Analysis I

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1 Logic, Sets and Numbers

1.1 Logic

Definition 1.1. A **mathematical Statement** is a well-formulated assertion that is strictly either true or false.

Example 1.2. "Every prime number is odd." is a false statement.

Definition 1.3. A **Statement** that depends on one or multiple variables is called a **predicate** and denoted by $P(x), Q(x), \dots$.

Example 1.4.

$$P(x) \coloneqq x > 2$$

Definition 1.5. The negation of a mathematical Statement A is defined as $\neg A \coloneqq A$ is false.

Example 1.6. $A \coloneqq \forall x \in \mathbb{R} : x > 2$ $\neg A \coloneqq \exists x \in \mathbb{R} : x \leq 2$

Definition 1.7. **Quantifiers** are short hand notation for making statements about predicates.

$$\forall x \in \mathbb{R} : P(x) \iff \text{for all} x \in \mathbb{R} \text{ holds } P(x)$$

$$\exists x \in \mathbb{R} : P(x) \iff \text{there exists} x \in \mathbb{R} \text{ s.t. } P(x)$$

$$\exists! x \in \mathbb{R} : P(x) \iff \text{there exists exactly one} x \in \mathbb{R} \text{ s.t. } P(x)$$

$$\nexists x \in \mathbb{R} : P(x) \iff \text{there exists no} x \in \mathbb{R} \text{ s.t. } P(x)$$

Example 1.8.

$$\forall x \in \mathbb{R} : x^2 \geq 0$$

$$\forall x \in \mathbb{R} \exists y \in \mathbb{R} : x = y^2$$

Important 1.9. The oderder of quantifiers matters. $\forall x \exists y$ is not equivalent to $\exists y \forall x$.

Definition 1.10. Basic **logical connectives** are defined as:

$$A \wedge B \coloneqq A \text{ and } B \text{ are true}$$

$$A \vee B \coloneqq A \text{ or } B \text{ is true}$$

$$A \implies B \coloneqq A \text{ implies } B \text{ is true}$$

$$A \iff B := A \text{ is equivalent to } B \text{ is true}$$

Remark 1.11. Observe that equivalence holds iff. $A \implies B \land B \implies A$

Lemma 1.12. From the above definition it follows that:

$$\neg A \lor B \iff A \implies B$$

This can be shown by evaluating the truth table for both statements.

Theorem 1.13. The following equivalences hold:

$$\neg(\forall x P(x)) \iff \exists x \neg P(x)$$

$$\neg(\exists x P(x)) \iff \forall x \neg P(x)$$

aswell as

$$\forall x (P(x) \land Q(x)) \iff (\forall x P(x)) \land (\forall x Q(x))$$
$$\exists x (P(x) \lor Q(x)) \iff (\exists x P(x)) \lor (\exists x Q(x))$$

1.2 Sets

Definition 1.14. A **set** is a collection of elements, that are listed or characterized by a certain property. The empty set is denoted by $\emptyset$ and is the only set that contains no elements.

Definition 1.15. An element x is **contained in a set** A iff. $x \in A$ else it is not contained in A denoted by $x \notin A$.

Definition 1.16. Let A and B be sets. A is a **subset** of B denoted by $A \subset B$ iff.

$$\forall x (x \in A \implies x \in B)$$

A is a **proper subset** of B denoted by $A \subsetneq B$ iff.

$$A \subset B \land A \neq B$$

Definition 1.17. Let A and B be sets.

- The **union** of A and B is defined as $A \cup B \coloneqq \{x \mid x \in A \lor x \in B\}$.
- The **intersection** of A and B is defined as $A \cap B \coloneqq \{x \mid x \in A \land x \in B\}$.
- The **difference** of A and B is defined as $A \setminus B \coloneqq \{x \mid x \in A \land x \notin B\}$.
- The **Cartesian product** of A and B is defined as $A \times B \coloneqq \{(a, b) \mid a \in A \land b \in B\}$.

Remark 1.18. Note that if $B \subset A$ then the diffrence $A \setminus B$ is called the complement of B in A and is denoted by B^c.

1.3 Numbers

The basic and often used sets of numbers are:

- $\mathbb{N}$: The set of natural numbers 1, 2, 3, 4, ...
- $\mathbb{Z}$: The set of integers ... -2, -1, 0, 1, 2, ...
- $\mathbb{Q}$: The set of rational numbers $\frac{p}{q}$ where $p, q \in \mathbb{Z}$
- $\mathbb{R}$: The set of real numbers

Remark 1.19. Some important subsets of the real numbers are $\mathbb{R}_0^+/\mathbb{R}^+$ and $\mathbb{R}_0^-/\mathbb{R}^-$ which define the intervalls $[0, \infty)/(0, \infty)$ and $(-\infty, 0]/(-\infty, 0)$.

Definition 1.20. Continuous subsets are called **intervalls.** An intervall is called

- **open** if it does not contain its endpoints $I = (a, b) \coloneqq \{x \mid a < x < b\}$
- **closed** if it contains its endpoints $I = [a, b] \coloneqq \{x \mid a \leq x \leq b\}$
- **half-open** if it contains one endpoint and not the other $I = [a, b) \coloneqq \{x \mid a \leq x < b\}$

Remark 1.21. An arbitrary subset of $\mathbb{R}$ is called open if it is a union of open intervalls and closed if its complement is open.

Definition 1.22. An **upper (lower) bound** of a subset $X \subset \mathbb{R}$ is an element $y \in \mathbb{R}$ such that:

$$\forall x \in X : x \leq (\geq) y$$

The **maximum (minimum)** of a subset $X \subset \mathbb{R}$ is the element $x_0 \in X$ such that:

$$\forall x \in X : x \leq (\geq) x_0$$

Often denoted as **max(X)** and **min(X)**.

Definition 1.23. An intervall $I = (a, b)$ is **bounded** if there exist upper and lower bounds.

Remark 1.24. A bounded and closed intervall is often refered to as **compact**.

Definition 1.25. The **Supremum** $sup(X)$ of a subset $X \subset \mathbb{R}$ is the least upper bound of X. and can be characterized by:

$$\forall x \in X : x \leq sup(X) \wedge \forall \epsilon > 0 : \exists x \in X : x > sup(X) - \epsilon$$

Definition 1.26. The **Infimum** $inf(X)$ of a subset $X \subset \mathbb{R}$ is the greatest lower bound of X. and can be characterized by:

$$\forall x \in X : x \geq inf(X) \wedge \forall \epsilon > 0 : \exists x \in X : x < inf(X) + \epsilon$$

Example 1.27. Let $X = [-2, 1)$ we have a minimum of $x_0 = -1$ but no maximum. Observe that it is bounded from above by 1 and 2 or any number larger than 1 but only 1 is the least upper bound (Supremum).

Theorem 1.28. The **maximum and minimum are unique** aslong as they exist.

Proof. Assume there exists $x_1, x_2 \in X$ such that $x_1 = max(X)$ and $x_2 = max(X)$. As both x_1 and x_2 are maximum of X we have

$$x_1 \geq x_2 \wedge x_2 \geq x_1 \implies x_1 = x_2$$

□

Definition 1.29. Axioms of Addition

- Commutativity: $x + y = y + x$
- Associativity: $x + (y + z) = (x + y) + z$
- Identity element: There exists an element $e \in X$ such that $x + e = x$ for all $x \in X$.
- Inverse element: For every $x \in X$ there exists an element $x^{-1} \in X$ such that

$x + x^{-1} = e$.

Remark 1.30. A set X together with an operation $+$ that satisfies the axioms of addition is called a **abelian group**.

Example 1.31. $(\mathbb{Z}, +)$ is an abelian group.

Definition 1.32. Axioms of Multiplication

- Commutativity: $x \cdot y = y \cdot x$
- Associativity: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- Identity element: There exists an element $e \in X$ such that $x \cdot e = x$ for all $x \in X$.
- Inverse element: For every $x \in X$ there exists an element $x^{-1} \in X$ such that $x \cdot x^{-1} = e$.

Definition 1.33. Distributivity of multiplication over addition:

$$\forall x, y, z \in X : x \cdot (y + z) = x \cdot y + x \cdot z$$

Remark 1.34. A set X together with operations $+$ and $\cdot$ that satisfies the axioms of addition and multiplication and distributivity is called a **field**.

Definition 1.35. Order Axioms

- reflexivity: $\forall x \in X : x \leq x$
- antisymmetry: $\forall x, y \in X : x \leq y \land y \leq x \implies x = y$
- transitivity: $\forall x, y, z \in X : x \leq y \land y \leq z \implies x \leq z$
- totality: $\forall x, y \in X : x \leq y \lor y \leq x$

Definition 1.36. A set X is said to be **(order)complete** iff for every non empty subsets $A, B \subseteq X$

$$\forall a \in A \forall b \in B : a \leq b$$
$$\implies \exists c \in X \quad \text{s.t.} \quad \forall a \in A : a \leq c \land \forall b \in B : c \leq b$$

Example 1.37. The real numbers $\mathbb{R}$ satisfy axiomatic properties for addition, multiplication, distributivity and order. Furthermore $\mathbb{R}$ is an order complete field.

Example 1.38. $\mathbb{Q}$ is not order complete. Let $A = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$ and $B = \{x \in \mathbb{Q} \mid x^2 \geq 2\}$. Then $\forall a \in A \forall b \in B : a \leq b$ but there is no $c \in \mathbb{Q}$ such that $c^2 = 2$.

Theorem 1.39. Every **non empty, bounded from above (below)** subset $X \subset \mathbb{R}$ has a **supremum (infimum)**.

