1. Logic, Sets and Numbers

1.1 Logic

Def. 1. A **mathematical Statement** is a well-formulated assertion that is strictly either true or false.

Def. 2. A **Statement** that depends on one or multiple variables is called a **predicate** and denoted by $P(x), Q(x), \ldots$.

Def. 3. **Quantifiers** for predicates:

- $\forall x \in \mathbb{R} : P(x)$ (for all x holds $P(x)$)
- $\exists x \in \mathbb{R} : P(x)$ (there exists x s.t. $P(x)$)
- $\exists! x \in \mathbb{R} : P(x)$ (exists exactly one x)
- $\nexists x \in \mathbb{R} : P(x)$ (exists no x s.t. $P(x)$)

Imp. 4. The order of quantifiers matters. $\forall x \exists y$ is not equivalent to $\exists y \forall x$.

Def. 5. Basic **logical connectives** are defined as:

$$A \wedge B := A \text{ and } B \text{ are true}$$

$$A \vee B := A \text{ or } B \text{ is true}$$

$$A \implies B := A \text{ implies } B \text{ is true}$$

$$A \iff B := A \text{ is equivalent to } B \text{ is true}$$

Lem. 6.

$$\neg A \vee B \iff A \implies B$$

Thm. 7. The following equivalences hold:

$$\neg(\forall x P(x)) \iff \exists x \neg P(x)$$

$$\neg(\exists x P(x)) \iff \forall x \neg P(x)$$

aswell as

$$\forall x (P(x) \wedge Q(x)) \iff (\forall x P(x)) \wedge (\forall x Q(x))$$

$$\exists x (P(x) \vee Q(x)) \iff (\exists x P(x)) \vee (\exists x Q(x))$$

1.2 Sets

Def. 8. A **set** is a collection of elements, that are listed or characterized by a certain property. The empty set $:= \emptyset$ and is the only set that contains no elements.

Def. 9. Let A and B be sets. A is a **subset** of B denoted by $A \subset B$ iff.

$$\forall x (x \in A \implies x \in B)$$

A is a **proper subset** of B denoted by $A \subsetneq B$ iff.

$$A \subset B \wedge A \neq B$$

Def. 10. Let A and B be sets. The **union** of A and $B := A \cup B := \{x \mid x \in A \vee x \in B\}$.

Def. 11. The **intersection** of A and $B := A \cap B := \{x \mid x \in A \wedge x \in B\}$.

Def. 12. The **difference** of A and $B := A \setminus B := \{x \mid x \in A \wedge x \notin B\}$.

Def. 13. The **Cartesian product** of A and $B := A \times B := \{(a, b) \mid a \in A \wedge b \in B\}$.

Rem. 14. Note that if $B \subset A$ then the diffrence $A \setminus B$ is called the complement of B in A and $:= B^c$.

1.3 Numbers

Rem. 15. Some important subsets of the real numbers are

$$\mathbb{R}_0^+ / \mathbb{R}^+$$

$$\mathbb{R}_0^- / \mathbb{R}^-$$

which define the intervalls

$$[0, \infty) / (0, \infty)$$

$$(-\infty, 0] / (-\infty, 0)$$

Def. 16. Continuous subsets are called **intervalls.**

Def. 17. An intervall is called **open** if it does not contain its endpoints

$$I = (a, b) := \{x \mid a < x < b\}$$

closed if it contains its endpoints

$$I = [a, b] := \{x \mid a \leq x \leq b\}$$

half-open if it contains one endpoint and not the other

$$I = [a, b) := \{x \mid a \leq x < b\}$$

Rem. 18. An arbitrary subset of $\mathbb{R}$ is called open if it is a union of open intervalls and closed if its complement is open.

Def. 19. An **upper (lower) bound** of a subset $X \subset \mathbb{R}$ is an element $y \in \mathbb{R}$ such that:

$$\forall x \in X : x \leq (\geq) y$$

The **maximum (minimum)** of a subset $X \subset \mathbb{R}$ is the element $x_0 \in X$ such that:

$$\forall x \in X : x \leq (\geq) x_0$$

Often denoted as **max(X)** and **min(X)**.

Def. 20. An intervall $I = (a, b)$ is **bounded** if there exist upper and lower bounds hence

$$a > -\infty \quad \text{and} \quad b < \infty$$

Rem. 21. A bounded and closed intervall is often refered to as **compact**.

Def. 22. The **Supremum** $sup(X)$ of a subset $X \subset \mathbb{R}$ is the least upper bound of X. and can be characterized by:

$$\forall x \in X : x \leq sup(X) \wedge$$

$$\forall \epsilon > 0 : \exists x \in X : x > sup(X) - \epsilon$$

Def. 23. The **Infimum** $inf(X)$ of a subset $X \subset \mathbb{R}$ is the greatest lower bound of X. and can be characterized by:

$$\forall x \in X : x \geq inf(X) \wedge$$

$$\forall \epsilon > 0 : \exists x \in X : x < inf(X) + \epsilon$$

Thm. 24. The **maximum and minimum are unique** aslong as they exist.

Def. 25. **Order Axioms** reflexivity:

$$\forall x \in X : x \leq x$$

antisymmetry:

$$\forall x, y \in X : x \leq y \wedge y \leq x \implies x = y$$

transitivity:

$$\forall x, y, z \in X : x \leq y \wedge y \leq z \implies x \leq z$$

totality:

$$\forall x, y \in X : x \leq y \vee y \leq x$$

Def. 26. A set X is said to be **(order)complete** iff for every non empty subsets $A, B \subseteq X$

$$\forall a \in A \forall b \in B : a \leq b$$

$$\implies \exists c \in X \quad \text{s.t.}$$

$$\forall a \in A : a \leq c \wedge$$

$$\forall b \in B : c \leq b$$

Ex. 27. The real numbers $\mathbb{R}$ satisfy axiomatic properties for addition, multiplication, distributivity and order. Furthermore $\mathbb{R}$ is an order complete field.

Ex. 28. $\mathbb{Q}$ is not order complete. Let $A = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$ and $B = \{x \in \mathbb{Q} \mid x^2 \geq 2\}$. Then $\forall a \in A \forall b \in B : a \leq b$ but there is no $c \in \mathbb{Q}$ such that $c^2 = 2$.

Thm. 29. Every **non empty, bounded from above (below)** subset $X \subset \mathbb{R}$ has a **supremum (infimum)**.

Thm. 30. The **Archimedian Principle** says that For $x \in \mathbb{R}$ and $y > 0$ there exists an element

$$n \in \mathbb{N} \quad \text{s.t.} \quad n \cdot y > x$$

For every $\epsilon > 0$ there exists an element $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \epsilon$$

Thm. 31. The Young Inequality states that for all $\epsilon > 0$ and all $x, y \in \mathbb{R}$:

$$2|xy| \leq \epsilon x^2 + \frac{1}{\epsilon} y^2$$

1.3.1 Vectors and Vector Spaces

Def. 32. The **euclidian space** $\mathbb{R}^n$ is the set of all vectors $x = (x_1, x_2, \ldots, x_n)$ where $x_i \in \mathbb{R}$.

