Rigorous Size–Expansion Tradeoffs for Tanner Codes: Infinite and Finite-Length Certificates

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Abstract

The size–expansion tradeoff converts vertex expansion for small left-vertex sets into guarantees for larger sets. A recent Tanner-code analysis expresses the degree-independent tradeoff as a countably infinite linear program and states a piecewise closed form. Its derivation assumes primal and dual optimizers and invokes complementary slackness without establishing attainment or strong duality, which are not automatic for an infinite program. We close this gap by constructing matching primal and dual feasible solutions. They are supported on two adjacent indices and prove the stated formula, attainment, and optimality using weak duality alone. We then retain the exact hypergeometric sampling coefficients and solve the resulting finite-length program by a second adjacent-support certificate. This yields an exact bound at every admissible pair of set sizes and, uniformly, the asymptotic tradeoff with an explicit $O(1/n)$ loss. We propagate that loss through minimum-distance and decoder-threshold consequences, replace the ordinary inverse by a generalized inverse, and handle $\delta = 1$ separately. A final conditioning result closes the passage from bounded-degree configuration multigraphs to simple Tanner graphs.

1 Introduction

Let $G = (L \cup R, E)$ be a sparse bipartite graph with $|L| = n$, left degree c, and right degree d. A Tanner code assigns an inner code $C_0 \subseteq \mathbb{F}_2^d$ to every right vertex and requires each local restriction to belong to C_0 [1]. Small-set vertex expansion of G gives linear minimum distance and supports efficient decoding; this principle goes back to expander codes [2] and has since been refined in several directions [3–5].

The size–expansion tradeoff asks how much expansion must remain when the allowed left-set size is increased by a factor $k > 1$. Chen, Cheng, Li, and Ouyang introduced the size–expansion optimization in their analysis of expander codes [4]. Zhou and Guo subsequently formulated the countably infinite, degree-independent relaxation, gave its piecewise formula, and applied it to distance and decoding statements for Tanner codes [6]. The closed-form expression appeared there, but its derivation does not establish the infinite-program hypotheses on which it relies. Specifically, arXiv version 4 first assumes that the countably infinite primal program has an optimizer, then writes an algebraic dual and applies complementary slackness [6, App. A.3]. Primal attainment, dual attainment, and zero duality gap are not consequences of finite-dimensional linear-programming duality in this setting. Thus the adjacent-support assertion and the resulting formula are not proved by that argument. The expression itself is not claimed as new here; its rigorous validation is.

This gap affects the interface to the coding statements. The sampling calculation at finite block length has hypergeometric coefficients, whereas the displayed tradeoff uses their Bernoulli limits. The resulting normalized loss is $O(1/n)$ [6, Sec. 5.1 and App. A.3]; it cannot be dropped when a distance or decoding theorem requires a strict expansion threshold. Continuity and monotonicity

also do not by themselves make an ordinary inverse single-valued, and the endpoint $\delta = 1$ does not share the claimed vanishing limit. The tightness statement also uses a random construction generated by a configuration procedure that may produce parallel edges [6, Sec. 5.3 and App. A.1], while a Tanner graph is taken here to be simple.

We resolve these points as follows.

1. We prove the stated piecewise formula by constructing an explicit feasible primal solution on two adjacent indices and an explicit feasible dual solution with the same value. Weak duality alone proves optimality and attainment, without presupposing infinite-dimensional strong duality.

2. We solve the exact finite-length program with hypergeometric coefficients by a second adjacent-support certificate. Its uniform consequence applies to every set of size at most $k\alpha n$ and retains the $O(1/n)$ loss needed in the threshold corollaries.

3. We define a generalized inverse of the non-increasing tradeoff function, treat $\delta = 1$ separately, and retain the finite-length loss until the strict distance or decoding threshold is applied.

4. We isolate a general conditioning lemma for bounded-degree bipartite configuration models. It converts a high-probability expansion event into the existence of a simple graph with the same event.

The decoder constructions and the probabilistic expansion estimates of the public preprint are not reproved here. They enter, with attribution, through the threshold parameter in Proposition 5.3 and the event used in Proposition 6.1. The results below repair the optimization and graph-model interfaces on which those applications depend.

2 Preliminaries

All graphs are finite. A bipartite graph is written $G = (L \cup R, E)$, with $|L| = n$. Unless a multigraph is explicitly named, the graph is simple.