Proof. Let B be the set of all upper bounds of X. As X is bounded from above and non empty, B is non empty. Therfore

$$\forall x \in X : x \leq b \quad \forall b \in B$$

By the order completenes of $\mathbb{R}$ there exists a $c \in \mathbb{R}$ such that

$$\forall x \in X : x \leq c \land \forall b \in B : c \leq b$$

Hence c is the least upper bound of X and therefore $c = sup(X)$. Assume there exists $c' \in \mathbb{R}$ such that $c' = sup(X)$ and $c' \neq c$. Then by definition of $sup(X)$ we have

$$c \leq c' \land c' \leq c \implies c = c'$$

□

Theorem 1.40. The **Archimedian Principle** says that

- For $x \in \mathbb{R}$ and $y > 0$ there exists an element $n \in \mathbb{N}$ such that $n * y > x$.
- For every $\epsilon > 0$ there exists an element $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

Proof. We prove the first theorem by contradition. Assume it does not hold hence there exists $x \in \mathbb{R}$ and $y > 0$ such that

$$\forall n \in \mathbb{N} : n * y \leq x$$

This implies that x is an upper bound for the set $X = \{n * y \mid n \in \mathbb{N}\}$. As X is non empty and bounded from above, it has a supremum $c = sup(X)$. As $y > 0$ $c - y$ can not be an upper bound of X hence there exists $n_0 \in \mathbb{N}$ such that

$$c - y < n_0 * y \implies c < (n_0 + 1) * y$$

This contradicts the assumption that c is the supremum of X. □

Proof. The second theorem is a direct consequence of the first theorem. It can be shown by setting $x = 1$ in the first theorem. □

Theorem 1.41. The Young Inequality states that for all $\epsilon > 0$ and all $x, y \in \mathbb{R}$:

$$2|xy| \leq \epsilon x^2 + \frac{1}{\epsilon} y^2$$

Proof.

$$\left(\sqrt{\epsilon} x - \frac{1}{\sqrt{\epsilon}} y\right)^2 \geq 0 \implies \epsilon x^2 - 2xy + \frac{1}{\epsilon} y^2 \geq 0 \implies \epsilon x^2 + \frac{1}{\epsilon} y^2 \geq 2xy$$

□

1.3.1 Vectors and Vector Spaces

Definition 1.42. The **euclidian space** $\mathbb{R}^n$ is the set of all vectors $x = (x_1, x_2, \dots, x_n)$ where $x_i \in \mathbb{R}$.

Definition 1.43. **Addition** and **scalar multiplication** are defined as:

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

$$\lambda x = (\lambda x_1, \lambda x_2, \dots, \lambda x_n)$$

Definition 1.44. A **vector space** has to satisfy the following axioms:

- Commutativity: $x + y = y + x$
- Associativity: $x + (y + z) = (x + y) + z$
- Identity element: There exists an element $e \in X$ such that $x + e = x$ for all $x \in X$.
- Inverse element: For every $x \in X$ there exists an element $x^{-1} \in X$ such that $x + x^{-1} = e$.
- Distributivity: $x \cdot (y + z) = x \cdot y + x \cdot z$

Definition 1.45. For $x, y \in \mathbb{R}^n$, the **standard scalar product** is defined as:

$$x \cdot y = \langle x, y \rangle := \sum_{i=1}^{n} x_i y_i$$

The **euclidean norm** is defined as:

$$||x|| := \sqrt{\sum_{i=1}^{n} x_i^2}$$

The **euclidean distance** between x and y is:

$$d(x, y) := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Theorem 1.46. The triangle- and the cauch-schwarz inequality hold for all $x, y, z \in$

$\mathbb{R}^n$:

$$||x - z|| \leq ||x - y|| + ||y - z||$$
$$|\langle x, y \rangle| \leq ||x|| ||y||$$

1.3.2 Complex Numbers

Definition 1.47. The **imaginary unit** i is defined as $i^2 = -1$.

Definition 1.48. The **complex numbers** $\mathbb{C}$ are the set of numbers

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$$

where a is the **real part** $\text{Re}(z)$, and b is the **imaginary part** $\text{Im}(z)$.

Definition 1.49. Arithmetic of complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$:

- Addition: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$
- Multiplication: $z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$

Definition 1.50. The **complex conjugate** of $z = x + iy$ is $\bar{z} = x - iy$.

Important 1.51. Division is calculated by expanding with the conjugate of the denominator:

$$\frac{z}{w} = \frac{z \cdot \bar{w}}{w \cdot \bar{w}} = \frac{z \cdot \bar{w}}{|w|^2}$$

Remark 1.52. Some usfull properties of the conjugate:

- $z \cdot \bar{z} = |z|^2$
- $z + \bar{z} = 2\text{Re}(z)$ and $z - \bar{z} = 2i\text{Im}(z)$
- $\overline{z + w} = \bar{z} + \bar{w}$ and $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

Definition 1.53. Geometrically, complex numbers can be represented in the Gaussian plane. The **absolute value** of $z = a + ib$ is its **distance from the origin**:

$$|z| = \sqrt{a^2 + b^2}$$

The distance between z_1 and z_2 is $d = |z_1 - z_2|$.

Remark 1.54. Observe that the triangle inequality holds in $\mathbb{C}$: $|z + w| \leq |z| + |w|$.

Theorem 1.55. The Euler's formula states that for $t \in \mathbb{R}$:

$$e^{it} = \cos(t) + i\sin(t)$$

This allows writing trigonometric functions using complex exponentials:

$$\cos(t) = \frac{e^{it} + e^{-it}}{2}, \quad \sin(t) = \frac{e^{it} - e^{-it}}{2i}$$

Remark 1.56. The **Intuition** behind eulers formula is we can use Taylor series to show that:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Using $x = it$ we get:

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \cdots = \cos(t) + i\sin(t)$$

Eulers formula leads us to a new representation of complex numbers: **the polarcoordinates**.

Definition 1.57. For $z \in \mathbb{C}$, the polar-coordinates are defined as:

$$z = r(\cos(t) + i\sin(t)) = re^{it}$$

where $r = |z|$ is the absolute value and $t = \arg(z)$ is the argument.

Remark 1.58. To convert from cartesian to polar coordinates:

$$r = \sqrt{a^2 + b^2}, \quad t = \arctan\left(\frac{b}{a}\right)$$

Note that the arctan function only returns values in $(-\frac{\pi}{2}, \frac{\pi}{2})$, so we have to adjust the angle based on the quadrant of the complex number.

Example 1.59. Multiplication and division in polar-coordinates is much easier than in cartesian coordinates:

$$z_1 \cdot z_2 = r_1 e^{it_1} \cdot r_2 e^{it_2} = r_1 r_2 e^{i(t_1 + t_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{it_1}}{r_2 e^{it_2}} = \frac{r_1}{r_2} e^{i(t_1 - t_2)}$$

Example 1.60. So is exponentiation or taking roots:

$$z^n = (re^{it})^n = r^n e^{int}$$

$$\sqrt[n]{z} = \sqrt[n]{r} e^{i\frac{t+2\pi k}{n}}, \quad k \in \{0, 1, \ldots, n-1\}$$

Definition 1.61. The n-th roots of unity are the solutions to $z^n = 1$, which are given by:

$$z_k = e^{i\frac{2\pi k}{n}}, \quad k \in \{0, 1, \ldots, n-1\}$$

Theorem 1.62. The sum of the roots of unit is always zero.

$$\sum_{k=0}^{n-1} e^{i\frac{2\pi k}{n}} = 0$$

Theorem 1.63. The **fundamental theorem of algebra** states that every polynomial of degree $n \geq 1$ has exactly n roots in $\mathbb{C}$ (counting multiplicity). These roots appear in konplex-conjugate pairs.

2 Sequences and Series

2.1 Sequences

Definition 2.1. A **sequence** (Folge) $(a_n)_{n\in\mathbb{N}_0}$ is an infinite list of numbers, mathematically a mapping $f : \mathbb{N}_0 \to X$. It can be given explicitly or recursively as

$$a_n = f(n) \quad \text{or} \quad a_n = g(a_{n-1}, a_{n-2}, \dots)$$

Example 2.2. A famous example is the Fibonacci sequence defined by $a_0 = 0, a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. The first few terms are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$

Definition 2.3. A sequence (a_n) is called **convergent** with limit $L \in \mathbb{R}$ if:

$$\forall \epsilon > 0 \exists N > 0 \quad \text{s.t.} \quad \forall n > N : |a_n - L| < \epsilon$$

Written as $\lim_{n\to\infty} a_n = L$.If no such L exists, the sequence is **divergent**.

> **Remark 2.4.** A special case of divergent sequences are sequences which diverge to $\pm\infty$. Their limits are called **improper limits**.

Theorem 2.5. The limit of a sequence is unique aslong as it exists.

Proof. Assume (a_n) converges to L_1 and L_2. Then for any $\epsilon > 0$ there exists N_1 and N_2 such that:

$$|a_n - L_1| < \epsilon \quad \forall n > N_1$$
$$|a_n - L_2| < \epsilon \quad \forall n > N_2$$

Then for any $n > \max(N_1, N_2)$:

$$|L_1 - L_2| = |L_1 - a_n + a_n - L_2| \leq |L_1 - a_n| + |a_n - L_2| < 2\epsilon$$

Since this holds for any $\epsilon > 0$, we must have $L_1 = L_2$. □

Theorem 2.6. Any subsequence $(a_{n_k})_{k\in\mathbb{N}_0}$ of a convergent sequence converges to the same limit L.

Definition 2.7. An **accumulation point** (Häufungspunkt) $A \in \mathbb{R}$ of a sequence fulfills:

$$\forall \epsilon > 0 \forall N \in \mathbb{N}_0 \exists n \geq N \text{ such that } |a_n - A| < \epsilon$$

Example 2.8. The sequence $a_n = \frac{1}{2}(-1)^n$ has the accumulation points $A_1 = \frac{1}{2}$ and $A_2 = -\frac{1}{2}$.

Theorem 2.9. A is an accumulation point iff there exists a convergent subsequence with limit A.

Proof. Let (a_n) be a sequence and A an accumulation point. We can easily construct a subsequence (a_{n_k}) that converges to A by selecting n_k such that $|a_{n_k} - A| < 1/k$ this is given to exist by the definition of an accumulation point. Now let (a_{n_k}) be a convergent subsequence with limit A then for any $\epsilon > 0$ there exists K such that $|a_{n_k} - A| < \epsilon$ for all $k > K$ and since $n_k \to \infty$ as $k \to \infty$ by definition it follows that A is an accumulation point. □

Theorem 2.10. Every convergent sequence has exactly one accumulation point (its limit).

Proof. For A to be an accumulation point of a_n we need that there are indefinitely many a_n that approach A. If a_n converges to L then we can intuitively say that that A has to be L. □

Definition 2.11. A sequence is **upper (lower) bounded** if there exists Msuch that $|a_n| \leq (\geq) M \quad \forall n \in \mathbb{N}_0$.