Def. 33. **Addition** and **scalar multiplication** are defined as:

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

$$\lambda x = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$$

Def. 34. A **vector space** must satisfy the following axioms:
Commutativity: $x + y = y + x$
Associativity: $x + (y + z) = (x + y) + z$
Identity: $\exists e \in X$ s.t. $x + e = x \quad \forall x \in X$
Inverse: $\forall x \in X \exists x^{-1} \in X$ s.t. $x + x^{-1} = e$
Distributivity: $x \cdot (y + z) = x \cdot y + x \cdot z$

Def. 35. For $x, y \in \mathbb{R}^n$, the **standard scalar product** is :=:

$$x \cdot y = \langle x, y\rangle := \sum_{i=1}^{n} x_i y_i$$

The **euclidean norm** is:

$$||x|| := \sqrt{\sum_{i=1}^{n} x_i^2}$$

The **euclidean distance** between x and y is:

$$d(x, y) := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Thm. 36. The triangle- and the cauch-schwarz inequality hold for all $x, y, z \in \mathbb{R}^n$:

$$||x - z|| \leq ||x - y|| + ||y - z||$$

$$|\langle x, y\rangle| \leq ||x|| ||y||$$

1.3.2 Complex Numbers

Def. 37. The **complex numbers** $\mathbb{C}$ are the set of numbers

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$$

where a is the **real part** $\mathrm{Re}(z)$, and b is the **imaginary part** $\mathrm{Im}(z)$.

Def. 38. Arithmetic of complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$: Addition:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

Multiplication:

$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

Def. 39. The **complex conjugate** of $z = x + iy$ is $\bar{z} = x - iy$.

Imp. 40. Division is calculated by expanding with the conjugate of the denominator:

$$\frac{z}{w} = \frac{z \cdot \bar{w}}{w \cdot \bar{w}} = \frac{z \cdot \bar{w}}{|w|^2}$$

Lem. 41. Some usfull properties of the conjugate:

$$z \cdot \bar{z} = |z|^2$$

$$z + \bar{z} = 2\mathrm{Re}(z)$$

$$z - \bar{z} = 2i\mathrm{Im}(z)$$

$$\overline{z + w} = \bar{z} + \bar{w}$$

$$\overline{z \cdot w} = \bar{z} \cdot \bar{w}$$

Def. 42. Geometrically, complex numbers can be represented in the Gaussian plane. The **absolute value** of $z = a + ib$ is its **distance from the origin**:

$$|z| = \sqrt{a^2 + b^2}$$

The distance between z_1 and z_2 is $d = |z_1 - z_2|$.

Thm. 43. The Euler's formula states that for $t \in \mathbb{R}$:

$$e^{it} = \cos(t) + i\sin(t)$$

This allows writing trigonometric functions using complex exponentials:

$$\cos(t) = \frac{e^{it} + e^{-it}}{2}$$

$$\sin(t) = \frac{e^{it} - e^{-it}}{2i}$$

Rem. 44. The **Intuition** behind eulers formula is we can use Taylor series to show that:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Using $x = it$ we get:

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \cdots = \cos(t) + i\sin(t)$$

Def. 45. For $z \in \mathbb{C}$, the polar-coordinates are defined as:

$$z = r(\cos(t) + i\sin(t)) = re^{it}$$

where $r = |z|$ is the absolute value and $t = \arg(z)$ is the argument.

Rem. 46. To convert from cartesian to polar coordinates:

$$r = \sqrt{a^2 + b^2}$$

$$t = \arctan\left(\frac{b}{a}\right)$$

Note that the arctan function only returns values in $(-\frac{\pi}{2}, \frac{\pi}{2})$, so we have to adjust the angle based on the quadrant of the complex number.

Ex. 47. Multiplication and division in polar-coordinates is much easier than in cartesian coordinates:

$$z_1 \cdot z_2 = r_1e^{it_1} \cdot r_2e^{it_2} = r_1r_2e^{i(t_1+t_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1e^{it_1}}{r_2e^{it_2}} = \frac{r_1}{r_2}e^{i(t_1-t_2)}$$

Ex. 48. So is exponentiation or taking roots:

$$z^n = (re^{it})^n = r^n e^{int}$$

$$\sqrt[n]{z} = \sqrt[n]{r}e^{i\frac{t+2\pi k}{n}}, \quad k \in \{0, 1, \ldots, n-1\}$$

Def. 49. The n-th roots of unity are the solutions to $z^n = 1$, which are given by:

$$z_k = e^{i\frac{2\pi k}{n}}, \quad k \in \{0, 1, \ldots, n-1\}$$

Thm. 50. The sum of the roots of unit is always zero.

$$\sum_{k=0}^{n-1} e^{i\frac{2\pi k}{n}} = 0$$

Thm. 51. The **fundamental theorem of algebra** states that every polynomial of degree $n \geq 1$ has exactly n roots in $\mathbb{C}$ (counting multiplicity). These roots appear in konplex-conjugate pairs.

1.4 Landau Notation (Asymptotics)

Def. 52. Let f, g be functions.

- $f(x) = O(g(x))$ as $x \to x_0$ iff $\exists C, \delta > 0$ s.t. $|f(x)| \leq C|g(x)|$ for $|x - x_0| < \delta$.
- $f(x) = o(g(x))$ as $x \to x_0$ iff $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$.
- $f(x) \sim g(x)$ as $x \to x_0$ iff $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$.

Rem. 53. Useful for Taylor: $f(x) = P_n(x) + O(|x - x_0|^{n+1})$.

1.5 Exam Strategy: Sets and Topology

Open: Every point is an interior point (ball $B_\epsilon(x) \subset A$). Check: Is the boundary excluded?
Closed: Complement is open OR contains all limit points. Check: Is the boundary included?
Bounded: $A \subset B_R(0)$ for some R.
Compact: In $\mathbb{R}^n$, compact $\iff$ closed and bounded.
Supremum/Infimum Tips: For $A = \{a_n : n \in \mathbb{N}\}$, check if a_n is monotonic. If $a_n \nearrow L$, then $\sup A = L$, $\inf A = a_1$. If a_n oscillates, check even/odd indices separately.

1.6 Exam Strategy: Complex Equations

Circles: $|z - z_0| = R$ is a circle of radius R around z_0.
Lines: $|z - a| = |z - b|$ is the perpendicular bisector of a and b.
Polynomials: If $P(z) = 0$ and coefficients are real, then $z \in \mathbb{C}$ is a root $\implies$ $\bar{z}$ is also a root.
Roots of Unity Sum: $\sum_{k=0}^{n-1} z_k = 0$ for $n > 1$.
Number of Solutions: By the Fundamental Theorem of Algebra, a polynomial of degree $n \geq 1$ has exactly n complex roots.
Vieta's Formulas: For polynomial $P(z) = a_nz^n + \cdots + a_0$ with roots $z_1, \ldots, z_n$: **Sum:** $\sum z_i = -\frac{a_{n-1}}{a_n}$. **Product:** $\prod z_i = (-1)^n \frac{a_0}{a_n}$.

