this into the wave equation and find a differential equation for* $a_m(t)$ *for all* m*.*

(c). *The general solution to* $h''_m(t) + m^2 h_m(t) = 0$ *is* $h_m(t) = A_m \cos mt + B_m \sin mt$*. Using the fact that* $\frac{\partial}{\partial t} u(x, 0) = 0$*, show that all of the* B_m *are zero.*

(d). *Find a formula for* A_m *for* $m = 1, 2, \ldots$ *in terms of* $f(x)$*.*

For those of you who have played stringed instruments, you'll notice that the $m = 1$ *term corresponds to the note of the open string. When you play a harmonic (lightly touching the string kills all the terms in the sum that aren't zero at that point), you'll notice that the positions where harmonics occur all correspond to nice fractions of the string length. Notice that the coefficient's dependence on* t *is such that a harmonic corresponding to a given* m *vibrates at a frequency* m *times that of the open string.*

Exercise 7.20 *We have defined a left eigenvector to be a vector such that* $\vec{w}^T \mathrm{A} = \lambda \vec{w}^T$*. An equivalent definition for a left eigenvector is that* $\vec{w}^T$ *is a left eigenvector of* A *iff for any vector* $\vec{x}$*,* $(\vec{w}^T \mathrm{A})^T \cdot \vec{x} = \lambda \vec{w} \cdot \vec{x}$*. Define an "left eigenfunction" of* $\frac{\partial^2}{\partial x^2}$ *to be any (nonzero)* $h(x)$ *such that for any function* $f(x)$ *that satisfies the boundary conditions,* $\int_{-\pi}^{\pi} h(x) \frac{\partial^2}{\partial x^2} f(x) \, \mathrm{d}x = \int_{-\pi}^{\pi} \lambda h(x) f(x) \, \mathrm{d}x$*.*

(a). *Show that if* $\vec{w}^T \mathrm{A} = \lambda \vec{w}^T$*, then* $(\vec{w}^T \mathrm{A})^T \cdot \vec{x} = \lambda \vec{w} \cdot \vec{x}$ *for any* $\vec{x}$*.*

(b). *Take* $\int_{-\pi}^{\pi} h(x) \frac{\partial^2}{\partial x^2} f(x) \, \mathrm{d}x$ *and integrate it by parts to get a term like* $\int_{-\pi}^{\pi} f(x) \frac{\partial^2}{\partial x^2} h(x) \, \mathrm{d}x$ *plus some boundary terms.*

(c). *Note that* $f(-\pi) = f(\pi)$ *and* $f'(-\pi) = f'(\pi)$*. Explain why if the boundary terms are always zero for any* f*, then we must have* $h(-\pi) = h(\pi)$ *and* $h'(-\pi) = h'(\pi)$*.*

(d). *Explain why* $\int_{-\pi}^{\pi} h(x) \frac{\partial^2}{\partial x^2} f(x) \, \mathrm{d}x = \int_{-\pi}^{\pi} f(x) \lambda h(x)$ *for every* f *iff* h *satisfies periodic boundary conditions and* h *is itself an eigenfunction of* $\frac{\partial^2}{\partial x^2}$*.*

(e). *Why is this analagous to a symmetric matrix?*

Exercise 7.21 *Let* $f(x) = x$ *from* $-\pi$ *to* π*.*

(a). *Show that* $a_0 = 0$ *and that* $a_m = 0$ *for* $m > 0$ *as well.*

(b). *Show that* $b_m = 1/m$*.*

(c). *Find* $\int_{-\pi}^{\pi} f(x) f(x) \, \mathrm{d}x$ *directly by using* $f(x) = x$*.*

(d). *Using the fact that* $\int_{-\pi}^{\pi} \sin^2(mx) \, \mathrm{d}x = \pi$*, show that substituting the series in for* $f(x)$ *in the integral* $\int_{-\pi}^{\pi} f(x) f(x) \, \mathrm{d}x$ *gives* $\frac{1}{\pi} \sum \frac{1}{m^2}$*.*

(e). *Since these two are equal, what is* $\sum_{m=1}^{\infty} \frac{1}{m^2}$*?*

End of lecture 28

Appendix A

Applications of theorem 3.64

In Theorem 3.64, we saw that given a matrix A, then given one solution $\vec{x}_p$ to $A\vec{x} = \vec{b}$, the general solution of $A\vec{x} = \vec{b}$ is made up of vectors of the form $\vec{x}_p + \vec{w}$ where $\vec{w}$ is any vector for which $A\vec{w} = \vec{0}$.