Theorem 2.12. Every convergent sequence is bounded.

Proof. By the definition of convergence there exists N sucht that for all $n \geq N$ we have $|a_n - L| < 1$. Thus $|a_n| < |L| + 1$ for all $n \geq N$. Further we have a finite number of terms $a_0, \ldots, a_{N-1}$ which are also bounded. Therefore the sequence is bounded by $M = \max(|a_0|, \ldots, |a_{N-1}|, |L| + 1)$. □

Theorem 2.13. Bolzano-Weierstrass Theorem: Every bounded sequence has an accumulation point.

Theorem 2.14. Limit arithmetic: For $\lim a_n = K$, $\lim b_n = L$ and $C \in \mathbb{R}$:

- $\lim(a_n \pm b_n) = K \pm L$
- $\lim(C \cdot a_n \cdot b_n) = C \cdot K \cdot L$
- $\lim(a_n/b_n) = K/L$ (if $L \neq 0$ and $b_n \neq 0$)

Remark 2.15. Important limits:

- $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

- $\lim_{n\to\infty} n^{1/n} = 1$
- $\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$
- $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$
- $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

Theorem 2.16. Sandwich-Theorem: If $a_n \leq b_n \leq c_n$ for $n \geq N_0$, and $\lim a_n = \lim c_n = L$, then $\lim b_n = L$.

Proof. Let $\lim a_n = \lim c_n = L$ and $\lim b_n = M$. From the above arithmetic we can sa that $a_n \leq b_n \leq c_n$ implies $L \leq M \leq L$ so we have that b_n is squeezed between a_n and c_n and thus must converge to $M = L$. $\square$

Example 2.17. We want to calculat $\lim_{n\to\infty}(2^n + 5^n)^{\frac{1}{n}}$. We can bound the sequence by $5^n \leq 2^n + 5^n \leq 2 \cdot 5^n$. Taking the n-th root we get $5 \leq (2^n + 5^n)^{\frac{1}{n}} \leq 2^{\frac{1}{n}} \cdot 5$. As $n \to \infty$ we have $2^{\frac{1}{n}} \to 1$ so by the sandwich theorem we have $\lim_{n\to\infty}(2^n + 5^n)^{\frac{1}{n}} = 5$.

Remark 2.18. The sandwich theorem is often used to find upper and lower bounds for a sequence that converge to the same limit and therfore conclude that the original sequence does too.

Definition 2.19. A sequence is **monotonically increasing (decreasing)** if $a_n \leq (\geq) a_m$ for $m > n$.

Theorem 2.20. Monotone Convergence Theorem: Every bounded monotonic sequence converges to its supremum (infimum).

Proof. Wlog let (a_n) be a bounded monotonically increasing sequence. Hence there exists $S = \sup\{a_n : n \in \mathbb{N}_0\}$ such that $a_n \leq S$ for all n and for any $\epsilon > 0$ there exists N such that $a_N > S - \epsilon$ and therfore $|a_n - S| < \epsilon$ for all $n \geq N$. $\square$

Definition 2.21. The limit of the supremum is called **limes superior** and denoted as:

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\geq n} a_k$$

The limit of the infimum is called **limes inferior** and denoted as:

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{k\geq n} a_k$$

Definition 2.22. A sequence is a **Cauchy sequence** iff

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall m, n \geq N : |a_n - a_m| < \epsilon$$

Theorem 2.23. Every Cauchy sequence is bounded.

Proof. Choose $\epsilon = 1$, then there exists N such that $|a_n - a_N| < 1$ for all $n \geq N$. Thus $|a_n| < |a_N| + 1$ for $n \geq N$, meaning (a_n) is bounded by $M = \max(|a_0|, \ldots, |a_{N-1}|, |a_N| + 1)$. □

Theorem 2.24. A sequence of real numbers converges iff it is a cauchy sequence.

Proof. $\implies$: Let (a_n) converge to L. For any $\epsilon > 0$, there exists N such that $\forall n \geq N$: $|a_n - L| < \epsilon/2$. Then for any $m, n \geq N$:

$$|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |L - a_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, (a_n) is a Cauchy sequence.

$\impliedby$: Let (a_n) be a Cauchy sequence therfore it is bounded. By the Bolzano-Weierstrass theorem, the bounded sequence (a_n) has an accumulation point L, so there exists a subsequence (a_{n_k}) converging to L. For any $\epsilon > 0$, since (a_n) is Cauchy, there exists N' such that $\forall m, n \geq N'$: $|a_n - a_m| < \epsilon/2$. Since $a_{n_k} \to L$, choose a k such that $n_k \geq N'$ and $|a_{n_k} - L| < \epsilon/2$. Then for any $n \geq N'$:

$$|a_n - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, the sequence converges to L. □

2.2 Series

Definition 2.25. A **series** (Reihe) is an infinite sum $\sum_{i=0}^{\infty} a_i$ of the terms of a sequence (a_n).

Definition 2.26. A series is said to converge iff the sequence of partialsums converges.

$$\sum_{i=0}^{\infty} a_i = \lim_{n \to \infty} \sum_{i=0}^{n} a_i$$

Remark 2.27. Some examples of series are:

- **geometric series**: $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ iff $|x| < 1$
- **harmonic series**: $\sum_{n=1}^{\infty} \frac{1}{n}$ converges for $s > 1$

Theorem 2.28. Necessary condition for convergence: for $\sum_{i=0}^{\infty} a_i$ to converge we need $\lim_{n\to\infty} a_n = 0$ i.e. it is a null sequence.

Theorem 2.29. For **series with non-negative terms** $a_n \geq 0$, the sequence of partial sums is monotonically increasing and **converges iff it is bounded.**

Definition 2.30. A series is **absolutely convergent** if $\sum |a_n|$ converges. A **conditionally convergent** series converges, but not absolutely.

Remark 2.31. Absolute convergence implies normal convergence.

Theorem 2.32. Riemann's rearrangement theorem: Let $\sum a_n$ be a conditionally convergent series. Then for any $x \in \mathbb{R}$, there exists a rearrangement of the series that converges to x.

Theorem 2.33. Comparison test (Vergleichskriterium): If $c_n \leq b_n \leq a_n$ for $n \geq N$:

- If $\sum a_n$ converges, then $\sum b_n$ converges (**Majorantenkriterium**).
- If $\sum c_n$ diverges, then $\sum b_n$ diverges (**Minorantenkriterium**).

Example 2.34.

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots$$

We can compare this series to the geometric series

$$1 + \sum_{n=1}^{\infty} (\frac{1}{2})^n = 1 + 2 = 3$$

Therfore accordint to the Majorantenkriterium the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

Theorem 2.35. Leibniz Criterion: For an alternating series $\sum(-1)^n a_n$ with a monotonically decreasing null sequence (a_n), the series converges.

Proof. Let (a_n) be a non-negative, monotonically decreasing sequence converging to 0. Let $s_n = \sum_{k=0}^{n}(-1)^k a_k$ be the n-th partial sum. For the even partial sums (s_{2n}), we have

$$s_{2n+2} - s_{2n} = -a_{2n+1} + a_{2n+2} \leq 0$$

since $a_{2n+2} \leq a_{2n+1}$. Thus, (s_{2n}) is monotonically decreasing. Similarly for the odd partial sums (s_{2n+1}), we have

$$s_{2n+3} - s_{2n+1} = a_{2n+2} - a_{2n+3} \geq 0$$

meaning (s_{2n+1}) is monotonically increasing. Note that

$$s_{2n+1} = s_{2n} - a_{2n+1} \leq s_{2n} \leq s_0 = a_0$$

This limits both bounds: $0 \leq s_{2n+1} \leq s_{2n} \leq a_0$. By the Monotone Convergence Theorem, both bounded monotonic subsequences converge: $\lim_{n\to\infty} s_{2n} = L_1$ and $\lim_{n\to\infty} s_{2n+1} = L_2$. Finally,

$$L_1 - L_2 = \lim_{n\to\infty}(s_{2n} - s_{2n+1}) = \lim_{n\to\infty} a_{2n+1} = 0$$

So $L_1 = L_2 = L$. Since both the even and odd subsequences converge to the same limit L, the entire sequence of partial sums (s_n) converges to L, thus the series converges. ☐

Theorem 2.36. Cauchy Criterion: The series $\sum a_n$ converges iff

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall m, n \geq N : |\sum_{k=n}^{m} a_k| < \epsilon$$

Theorem 2.37. Root Criterion (Wurzelkriterium): Let $\rho = \limsup \sqrt[n]{|a_n|}$. If $\rho < 1$, abs conv; if $\rho > 1$, div.

Theorem 2.38. Ratio Criterion (Quotientenkriterium): Let $\rho = \lim \left|\frac{a_{n+1}}{a_n}\right|$. If $\rho < 1$, abs conv; if $\rho > 1$, div.

Example 2.39.

$$\sum_{n=1}^{\infty} n^4 e^{-n}$$

$$\rho = \lim_{n\to\infty} \frac{(n+1)^4 e^{-(n+1)}}{n^4 e^{-n}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^4 e^{-1} = e^{-1} < 1$$

Therfore the series converges.

Definition 2.40. A **power series** (Potenzreihe) centered at a has the form

$$\sum_{k=0}^{\infty} c_k(x-a)^k$$

Theorem 2.41. Every power series has a **convergence radius** R, with the **interval of convergence** $(a-R, a+R)$. R is computed by the formula $R = \frac{1}{\limsup |c_n|^{1/n}}$.

Example 2.42.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}x^n$$

$$\rho = \limsup \frac{1}{n^{1/n}} = 1 \implies R = 1$$

Therfore the series converges for

$$x \in (-1, 1)$$

.

Remark 2.43. For $x = 1$ we have the harmonic series, which diverges. For $x = -1$ we have the alternating harmonic series, which converges.

3 Functions

Definition 3.1. A **function** $f : D \to C$ is a map from a domian D to a codomain C s.t. for each $x \in D$ there is exactly one $y \in C$ st. $f(x) = y$.

Remark 3.2. The input is called the **independent variable** (argument) while the output is called the **dependent variable**.