Definition 2.1 (Bipartite vertex expander)**.** *For integers $c, d \geq 1$, $0 < \alpha < 1$, and $0 < \delta \leq 1$, a (c, d, α, δ)-bipartite expander is a (c, d)-regular bipartite graph such that*

$$|N(S)| \geq \delta c|S| \qquad \text{for every } S \subseteq L \text{ with } |S| \leq \alpha n.$$

Fix, for each $v \in R$, an ordering of its d neighbors. For a binary linear code $C_0 \subseteq \mathbb{F}_2^d$, the Tanner code is

$$T(G, C_0) := \{x \in \mathbb{F}_2^L : x|_{N(v)} \in C_0 \text{ for every } v \in R\}.$$

Write d_0 for the minimum distance of C_0.

For $k > 1$, set

$$q = q(k) := 1 - \frac{1}{k}, \qquad a_i(k) := 1 - q^i, \qquad i \geq 1.$$

The quantity $a_i(k)$ is the probability that an independent sample which keeps each element with probability $1/k$ meets a fixed i-set.

Definition 2.2 (Universal size–expansion function)**.** *For $k > 1$ and $0 < \delta \le 1$, define $f_\delta(k)$ as the infimum of*

$$\frac{1}{k}\sum_{i\ge 1}\beta_i \tag{1}$$

over nonnegative sequences $(\beta_i)_{i\ge 1}$ satisfying

$$\sum_{i\ge 1} i\beta_i = k, \qquad \sum_{i\ge 1} a_i(k)\beta_i \ge \delta. \tag{2}$$

The degree-dependent function $f_{d,\delta}(k)$ is defined by restricting the variables to $1 \le i \le d$.

The sequence with $\beta_1 = k$ is feasible, so the programs are nonempty. Since $a_i(k) \le 1$, every feasible sequence satisfies $\sum_i \beta_i \ge \delta$. Thus $f_\delta(k) \ge \delta/k > 0$.

3 Rigorous solution of the universal program

Define

$$r_i(k) := \frac{ka_i(k)}{i} = \frac{1 + q + \cdots + q^{i-1}}{i}.$$

Lemma 3.1. *For every $k > 1$, the sequence $(r_i(k))_{i\ge 1}$ decreases strictly from $r_1(k) = 1$ to 0.*

Proof. The displayed expression is the average of the first i terms of the strictly decreasing sequence $1, q, q^2, \ldots$. Adding the next term, which is below the current average, strictly decreases the average. The limit is zero because the numerator is bounded by $(1-q)^{-1} = k$, whereas the denominator tends to infinity. □

For $0 < \delta < 1$, Lemma 3.1 gives a unique integer $m \ge 1$, apart from the harmless equality convention at a breakpoint, such that

$$r_{m+1}(k) \le \delta \le r_m(k). \tag{3}$$

Let

$$D_m := (m+1)a_m(k) - ma_{m+1}(k).$$

Since $a_{j+1}(k) - a_j(k) = q^j/k$,

$$D_m = \frac{1}{k}\left(\sum_{j=0}^{m-1} q^j - mq^m\right) > 0.$$

The strict inequality holds because every term in the sum is larger than q^m.

Theorem 3.2 (Adjacent-support certificate)**.** *Let $k > 1$ and $0 < \delta < 1$, and choose m by (3). The universal program attains its optimum at*

$$\beta_m = \frac{(m+1)\delta - ka_{m+1}(k)}{D_m}, \tag{4}$$

$$\beta_{m+1} = \frac{ka_m(k) - m\delta}{D_m}, \tag{5}$$

with all other coordinates zero. Its value is

$$f_\delta(k) = \frac{\delta - q^m}{k - (k+m)q^m}. \tag{6}$$

No infinite-dimensional strong-duality theorem is needed for this conclusion.

Proof. The inequalities in (3) imply that the two quantities in (4)–(5) are nonnegative. Direct substitution gives

$$m\beta_m + (m+1)\beta_{m+1} = k, \qquad a_m\beta_m + a_{m+1}\beta_{m+1} = \delta,$$

so the sequence is primal feasible.