For solving matrix equations, this is not terribly useful for applications — we already have an algorithm, Gaussian Elimination, which will give us all solutions to $A\vec{x} = \vec{b}$. It does have some use for theory: it shows that if the nullspace is just $\vec{0}$, then whenever a solution exists, it is unique. We could get this however, just by counting free variables. More generally however, if you analyze the proof of the theorem, it's clear that there is no requirement that this be a matrix. Rather, any linear function from any vector space to any other vector space has this property. Where this is really useful is in solving linear differential equations.

Let T be a "linear differential operator", that is, a linear function of functions that involves derivatives. So for example: $T(f(x)) = f''(x) - 2(x+1)f'(x) + f(x)$. It may be hard to solve $T(f(x)) = y(x)$ for some given y. We don't have something like Gaussian Elimination for functions (they are an infinite dimensional space, and if you look at Gaussian Elimination, it sort of depends on the fact that at some point the matrix is finally in echelon form and we can find solutions going from the bottom up. This doesn't happen if the matrix has infinitely many rows.). Our theorem tells us that the general solution to $T(f(x)) = y(x)$ comes from any single particular solution, say $f_p(x)$ which we find by whatever means necessary and the solution to $T(f(x)) = 0$. So most methods we learn in differential equations courses for linear differential equations boil down to finding all solutions to $T(f(x)) = 0$ and just one solution to $T(f(x)) = y(x)$. Often this is a non-trivial task, and for some equations there isn't really a method known.

Example A.1 *Find all solutions to the differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5x + 3$$

Here $T(y(x)) = y'(x)$, and we are solving $T(y(x)) = 5x + 3$ We can easily see that $5x^2/2 + 3x$ is a solution to this equation. In fact, it's a common mistake to only come up with this solution. But the null space of T is the set of constant functions. So the set of all solutions is $5x^2/2 + 3x + C$ where C is any constant.

Example A.2 *Find all solutions to the differential equation*

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = x$$

We start by trying to find a single solution to the equation. We might guess that y should be something like ax. If we plug that in we get $2a + ax = x$, which clearly doesn't work. But it gives a hint that perhaps $ax + b$ would work. We plug this in and we have $2a + ax + b = x$, so $a = 1$ and $b = -2$ gives

a solution: $y = x - 2$. (you will learn more efficient ways to do this in a differential equations class.) We have one solution.
Now we need to find all solutions to $y'' + 2y' + y = 0$. There are many techniques available to solve this equation.[1] *It turns out that the solution looks like $C_1e^{-x} + C_2xe^{-x}$. (you will learn how to find this in a differential equations class.)*
So the general solution to this equation is $(C_1 + C_2x)e^{-x} + x - 2$.

