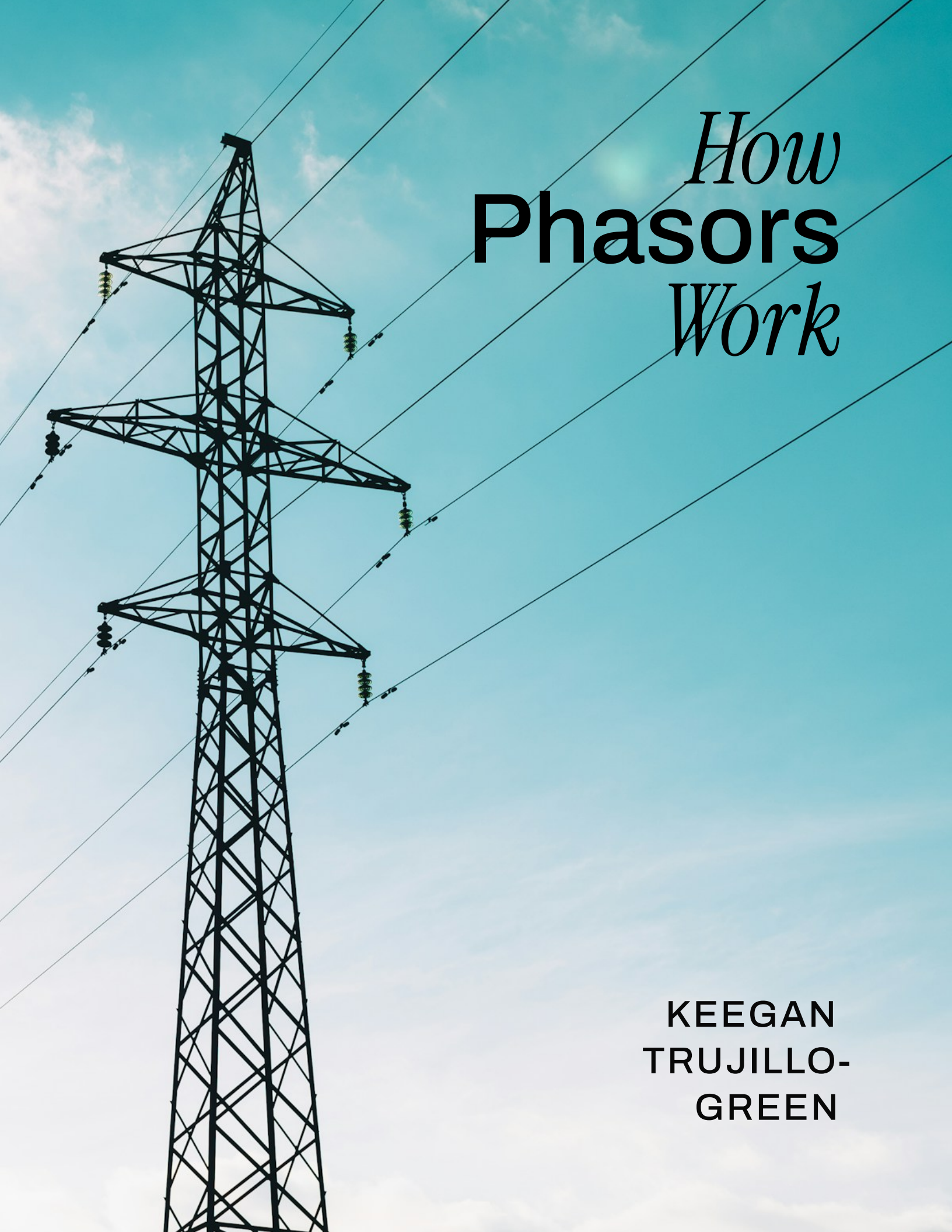


How Phasors *Work*

KEEGAN TRUJILLO-GREEN

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KEEGAN TRUJILLO-GREEN
BASc. Mechatronic Systems Engineering

This book is a work in progress. Existing chapters will be revised, and new chapters will be added. Readers are encouraged to provide feedback at:

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Chapter 1

Introduction

Many industries and fields of study have concepts or ways of doing things that are taken for granted internally but are foreign or unknown to outsiders. Lawyers have legalese and engineers have jargon, but few fields have as large a barrier to understanding as electrical engineering. The phasor may be a staple of the electrical engineer, but being able to do phasor calculations does not mean that one understands the abstractions at play. Indeed, the concept of phasors is a difficult one to grapple with for seasoned engineers—let alone students. However, phasors are simply an excellent tool for analyzing AC power systems—a true gift from mathematics. Clearly understanding this tool and being comfortable applying it is worthwhile, and—with the right learning material—need not be a daunting task.

What is AC Power?

Alternating current, or AC, simply describes electric current that is oscillating. However, the term is also often used to describe oscillating voltage and electric power. This is perhaps more useful, as AC voltage is what drives an AC current, and the result—AC power—is what we use to do useful work. AC means that the voltage and current oscillates a fixed number of times per second, that is, at a constant frequency f. In the vast majority of cases, this oscillation follows a sinusoidal wave.

Why do we use AC power? The key is in the name—alternating current is used in high-power applications, whether on the supply side or on the demand side.

On the supply side, we have the generation of electric power. Electric generators usually operate using rotating components. As a result, the most efficient types of generators—namely, *synchronous generators*—produce AC power, not DC power. Historically, when DC was favored, *dynamos* were used to generate DC power, but they are less efficient. Even now when DC power is needed, synchronous generators are used, and their output is efficiently converted to DC. Examples of synchronous generators include the alternator in your car (or your EV's motor when used for regenerative braking), diesel generators such as those used by homeowners to power homes during power outages, and the generator in a power plant. These all produce AC power.

On the demand side, we have the consumption of electric power. The

largest standalone electric loads are electric motors, the most efficient types of which consume AC power. For example, the electric motors in a diesel–electric ship or train, or in a large industrial plant, all consume AC power.

It is not necessarily enough that AC power is practical to generate and practical to use. In the case of the electrical grid, it must also be practical to transmit over long distances from where power is traditionally generated—large, centralized power plants—to the cities where it is consumed. However, transmitting power over long distances incurs power losses due to the resistance of transmission lines. Minimizing the power losses is achieved by transmitting power at high voltages. High voltages, however, are obviously unsafe and impractical for most consumers, so the voltage must be lowered as we go from transmission lines to distribution lines, into cities and neighborhoods. Furthermore, the voltage must typically be stepped up at its source—the power plant—prior to transmission. Transformers are the easiest and most efficient way to convert between voltage levels, but require AC instead of DC. So, in order to transmit power over long distances to where it is needed, while minimizing losses and delivering power at a safe and convenient voltage, AC power must be used.

Loads of All Kinds

An electric circuit is only useful if it accomplishes a task. In general, that task may be concerned with logic (as in a computer), signals (as in audio equipment or radio-frequency communications), or the transmission and conversion of energy (as in the electrical grid). Discussion will focus on the electrical grid, but also applies to "off-the-grid" applications such as diesel–electric vehicles. As the previous explanation of why we use AC power indicated, being able to transmit energy from one location to another is among the strengths of AC power. However, electrical energy is useless in itself unless it can be generated in the first place and can then go on to do something useful. A different form of energy must be converted to electrical energy (whether directly or indirectly) from an energy source. For example, nuclear, hydro, gas, and solar are sources of atomic, kinetic, chemical, and light energy, respectively. In most cases, the final conversion to electrical energy takes place in an electric generator. But the ability to generate electricity is useless unless electric loads are plugged-in and consuming power; without loads, the electrical grid would technically no longer be a circuit and would serve no purpose. Thus, the electrical grid, as a circuit, includes the loads plugged into it. The electrical energy that is generated is only useful if it can be converted, once again, into another form of energy. For example, a ceiling fan converts electrical energy to mechanical energy. Even a laptop computer—which when on battery power is just a huge, standalone logic circuit—generates heat as a necessary byproduct of its operation. When plugged in, from the grid's perspective, a computer is little more than an electric heater.

Any load that can be plugged into the grid can be modeled by one of three simple discrete components—resistor, inductor, capacitor—or a combination thereof. For example, a laptop computer can be modeled by a resistor. A ceiling fan can be modeled by a combination of resistors and inductors. Furthermore, on the supply side, an alterna-

tor can be modeled by an AC voltage source, resistor, and inductor. Being able to model any load or generator in terms of just a few simple, standard electric components means that the methods used to analyze one AC circuit can be applied just as well to another.

A resistor dissipates electrical power. It typically does so by converting electrical energy to thermal energy, but in the case of loads such as a motor, most of it is converted to something other than heat (in this case, mechanical energy). Inductors and capacitors are also able to convert electrical energy. But, unlike resistors, that conversion is bidirectional; they can store their energy and convert it back to electrical energy. An inductor stores magnetic potential energy in a magnetic field, which can be used to temporarily sustain current, even when external voltage is removed. A capacitor stores electric potential energy (and thus electric potential, a voltage) in an electric field, which can be used to temporarily sustain voltage. This is shown in Figure 1.1.

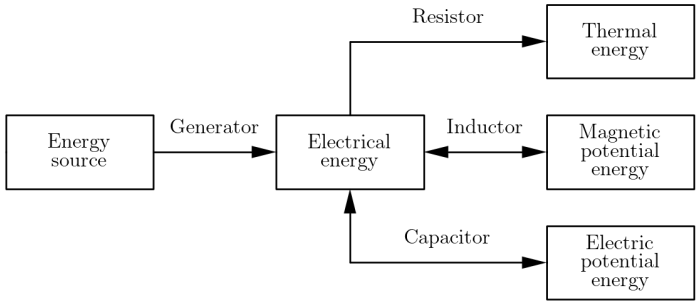


Figure 1.1: Conversion of energy by basic electric components.

DC Versus AC, Resistive Versus Reactive

Consider a circuit with a voltage source and a purely resistive load R. If the voltage source is a DC source, of constant voltage V, the circuit is DC. Current flows according to Ohm's law, $I = V/R$. As we will see in What is a Phasor?, if the voltage source is AC, its voltage $v(t)$ can be expressed as $V\cos(2\pi ft + \theta)$. Ohm's law still applies, and would now be written as $i(t) = v(t)/R$. This means that as the voltage sinusoid $v(t)$ reaches a peak, so does the current sinusoid $i(t)$—they are *in phase*. Figure 1.2 illustrates this. However, if we were to replace the resistor with an inductor or capacitor, the current would no longer be in phase with the voltage. A capacitor's current lags behind the AC voltage, and an inductor's leads it. As will be discussed in , this is closely related to the rate at which energy is converted in inductors and capacitors.

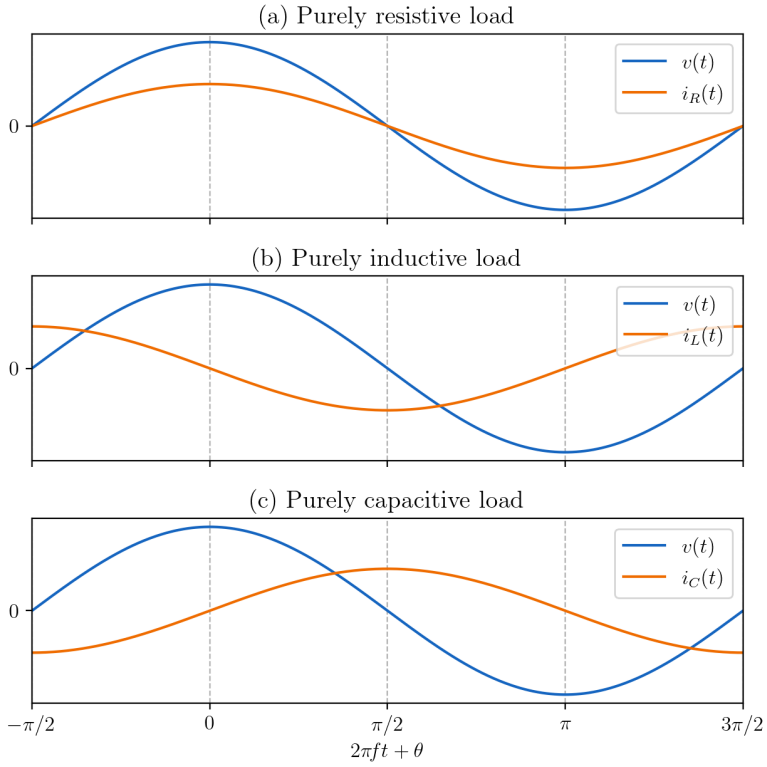


Figure 1.2: In-phase, leading, and lagging current waveforms in purely resistive, inductive, and capacitive loads.

Chapter 2

What is a Phasor?

Phasors: A Notation and Way to Make Calculations Straightforward

A phasor is analogous to a negative number, in the sense that both negative numbers and phasors are mathematical representations, notations, and ways to make calculations straightforward.

A negative number is a *mathematical representation* of the opposite of a quantity. It is distinct from the number zero which represents the absence or lack of a quantity. You can have no fewer than zero apples. Similarly, you cannot have a negative quantity of dollar bills. However, if the quantity of dollars refers not to physical legal tender but to an accounting of how many units of currency you have at your disposal, then a negative quantity represents the opposite of having dollars at your disposal—that is, owing dollars. In addition to this representation, negative numbers offer:

- A *notation*. By prefixing a regular number (x) with a minus sign ($-$), it becomes negative ($-x$).
- A *way to make calculations straightforward*. A negative number is like a different kind of quantity on which arithmetic operations can be done by treating them accordingly; to subtract a negative quantity you add its positive counterpart, to multiply you multiply by the positive counterpart and negate the product, and so on.

Similarly, a phasor is a *mathematical representation* of a sinusoid, in terms of its amplitude A and phase θ. A sinusoid already has a mathematical representation in the time domain with the notation $A\cos(\omega t + \theta)$. However, while this representation makes calculations possible, it does not make them straightforward and does not have a concise notation. Representing sinusoids in the phasor domain offers:

- A *notation*. Phasors can be written concisely, such as in polar form $A\angle\theta$. The "ωt" is implicit.
- A *way to make calculations straightforward*. As we will see in Life With and Without Phasors: Example Circuit Analysis, working in the phasor domain allows you to write and solve AC circuit analysis equations more quickly and easily than working in the time domain. Working in the time domain requires writing and solving differential equations from scratch for each circuit, even though the differential equations and their solutions are

similar from one circuit to another. It turns out that the calculations that can be done with phasors are the bare minimum calculations that must be done in order to solve an AC circuit.

Back to Basics: Treating Sinusoids Like Vectors

Consider the right triangle XYZ in Figure 2.1(a). This triangle (like every triangle) has a ratio between length of side XZ (adjacent to ϕ_X) and the length of side XY (the hypotenuse). The cosine function represents the relationship between this ratio and angle ϕ_X:

$$\cos \phi_X = \frac{XZ}{XY} \tag{2.1}$$

As ϕ_X ranges from 0 to 2π radians (or 0 to 360 degrees), point Y draws a circle centered around X whose radius A is the hypotenuse's length, as in Figure 2.1(b).

If we were to plot the length of side XZ as a function of ϕ_X, we would have a cosine wave of amplitude $A = XY$, as in Figure 2.1(c), that repeats every multiple of 2π radians. This cosine wave may represent AC voltage or current.

$$A \cos \phi_X \tag{2.2}$$

If we wanted the drawing of our circle to depend on time t, and we wanted f circles to be drawn per second, we would replace the independent variable ϕ_X with t:

$$A \cos 2\pi f t \tag{2.3}$$

where f is frequency in units of cycles per second (or *hertz*, Hz). For conciseness, we can replace $2\pi f$ with ω, the *natural frequency*, which is in units of radians per second:

$$A \cos \omega t \tag{2.4}$$

If we wanted to shift the drawing of our circle in time, we could add or subtract a constant from t. However, the mathematically equivalent convention is to add or subtract a constant θ, called the *phase*, from ωt. Now we're drawing our circle with a head start of θ, from θ to $\theta+2\pi$ radians, as in Figure 2.1(d). This corresponds to a phase-shifted cosine wave:

$$A \cos(\omega t + \theta) \tag{2.5}$$

What if we wanted to add cosine waves? If the cosine wave represents AC voltage or current, being able to do so is important for AC circuit analysis. We can prove that the sum of two phase-shifted cosine waves is another phase-shifted cosine wave, and apply trigonometric identities to calculate it. However, the geometric approach shown in Figure 2.2 is more intuitive. Consider that point X of the first wave's triangle is at the origin of its two-dimensional space, and that the second cosine wave is similarly formed at any moment in time by a second triangle $X'Y'Z'$. If the two waves were added, then the second

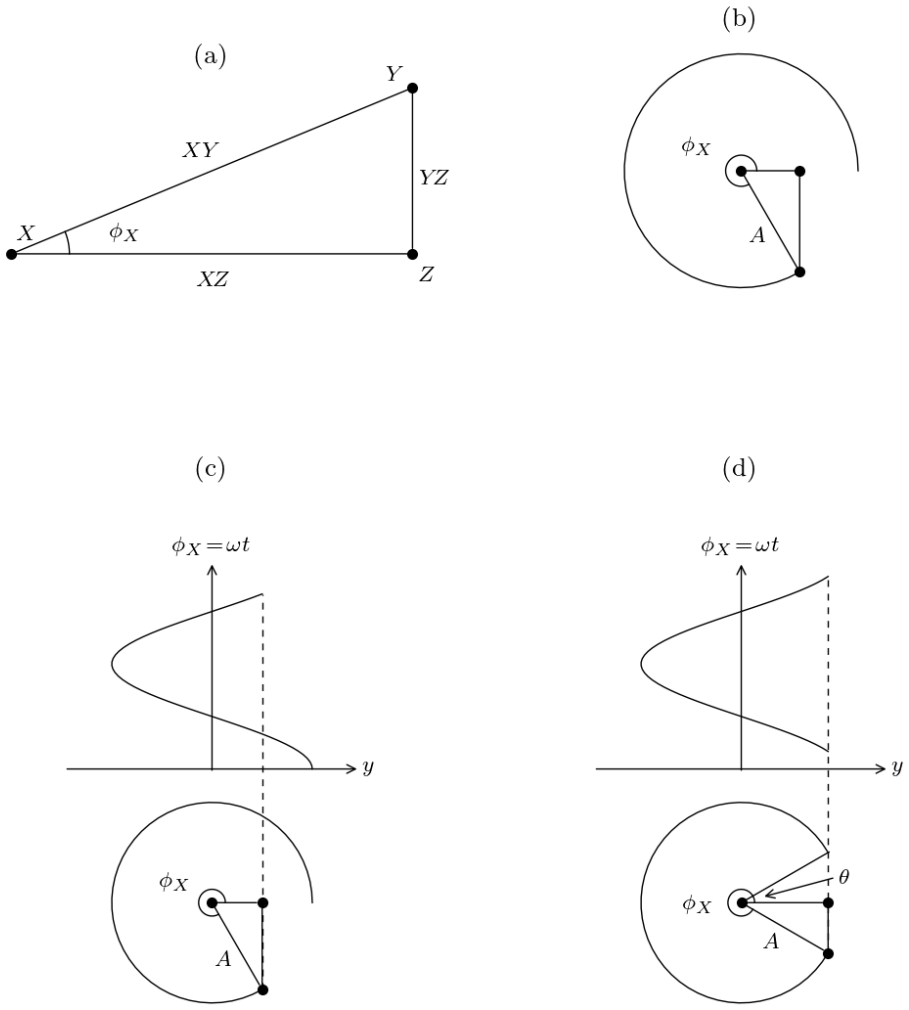


Figure 2.1: When the angle ϕ_X of a triangle (a) is swept from 0 to 2π radians, it draws a circle (b), the horizontal displacement of whose point Y draws out a cosine wave (c) as a function of ϕ_X. Shifting ϕ_X by θ results in a phase shift (d).

triangle would be translated such that its point X' would be situated at the first triangle's point Y. Thus, the second triangle's point Y' would have the coordinates $(XZ + X'Z', YZ + Y'Z')$, representing the sum of the two waves. Similarly, if we were instead subtracting the second wave from the first, the point Y' would have the coordinates $(XZ - X'Z', YZ - Y'Z')$. Clearly, the addition and subtraction of the triangles by which cosine waves are formed satisfies the properties of vector addition and subtraction.

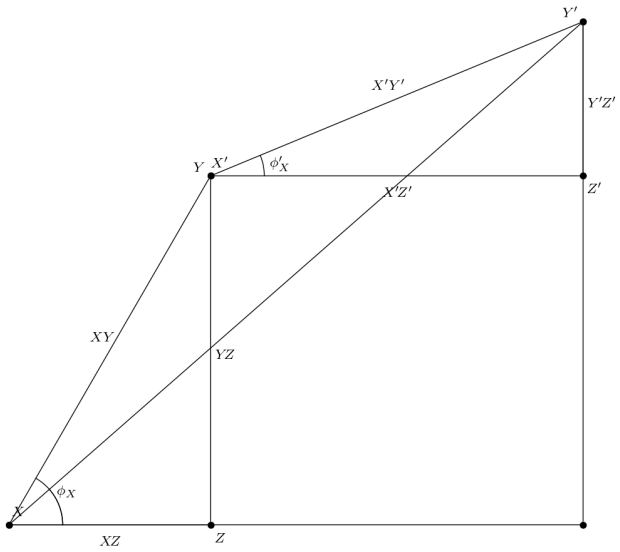


Figure 2.2: Phasors satisfy the properties of vector addition and subtraction.

Phasors satisfy properties of vectors—addition, subtraction, multiplication by a scalar, and so on. However, phasors are not vectors. Vectors exist in n-dimensional space. As the geometric analogy suggested, phasors exist exclusively in two-dimensional space, and as we will learn in the next section, phasors must satisfy additional properties that vectors do not have.

Deriving the Concept of Phasors

Let's reverse-engineer the concept of phasors. Consider the circuit of Figure 2.3.

The voltage source generates a cosine wave $v_S(t) = |V_S|\cos(\omega t + \theta_S)$. If using the geometric analogy to represent the voltage source's cosine wave, it would have coordinates $(V_{Sx}, V_{Sy}) = (|V_S|\cos\theta_S, |V_S|\sin\theta_S)$. Say that we want to solve for the steady-state current $i(t)$ through the circuit. We would start by writing the integro-differential equation:

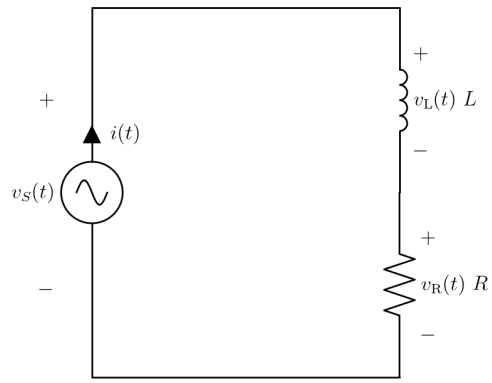


Figure 2.3: A series RL circuit consisting of a voltage source S, a resistor R, and an inductor L.

$$\begin{aligned} i(t) &= \frac{1}{L}\int_0^t v_L(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{L}\int_0^t (v_S(\tau) - v_R(\tau))\,\mathrm{d}\tau \\ &= \frac{1}{L}\int_0^t (v_S(\tau) - i(\tau)R)\,\mathrm{d}\tau \end{aligned} \tag{2.6}$$

The steady-state solution to this integro-differential equation is as follows. Note that setting up and solving AC circuits' integro-differential equations will be discussed further in Chapter 3.

$$i(t) = \frac{|V_S|}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta_S - \arctan\frac{\omega L}{R}\right) \tag{2.7}$$

Using the geometric analogy, we can represent the sinusoid of $i(t)$ using coordinates (I_x, I_y). We can write I_x as:

$$\begin{aligned} I_x &= \frac{|V_S|}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\theta_S - \arctan\frac{\omega L}{R}\right) \\ &= \frac{|V_S|}{\sqrt{R^2 + (\omega L)^2}} \left(\cos\theta_S \cdot \cos\left(\arctan\frac{\omega L}{R}\right) + \sin\theta_S \cdot \sin\left(\arctan\frac{\omega L}{R}\right)\right) \\ &= \frac{|V_S|}{\sqrt{R^2 + (\omega L)^2}} \left(\cos\theta_S \cdot \frac{R}{\sqrt{R^2 + (\omega L)^2}} + \sin\theta_S \cdot \frac{\omega L}{\sqrt{R^2 + (\omega L)^2}}\right) \\ &= \frac{|V_S|\cos\theta_S \cdot R + |V_S|\sin\theta_S \cdot \omega L}{R^2 + (\omega L)^2} \end{aligned} \tag{2.8}$$

Similarly, I_y can be written as follows.

$$\begin{aligned} I_y &= \frac{|V_S|}{\sqrt{R^2 + (\omega L)^2}} \sin\left(\theta_S - \arctan\frac{\omega L}{R}\right) \\ &= \frac{|V_S| \sin\theta_S \cdot R - |V_S| \cos\theta_S \cdot \omega L}{R^2 + (\omega L)^2} \end{aligned} \tag{2.9}$$

Let's substitute V_{Sx} for $|V_S| \cos\theta_S$ and V_{Sy} for $|V_S| \sin\theta_S$. Let's also replace R and ωL with new quantities Z_x and Z_y, whose significance will become apparent shortly.

$$(I_x, I_y) = \left(\frac{V_{Sx}Z_x + V_{Sy}Z_y}{Z_x^2 + Z_y^2}, \frac{V_{Sy}Z_x - V_{Sx}Z_y}{Z_x^2 + Z_y^2}\right) \tag{2.10}$$

Making use of the properties of complex numbers (in the box below), we discover what is important about being able to write (I_x, I_y) in this way. The complex number $I_x + j\,I_y$ (where j is the imaginary unit) is the result of the following complex number division:

$$I_x + j\,I_y = \frac{V_{Sx} + j\,V_{Sy}}{Z_x + j\,Z_y} \tag{2.11}$$

Properties of complex numbers
Addition and subtraction:

$$(a + j\,b) \pm (c + j\,d) = (a \pm c) + j\,(b \pm d) \tag{2.12}$$

Multiplication:

$$\begin{aligned} (a + j\,b)(c + j\,d) &= ac + j\,da + j\,bc + (j\,b)(j\,d) \\ &= (ac - bd) + j\,(bc + ad) \end{aligned} \tag{2.13}$$

Division (by multiplying both numerator and denominator by denominator's conjugate):

$$\begin{aligned} \frac{a + j\,b}{c + j\,d} &= \frac{a + j\,b}{c + j\,d}\frac{c - j\,d}{c - j\,d} \\ &= \frac{ac - j\,da + j\,bc - (j\,b)(j\,d)}{c^2 - j\,dc + j\,dc - (j\,d)(j\,d)} \\ &= \frac{(ac + bd) + j\,(bc - ad)}{c^2 + d^2} \end{aligned} \tag{2.14}$$

Knowing that $Z_x = R$, Equation (2.11) feels like Ohm's law. If we were to remove the inductor L from the circuit and thus set $Z_y = \omega L$ to zero, the equation indeed becomes Ohm's law for a purely resistive circuit:

$$\begin{aligned} I_x + j\,I_y &= \frac{V_{Sx} + j\,V_{Sy}}{R} = \frac{|V_S|\cos\theta_S}{R} + j\,\frac{|V_S|\sin\theta_S}{R} \\ \implies (I_x, I_y) &= \left(\frac{|V_S|\cos\theta_S}{R}, \frac{|V_S|\sin\theta_S}{R}\right) \\ \implies i(t) &= \frac{v_S(t)}{R} \end{aligned} \tag{2.15}$$

In fact, Equation (2.11) is the generalization of Ohm's law for an AC circuit with inductive and capacitive components, and $Z_x + j\,Z_y$, called *impedance*, is the generalization of electrical resistance R for such a circuit. Just as we could write $i(t) = v_S(t)/R$ for a purely resistive AC circuit, we can write $I_x + j\,I_y = (V_{Sx} + j\,V_{Sy})/(Z_x + j\,Z_y)$ for an AC circuit with inductive and capacitive components.