The algebraic dual of (1)–(2) is

$$\begin{aligned} \text{maximize} \quad & kx + \delta y, \\ \text{subject to} \quad & ix + a_i(k)y \le \frac{1}{k} \quad (i \ge 1), \\ & y \ge 0, \end{aligned} \tag{7}$$

where x is unrestricted. Put

$$\Delta_m := a_{m+1}(k) - a_m(k) = \frac{q^m}{k}, \qquad y := \frac{1}{kD_m}, \qquad x := -\frac{\Delta_m}{kD_m}.$$

For $h_i := a_i(k) - i\Delta_m$,

$$h_{i+1} - h_i = \frac{q^i - q^m}{k}.$$

Thus h_i increases up to $i = m$, is equal at m and $m+1$, and then decreases. Its maximum is $h_m = h_{m+1} = D_m$. Consequently

$$ix + a_i(k)y = \frac{h_i}{kD_m} \le \frac{1}{k}$$

for every $i \ge 1$, so (x, y) is dual feasible. The dual constraints are equalities at the two indices supporting the primal solution. Therefore

$$\frac{\beta_m + \beta_{m+1}}{k} = kx + \delta y.$$

For completeness, weak duality is valid here without any topological duality assumption. If β is primal feasible, then all relevant series converge absolutely because $a_i(k) \le i/k$, and summing the dual inequalities after multiplication by β_i gives

$$\frac{1}{k}\sum_i \beta_i \ge x\sum_i i\beta_i + y\sum_i a_i(k)\beta_i \ge kx + \delta y.$$

The matching feasible values therefore prove optimality and attainment. Summing (4)–(5) and using $D_m = [k - (k+m)q^m]/k$ gives (6). □

The proof does not infer adjacent support from complementary slackness. Instead it exhibits the adjacent-support point and a supporting affine function valid at every index. Equality of their objective values proves all duality facts needed for this particular countably infinite program.

Corollary 3.3 (Finite right degree)**.** *For $d \ge 1$, $k > 1$, and $0 < \delta < 1$,*

$$f_{d,\delta}(k) = \max\left\{ f_\delta(k), \frac{1}{d} \right\}.$$

Proof. Every feasible point of the finite program satisfies $\sum_{i=1}^{d} \beta_i \ge k/d$, giving the lower bound $1/d$. If $\delta \le r_d(k)$, the point $\beta_d = k/d$ is feasible and attains that value. If $\delta > r_d(k)$, the index selected in (3) satisfies $m < d$, and the optimizer from Theorem 3.2 is feasible for the finite program. □

Proposition 3.4 (Monotonicity, limits, and the endpoint)**.** *For fixed $0 < \delta < 1$, the function f_δ is positive, continuous, and non-increasing on $(1, \infty)$, with*

$$\lim_{k \downarrow 1} f_\delta(k) = \delta, \qquad \lim_{k \to \infty} f_\delta(k) = 0.$$

At the endpoint $\delta = 1$, however,

$$f_1(k) = 1 \qquad \text{for every } k > 1.$$

Proof. For monotonicity, observe that $ka_i(k)$ is non-decreasing in k. Indeed, with $q = 1 - 1/k$, its derivative is

$$1 - q^i - i(1-q)q^{i-1} \geq 0;$$

the inequality follows because $q^{i-1}(i - (i-1)q) \leq 1$ on $[0, 1]$. The case $i = 1$ is immediate; for $i \geq 2$, its derivative is $i(i-1)q^{i-2}(1-q) \geq 0$, and its value at $q = 1$ is one. If $k' > k$ and β is feasible at k, then $(k'/k)\beta$ is feasible at k' and has the same objective value. Hence $f_\delta(k') \leq f_\delta(k)$.

The formula in (6) is continuous within each interval on which m is fixed. At a breakpoint $\delta = r_{m+1}(k)$, both adjacent formulas equal $1/(m+1)$, so the pieces join continuously. As $k \downarrow 1$, the selected m remains bounded: for example, $r_i(k) \leq k/i < 2/i$ when $k < 2$, so $m < 2/\delta + 1$. Hence $q^m \to 0$, which gives the first limit.

For the second limit, choose a constant $z > 0$ small enough that $(1 - e^{-z})/z > \delta$, and set $M = \lfloor zk \rfloor$. For all sufficiently large k, $r_M(k) > \delta$; hence the one-coordinate point $\beta_M = k/M$ is feasible and gives $f_\delta(k) \leq 1/M \to 0$.