A.1 Solving Constant Coefficient Linear ODEs

One of the most frequently encountered classes of differential equations is "constant coefficient linear differential equations". This is something like $ay'' + by' + cy = f(x)$. These come up in physics for example for a driven spring, or for current in an electrical system. The method that is usually used to find all soltions to these equations boils down to first, finding just one solution (any solution will do) to $ay'' + by' + cy = f(x)$, and call it y_p. Then we try to find all solutions to $ay'' + by' + cy = 0$. To do this, we know that the solution is a vector space, so we try to find a basis. We guess that the basis might have functions like e^{rx}. Plug this in to the equation to figure out what r could be: $ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$. So if $ar^2 + br + c = 0$, we have a solution. Usually this has two solutions, r_1 and r_2 with $r_1 \neq r_2$. In this case e^{r_1x} and e^{r_2x} form a basis for the null space. So our general solution is of the form $y_p + C_1e^{r_1x} + C_2e^{r_2x}$. I made a few assumptions here: one is that I can find y_p, another is that the roots r_1 and r_2 were not equal, and finally that the null space has dimension 2. Sometimes we cannot find y_p. It is possible that the root may be repeated (in which case it turns out that the basis is e^{rx} and xe^{rx}, but that takes a bit more explanation), and finally — we do not have to worry about the dimension issue. The dimension of the null space is always the highest number of derivatives.

A further comment is in order. This is a large part of why we care about vector spaces in general and null spaces in particular. For any linear transformation, the null space of the transformation is a vector space. Since there are many cases in mathematics where we want to solve $T(x) = b$ for a linear function T, this means that we are frequently working with the null space of T.

A.2 An alternate way to solve constant coefficient linear systems

The usual approach to solving the constant coefficient linear systems I mentioned above relies on some guessing. We guess that the null space is made up of exponentials. We usually have to guess what y_p is and check that it works. I don't much care for the "guess and check" approach, and in particular in some cases there isn't a clear way to guess y_p.

So here is the approach I prefer. You may want to refer back to it when you learn the guess and check method in a later course.

A.2.1 Integrating factors

First, let me introduce "integrating factors", and let's solve just "first-order equations" $y' + ay = f(x)$. I'm going to multiply the equation by a function $h(x)$ to get $h(x)y' + ah(x)y = h(x)f(x)$. If I can do a change of variables to make this $z' = h(x)f(x)$, then z is simply the integral of $h(x)f(x)$ (plus a constant). What could I make h be in order to find something simple for z? Think back to your product rule. If $h(x) = e^{ax}$,

[1] When I was applying for scholarships for graduate school, one of the scholarships I received had an interview process, and in the first interview they gave me this equation and asked how many ways I could solve it. They also asked me to derive a formula for the n-th Fibonacci number.

then $[yh(x)]' = yh'(x) + ah(x)y$. So if we choose $h(x) = e^{ax}$, then our system becomes

$$[ye^{ax}]' = f(x)e^{ax}$$

Let $F(x)$ be an integral of $f(x)e^{ax}$. Then $ye^{ax} = F(x) + C$. Or:

$$y(x) = F(x)e^{-ax} + Ce^{-ax}$$

Obviously, if we have a nasty f, then the integration might be difficult. But at least we can express the answer in ters of an integral.

Note that this approach can be modified to work even if the coefficient a is a function of x, but I'm not dealing with that case here.

A.2.2 Differential Operators

Let's now denote the derivative operation by D. So $\mathrm{D}y = y'$. Think of this as being like a matrix. In fact, if we think of the function y in terms of the (infinite) vector of its Taylor Series, then D can be thought of as the (infinite) matrix that takes the vector for y to the vector for its derivative. Notice that D doesn't really exist on its own — it's only defined in terms of what it does to a function. We can talk about $\mathrm{D}+3$ as being the "operator" or linear transformation which produces $y' + 3y$ when we apply it to y. Some texts will use $\mathrm{D} + 3\mathrm{I}$ for this (and this is technically more correct), but I won't.

Some observations are in order: $\mathrm{D}ay = a\mathrm{D}y$; $\mathrm{D}[(\mathrm{D}+1)y] = (\mathrm{D}^2 + \mathrm{D})y = \mathrm{D}[(\mathrm{D}+1)y]$. In general, these differential operators commute with one another. Just like a single square matrix commutes with itself and the identity. This will allow us to factor a differential operator.

If $r^2 + br + c = (r - r_1)(r - r_2)$, then $\mathrm{D}^2 + b\mathrm{D} + c = (\mathrm{D} - r_1)(\mathrm{D} - r_2)$ (and in fact the order doesn't matter).