This indicates to us that the coordinates we've been working with in our geometric analogy are actually complex numbers. They satisfy all the properties of complex numbers, including addition, subtraction, multiplication by a scalar, multiplication, and division (as we've just seen). A complex number, as used to represent a sinusoidal voltage or current, is what we call a phasor. Furthermore, a complex number, as used to represent the ratio between a voltage phasor and a current phasor, is what we call impedance. Impedance, however, is not itself a phasor because it does not represent a sine wave.

The most important feature of phasors is that they allow us to do calculations using complex numbers rather than solving differential equations. For example, rather than solving differential equation (2.6), we can quickly convert $v_S(t)$ to a phasor, solve a regular equation in the phasor domain, and convert the solution to $i(t)$ back in the time domain.

You may be thinking that complex numbers are an unintuitive or downright far-fetched way to represent solving differential equations. In answer to this, consider that the behavior of multiplying a complex number of unit magnitude by j is identical to the behavior of differentiating a sinusoid of unit amplitude with respect to time, as shown in the following diagram. Similarly, by reversing the diagram, the behavior of dividing a complex number of unit magnitude by j is identical to the behavior of integrating a sinusoid of unit amplitude with respect to time. So, instead of writing derivatives and integrals to model capacitors and inductors, we can simply multiply and divide by j.

$$\begin{array}{lcll} & 1 & \cos t & \\ \text{Multiply by } j & \downarrow & \downarrow & \text{Differentiate w.r.t. } t \\ & j & -\sin t & \\ \text{Multiply by } j & \downarrow & \downarrow & \text{Differentiate w.r.t. } t \\ & -1 & -\cos t & \\ \text{Multiply by } j & \downarrow & \downarrow & \text{Differentiate w.r.t. } t \\ & -j & \sin t & \\ \text{Multiply by } j & \downarrow & \downarrow & \text{Differentiate w.r.t. } t \\ & 1 & \cos t & \end{array} \tag{2.16}$$

Notice too how there is a $+\pi/2$ (or $+90^\circ$) phase shift from $\cos t$ to $-\sin t$, and from $-\sin t$ to $-\cos t$, and so on. Thus, multiplying or dividing by j in the phasor domain is also equivalent to shifting the corresponding sinusoid in the time domain by a quarter cycle. This is illustrated in Figure 2.4.

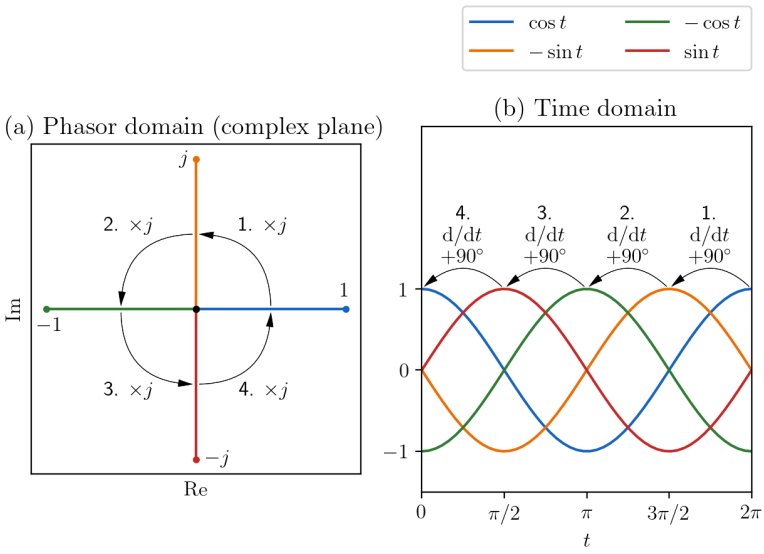


Figure 2.4: Each multiplication by the imaginary unit j in the phasor domain corresponds to a $+90^\circ$ phase shift in the time domain.

Notation for Phasors

Previously, we treated phasors as coordinates in two-dimensional space, with x and y components. For example, we had:

$$(V_{Sx}, V_{Sy}) \qquad (I_x, I_y) \tag{2.17}$$

However, now that we know that phasors (and impedances) are just complex numbers, we can replace this notation with complex number notation, with a real and imaginary component. For example:

$$V_S = \mathrm{Re}(V_S) + j\,\mathrm{Im}(V_S) \qquad I = \mathrm{Re}(I) + j\,\mathrm{Im}(I) \qquad Z = \mathrm{Re}(Z) + j\,\mathrm{Im}(Z) \tag{2.18}$$

Polar Form

Let's introduce a polar form of complex number notation, which is used to write a phasor (or impedance) in terms of amplitude and phase, rather than in terms of its real and imaginary components. For example, using Euler's formula $e^{jx} = \cos x + j \sin x$, we can write $V_S = |V_S| \cos\theta_S + j\,|V_S| \sin\theta_S$ as:

$$|V_S|\, e^{j\theta_S} \tag{2.19}$$

Or using *Steinmetz notation*:

$$|V_S| \angle \theta_S \tag{2.20}$$

We can just as well apply this polar form to equations (2.8) and (2.9):

$$\begin{aligned} I &= \frac{|V_S|}{\sqrt{R^2+(\omega L)^2}} \cos\left(\theta_S - \arctan\frac{\omega L}{R}\right) \\ &+ j\,\frac{|V_S|}{\sqrt{R^2+(\omega L)^2}} \sin\left(\theta_S - \arctan\frac{\omega L}{R}\right) \\ &= \frac{|V_S|}{\sqrt{R^2+(\omega L)^2}} \angle\left(\theta_S - \arctan\frac{\omega L}{R}\right) \end{aligned} \tag{2.21}$$

Previously, equations (2.8) and (2.9) led us to see a way that I can be calculated from V_S by dividing by impedance Z using division of complex numbers. If we take a close look at Equation (2.21), we can find an additional, equivalent way in which I can be calculated from V_S. Notice how V_S can be converted to I by dividing its amplitude $|V_S|$ by $\sqrt{R^2+(\omega L)^2}$ and by subtracting $\arctan(\omega L/R)$ from θ_S. This demonstrates a useful property of phasors and complex numbers in general in polar form—two complex numbers can easily be divided by dividing their amplitudes and subtracting their phases:

$$\frac{A_1\angle\theta_1}{A_2\angle\theta_2} = \frac{A_1}{A_2}\angle(\theta_1-\theta_2) \tag{2.22}$$

Of course, all properties of complex numbers can be written in polar form, but multiplying and dividing complex numbers is easiest when they are in polar form. On the other hand, adding and subtracting complex numbers is easiest when they are in the non-polar form.

Properties of complex numbers (polar form)
Addition and subtraction:

$$\begin{aligned} &A_1\angle\theta_1 \pm A_2\angle\theta_2 = A\angle\theta \\ &\text{where:} \\ &A = \sqrt{(A_1\cos\theta_1 \pm A_2\cos\theta_2)^2 + (A_1\sin\theta_1 \pm A_2\sin\theta_2)^2} \\ &\theta = \arctan\frac{A_1\sin\theta_1 \pm A_2\sin\theta_2}{A_1\cos\theta_1 \pm A_2\cos\theta_2} \end{aligned} \tag{2.23}$$

Multiplication:

$$A_1\angle\theta_1 \cdot A_2\angle\theta_2 = A_1A_2\angle(\theta_1+\theta_2) \tag{2.24}$$

Division:

$$\frac{A_1\angle\theta_1}{A_2\angle\theta_2} = \frac{A_1}{A_2}\angle(\theta_1-\theta_2) \tag{2.25}$$

Complex numbers can be converted to and from polar form using the following formulae:

$$\begin{aligned} a + j\,b &= \sqrt{a^2 + b^2}\,\angle \arctan\frac{b}{a} \\ A\angle\theta &= A\cos\theta + j\,A\sin\theta \end{aligned} \tag{2.26}$$

The Phasor Transformation

Thus far, we've looked at how phasors fit into AC circuit mathematics to see where they come from. At this point we can provide a formal definition for a phasor and how it is obtained from a sinusoid in the time domain. Consider the sinusoid $A\cos(\omega t + \theta)$. Adding an imaginary component $j\,A\sin(\omega t + \theta)$ gives us:

$$A(\cos(\omega t + \theta) + j\ \sin(\omega t + \theta)) \tag{2.27}$$

By setting $t = 0$ and thus removing the time-varying part ωt, we can convert this to a phasor as follows. Setting t to a specific value allows us to produce a "snapshot" of the sinusoid. Any specific value for t will do, however $t = 0$ is most convenient.

$$A(\cos\theta + j\ \sin\theta) \tag{2.28}$$

If we convert to polar form,

$$A\mathrm{e}^{j(\omega t+\theta)} = A\mathrm{e}^{j\omega t}\mathrm{e}^{j\theta}, \tag{2.29}$$

we can accomplish the same thing by dividing by the time-varying factor $\mathrm{e}^{j\omega t}$ to obtain:

$$A\mathrm{e}^{j\theta} \tag{2.30}$$

Chapter 3

Life With and Without Phasors: Example Circuit Analysis

One of the main reasons for many people being confused about phasors is that phasors are abstract, but they don't understand why the abstraction exists. In other words, they may know what phasors *represent*, but they don't know what they *replace*—they don't know of any alternative approach for analyzing AC circuits that can be used to put phasors in perspective. This is understandable as many electrical engineering textbooks poorly explain why we use phasors.

With the aid of an example AC circuit (Figure 3.1), this chapter goes over what's hidden behind the phasor abstraction. By first trying to solve the circuit without phasors, we will better understand how AC circuit analysis works at its core. And by then solving the circuit with phasors, we will see that it is not only a more concise notation, but also a simpler mathematical representation and a faster method for analyzing AC circuits.

Say that we want to solve for voltages $v_a(t)$ and $v_b(t)$. Doing so would allow us to solve for the voltage across—and therefore the current through—any one component. Just as for a purely resistive DC network, Kirchhoff's Current Law (KCL) applies and we can do nodal analysis to determine the voltages at nodes a and b. In particular we must use the supernode technique with a single node ab because the current through voltage source v_{s2} cannot be put in terms of $v_a(t)$ or $v_b(t)$. This loses us an equation, but we gain one in its place with the following relationship:

$$v_{s2}(t) = v_a(t) - v_b(t) \tag{3.1}$$

Let's try solving the circuit using node voltage analysis, with and without phasors.

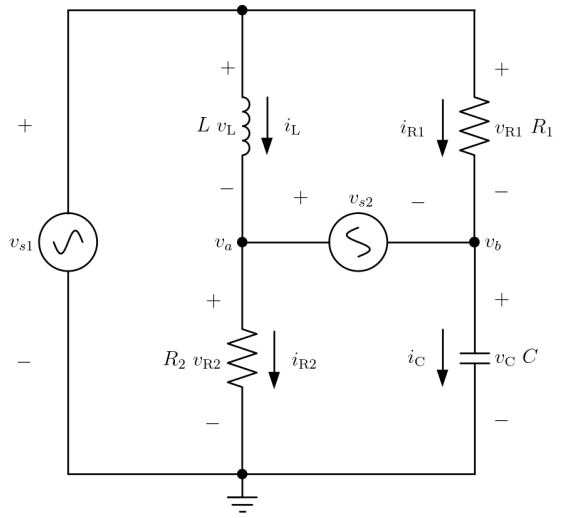


Figure 3.1: Example circuit with multiple AC sources and a bridge-type topology that does not allow us to consolidate impedances using formulae for impedances in series or in parallel.

Note
While the parameters $|V_{s1}|$, ω, R_1, etc. can take on any numerical values, at certain points in the chapter we will substitute the following very arbitrary values to simplify or be able to plot our equations.

$$\begin{gathered} |V_{s1}| = 1\ \text{V} \qquad |V_{s2}| = 2\ \text{V} \qquad \omega = 3\ \text{rad/s} \\ R_1 = 4\ \Omega \qquad R_2 = 5\ \Omega \qquad L = 6\ \text{H} \qquad C = 7\ \text{F} \end{gathered} \tag{3.2}$$

Solving the Circuit Without Phasors

Using KCL, the sum of currents flowing out of supernode ab must sum to zero. We can therefore write:

$$-i_L(t) + i_{R2}(t) - i_{R1}(t) + i_C(t) = 0 \tag{3.3}$$

Substituting in the components' characteristic equations gives us:

$$-\frac{1}{L}\int_0^t v_L(\tau)\,\mathrm{d}\tau + \frac{v_{R2}(t)}{R_2} - \frac{v_{R1}(t)}{R_1} + C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} = 0 \tag{3.4}$$

Putting everything in terms of the node voltages yields:

$$-\frac{1}{L}\int_0^t (v_{s1}(\tau) - v_a(\tau))\,\mathrm{d}\tau + \frac{v_a(t)}{R_2} - \frac{v_{s1}(t) - v_b(t)}{R_1} + C\frac{\mathrm{d}v_b(t)}{\mathrm{d}t} = 0 \tag{3.5}$$

Paired with Equation (3.1), this represents a system of two integro-differential equations. By differentiating, we put this in the more fa-

miliar form of a system of differential equations (in which the highest-order derivative is two):

$$\begin{aligned} &-\frac{1}{L}(v_{s1}(t)-v_a(t))+\frac{1}{R_2}\frac{\mathrm{d}v_a(t)}{\mathrm{d}t}-\frac{1}{R_1}\frac{\mathrm{d}}{\mathrm{d}t}(v_{s1}(t)-v_b(t))+C\frac{\mathrm{d}^2v_b(t)}{\mathrm{d}t^2}=0\\ &v_{s2}(t)=v_a(t)-v_b(t)\end{aligned} \tag{3.6}$$

We put this into a canonical form of a system of first-order differential equations by introducing an auxiliary variable $v_b'(t)$ and by using Equation (3.1) to replace the need for variable $v_a(t)$ with $v_b'(t)$:

$$\begin{aligned} \frac{\mathrm{d}v_b(t)}{\mathrm{d}t}&=v_b'(t)\\ \frac{\mathrm{d}v_b'(t)}{\mathrm{d}t}&=\frac{1}{C}\left[\frac{v_{s1}(t)-(v_{s2}(t)+v_b(t))}{L}-\frac{v_{s2}'(t)+v_b'(t)}{R_2}+\frac{v_{s1}'(t)-v_b'(t)}{R_1}\right]\end{aligned} \tag{3.7}$$

We specify that $v_{si}(t)=V_{si}\sin\omega t$ (and $v_{si}'(t)=\omega V_{si}\cos\omega t$):

$$\begin{aligned} \frac{\mathrm{d}v_b(t)}{\mathrm{d}t}&=v_b'(t)\\ \frac{\mathrm{d}v_b'(t)}{\mathrm{d}t}&=\frac{1}{C}\left[\begin{array}{l}\dfrac{(|V_{s1}|-|V_{s2}|)\sin\omega t-v_b(t)}{L}\\ -\dfrac{\omega|V_{s2}|\cos\omega t+v_b'(t)}{R_2}\\ +\dfrac{\omega|V_{s1}|\cos\omega t-v_b'(t)}{R_1}\end{array}\right]\end{aligned} \tag{3.8}$$

Our system is now in terms of two unknowns—$v_b(t)$ and $v_b'(t)$—two functions to solve for. We start to solve our system of differential equations by assuming a solution of the form:

$$\begin{aligned} v_b(t)&=V_b\cos(\omega t+\theta_b)\\ v_b'(t)&=-\omega V_b\sin(\omega t+\theta_b)\end{aligned} \tag{3.9}$$

By substituting into our system of differential equations, we convert it to a regular system of equations—or, in this case, just one equation:

$$\begin{aligned} &-\omega^2V_b\cos(\omega t+\theta_b)\\ &=\frac{1}{C}\left[\begin{array}{l}\dfrac{(|V_{s1}|-|V_{s2}|)\sin\omega t-V_b\cos(\omega t+\theta_b)}{L}\\ +\omega\left[\left(\dfrac{|V_{s1}|}{R_1}-\dfrac{|V_{s2}|}{R_2}\right)\cos\omega t+\left(\dfrac{1}{R_1}+\dfrac{1}{R_2}\right)V_b\sin(\omega t+\theta_b)\right]\end{array}\right]\end{aligned} \tag{3.10}$$

This equation is in terms of two unknowns—V_b and θ_b. While having only one equation may appear to be insufficient to solve for two unknowns, the equation is parameterized by t and must be satisfied for all values of t. So, in this sense, we actually have an infinite number

of equations at our disposal. We can pick two 'easy' values of t that result in two relatively simple equations:

$t = 0$:

$$-\omega^2 C V_b \cos\theta_b = \frac{-V_b \cos\theta_b}{L} + \omega\left[\left(\frac{|V_{s1}|}{R_1} - \frac{|V_{s2}|}{R_2}\right) + \left(\frac{1}{R_1} + \frac{1}{R_2}\right) V_b \sin\theta_b\right] \tag{3.11}$$

$t = -\theta_b/\omega$:

$$-\omega^2 C V_b = \frac{-(|V_{s1}| - |V_{s2}|)\sin\theta_b - V_b}{L} + \omega\left(\frac{|V_{s1}|}{R_1} - \frac{|V_{s2}|}{R_2}\right)\cos\theta_b \tag{3.12}$$

This nonlinear system is inconvenient to solve. We can plot these two equations as shown in Figure 3.2 and solve the system numerically. However, numerical methods are less efficient than being able to directly calculate the solution. We should be able to solve AC circuits very quickly.

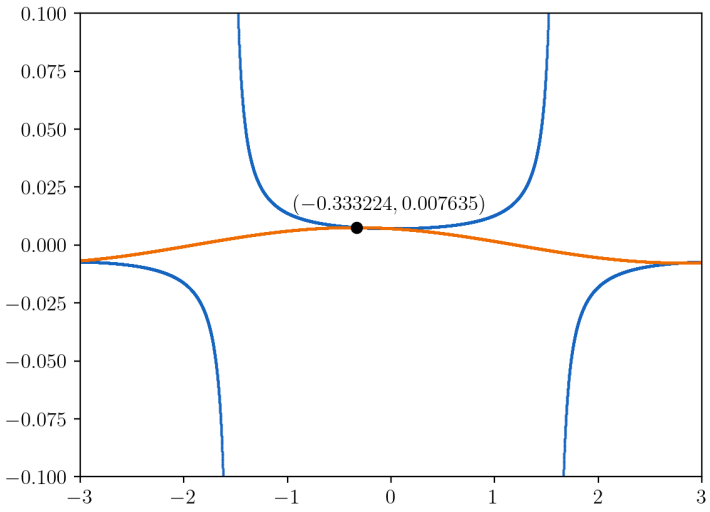


Figure 3.2: Solving the system of equations (3.11) and (3.12) graphically.

Note
There are technically an infinite number of solutions for θ_b, because any multiple of 2π can be added to (or subtracted from) θ_b, resulting in an identical sinusoid with a different equation.

Furthermore, as we have seen, the process to derive the system of equations is long and notation-heavy despite being consistent between different AC circuits.

Function	Laplace transform
Sine wave	$\mathcal{L}\{\sin(\omega t)\,u(t)\} = \frac{\omega}{s^2+\omega^2}$
Phase-shifted cosine wave	$\mathcal{L}\{\cos(\omega t+\theta)\,u(t)\} = \frac{s\cos\theta-\omega\sin\theta}{s^2+\omega^2}$
Exponential decay	$\mathcal{L}\{e^{-\alpha t}\,u(t)\} = \frac{1}{s+\alpha}$
Exponentially decaying cosine wave	$\mathcal{L}\{e^{-\alpha t}\cos(\omega t)\,u(t)\} = \frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Table 3.1: Selected Laplace transforms.

Before we see how the circuit can be solved with phasors, let's see how we can solve its system of integro-differential equations with the aid of the Laplace transform.

Solving the Circuit with the Laplace Transform

The Laplace transform $\mathcal{L}$ converts a function $f(t)$ (in the *time domain*) into a function $F(s)$ (in the *complex frequency domain*) through the following integration:

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty f(t)\,e^{-st}\,dt \tag{3.13}$$

The Laplace transform is closely related to the Fourier transform. Both transforms convert to the frequency domain. However, the Laplace transform converts to a frequency domain whose variable s is a *complex number*, which can simplify calculations. This domain is known as the *s-domain* or *Laplace domain*.

The Laplace transform has a number of favorable properties. One such property is that differentiating with respect to t in the time domain becomes multiplication by s in the Laplace domain. Similarly, integrating with respect to t in the time domain becomes division by s in the Laplace domain. Because of this, the Laplace transform is often used to convert a system of differential equations into a regular system of equations which can be solved more easily.

Furthermore, many useful functions in the time domain become simple, rational functions in the Laplace domain, for example those in Table 3.1.

Listings of the Laplace transforms of common functions, such as those above, are called Laplace tables.

$u(t)$ is the Heaviside step function, which is 1 for $t \geq 0$ and 0 otherwise. The time-domain functions are multiplied by $u(t)$—that is, only nonzero for $t \geq 0$, for the same reason that the lower bound of integration in Equation (3.13) is 0. The Laplace transform only looks at $t \geq 0$, and is thus useful for solving differential equations involving functions that are only nonzero for $t \geq 0$. This aspect allows us to answer questions about a system—particularly an electric circuit—that is modeled by differential equations, such as: "what happens when the system is turned on?", "what happens when the value of this component suddenly changes?", and "what happens when the circuit's topology changes in this way?" To consider functions that are nonzero for $t < 0$, and thus model no sudden change at $t = 0$, the *two-sided* or *bilateral* Laplace transform can be used.

We want to solve the system of two integro-differential equations (3.1) and (3.5) that we derived in the previous section. That system was in

terms of $v_a(t)$ and $v_b(t)$, in the time domain. First, we will apply the Laplace transform to convert it to a regular system of equations in terms of $V_a(s)$ and $V_b(s)$, in the Laplace domain. Then, we will be able to solve for $V_a(s)$ and $V_b(s)$ very easily. Finally, we will convert our solution back to the time domain using the inverse Laplace transform.