2. Sequences and Series

2.1 Theory

Def. 54. A **sequence** (Folge) $(a_n)_{n\in\mathbb{N}_0}$ is an infinite list of numbers, mathematically a mapping $f : \mathbb{N}_0 \to X$. It can be given explicitly or recursively as

$$a_n = f(n) \quad \text{or} \quad a_n = g(a_{n-1}, a_{n-2}, \ldots)$$

Def. 55. A sequence (a_n) is called **convergent** with limit $L \in \mathbb{R}$ if:

$$\forall \epsilon > 0 \exists N > 0 \quad \text{s.t.}$$

$$\forall n > N : |a_n - L| < \epsilon$$

Written as $\lim_{n\to\infty} a_n = L$.If no such L exists, the sequence is **divergent**.

Rem. 56. A special case of divergent sequences are sequences which diverge to $\pm\infty$. Their limits are called **improper limits**.

Thm. 57. The limit of a sequence is unique aslong as it exists.

Thm. 58. Any subsequence $(a_{n_k})_{k\in\mathbb{N}_0}$ of a convergent sequence converges to the same limit L.

Def. 59. An **accumulation point** $A \in \mathbb{R}$ of a sequence fulfills:

$$\forall\epsilon > 0 \forall N \in \mathbb{N}_0 \exists n \geq N$$

$$|a_n - A| < \epsilon$$

Thm. 60. A is an accumulation point $\iff$ there exists a convergent subsequence with limit A.

Thm. 61. Every convergent sequence has exactly one accumulation point (its limit).

Def. 62. A sequence is **upper (lower) bounded** if there exists M such that $|a_n| \leq (\geq)M \quad \forall n \in \mathbb{N}_0$.

Thm. 63. Every convergent sequence is bounded.

Thm. 64. Bolzano-Weierstrass Theorem: Every bounded sequence has an accumulation point.

Thm. 65. Limit arithmetic: For $\lim a_n = K$, $\lim b_n = L$ and $C \in \mathbb{R}$:

$$\lim(a_n \pm b_n) = K \pm L$$

$$\lim(C \cdot a_n \cdot b_n) = C \cdot K \cdot L$$

$$\lim(a_n/b_n) = K/L$$

Thm. 66. Sandwich-Theorem: If $a_n \leq b_n \leq c_n$ for $n \geq N_0$, and $\lim a_n = \lim c_n = L$, then $\lim b_n = L$.

Thm. 67. Useful Inequalities (for Sandwich Theorem & Bounding): Bernoulli's Inequality: $(1+x)^n \geq 1+nx$ (for $x \geq -1$ and $n \in \mathbb{N}$). **Reverse Triangle Inequality:** $\big||x| - |y|\big| \leq |x-y|$. **Trigonometric Bounding:** $\sin(x) \leq x \leq \tan(x)$ (for $x \in (0, \pi/2)$).

Def. 68. A sequence is **monotonically increasing (decreasing)** if $a_n \leq (\geq) a_m$ for $m > n$.

Thm. 69. Monotone Convergence Theorem: Every bounded monotonic sequence converges to its supremum (infimum).

Def. 70. The limit of the supremum is called **limes superior** and denoted as:

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\geq n} a_k$$

The limit of the infimum is called **limes inferior** and denoted as:

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{k\geq n} a_k$$

Def. 71. A sequence is a **Cauchy sequence** iff

$$\forall\epsilon > 0 \exists N \in \mathbb{N} \text{ such that}$$

$$\forall m, n \geq N : |a_n - a_m| < \epsilon$$

Thm. 72. Every Cauchy sequence is bounded.

Thm. 73. A sequence of real numbers converges iff it is a cauchy sequence.

Rem. 74. Important limits (for $n \to \infty$):
$\lim \frac{\ln n}{n} = 0$
$\lim n^{1/n} = 1$, $\lim x^{1/n} = 1 \quad (x > 0)$
$\lim \left(1 + \frac{x}{n}\right)^n = e^x$, $\lim \left(1 + \frac{1}{n}\right)^n = e$
$\lim \sqrt{n^2+n} - n = \frac{1}{2}$
$\lim \frac{n^k}{a^n} = 0 \quad (a > 1)$
$\lim \frac{\log_a n}{n^k} = 0 \quad (a > 1, k > 0)$
$\lim \frac{a^n}{n!} = \lim \frac{x^n}{n!} = 0$
$\lim \frac{n!}{n^n} = 0$

Thm. 75. Growth Hierarchy: For $x \to \infty$ or $n \to \infty$:

$$\ln(\ln n) \ll \ln n \ll n^\epsilon \ll n^k$$

$$\ll a^n \ll n! \ll n^n \quad (\epsilon, k > 0, a > 1)$$

2.1.1 Series

Def. 76. A **series** is an infinite sum $\sum_{i=0}^\infty a_i$ of the terms of a sequence (a_n).

Def. 77. A series is said to converge iff the sequence of partialsums converges.

$$\sum_{i=0}^\infty a_i = \lim_{n\to\infty} \sum_{i=0}^n a_i$$

Rem. 78. Some important series values:
Geometric: $\sum_{n=0}^\infty x^n = \frac{1}{1-x}$ (for $|x| < 1$).
Shifted Geometric: $\sum_{k=m}^\infty q^k = \frac{q^m}{1-q}$ (for $|q| < 1, m \geq 0$).
Basel Problem: $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$.
Alternating Harmonic: $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} = \ln(2)$.
Leibniz Formula: $\sum_{n=0}^\infty \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$.
p-Series: $\sum_{n=1}^\infty \frac{1}{n^p}$ converges iff $p > 1$.

Thm. 79. Necessary condition for convergence: for $\sum_{i=0}^\infty a_i$ to converge we need $\lim_{n\to\infty} a_n = 0$ i.e. it is a null sequence.

Thm. 80. For **series with non-negative terms** $a_n \geq 0$, the sequence of partial sums is monotonically increasing and **converges iff it is bounded.**

Def. 81. A series is **absolutely convergent** if $\sum |a_n|$ converges. A **conditionally convergent** series converges, but not absolutely.

Thm. 82. Riemann's rearrangement theorem: Let $\sum a_n$ be a conditionally convergent series. Then for any $x \in \mathbb{R}$, there exists a rearrangement of the series that converges to x.

Thm. 83. Comparison test (Vergleichskriterium): If $c_n \leq b_n \leq a_n$ for $n \geq N$: If $\sum a_n$ converges, then $\sum b_n$ converges (**Majorantenkriterium**). If $\sum c_n$ diverges, then $\sum b_n$ diverges (**Minorantenkriterium**).

Thm. 84. Leibniz Criterion: For an alternating series $\sum(-1)^n a_n$ with a monotonically decreasing null sequence (a_n), the series converges. **Error Bound**: The error of the n-th partial sum S_n is bounded by the next term: $|S - S_n| \leq a_{n+1}$.

Thm. 85. Cauchy Criterion: The series $\sum a_n$ converges iff

$$\forall\epsilon > 0 \exists N \in \mathbb{N} \text{ such that}$$

$$\forall m, n \geq N : |\sum_{k=n}^m a_k| < \epsilon$$

Thm. 86. Root Criterion (Wurzelkriterium): Let $\rho = \limsup \sqrt[n]{|a_n|}$. If $\rho < 1$, abs conv; if $\rho > 1$, div.