So now look at $y'' + by' + cy = f(x)$ — we can assume the first coefficient is 1: if not, divide both sides by it (and it's not zero because then it wouldn't be the first coefficient). This becomes

$$(\mathrm{D} - r_1)(\mathrm{D} - r_2)y = f(x)$$

If we set $v = (\mathrm{D} - r_2)y$, then this becomes

$$(\mathrm{D} - r_1)v = f(x)$$

or

$$v' - r_1 v = f(x)$$

We've learned how to solve this. Multiply by $e^{-r_1 x}$.

$$[ve^{-r_1 x}]' = f(x)e^{-r_1 x}$$

Let $F(x)$ be an integral of the right hand side

$$ve^{-r_1 x} = F(x) + K$$

or

$$v = F(x)e^{r_1 x} + Ke^{r_1 x}$$

Now $v = (\mathrm{D} - r_2)y$, so we have

$$y' - r_2 y = v = F(x)e^{r_1 x} + Ke^{r_1 x}$$

Solving by the integrating factor $e^{-r_2 x}$, we get

$$[e^{-r_2 x}y]' = F(x)e^{(r_1 - r_2)x} + Ke^{(r_1 - r_2)x}$$

We integrate this. If $r_1 \neq r_2$ we get

$$e^{-r_2 x} y = G(x) + C_1 e^{(r_1 - r_2)x} + C_2$$

where G is the appropriate integral and $C_1 = K/(r_1 - r_2)$. This becomes

$$y = G(x) e^{r_2 x} + C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

If $r_1 = r_2 = r$, then instead we integrate and get

$$y = G(x) e^{rx} + C_1 x e^{rx} + C_2 e^{rx}$$

So this approach naturally handles the repeated root without having to think of it as a special case. We just do whatever integral the problem presents us with. It also handles an arbitrary $f(x)$, where we perhaps end up with a solution having $G(x)$ expressed as an integral.

If you understand everything I just did, then you will already know how to handle several lectures of a differential equations course.

Appendix B

Complex Numbers

Complex numbers are numbers of the form $a+bi$ where $i^2 = -1$. With respect to addition and multiplication by real numbers, they form a 2-dimensional vector space over $\mathbb{R}$, with a basis of 1 and i. So any $z \in \mathbb{C}$ can be written in one and only one way as a linear combination of 1 and i.

On top of addition and multiplication by real numbers, they have additional operations: most obviously multiplication of two complex numbers gives a new complex number and respects all the rules we would expect, subject to $i^2 = -1$.

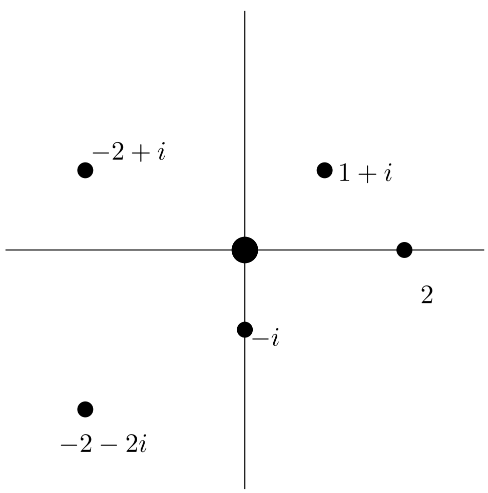
$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = ac - bd + (ad + bc)i$$

Another operation exists, namely conjugation. We denote this by overlining a variable. When we conjugate a number, we replace i by $-i$ everywhere

$$\overline{a + bi} = a - bi$$

It is useful to be aware that for any complex number $z = a + bi$, we have $z\overline{z} \in \mathbb{R}$, with $z\overline{z} = a^2 + b^2$.

We frequently discuss the complex plane. This is a two-dimensional plane. Each point on the plane represents a complex number. The horizontal axis represents the real part of a number, and the vertical axis the imaginary part. So the point whose coordinates are (a, b) represents the number $a + bi$.