Applying the Laplace transform to equations (3.1) and (3.5) gives us:

$$\begin{aligned} &-\frac{|V_{s1}|(s) - V_a(s)}{Ls} + \frac{V_a(s)}{R_2} - \frac{|V_{s1}|(s) - V_b(s)}{R_1} + CsV_b(s) = 0 \\ &|V_{s2}|(s) = V_a(s) - V_b(s) \end{aligned} \tag{3.14}$$

Here, $|V_{s1}|(s)$ and $|V_{s2}|(s)$ are the Laplace transforms of $v_{s1}(t)$ and $v_{s2}(t)$—not to be confused with the amplitudes $|V_{s1}|$ and $|V_{s2}|$ of $v_{s1}(t)$ and $v_{s2}(t)$.

Solving for $V_a(s)$ in the second equation and substituting it into the first equation yields:

$$\begin{aligned} &-\frac{|V_{s1}|(s) - (|V_{s2}|(s) + V_b(s))}{Ls} + \frac{|V_{s2}|(s) + V_b(s)}{R_2} \\ &\quad - \frac{|V_{s1}|(s) - V_b(s)}{R_1} + CsV_b(s) = 0 \end{aligned} \tag{3.15}$$

Solving for $V_b(s)$ gives us:

$$V_b = \frac{R_2(R_1 + Ls)|V_{s1}|(s) - R_1(R_2 + Ls)|V_{s2}|(s)}{R_1R_2 + (R_1 + R_2)Ls + R_1R_2LCs^2} \tag{3.16}$$

Substituting in $V_{si}(s) = \mathcal{L}\{V_{si}\sin(\omega t)\,u(t)\} = V_{si}\,\omega/(s^2 + \omega^2)$ yields the following. Note that, because the voltage sources are multiplied by $u(t)$, we are essentially answering the question of how the circuit responds when turned on at $t = 0$.

$$V_b = \frac{R_2(R_1 + Ls)|V_{s1}|\omega - R_1(R_2 + Ls)|V_{s2}|\omega}{(s^2 + \omega^2)(R_1R_2 + (R_1 + R_2)Ls + R_1R_2LCs^2)} \tag{3.17}$$

At this point, the algebra will be significantly easier if we substitute in known values for our parameters. Let's introduce numerical values (3.2), after which we will omit units for brevity. However, $V_b(s)$ is still in units of volts and s is still, like ω, in units of radians per second.

$$\begin{aligned} &V_b(s) \\ &= \frac{(5\,\Omega)(4\,\Omega + (6\,\mathrm{H})s)(1\,\mathrm{V})(3\,\mathrm{rad/s}) - (4\,\Omega)(5\,\Omega - (6\,\mathrm{H})s)(2\,\mathrm{V})(3\,\mathrm{rad/s})}{(3\,\mathrm{rad/s})(-(4\,\Omega)(5\,\Omega) + (4\,\Omega - 5\,\Omega)(6\,\mathrm{H})s + (4\,\Omega)(5\,\Omega)(6\,\mathrm{H})(7\,\mathrm{F})s^2)} \\ &= \frac{-54s - 60}{(s^2 + 9)(840s^2 + 54s + 20)} \end{aligned} \tag{3.18}$$

We need to convert this solution for $V_b(s)$ back to the time domain. Converting a function from the Laplace domain to the time domain

can be done by looking it up in a Laplace table. However, $V_b(s)$ is too complicated in its current form to find a matching function in a Laplace table—in particular, because the order of the polynomial in the denominator is four. This is higher than the highest-order denominator of two that can be found in typical Laplace tables. We can handle this by rewriting our solution as a linear combination of simpler functions that we will be able to find in the Laplace table. To determine such simpler functions, we factor the denominator then apply *partial fraction decomposition*. The denominator needs to be split into factors whose order is at most two. This is already the case. Next, we assume the following modified format of our V_b solution and find the values A_1 through A_4 that make it true:

$$\begin{aligned}
V_b(s) &= \frac{A_1 s + A_2}{s^2 + 9} + \frac{A_3 s + A_4}{840s^2 + 54s + 20} \\
&\implies \frac{-54s - 60}{(s^2 + 9)(840s^2 + 54s + 20)} = \frac{A_1 s + A_2}{s^2 + 9} + \frac{A_3 s + A_4}{840s^2 + 54s + 20} \\
&\implies -54s - 60 = (A_1 s + A_2)(840s^2 + 54s + 20) + (A_3 s + A_4)(s^2 + 9) \\
&\implies (-840A_1 - A_3)s^3 \\
&\qquad + (-54A_1 - 840A_2 - A_4)s^2 \\
&\qquad + (-20A_1 - 54A_2 - 9A_3 - 54)s \\
&\qquad + (-20A_2 - 9A_4 - 60) \\
&\qquad = 0 \\
&\implies \begin{cases} -840A_1 - A_3 = 0 \\ -54A_1 - 840A_2 - A_4 = 0 \\ -20A_1 - 54A_2 - 9A_3 - 54 = 0 \\ -20A_2 - 9A_4 - 60 = 0 \end{cases} \\
&\implies \begin{cases} A_1 = 102600/14219461 \\ A_2 = 106539/14219461 \\ A_3 = -86184000/14219461 \\ A_4 = -95033160/14219461 \end{cases}
\end{aligned} \tag{3.19}$$

Substituting A_1 through A_4 gives us:

$$V_b(s) = \underbrace{\frac{\frac{102600}{14219461}s + \frac{106539}{14219461}}{s^2 + 9}}_{\text{First term}} + \underbrace{\frac{-\frac{86184000}{14219461}s - \frac{95033160}{14219461}}{840s^2 + 54s + 20}}_{\text{Second term}} \tag{3.20}$$

Inverse Laplace transform of the first term. The first term matches the following Laplace table function:

$$\mathcal{L}\{\cos(\omega t + \theta)\, u(t)\} = \frac{s\cos\theta + \omega\sin\theta}{s + \omega^2} \tag{3.21}$$

Therefore:

$$\text{First term} = \frac{\frac{102600}{14219461}s + \frac{106539}{14219461}}{s^2 + 9} = B_1 \frac{s\cos\theta_1 - \omega_1\sin\theta_1}{s + \omega_1^2} \tag{3.22}$$

The ω_1 here is (not coincidentally) the same as our natural frequency $\omega = 3$ rad/s. Now, we need to solve for θ_1 and B_1, which we can do by setting up the following system of equations:

$$\begin{aligned} B_1 \cos\theta_1 &= 102600/14219461 \\ -3B_1 \sin\theta_1 &= 106539/14219461 \end{aligned} \tag{3.23}$$

We can solve for θ_1 by dividing the second equation by the first:

$$\begin{aligned} &\frac{106539/14219461}{102600/14219461} = -3\tan\theta_1 \\ &\implies \theta_1 = \arctan\left(-\frac{35513}{102600}\right) = -19.092^\circ \end{aligned} \tag{3.24}$$

And we can solve for B_1 by dividing the second equation by -3 and adding its square to the square of the first:

$$\begin{aligned} &\left(\frac{102600}{14219461}\right)^2 + \left(\frac{106539}{-3\cdot 14219461}\right)^2 = B_1^2\cos^2\theta_1 + B_1^2\sin^2\theta_1 = B_1^2 \\ &\implies B_1 = \sqrt{\left(\frac{102600}{14219461}\right)^2 + \left(\frac{35513}{14219461}\right)^2} = 0.007635 \end{aligned} \tag{3.25}$$

Therefore, the inverse Laplace transform of the first term is:

$$\mathcal{L}^{-1}\left\{\frac{\frac{102600}{14219461}s + \frac{106539}{14219461}}{s^2+9}\right\} = 0.007635\cos(3t - 19.092^\circ)\,u(t) \tag{3.26}$$

Derivation: Laplace transform of an exponentially decaying, phase-shifted cosine wave

$$\begin{aligned} &\mathcal{L}\{e^{-\alpha t}\cos(\omega t+\theta)\,u(t)\} \\ &\quad= \mathcal{L}\{e^{-\alpha t}\left[\cos\omega t\cos\theta - \sin\omega t\sin\theta\right]u(t)\} \\ &\quad= \cos\theta\,\mathcal{L}\{e^{-\alpha t}\cos\omega t\,u(t)\} - \sin\theta\,\mathcal{L}\{e^{-\alpha t}\sin\omega t\,u(t)\} \\ &\quad= \cos\theta\frac{s+\alpha}{(s+\alpha)^2+\omega^2} - \sin\theta\frac{\omega}{(s+\alpha)^2+\omega^2} \\ &\quad= \frac{(s+\alpha)\cos\theta - \omega\sin\theta}{(s+\alpha)^2+\omega^2} \end{aligned} \tag{3.27}$$

Inverse Laplace transform of the second term. The second term matches the following function from (3.27):

$$\mathcal{L}\{e^{-\alpha t}\cos(\omega t+\theta)\,u(t)\} = \frac{(s+\alpha)\cos\theta - \omega\sin\theta}{(s+\alpha)^2+\omega^2} \tag{3.28}$$

Therefore:

$$\begin{aligned}\text{Second term} &= \frac{-\frac{86184000}{14219461}s - B_2\frac{95033160}{14219461}}{840s^2+54s+20} = \frac{(s+\alpha_2)\cos\theta_2 - \omega_2\sin\theta_2}{(s+\alpha_2)^2+\omega_2^2} \\ \implies & \frac{-\frac{102600}{14219461}s - \frac{791943}{99536227}}{s^2+\frac{9}{140}s+\frac{1}{42}} = B_2\frac{s\cos\theta_2 + (\alpha_2\cos\theta_2 - \omega_2\sin\theta_2)}{s^2+2\alpha_2 s+(\alpha_2^2+\omega_2^2)}\end{aligned} \tag{3.29}$$

We need to solve for α_2, ω_2, θ_2, and B_2:

$$\begin{aligned}&9/140 = 2\alpha_2 \implies \alpha_2 = 9/280 = 0.03214 \\ &\alpha_2^2+\omega_2^2 = \frac{1}{42} \implies \omega_2 = \sqrt{\frac{1}{42}-\left(\frac{9}{280}\right)^2} = 0.15092 \text{ rad/s}\end{aligned} \tag{3.30}$$

We can solve for θ_2 and B_2 by setting up and solving the following system of equations:

$$\begin{aligned}&\left\{\begin{array}{l} -102600/14219461 = B_2\cos\theta_2 \\ -791943/99536227 = B_2\,(0.03214\cos\theta_2 - 0.15092\sin\theta_2)\end{array}\right\} \\ &\implies \left\{\begin{array}{l}\theta_2 = -81.9756^\circ \\ B_2 = -0.05169\end{array}\right\}\end{aligned} \tag{3.31}$$

$$B_2 = -\frac{102600}{14219461\cos\theta_2} \tag{3.32}$$

Therefore, the inverse Laplace transform of the second term is:

$$\begin{aligned}&\mathcal{L}^{-1}\left\{\frac{-\frac{86184000}{14219461}s - \frac{95033160}{14219461}}{840s^2+54s+20}\right\} \\ &\quad = -0.05169\,\mathrm{e}^{-0.03214t}\cos(0.15092\,t - 81.9756^\circ)\,u(t)\end{aligned} \tag{3.33}$$

And finally, substituting in the inverse Laplace transforms of the first and second terms yields the following full solution for $v_b(t)$ in the time domain:

$$\begin{aligned}v_b(t) = (&\underbrace{0.007635\cos(3t - 19.092^\circ)}_{\text{Steady-state response, } v_b^{SS}(t)} \\ &- \underbrace{0.05169\,\mathrm{e}^{-0.03214t}\cos(0.15092\,t - 81.9756^\circ)}_{\text{Transient response}})\,u(t)\end{aligned} \tag{3.34}$$

From this we can calculate $v_a(t)$ via trigonometric identities. Its transient response is the same as for $v_b(t)$, and its steady-state response is:

$$\begin{aligned}v_a^{SS}(t) &= v_{s2}(t) + v_b^{SS}(t) \\ &= 0.007635\cos(3t - 19.092) + 2\sin(3t) \\ &= 1.99752\cos(3t - 89.7930^\circ)\end{aligned} \tag{3.35}$$

The solution for $v_b(t)$ (and $v_a(t)$), plotted in Figure 3.3, has two distinct terms. It is a linear combination of cosine waves, each with their own natural frequency and phase, but one of the cosine waves is exponentially decaying. The exponentially decaying cosine models how the circuit responds when turned on at $t = 0$—its *transient response*. The non-decaying cosine is how the circuit responds in the longer term—its *steady-state response* as t tends to infinity (but practically much sooner as the transient quickly dies out).

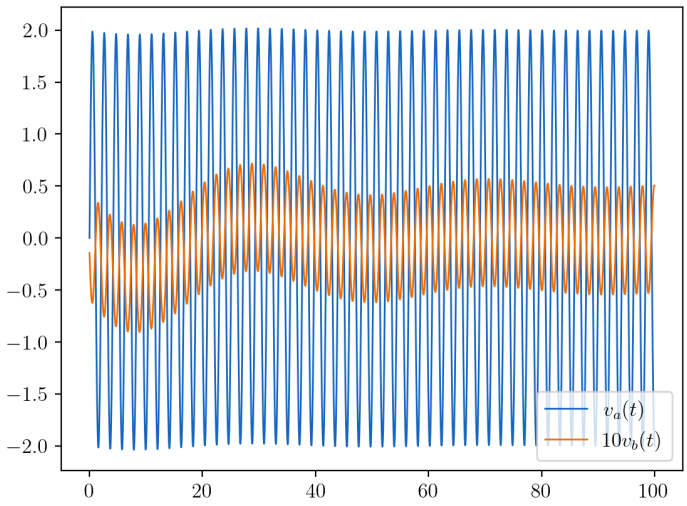


Figure 3.3: Transient and steady-state solution for $v_a(t)$ and $v_b(t)$.

Solving the Circuit with Phasors

Using KCL, the sum of currents flowing out of supernode ab must sum to zero. We can therefore write:

$$-I_{\mathrm{L}} + I_{\mathrm{R2}} - I_{\mathrm{R1}} + I_{\mathrm{C}} = 0 \tag{3.36}$$

$$-\frac{V_{\mathrm{L}}}{Z_{\mathrm{L}}} + \frac{V_{\mathrm{R2}}}{Z_{\mathrm{R2}}} - \frac{V_{\mathrm{R1}}}{Z_{\mathrm{R1}}} + \frac{V_{\mathrm{C}}}{Z_{\mathrm{C}}} = 0 \tag{3.37}$$

Substituting in the components' impedances and putting everything in terms of the node voltages gives us:

$$-\frac{V_{s1} - V_a}{j\,\omega L} + \frac{V_a}{R_2} - \frac{V_{s1} - V_b}{R_1} + \frac{V_b}{j\frac{-1}{\omega C}} = 0 \tag{3.38}$$

Paired with Equation (3.1), $V_a - V_b = V_{s2}$, this represents a system of two linear equations in two unknowns V_a and V_b. Substituting Equation (3.1) into the above yields:

$$j\frac{V_{s1} - (V_{s2} + V_b)}{\omega L} + \frac{V_{s2} + V_b}{R_2} - \frac{V_{s1} - V_b}{R_1} + j\,\omega C V_b = 0 \tag{3.39}$$

Solving for V_b yields the following, and we can solve for V_a from this using Equation (3.1).

$$V_b = \frac{\omega L(R_2 V_{s1} - R_1 V_{s2}) + j\, R_1 R_2 (V_{s2} - V_{s1})}{\omega L(R_1 + R_2) + j\,(\omega^2 CL - 1) R_1 R_2} \tag{3.40}$$

Getting here required some algebra involving complex numbers. However, when solving a real circuit whose parameters have known values, this process would have been aided significantly by being able to evaluate/simplify expressions at each step. Let's introduce numerical values (3.2) to see how V_b and V_a can be evaluated.

$$\begin{aligned}
&V_b \\
&= \frac{(3\text{rad/s})(6\text{H})((5\Omega)(j\,1\text{V}) - (4\Omega)(j\,2\text{V})) + j\,(4\Omega)(5\Omega)((j\,2\text{V}) - (j\,1\text{V}))}{(3\text{rad/s})(6\text{H})((4\Omega) + (5\Omega)) + j\,((3\text{rad/s})^2(7\text{F})(6\text{H}) - 1)(4\Omega)(5\Omega)} \\
&= \frac{-20 - j\,54}{162 + j\,7540}\ \text{V} \\
&= \frac{-20 - j\,54}{162 + j\,7540}\,\frac{162 - j\,7540}{162 - j\,7540}\ \text{V} \\
&= \frac{(-20 - j\,54)(162 - j\,7540)}{162^2 + 7540^2}\ \text{V} \\
&= \frac{-3240 + j\,150800 - j\,8748 - 407160}{26244 + 56851600}\ \text{V} \\
&= \frac{-410400 + j\,142052}{56877844}\ \text{V} \\
&= \boxed{(-0.007215 + j\,0.002497)\ \text{V}}
\end{aligned}$$

$$\begin{aligned}
V_a &= V_b + V_{s2} \\
&= (-0.007215 + j\,0.002497)\ \text{V} + j\,2\ \text{V} \\
&= \boxed{(-0.007215 + j\,2.002497)\ \text{V}}
\end{aligned} \tag{3.41}$$

By calculating the amplitude A and phase θ, we can convert from complex number phasor notation $a + j\,b$ to either polar phasor notation $A\angle\theta$ or back into the time domain as $A\cos(\omega t + \theta)$. The amplitude and phase are calculated using:

$$A = \sqrt{a^2 + b^2}, \qquad \theta = \arctan\frac{b}{a} \tag{3.42}$$

$$\begin{aligned}
V_b &= 0.007635\,\angle{-19.092^\circ} \\
V_a &= 2.002510\,\angle{-89.794^\circ}
\end{aligned} \tag{3.43}$$

With the amplitude and phase calculated, it becomes very easy to plot the solution for $v_a(t)$ and $v_b(t)$, as in Figure 3.4.

Checking the Phasor Solution

At this point, we have derived the equations representing the circuit with and without phasors, and we have solved the circuit with phasors. To check our work, we can plug the solution obtained using phasors into the differential equation (3.8) obtained without using phasors, and verify that the equation holds true (that the left-hand side equals the right-hand side). This can be done conveniently by

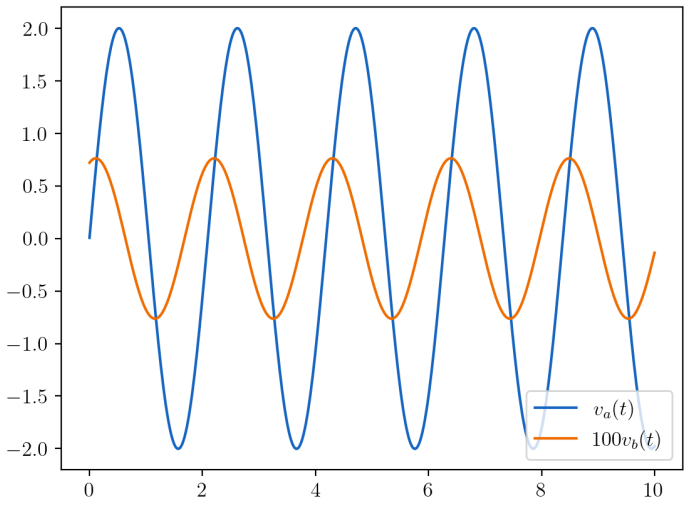


Figure 3.4: Steady-state solution for $v_a(t)$ and $v_b(t)$.

plotting both the left- and right-hand sides, as in Figure 3.5. As can be seen, the left- and right-hand sides are equal.

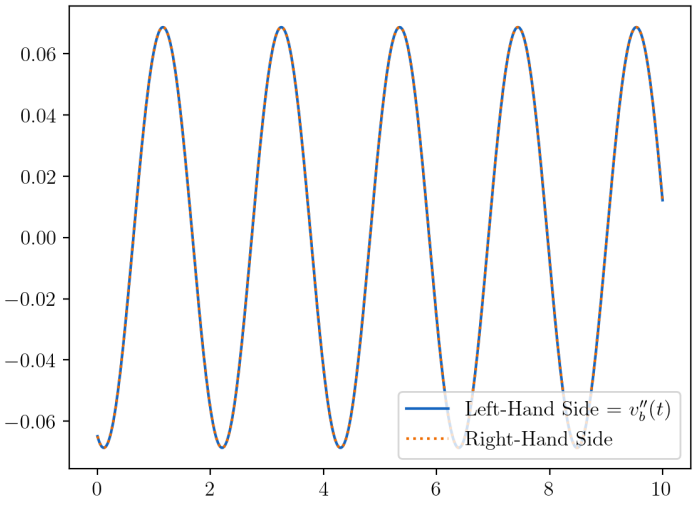


Figure 3.5: Checking the solution for $v_b(t)$ by plugging it into Equation (3.8) and plotting the left- and right-hand sides.

Chapter 4

Tutorial: Building an AC Circuit Analyzer

This chapter is a practical tutorial that will build your AC circuit analysis and programming skills. The goal is to develop a reusable software tool that can solve for all voltages and currents of any steady-state AC circuit consisting of passive components: sinusoidal voltage and current sources, resistors, inductors, and capacitors. To keep the program simple, there is no need to build a user interface. The input circuit to analyze can be built programmatically, by defining components and how they're connected.

Let's first discuss how we can model the problem and design our solution. Jumping right into a solution would not be as interesting, useful, or realistic; the solutions to real-world problems are not handed out and often require thinking through the problem from the beginning before writing any code.

Design Discussion

Let's start with the building blocks that will be used to construct and represent the circuit. It makes sense to use classes (or structs, in programming languages like Rust and Go) to represent our two-terminal electric components, but we also need to think about how to represent the connections between those components. Consider:

- Does the orientation with which components are connected (positive versus negative terminals) matter?

  Yes; it matters in order to establish a convention for voltage polarity and current direction, for the sake of "measuring" the solved voltage/current of a component, as well as for specifying the voltage/current of a sinusoidal voltage/current source.

- Is it best to model the components as being connected by wires, or by nodes?

  Wires are the more intuitive physical representation, but if wires were to be used, at least $n - 1$ of them would be needed to connect n component terminals to each other. The situation can be simplified by using a "wire" with n ends to connect to n terminals. This "n-ended wire" is simply a node. Using nodes is also the clear choice to be able to apply KCL easily.

For convenience when applying KCL, each node can list all components attached to it. However, that would not allow us to represent which of the component's terminals—positive or negative—are attached to a given node. To address this, each component can have attributes `positive` and `negative`, specifying which terminal each node is attached to.

We know that any AC circuit can be analyzed by applying KCL and solving a linear system of complex-valued phasor equations in terms of the node voltages. Formulating that system of equations is routine when analyzing circuits by hand, but formalizing that process to be done by a computer requires more thought. Consider:

- What does "summing the currents flowing out of each node" mean when developing a program? Does that mean finding a way to represent equations programmatically and then convert them to the standard format for linear systems, $A\mathbf{x} = \mathbf{b}$, with which solvers are compatible? Or does that mean populating the matrix A and vector $\mathbf{b}$ directly?

  For convenience, we can indeed use a symbolic math library (such as Python's `sympy` and its `nsolve` solver) to "write" and solve the equations, without needing to manually put them into the $A\mathbf{x} = \mathbf{b}$ format; `sympy` can do the heavy lifting.

- How does one determine when the supernode technique must be used, and apply it programmatically?

  The system of equations can be generalized—boiling down the circuit analysis problem to its simplest laws—such that the supernode technique need not be explicitly applied.

  In regular KCL, we sum the currents out of each node *in terms of the node voltages*. When this cannot be done, such as for nodes a and b connected by voltage source V_{s2} in Figure 3.1, we must combine them and sum the currents out of the resulting supernode ab. The voltage source's characteristic equation, $V_{s2} = V_a - V_b$, makes up for the missing KCL equation.

  Instead, let's sum the currents out of each node, and—rather than putting those currents in terms of node voltages—add the characteristic equation for each connected component to the system of equations. This eliminates the need for the supernode technique. This approach is atypical when solving equations by hand, as it requires writing out a larger initial system of equations, but this is a non-issue for a computer.

For every node and component, there will be one unknown (the node's voltage or the component's current) and one equation (KCL or the component's characteristic equation). However, there are two exceptions to this. Firstly, in an n-node circuit, only $n - 1$ KCL equations are required; the nth adds no new information to the system of equations. And secondly, because voltages are relative, a voltage reference is needed to prevent the system from having an infinite number of solutions. Both these points can be addressed by making the nth node a ground node, which contributes $V = 0$ to the system of equations rather than KCL.

Now that we have formalized the problem and considered the design options, we're ready to implement. Give it a shot, consulting the following sections for ideas if desired (they use Python, but feel free to use any programming language).