Thm. 87. Ratio Criterion (Quotientenkriterium): Let $\rho = \lim \left|\frac{a_{n+1}}{a_n}\right|$. If $\rho < 1$, abs conv; if $\rho > 1$, div.

Def. 88. A **power series** (Potenzreihe) centered at a has the form

$$\sum_{k=0}^\infty c_k (x-a)^k$$

Thm. 89. Every power series has a **convergence radius** R, with the **interval of convergence** $(a-R, a+R)$. R is computed by the formula $R = \frac{1}{\limsup |c_n|^{1/n}}$.

2.2 Lookup Tables

Test	Condition	Conclusion
Null seq.	$\lim a_n \neq 0$	Divergent
Geometric	$\sum q^n$	$\lvert q\rvert < 1$ Conv, sum $\frac{1}{1-q}$
Harmonic	$\sum \frac{1}{n^s}$	$s > 1$ Conv, $s \leq 1$ Div
Ratio	$\lim \lvert a_{n+1}/a_n\rvert = \rho$	$\rho < 1$ Conv, $\rho > 1$ Div
Root	$\lim \lvert a_n\rvert^{1/n} = \rho$	$\rho < 1$ Conv, $\rho > 1$ Div
Leibniz	Alt. $\sum(-1)^n b_n$, $b_n \searrow 0$	Convergent
Integral	$a_n = f(n)$, $f \geq 0$, $\searrow$	Conv $\iff$ $\int_1^\infty f(x)dx$ conv
Comp. I	$\lvert a_n\rvert \leq b_n$	$\sum b_n$ conv $\implies$ $\sum a_n$ abs. conv
Comp. II	$a_n \geq b_n \geq 0$	$\sum b_n$ div $\implies$ $\sum a_n$ div

2.2.1 Common Sequence Limits

$\lim n^{1/n}$	1
$\lim \frac{\ln n}{n}$	0
$\lim x^{1/n}$ $(x > 0)$	1
$\lim(1+\frac{x}{n})^n$	e^x
$\lim(\sqrt{n^2+n-n})$	1/2
$\lim \frac{a^n}{n!}$	0
$\lim(1-\frac{1}{n})^n$	$1/e$
$\lim(1+\frac{a}{n})^n$	e^a
$\lim n(\sqrt[n]{x}-1)$	$\ln x$
$\lim n\ln(1+\frac{x}{n})$	x
$\lim \sqrt[n]{a^n+b^n}$	$\max(a,b)$
$\lim(1+\frac{x}{n^2})^n$	1
$\lim \frac{\sqrt[n]{n!}}{n}$	$1/e$
$\lim \frac{n!}{(n/e)^n\sqrt{2\pi n}}$	1
$\lim n\sin(\frac{x}{n})$	x
$\lim n\tan(\frac{x}{n})$	x
$\lim \cos(\frac{x}{n})^n$	1
$\lim \cos(\frac{x}{\sqrt{n}})^n$	$e^{-x^2/2}$
$\lim \sum_{k=1}^n \frac{1}{k} - \ln n$	$\gamma \approx 0.577$
$\lim \frac{\sin(1/n)}{1/n}$	1
$\lim(1+\frac{a}{n})^{bn}$	e^{ab}
$\lim \frac{n^k}{e^n}$ $(k > 0)$	0
$\lim n^p a^n$ $(\lvert a\rvert < 1)$	0
$\lim \frac{\ln(n^k)}{n}$	0

2.3 Most Common Proofs

2.3.1 Proof Outline: Uniqueness of limits

Goal: If $a_n \to L_1$ and $a_n \to L_2$, then $L_1 = L_2$.
Step 1: Assume $L_1 \neq L_2$. Let $\epsilon = \frac{|L_1-L_2|}{2} > 0$.
Step 2: $\exists N_1$ s.t. $\forall n > N_1 : |a_n - L_1| < \epsilon$.
Step 3: $\exists N_2$ s.t. $\forall n > N_2 : |a_n - L_2| < \epsilon$.
Step 4: For $n > \max(N_1, N_2)$, $|L_1-L_2| = |L_1 - a_n + a_n - L_2| \leq |L_1 - a_n| + |a_n - L_2| < 2\epsilon = |L_1 - L_2|$.
Step 5: This yields a contradiction. Thus $L_1 = L_2$.

2.3.2 Proof Outline: Algebraic rules for limits

Goal: If $a_n \to a$ and $b_n \to b$, then $\lim(a_n + b_n) = a + b$ and $\lim(a_n b_n) = ab$.
Sum Rule: Let $\epsilon > 0$. Use triangle inequality: $|(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$. Since $a_n \to a, b_n \to b$, choose N s.t. for $n > N$, both differences are $< \epsilon/2$. Thus $|(a_n+b_n)-(a+b)| < \epsilon$.
Product Rule: Let $\epsilon > 0$. Add and subtract $a_n b$: $|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab| \leq |a_n||b_n - b| + |b||a_n - a|$. Since (a_n) converges, it is bounded by M. Choose N s.t. $|b_n - b| < \frac{\epsilon}{2M}$ and $|a_n - a| < \frac{\epsilon}{2(|b|+1)}$. The sum is then $< \epsilon$.

2.3.3 Proof Outline: Monotone convergence theorem

Goal: Every bounded monotonically increasing sequence (a_n) converges.
Step 1: Let $S = \{a_n : n \in \mathbb{N}\}$. S is bounded above, so $L = \sup(S)$ exists.
Step 2: Let $\epsilon > 0$. $L - \epsilon$ is not an upper bound of S.
Step 3: Thus $\exists N$ s.t. $a_N > L - \epsilon$.
Step 4: Since (a_n) is monotonically increasing, $\forall n \geq N$, $L - \epsilon < a_N \leq a_n \leq L < L + \epsilon$.
Step 5: Thus $|a_n - L| < \epsilon$, meaning $a_n \to L$.

2.3.4 Proof Outline: Bolzano-Weierstrass Theorem

Goal: Every bounded sequence (a_n) has a convergent subsequence.
Interval Bisection: (a_n) is bounded, so it lies in $I_0 = [a,b]$. Halve I_0. One subinterval must contain infinitely many terms; call it I_1. Pick $a_{n_1} \in I_1$. Halve I_1 to find I_2 with infinitely many terms, pick $a_{n_2} \in I_2$ with $n_2 > n_1$. Repeat to get nested intervals I_k with lengths $\frac{b-a}{2^k} \to 0$ and a subsequence a_{n_k}. By completeness, $\bigcap I_k = \{L\}$. Since $a_{n_k} \in I_k$, $a_{n_k} \to L$.

2.3.5 Proof Outline: Cauchy Criterion

Goal: A sequence is convergent iff it is a Cauchy sequence.
Convergent implies Cauchy: Assume $a_n \to L$. For large n, m, $|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |L - a_m| < \epsilon/2 + \epsilon/2 = \epsilon$.
Cauchy implies Convergent: Assume (a_n) is Cauchy. It must be bounded (choose $\epsilon = 1 \implies |a_n - a_N| < 1$). By Bolzano-Weierstrass, it has a convergent subsequence $a_{n_k} \to L$. Then $|a_n - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L|$. The Cauchy property bounds the first term, and subsequence convergence bounds the second, so $a_n \to L$.