Finally, Bernoulli's inequality gives $a_i(k) \leq i/k$, strictly for $i > 1$. When $\delta = 1$, the two primal constraints force equality in this inequality, so all mass lies at $i = 1$. Thus $\beta_1 = k$ and the objective is one. □

Strict monotonicity is unnecessary for applications. The following inverse is stable even if a level set contains an interval.

Definition 3.5 (Generalized inverse)**.** *For $0 < \delta < 1$ and $0 < \tau < \delta$, define*

$$K_\delta(\tau) := \sup\{k > 1 : f_\delta(k) \geq \tau\}. \tag{8}$$

By Proposition 3.4, $1 < K_\delta(\tau) < \infty$ and

$$f_\delta(K_\delta(\tau)) = \tau. \tag{9}$$

4 A finite-length size–expansion theorem

The asymptotic coefficient $a_i(k)$ comes from sampling without replacement. The next elementary estimate records the needed uniform approximation.

Lemma 4.1 (Hypergeometric comparison)**.** *Fix d and a compact interval $1 < \kappa_- \leq \kappa \leq \kappa_+ < \infty$. Let s, t be positive integers with $s/t = \kappa$, and let $1 \leq i \leq d$. If a t-subset is chosen uniformly from an s-set, then the probability $p_i(s,t)$ that it meets a fixed i-subset satisfies*

$$a_i(\kappa) \leq p_i(s,t) \leq a_i(\kappa) + \frac{C}{s},$$

where C depends only on d, κ_-, κ_+.

Proof. The avoidance probability is

$$1 - p_i(s,t) = \frac{\binom{s-i}{t}}{\binom{s}{t}} = \prod_{j=0}^{i-1} \left(1 - \frac{t}{s-j}\right).$$

Each factor is at most $1 - t/s$, giving the lower bound on p_i. Write $u = t/s$. For all sufficiently large s, all factors below lie in $[0,1]$, and

$$0 \le (1-u) - \left(1 - \frac{t}{s-j}\right) = \frac{tj}{s(s-j)} \le \frac{C_0 j}{s} \qquad (0 \le j < i).$$

The telescoping product inequality $|\prod_j x_j - \prod_j y_j| \le \sum_j |x_j - y_j|$ for factors in $[0,1]$ now gives

$$0 \le p_i(s,t) - a_i(\kappa) \le \frac{C_1}{s}.$$

Only finitely many smaller values of s remain for each $i \le d$; an enlargement of the constant covers them uniformly. □

The sampling identity behind the size–expansion method is already exact before its coefficients are replaced by their Bernoulli limits [4, Sec. 3]. Retaining those hypergeometric coefficients defines the following finite-length program. For $0 < \delta \le 1$ and integers $2 \le t < s$ with $s \ge t + d$, put $\kappa = s/t$ and

$$\Phi_{d,\delta}(s,t) := \min \left\{ \frac{1}{\kappa} \sum_{i=1}^{d} \beta_i : \begin{array}{l} \beta_i \ge 0, \quad \sum_{i=1}^{d} i\beta_i = \kappa, \\ \sum_{i=1}^{d} p_i(s,t)\beta_i \ge \delta \end{array} \right\}. \tag{10}$$

Proposition 4.2 (Exact finite-length certificate)**.** *Define*

$$R_i(s,t) := \frac{\kappa p_i(s,t)}{i}, \qquad 1 \le i \le d.$$

The sequence $R_1(s,t), \dots, R_d(s,t)$ is strictly decreasing and $R_1(s,t) = 1$. If $\delta \le R_d(s,t)$, then

$$\Phi_{d,\delta}(s,t) = \frac{1}{d}.$$

Otherwise, let $m < d$ be determined by

$$R_{m+1}(s,t) \le \delta \le R_m(s,t)$$

and set

$$B_m := (m+1)p_m(s,t) - m p_{m+1}(s,t).$$

Then $B_m > 0$, and the exact optimum is supported on $m, m+1$, with

$$\beta_m = \frac{(m+1)\delta - \kappa p_{m+1}(s,t)}{B_m}, \tag{11}$$

$$\beta_{m+1} = \frac{\kappa p_m(s,t) - m\delta}{B_m}, \tag{12}$$

and value $\Phi_{d,\delta}(s,t) = (\beta_m + \beta_{m+1})/\kappa$.