Definition 3.3. A function f is called **bouned** if there exists $M \in \mathbb{R}$ such that $|f(x)| \leq (\geq) M$ for all $x \in D$.

Definition 3.4. A function f is called

- **injective** iff $\forall x_1, x_2 \in A : f(x_1) = f(x_2) \implies x_1 = x_2$.
- **surjective** iff $\forall y \in B \exists x \in A : f(x) = y$.
- **bijective** iff f is injective and surjective.

Example 3.5. Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$. Observe that f is not surjective as no output is negative, nor is it injective as for ex. $f(1) = f(-1) = 1$. But $g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $g(x) = x^2$ is both injective and surjective.

Definition 3.6. The **restriction** of f under $D' \subset \mathbb{D}$ is defined as $f \restriction_{D'}: D' \to C, x \mapsto f(x) \quad \forall x \in \mathbb{D}$.

Remark 3.7. Note that $f \restriction_{D'}$ and f are two diffrent functions.

Definition 3.8. Let $f : D \to C$ and $g : C \to B$ be functions then

$$g \circ f : D \to B \quad \text{with} \quad g \circ f(x) = g(f(x)) \quad \forall x \in D$$

is called the **composition** of f and g.

Definition 3.9. The **identity function** $id_D : D \to D$ is defined as $id_D(x) = x \quad \forall x \in D$.

Definition 3.10. If a function $f : D \to C$ is bijective then there exists a unique function $f^{-1} : C \to D$ such that

$$f(f^{-1}(y)) = id_C = y \quad \forall y \in C$$

and

$$f^{-1}(f(x)) = id_D = x \quad \forall x \in D$$

f^{-1} is called the **inverse function** of f.

Definition 3.11. A function $f : \mathbb{R} \to \mathbb{R}$ is called

- **even** iff $\forall x \in \mathbb{R} : f(-x) = f(x)$
- **odd** iff $\forall x \in \mathbb{R} : f(-x) = -f(x)$

Example 3.12. The function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^3$ is odd as $f(-x) = (-x)^3 = -x^3 = -f(x)$ while the function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = x^2$ is even as $g(-x) = (-x)^2 = x^2 = g(x)$.

Definition 3.13. A function $f : \mathbb{R} \to \mathbb{R}$ is called

- **(strictly) monotonically increasing** iff $\forall x_1, x_2 \in \mathbb{R} : x_1 < x_2 \implies f(x_1) < f(x_2)$
- **(strictly) monotonically decreasing** iff $\forall x_1, x_2 \in \mathbb{R} : x_1 < x_2 \implies f(x_1) > f(x_2)$
- **(strictly) monotonic** iff f is either (strictly) monotonically increasing or (strictly) monotonically decreasing

Theorem 3.14. Every strictly monotonic function is injective.

Proof. Let f be strictly monotonic (wlog increasing) and assume it is not injective then there exists $x_1 \neq x_2$ such tat $f(x_1) = f(x_2)$. Wlog assume $x_1 < x_2$ then $f(x_1) < f(x_2)$ which contradicts the assumption that $f(x_1) = f(x_2)$. □

Remark 3.15. Observe that functions do not have to be even or odd.

Definition 3.16. The **graph** of a function $f : D \to C$ is the set of all ordered pairs $(x, f(x))$ for $x \in D$. It is **konvex** if it lies below the line segment connecting any two points on the graph. Similarly it is called **konkav** if it lies above the line segment connecting any two points on the graph.

3.1 Limits in the context of functions

Definition 3.17. Let $f : \mathbb{D} \to \mathbb{R}$ be a function satisfying $\mathbb{D} \cap (x_0 - \delta, x_0 + \delta) \neq \emptyset$ for all $\delta > 0$ and $x_0 \in \mathbb{R}$. We say that $\lim_{x \to x_0} f(x) = L$ iff

$$\forall \varepsilon > 0 \exists \delta > 0 \text{s.t.}$$

$$x \in \mathbb{D} \cap (x_0 - \delta, x_0 + \delta) \implies |f(x) - L| < \varepsilon$$

The limit L is unique.

Remark 3.18. The intuition behind the definition of the limit is that for any given distance $\varepsilon > 0$ around L, we can find a $\delta > 0$ such that for all $x \in \mathbb{D}$ that are at least δ close to x_0, we have $|f(x) - L| < \varepsilon$. So for any boound on the output (ε) there exists a bound on the input (δ) such that all elements in the input bound $(x_0 - \delta, x_0 + \delta)$ are mapped to the output bound $(L - \varepsilon, L + \varepsilon)$.

Definition 3.19. A function $f : \mathbb{D} \to \mathbb{R}$ has in x_0 a

- **right-sided limit** $\lim_{x \searrow x_0} f(x) = \lim_{x \to x_0^+} f(x) = L$ iff

$$\forall \varepsilon > 0 \exists \delta > 0 \quad \text{s.t.} \quad \forall x \in \mathbb{D}(x_0 < x < x_0 + \delta \implies |f(x) - L| < \varepsilon)$$

- **left-sided limit** $\lim_{x \nearrow x_0} f(x) = L$ iff

$$\forall \varepsilon > 0 \exists \delta > 0 \quad \text{s.t.} \quad \forall x \in \mathbb{D}(x_0 - \delta < x < x_0 \implies |f(x) - L| < \varepsilon)$$

Definition 3.20. A function $f : \mathbb{D} \to \mathbb{R}$ has a

- **limit for x approaching infinity** $\lim_{x \to \infty} f(x) = L$ iff

$$\forall \varepsilon > 0 \exists M > 0 \quad \text{s.t.} \quad \forall x \in \mathbb{D}(x > M \implies |f(x) - L| < \varepsilon)$$

- **limit for x approaching negative infinity** $\lim_{x \to -\infty} f(x) = L$ iff

$$\forall \varepsilon > 0 \exists M > 0 \quad \text{s.t.} \quad \forall x \in \mathbb{D}(x < M \implies |f(x) - L| < \varepsilon)$$

Definition 3.21. A function $f : \mathbb{D} \to \mathbb{R}$ has in x_0 a

- **improper limit approaching infinity** $\lim_{x \to x_0} f(x) = \infty$ iff

$$\forall N > 0 \exists \delta > 0 \quad \text{s.t.} \quad \forall x \in \mathbb{D}(0 < |x - x_0| < \delta \implies f(x) > N)$$

- **improper limit approaching negative infinity** $\lim_{x \to x_0} f(x) = -\infty$ iff

$$\forall N > 0 \exists \delta > 0 \quad \text{s.t.} \quad \forall x \in \mathbb{D}(0 < |x - x_0| < \delta \implies f(x) < -N)$$

Theorem 3.22. Let $A \in \mathbb{R}$ and assume $\lim_{x \to x_0} f(x) = K(\neq \pm\infty)$ and $\lim_{x \to x_0} g(x) = L(\neq \pm\infty)$.

- $\lim_{x \to x_0}(f(x) + g(x)) = K + L$
- $\lim_{x \to x_0}(f(x) - g(x)) = K - L$
- $\lim_{x \to x_0}(f(x)g(x)) = KL$
- $\lim_{x \to x_0}(f(x)/g(x)) = K/L$
- $\lim_{x \to x_0}(Af(x)) = AK$
- $\lim_{x \to x_0}(f(x)/g(x)) = K/L$ if $L \neq 0$

3.2 Continuity of functions

Definition 3.23. A function $f : D \to \mathbb{R}$ is called **continous** at x_0 if

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that } \forall x \in I(\mid x - x_0 \mid < \delta \implies \mid f(x) - f(c) \mid > \epsilon)$$

The function is called continuous if it is continuous at every point in its domain.

Remark 3.24. The **intuition** behind this definition is that for any given distance $\epsilon > 0$ around $f(x_0)$, we can find a $\delta > 0$ such that for all $x \in D$ that are at least δ close to x_0, we have $|f(x) - f(x_0)| < \epsilon$. So for any bound on the output (ϵ) there exists a bound on the input (δ) such that all elements in the input bound $(x_0 - \delta, x_0 + \delta)$ are mapped to the output bound $(f(x_0) - \epsilon, f(x_0) + \epsilon)$.

Definition 3.25. Let $f : D \to \mathbb{R}$ be a function. Then f is called **left(right)-continuous** at x_0 if

$$\lim_{x \to x_0^{-(+)}} f(x) = f(x_0)$$

Theorem 3.26. Let $f : D \subset \mathbb{D}(f) \subset \mathbb{R}$ be a function and $x_0 \in D$. f is continuous at x_0 iff for every sequence $(x_n)_{n \in \mathbb{N}} \subset D$ with $\lim_{n \to \infty} x_n = x_0$ it holds that $\lim_{n \to \infty} f(x_n) = f(x_0)$.

Remark 3.27. Continuity can also be viewed in terms of intervals, where a function f is continuous iff for every open interval $I \subset \mathbb{R}$ it holds that $f^{-1}(I)$ is open.

Lemma 3.28. Let f, g be continuous functions, the follwing holds:

- $f + g$ is continuous
- $f - g$ is continuous
- $f \cdot g$ is continuous
- $c \cdot f$ is continuous for all $c \in \mathbb{R}$
- $\frac{f}{g}$ is continuous if $g(x) \neq 0$
- $f \circ g$ is continuous

Proof. Addition/Subtraction/Multiplication/Division follow from the sequence-definition of continuity and the properties of limits. It is left to prove that $f \circ g$ is continuous if f and g are. Let g be continuous at a point c, and let f be continuous at the point $g(c)$.

By the limit definition of continuity, since g is continuous at c, we know:

$$\lim_{x \to c} g(x) = g(c)$$

Similarly, since f is continuous at $g(c)$, for any variable y approaching $g(c)$, we know:

$$\lim_{y \to g(c)} f(y) = f(g(c))$$

To find the limit of the composite function $f(g(x))$ as $x \to c$, we substitute $g(x)$ for our input y. Because f is continuous, we are allowed to pass the limit directly inside the function f:

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right)$$

Now, we just substitute the known limit of $g(x)$:

$$f\left(\lim_{x \to c} g(x)\right) = f(g(c))$$

Therefore, $\lim_{x \to c}(f \circ g)(x) = (f \circ g)(c)$. This perfectly satisfies the definition of continuity, proving that $f \circ g$ is continuous at c ◻

Theorem 3.29. Let I be an interval and let $f : I \to J$ be a continuous and bijective function. Then J is an interval too and the inverse function $f^{-1} : J \to I$ is continuous.