2.4 Exam Strategies

Type $\frac{\ln^\alpha n}{n^\beta}$: Converges for $\beta > 0$ (any α), or $\beta = 0, \alpha < 0$.
Type $\sum \frac{1}{n \ln^\alpha n}$ (Bertrand): Converges $\iff$ $\alpha > 1$.
Finding $\limsup / \liminf$: List accumulation points. If a_n is defined piecewise or with $(-1)^n$, look at sub-sequences.
Radius of Convergence R: Use $R = \lim |\frac{c_n}{c_{n+1}}|$ or $R = \frac{1}{\lim \sqrt[n]{|c_n|}} \in [0, \infty]$.
Boundary Check: Always check $|x - x_0| = R$ separately. Use Leibniz or Comparison.
Telescoping Series: Write $a_n = b_n - b_{n+1}$. Partial sum $S_N = b_1 - b_{N+1}$.

3. Functions

3.1 Theory

Def. 90. A **function** $f : D \to C$ is a map from a domian D to a codomain C s.t. for each $x \in D$ there is exactly one $y \in C$ st. $f(x) = y$ (well defined).

Def. 91. A function f is called **bouned** if there exists $M \in \mathbb{R}$ such that $|f(x)| \leq (\geq) M$ for all $x \in D$.

Def. 92. A function f is called **injective** iff

$$\forall x_1, x_2 \in A : f(x_1) = f(x_2) \implies x_1 = x_2$$

surjective iff

$$\forall y \in B \exists x \in A : f(x) = y$$

and bijective iff f is injective and surjective.

Def. 93. The **restriction** of f under $D' \subset \mathbb{D}$:=

$$f\restriction_{D'}: D' \to C, x \mapsto f(x) \quad \forall x \in \mathbb{D}$$

Def. 94. Let $f : D \to C$ and $g : C \to B$ be functions then

$$g \circ f : D \to B \quad \text{with}$$

$$g \circ f(x) = g(f(x)) \quad \forall x \in D$$

is called the **composition** of f and g.

Def. 95. The **identity function** $id_D : D \to D := id_D(x) = x \quad \forall x \in D$.

Def. 96. If a function $f : D \to C$ is bijective then there exists a unique function $f^{-1} : C \to D$ such that

$$f(f^{-1}(y)) = id_C = y \quad \forall y \in C$$

and

$$f^{-1}(f(x)) = id_D = x \quad \forall x \in D$$

f^{-1} is called the **inverse function** of f.

Def. 97. A function $f : \mathbb{R} \to \mathbb{R}$ is called **even** iff

$$\forall x \in \mathbb{R} : f(-x) = f(x)$$

and **odd** iff

$$\forall x \in \mathbb{R} : f(-x) = -f(x)$$

Def. 98. A function $f : \mathbb{R} \to \mathbb{R}$ is called **(strictly) monotonically increasing** iff

$$\forall x_1, x_2 \in \mathbb{R} : x_1 < x_2 \implies f(x_1) < f(x_2)$$

and **(strictly) monotonically decreasing** iff

$$\forall x_1, x_2 \in \mathbb{R} : x_1 < x_2 \implies f(x_1) > f(x_2)$$

Thm. 99. Every strictly monotonic function is injective.

3.1.1 Limits in the context of functions

Def. 100. Let $f : \mathbb{D} \to \mathbb{R}$ be a function satisfying $\mathbb{D} \cap (x_0 - \delta, x_0 + \delta) \neq \emptyset$ for all $\delta > 0$ and $x_0 \in \mathbb{R}$. We say that $\lim_{x \to x_0} f(x) = L$ iff

$$\forall \varepsilon > 0 \exists \delta > 0 \text{s.t.}$$

$$x \in \mathbb{D} \cap (x_0 - \delta, x_0 + \delta) \implies |f(x) - L| < \varepsilon$$

The limit L is unique.

Def. 101. A function $f : \mathbb{D} \to \mathbb{R}$ has in x_0 a **right-sided limit** $\lim_{x \searrow x_0} f(x) = \lim_{x \to x_0^+} f(x) = L$ iff

$$\forall \varepsilon > 0 \exists \delta > 0 \quad \text{s.t.}$$

$$\forall x \in \mathbb{D}(x_0 < x < x_0 + \delta \implies |f(x) - L| < \varepsilon)$$

Def. 102. A function $f : \mathbb{D} \to \mathbb{R}$ has a **limit for x approaching infinity** $\lim_{x \to \infty} f(x) = L$ iff

$$\forall \varepsilon > 0 \exists M > 0 \quad \text{s.t.}$$

$$\forall x \in \mathbb{D}(x > M \implies |f(x) - L| < \varepsilon)$$

Def. 103. A function $f : \mathbb{D} \to \mathbb{R}$ has in x_0 a **improper limit approaching infinity** $\lim_{x \to x_0} f(x) = \infty$ iff

$$\forall N > 0 \exists \delta > 0 \quad \text{s.t.}$$

$$forall x \in \mathbb{D}(0 < |x - x_0| < \delta \implies f(x) > N)$$

Thm. 104. Let $\lim_{x\to x_0} f(x) = K(\neq \pm\infty)$ and $\lim_{x\to x_0} g(x) = L(\neq \pm\infty)$

$$\lim_{x\to x_0} (f(x)+g(x)) = K+L$$

$$\lim_{x\to x_0} (f(x)-g(x)) = K-L$$

$$\lim_{x\to x_0} (f(x)g(x)) = KL$$

$$\lim_{x\to x_0} (f(x)/g(x)) = K/L$$

$$\lim_{x\to x_0} (Af(x)) = AK$$

$$\lim_{x\to x_0} (f(x)/g(x)) = K/L \text{if} L \neq 0$$

3.1.2 Continuity of functions

Def. 105. A function $f : D \to \mathbb{R}$ is called **continous** at x_0 if

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that}$$

$$\forall x \in I(\mid x - x_0 \mid < \delta \implies \mid f(x) - f(c) \mid > \epsilon)$$

The function is called continuous if it is continuous at every point in its domain.

Def. 106. A function f is **Lipschitz continuous** if there exists $L \geq 0$ such that $|f(x) - f(y)| \leq L|x-y|$ for all $x, y \in D$.
Properties: A bounded derivative $|f'(x)| \leq L$ implies Lipschitz continuity. Lipschitz continuity strictly implies uniform continuity.

Rem. 107. The **intuition** behind this definition is that for any given distance $\epsilon > 0$ around $f(x_0)$, we can find a $\delta > 0$ such that for all $x \in D$ that are at least δ close to x_0, we have $|f(x) - f(x_0)| < \epsilon$. So for any bound on the output (ϵ) there exists a bound on the input (δ) such that all elements in the input bound $(x_0-\delta, x_0+\delta)$ are mapped to the output bound $(f(x_0) - \epsilon, f(x_0) + \epsilon)$.