Proof. Write $p_0 = 0$. Pascal's identity gives

$$p_{i+1}(s,t) - p_i(s,t) = \frac{\binom{s-i-1}{t-1}}{\binom{s}{t}}, \qquad 0 \le i \le d-1.$$

These first differences are positive and strictly decreasing under $2 \le t$ and $s \ge t+d$. Thus p_i/i, the average of the first i differences, is strictly decreasing. Also $p_1 = t/s = 1/\kappa$, proving the assertions about R_i.

If $\delta \le R_d$, the point $\beta_d = \kappa/d$ is feasible. The moment constraint implies $\sum_i \beta_i \ge \kappa/d$, so its value is optimal. Suppose instead that $\delta > R_d$. The inequalities defining m make (11)–(12) nonnegative, and direct substitution verifies both constraints with equality.

It remains to certify optimality. Put

$$\Delta_m := p_{m+1}(s,t) - p_m(s,t), \qquad y := \frac{1}{\kappa B_m}, \qquad x := -\frac{\Delta_m}{\kappa B_m}.$$

The decreasing first differences imply that $p_i(s,t) - i\Delta_m$ is maximized at $i = m, m+1$, where it equals B_m. Moreover, $p_m = \sum_{j=0}^{m-1}(p_{j+1} - p_j) > m\Delta_m$, so $B_m > 0$. Consequently

$$ix + p_i(s,t)y \le \frac{1}{\kappa} \qquad (1 \le i \le d).$$

This is a feasible dual solution to (10), and its constraints are tight at the two primal support indices. The primal and dual values agree, proving the stated formula. □

Corollary 4.3 (Exact expansion at a prescribed size)**.** *Let G be left-regular of degree c and have maximum right degree at most d. Suppose every t-subset of L has at least δct neighbors. Under the hypotheses of Proposition 4.2, every s-subset $S \subseteq L$ satisfies*

$$\frac{|N(S)|}{c|S|} \ge \Phi_{d,\delta}(s,t).$$

Proof. Choose a uniformly random t-subset $S' \subseteq S$. If $N_i(S)$ denotes the right vertices with exactly i neighbors in S, set $\beta_i = |N_i(S)|/(ct)$. Double counting gives $\sum_i i\beta_i = s/t = \kappa$, while

$$\sum_i p_i(s,t)\beta_i = \frac{\mathbb{E}|N(S')|}{ct} \ge \delta.$$

Thus β is feasible in (10), and its objective is $|N(S)|/(c|S|)$. □

Theorem 4.4 (Finite-length tradeoff)**.** *Fix c, d, $0 < \alpha < 1$, $0 < \delta < 1$, and $1 < k < 1/\alpha$. There is a constant $C = C(c,d,\alpha,\delta,k)$ such that every (c,d,α,δ)-bipartite expander with n left vertices is a*

$$\left(c, d, k\alpha, f_\delta(k) - \frac{C}{n}\right)$$

bipartite expander for all sufficiently large n.

Proof. Let $S \subseteq L$ and write $s = |S|$. If $s \le \alpha n$, the original expansion hypothesis applies, and $\delta \ge f_\delta(k)$ by Proposition 3.4.

Put $A = \lfloor \alpha n \rfloor$. We first consider $A < s \le kA$. Set $t = \lceil s/k \rceil$ and $\kappa = s/t$. Then $t \le A$, and for large n the value of κ belongs to a fixed compact subinterval of $(1, k]$.

For sufficiently large n, the integer conditions of Corollary 4.3 hold, and hence

$$\frac{|N(S)|}{cs} \geq \Phi_{d,\delta}(s,t).$$

For every feasible point of the exact program, Lemma 4.1 and $\sum_i \beta_i \leq \sum_i i\beta_i = \kappa$ give $\sum_i a_i(\kappa)\beta_i \geq \delta - C_1/n$, because $s \geq \alpha n$. Taking the minimum yields

$$\Phi_{d,\delta}(s,t) \geq f_{d,\delta-C_1/n}(\kappa). \tag{13}$$

We next make the dependence on the right-hand side explicit. The finite program has an attained dual optimum (x,y), with $y \geq 0$. Its constraint at $i = 1$ gives $x \leq (1-y)/\kappa$, and hence