Data Model

To start developing our program, we need a way to represent voltage and current values—for the sake of representing solved values, as well as for specifying the voltage/current of a sinusoidal voltage/current source. Python has built-in support for complex numbers, with a dedicated `complex` type and `1j` literal. However, it is more common outside of numerical computation to work with phasors in polar form. As such, we create a `Phasor` class (polar form), with methods for converting to and from complex numbers:

```python
@dataclass
class Phasor:
    magnitude: float
    phase_deg: float = 0.0

    @property
    def phase_rad(self) -> float:
        return radians(self.phase_deg)

    @classmethod
    def from_complex(cls, value: complex) -> Self:
        return cls(
            magnitude=abs(value),
            phase_deg=degrees(atan2(value.real, value.imag)),
        )

    def to_complex(self) -> complex:
        return self.magnitude * (
            cos(self.phase_deg) + 1j * sin(self.phase_deg)
        )

    def __repr__(self) -> str:
        return f"{self.magnitude:.6f} @ {self.phase_deg:.3f}°"
```

We also need a way to represent the duality of voltages and currents. They are either unknown variables (`Symbol`s) in our `sympy` equations to solve for, or solved output values. To represent this, we create `Voltage` and `Current` classes, each with both a `variable` (of type `Symbol`) and a `value` (of type `Phasor`). The `Symbol` is automatically named depending on whether it represents a node's voltage or a component's current, allowing both us and `sympy` to meaningfully distinguish them. These traits are common to both the `Voltage` and `Current` classes, so we first create a shared base class for them called `_Symbolic`.

Note that a positive value of current indicates flow from the positive terminal to the negative terminal, by convention.

```python
@dataclass(kw_only=True)
class _Symbolic:
    value: Phasor | None = None

    @property
    def variable(self) -> Symbol:
        return Symbol(str(self))
```

```python
@dataclass
class Voltage(_Symbolic):
    node_name: str

    def __str__(self) -> str:
        return f"(node {self.node_name} voltage)"

@dataclass
class Current(_Symbolic):
    component_name: str

    def __str__(self) -> str:
        return f"(component {self.component_name} current)"
```

Building Blocks

Now we create the building blocks that we'll use to represent the circuit: nodes and components. First we create the `Node` class. Every node contributes a KCL equation and has a `Voltage` that is free to vary, except the ground node (`is_ground=True`), whose voltage is fixed at zero.

```python
@dataclass(repr=False)
class Node:
    name: str
    is_ground: bool = False
    voltage: Voltage = field(init=False)
    connected_components: list[_BaseComponent] = field(
        init=False
    )

    def __post_init__(self):
        self.voltage = Voltage(node_name=self.name)
        # Appended to by `BaseComponent.__post_init__`s:
        self.connected_components = []

    def __hash__(self) -> int:
        return hash(self.name)

    @property
    def equation(self):
        if self.is_ground:
            return self.voltage.variable
        else:
            return sum(
                c.current.variable
                * (1 if c.positive is self else -1)
                for c in self.connected_components
            )
```

Unlike nodes, which are universal, we have multiple types of component. However, they have much in common: they all have current flowing from a positive terminal to a negative terminal, and they generally have a characteristic equation describing the relationship between that electric current and the voltage difference across the component (that is, between the two connected nodes). The only exception to this will be the voltage and current sources, whose voltage and current variables are obviously fixed. To avoid having to write the same code over and over, we create a `_BaseComponent`, of which every type of component will be a subclass. Each component must implement the `equation` method (which really returns a `sympy` *ex-*

pression, used as the LHS in an equation whose RHS is zero). The frequency (`frequency_rad_per_s`) is passed to each `equation` method because it is a global property of the circuit rather than each component, but will be needed by the `Inductor` and `Capacitor` components to calculate reactance and admittance, as we will see in the next section.

```
@dataclass(repr=False)
class _BaseComponent(abc.ABC):
   name: str
   negative: Node
   positive: Node
   current: Current = field(init=False)

   def __post_init__(self):
      self.current = Current(component_name=self.name)
      for node in self._connected_nodes:
         node.connected_components.append(self)

   @property
   def _connected_nodes(self) -> list[Node]:
      return [self.negative, self.positive]

   @abc.abstractmethod
   def equation(self, frequency_rad_per_s: float):
      raise NotImplementedError

   @property
   def voltage_difference(self):
      return (
         self.positive.voltage.variable
         - self.negative.voltage.variable
      )
```

Component Library

Now that we have created our framework, we are able to implement our components and focus on the characteristic equation of each:

```
@dataclass(repr=False)
class VoltageSource(_BaseComponent):
   value: Phasor

   def equation(self, frequency_rad_per_s: float):
      return self.voltage_difference - self.value.to_complex()

@dataclass(repr=False)
class CurrentSource(_BaseComponent):
   value: Phasor

   def equation(self, frequency_rad_per_s: float):
      return self.current.variable - self.value.to_complex()

@dataclass(repr=False)
class Resistor(_BaseComponent):
   resistance_ohm: float

   def equation(self, frequency_rad_per_s: float):
      return (
         self.voltage_difference
         - self.current.variable * self.resistance_ohm
      )
```

```python
@dataclass(repr=False)
class Inductor(_BaseComponent):
    inductance_h: float

    def equation(self, frequency_rad_per_s: float):
        reactance = 1j * frequency_rad_per_s * self.inductance_h
        return (
            self.voltage_difference
            - self.current.variable * reactance
        )

@dataclass(repr=False)
class Capacitor(_BaseComponent):
    capacitance_f: float

    def equation(self, frequency_rad_per_s: float):
        admittance = 1j * (
            frequency_rad_per_s * self.capacitance_f
        )
        return (
            self.voltage_difference * admittance
            - self.current.variable
        )
```

Rather than using reactance for the capacitor, we use its inverse—admittance—as it conveniently allows the circuit's frequency to be set to zero to model the circuit under DC conditions, without encountering a division-by-zero error.

The Circuit Solver

All that's left to do is create a circuit solver to determine the voltage and current values. This is done by:

1. Passing all the circuit's components to the solver.
2. Extracting all the circuit's nodes (taken from the components' `positive` and `negative` terminals).
3. Collecting the unknowns (voltages and currents) and equations (KCL and characteristic equations) from the nodes and components.
4. Solving the equations for the unknowns using `sympy`'s `nsolve`.
5. Assigning the solved values to their respective nodes and components.

```python
@dataclass(repr=False)
class AcCircuitSolver:
    frequency_hz: float
    components: list[_BaseComponent]

    def solve(self) -> None:
        connected_nodes = self._get_connected_nodes()
        symbolics = self._get_symbolics(connected_nodes)
        equations = self._get_equations(connected_nodes)
        unknowns = [s.variable for s in symbolics]
        initial_guess = [0.0 for s in symbolics]
        solution = nsolve(equations, unknowns, initial_guess)
        for unknown, solved_value in zip(symbolics, solution):
            unknown.value = Phasor.from_complex(
                complex(solved_value)
            )
```

```
    def _get_symbolics(
        self, connected_nodes: list[Node]
    ) -> list[_Symbolic]:
        symbolics: list[_Symbolic] = []
        for n in connected_nodes:
            symbolics.append(n.voltage)
        for c in self.components:
            symbolics.append(c.current)
        return symbolics

    def _get_equations(self, connected_nodes: list[Node]):
        equations = []
        for n in connected_nodes:
            equations.append(n.equation)
        frequency_rad_per_s = 2 * pi * self.frequency_hz
        for c in self.components:
            equations.append(c.equation(frequency_rad_per_s))
        return equations

    def _get_connected_nodes(self) -> list[Node]:
        nodes: list[Node] = []
        for component in self.components:
            nodes += component._connected_nodes
        # Deduplicate (requires `Node.__hash__`):
        nodes = list(set(nodes))
        return nodes
```

Example Usage

We can apply our solver to the example circuit of Figure 3.1 from the last chapter. We specify the circuit's nodes—the positive and negative terminals of both AC voltage sources—giving each a unique name:

```
vs1p = Node("Vs1p")
vs1n = Node("Vs1n", is_ground=True)
a = Node("Vs2p")
b = Node("Vs2n")
```

Now we specify the two AC voltage sources, the two resistors, the inductor, and the capacitor:

```
vs1 = VoltageSource("Vs1", vs1n, vs1p, Phasor(magnitude=1))
vs2 = VoltageSource("Vs2", b, a, Phasor(magnitude=2))
r1 = Resistor("R1", b, vs1p, resistance_ohm=4)
r2 = Resistor("R2", vs1n, a, resistance_ohm=5)
l = Inductor("L", a, vs1p, inductance_h=6)
c = Capacitor("C", vs1n, b, capacitance_f=7)
```

Next, we pass the components to the solver, specify the circuit's frequency (or 0 Hz for DC conditions), and run the solver:

```
solver = AcCircuitSolver(
    frequency_hz=(3 / (2 * pi)),
    components=[vs1, vs2, r1, r2, l, c],
)
solver.solve()
```

We can access the solved voltage and current values through the original node and component objects:

```
print(f"{a.voltage.value = }")
print(f"{b.voltage.value = }")
print(f"{vs2.current.value = }")
```

```
a.voltage.value = 2.003 @ 89.794°
b.voltage.value = 0.008 @ 19.092°
vs2.current.value = 0.405 @ -82.293°
```

This chapter is available as a runnable Python notebook and as a Python script.

Additional Exercises

1. Add a plotting feature to the circuit analyzer (e.g., using Python's `matplotlib`).
2. Add the ability to import and analyze the circuit from a file or from a circuit diagram.
3. Consider how you would adapt the circuit analyzer to support transient simulations and active components (diodes, transistors, etc.).

See here for some inspiration.

Chapter 5

Complex Power and the Power Factor

In the previous chapters, we learned how complex numbers can simplify calculations involving voltage and current. We will see how complex numbers can also provide simplified power calculations. But first, in order to understand the nuances of AC power, we will see how power can be derived and quantified for a general AC circuit in the time domain.

Power Derivations in the Time Domain

Consider the circuit of Figure 5.1. Such a circuit is useful because it can be used to model any series connection between one or more impedances and one or more voltage sources—as long as the voltage sources can be lumped together as one $v_S(t)$ and the impedances can be lumped together as one Z.

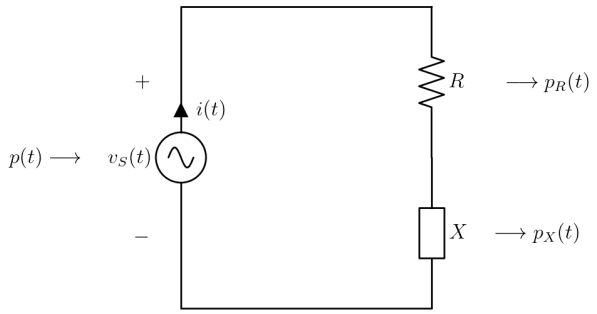


Figure 5.1: AC circuit with voltage source $v_S(t) = |V_S| \cos(\omega t + \theta_S)$ and impedance $Z = R + jX$.

Using the approaches from the previous two chapters, we can determine the current through the circuit in the time domain:

$$i(t) = \frac{|V_S|}{\sqrt{R^2 + X^2}} \cos\left(\omega t + \theta_S - \arctan\frac{X}{R}\right) \tag{5.1}$$

Thus, the instantaneous power at time t is:

$$p(t) = v_S(t)\, i(t) = \frac{|V_S|^2}{\sqrt{R^2 + X^2}} \cos(\omega t + \theta_S) \cos\left(\omega t + \theta_S - \arctan\frac{X}{R}\right) \tag{5.2}$$

Applying the trigonometric identity $\cos\theta_a \cos\theta_b = (\cos(\theta_a - \theta_b) + \cos(\theta_a + \theta_b))/2$ gives us the following expression for $p(t)$. We see that $p(t)$ is a sinusoid, just like $v_S(t)$ and $i(t)$. However, $p(t)$ oscillates with twice the frequency of the voltage and current (2ω, instead of ω), and does not necessarily oscillate around zero. The time-invariant part of $p(t)$ is the "centerline" around which the sinusoid oscillates, which is also equal to the sinusoid's time-averaged value denoted by $\bar{p}$.

$$\begin{aligned}
&p(t) \\
&= \frac{|V_S|^2}{\sqrt{R^2 + X^2}} \\
&\cdot \frac{1}{2}\left[\cos\left(\omega t + \theta_S - \left(\omega t + \theta_S - \arctan\frac{X}{R}\right)\right) + \cos\left(2(\omega t + \theta_S) - \arctan\frac{X}{R}\right)\right] \\
&= \frac{1}{2}\frac{|V_S|^2}{\sqrt{R^2 + X^2}} \cos\left(\arctan\frac{X}{R}\right) \\
&+ \frac{1}{2}\frac{|V_S|^2}{\sqrt{R^2 + X^2}} \cos\left(2(\omega t + \theta_S) - \arctan\frac{X}{R}\right) \\
&= \underbrace{\frac{1}{2}\frac{|V_S|^2}{R^2 + X^2} R}_{\bar{p},\ \text{time-invariant}} + \underbrace{\frac{1}{2}\frac{|V_S|^2}{\sqrt{R^2 + X^2}} \cos\left(2(\omega t + \theta_S) - \arctan\frac{X}{R}\right)}_{\text{time-variant}}
\end{aligned} \tag{5.3}$$

We can derive $p(t)$ for a purely resistive AC circuit as follows, by setting $X = 0$. We can see that $p(t)$ is always greater than (or equal to) zero, as illustrated in Figure 5.2(a). Because resistor R always dissipates power and in the real world likely represents a device that can do useful work—such as a light bulb or laptop—we call this *real* or *active* power. In the $X = 0$ case, we can quantify our circuit's active power by its peak value (relative to $\bar{p}$) of $|V_S|^2 / 2R$.

$$p(t) = \underbrace{\frac{1}{2}\frac{|V_S|^2}{R}}_{\bar{p}} + \underbrace{\underbrace{\frac{1}{2}\frac{|V_S|^2}{R}}_{\text{peak value}} \cos(2(\omega t + \theta_S))}_{\text{time-variant part}} \tag{5.4}$$

Similarly, we can derive $p(t)$ for an AC circuit with only inductors and capacitors as follows, by setting $R = 0$. The average power $\bar{p}$ is now zero; $p(t)$ oscillates around zero as illustrated in Figure 5.2(b) and (c). Recall from Chapter 1 that inductive and capacitive components store energy. The rate at which the component stores or releases electrical energy is what we call its electric power—positive when converting *from* electrical energy and negative when converting back *to* electrical energy. When an AC voltage is applied to an inductive or capacitive component, it stores and releases energy in a cycle. The average power going into the component is equal to the average power going out of it over time. Power is exchanged back and forth between

the AC voltage source and the inductor or capacitor, but none is dissipated or does any useful work. We call this *reactive* power. In the $R = 0$ case, we can quantify our circuit's reactive power by its peak value of $|V_S|^2 / 2X$.

$$p(t) = \underbrace{\underbrace{\frac{1}{2}\frac{|V_S|^2}{X}}_{\text{peak value}} \cos\left(2(\omega t + \theta_S) + \frac{\pi}{2}\right)}_{\text{time-variant}} \tag{5.5}$$

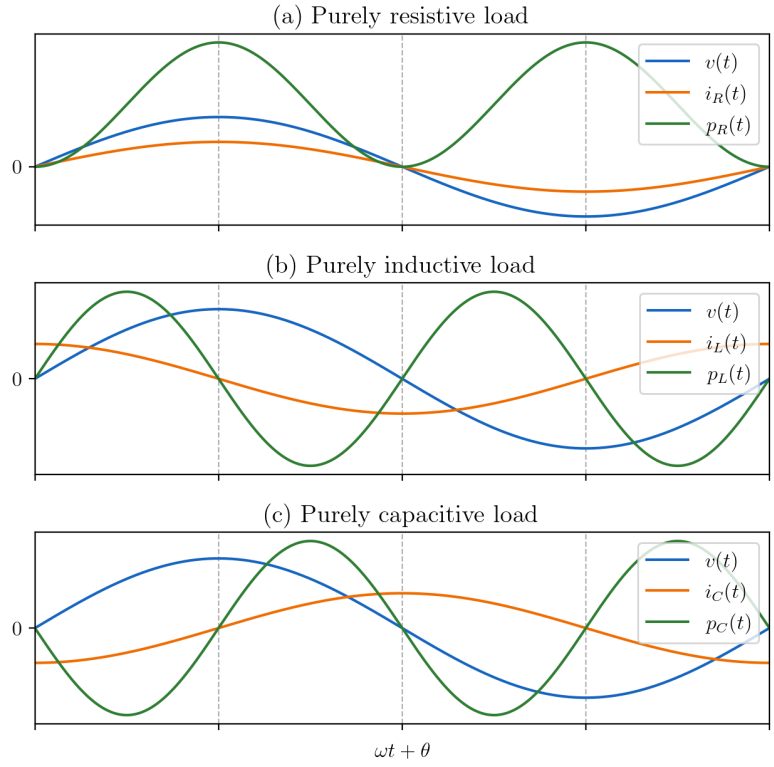


Figure 5.2: Electrical current and power waveforms for purely resistive, inductive, and capacitive loads.

But how do we quantify the power for a circuit with nonzero R *and* X? There are two components—power being dissipated, and power being exchanged. Firstly, the power dissipated by the resistive load is:

$$p_R(t) = i^2(t)R = \frac{V_S^2}{R^2 + X^2} \cos^2\left(\omega t + \theta_S - \arctan\frac{X}{R}\right)R \tag{5.6}$$

Applying the trigonometric identity $\cos^2\theta = (\cos 2\theta + 1)/2$ gives us the following—the circuit's active power—whose peak value (relative to $\bar{p}_R$) is $V_S^2 R \,/\, 2(R^2 + X^2)$. Note that even though this is only active power and is only dissipated by the resistor, it is affected by X in addition to R.

$$p_R(t) = \underbrace{\frac{1}{2}\frac{V_S^2}{R^2+X^2}R}_{\bar{p}_R,\ \text{time-invariant}} + \underbrace{\underbrace{\frac{1}{2}\frac{V_S^2}{R^2+X^2}R}_{\text{peak value}}\cos\left(2\left(\omega t + \theta_S - \arctan\frac{X}{R}\right)\right)}_{\text{time-variant}} \tag{5.7}$$

And secondly, the power exchanged by the reactive load is:

$$\begin{aligned} p_X(t) &= p(t) - p_R(t) \\ &= \frac{1}{2}\frac{V_S^2}{R^2+X^2}R + \frac{1}{2}\frac{V_S^2}{\sqrt{R^2+X^2}}\cos\left(2(\omega t + \theta_S) - \arctan\frac{X}{R}\right) \\ &\quad - \frac{1}{2}\frac{V_S^2}{R^2+X^2}R - \frac{1}{2}\frac{V_S^2}{R^2+X^2}R\cos\left(2\left(\omega t + \theta_S - \arctan\frac{X}{R}\right)\right) \\ &= \frac{1}{2}\frac{V_S^2}{R^2+X^2}\left[\begin{array}{r}\sqrt{R^2+X^2}\cos\left(2(\omega t + \theta_S) - \arctan\frac{X}{R}\right) \\ -R\cos\left(2\left(\omega t + \theta_S - \arctan\frac{X}{R}\right)\right)\end{array}\right] \end{aligned} \tag{5.8}$$

To the expression enclosed in square brackets, we now apply the trigonometric identity $a\cos(x+\theta_a) + b\cos(x+\theta_b) = c\cos(x+\theta_c)$, where $c = \sqrt{a^2+b^2+2ab\cos(\theta_a-\theta_b)}$ and $\theta_c = \arctan[(a\sin\theta_a + b\sin\theta_b)/(a\cos\theta_a + b\cos\theta_b)]$.

$$x = 2(\omega t + \theta_S)$$

$$\begin{aligned} c &= \sqrt{\begin{array}{l}\left(\sqrt{R^2+X^2}\right)^2 + R^2 \\ + 2\sqrt{R^2+X^2} \\ \quad \cdot(-R)\cos\left(2(\omega t+\theta_S) - \arctan\frac{X}{R} - 2\left(\omega t + \theta_S - \arctan\frac{X}{R}\right)\right)\end{array}} \\ &= \sqrt{2R^2 + X^2 - 2R\sqrt{R^2+X^2}\cos\left(\arctan\frac{X}{R}\right)} \\ &= \sqrt{2R^2 + X^2 - 2R^2} \\ &= X \\ \theta_c &= \arctan\frac{\sqrt{R^2+X^2}\sin(-\arctan\frac{X}{R}) - R\sin(-2\arctan\frac{X}{R})}{\sqrt{R^2+X^2}\cos(-\arctan\frac{X}{R}) - R\cos(-2\arctan\frac{X}{R})} \\ &= \arctan\frac{-\sqrt{R^2+X^2}\sin(\arctan\frac{X}{R}) + 2R\sin(\arctan\frac{X}{R})\cos(\arctan\frac{X}{R})}{\sqrt{R^2+X^2}\cos(\arctan\frac{X}{R}) - R\left(2\cos^2(\arctan\frac{X}{R}) - 1\right)} \\ &= \arctan\frac{-X + 2R\frac{XR}{R^2+X^2}}{R - R\left(2\frac{R^2}{R^2+X^2} - 1\right)} \\ &= \arctan\frac{-X(R^2+X^2) + 2R(XR)}{2R(R^2+X^2) - 2RR^2} \\ &= \arctan\frac{R^2 - X^2}{2RX} \end{aligned} \tag{5.9}$$

Thus, reactive load's power is the following—the circuit's reactive power—whose peak value is $V_S^2 X / 2(R^2 + X^2)$. Note that even though this is only reactive power and is only exchanged by the reactive load, it is affected by R in addition to X.

$$p_X(t) = \underbrace{\frac{1}{2}\frac{V_S^2}{R^2+X^2}X}_{\text{peak value}} \underbrace{\cos\left(2(\omega t + \theta_S) + \arctan\frac{R^2 - X^2}{2RX}\right)}_{\text{time-variant}} \tag{5.10}$$

Now we can determine the active and reactive power for any circuit like that in Figure 5.1.

Power as a Phasor?

The previous section informed us of the nuances of AC power, but the calculations were tedious. We ought to be able to determine the active and reactive power of a circuit—or of any of its components—more easily. Perhaps if AC voltage and current calculations are simplified by working in the phasor domain, then so would AC power calculations. If $p(t) = v^2(t)/R$ for a purely resistive circuit, then surely the equivalent for a circuit that includes reactive components would be $P = V_S^2/Z$? Let's attempt to use this definition:

$$\begin{aligned} P &= \frac{V_S^2}{Z} \\ &= \frac{(|V_S|\angle\theta_S)^2}{\sqrt{R^2+X^2}\angle\arctan\frac{X}{R}} \\ &= \frac{|V_S|^2\angle 2\theta_S}{\sqrt{R^2+X^2}\angle\arctan\frac{X}{R}} \\ &= \frac{|V_S|^2}{\sqrt{R^2+X^2}}\angle\left(2\theta_S - \arctan\frac{X}{R}\right) \end{aligned} \tag{5.11}$$

If we were to naively convert this phasor into the time domain as follows, we would find that it does not match Equation (5.3) for $p(t)$ which we derived in the previous section. Firstly, because it oscillates around zero rather than around $|V_S|^2 R \,/\, 2(R^2 + X^2)$. Secondly, because its amplitude is $|V_S|^2/\sqrt{R^2+X^2}$ rather than $|V_S|^2/\,2\sqrt{R^2+X^2}$. And thirdly, because it oscillates at natural frequency ω rather than 2ω. Clearly, this does not work—power cannot be represented by a phasor.

$$\begin{aligned} &\frac{|V_S|^2}{\sqrt{R^2+X^2}}\cos\left(\omega t + 2\theta_S - \arctan\frac{X}{R}\right) \\ &\neq \underbrace{\frac{1}{2}\frac{|V_S|^2}{R^2+X^2}R + \frac{1}{2}\frac{|V_S|^2}{\sqrt{R^2+X^2}}\cos\left(2(\omega t + \theta_S) - \arctan\frac{X}{R}\right)}_{\text{From Equation 4.3}} \end{aligned} \tag{5.12}$$

However, this does not necessarily mean that AC power cannot be quantified by a complex number or that complex numbers cannot somehow be used to simplify AC power calculations. Note that active and reactive power do not depend on θ_S. In Equation (5.11), if we were to treat θ_S as if it were zero, we could write:

$$
\begin{aligned}
P &= \frac{|V_S|^2}{\sqrt{R^2+X^2}} \angle -\arctan\left(\frac{X}{R}\right) \\
&= \frac{|V_S|^2}{\sqrt{R^2+X^2}} \cos\left(-\arctan\frac{X}{R}\right) + j\,\frac{|V_S|^2}{\sqrt{R^2+X^2}} \sin\left(-\arctan\frac{X}{R}\right) \\
&= \frac{|V_S|^2}{\sqrt{R^2+X^2}} \frac{R}{\sqrt{R^2+X^2}} - j\,\frac{|V_S|^2}{\sqrt{R^2+X^2}} \frac{X}{\sqrt{R^2+X^2}} \\
&= \frac{|V_S|^2}{R^2+X^2} R - j\,\frac{|V_S|^2}{R^2+X^2} X
\end{aligned}
\tag{5.13}
$$

Note how the real part, $|V_S|^2 R\,/(R^2+X^2)$, and the imaginary part, $|V_S|^2 X/(R^2+X^2)$, are two times the peak values of the circuit's active and reactive power that we derived in the previous section. These are the *peak-to-peak* values of the active and reactive power. This means that we are en route to a way to use complex numbers to quantify AC power. In practice, it is not the entire expression $|V_S|^2 R\,/(R^2+X^2) + j\,|V_S|^2 X/(R^2+X^2)$ that we denote P, but just half of the real part $|V_S|^2 R\,/\,2(R^2+X^2)$. We denote a negative half of the imaginary part $-|V_S|^2 X/\,2(R^2+X^2)$ by Q, and the expression as a whole by complex number S. Thus, $S = P + j\,Q$. S is what we call *complex power*.