Theorem 3.30. Let $f : D \to \mathbb{R}$ be a continuous function. Let $c \in [f(a), f(b)]$ then there exists a $x_0 \in [a, b]$ such that $f(x_0) = c$.

Proof. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Without loss of generality, assume $f(a) < f(b)$. We choose an arbitrary $c \in \mathbb{R}$ such that $f(a) \le c \le f(b)$ and define the set:

$$X := \{t \in [a, b] \mid f(t) \le c\}$$

Observe that $X \neq \emptyset$ (since $a \in X$ as $f(a) \le c$) and X is bounded above by b. Thus, by the completeness axiom of the real numbers, the supremum $s := \sup X$ exists, and clearly $s \in [a, b]$.

By the definition of the supremum, for all $n \in \mathbb{N}$, there exists $t_n \in X \cap [s - \frac{1}{n}, s]$. Because $t_n \to s$ as $n \to \infty$ and f is continuous, we have $\lim_{n\to\infty} f(t_n) = f(s)$. Since $t_n \in X$, we know $f(t_n) \le c$ for all $n \in \mathbb{N}$. By the limit-preserving properties of inequalities, it follows that $f(s) \le c$.

Now, assume for the sake of contradiction that $f(s) < c$. Since $f(s) < c \le f(b)$, we know $s \neq b$, which implies $s < b$.

Let $\epsilon = c - f(s) > 0$. By the continuity of f at s, there exists a $\delta > 0$ such that for all $y \in [a, b]$:

$$|y - s| < \delta \implies |f(y) - f(s)| < \epsilon$$

This implies $f(y) < f(s) + \epsilon = c$.

Now, define $y := \min(b, s + \frac{\delta}{2})$. Since $s < b$ and $\delta > 0$, we strictly have $y > s$. Furthermore, $y \in [a, b]$ and $|y - s| = \frac{\delta}{2} < \delta$. Therefore, by our continuity implication, $f(y) < c$, which means $y \in X$.

However, we have found an element $y \in X$ such that $y > s$, which directly contradicts the fact that s is an upper bound (the supremum) of X.

Thus, our assumption that $f(s) < c$ must be false. Since we have already established that $f(s) \le c$, we conclude that $f(s) = c$. $\square$

Definition 3.31. A bounded and closed interval is called **compact**.

Example 3.32. The interval $[0, 1]$ is compact while $[0, \infty)$ is not.

Lemma 3.33. Let $[a, b]$ be a compact interval and let $(x_n)_{n\in\mathbb{N}} \subset [a, b]$ be a sequence. Then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} x_{n_k} = x_0 \text{ for some } x_0 \in [a, b].$$

Definition 3.34. Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a function

- If for $x_0 \in D$ it holds that $f(x_0) \ge f(x) \forall x \in D$, then f has a local maximum at x_0.

- If for $x_0 \in D$ it holds that $f(x_0) \leq f(x) \forall x \in D$, then f has a local minimum at x_0.
- Local maxima and minima are called local extrema.

Theorem 3.35. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a continous function and I be a compact interval. Then f is bounded and attains its maximum and minimum value on I. Hence

$$\exists x_{min}, x_{max} \in I \text{ such that } f(x_{min}) \leq f(x) \leq f(x_{max}) \forall x \in I$$

4 Differential calculus

Definition 4.1. The **average rate of change** of a function f in the domain between a and $a+h$ is defined as

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$$

and also called the **difference quotient**.

Definition 4.2. The **derivative** f' of a function f at the coordinate a is defined as

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = lim_{b \to a} \frac{f(b) - f(a)}{b - a} = \frac{df}{dx}|_{x=a} = f'(a)$$

Remark 4.3. The derivative $f'(a)$ can geometrically interpreted as the incline/decline of the graph of f at the point $(a, f(a))$.

Definition 4.4. If for a function f at the cooridnate a the derivative $f'(a)$ exists then f is called **differentiable** at a. If f is differentiable for every $a \in \mathbb{D}$ then f is called differentiable.

Example 4.5. let $f(x) = |x|$ we can see that for $a = 0$ there is no derivative (This can be seen in the graph as a "sharp" corner)

Remark 4.6. We can generally expect that there are no derivatives at undefined points $a \notin \mathbb{D}$.

Lemma 4.7. If f is differentiable at x_0 then f is continous at x_0.

Proof. Assume f differentiable then

$$\lim_{x \to x_0} f(x) = \lim_{h \to 0} f(x_0 + h) = \lim_{h \to 0} f(x_0 + h) - f(x_0) + f(x_0)$$

$$= f(x_0) + \lim_{h \to 0} f(x_0 + h) - f(x_0) = f(x_0) + \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \cdot h$$

$$= f(x_0) + \lim_{h \to 0} h \cdot \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Observe that the last term exists by the assumption hence

$$= f(x_0) + 0 \cdot f'(x_0) = f(x_0)$$

□

Definition 4.8. If the limit

$$\lim_{h>0\to 0} \frac{f(a+h)-f(a)}{h}$$

existst then we call it the **right-sided derivative** (or the left-sided derivative analogue)

Definition 4.9. For a function f we can iteratively define higher derivatives for $n \in \mathbb{N}_0$

$$f^(0) = f, f^(1) = f', f^(2) = f'', \dots f^(n+1) = (f^(n))'$$

if $f^(n)$ exists we call it n**-times differentiable**.

Definition 4.10. if $f^(n)$ is continous we call f n**-times continous differentiable**. The set of all n-times continous differentiable functions on D is denoted as $C^n(D)$

Definition 4.11. We call $C^\infty(D) := \cap_{n=0}^{\infty} C^n(D)$ the set of **smooth functions**.

Definition 4.12. If for a function f and some $x_0 \in \mathbb{D}$ and $\delta > 0$ we have

$$f(x_0) \geq (\leq) f(x) \quad \forall x \in \mathbb{D} \cap (x_0 - \delta, x_0 + \delta)$$

then x_0 is called a **local maximum (minimum)**.

4.1 Derivation rules

Theorem 4.13. For functions $f(x)$ we have

- $f(x) = ax^p \quad \Longrightarrow \quad f'(x) = apx^{p-1}$
- $f(x) = a^x \quad \Longrightarrow \quad f'(x) = ln(a) \cdot a^x$
- $f(x) = sin(x) \quad \Longrightarrow \quad f'(x) = cos(x)$
- $f(x) = cos(x) \quad \Longrightarrow \quad f'(x) = -sin(x)$
- $f(x) = tan(x) \quad \Longrightarrow \quad f'(x) = \frac{1}{cos^2(x)} = 1 + tan^2(x)$
- $f(x) = sinh(x) \quad \Longrightarrow \quad f'(x) = cosh(x)$
- $f(x) = cosh(x) \quad \Longrightarrow \quad f'(x) = sinh(x)$
- $f(x) = ln(x) \quad \Longrightarrow \quad f'(x) = \frac{1}{x}$

- $f(x) = arcsin(x) \quad \implies \quad f'(x) = \frac{1}{\sqrt{1-x^2}}$
- $f(x) = arccos(x) \quad \implies \quad f'(x) = \frac{-1}{\sqrt{1-x^2}}$
- $f(x) = arctan(x) \quad \implies \quad f'(x) = \frac{1}{1+x^2}$

Theorem 4.14. Sum Rule: For the n-th derivative of a sum:

$$(f+g)^{(n)} = f^{(n)} + g^{(n)}$$

Proof. This can be proved inductively by applying the definition of the derivative n times.

$$(f+g)'(x) = lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$$

$$= lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= f'(x) + g'(x)$$

For $n = 2$ we have

$$(f+g)''(x) = (f'+g')'(x) = f''(x) + g''(x)$$

and so on. □

Theorem 4.15. General Leibniz Rule (Product): The n-th derivative of a product is given by:

$$(f\cdot g)^{(n)} = \sum_{k=0}^{n}\binom{n}{k} f^{(n-k)}g^{(k)}$$

Proof. This is proved analogously to the sum rule.

$$(f\cdot g)'(x) = lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$= lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}$$

$$= lim_{h\to 0}\frac{g(x+h)(f(x+h)-f(x))}{h} + lim_{h\to 0}\frac{f(x)(g(x+h)-g(x))}{h}$$

$$= g(x)f'(x) + f(x)g'(x)$$

For $n = 2$ we have

$$(f\cdot g)''(x) = (f'g+fg')'(x) = f''g + f'g' + f'g' + fg'' = f''g + 2f'g' + fg''$$

and so on. □

Theorem 4.16. Chain Rule: For the composition of functions:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Proof. Observe that

$$g(y) = g(y_0) + g'(y_0)(y - y_0) + g(y) - g(y_0) - g'(y_0)(y - y_0)$$

with the first two summands being a linear (tangential) approximation of $g(y)$ near y_0

$$g(y) \approx g(y_0) + g'(y_0)(y - y_0)$$

By taking out $(y - y_0)$ of the last two summands in the original equation we get:

$$g(y) = g(y_0) + g'(y_0)(y - y_0) + \left[\frac{g(y) - g(y_0)}{y - y_0} - g'(y_0)\right](y - y_0)$$

We define $\epsilon = \frac{g(y)-g(y_0)}{y-y_0} - g'(y_0)$ and get

$$\lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0}$$

$$= \lim_{x \to x_0} \left(f(f(x_0)) + g'(f(x_0))[f(x) - f(x_0)] + \epsilon f(x)[f(x) - f(x_0)] - g(f(x_0))\right) \frac{1}{x - x_0}$$

$$= g'(y_0) f'(x_0)$$

□

Theorem 4.17. Quotient Rule: For $g(x) \neq 0$:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Remark 4.18. This follows directly from the product and the chain rule.

Theorem 4.19. Inverse Functions: If f is differentiable and injective:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

4.2 Bernoulli-L'Hopital

Theorem 4.20. Let $I \subset \mathbb{R}$ be a an intervall, $f : I \to \mathbb{R}$, and let x_0 be a local Extrema of f. Then one of the following holds:

- x_0 is an endpoint of I
- f is not differentiable at x_0

- f is differentiable at x_0 and $f'(x_0) = 0$

Remark 4.21. From $f'(x_0) = 0$ we can not conclude that x_0 is an extremum.