$$f_{d,\delta}(\kappa) = \kappa x + \delta y \leq 1 - (1-\delta)y.$$

Since the primal value is nonnegative, an optimal multiplier satisfies $y \leq 1/(1-\delta)$. The same dual point is feasible after replacing δ by $\delta - \eta$, so for $0 \leq \eta < \delta$,

$$f_{d,\delta-\eta}(\kappa) \geq f_{d,\delta}(\kappa) - \frac{\eta}{1-\delta}.$$

Combining this bound with Corollary 3.3, monotonicity in $\kappa \leq k$, and (13) gives

$$\frac{|N(S)|}{cs} \geq f_\delta(\kappa) - \frac{C_2}{n} \geq f_\delta(k) - \frac{C_2}{n}.$$

If $kA < s \leq k\alpha n$, delete fewer than $k+1$ vertices to obtain a subset $S_0 \subseteq S$ with $|S_0| \leq kA$. Since $N(S_0) \subseteq N(S)$ and $|S| - |S_0| = O(1)$, dividing the bound for S_0 by $|S|$ changes its expansion factor by only $O(1/n)$. The constant is therefore independent of S, completing the proof. □

Remark 4.5. *The normalized error in Theorem 4.4 is $O(1/n)$. Before normalization, the expectation is a sum over $O(n)$ right vertices, each with an $O(1/n)$ probability approximation; its additive error is therefore $O(1)$, not $O(1/n)$.*

5 Consequences for Tanner codes

We first recall the direct distance argument.

Lemma 5.1 (Expansion implies distance)**.** *If G is a (c,d,ρ,Δ)-bipartite expander and $\Delta d_0 > 1$, then*

$$d_{\min}(T(G,C_0)) > \rho n.$$

Proof. Let $x \neq 0$ be a Tanner-code word and let $S = \mathrm{supp}(x)$. Every right vertex in $N(S)$ sees a nonzero word of C_0, hence at least d_0 positions of S. If $|S| \leq \rho n$, then

$$c|S| \geq d_0|N(S)| \geq d_0\Delta c|S| > c|S|,$$

a contradiction. □

Theorem 5.2 (Distance through the generalized inverse)**.** *Let $0 < \delta < 1$, let $d_0 \geq 2$, and assume $\delta d_0 > 1$. Choose*

$$0 < \varepsilon < \delta - \frac{1}{d_0}, \qquad k_* := K_\delta\left(\frac{1}{d_0} + \varepsilon\right),$$

and suppose $k_*\alpha < 1$. Then every (c,d,α,δ)-bipartite expander satisfies

$$d_{\min}(T(G,C_0)) > k_*\alpha n$$

for all sufficiently large n.

Proof. By (9), $f_\delta(k_*) = 1/d_0 + \varepsilon$. The finite-length loss in Theorem 4.4 is smaller than $\varepsilon/2$ for large n. Thus G expands all sets of size at most $k_*\alpha n$ by a factor Δ satisfying

$$\Delta d_0 > \left(\frac{1}{d_0} + \frac{\varepsilon}{2}\right) d_0 > 1.$$

Apply Lemma 5.1. □

The same calculation transfers any decoder theorem whose hypothesis depends on an expansion threshold.

Proposition 5.3 (Decoder-threshold transfer)**.** *Suppose an algorithm has the following guarantee for a fixed $\theta > 0$: on every (c, d, ρ, Δ)-bipartite expander with $\Delta d_0 > \theta$, it corrects every error pattern of weight at most ρn. Let $\delta d_0 > \theta$, choose*

$$0 < \varepsilon < \delta - \frac{\theta}{d_0}, \qquad k_* := K_\delta\left(\frac{\theta}{d_0} + \varepsilon\right),$$

and assume $k_*\alpha < 1$. Then, for sufficiently large n, the same algorithm corrects every error pattern of weight at most $k_*\alpha n$ on the original (c, d, α, δ)-expander.

Proof. The finite-length expansion factor at scale $k_*\alpha n$ is at least $\theta/d_0 + \varepsilon - O(1/n)$. Its product with d_0 is larger than θ for all sufficiently large n, so the assumed decoder theorem applies with $\rho = k_*\alpha$. □

For example, the public preprint of Zhou and Guo gives randomized and deterministic linear-time decoders under the threshold $\theta = 2$ [6]. Proposition 5.3 supplies the precise finite-length interpretation of the corresponding size–expansion radius.