Def. 108. Let $f : D \to \mathbb{R}$ be a function. Then f is called **left(right)-continuous** at x_0 if

$$\lim_{x\to x_0^{-(+)}} f(x) = f(x_0)$$

Thm. 109. Let $f : D \subset \mathbb{D}(f) \subset \mathbb{R}$ be a function and $x_0 \in D$. f is continuous at x_0 iff for every sequence $(x_n)_{n\in\mathbb{N}} \subset D$ with $\lim_{n\to\infty} x_n = x_0$ it holds that $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Rem. 110. Continuity can also be viewed in terms of intervals, where a function f is continuous iff for every open interval $I \subset \mathbb{R}$ it holds that $f^{-1}(I)$ is open.

Lem. 111. Let f, g be continuous functions then

$$f+g, \quad f-g, \quad f\cdot g, \quad c\cdot f, \quad \frac{f}{g}, \quad f\circ g$$

are continuous

Thm. 112. Let I be an interval and let $f : I \to J$ be a continuous and bijective function. Then J is an interval too and the inverse function $f^{-1} : J \to I$ is continuous.

Thm. 113. Intermediate Value Theorem (IVT): Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Let c be a value between $f(a)$ and $f(b)$. Then there exists a $x_0 \in [a,b]$ such that $f(x_0) = c$.

Lem. 114. Let $[a,b]$ be a compact interval and let $(x_n)_{n\in\mathbb{N}} \subset [a,b]$ be a sequence. Then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} x_{n_k} = x_0 \quad \text{for some } x_0 \in [a,b].$$

Def. 115. Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a function If for $x_0 \in D$ it holds that $f(x_0) \geq f(x) \forall x \in D$, then f has a local maximum at x_0. If for $x_0 \in D$ it holds that $f(x_0) \leq f(x) \forall x \in D$, then f has a local minimum at x_0.

Thm. 116. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a contnous function and I be a compact interval. Then f is bounded and attains its maximum and minimum value on I. Hence

$$\exists x_{min}, x_{max} \in I \text{ such that}$$

$$f(x_{min}) \leq f(x) \leq f(x_{max}) \quad \forall x \in I$$

3.2 Lookup Tables

3.2.1 Trigonometric Identities

Category	**Formula**
Pythagoras	$\sin^2 x + \cos^2 x = 1$ $1 + \tan^2 x = \frac{1}{\cos^2 x}$
Addition	$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$ $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$ $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
Double Angle	$\sin(2x) = 2\sin x \cos x$ $\cos(2x) = \cos^2 x - \sin^2 x$ $= 2\cos^2 x - 1 = 1 - 2\sin^2 x$
Half Angle	$\sin^2 x = \frac{1-\cos(2x)}{2}$ $\cos^2 x = \frac{1+\cos(2x)}{2}$
Subst. trick	$\tan(x/2) = t \implies$ $\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$ $dx = \frac{2dt}{1+t^2}$
Prod-to-Sum	$\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$ $\cos x \cos y = \frac{1}{2}(\cos(x-y) + \cos(x+y))$ $\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$

3.2.2 Hyperbolic Functions

Defs: $\cosh x = \frac{e^x+e^{-x}}{2}$, $\sinh x = \frac{e^x-e^{-x}}{2}$, $\tanh x = \frac{e^x-e^{-x}}{e^x+e^{-x}}$.
Identities: $\cosh^2 x - \sinh^2 x = 1$, $\sinh(2x) = 2\sinh x \cosh x$, $\cosh(2x) = \cosh^2 x + \sinh^2 x$.
Derivatives: $(\sinh x)' = \cosh x$, $(\cosh x)' = \sinh x$, $(\tanh x)' = \frac{1}{\cosh^2 x}$.
Inverses: $\operatorname{arsinh}(x) = \ln(x + \sqrt{x^2+1})$, $\operatorname{arcosh}(x) = \ln(x + \sqrt{x^2-1})$.

3.3 Most Common Proofs

3.3.1 Proof Outline: Intermediate Value Theorem (IVT)

Goal: $\exists x_0 \in [a,b]$ with $f(x_0) = c$.
Step 1: Assume $f(a) < c < f(b)$. Define $S = \{x \in [a,b] : f(x) \leq c\}$.
Step 2: S is non-empty ($a \in S$) and bounded above (by b). By completeness of $\mathbb{R}$, $x_0 = \sup(S)$ exists.
Step 3: Show $f(x_0) \leq c$ using continuity and a sequence in S converging to x_0.
Step 4: Show $f(x_0) \geq c$ using continuity and a sequence outside S converging to x_0. Conclude $f(x_0) = c$.
Alternative: Bisection method. Halve $[a,b]$, pick the half where f changes sign across c. Nested intervals shrink to x_0.

3.3.2 Proof Outline: Continuity

Goal: Show f is continuous at x_0.
$\epsilon - \delta$ **Method:** Let $\epsilon > 0$. We need $|f(x) - f(x_0)| < \epsilon$. Manipulate $|f(x) - f(x_0)|$ to factor out $|x - x_0|$. Bound the remaining factor by M, assuming $|x - x_0| < 1$. Choose $\delta = \min(1, \frac{\epsilon}{M})$. Then $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$.
Sequence Method: Let $x_n \to x_0$. Show $\lim f(x_n) = f(x_0)$ using algebraic limit theorems.

3.3.3 Proof Outline: Extreme Value Theorem

Goal: Continuous $f : [a,b] \to \mathbb{R}$ attains its max and min.
Boundedness: Assume f is not bounded above. Then $\exists x_n \in [a,b]$ with $f(x_n) > n$. By Bolzano-Weierstrass, (x_n) has a convergent subsequence $x_{n_k} \to c \in [a,b]$. By continuity, $f(x_{n_k}) \to f(c)$, contradicting $f(x_{n_k}) > n_k \to \infty$. Thus f is bounded above. Let $M = \sup f([a,b])$.
Attainment: By definition of supremum, $\exists y_n \in [a,b]$ s.t. $f(y_n) \to M$. By Bolzano-Weierstrass, y_n has a convergent subsequence $y_{n_k} \to d \in [a,b]$. By continuity, $f(y_{n_k}) \to f(d) = M$. Thus max is attained at d.

3.3.4 Proof Outline: Uniform Continuity (Heine-Cantor)

Goal: Continuous f on compact $K = [a,b]$ is uniformly continuous.
Step 1: Assume f is not uniformly continuous on K.
Step 2: $\exists \epsilon > 0$ s.t. for all $\delta = 1/n$, there exist $x_n, y_n \in K$ with $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \epsilon$.
Step 3: By Bolzano-Weierstrass, (x_n) has convergent subsequence $x_{n_k} \to c \in K$.
Step 4: Since $|x_{n_k} - y_{n_k}| < 1/n_k \to 0$, $y_{n_k} \to c$ as well.
Step 5: By continuity of f, $f(x_{n_k}) \to f(c)$ and $f(y_{n_k}) \to f(c)$. Thus $|f(x_{n_k}) - f(y_{n_k})| \to 0$, contradicting $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon$.

3.4 Exam Strategies

Prove and Disprove Uniform Continuity:
To PROVE: Use the Heine-Cantor Theorem (a continuous function on a compact interval is always uniformly continuous) or show the function is Lipschitz continuous.
To DISPROVE: Find two sequences x_n and y_n such that $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \epsilon_0 > 0$.