Theorem 4.22. Rolle's Theorem: Let $a < b$ with $f(a) = f(b)$ and $f : [a, b] \to \mathbb{R}$ a completely continous function on $[a, b]$ and differentiable on (a, b) then $\exists \xi \in (a, b)$ with $f'(\xi) = 0$

Proof. As $[a, b]$ is compact we can conclud that there exists ξ such that $f(\xi)$ is max or min hence $f'(\xi) = 0$. If $\xi \in (a, b)$ we found our ξ else the max and min is at a therfore also at b hence the function is constant which implies that $f'(x) = 0 \forall x \in [a, b]$. □

Theorem 4.23. Mean-Value-Theorem: Let $a < b, f : [a, b] \to \mathbb{R}$ a completely continous function on $[a, b]$ and differentiable on (a, b) then $\exists \xi \in (a, b)$ with

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

Proof. We define a function g with

$$g(x) := f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

and observe that g fulfilles the prequisites to apply rolles theorem. Hence

$$\begin{aligned} &\exists \xi g'(\xi) = 0 \\ \implies &\exists \xi f'(\xi) - \frac{f(b) - f(a)}{b - a}\xi = 0 \\ \implies &\exists \xi f'(\xi) = \frac{f(b) - f(a)}{b - a} \end{aligned}$$

□

Theorem 4.24. Cauchy's Mean-Value-Theorem: Let $a < b, f : [a, b] \to \mathbb{R}$ a completely continous function on $[a, b]$ and differentiable on (a, b) then $\exists \xi \in (a, b)$ with

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

Proof. We define a function

$$F(x) := g(x)(f(b) - f(a)) - f(x)(g(b) - g(a))$$

and observe that F fulfilles the prequisites to apply rolles theorem. Hence

$$\exists \xi F'(\xi) = 0$$
$$\implies \exists \xi g'(\xi)(f(b) - f(a)) - f'(\xi)(g(b) - g(a)) = 0$$
$$\implies \exists \xi \frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$\square$

Theorem 4.25. Bernoulli-L'Hopital's Rule: Let $f, g : I \to \mathbb{R}$ be differentiable functions on (a, b) and let $x_0 \in (a, b)$. If $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in (a, b)$. If

$$\lim_{x \to x_0} |f(x)| = \lim_{x \to x_0} |g(x)| \in \{0, \infty\} \quad \text{and} \quad \exists L = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$$

Example 4.26. Let $f(x) = sin(x)$ and $g(x) = x$ we have $sin(0) = 0$ and $g(0) = 0$ further we have $f'(x) = cos(x)$ and $g'(x) = 1 \neq 0$ hence

$$\lim_{x \to 0} \frac{sin(x)}{x} = \lim_{x \to 0} \frac{cos(x)}{1} = 1$$

Remark 4.27. This example can also be solved via the definition of the derivative.

$$\lim_{x \to 0} \frac{sin(x)}{x} = \lim_{x \to 0} \frac{sin(x) - sin(0)}{x - 0} = sin'(0) = cos(0) = 1$$

4.3 Convex and Concave graphs

Definition 4.28. A function $f : I \to \mathbb{R}$ is called **convex** iff

$$\forall a, b \in I \forall t \in (0, 1) : f((1 - t)a + bt) \leq (1 - t)f(a) + tf(b)$$

Remark 4.29. This is equivalent to saying that for any $x \in (a, b)$ we have

$$\frac{f(x) - f(a)}{x - a} \geq \frac{f(b) - f(x)}{b - x}$$

Definition 4.30. A function $f : I \to \mathbb{R}$ is called **concave** iff

$$\forall a, b \in I \forall t \in (0,1) : f((1-t)a + bt) \geq (1-t)f(a) + tf(b)$$

Theorem 4.31. Let $(a,b) \subset \mathbb{R}$ and $f : (a,b) \to \mathbb{R}$ be differentiable. Then

$$\forall x : f'(x) \geq 0 \quad \iff \quad f \text{ is increasing}$$

Remark 4.32. If $f'(x) = 0$ for all x then f is constant.

Theorem 4.33. Let $(a,b) \subset \mathbb{R}$ and $f : (a,b) \to \mathbb{R}$ be differentiable. Then

$$f \text{ is convex} \quad \iff \quad f' \text{ is increasing}$$

Proof. $\Longleftarrow$: Assume f' is increasing and let $x \in (a,b)$ be arbitrary. Then we have have by the mean value theorem that there exists $\psi \in (a,x)\xi \in (x,b)$ s.t.:

$$f'(\psi) = \frac{f(x) - f(a)}{x - a} \quad \text{and} \quad f'(\xi) = \frac{f(b) - f(x)}{b - x}$$

As f' is increasing we know that $f'(\psi) \leq f'(\xi)$ and therefore

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(x)}{b - x}$$

$\Longrightarrow$: Assume f is convex and let h be small enough s.t. $x = a + h \leq b - h$ then by the convexity of f we have

$$\frac{f(a+h) - f(a)}{h} \leq \frac{f(b-h) - f(a+h)}{(b-h) - (a+h)} \leq \frac{f(b) - f(b-h)}{h}$$

and as f is differentiable they exists and are comparable therfore $f'(a) \leq f'(b)$ □

Theorem 4.34. Let $(a,b) \subset \mathbb{R}$ and $f : (a,b) \to \mathbb{R}$ be twice differentiable. Then

$$f \text{ is convex} \quad \iff \quad f'' \geq 0$$

4.4 Taylor Series

Lemma 4.35. We can localy approximate the graph of f at x_0 with the tangential line $t(x)$

$$f(x) \approx t(x) = f(x_0) + f'(x_0)(x - x_0)$$

Example 4.36. Let $f(x) = x^2$ and $x_0 = 2$ then

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) = 2^2 + 2(2)(x - 2) = 4 + 4(x - 2) = 4x - 4$$

Lemma 4.37. To even more accurately approximate the function f we can add further terms to the tangential line expansion

$$f(x) \approx t(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

Remark 4.38. The **Intuition** behind this is that we want a function that looks something like this:

$$P(x) = A + B(x - x_0) + C(x - x_0)^2$$

if $P(x_0) = f(x_0)$ we have $A = f(x_0)$. Further if $P'(x_0) = f'(x_0)$ we have $B = f'(x_0)$ and if $P''(x_0) = f''(x_0)$ we have $C = f''(x_0)/2$.

Example 4.39. Let $f(x) = x^2$ and $x_0 = 2$ then

$$f(x) \approx f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2}(x-x_0)^2 = 2^2+2(2)(x-2)+\frac{2}{2}(x-2)^2 = 4+4(x-2)+(x-2)^2$$

Theorem 4.40. Taylor's Theorem: Let $f : (a, b) \to \mathbb{R}$ be $n+1$ times differentiable. Then

$$f(x) = P_n(x) + R(x)$$

with $P_n(x)$ the Taylor Polynomial of degree n and $R(x)$ the remainder term

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

$$R(x) = \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - x_0)^{n+1} \quad \text{for some } \xi \in (x_0, x) \text{ or } (x, x_0)$$

Remark 4.41. There are other forms of the remainder term, such as the Lagrange form and the Cauchy form.

Example 4.42. Assume we know that the equation

$$\sqrt{3x - 4} + 2 \sin x - x = \frac{1}{2}$$

has a solution near $x_0 = 2$. We can use Taylor's Theorem to approximate the solution. Let $f(x) = \sqrt{3x - 4} + 2 \sin x - x$. Then

$$f(2) = \sqrt{2} + 2 \sin 2 - 2$$

$$f'(2) = \frac{3}{2\sqrt{2}} + 2\cos 2 - 1$$

Therefore the Taylor Approximation up to degree 2 is

$$f(x) \approx f(2) + f'(2)(x-2)$$

Setting this to $\frac{1}{2}$ and solving for x gives us the linear approximation of the solution.

Definition 4.43. The **Taylor series expansion** of an infinitely differentiable function $f : (a, b) \to \mathbb{R}$ at x_0 is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Example 4.44. The following Taylor series converge for all $x \in \mathbb{R}$:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Important 4.45. Not all infinitely differentiable functions have a Taylor series expansion that converges to the function itself. For example $ln(1 + x)$, $\frac{1}{1-x}$, $(1 + x)^p$ only converge for $-1 < x < 1$

Theorem 4.46. Let $f(x)$ and $g(x)$ be two powerseries with x in their Radius of Convergence R.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$g(x) = \sum_{n=0}^{\infty} b_n x^n$$

Then

$$f(x)g(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n$$

Theorem 4.47. Let $f(x)$ be a power series with x in its Radius of Convergence R.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

then

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Theorem 4.48. Let $f(x)$ be a power series with x in its Radius of Convergence R.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

then

$$\int f(x)dx = \sum_{n=0}^{\infty} \int a_n x^n dx$$

Example 4.49. Assume we know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$ and we want to calculate the Taylor-Expansion of $g(x) = (1-x)^{-2}$. We observe that $g(x) = (\frac{d}{dx})(1-x)^{-1}$. Therefore

$$g(x) = \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$$

5 Integration

5.1 Indefinite Integrals and Techniques

Definition 5.1. Functions $F : I \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in (a, b)$ are called **antiderivatives** of f.

Definition 5.2. The family of all anti-derivatives of a function $f : I \to \mathbb{R}$ for $I \subseteq \mathbb{R}$ an open interval is called the **indefinite integral** and is denoted by

$$\int f(x)dx = \{F : I \to \mathbb{R} : F'(x) = f(x)\} = F(x) + C$$

Theorem 5.3. Let $f, g : I \to \mathbb{R}$ be integrable functions and let $c \in \mathbb{R}$. Then

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$$

$$\int cf(x)dx = c \int f(x)dx$$

Remark 5.4. Some important basic antiderivatives:

- $f(x) = 0 \quad \implies \quad F(x) = C$
- $f(x) = 1 \quad \implies \quad F(x) = x + C$
- $f(x) = x^p \quad \implies \quad F(x) = \frac{1}{p+1}x^{p+1} + C \quad (p \neq -1)$
- $f(x) = e^x \quad \implies \quad F(x) = e^x + C$
- $f(x) = a^x \quad \implies \quad F(x) = \frac{a^x}{\ln(a)} + C$
- $f(x) = \sin(x) \quad \implies \quad F(x) = -\cos(x) + C$
- $f(x) = \cos(x) \quad \implies \quad F(x) = \sin(x) + C$
- $f(x) = \frac{1}{x} \quad \implies \quad F(x) = \ln|x| + C$
- $f(x) = \frac{1}{1+x^2} \quad \implies \quad F(x) = \arctan(x) + C$

Theorem 5.5. Integration by Parts: For differentiable functions $f(x)$ and $g(x)$ we have

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

Remark 5.6. The equation is derived from the product rule for differentiation.