6 From configuration multigraphs to simple graphs

The bipartite configuration model pairs left and right half-edges uniformly. It naturally produces a multigraph. The following observation is useful when an expansion proof is carried out in that model but the code definition requires a simple graph.

Proposition 6.1 (Simplicity transfer)**.** *Consider a sequence of bipartite configuration models with $\Theta(n)$ half-edges and maximum left and right degrees bounded independently of n. Let $\mathcal{E}_n$ be an event such that $\mathbb{P}(\mathcal{E}_n) = 1 - o(1)$, and let $\mathcal{S}_n$ be the event that the realization is simple. Then*

$$\mathbb{P}(\mathcal{E}_n \mid \mathcal{S}_n) = 1 - o(1).$$

In particular, for all sufficiently large n, there exists a simple realization for which $\mathcal{E}_n$ holds.

The conclusion also holds for finitely many independent configuration models and their joint high-probability event.

Proof. Janson's bipartite configuration-model theorem implies that, under the stated bounded-degree assumptions, the probability $\mathbb{P}(\mathcal{S}_n)$ of simplicity is bounded below by a constant $p > 0$ [7, Thm. 5.1]. The theorem's degree-sequence conditions hold because the total number of half-edges is $\Theta(n)$ and both maximum degrees are bounded. Therefore

$$\mathbb{P}(\mathcal{E}_n \cap \mathcal{S}_n) \geq \mathbb{P}(\mathcal{S}_n) - \mathbb{P}(\mathcal{E}_n^c) \geq p - o(1) > 0.$$

Moreover,

$$\mathbb{P}(\mathcal{E}_n^c \mid \mathcal{S}_n) \le \frac{\mathbb{P}(\mathcal{E}_n^c)}{\mathbb{P}(\mathcal{S}_n)} = o(1),$$

which proves the conditional high-probability assertion. For finitely many independent models, joint simplicity still has probability bounded below by a positive constant, and the same argument applies. □

Corollary 6.2 (Application to a merged Tanner construction)**.** *Suppose two bounded-degree bipartite configuration models have disjoint left vertex sets, their joint expansion event holds with probability* $1 - o(1)$*, and selected right vertices are merged pairwise. For all sufficiently large admissible block lengths, the components may be chosen simple while retaining the expansion event. The merged graph is also simple.*

Proof. Choose component realizations from the positive-probability event supplied by Proposition 6.1. Simplicity within each component rules out parallel edges before merging. After merging, an edge from the first component and an edge from the second cannot become parallel because their left endpoints lie in disjoint sets. □

Zhou and Guo's public preprint analyzes a two-component tightness construction by sampling two bounded-degree configurations and then merging selected right vertices [6, Sec. 5.3 and App. A.1]. Its joint expansion event has probability $1 - o(1)$. Because the stated sampling procedure may produce parallel edges, that estimate alone does not yet give the simple graph used in the Tanner-code definition. Proposition 6.1 and Corollary 6.2 close this graph-model step: conditioning both components on simplicity preserves the expansion estimate. For completeness, the local-code ordering does not require a minimum word to have consecutive support. Given a minimum-weight word $w \in C_0$, the neighbors from the first component can be assigned to the coordinates in $\mathrm{supp}(w)$; the indicator of the first left part then satisfies every merged local constraint. This observation uses, but does not replace, the cited probabilistic expansion estimate. In that construction, c, d, and α may depend on the prescribed parameters; it is not a fixed-degree or fixed-α optimality statement.

7 Conclusion

The previously stated degree-independent size–expansion formula is now a theorem rather than a formal consequence of infinite-dimensional complementary slackness. Two adjacent primal coordinates and an explicit supporting affine function of the strictly concave sequence $a_i(k)$ give matching feasible values, proving attainment and optimality by weak duality. The same geometry solves the genuinely finite-length program after the Bernoulli coefficients are replaced by their exact hypergeometric counterparts. This produces exact prescribed-size bounds and a uniform $O(1/n)$ theorem whose loss is retained through the strict distance and decoder thresholds. The generalized inverse, the separate $\delta = 1$ analysis, and the simplicity-conditioning result close the remaining analytic and graph-model interfaces required by Tanner-code applications.

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