We can see that $|z| = \sqrt{z\overline{z}}$ gives the geometric length of the distance from 0 to z.

One very useful thing we see with complex numbers is to consider e^z. If $z = a + bi$, this becomes $e^a e^{bi}$. We know how to handle e^a. What about e^{bi}? We'll do that in a second, but first, let me ask, what is $|e^{bi}|$? It's

$\sqrt{e^{bi}e^{-bi}} = \sqrt{e^0} = \sqrt{1} = 1$. So we know already that it must lie on what is called the "unit circle". That is, regardless of what (real number) b is, e^{bi} lies on the circle of radius 1 centered at the origin. Now, let's work on e^{bi}, but because it will be better to think of b as an angle, let's look at $e^{i\theta}$: We know that

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

Plugging in θi we get

$$e^{\theta i} = 1 + i\theta - \frac{1}{2!}\theta^2 - i\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + i\frac{1}{5!}\theta^5 - \frac{1}{6!}\theta^6 - \cdots$$

We can separate this into real and imaginary parts:

$$\begin{aligned} e^{\theta i} &= \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + \cdots\right) \\ &\quad + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7 + \cdots\right) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

Where does this sit on the complex plane? It's at the point whose x coordinate is $\cos\theta$ and y coordinate is $\sin\theta$. This is the point on the unit circle that corresponds to θ radians from the horizontal axis. That is, if we use polar coordinates, it's $(1, \theta)$. More generally, the point $re^{i\theta}$ where $r, \theta \in \mathbb{R}$ is the point at distance r from the origin with an angle of θ from the horizontal.

This has a huge number of applications. As just one: we'll derive the sum identities for cos and sin: Consider e^{Ai+Bi}. We will write it in two different ways, and then use the fact that $\mathbb{C}$ is a vector space over $\mathbb{R}$ to tell us the real and imaginary parts of both expressions must be the same.

$$\begin{aligned} \cos(A+B) + i\sin(A+B) &= e^{(A+B)i} \\ &= e^{Ai}e^{Bi} \\ &= (\cos A + i\sin A)(\cos B + i\sin B) \\ &= \cos A\cos B - \sin A\sin B + i(\cos A\sin B + \cos B\sin A) \end{aligned}$$

So $\cos(A+B) = \cos A\cos B - \sin A\sin B$ and $\sin(A+B) = \cos A\sin B + \cos B\sin A$.

And anyone who forced you to memorize these two formulas rather than learning how to derive them took time from your live you'll never get back. Noone in their right mind would put effort into memorizing these formulas when there is such an easy derivation.

Appendix C

The quadratic formula

The quadratic function is very useful, and something that you should memorize. If you struggle to memorize it, then you should at least memorize how to derive it. It can be derived through "completing the square", and I'll basically do this, but in a different manner from what you may have learned back in Algebra.

It's hard to immediatly see how to solve $ax^2 + bx + c = 0$. However, if we instead have $d(x + e)^2 + f = 0$, then we can solve this. We find $d(x + e)^2 = -f$ so $(x + e)^2 = -f/d$, and $(x + e) = \pm\sqrt{-f/d}$. Finally, we have $x = -e \pm \sqrt{-f/d}$.

So our goal is to figure out d, e, and f such that $d(x + e)^2 + f = ax^2 + bx + c$. We expand out the square: $dx^2 + 2dex + de^2 + f = ax^2 + bx + c$. Recall that x^2, x, and 1 form a basis for the quadratic polynomials. So if these two polynomials are equal, so are their coefficients.

So $d = a$ from the x^2 terms. Looking at the x terms, $2de = b$. Since $d = a$, we conclude $e = b/2a$. Looking at the constant term, we have $de^2 + f = c$. Substituting for d and e, we conclude $f = c - b^2/4a$.

So in other words:

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Thus if $ax^2 + bx + c = 0$ then

$$\begin{aligned}
a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a} - c \\
a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
\left(x + \frac{b}{2a}\right) &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
\left(x + \frac{b}{2a}\right) &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

Index