Complex Power

By quantifying AC power as

$$
S = \frac{1}{2}\frac{|V_S|^2}{Z^*} \tag{5.14}
$$

rather than as V_S^2/Z, we can remove θ_S from our calculations and make them simpler. For example, for a circuit like that in Figure 5.1, we can calculate complex power S easily using only complex number division:

$$
S = \frac{1}{2}\frac{|V_S|^2}{R - jX} = \underbrace{\frac{1}{2}\frac{|V_S|^2}{R^2+X^2}R}_{P} + j\,\underbrace{\frac{1}{2}\frac{|V_S|^2}{R^2+X^2}X}_{Q} \tag{5.15}
$$

From Equation (5.14), we can derive the following—the AC power equivalent of the formula $P = I^2 R$.

$$
\begin{aligned}
S &= \frac{1}{2}\frac{|V_S|^2}{Z^*} = \frac{1}{2}\frac{|V_S|^2}{Z^*}\frac{Z}{Z} = \frac{1}{2}\frac{|V_S|^2 Z}{|Z|^2} = \frac{1}{2}\left(\frac{|V_S|}{|Z|}\right)^2 Z \\
&= \frac{1}{2}|I|^2 Z
\end{aligned}
\tag{5.16}
$$

From which we can derive the AC power equivalent of $P = V_S I$:

$$
\begin{aligned}
S &= \frac{1}{2}|I|^2 Z = \frac{1}{2}|I|^2\frac{V_S}{I} = \frac{1}{2}|I|^2\frac{V_S}{I}\frac{I^*}{I^*} = \frac{1}{2}|I|^2\frac{V_S I^*}{|I|^2} \\
&= \frac{1}{2}V_S I^*
\end{aligned}
\tag{5.17}
$$

Clearly the $\frac{1}{2}$ coefficients are tedious to work with. In practice, electrical engineers often work with the root mean square (RMS) values

Load	Active power P	Reactive power Q	Complex power S	Phase angle ϕ
Purely resistive	> 0	$= 0$	$= P$	$= 0$
Purely inductive	$= 0$	> 0	$= jQ$	> 0
Purely capacitive	$= 0$	< 0	$= jQ$	< 0

Table 5.1: Active power, reactive power, complex power, and phase angle of purely resistive, inductive, and capacitive loads.

of voltage, $V_{\text{RMS}} = V_S/\sqrt{2}$, and current, $I_{\text{RMS}} = I/\sqrt{2}$, such that the above complex power formulae become:

$$\boxed{S = \frac{|V_{\text{RMS}}|^2}{Z^*}} \qquad \boxed{S = |I_{\text{RMS}}|^2 Z} \qquad \boxed{S = V_{\text{RMS}}\, I_{\text{RMS}}^*} \tag{5.18}$$

Looking at complex power in polar form, its magnitude $|S|$ is what we call *apparent power*, and its angle $\phi = \arctan(Q/P)$ is the same as the phase angle $\arctan(X/R)$ by which the current lags (or leads) the voltage. Table 6.1 summarizes the values taken on by P, Q, S, and ϕ for different types of load.

The Power Triangle

In What is a Phasor?, we saw how a triangle could be used to depict a phasor (and thus any complex number) in the complex plane. Accordingly, complex power is often depicted as a *power triangle* (Figure 5.3), which aids in remembering the relationships between P, Q, S, and ϕ.

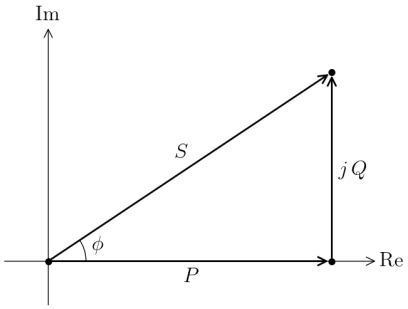


Figure 5.3: The power triangle.

Even though power $p(t)$ is a sinusoid, and a phasor rotating in the complex plane traces out a sinusoid, S is not a phasor and would not form $p(t)$ if rotated about the origin. However, if S were rotated about P, it would correctly trace out $p(t)$ as illustrated in Figure 5.4(a). Figure 5.4(b), (c), and (d) are for purely resistive, inductive, and capacitive loads and correspond to Figure 5.2(a), (b), and (c).

The Power Factor

This section introduces the concept of the power factor. The power factor is a general concept in AC systems, but is particularly consequential for the electrical grid and its users. So, understanding why

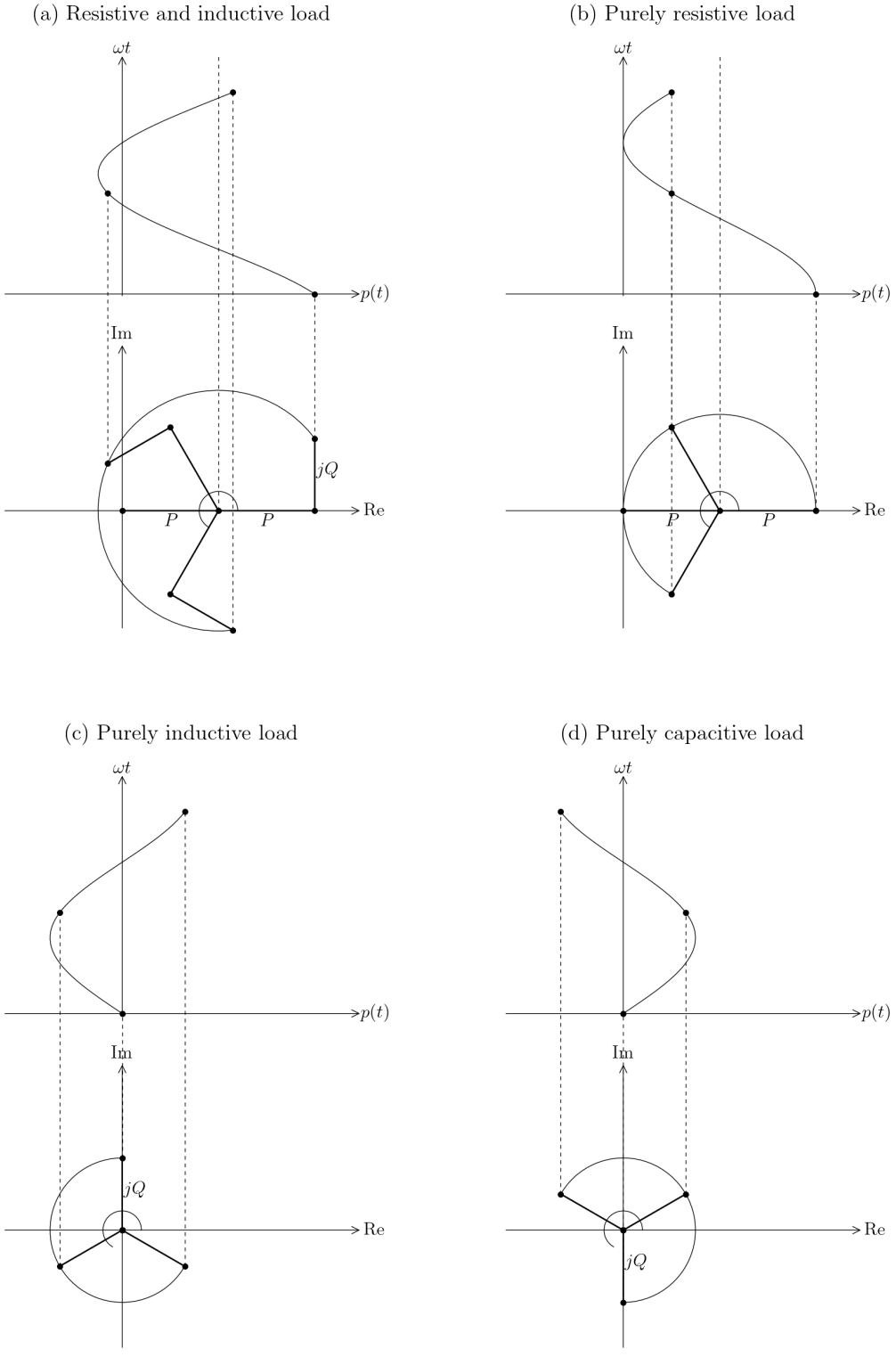


Figure 5.4: Instantaneous power waveforms drawn by rotating S about P for different loads.

the concept of the power factor exists and its implications first requires a high-level understanding of the electrical grid. The electrical grid can be roughly modeled at a very high level by the circuit shown in Figure 5.5. Voltage source v_S represents a generating station. The resistances of value R_i and the reactance X represent loads connected to the grid in parallel, from multiple grid users. Whereas the resistances of value R_i may represent loads such as light bulbs or laptops, X represents a reactive load, such as an inductive ballast in a fluorescent light, or a large industrial induction motor. Resistance r represents the resistance of transmission lines between the generating station and the grid loads. Note that liberties have been taken at every part of this model. In reality, grids are complex circuits with multiple generating stations and loads, all connected with a network of transmission and distribution lines. Generating stations cannot just be represented by voltage sources, and more accurate transmission line models incorporate reactances. Nonetheless, the circuit shown in Figure 5.5 is a sufficient model for the purposes of understanding the power factor.

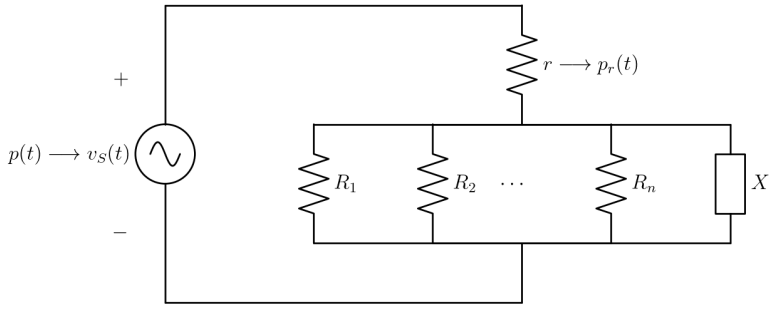


Figure 5.5: A rough, high-level model of the electrical grid for the purposes of understanding the power factor.

Using the formulae for impedances in series and in parallel, the total impedance looking into the grid loads from the generating station is as follows. We can see that the presence of reactive load X reduces the magnitude of the impedance $|Z|$.

$$Z = r + \frac{1}{n \cdot \frac{1}{R} + \frac{1}{jX}} \tag{5.19}$$

Adapting Equation (5.3), the power $p(t)$ from the generating station is as follows. The average and peak-to-peak values of $p(t)$ are $\bar{p}$ and $|V_S|^2/|Z|$, respectively.

$$p(t) = \underbrace{\frac{1}{2}\frac{|V_S|^2}{|Z|^2}\operatorname{Re}(Z)}_{\bar{p}} + \frac{1}{2}\frac{|V_S|^2}{|Z|}\cos\left(2(\omega t + \theta_S) - \arctan\frac{\operatorname{Im}(Z)}{\operatorname{Re}(Z)}\right) \tag{5.20}$$

And adapting Equation (5.7), the power $p_r(t)$ dissipated by resistance r is as follows. The average and peak-to-peak values of $p_r(t)$ are $\bar{p}_r$ and $|V_S|^2 r/|Z|^2$, respectively.

$$p_r(t) = \underbrace{\frac{1}{2}\frac{|V_S|^2}{|Z|^2}\,r}_{\bar{p}_r} + \frac{1}{2}\frac{|V_S|^2}{|Z|^2}\,r\cos\left(2\left(\omega t + \theta_S - \arctan\frac{\mathrm{Im}(Z)}{\mathrm{Re}(Z)}\right)\right) \tag{5.21}$$

If $|Z|$ is reduced due to the presence of reactive load X, then this will increase the peak-to-peak value of $p(t)$, the peak-to-peak value of $p_r(t)$, and the average value of $p_r(t)$. This has a number of consequences. Firstly, the generating station V_S must be able to produce higher peak instantaneous power. Secondly, the transmission line r must be able to dissipate a higher average power without overheating. And thirdly, because the transmission line r is dissipating more power, its voltage drop is higher. Thus, the generating station must operate at higher voltage and power in order to maintain the same voltage and power for each connected load. Thus, reactive loads tend to incur higher infrastructure costs, in the form of higher-capacity generators and transmission lines, and higher operating costs. This extra demand on the grid would exist even if reactive load X were instead a resistive load—but, with a reactive load, the extra demand is unnecessary. With a reactive load, extra power is being transferred back and forth needlessly between the load and the generating station. Whenever the reactive load stores energy, the generating station must be providing it, and whenever the reactive load releases energy, the generating station must be absorbing it.

It would not be fair for the generating station or grid operator to pay for the higher infrastructure costs that reactive loads require, nor would it be fair for the generating station to pay for the higher operating costs. So, the cost of the extra demand on the grid is often passed onto the grid user who is responsible for it according to their power meter—for example, an industrial utility customer. This is done by having grid users' power meters report the apparent power $|S|$, rather than just the active power P, over time. Grid users can reduce their electricity costs, without reducing their active power usage, by reducing their reactive loads. This reduces their reactive power Q, which in turn reduces $|S|$.

> **Note**
> If your power factor is low, you are paying extra to have power needlessly transmitted back and forth between your reactive loads and the grid's generating stations, potentially over hundreds of kilometers of transmission lines, sixty times a second.

The ratio between P and $|S|$ is what is known as the power factor (PF):

$$\boxed{\mathrm{PF} = \frac{P}{|S|}} \tag{5.22}$$

The power factor ranges from 0 for a purely reactive load to 1 for a purely resistive load. However, a negative power factor can arise when the "load" (the grid user) is actually supplying active power rather than consuming it. The power factor is described as *lagging* for a more inductive load with $\phi > 0$, or *leading* for a more capacitive load

with $\phi < 0$. Because the power factor is the ratio between the adjacent side and hypotenuse of the power triangle, it can be reformulated as $\cos\phi$.

Increasing the power factor reduces electricity costs. This can be done by choosing resistive loads over reactive loads where possible (e.g., LEDs over fluorescent lights), by implementing *power factor correction (PFC)*, or by choosing loads that have power factor correction built-in. Implementing power factor correction is important when installing loads that necessarily have a low power factor (such as an induction motor) or when a grid user seeks to reduce the costs incurred by low-power-factor loads that they already have and want to keep using (such as fluorescent lights).

Power Factor Correction

Most loads with a low power factor are inductive. Because of this, their lagging power factor (inductive reactance) can be counteracted using loads with a leading power factor—such as synchronous motors, or large banks of capacitors whose sole purpose is to do power factor correction. The reactive power exchanged by a capacitor is exactly 180° out-of-phase with an inductive load, meaning that when the the inductive load stores energy, the capacitor discharges to provide that energy, and when the inductive load releases energy, the capacitor charges to absorb that energy. The required capacitance C can be calculated such that the power exchanged by the capacitor is exactly opposite to that exchanged by the inductive load. This can be done by calculating what value of C sets the total reactance to zero. Let the impedance of the inductive load be $\mathrm{Re}(Z_L) + j\,\mathrm{Im}(Z_L)$ rather than just ωL to account for the fact that its equivalent circuit likely includes resistances. The capacitor can be placed in series or in parallel with the inductive load, but if in parallel, the power-factor-corrective capacitors do not need to be co-located with a facility's inductive loads.

For a capacitor C in series with inductive load Z_L:

$$\begin{aligned} Z_{\text{tot.}} = Z_L + Z_C &= \mathrm{Re}(Z_L) + j\,\mathrm{Im}(Z_L) + jX_C \\ \mathrm{Im}(Z_{\text{tot.}}) = 0 &\implies \mathrm{Im}(Z_L) + X_C = 0 \\ &\implies \mathrm{Im}(Z_L) - \frac{1}{\omega C} = 0 \\ &\implies C = \frac{1}{\omega\,\mathrm{Im}(Z_L)} \end{aligned} \tag{5.23}$$

For a capacitor C in parallel with inductive load Z_L:

$$\begin{aligned} Z_{\text{tot.}} = \frac{1}{Z_L} + \frac{1}{Z_C} &= \frac{1}{\mathrm{Re}(Z_L) + j\,\mathrm{Im}(Z_L)} + \frac{1}{jX_C} \\ &= \frac{\mathrm{Re}(Z_L) - j\,\mathrm{Im}(Z_L)}{|Z_L|^2} - j\frac{1}{X_C} \\ \mathrm{Im}(Z_{\text{tot.}}) = 0 &\implies -\frac{\mathrm{Im}(Z_L)}{|Z_L|^2} - \frac{1}{X_C} = 0 \\ &\implies -\frac{\mathrm{Im}(Z_L)}{|Z_L|^2} - \frac{1}{-1/\omega C} = 0 \\ &\implies C = \frac{\mathrm{Im}(Z_L)}{\omega|Z_L|^2} \end{aligned} \tag{5.24}$$

The capacitor is taking the place of the generating station in terms of providing and absorbing the extra energy that the inductive load must store and release. But, because this extra energy no longer needs to be transmitted over long distances to and from a generating station, the infrastructure and other costs are reduced and no longer passed on to the grid user. The grid users with the highest cost savings from power factor correction are industrial plants with many, large induction motors. Residential utility customers are usually not metered or charged for their reactive power, and so would not typically benefit from power factor correction. There do exist small capacitive devices that plug directly into wall outlets and supposedly do power factor correction for residents, but they would save nothing for most residential utility customers—that is, if they do any power factor correction at all.

Chapter 6

Three-Phase Power

Introduction to Three-Phase Power

The means of generation, transfer, and consumption of AC power that we have discussed so far is known as *single-phase* power. If not already familiar with three-phase power, you will be surprised to learn that single-phase power is not the main way that power is transferred in the electrical grid. In a single-phase system, a sinusoidal voltage $v(t)$ is generated between a set of two power lines such that power is transferred to loads placed across those lines (in parallel). This is illustrated by Figure 6.1(a). As we learned in the previous chapter, the power delivered by a single-phase circuit oscillates sinusoidally with time, which can be impractical.

On the other hand, three-phase power, sometimes abbreviated 3ϕ, is a way of transmitting power for which the power delivered is constant over time. Steady power is advantageous for heavy industrial loads, such as large electric motors, and for long-distance transmission. This is one reason why three-phase power is carried by the grid's transmission lines, besides the fact that three-phase power can reduce the amount of conductor material needed and thus results in reduced infrastructure costs.

A three-phase circuit has three voltage lines, each of which carries a sinusoidally oscillating voltage. The three voltages are called *phase voltages* and are denoted $v_a(t)$, $v_b(t)$, and $v_c(t)$. The three voltage sinusoids are of equal magnitude $|V|$, but their phase angles are one third of a cycle ($2\pi/3$ radians) apart, meaning that the three voltage sinusoids sum to zero at any given time. The three phase voltages can be represented by phasors as in Figure 6.2.

The generators that supply the phase voltages (and thus power the circuit) are called three-phase generators, and the loads that are powered by the circuit are called three-phase loads. Each of these generators and loads is connected to all three lines. This is illustrated by Figure 6.1(b). Each load actually consists of three impedances. If a given load's three impedances are of equal value (Z), then the load is called a *balanced* three-phase load. A three-phase circuit in which all loads are balanced is called a balanced three-phase circuit.

For a balanced load, the fact that the voltage sinusoids sum to zero also means that the sinusoidal currents through the three impedances sum to zero. In other words, the three currents cancel out. Thus,

no extra line is required to carry current back to the generator(s), as is required in a single-phase circuit. This is why a balanced three-phase circuit can transfer the same amount of power as a single-phase circuit, but with less conductor material. An unbalanced three-phase circuit, on the other hand, requires a fourth *neutral line* to which each unbalanced load connects.

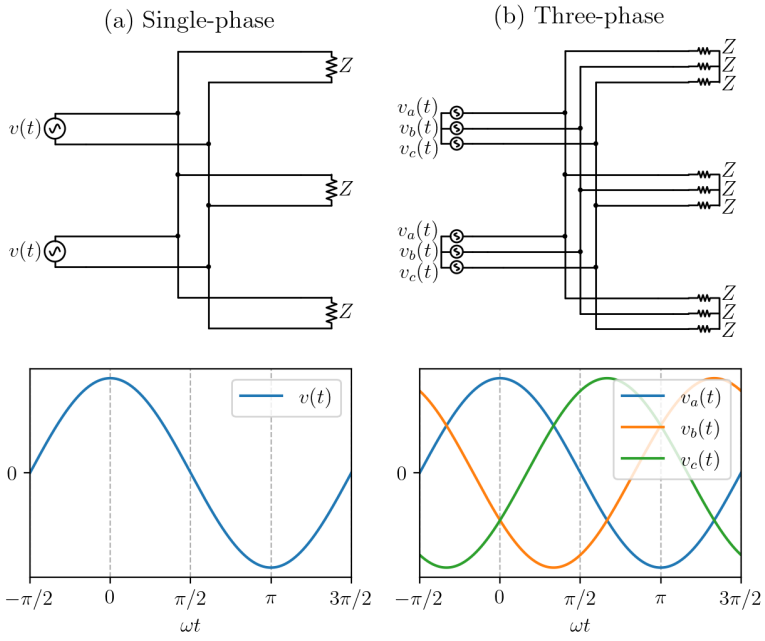


Figure 6.1: Single- and three-phase systems.

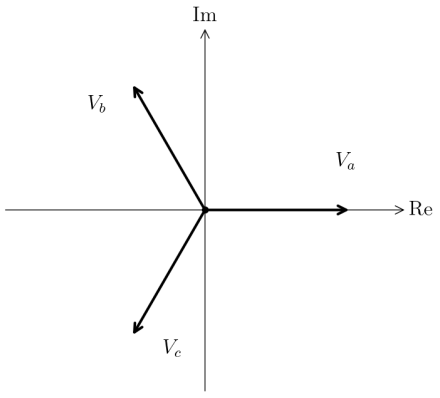


Figure 6.2: Three-phase voltage phasors.

Comparison of *n*-Phase Circuits

Circuits with a number of phases other than one or three are possible. Circuits with more than one phase are known as polyphase systems. But why is three-phase power more common than other polyphase systems, and more common than even single-phase systems? Why not two-phase power, or four? This section will compare circuits with an arbitrary number of phases n. We will determine the

advantages and disadvantages of circuits with different numbers of phases. We will assume balanced three-phase loads with, for simplicity, impedance value R. However, the conclusions would be the same for an arbitrary impedance Z.

Consider a **circuit with a single phase** a, such as that shown in Figure 6.3(a). It has one impedance $R_a = R$, supplied by one phase voltage:

$$v_a(t) = |V| \cos \omega t \tag{6.1}$$

Recall the instantaneous power delivered over time in a single-phase circuit from Chapter 5. In a single-phase circuit, the total power $p_{\text{tot.}}(t)$ delivered in the circuit is just the power $p_a(t)$ delivered to impedance R_a, the time-average of which is $\bar{p}_{\text{tot.}} = |V|^2 / 2R$.

$$p_{\text{tot.}}(t) = p_a(t) = \frac{|V|^2}{R} = \frac{|V|^2}{R} \cos^2 \omega t = \underbrace{\frac{1}{2}\frac{|V|^2}{R}}_{\bar{p}_{\text{tot.}}} + \frac{1}{2}\frac{|V|^2}{R} \cos 2\omega t \tag{6.2}$$

Similarly, the sum of the phase currents $i_{\text{tot.}}(t)$ is just the current $i_a(t)$ delivered through impedance R_a, as follows:

$$i_{\text{tot.}}(t) = i_a(t) = \frac{|V|}{R} \cos \omega t \tag{6.3}$$

Of course, the sum of the phase currents is nonzero (except where it crosses the ωt-axis) and we need a 'return' line (the neutral line) connected to the same 0-V potential as voltage source v_a. While these conclusions may seem obvious for a single-phase circuit, we will see that they do not necessarily apply to higher-phase circuits.

In a **two-phase circuit** such as in Figure 6.3(b), we have two impedances, $R_a = R_b = R$. The impedances' respective voltages are $v_a(t)$ and $v_b(t)$, whose magnitudes are both $|V|$ and whose phases are $\pi/2$ radians out-of-phase relative to each other:

$$\begin{aligned} v_a(t) &= |V| \cos \omega t \\ v_b(t) &= |V| \cos(\omega t + \pi/2) \end{aligned} \tag{6.4}$$

> **In a two-phase circuit, why are the phase voltages out-of-phase by $\pi/2$ radians?**
> Looking at a three-phase circuit and trying to simplify it to two phases, one might expect the two phase voltages to be out-of-phase by π radians such that they sum to zero. However, this is not what we mean when we talk about two-phase circuits. The two phase voltages being out-of-phase by π radians would just be a version of the single-phase circuit in which $|V_a|$ is replaced by $|V|/2$ and the 0-V potential is replaced by $-|V|/2$.