Example 5.7. Let $f'(x) = e^x \implies f(x) = e^x$ and $g(x) = x \implies g'(x) = 1$. Then

$$\int xe^x dx = xe^x - \int e^x \cdot 1 dx = xe^x - e^x + C$$

Theorem 5.8. Substitution Rule: For functions $f(x)$ and $g(x)$ we have

$$\int f(g(x))g'(x)dx = \int f(u)u'dx = \int f(u)\frac{du}{dx}dx = \int f(u)du = F(g(x)) + C$$

where F is an antiderivative of f.

Remark 5.9. The equation is derived from the chain rule for differentiation.

Example 5.10. Evaluate $\int 2xe^{x^2} dx$. Let $u = x^2 \implies \frac{du}{dx} = 2x \implies du = 2xdx$. Substituting gives:

$$\int e^u du = e^u + C = e^{x^2} + C$$

5.2 The Definite Integral (Riemann Integral)

Remark 5.11. Upper and Lower Sums (Stair Functions): One can partition the integration interval into subintervals. For each subinterval we can approximate the area of the function with the area of a rectangle. The sum of the areas of these rectangles is an approximation of the area under the function. Taking the infimum of the function on each subinterval yields the lower sum L. Taking the supremum yields the upper sum U.

Definition 5.12. Let U and L denote the upper and lower sums of a bounded function f. Then f is called **Riemann-integrable** iff $\inf U = \sup L$. The value of the definite integral $\int_a^b f(x)dx$ is then this common value.

Theorem 5.13. Integrability Conditions:

- All continuous functions on $[a, b]$ are Riemann-integrable.
- All monotone functions on $[a, b]$ are Riemann-integrable.

Theorem 5.14. Let f and g be integrable and $c \in \mathbb{R}$ then:

- Linearity: $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- Monotonicity: $f(x) \leq g(x) \implies \int_a^b f(x)dx \leq \int_a^b g(x)dx$
- Triangle inequality: $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$
- Inversion of bounds: $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- Partition of interval: $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$

Theorem 5.15. Mean Value Theorem for Integrals: Let $f : [a,b] \to \mathbb{R}$ be continuous. Then there exists a $c \in [a,b]$ such that

$$\int_a^b f(x)dx = (b-a)f(c)$$

Theorem 5.16. Fundamental Theorem of Integral Calculus: Let $f : [a,b] \to \mathbb{R}$ be continuous. Then for all $C \in \mathbb{R}$, $F(x)$ is an antiderivative of f where:

$$F(x) = \int_a^x f(y)dy + C$$

Proof. As f is continuous it is integrable on $[a,x]$ for any $x \in [a,b]$. Let $x_0 \in [a,b]$ and $\varepsilon > 0$ arbitrary. Then there exist a $\delta > 0$ such that for any $x \in [a,b]$ with $0 < |x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \varepsilon$. Then for $x \in [a,b]$ with $0 < |x - x_0| < \delta$:

$$\begin{aligned}
\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| &= \left|\frac{1}{x - x_0}\left(\int_a^x f(y)dy - \int_a^{x_0} f(y)dy\right) - f(x_0)\right| \\
= \left|\frac{1}{x - x_0}\int_{x_0}^x f(y)dy - \frac{1}{x - x_0}\int_{x_0}^x f(x_0)dy\right| &= \left|\frac{1}{x - x_0}\int_{x_0}^x (f(y) - f(x_0))dy\right| \\
&\leq \frac{1}{|x - x_0|}\left|\int_{x_0}^x |f(y) - f(x_0)|dy\right| \leq \frac{1}{|x - x_0|}\left|\int_{x_0}^x \varepsilon dy\right| = \varepsilon
\end{aligned}$$

□

Proof. **Uniqueness of Antiderivatives:** Let $G(x)$ be another antiderivative of $f : [a,b] \to \mathbb{R}$ where $F(x) = \int_a^x f(y)dy + C$. We define $H(x) = F(x) - G(x)$ and get

$$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0$$

Hence $H(x)$ must be a constant C' and $F(x) = G(x) + C'$. Thus, antiderivatives are unique up to a constant. □

5.3 Sequences of Functions and Integrals

Definition 5.17. Let $(f_n)_{n\in\mathbb{N}_0}$ be a sequence of functions.

- **Pointwise Convergence**: It converges pointwise to f if for every x, the sequence of real numbers $(f_n(x))$ converges to $f(x)$.
- **Uniform Convergence**: It converges uniformly to f if for every $\epsilon > 0$, there exists an N such that for all $n \geq N$ and **for all** x simultaneously, $|f_n(x)-f(x)| < \epsilon$.

Theorem 5.18. Swapping Limit and Integral: If a sequence of integrable functions (f_n) converges **uniformly** to f on $[a, b]$, then the limit function f is also integrable, and we can swap the limit and the integral:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$$

5.4 Definite Integration Techniques

Lemma 5.19. If $F : [a, b] \to \mathbb{R}$ is continuously differentiable, then:

$$F(x) = F(a) + \int_a^x F'(t)dt \quad \text{and} \quad \int_a^b f(t)dt = F(b) - F(a) = \Big[F(x)\Big]_a^b$$

Lemma 5.20. Substitution for Definite Integrals: Let f be continuous and g continuously differentiable.

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Version 2: If $f'(x) \neq 0$ and f^{-1} is the inverse of f:

$$\int_a^b g(f(x))dx = \int_{f(a)}^{f(b)} g(y)(f^{-1})'(y)dy$$

Example 5.21. Evaluate $\int_0^{\pi/2} \sin(x)\cos(x)dx$. Let $u = \sin(x)$. Then $\frac{du}{dx} = \cos(x) \implies du = \cos(x)dx$. For definite integrals, we must also substitute the bounds: Lower bound: $x = 0 \implies u = \sin(0) = 0$. Upper bound: $x = \pi/2 \implies u = \sin(\pi/2) = 1$. Substituting gives:

$$\int_0^1 u\,du = \left[\frac{1}{2}u^2\right]_0^1 = \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 = \frac{1}{2}$$

Lemma 5.22. Integration by Parts for Definite Integrals: Let f, g be continuously differentiable.

$$\int_a^b f'(x)g(x)dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b f(x)g'(x)dx$$

Example 5.23. Evaluate $\int_1^e \ln(x)dx$. We can write this as $\int_1^e 1 \cdot \ln(x)dx$. Let $f'(x) = 1 \implies f(x) = x$. Let $g(x) = \ln(x) \implies g'(x) = \frac{1}{x}$. Using integration by parts for definite integrals:

$$\int_1^e 1 \cdot \ln(x)dx = \Big[x\ln(x)\Big]_1^e - \int_1^e x \cdot \frac{1}{x}dx$$

$$= (e\ln(e) - 1\ln(1)) - \int_1^e 1dx = e - \Big[x\Big]_1^e = e - (e-1) = 1$$

6 Differential Equations

6.1 Modelling and Definitions

Example 6.1. Assume at time $t = 0$ one takes in a certain amount of alcohol. The alcohol first flows to the stomach and is then absorbed into the blood. We want to find a description of the current amount of alcohol in the stomach $m_S(t)$ and in the blood $m_B(t)$.

We assume the following:

- the amount of alcohol in the stomach decreases by $\alpha\Delta t \cdot m_S(t)$

$$m_S(t + \Delta t) = m_S(t) - \alpha\Delta t \cdot m_S(t)$$
$$\implies \frac{m_S(t + \Delta t) - m_S(t)}{\Delta t} = -\alpha \cdot m_S(t)$$
$$\implies m_S'(t) = -\alpha \cdot m_S(t)$$

- the amount of alcohol in the blood is decreased by a constant amount of $\beta\Delta t$

$$m_B'(t) = \alpha \cdot m_S(t) - \beta$$

Example 6.2. A differential equation describing bounded growth, i.e., a quantity that cannot grow arbitrarily large, is modelled as:

$$u'(t) = ku(t)\left(1 - \frac{u(t)}{L}\right)$$

where L and k are positive constants. The value L represents the maximum capacity. This equation can be simplified into a separable differential equation using the substitution $v(t) = \frac{1}{u(t)}$.

Example 6.3. A differential equation describing a bimolecular reaction $A + B \to C$, where $u(t)$ is the amount of substance C at time t, is given by:

$$u'(t) = k(a - u(t))(b - u(t))$$

where a, b, and k are positive real constants representing the initial concentrations and the reaction rate, respectively.

Definition 6.4. A differential equation is an equation which contains derivatives of functions.

Definition 6.5. A differential equation is called

- of **n-th order**, if the n-th derivative is the highest occurring derivative.
- **linear**, if the unknown function $u(t)$ and its derivatives appear in the equation in the form of a linear combination.
- **homogeneous**, if a linear ODE only contains multiples of the unknown function $u(t)$ and its derivatives (i.e. right hand side is 0). It is **inhomogeneous** if there is a term (the disturbance function) that only depends on the independent

variable.

Definition 6.6. An ordinary differential equation (ODE) is a differential equation which contains only derivatives of one independent variable.

Theorem 6.7 (Picard-Lindelöf - Existence and Uniqueness)**.** If $f(x, y)$ is continuous and Lipschitz continuous with respect to y, then the initial value problem $y' = f(x, y)$ with $y(x_0) = y_0$ has a unique solution in some interval around x_0.

6.2 First-Order ODEs

Theorem 6.8. For an ODE of the form $y' = f(x)g(y)$, we can separate the variables and integrate:

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{1}{g(y)}dy = \int f(x)dx + C$$

Example 6.9. Solve $y' = 2xy$. We separate: $\frac{1}{y}dy = 2xdx$. Integrating both sides gives $\ln|y| = x^2 + C$. Solving for y yields $y = \pm e^C e^{x^2} = Ke^{x^2}$.