The total power delivered in the two-phase circuit is the sum of power delivered to both loads, as follows. Note that it is no longer sinusoidal,

as in was in the single-phase circuit, but *constant*. In fact, the time-average total power delivered by any circuit with $n \geq 2$ phases is always constant, making such circuits strong contenders for being able to deliver a steady supply of total power in a power system.

$$\begin{aligned} p_{\text{tot.}}(t) &= p_a(t) + p_b(t) \\ &= \frac{V_a^2 + V_b^2}{R} \\ &= \frac{(|V|\cos\omega t)^2 + \left(|V|\cos\left(\omega t + \frac{\pi}{2}\right)\right)^2}{R} \\ &\text{Apply Pythagorean identity, } \sin^2 x + \cos^2 x = 1: \\ &= \frac{|V|^2((\cos\omega t)^2 + (-\sin\omega t)^2)}{R} \\ &= \underbrace{\frac{|V|^2}{R}}_{\bar{p}_{\text{tot.}}} \end{aligned} \tag{6.5}$$

Just as in the single-phase circuit, the sum of the phase currents in the two-phase circuit is nonzero and a neutral line is indeed needed:

$$\begin{aligned} i_{\text{tot.}}(t) &= i_a(t) + i_b(t) \\ &= |V|\cos\omega t + |V|\cos\left(\omega t + \frac{\pi}{2}\right) = \sqrt{2}\,\frac{|V|}{R}\cos\left(\omega t + \frac{\pi}{4}\right) \end{aligned} \tag{6.6}$$

In a **three-phase circuit** such as in Figure 6.3(c), the three phase voltages are $2\pi/3$ radians out-of-phase relative to each other:

$$\begin{aligned} v_a(t) &= |V|\cos\omega t \\ v_b(t) &= |V|\cos(\omega t + 2\pi/3) \\ v_c(t) &= |V|\cos(\omega t - 2\pi/3) \end{aligned} \tag{6.7}$$

Just as in the two-phase circuit, the total power delivered in the three-phase circuit is constant:

$$p_{\text{tot.}}(t) = p_a(t) + p_b(t) + p_c(t) = \underbrace{\frac{3}{2}\frac{|V|^2}{R}}_{\bar{p}_{\text{tot.}}} \tag{6.8}$$

However, unlike the single- and two-phase circuits, the sum of the phase currents in the three-phase circuit is zero at all times (as long as the loads are balanced). Thus, no neutral line is needed. In fact, no neutral line is needed for any circuit with $n \geq 3$ phases. This means that a three-phase circuit can deliver 3 times as much power as a single-phase circuit using only 1.5 times as many lines and only 1.5 times as much conductor material.

Indeed, in **circuits with more than three phases**, such as in Figure 6.3 (d) and (e), the time-average total power delivered is constant and no neutral line is needed—the same as for a three-phase circuit. Circuits with $n > 3$ phases have no advantage over three-phase circuits and are avoided because they need more conductors that would have to

n	Min. # of lines	Total power, $p_{\text{tot.}}(t)$	Avg. total power, $\bar{p}_{\text{tot.}}$	Avg. total power per line
1	2	Sinusoidal (Eq. (6.2))	$= \lvert V\rvert^2 / 2R$	$= \lvert V\rvert^2 / 4R$
2	3	Constant	$= \lvert V\rvert^2 / R$	$= \lvert V\rvert^2 / 3R$
3	3	Constant	$= 3\lvert V\rvert^2 / 2R$	$= \lvert V\rvert^2 / 2R$
4	4	Constant	$= 2\lvert V\rvert^2 / R$	$= \lvert V\rvert^2 / 2R$
Any $n \geq 3$	n	Constant	$= n\lvert V\rvert^2 / 2R$	$= \lvert V\rvert^2 / 2R$

Table 6.1: Summary of circuits with different numbers of phases, in terms of the minimum number of lines needed, whether the total power is sinusoidal or constant, the time-average total power, and the time-average total power per line.

be installed and maintained in the power system (even if they need the same number of conductors per unit power). Thus, three-phase power is the clear winner out of the candidates tabulated in Table 6.1.

Wye Versus Delta Configurations

So far, we have assumed three-phase generators and loads to be in the Y configuration (pronounced, and often written as, "wye"). Also called the "star" configuration, the wye configuration includes a 0-V neutral node which is normally grounded and will be denoted n. Each of the three-phase lines a, b, and c are indirectly connected to n. In a three-phase generator following the wye configuration, each of the three lines is connected to node n through a voltage source v_i, resulting in the three phase voltages $v_i(t)$. In a wye-configured three-phase load, each of the three lines is connected to node n through an impedance Z. If the load is unbalanced, a fourth, neutral line is required to carry extra current at node n of the load back to node n of the generator. The different cases are illustrated in Figure 6.4.

There is a second three-phase configuration called Δ, or delta. In the delta configuration, components—whether voltage sources or impedances—are not connected between each line and neutral. Instead, components are connected across each pair of lines: (a, b), (b, c), and (c, a). There is no node n and no possibility of a neutral line, so loads following the delta configuration must be balanced. The voltage across each component is the difference between the voltages of the two lines to which it is connected, e.g., $v_{ba}(t) = v_b(t) - v_a(t)$. The magnitude of the voltage across each component is $\sqrt{3}|V|$, or $\sqrt{3}$ times as much that of a wye-connected counterpart.

Both wye- and delta-configured generators and loads can be connected to the same three-phase circuit.

What about transformers? A transformer is like a generator and a load, simultaneously. A regular transformer has two windings—primary and secondary. A three-phase transformer typically has two sets of three windings apiece. If we're talking about a transformer in a grid substation, whose role is to step down voltage levels, then the primary set of windings acts as a load on the high-voltage side, and the secondary set acts as a generator for the low-voltage side. A three-phase transformer can have two wye-configured sets of windings, two

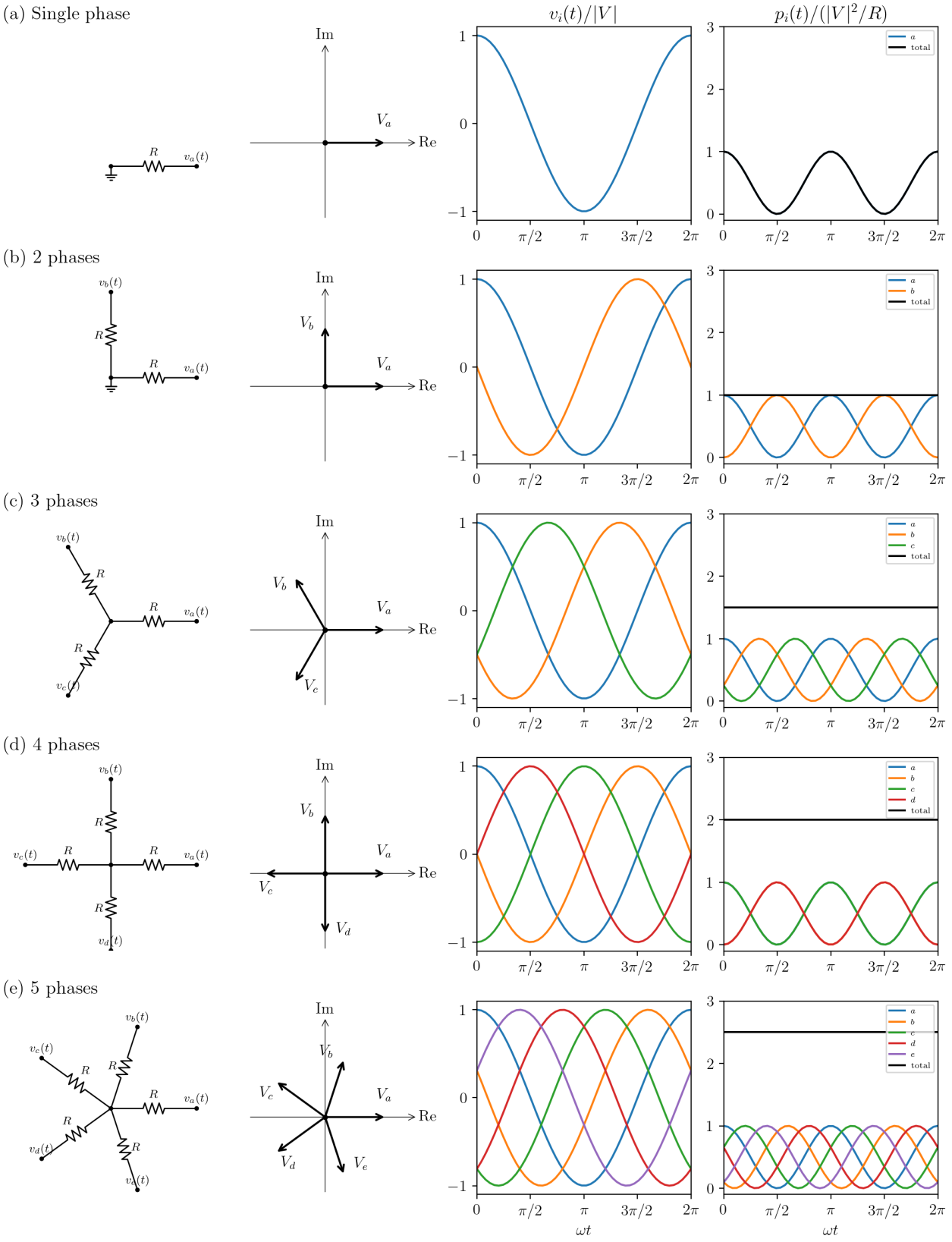


Figure 6.3: Comparison of n-phase systems.

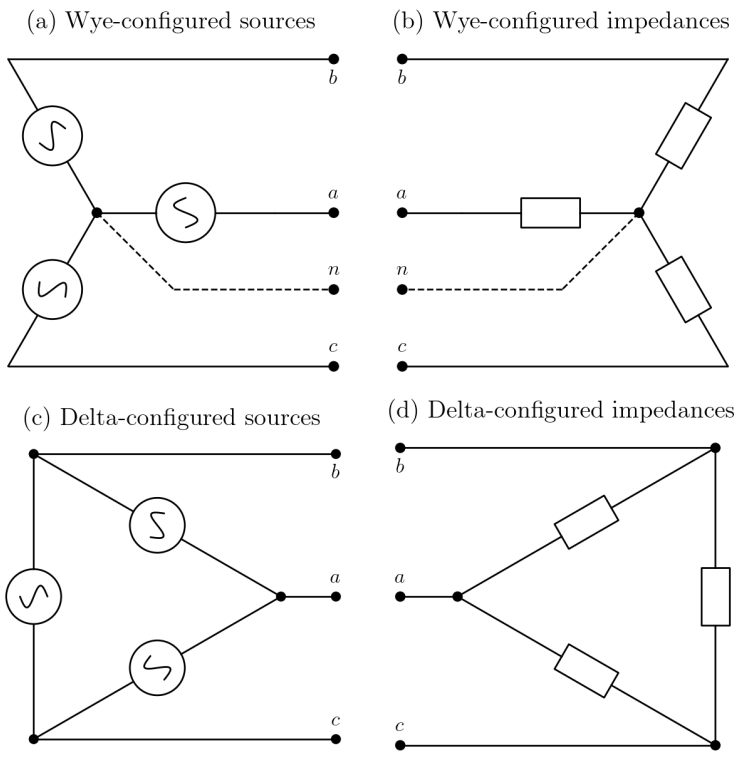


Figure 6.4: Wye- and delta-configured sources and impedances.

delta-configured sets of windings, or one wye-configured set and one delta-configured set, as illustrated in Figure 6.5. As can be seen in Figure 6.5, a three-phase transformer can be constructed from three single-phase transformers.

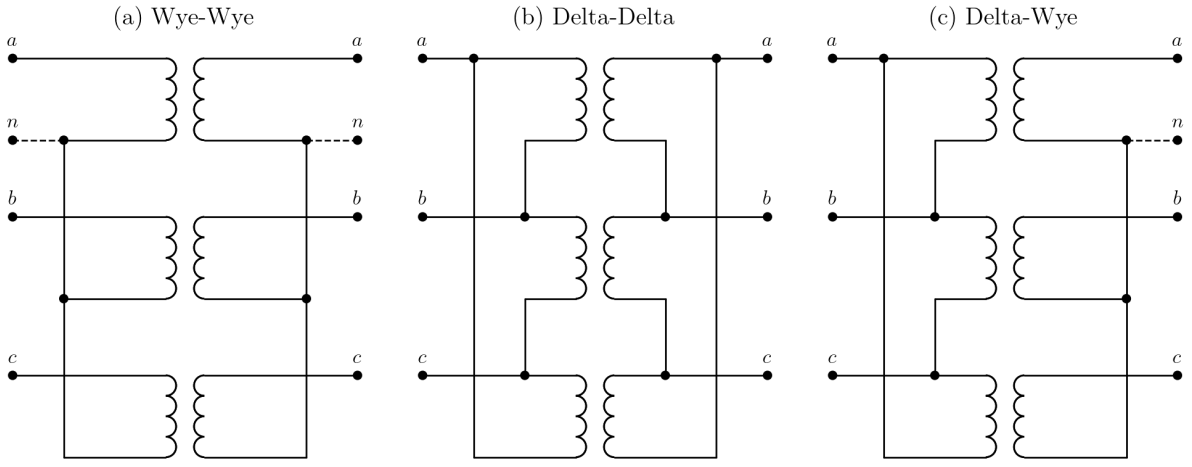


Figure 6.5: Three-phase transformer configurations.

In a delta-wye transformer, $\sqrt{3}$ times the phase voltage on the primary side is converted to the phase voltage on the secondary side. In a wye-delta transformer, the inverse is true: the primary side's phase voltage is converted to $\sqrt{3}$ times the secondary side's phase voltage. Thus, even with the same number of primary windings as secondary windings, a delta-wye or wye-delta transformer steps voltage up or down, respectively, by a factor of $\sqrt{3} \approx 1.73$.

A transformer provides isolation between the primary and secondary sides, allowing a neutral line to exist on a wye-configured set of windings while being absent on the other set of windings, as shown in Figure 6.5(a) and (c).

Three-Phase Power in the Grid

Most appliances—even high-power appliances like ovens, clothes dryers, and many residential EV chargers—use single-phase power. Thus, most residential and commercial buildings are supplied with single-phase power. How is this possible when the rest of the grid is three-phase? In reality, a building's single-phase supply represents one part of a larger, three-phase load formed at the distribution level of the grid. Each of these high-level three-phase loads includes the electrical supply to multiple buildings.

Efforts are made to design the topology of the grid such that the impedances between the three branches are balanced. Unbalanced impedances would mean that the lines and transformers that feed them are underutilized. However, there is only so much that can be done. The impedance on a given branch can change at the press of a button—when someone turns on their clothes dryer, for example. Thus, the wye configuration, with a neutral line, must be used.

As we move up in the distribution grid, transformers are encountered to convert between voltage levels.

Chapter 7

Grid Modeling

If the example circuits we've seen so far seem complicated to analyze, then this pales in comparison to the complexity of analyzing and operating the electrical grid. The electrical grid may be just another circuit, but it has rapid, far-reaching, and expensive consequences if it fails. If the grid is operated incorrectly, its voltages and frequencies can be put in jeopardy in a matter of seconds or less, creating a risk of cascading failure. Understanding the grid and how it behaves depending on how it's operated is crucial to be able to control it correctly. Grid modeling allows operators to not only stay in control of grid voltages and frequencies, but also to dispatch generators in a way that is cost-optimal.

The grid is modeled as a network of N electrical buses connected by L branches. [Pow, b] A bus is a shared point of connection that has minimal electrical impedance. Generators and loads are attached to the buses. Each pair of buses (i, k)[1] may be connected by a branch—such as a power line or transformer—which has nonzero impedance and can result in voltage and phase differences between buses i and k. In the grid, each bus has a specific nominal voltage, typically 4 kV to 765 kV per phase depending on whether it is in the transmission or distribution part of the grid. The sets of buses and branches are denoted $\mathcal{N}$ and $\mathcal{L}$, respectively.

For example, consider Figure 7.1. It shows a simple electrical grid with three buses $\mathcal{N} = \{1, 2, 3\}$, two generators (at buses 1 and 2), and a load (at bus 3). Buses 1 and 2 are connected by a line branch, and buses 2 and 3 are connected by a transformer branch. This type of diagram is called a *single-line diagram* because it depicts the parallel conductors in a single-phase system or balanced three-phase system—like the electrical grid—as single lines for the sake of simplicity.

Figure 7.2 shows the circuit diagram that corresponds to the above single-line diagram for a three-phase system. Each bus in the single-line diagram corresponds to a set of three nodes in the circuit (one per phase).

Each branch ik has a known series impedance z_{ik} and shunt impedance z_{ik}^{Sh} according to the Π (Pi) branch model. In grid modeling, these pa-

[1] Index k is used instead of j to distinguish it from the imaginary unit.

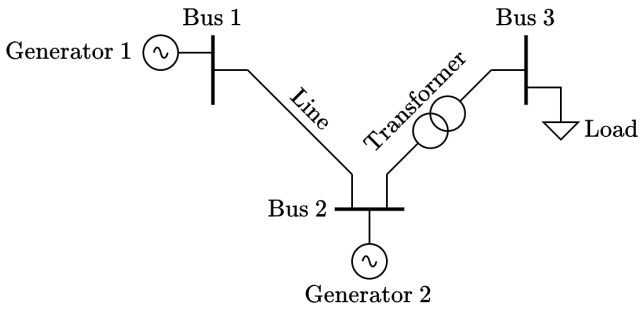


Figure 7.1: Single-line diagram of an example electrical grid. (The symbol for "Load" should not be confused with the symbol for electrical ground.)

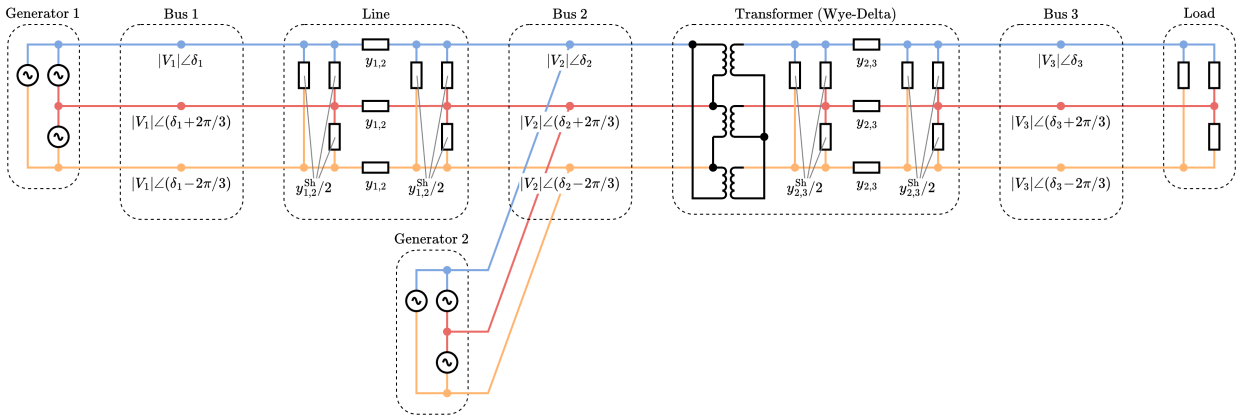


Figure 7.2: Circuit diagram of the example electrical grid in Figure 7.1.

rameters are often expressed as *admittance*: series admittance $y_{ik} = 1/z_{ik}$ and shunt admittance $y_{ik}^{\text{Sh}} = 1/z_{ik}^{\text{Sh}}$. The Π branch model applies the shunt admittance in equal parts on the left and right sides, as shown in Figure 7.3. This results in a single/per-phase circuit diagram that looks like the Greek letter Π, hence the name.

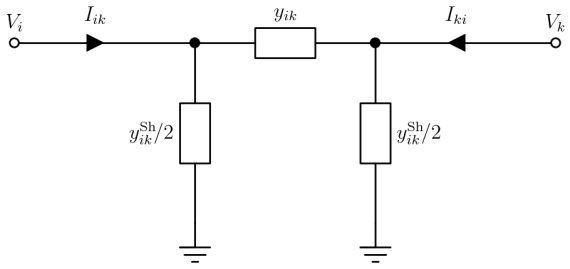


Figure 7.3: Per-phase power line model (the Π branch model).

Transformer branches build upon this in order to model nonideal transformers. Transformer branches are modeled as ideal transformers paired with the Π branch model, as shown in Figure 7.4. A complex-valued voltage ratio a_{ik} is used to model both the transformer's voltage ratio $T_{ik} = |a_{ik}|$ and, in the case of a phase-shifting transformer such as a zigzag transformer, its phase shift $\varphi_{ik} = \arg(a_{ik})$. With the ideal transformer adjacent to bus i, bus i is known as the *tap bus* and bus k as the *impedance bus* or *Z bus*.

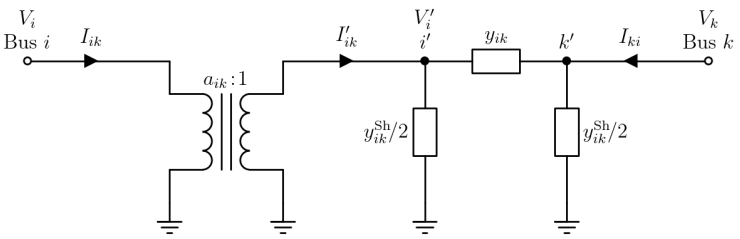


Figure 7.4: Per-phase transformer model (the Π branch model + ideal transformer).

The Power Flow (PF) Problem

At any given time, a grid operator must dispatch generators to serve the loads in its grid. Each load l consists of a given active power P_l and reactive power Q_l. A dispatch specifies the active power P_g at which to operate each generator g. Once a grid operator has decided the generator setpoints, they must be able to validate those setpoints subject to how power will flow through the grid. The grid operator must do this at regular operating intervals (typically 5-minute or 1-hour intervals) because the predicted load and available generation change over time. This consists of:

1. Validating that power generation will satisfy the loads.
2. Validating that not too much power is flowing through a given line or transformer. This is to avoid overloading/overheating it.
3. Validating that voltages are within acceptable margins. This is for the sake of the loads.

The task of determining how power will flow through the grid and whether it will satisfy the loads is known as the power flow (PF) problem [Mones, 2020, Pow, a].

Because the grid is a circuit, we can solve the power flow problem the same as we solve any other circuit, by doing nodal analysis and solving a system of KCL equations to determine the voltage at each bus, denoted $V_i = |V_i|\angle\delta_i$. In the three-phase case, the voltage magnitude $|V_i|$ can be assumed to be the same across the three phases at each bus, and the voltage angle δ_i can be taken for the first phase only, knowing that the other two phases will be $\pm 2\pi/3$ radians apart.

For generality, we will model all branches as transformer branches. For line branches, we can simply set $T_{ik} = 1$ and $\varphi_{ik} = 0$. We will call branch ik an *i-forward branch* if bus i is the tap bus, or an *i-reverse branch* if bus i is the impedance bus. In this sense, the grid model becomes a directed graph rather than an undirected graph, and we must account for the forward and reverse branch directions distinctly. We do this by deriving the current I_{ik} flowing into branch ik from bus i (that is, flowing directly into the ideal transformer), as well as the current I_{ki} flowing into the branch from the opposite bus k. To determine I_{ik}, we apply KCL by summing the currents out of node i' in Figure 7.4:

$$\begin{aligned}
&-I'_{ik} + V'_i \frac{y_{ik}^{\text{Sh}}}{2} + (V'_i - V_k)\, y_{ik} = 0 \\
&\implies I'_{ik} = V'_i \left(\frac{y_{ik}^{\text{Sh}}}{2} + y_{ik}\right) - V_k\, y_{ik} \\
&\implies I_{ik} a_{ik}^* = V_i \frac{1}{a_{ik}} \left(\frac{y_{ik}^{\text{Sh}}}{2} + y_{ik}\right) - V_k\, y_{ik} \\
&\implies I_{ik} = V_i \frac{1}{|a_{ik}|^2} \left(\frac{y_{ik}^{\text{Sh}}}{2} + y_{ik}\right) - V_k \frac{1}{a_{ik}^*} y_{ik}
\end{aligned} \tag{7.1}$$

And to determine I_{ki}, we apply KCL by summing the currents out of node k':

$$\begin{aligned}
&-I_{ki} + (V_k - V'_i)\, y_{ik} + V_k \frac{y_{ik}^{\text{Sh}}}{2} = 0 \\
&\implies I_{ki} = V_k \left(y_{ik} + \frac{y_{ik}^{\text{Sh}}}{2}\right) - V'_i\, y_{ik} \\
&\implies I_{ki} = V_k \left(y_{ik} + \frac{y_{ik}^{\text{Sh}}}{2}\right) - V_i \frac{1}{a_{ik}} y_{ik}
\end{aligned} \tag{7.2}$$

The Bus Injection Model

To define the power flow problem, we need to formulate a set of power flow equations. The most common formulation is known as the *bus injection model*. We start with KCL, which tells us that at a bus i with nominal voltage V_i, the current $(S_i/V_i)^*$ injected due to attached generation and/or load S_i must equal the sum of currents flowing out of the bus:

$$\begin{aligned}(S_i/V_i)^* &= \text{total current out of bus } i \text{ via } i\text{-forward branches} \\ &\quad + \text{total current out of bus } i \text{ via } i\text{-reverse branches} \\ &= \sum_{k:(i,k)\in\mathcal{L}} I_{ik} + \sum_{k:(k,i)\in\mathcal{L}} I_{ik}\end{aligned} \tag{7.3}$$

The subscript "$k:(i,k)\in\mathcal{L}$" means "for each bus k to which bus i is connected by an i-forward branch" and the subscript "$k:(k,i)\in\mathcal{L}$" means "for each bus k to which bus i is connected by an i-reverse branch". Substituting I_{ik} and I_{ki} from equations (7.1) and (7.2) gives us:

$$\begin{aligned}(S_i/V_i)^* = &\sum_{k:(i,k)\in\mathcal{L}} \left(V_i \frac{1}{|a_{ik}|^2}\left(\frac{y_{ik}^{\text{Sh}}}{2} + y_{ik}\right) - V_k \frac{1}{a_{ik}^*} y_{ik} \right) \\ &+ \sum_{k:(k,i)\in\mathcal{L}} \left(V_i \left(\frac{y_{ki}^{\text{Sh}}}{2} + y_{ki}\right) - V_k \frac{1}{a_{ki}} y_{ki} \right)\end{aligned} \tag{7.4}$$

> **Note**
> Branch ik is the same as a branch ki, admittance y_{ik} is the same as y_{ki}, and transformer ratio a_{ik} is the same as a_{ki}; the order of the subscript simply indicates whether the bus i for which the KCL equation is written is considered the start or end of the branch.

Splitting up the summations and rearranging:

$$\begin{aligned}(S_i/V_i)^* = V_i &\left(\sum_{k:(i,k)\in\mathcal{L}} \frac{1}{|a_{ik}|^2}\left(\frac{y_{ik}^{\text{Sh}}}{2} + y_{ik}\right) + \sum_{k:(k,i)\in\mathcal{L}} \left(\frac{y_{ki}^{\text{Sh}}}{2} + y_{ki}\right)\right) \\ &- \sum_{k:(i,k)\in\mathcal{L}} V_k \frac{1}{a_{ik}^*} y_{ik} - \sum_{k:(k,i)\in\mathcal{L}} V_k \frac{1}{a_{ki}} y_{ki}\end{aligned} \tag{7.5}$$

The bus injection model is typically expressed in a way that allows some complexity to be moved into a new $N{\times}N$ matrix Y, called the *bus admittance matrix*, which allows the above equation to be rewritten succinctly as:

$$(S_i/V_i)^* = \sum_{k\in\mathcal{N}} V_k \, Y_{ik} \tag{7.6}$$

Or, even more simply, as a matrix equation:

$$S = (YV)^* \circ V \tag{7.7}$$

where "$\circ$" indicates element-wise vector multiplication and Y is defined as having the following diagonal and off-diagonal elements:

$$\begin{aligned} Y_{ii} &= \sum_{k:(i,k)\in\mathcal{L}} \frac{1}{|a_{ik}|^2}\left(\frac{y_{ik}^{\text{Sh}}}{2} + y_{ik}\right) + \sum_{k:(k,i)\in\mathcal{L}}\left(\frac{y_{ki}^{\text{Sh}}}{2} + y_{ki}\right) \\ Y_{ik} &= \begin{cases} -y_{ik}/a_{ik}^* & \text{if branch } ik \text{ is an } i\text{-forward branch} \\ -y_{ki}/a_{ki} & \text{if branch } ik \text{ is an } i\text{-reverse branch} \\ 0 & \text{if branch } ik \text{ does not exist (no branch)} \end{cases} \end{aligned} \tag{7.8}$$

> **Note**
> Although y_{ik} is the same as y_{ki}, Y_{ik} is not necessarily equal to Y_{ki}.

Thus, the admittance matrix does not only specify the admittance values between buses in its off-diagonal elements. Its off-diagonal elements additionally specify whether a branch is present between any ik-pair of buses ($Y_{ik} \neq 0$), whether a branch is a transformer ($a_{ik} \neq 1$), and its voltage ratio if it is a transformer ($T_{ik} \neq 1$), and even its phase shift if it is also a phase-shifting transformer ($\varphi_{ik} \neq 0$). Furthermore, the diagonal elements specify, for each bus, the shunt admittances of neighboring branches.

Just as how a branch admittance y can be split into real and imaginary parts $g + jb$, where g is conductance[2] and b is susceptance, the admittance matrix Y can be split into $G + jB$, which we will leverage shortly.

The bus injection equations are typically arranged as equations for active and reactive power $P + jQ = S$. To describe the power at a given bus i, we take the conjugate of Equation (7.6) and multiply both sides by V_i:

$$P_i + jQ_i = \sum_{k\in\mathcal{N}} V_i V_k^* Y_{ik}^* \tag{7.9}$$

Solver software often expects real-valued equations, so we work towards splitting this equation into a real part and an imaginary part. Furthermore, the bus injection model most commonly uses polar coordinates for voltage and rectangular coordinates for admittance. To satisfy this, we substitute $V = |V|e^{j\delta} = |V|\angle\delta$ and $Y = G + jB$ in the above equation, giving us the following. We will eventually be able to take the imaginary unit j out of the picture.

[2] Not to be confused with generator index g.

$$\begin{aligned} P_i + jQ_i &= \sum_{k\in\mathcal{N}} \left(|V_i|\angle\delta_i\right)\left(|V_k|\angle-\delta_k\right)\left(G_{ik} - jB_{ik}\right) \\ &= \sum_{k\in\mathcal{N}} \left[|V_i|\,|V_k|\angle\left(\delta_i - \delta_k\right)\right]\left(G_{ik} - jB_{ik}\right) \\ &= \sum_{k\in\mathcal{N}} |V_i|\,|V_k|\left(\cos(\delta_i - \delta_k) + j\sin(\delta_i - \delta_k)\right)\left(G_{ik} - jB_{ik}\right) \end{aligned} \tag{7.10}$$

Expanding yields:

$$P_i + jQ_i = |V_i| \sum_{k\in\mathcal{N}} |V_k| \begin{pmatrix} G_{ik}\cos(\delta_i - \delta_k) - jB_{ik}\cos(\delta_i - \delta_k) \\ + jG_{ik}\sin(\delta_i - \delta_k) + B_{ik}\sin(\delta_i - \delta_k) \end{pmatrix} \tag{7.11}$$

Finally, we are able to split this complex-valued equation into the following real-valued *power flow equations*, for each bus i. These are suitable for use with solver software.

$$\boxed{\begin{aligned} P_i &= |V_i| \sum_{k\in\mathcal{N}} |V_k| \left(G_{ik}\cos(\delta_i - \delta_k) + B_{ik}\sin(\delta_i - \delta_k)\right) \\ Q_i &= |V_i| \sum_{k\in\mathcal{N}} |V_k| \left(G_{ik}\sin(\delta_i - \delta_k) - B_{ik}\cos(\delta_i - \delta_k)\right) \end{aligned}} \tag{7.12}$$

For a branch l from bus i to bus k, the power flowing into the branch from bus i is:

$$\begin{aligned} P_{ik} &= |V_i|\,|V_k|\left(G_{ik}\cos(\delta_i - \delta_k) + B_{ik}\sin(\delta_i - \delta_k)\right) + |V_i|^2 G^{\mathrm{f}}_{l,i} \\ Q_{ik} &= |V_i|\,|V_k|\left(G_{ik}\sin(\delta_i - \delta_k) - B_{ik}\cos(\delta_i - \delta_k)\right) + |V_i|^2 B^{\mathrm{f}}_{l,i} \end{aligned} \tag{7.13}$$

And the power flowing into the branch from bus k is:

$$\begin{aligned} P_{ki} &= |V_k|\,|V_i|\left(G_{ki}\cos(\delta_k - \delta_i) + B_{ki}\sin(\delta_k - \delta_i)\right) + |V_k|^2 G^{\mathrm{t}}_{l,k} \\ Q_{ki} &= |V_k|\,|V_i|\left(G_{ki}\sin(\delta_k - \delta_i) - B_{ki}\cos(\delta_k - \delta_i)\right) + |V_k|^2 B^{\mathrm{t}}_{l,k} \end{aligned} \tag{7.14}$$

Where G^{f}, B^{f}, G^{t}, and B^{t} are the real and imaginary components of the branch-from-bus admittance matrix Y^{f} and the branch-to-bus admittance matrix Y^{t}, detailed in this chapter's appendix.

Applying the Per-Unit System

The per-unit system is the practice in electrical engineering of normalizing quantities like voltage and power to a dimensionless value between 0 and 1. Each quantity x is expressed as a fraction, denoted x^{pu}, of some base quantity, x^{base}, such that $x = x^{\mathrm{pu}}x^{\mathrm{base}}$. This can be done with many electrical engineering quantities:

$$\begin{gathered} V = V^{\mathrm{pu}}V^{\mathrm{base}} \qquad S = S^{\mathrm{pu}}S^{\mathrm{base}} \qquad I = I^{\mathrm{pu}}I^{\mathrm{base}} \\ y = y^{\mathrm{pu}}y^{\mathrm{base}} \qquad Z = Z^{\mathrm{pu}}Z^{\mathrm{base}} \end{gathered} \tag{7.15}$$

For example, in the per-unit system with a base of $S^{\text{base}} = 100$ MVA, a quantity of $S = 90$ MVA would become $S^{\text{pu}} = 0.9$ per unit or 0.9 pu.

The per-unit system makes it easier to interpret quantities relative to the voltage and power ratings of equipment such as buses and generators. And when applied to the power flow problem, the per-unit system offers additional advantages [Per]:

- It improves the problem's stability when solving using numerical methods.
- As we will see, it allows most transformer branches to be treated simply as line branches because the transformer voltage ratio becomes $1:1$ in the per-unit system.

To express the power flow equations in the per-unit system, we select the nominal bus voltage as the base voltage V_i^{base} at each bus i, and an arbitrary value S^{base} as the base power everywhere. We substitute $S_i^{\text{pu}} S^{\text{base}}$ for S_i and $V_i^{\text{pu}} V_i^{\text{base}}$ for V_i in Equation (7.5) as follows. We also split the transformer voltage ratio a_{ik} into nominal voltage ratio $V_i^{\text{base}}/V_k^{\text{base}}$ (which we denote a_{ik}^{base}) times an off-nominal factor (which we denote a_{ik}^{pu}). It is not typical to represent transformer voltage ratios in the per-unit system as such—they are dimensionless quantities to begin with—but we use the notation $a_{ik} = a_{ik}^{\text{pu}} a_{ik}^{\text{base}}$ nonetheless for consistency.

$$\begin{aligned}
&\left(\frac{S_i^{\text{pu}} S^{\text{base}}}{V_i^{\text{pu}} V_i^{\text{base}}}\right)^* \\
&= V_i^{\text{pu}} V_i^{\text{base}} \left(\sum_{k:(i,k)\in\mathcal{L}} \frac{1}{|a_{ik}^{\text{pu}} a_{ik}^{\text{base}}|^2} \left(\frac{y_{ik}^{\text{Sh}}}{2} + y_{ik}\right) + \sum_{k:(k,i)\in\mathcal{L}} \left(\frac{y_{ki}^{\text{Sh}}}{2} + y_{ki}\right)\right) \\
&- \sum_{k:(i,k)\in\mathcal{L}} V_k^{\text{pu}} V_k^{\text{base}} \frac{1}{(a_{ik}^{\text{pu}} a_{ik}^{\text{base}})^*} y_{ik} - \sum_{k:(k,i)\in\mathcal{L}} V_k^{\text{pu}} V_k^{\text{base}} \frac{1}{a_{ki}^{\text{pu}} a_{ki}^{\text{base}}} y_{ki}
\end{aligned} \tag{7.16}$$

Substituting $a_{ik}^{\text{base}} = V_i^{\text{base}}/V_k^{\text{base}}$, multiplying both sides by $V_i^{\text{base}}/S^{\text{base}}$ and simplifying:

$$\begin{aligned}
&\left(\frac{S_i^{\text{pu}}}{V_i^{\text{pu}}}\right)^* \\
&= V_i^{\text{pu}} \left(\sum_{k:(i,k)\in\mathcal{L}} \frac{1}{|a_{ik}^{\text{pu}}|^2} \frac{(V_k^{\text{base}})^2}{S^{\text{base}}} \left(\frac{y_{ik}^{\text{Sh}}}{2} + y_{ik}\right) + \sum_{k:(k,i)\in\mathcal{L}} \frac{(V_i^{\text{base}})^2}{S^{\text{base}}} \left(\frac{y_{ki}^{\text{Sh}}}{2} + y_{ki}\right)\right) \\
&- \sum_{k:(i,k)\in\mathcal{L}} V_k^{\text{pu}} \frac{(V_k^{\text{base}})^2}{S^{\text{base}}} \frac{1}{(a_{ik}^{\text{pu}})^*} y_{ik} - \sum_{k:(k,i)\in\mathcal{L}} V_k^{\text{pu}} \frac{(V_k^{\text{base}})^2}{S^{\text{base}}} \frac{1}{a_{ki}^{\text{pu}}} y_{ki}
\end{aligned} \tag{7.17}$$

We now apply the per-unit system to the admittances, defining $y_{ik} = y_{ik}^{\text{pu}} y_{ik}^{\text{base}}$. If we select $S^{\text{base}}/(V_k^{\text{base}})^2$ as the base admittance y_{ik}^{base}, all base terms conveniently cancel out, leaving something that looks exactly like Equation (7.5), but with "pu" scripts:

$$\left(\frac{S_i^{\text{pu}}}{V_i^{\text{pu}}}\right)^* = V_i^{\text{pu}}\left(\sum_{k:(i,k)\in\mathcal{L}} \frac{1}{|a_{ik}^{\text{pu}}|^2}\left(\frac{y_{ik}^{\text{Sh,pu}}}{2} + y_{ik}^{\text{pu}}\right) + \sum_{k:(k,i)\in\mathcal{L}}\left(\frac{y_{ki}^{\text{Sh,pu}}}{2} + y_{ki}^{\text{pu}}\right)\right) - \sum_{k:(i,k)\in\mathcal{L}} V_k^{\text{pu}} \frac{1}{(a_{ik}^{\text{pu}})^*} y_{ik}^{\text{pu}} - \sum_{k:(k,i)\in\mathcal{L}} V_k^{\text{pu}} \frac{1}{a_{ki}^{\text{pu}}} y_{ki}^{\text{pu}} \tag{7.18}$$

When the voltage ratio of a transformer ik is *nominal*—that is, equal to the ratio between the nominal voltage at bus i and the nominal voltage at bus k, then $a_{ik}^{\text{pu}} = 1$ and the equation's terms concerning the transformer branch simplify to those of a line branch. This is because the differing bus voltages $V_i^{\text{base}} \neq V_k^{\text{base}}$ are now accounted for as part of $y_{ki}^{\text{base}} \neq y_{ik}^{\text{base}}$, respectively. The power flow problem is typically expressed in the per-unit system because of this advantage, as well as the advantage of improved numerical stability.

Bus Classifications

Our grid model, expressed by the power flow equations, currently has too many unknown variables to be able to solve it. We know the branch admittances, but we do not yet know how the loads should behave or how the bus voltages can be controlled. We now have a model of the grid, but as-is, we cannot use this model to determine how power will flow through the grid. In this section, we will narrow down our definition of the power flow problem.

Each bus and thus each pair of (P_i, Q_i) equations (7.12) has four variables: voltage magnitude $|V_i|$, voltage angle δ_i, active power P_i, and reactive power Q_i. Each bus contributes two equations, for a total of $2N$ independent equations making up the system of equations. There are $4N$ unique variables, so half of them must be fixed in order for the system of equations to have an exactly determined solution. The other half of the variables must be left as free variables that can take on whatever values are required to satisfy the system of equations. Which variables we fix and which variables we allow to vary depends on the type of each bus, as we will discuss. In order to support different numbers of buses of each type, half of the variables *at each bus* must be fixed and the other half free. At most buses, we don't care about the voltage angle, so we leave δ_i as a free variable.

Load bus. A bus to which only load(s) are attached is known as a load bus. At a load bus, P_i and Q_i are fixed based on the demands of the load (or loads). Although we wish we could fix $|V_i|$ to the exact nominal voltage of the bus for the sake of the attached load(s), this is not generally possible without making the system of equations under- or over-determined. Because of this, loads generally accept a voltage range, and system operators are required to keep $|V_i|$ within an even narrower range. Because only variables P_i and Q_i are fixed at a load bus, a load bus is also known as a *PQ bus*.

Generator bus. A bus to which only generator(s) are attached is known as a generator bus. At a generator bus, P_i is fixed based on the generator setpoints and $|V_i|$ is fixed to the bus nominal voltage. Q_i is allowed to vary at a generator bus to whatever value satisfies the system. Because only variables P_i and $|V_i|$ are fixed at a generator bus, a generator bus is also known as a *PV bus* or *voltage-controlled bus*.

Bus type	P_i	Q_i	V_i	δ_i
Load/PQ	Fixed by load	Fixed by load	Free	Free
Generator/PV	Fixed by dispatch	Free	Fixed by grid operating requirement	Free
Slack/Vδ	Free	Free	Fixed by grid operating requirement	Fixed to 0° reference

Table 7.1: Bus types and their variable classifications.

What if a bus has both generator(s) and load(s) attached? The net active power P_i must be fixed because the active powers of both the generator(s) and loads(s) are fixed. The net reactive power Q_i must be free to vary because, while the reactive powers of the load(s) are fixed, those of the generator(s) are not. And $|V_i|$ is fixed, just as in a generator bus. Indeed, if we were to model a combined generator/load bus as a separate generator bus i and a separate load bus k, connected by a line ik with zero impedance, then $|V_i|$ would end up equalling $|V_k|$ anyway. So, a combined generator/load bus is treated as a generator bus.

And what if a bus has neither generators nor loads attached? In this case, the bus is treated as a load bus with fixed $P_i = 0$ and $Q_i = 0$.

Slack bus. So far, the classifications of variables as fixed versus free means that the number of free variables equals the number of equations. However, this does not necessarily mean that the system of equations has a solution that is feasible or that it has a solution that is unique. Indeed, with only the two bus types we've defined so far, there is a feasibility issue and a uniqueness issue. Firstly, active power losses due to electrical resistance in the power lines and transformers mean that the total power supplied by the generators does not necessarily equal the total power demanded by the loads plus the power losses. The deficit must be made up for by one or more generators at one or more buses, at which P must thus be free to vary. Otherwise, there would not necessarily be a feasible solution. Secondly, voltage angles are relative, so at one bus, δ_i must be fixed to 0° as a reference against which all other δ_is are measured. Otherwise, there would be no unique solution. By convention, both the feasibility issue and the uniqueness issue are solved by the same bus. This third type of bus is known as a slack bus, also known as a *swing bus*, *reference bus*, or *Vδ bus*.

Table 7.1 summarizes the bus types and their variable classifications.

As an example of how these bus classifications would apply, consider the electrical grid of Figure 7.5. Buses 2 and 3 are both load buses (even though bus 2 has no attached load). Either of buses 1 and 4 can be a generator bus (even though bus 4 has an attached load), but one of them must be a slack bus.

Power Flow Validation

After solving the power flow problem, a grid operator is able to validate their generator dispatch according to the three requirements listed under The Power Flow (PF) Problem:

1. Validating that power generation will satisfy the loads. More

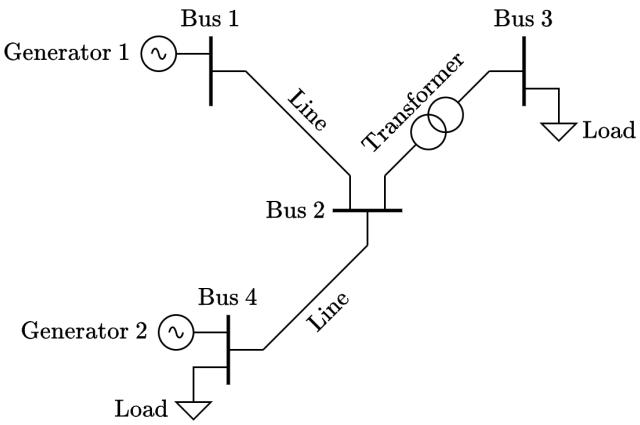


Figure 7.5: Single-line diagram of another example electrical grid.

specifically, validating that the slack generation—which is free to vary to whatever value is required to satisfy the loads—is within the maximum power of slack generator g:

$$S_g \leq S_g^{\max} \; \forall \; g \in \mathcal{G} \tag{7.19}$$

2. Validating that not too much power is flowing into or out of a branch (a line or transformer):

$$|S_{ik}| \leq S_l^{\max}, \; |S_{ki}| \leq S_l^{\max} \; \forall \; (i,k) \in \mathcal{L} \tag{7.20}$$

We refer to this as the *branch flow constraint*.

3. Validating that voltages are within acceptable margins:

$$V_i^{\min} \leq |V_i| \leq V_i^{\max} \; \forall \; i \in \mathcal{N} \tag{7.21}$$

The branch flow constraint's $|S_{ik}|$ is derived as follows:

$$(S_{ik}/V_i)^* = V_i Y_{l,i}^{\mathrm{f}} - V_k Y_{l,k}^{\mathrm{f}} \tag{7.22}$$

$$S_{ik} = V_i \left(V_i Y_{l,i}^{\mathrm{f}} - V_k Y_{l,k}^{\mathrm{f}}\right)^* \tag{7.23}$$

$$
\begin{aligned}
&|S_{ik}| \\
&= |V_i|\,|V_i Y^{\mathrm{f}}_{l,i} - V_k Y^{\mathrm{f}}_{l,k}| \\
&= |V_i| \left| \begin{array}{l} |V_i|\,|Y^{\mathrm{f}}_{l,i}| \left(\cos(\delta_i + \arg(Y^{\mathrm{f}}_{l,i})) + j\sin(\delta_i + \arg(Y^{\mathrm{f}}_{l,i}))\right) \\ -|V_k|\,|Y^{\mathrm{f}}_{l,k}| \left(\cos(\delta_k + \arg(Y^{\mathrm{f}}_{l,k})) + j\sin(\delta_i + \arg(Y^{\mathrm{f}}_{l,k}))\right) \end{array} \right| \\
&= |V_i| \sqrt{\begin{array}{l} \left[|V_i|\,|Y^{\mathrm{f}}_{l,i}|\cos(\delta_i + \arg(Y^{\mathrm{f}}_{l,i})) + |V_k|\,|Y^{\mathrm{f}}_{l,k}|\cos(\delta_k + \arg(Y^{\mathrm{f}}_{l,k}))\right]^2 \\ + \left[|V_i|\,|Y^{\mathrm{f}}_{l,i}|\sin(\delta_i + \arg(Y^{\mathrm{f}}_{l,i})) + |V_k|\,|Y^{\mathrm{f}}_{l,k}|\sin(\delta_i + \arg(Y^{\mathrm{f}}_{l,k}))\right]^2 \end{array}}
\end{aligned}
\tag{7.24}
$$

The same is done for $|S_{ki}|$ by replacing Y^{f} with Y^{t}.

The Economic Dispatch (ED) Problem

The power flow problem is intended to determine *whether* generator setpoints are valid, but it alone does not help determine *what* generator setpoints to use in the first place. When a grid operator decides how to dispatch generators, their goal—within the requirement of satisfying the grid's loads—is to minimize their operating cost incurred during a given operating interval. Different types of generators have different costs per MWh of electrical energy; for example, operating a hydroelectric generator is less expensive than fuel for a natural-gas-powered generator. Furthermore, a given generator g can also have a cost per MWh that increases with the setpoint P_g, in MW, at which the generator is operated. Thus, each generator has its own cost function $C_g(P_g)$, and the total operating cost for a given interval of time depends on both on *which* generators are dispatched, and *what setpoint* each of those generators is operated at.

The grid operator can achieve their goal of minimizing their operating cost by solving an optimization problem in which the objective function—to be minimized—is the total operating cost, and the decision variables are the generator setpoints. This optimization problem is what is known as the economic dispatch (ED) problem:

$$
\boxed{\min \sum_{g \in \mathcal{G}} C_g(P_g)} \tag{7.25}
$$

How the cost function C_g is formulated depends on the application. If the application is research or planning purposes, then a quadratic fit is typically used to approximately model how the cost per MWh increases as P_g increases. For day-to-day and real-time dispatch by the grid operator, a piecewise linear function is often used for greater accuracy.