Theorem 6.10. If a differential equation is of the form $y' = f(ax + by + c)$, we can substitute $u = ax + by + c$. This transforms the equation into an autonomous differential equation with separable variables.

Example 6.11. Solve $y' = (x + y)^2$. Substitute $u = x + y$, then $u' = 1 + y'$. The ODE becomes $u' - 1 = u^2 \implies u' = u^2 + 1$. This can now be solved using separation of variables: $\int \frac{1}{u^2+1}du = \int 1dx \implies \arctan(u) = x + C$. Back-substituting gives $x + y = \tan(x + C) \implies y = \tan(x + C) - x$.

Theorem 6.12. If a differential equation can be written in the form:

$$y' = g\left(\frac{y}{x}\right)$$

it is called homogeneous in the variables. We can solve this by substituting $u = \frac{y}{x}$, which transforms it into an equation with separable variables.

Example 6.13. Solve the differential equation $y' = \frac{x^2+y^2}{xy}$.

First, we rewrite the right-hand side to show it is homogeneous in the variables (by dividing each term by xy):

$$y' = \frac{x^2}{xy} + \frac{y^2}{xy} = \frac{x}{y} + \frac{y}{x} = \frac{1}{\frac{y}{x}} + \frac{y}{x}$$

We use the substitution $u = \frac{y}{x}$, which implies $y = u \cdot x$. By the product rule, the derivative is $y' = u'x + u$.

Substituting y' and $\frac{y}{x}$ into our equation yields:

$$u'x + u = \frac{1}{u} + u$$

Subtracting u from both sides gives a separable differential equation:

$$u'x = \frac{1}{u} \implies u\,du = \frac{1}{x}\,dx$$

Integrating both sides:

$$\int u\,du = \int \frac{1}{x}\,dx \implies \frac{1}{2}u^2 = \ln|x| + C$$

Solving for u:

$$u^2 = 2\ln|x| + 2C \implies u = \pm\sqrt{2\ln|x| + K}$$

Finally, back-substituting $u = \frac{y}{x}$:

$$\frac{y}{x} = \pm\sqrt{2\ln|x| + K} \implies y = \pm x\sqrt{2\ln|x| + K}$$

6.3 Linear ODEs of n-th Order

Theorem 6.14. If y_1 and y_2 are solutions to a linear homogeneous differential equation so is

$$y(x) = C_1 y_1(x) + C_2 y_2(x), \quad C_1, C_2 \in \mathbb{R}$$

Theorem 6.15. Linear homogenous differential equations of n-th order have n linearly independent solutions $y_1, ..., y_n$ such that

$$y(x) = C_1 y_1(x) + ... + C_n y_n(x), \quad C_i \in \mathbb{R},\ \text{for } i = 1, ..., n$$

is called the general solution.

Lemma 6.16. Let $y(x) = e^{\lambda x}, \lambda \in \mathbb{C}$. By deriving n times and substituting into the homogeneous ODE with constant coefficients, we obtain the characteristic equation:

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 = 0$$

The corresponding polynomial is the characteristic polynomial $p(\lambda)$.

Theorem 6.17. We can find the general solution of a linear, homogeneous ODE of n-th order with constant coefficients by finding the roots of the characteristic polynomial:

- **Distinct real roots**: For each distinct root $\lambda_i \in \mathbb{R}$, we get a solution $y_i(x) = e^{\lambda_i x}$.
- **Multiple real roots**: If a root $\lambda_i \in \mathbb{R}$ has multiplicity m, the m linearly independent solutions are $e^{\lambda_i x}, xe^{\lambda_i x}, x^2 e^{\lambda_i x}, \ldots, x^{m-1} e^{\lambda_i x}$.
- **Complex roots**: Complex roots always come in conjugate pairs $\lambda = \alpha \pm i\beta \in \mathbb{C}$. If they have multiplicity 1, they yield two real solutions: $y_1(x) = e^{\alpha x}\cos(\beta x)$ and $y_2(x) = e^{\alpha x}\sin(\beta x)$.

Example 6.18. Solve $y'' - 5y' + 6y = 0$. Using the Euler Ansatz $y(x) = e^{\lambda x}$, we obtain the characteristic equation:

$$\lambda^2 - 5\lambda + 6 = 0$$

The roots are $\lambda_1 = 2$ and $\lambda_2 = 3$. Both are distinct real roots. The general solution is therefore $y(x) = C_1 e^{2x} + C_2 e^{3x}$.

6.4 Inhomogeneous Linear ODEs

Lemma 6.19. To solve general linear ODEs, we first find the general solution of the associated linear homogeneous ODE $y_h(x)$ of n-th order. Then we use an appropriate form $y_p(x)$ to find the particular solution of the inhomogeneous ODE. The general solution is then $y(x) = y_h(x) + y_p(x)$.

6.4.1 Method of Undetermined Coefficients (Ansatz)

This method works for ODEs with **constant coefficients** and specific types of right-hand sides (disturbance functions) $s(x)$.

Theorem 6.20. If $s(x) = a_0 + a_1 x + ... + a_m x^m$, then let $y_p(x) = C_0 + C_1 x + ... + C_m x^m$. Find the constants C_i by plugging $y_p(x)$ into the ODE.

Theorem 6.21. If $s(x) = Ae^{kx}$, then let $y_p(x) = Ce^{kx}$. Find C by plugging $y_p(x)$ into the ODE.

Theorem 6.22 (Trigonometric / Oscillation)**.** If $s(x) = A\sin(\omega x) + B\cos(\omega x)$, then let $y_p(x) = C_1 \sin(\omega x) + C_2 \cos(\omega x)$. Find the constants C_i by plugging $y_p(x)$ into the ODE. Note: You must always include both sine and cosine in the Ansatz!

Important 6.23. Resonance (Special Cases): If the structure of the right-hand side $s(x)$ matches a root of the characteristic polynomial (e.g. k is an m-fold root, or $i\omega$ is an m-fold root), then we must multiply our standard Ansatz by x^m. For example, if $s(x) = Ae^{kx}$ and k is a single root ($m = 1$) of $p(\lambda)$, the Ansatz becomes $y_p(x) = x \cdot Ce^{kx}$.

Example 6.24 (Undetermined Coefficients)**.** Find a particular solution for $y''-3y'+2y = e^{3x}$. The right-hand side is an exponential $s(x) = e^{3x}$, with $k = 3$. The roots of the characteristic polynomial are $\lambda_1 = 1, \lambda_2 = 2$. Since $k = 3$ is not a root, we have no resonance. We use the Ansatz $y_p(x) = Ce^{3x}$. Plugging it into the ODE gives:

$$y_p''(x) - 3y_p'(x) + 2y_p(x) = 9Ce^{3x} - 3(3Ce^{3x}) + 2Ce^{3x} = 2Ce^{3x}$$

We equate this to the right-hand side e^{3x}:

$$2Ce^{3x} = e^{3x} \implies 2C = 1 \implies C = \frac{1}{2}$$

The particular solution is $y_p(x) = \frac{1}{2}e^{3x}$.

Example 6.25 (Resonance)**.** Find a particular solution for $y'' - 3y' + 2y = e^{2x}$. The right-hand side is $s(x) = e^{2x}$, with $k = 2$. The roots of the characteristic polynomial are $\lambda_1 = 1, \lambda_2 = 2$. Since $k = 2$ matches a root, we have resonance ($m = 1$). Our Ansatz must be multiplied by x: $y_p(x) = x \cdot Ce^{2x}$. We plug $y_p(x)$, $y_p'(x) = C(1 + 2x)e^{2x}$, and $y_p''(x) = C(4 + 4x)e^{2x}$ into the ODE to solve for C:

$$C(4 + 4x)e^{2x} - 3C(1 + 2x)e^{2x} + 2Cxe^{2x} = e^{2x}$$

Dividing by e^{2x} gives: $4C + 4Cx - 3C - 6Cx + 2Cx = 1 \implies C = 1$. The particular solution is $y_p(x) = xe^{2x}$.

6.4.2 Variation of Constants (First-Order ODEs)

Theorem 6.26 (Variation of Constants)**.** For an inhomogeneous, linear differential equation of first order, we can find the particular solution by replacing the constant C from the homogeneous solution with a function $C(x)$.

If the homogeneous solution is $y_h(x) = C \cdot y_1(x)$, we set the particular solution to:

$$y_p(x) = C(x) \cdot y_1(x)$$

By plugging this $y_p(x)$ back into the original inhomogeneous ODE, we get a condition for $C'(x)$, which we can integrate to find the function $C(x)$.

6.5 Initial Value Problems and Boundary-Value Problems

The general solution $y(x)$ of an n-th order ODE contains n unknown constants. We need n additional conditions to specify a unique solution curve.

6.5.1 Initial Value Problem (IVP)

An Initial Value Problem specifies the conditions at the **same point** (e.g., t_0).

Example 6.27. Solve the IVP: $y'' + y = 0$ with $y(0) = 1$ and $y'(0) = 0$. The general solution is $y(x) = C_1 \cos(x) + C_2 \sin(x)$. Using the conditions: $y(0) = C_1 \cdot 1 + C_2 \cdot 0 = 1 \implies C_1 = 1$. $y'(x) = -C_1 \sin(x) + C_2 \cos(x) \implies y'(0) = C_2 \cdot 1 = 0 \implies C_2 = 0$. The specific solution is $y(x) = \cos(x)$.

6.5.2 Boundary-Value Problem (BVP)

A Boundary-Value Problem specifies the conditions at **different points** (often boundaries of an interval, $t_0 \neq t_1$).

Example 6.28. Solve the BVP: $y'' + y = 0$ with $y(0) = 0$ and $y(\pi/2) = 1$. The general solution is $y(x) = C_1 \cos(x) + C_2 \sin(x)$. Using the first boundary condition: $y(0) = C_1 \cdot 1 + 0 = 0 \implies C_1 = 0$. Using the second boundary condition: $y(\pi/2) = 0 + C_2 \cdot 1 = 1 \implies C_2 = 1$. The specific solution is $y(x) = \sin(x)$.