So far, we have assumed that all generators are dispatchable (controllable power) and that all loads are non-dispatchable (uncontrollable power). There do exist non-dispatchable generators whose power production *cannot* be controlled, as well as dispatchable loads (such as smart thermostats and smart EV charging in some cases) whose power consumption *can* be controlled by the grid operator. [Biddle, 2024] However, this does not mean that the grid model discussed so

far cannot represent these resources. Non-dispatchable generators can be modeled as loads whose fixed power is positive rather than negative. As for dispatchable loads, they can be modeled as generators whose maximum power is negative rather than positive, and with cost functions that intake and output negative values. This negative cost—associated with consuming, rather than producing—is called *utility* in economics. Utility quantifies the economic value delivered to consumers (loads) when they are dispatched by the grid operator, in contrast to the cost incurred by producers (generators) by being dispatched.[3]

The Optimal Power Flow (OPF) Problem

Not just any solution to the economic dispatch problem will do. Operating costs cannot be minimized without ensuring that the setpoints are valid per the power flow equations (7.12) and the criteria in the previous section (equations (7.19), (7.20), (7.21)). These become equality and inequality constraints in the optimization problem, respectively. The result of pairing the objective function for economic dispatch with the power flow constraints is known as the optimal power flow (OPF) problem [Frank and Rebennack, 2016, Mones, 2021]:

$$
\begin{aligned}
&\min_{\mathbf{P}} \sum_{g\in\mathcal{G}} C_g(P_g) \\
&\text{subject to:} \\
&P_i = |V_i| \sum_{k\in\mathcal{N}} |V_k| \left(G_{ik}\sin(\delta_i-\delta_k) - B_{ik}\cos(\delta_i-\delta_k)\right) && \forall\, i \in \mathcal{N} \\
&Q_i = |V_i| \sum_{k\in\mathcal{N}} |V_k| \left(G_{ik}\cos(\delta_i-\delta_k) + B_{ik}\sin(\delta_i-\delta_k)\right) && \forall\, i \in \mathcal{N} \\
&S_g^{\min} \le S_g \le S_g^{\max} && \forall\, g \in \mathcal{G} \\
&|S_{ik}| \le S_l^{\max},\ |S_{ki}| \le S_l^{\max} && \forall\, (i,k) \in \mathcal{L} \\
&V_i^{\min} \le |V_i| \le V_i^{\max} && \forall\, i \in \mathcal{N}
\end{aligned}
\tag{7.26}
$$

This OPF problem (7.26) must be solved for every operating interval, making a dispatch decision and minimizing the total cost incurred over that interval independent of any other interval. However, as we will see, there are OPF variants that require considering multiple intervals in one optimization problem, making a dispatch decision for every interval simultaneously and minimizing the total cost incurred over all intervals. For these purposes, we add an interval index t to all time-variant quantities and denote the interval duration as Δt.

Ramp rate limits. The first multi-interval OPF variant that we will discuss involves limits on generators' ramp rates. How quickly a generator's power output (in MW) changes from one interval to the next is referred to as its *ramp rate* (e.g., in MW per minute). If the power output is analogous to velocity, then ramp rate's analog is acceleration, and there are limits to both. Some generators may not be able

[3] To learn more about the economics of operating the electrical grid, see How Electricity Markets Work.

to change their power output very quickly. To keep the dispatch decision within such physical limitations, the following constraint is added to respect each generator's ramp rate limit when ramping up, $\dot{P}_g^{\mathrm{U}}$, and when ramping down, $\dot{P}_g^{\mathrm{D}}$:

$$\dot{P}_g^{\mathrm{D}} \leq \frac{P_{gt} - P_{g,t-1}}{\Delta t} \leq \dot{P}_g^{\mathrm{U}} \tag{7.27}$$

Ramp rate limits $\dot{P}_g^{\mathrm{U}}$ and $\dot{P}_g^{\mathrm{D}}$ are positive and negative values, respectively. For a given t, only one side of this two-sided inequality constraint can be active.

The Unit Commitment Problem

The basic OPF problem (7.26) accounts for the power cost that is incurred by operating a generator at a certain setpoint (e.g., incurred due to fuel use). However, there are additional *standby* costs which depend on whether a generator is in a running state versus an off state over a given operating interval, and there are additional *startup* and *shutdown* costs which depend on whether the generator is transitioning from an off state to a running state (or vice versa) from one interval to the next. In order to account for these additional costs in our objective function, we must introduce additional binary decision variables, w_{gt}, specifying whether generator g is running ($w_{gt} = 1$) or not ($w_{gt} = 0$) over interval t. This requires modifying constraint (7.19) as:

$$w_{gt} S_g^{\min} \leq S_{gt} \leq w_{gt} S_g^{\max} \tag{7.28}$$

The variant of the OPF problem that includes standby, startup, and shutdown costs is thus a scheduling problem, and is known as the unit commitment (UC) problem [Uni]. *Unit commitment* refers to whether a generating unit (that is, a generator) has committed to be running—and thus able to supply power—at a given time.

Standby cost. In addition to a power cost, a generator might have a cost that is incurred as long as it is running, regardless of the generator's setpoint. This is known as standby cost, C_g^{SB}, and it disincentivizes a generator from being run (and thus from generating *any* power) if there is a favorable alternative in terms of cost. For example, the grid operator may be willing to pay a disproportionately higher price to operate an already-running generator at a higher setpoint, rather than running a second generator to supply the extra power, if that second generator's standby cost is high. Accounting for standby cost, the objective becomes:

$$\min_{\mathbf{P},\mathbf{w}} \sum_{t\in\mathcal{T}} \sum_{g\in\mathcal{G}} w_{gt} \left(C_g(P_g) + C_g^{\mathrm{SB}} \right) \tag{7.29}$$

Startup and shutdown costs. In addition to a power cost and a standby cost, a generator might have costs associated with starting up and shutting down, denoted C_g^{SU} and C_g^{SD}, respectively. Startup and shutdown costs only matter if there are standby costs; if there were no standby costs, there would be no reason to shut down generators and thus no reason to incur startup or shutdown costs. Accounting for

startup and shutdown costs in addition to standby cost, the objective function becomes:

$$\sum_{t\in\mathcal{T}}\sum_{g\in\mathcal{G}}\left[w_{gt}\left(C_g(P_g)+C_g^{\mathrm{SB}}\right)+w_{gt}(w_{g,t-1}-1)C_g^{\mathrm{SU}}+(w_{gt}-1)w_{g,t-1}C_g^{\mathrm{SD}}\right] \tag{7.30}$$

where $w_{gt}(w_{g,t-1}-1)$ and $(w_{gt}-1)w_{g,t-1}$ indicate whether a generator has started up or shut down from one interval to the next, respectively.

Startup and shutdown ramp rate limits. Some generators have stricter ramp rate limits that apply specifically when starting up or shutting down, which can be accounted for using the following constraints:

$$\begin{aligned} \dot{P}_g^{\mathrm{SD}} &\leq w_{gt}(w_{g,t-1}-1)\frac{P_{gt}-P_{g,t-1}}{\Delta t} \\ (w_{gt}-1)w_{g,t-1}\frac{P_{gt}-P_{g,t-1}}{\Delta t} &\leq \dot{P}_g^{\mathrm{SU}} \end{aligned} \tag{7.31}$$

Minimum uptime and downtime. It may be required that a generator be in a standby/running state, or in an off state, for a minimum number of operating intervals. This can be accounted for using the following constraints:

$$\begin{aligned} \text{minimum uptime} &\leq \sum_{t\in\mathcal{T}} w_{gt} \\ \text{minimum downtime} &\leq \sum_{t\in\mathcal{T}}(1-w_{gt}) \end{aligned} \tag{7.32}$$

The Linear Optimal Power Flow (LOPF) Problem

The power flow equations (7.12) contain quadratic terms multiplied with trigonometric functions, and branch flow constraint (7.20) is even more complicated. The power flow equations being nonlinear is easy enough when solving them as a system of equations, and the branch flow constraint being nonlinear is easy enough when evaluating the power flow solution, but when used as constraints in OPF, the nonlinearity makes the optimization problem significantly more difficult. To simplify (linearize) the power flow equations and branch flow constraint, thereby making the optimization problem easier, the following approximations are applied:

1. Series resistance is negligible compared to series reactance: $g_{ik} \approx 0$ and $y_{ik} \approx jb_{ik}$. This approximation is reasonable for transmission lines and, combined with approximation 2, assumes a lossless network.
2. Shunt conductance and susceptance are negligible: $g_{ik}^{\mathrm{Sh}} \approx b_{ik}^{\mathrm{Sh}} \approx 0$.
3. Turns ratios are approximately nominal: $|a_{ik}| \approx 1$.
4. Voltage angle differences and transformer phase shifts are small: $\sin(\delta_i - \delta_k - \varphi_{ik}) \approx \delta_i - \delta_k - \varphi_{ik}$ and $\cos(\delta_i - \delta_k) \approx 1$.

5. Voltage magnitudes are approximately nominal: $|V_i| \approx |V_k| \approx 1$ pu everywhere. This supports approximation 3 and holds for a lossless network.

We start by applying approximations 1–3 to equation (7.5) as follows. We will apply approximations 4 and 5 later.

$$\begin{aligned}(S_i/V_i)^* = V_i \left(\sum_{k:(i,k)\in\mathcal{L}} jb_{ik} + \sum_{k:(k,i)\in\mathcal{L}} jb_{ki} \right) \\ - \sum_{k:(i,k)\in\mathcal{L}} V_k \left(jb_{ik} \angle \varphi_{ik}\right) - \sum_{k:(k,i)\in\mathcal{L}} V_k \left(jb_{ki} \angle -\varphi_{ki}\right)\end{aligned} \tag{7.33}$$

We redefine the bus admittance matrix Y as:

$$Y_{ii} = jB_{ii} \angle \Phi_{ii} = \sum_{k:(i,k)\in\mathcal{L}} jb_{ik} + \sum_{k:(k,i)\in\mathcal{L}} jb_{ki}$$

$$Y_{ik} = jB_{ik} \angle \Phi_{ik} = \begin{cases} -jb_{ik} \angle \varphi_{ik} & \text{if branch } ik \text{ is an } i\text{-forward branch} \\ -jb_{ki} \angle -\varphi_{ki} & \text{if branch } ik \text{ is an } i\text{-reverse branch} \\ 0 & \text{if branch } ik \text{ does not exist (no branch)} \end{cases} \tag{7.34}$$

Equation (7.33) becomes:

$$(S_i/V_i)^* = \sum_{k\in\mathcal{N}} V_k \left(jB_{ik} \angle \Phi_{ik}\right) \tag{7.35}$$

We take the conjugate and multiply both sides by V_i:

$$\begin{aligned} S_i &= \sum_{k\in\mathcal{N}} V_i V_k^* \left(jB_{ik} \angle -\Phi_{ik}\right) \\ &= \sum_{k\in\mathcal{N}} \left[|V_i| \, |V_k| \angle (\delta_i - \delta_k)\right] \left(jB_{ik} \angle -\Phi_{ik}\right) \\ &= \sum_{k\in\mathcal{N}} |V_i| \, |V_k| \, jB_{ik} \angle (\delta_i - \delta_k - \Phi_{ik}) \\ &= \sum_{k\in\mathcal{N}} |V_i| \, |V_k| \, jB_{ik} \left(\cos(\delta_i - \delta_k - \Phi_{ik}) + j \sin(\delta_i - \delta_k - \Phi_{ik})\right) \end{aligned} \tag{7.36}$$

Splitting S_i into active and reactive power gives us:

$$\begin{aligned} P_i &= -|V_i| \sum_{k\in\mathcal{N}} |V_k| \, B_{ik} \sin(\delta_i - \delta_k - \Phi_{ik}) \\ Q_i &= |V_i| \sum_{k\in\mathcal{N}} |V_k| \, B_{ik} \cos(\delta_i - \delta_k - \Phi_{ik}) \end{aligned} \tag{7.37}$$

For a branch between buses i and k, the power flowing into the branch from bus i is as follows. (P_{ik} assumes $\sin \Phi_{ik} \approx 0$.)

Linear power flow	DC circuit analysis
Conservation of power flow	Conservation of current flow (KCL)
P_i, P_{ik}	I_i, I_{ik}
θ_i, θ_k	V_i, V_k
Φ_{ik}	ΔV_{ik}
$-1/B_{ik}$	R_{ik}

Table 7.2: Linear power flow is analogous to DC circuit analysis.

$$\begin{aligned} P_{ik} &= -|V_i|\,|V_k|\,B_{ik}\sin(\delta_i - \delta_k - \Phi_{ik}) \\ Q_{ik} &= \;\;|V_i|\,|V_k|\,B_{ik}\cos(\delta_i - \delta_k - \Phi_{ik}) - |V_i|^2 B_{ik} \end{aligned} \tag{7.38}$$

Applying approximation 4 yields:

$$\begin{aligned} P_{ik} &= -|V_i|\,|V_k|\,B_{ik}(\delta_i - \delta_k - \Phi_{ik}) \\ Q_{ik} &= \;\;|V_i|\,|V_k|\,B_{ik} - |V_i|^2 B_{ik} \end{aligned} \tag{7.39}$$

And, finally, applying approximation 5 gives us:

$$\begin{aligned} P_{ik} &= B_{ik}(\delta_i - \delta_k - \Phi_{ik}) \\ Q_{ik} &= 0 \end{aligned} \tag{7.40}$$

Now, when we once again sum the power flowing out of every bus i, it includes only active power:

$$P_i = \sum_{k\in\mathcal{N}, k\neq i} P_{ik} = -\sum_{k\in\mathcal{N}, k\neq i} B_{ik}(\delta_i - \delta_k - \Phi_{ik}) \tag{7.41}$$

This is the linear power flow equation, which applies to every bus i and is much simpler than the original (nonlinear) power flow equation due to the approximations applied. Because the linear power flow equation approximates voltage magnitudes as nominal and assumes zero reactive power, there is no differentiating between PQ and PV buses. Furthermore, the lack of reactive power allows us to simplify the branch flow constraint (7.20):

$$-P_l^{\max} \leq P_{ik} \leq P_l^{\max} \;\forall\; (i,k) \in \mathcal{L} \tag{7.42}$$

When OPF uses the linear power flow equation and the correspondingly simplified branch flow constraint, it is known as *linear optimal power flow (LOPF)*.

> **Note**
> Equation (7.41) is analogous to DC circuit analysis using KCL ($I_i = \Sigma(V_i - V_k)/r_{ik}$) as shown in Table 7.2.

Cycle-Based Formulation

The LOPF problem uses bus voltage angles as decision variables, but there are advantages to using the branch power flows instead. For such an alternate formulation, we are still constraining the net power injected at each bus i to equal the total power flowing into the bus

from each connected branch, but we do so in terms of the branch power flow P_l. We refer to the *signed incidence matrix* $(C^{\mathrm{t}} - C^{\mathrm{f}})$ to indicate whether P_l is flowing into or out of bus i (+1 or -1, or 0 if l is not connected to it):

$$P_i = \sum_{k \in \mathcal{N}, k \neq i} P_{ik} = \sum_{l \in \mathcal{L}} (C^{\mathrm{t}}_{l,i} - C^{\mathrm{f}}_{l,i})\, P_l \tag{7.43}$$

This increases the number of variables while the number of equations in the system stays the same. To make the system solvable, we apply Kirchhoff's Voltage Law (KVL), which states that the net voltage difference around each loop or cycle in the circuit/grid is zero. This means that if we add up the voltage difference $V_i - V_k$ across every branch ik in a given cycle, they must add up to zero. Because we assume $V = 1$ pu, this becomes $1\angle\delta_i - 1\angle\delta_k$. Ensuring these sum to zero around a given cycle is equivalent to just ensuring that the cycle's voltage *angle* differences ($\Delta\delta_l = \delta_i - \delta_k$) sum to zero, or:

$$\sum_{l \in \mathcal{L}} C_{l,c}\, \Delta\delta_l = 0 \quad \forall\, c \in \mathcal{C} \tag{7.44}$$

where $\mathcal{C}$ is the set of cycles, and $C_{l,c}$ indicates whether branch l is in cycle c (+1 or -1 depending on the direction, or 0 if absent). Using Equation (7.40), we can express this in terms of the decision variables P_l as follows, thereby completing the system of equations.

$$\sum_{l \in \mathcal{L}} C_{l,c} \left(\frac{P_l}{b_l} + \varphi_l \right) = 0 \quad \forall\, c \in \mathcal{C} \tag{7.45}$$

Summary

We've introduced many concepts; Figure 7.6 provides a summary of these concepts and how they relate to each other.

Appendix: Branch-From-Bus and Branch-To-Bus Admittance Matrices

The bus admittance matrix Y is a convenient way to encapsulate the complexity of series admittances (y), shunt admittances (y^{Sh}), and transformer ratios (a). However, because each diagonal element Y_{ii} is a sum over every i-connected branch, the bus admittance matrix does not allow us to calculate individual branch current and power values (I_{ik} and S_{ik}). This can be overcome using the bus admittance matrix with a "branch-from-bus" admittance matrix Y^{f} and a "branch-to-bus" admittance matrix Y^{t}. Both of these matrices are sparse matrices of dimension $N \times L$, where N is the number of buses and L is the number of branches; both matrices have a row for each branch and a column for each bus. For a branch l from bus i to bus k, row l of each matrix has nonzero entries only in columns i and k:

$$\begin{aligned} Y^{\mathrm{f}}_{l,i} &= \frac{1}{|a_{ik}|^2} \left(\frac{y^{\mathrm{Sh}}_{ik}}{2} + y_{ik} \right) & Y^{\mathrm{f}}_{l,k} &= -\frac{y_{ik}}{a^*_{ik}} \\ Y^{\mathrm{t}}_{l,i} &= -\frac{y_{ik}}{a_{ik}} & Y^{\mathrm{t}}_{l,k} &= \frac{y^{\mathrm{Sh}}_{ik}}{2} + y_{ik} \end{aligned} \tag{7.46}$$

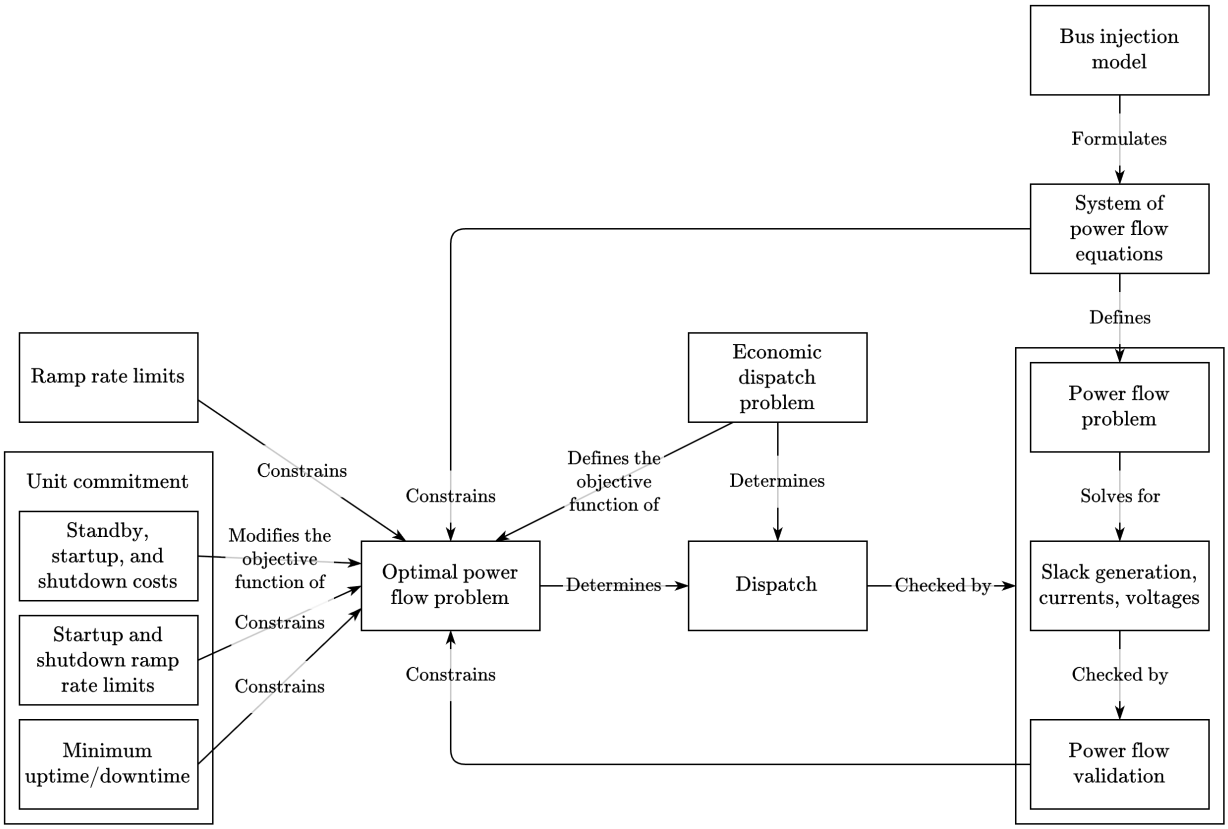


Figure 7.6: This chapter's concepts and how they inter-relate.

The familiar bus admittance matrix can be reconstructed from these as follows, using branch-from-bus and branch-to-bus *incidence* matrices C^{f} and C^{t}:

$$Y = (C^{\mathrm{f}})^{\top} Y^{\mathrm{f}} + (C^{\mathrm{t}})^{\top} Y^{\mathrm{t}} \tag{7.47}$$

Matrices C^{f} and C^{t} are also $N \times L$ sparse matrices, consisting mostly of zeroes. C^{f} contains a "1" entry indicating the bus that each branch originates from. Similarly, C^{t} indicates which bus each branch goes to.

Example
This is more intuitively understood using an example. Consider a 3-bus, 2-branch network.

Branches $l = 1$ originates from bus $i = 1$ and branch $l = 2$ originates from bus $i = 2$:

$$Y^{\mathrm{f}} = \begin{bmatrix} \frac{1}{|a_{12}|^2}\left(\frac{y_{12}^{\mathrm{Sh}}}{2} + y_{12}\right) & -\frac{y_{12}}{a_{12}^*} & 0 \\ 0 & \frac{1}{|a_{23}|^2}\left(\frac{y_{23}^{\mathrm{Sh}}}{2} + y_{23}\right) & -\frac{y_{23}}{a_{23}^*} \end{bmatrix} \tag{7.48}$$

$$C^{\mathrm{f}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{7.49}$$

Branches $l = 1$ goes to bus $k = 2$ and branch $l = 2$ goes to bus $k = 3$:

$$Y^{\mathrm{t}} = \begin{bmatrix} -\frac{y_{12}}{a_{12}} & \left(\frac{y_{12}^{\mathrm{Sh}}}{2} + y_{12}\right) & 0 \\ 0 & -\frac{y_{23}}{a_{23}} & \left(\frac{y_{23}^{\mathrm{Sh}}}{2} + y_{23}\right) \end{bmatrix} \tag{7.50}$$

$$C^{\mathrm{f}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.51}$$

The bus admittance matrix can be reconstructed as follows:

$$(C^{\mathrm{f}})^{\top} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{7.52}$$

$$(C^{\mathrm{f}})^{\top} Y^{\mathrm{f}} = \begin{bmatrix} \frac{1}{|a_{12}|^2}\left(\frac{y_{12}^{\mathrm{Sh}}}{2} + y_{12}\right) & -\frac{y_{12}}{a_{12}^*} & 0 \\ 0 & \frac{1}{|a_{23}|^2}\left(\frac{y_{23}^{\mathrm{Sh}}}{2} + y_{23}\right) & -\frac{y_{23}}{a_{23}^*} \\ 0 & 0 & 0 \end{bmatrix} \tag{7.53}$$

$$(C^{\mathrm{f}})^{\top} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.54}$$

$$(C^{\mathrm{f}})^{\top} Y^{\mathrm{f}} = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{y_{12}}{a_{12}} & \left(\frac{y_{12}^{\mathrm{Sh}}}{2} + y_{12}\right) & 0 \\ 0 & -\frac{y_{23}}{a_{23}} & \left(\frac{y_{23}^{\mathrm{Sh}}}{2} + y_{23}\right) \end{bmatrix} \tag{7.55}$$

$$\begin{aligned} Y &= (C^{\mathrm{f}})^{\top} Y^{\mathrm{f}} + (C^{\mathrm{t}})^{\top} Y^{\mathrm{t}} \\ &= \begin{bmatrix} \frac{1}{|a_{12}|^2}\left(\frac{y_{12}^{\mathrm{Sh}}}{2} + y_{12}\right) & -\frac{y_{12}}{a_{12}^*} & 0 \\ -\frac{y_{12}}{a_{12}} & \frac{1}{|a_{23}|^2}\left(\frac{y_{23}^{\mathrm{Sh}}}{2} + y_{23}\right) + \left(\frac{y_{12}^{\mathrm{Sh}}}{2} + y_{12}\right) & -\frac{y_{23}}{a_{23}^*} \\ 0 & -\frac{y_{23}}{a_{23}} & \left(\frac{y_{23}^{\mathrm{Sh}}}{2} + y_{23}\right) \end{bmatrix} \end{aligned} \tag{7.56}$$

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Basics of DC versus AC power and resistive versus reactive loads.	Relationship between sinusoids and phasors. Deriving the concept of phasors. Formal phasor transformation and notation.	Seeing the benefits of phasors in practice over differential equations and the Laplace transform.
Tutorial: Building an AC Circuit Analyzer	Complex Power and the Power Factor	Three-Phase Power
Developing AC circuit analysis skills by building a software tool to solve any AC circuit.	Deriving complex power and the power triangle from time-domain instantaneous power. Power factor and power factor correction.	Motivation for three-phase power. Wye versus delta configurations. Three-phase power in the grid.
Grid Modeling		
Bus injection model. Power flow, optimal power flow, economic dispatch, and unit commitment problems.		