4. Differential calculus

4.1 Theory

Def. 117. The **average rate of change** of a function f in the domain between a and $a+h$:=

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h)-f(a)}{h}$$

and also called the **difference quotient**.

Def. 118. The **derivative** f' of a function f at the coordinate a :=

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{b\to a}\frac{f(b)-f(a)}{b-a}$$

$$= \frac{df}{dx}|_{x=a} = f'(a)$$

Rem. 119. The derivative $f'(a)$ can geometrically interpreted as the incline/decline of the graph of f at the point $(a, f(a))$.

Def. 120. If for a function f at the coordinate a the derivative $f'(a)$ exists then f is called **differentiable** at a. If f is differentiable for every $a \in \mathbb{D}$ then f is called differentiable.

Thm. 121. Differentiability in Landau Notation: A function f is differentiable at x_0 iff there exists $K \in \mathbb{R}$ such that $f(x) - f(x_0) - K(x - x_0) = o(|x - x_0|)$ as $x \to x_0$. The constant K is exactly $f'(x_0)$.

Lem. 122. If f is differentiable at x_0 then f is continuous at x_0.

Def. 123. If the limit

$$\lim_{h>0\to 0}\frac{f(a+h)-f(a)}{h}$$

exists then we call it the **right-sided derivative** (or the left-sided derivative analogue)

Def. 124. For a function f we can iteratively define higher derivatives for $n \in \mathbb{N}_0$

$$f^{(1)} = f', f^{(2)} = f'', \dots f^{(n+1)} = (f^{(n)})'$$

if $f^{(n)}$ exists we call it n**-times differentiable**.

Def. 125. if $f^{(n)}$ is continuous we call f n**-times continuous differentiable**. The set of all n-times continuous differentiable functions on D is denoted as $C^n(D)$

Def. 126. If for a function f and some $x_0 \in \mathbb{D}$ and $\delta > 0$ we have

$$f(x_0) \geq (\leq) f(x) \quad \forall x \in \mathbb{D} \cap (x_0 - \delta, x_0 + \delta)$$

then x_0 is called a **local maximum (minimum)**.

4.1.1 Derivation rules

Thm. 127. Sum Rule: For the n-th derivative of a sum:

$$(f+g)^{(n)} = f^{(n)} + g^{(n)}$$

Thm. 128. General Leibniz Rule (Product): The n-th derivative of a product is given by:

$$(f\cdot g)^{(n)} = \sum_{k=0}^{n}\binom{n}{k} f^{(n-k)} g^{(k)}$$

Thm. 129. Chain Rule: For the composition of functions:

$$(f\circ g)'(x)$$

$$= f'(g(x))\cdot g'(x)$$

Thm. 130. Quotient Rule: For $g(x) \neq 0$:

$$\left(\frac{f}{g}\right)'$$

$$= \frac{f'g - fg'}{g^2}$$

Thm. 131. Inverse Functions: If f is differentiable and injective:

$$(f^{-1})'(x)$$

$$= \frac{1}{f'(f^{-1}(x))}$$

Thm. 132. Logarithmic Differentiation: To differentiate $h(x) = f(x)^{g(x)}$, rewrite it as $h(x) = e^{g(x)\ln(f(x))}$. The derivative is:

$$h'(x) = f(x)^{g(x)}\left[g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)}\right]$$

4.1.2 Bernoulli-L'Hopital

Thm. 133. Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, and let x_0 be a local extremum of f. Then one of the following holds: 1) x_0 is an endpoint of I. 2) f is not differentiable at x_0. 3) f is differentiable at x_0 and $f'(x_0) = 0$.

Thm. 134. Rolle's Theorem: Let $a < b$ with $f(a) = f(b)$, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then $\exists \xi \in (a, b)$ with

$$f'(\xi) = 0$$

Thm. 135. Mean-Value-Theorem: Let $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then $\exists \xi \in (a, b)$ with

$$f'(\xi) = \frac{f(b)-f(a)}{b-a}$$

Thm. 136. Cauchy's Mean-Value-Theorem: Let $a < b$, and let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then $\exists \xi \in (a, b)$ with

$$\frac{f(b)-f(a)}{g(b)-g(a)}$$

$$= \frac{f'(\xi)}{g'(\xi)}$$

Thm. 137. Bernoulli-L'Hopital's Rule: Let $f, g : I \to \mathbb{R}$ be differentiable functions on (a, b) and let $x_0 \in (a, b)$. If $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in (a, b)$. If

$$\lim_{x\to x_0}|f(x)| = \lim_{x\to x_0}|g(x)| \in \{0, \infty\}$$

$$\text{and} \quad \exists L = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)} = L$$

4.1.3 Convex and Concave graphs

Def. 138. A function $f : I \to \mathbb{R}$ is called **convex** iff $\forall a, b \in I \forall t \in (0, 1)$

$$f((1-t)a + bt) \leq (1-t)f(a) + tf(b)$$

Def. 139. A function $f : I \to \mathbb{R}$ is called **concave** iff $\forall a, b \in I \forall t \in (0, 1)$

$$f((1-t)a + bt) \geq (1-t)f(a) + tf(b)$$

Thm. 140. Let $(a, b) \subset \mathbb{R}$ and $f : (a, b) \to \mathbb{R}$ be differentiable. Then $\forall x$

$$f'(x) \geq 0 \quad \Longleftrightarrow \quad f \text{ is increasing}$$

Thm. 141. Let $(a, b) \subset \mathbb{R}$ and $f : (a, b) \to \mathbb{R}$ be differentiable. Then

$$f \text{ is convex} \quad \Longleftrightarrow \quad f' \text{ is increasing}$$

Thm. 142. Let $(a, b) \subset \mathbb{R}$ and $f : (a, b) \to \mathbb{R}$ be twice differentiable. Then

$$f \text{ is convex} \quad \Longleftrightarrow \quad f'' \geq 0$$

4.1.4 Taylor Series

Lem. 143. We can locally approximate the graph of f at x_0 with the tangent line $t(x)$:

$$f(x) \approx t(x)$$

$$= f(x_0) + f'(x_0)(x - x_0)$$

Lem. 144. To even more accurately approximate the function f we can add further terms to the tangent line expansion:

$$f(x) \approx t(x)$$

$$= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

Thm. 145. Taylor's Theorem: Let $f : (a, b) \to \mathbb{R}$ be $n+1$ times differentiable. Then

$$f(x) = P_n(x) + R(x)$$

with $P_n(x)$ the Taylor polynomial of degree n and $R(x)$ the remainder term:

$$P_n(x)$$

$$= \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

$$R(x)$$

$$= \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} \quad (\xi \text{ between } x_0, x)$$

Def. 146. The **Taylor series expansion** of an infinitely differentiable function $f : (a, b) \to \mathbb{R}$ at x_0 is given by:

$$f(x)$$

$$= \sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Thm. 147. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be two power series with radius of convergence R. Then:

$$f(x)g(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n$$

Thm. 148. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then:

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Thm. 149. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then:

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \int a_n x^n \, dx$$

4.2 Lookup Tables

4.2.1 Common Derivatives

$f(x)$	$f'(x)$
x^p	px^{p-1}
$\ln\|x\|$	$1/x$
e^x	e^x
a^x	$a^x \ln a$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$1 + \tan^2 x = \sec^2 x$
$\cot x$	$-(1 + \cot^2 x) = -\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\text{arccot}\ x$	$\frac{-1}{1+x^2}$
$\log_a x$	$\frac{1}{x \ln a}$
x^x	$x^x(1 + \ln x)$
$\text{arsinh}\ x$	$\frac{1}{\sqrt{1+x^2}}$
$\text{arcosh}\ x$	$\frac{1}{\sqrt{x^2-1}}$
$\text{artanh}\ x$	$\frac{1}{1-x^2}$
$\text{arcoth}\ x$	$\frac{1}{1-x^2}$
$\tanh x$	$\text{sech}^2 x$
$\coth x$	$-\text{csch}^2 x$
$\text{sech}\ x$	$-\text{sech}\ x \tanh x$
$\text{csch}\ x$	$-\text{csch}\ x \coth x$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\frac{1}{x}$	$-\frac{1}{x^2}$
$x \ln x$	$\ln x + 1$
$\ln(\ln x)$	$\frac{1}{x \ln x}$
$\|x\|$	$\frac{x}{\|x\|} = \text{sgn}(x)$
$\log_x(a)$	$-\frac{\ln a}{x(\ln x)^2}$
e^{ax}	ae^{ax}
a^{cx}	$ca^{cx} \ln a$
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\sin^2 x$	$\sin(2x)$
$\cos^2 x$	$-\sin(2x)$
$\arcsin(\frac{x}{a})$	$\frac{1}{\sqrt{a^2-x^2}}$
$\arctan(\frac{x}{a})$	$\frac{a}{a^2+x^2}$
$\text{arcsec}\ x$	$\frac{1}{\|x\|\sqrt{x^2-1}}$
$\text{arccsc}\ x$	$\frac{-1}{\|x\|\sqrt{x^2-1}}$
$W(x)$	$\frac{W(x)}{x(1+W(x))}$
$\int_0^x f(t)dt$	$f(x)$

4.2.2 Standard Taylor Series (at $x = 0$)

Func.	**Series**	**First Terms**	**Rad**
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2} + \ldots$	∞
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$x - \frac{x^3}{6} + \ldots$	∞
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$1 - \frac{x^2}{2} + \ldots$	∞
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + \ldots$	1
$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$	$x - \frac{x^2}{2} + \ldots$	1
$\arctan x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$	$x - \frac{x^3}{3} + \ldots$	1
$(1+x)^\alpha$	$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$	$1 + \alpha x + \ldots$	1
$\sinh x$	$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$	$x + \frac{x^3}{6} + \ldots$	∞
$\cosh x$	$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$	$1 + \frac{x^2}{2} + \ldots$	∞
$\tan x$	$\ldots$	$x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$	$\pi/2$
$\arcsin x$	$\ldots$	$x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots$	1
$-\ln(1-x)$	$\sum_{n=1}^{\infty} \frac{x^n}{n}$	$x + \frac{x^2}{2} + \ldots$	1

Imp. 150. Radii of Convergence for additional Taylor Series:
Logarithm: $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$ (for $x \in (-1, 1]$).
Geometric: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ (for $|x| < 1$).
Binomial: $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \ldots$ (for $|x| < 1$).

4.3 Most Common Proofs

4.3.1 Proof Outline: Mean Value Theorem (MVT)

Goal: $\exists c \in (a, b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.
Step 1: Define helper $h(x) = f(x) - \left(f(a) + \frac{f(b)-f(a)}{b-a}(x - a)\right)$.
Step 2: Verify $h(a) = 0$ and $h(b) = 0$. h is continuous on $[a, b]$ and differentiable on (a, b).
Step 3: Apply Rolle's Theorem to $h(x)$: $\exists c \in (a, b)$ s.t. $h'(c) = 0$.
Step 4: Calculate $h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. Setting $h'(c) = 0$ yields the MVT.

4.3.2 Proof Outline: Differentiability

Goal: Show $f(x)$ is differentiable at x_0.
Step 1: Set up the limit of the difference quotient: $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$.
Step 2: If piecewise, compute left-sided limit ($h \nearrow 0$) and right-sided limit ($h \searrow 0$) separately.
Step 3: Use algebraic manipulation, Taylor expansions, or L'Hôpital's rule to evaluate limits.
Step 4: Show both one-sided limits exist and are equal. The common value is $f'(x_0)$.

4.3.3 Proof Outline: Differentiability implies continuity

Goal: If $f'(x_0)$ exists, then f is continuous at x_0.
Step 1: We must show $\lim_{x\to x_0}(f(x) - f(x_0)) = 0$.
Step 2: For $x \neq x_0$, rewrite: $f(x) - f(x_0) = \frac{f(x)-f(x_0)}{x-x_0}(x - x_0)$.
Step 3: Take $\lim_{x\to x_0}$. Because f is diff. at x_0, $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$.
Step 4: Also, $\lim_{x\to x_0}(x - x_0) = 0$.
Step 5: By limit rules, the product limit is $f'(x_0) \cdot 0 = 0$. Thus $\lim_{x\to x_0} f(x) = f(x_0)$.

4.3.4 Proof Outline: Rolle's theorem

Goal: If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then $\exists c \in (a, b)$ with $f'(c) = 0$.
Step 1: If $f(x)$ is constant on $[a, b]$, then $f'(x) = 0$ for all $x \in (a, b)$, any c works.
Step 2: If not constant, by EVT, f attains a max and min on $[a, b]$.
Step 3: Since $f(a) = f(b)$, at least one extreme value must occur at an interior point $c \in (a, b)$.
Step 4: By Fermat's Theorem, if a function attains a local extremum at an interior point where differentiable, its derivative there is 0. Thus $f'(c) = 0$.

4.4 Exam Strategies

4.4.1 Piecewise Functions

To check continuity and differentiability at x_0:
Continuity: Check if $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = f(x_0)$.
Differentiability: If continuous, check if $\lim_{h\searrow 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\nearrow 0} \frac{f(x_0+h)-f(x_0)}{h}$.
Shortcut: If $f'(x)$ exists for $x \neq x_0$ and $\lim_{x\to x_0} f'(x) = L$ exists, then $f'(x_0) = L$.

4.4.2 Limits - Taylor vs. L'Hôpital

Use Taylor when: The expression has complicated products, powers, or nested functions (e.g., $e^{\sin x} - 1$).
Use L'Hôpital when: The expression is a simple fraction $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and derivatives are easy.
Important Continuous Limits:
$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$, $\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^a$
$\lim_{x\to 0} \frac{e^x-1}{x} = 1$, $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$, $\lim_{x\to\infty} x^{1/x} = 1$.

Equivalent Infinitesimals ($x \to 0$): $\sin(x) \sim x$, $\tan(x) \sim x$, $e^x - 1 \sim x$, $\ln(1+x) \sim x$, $1 - \cos(x) \sim \frac{x^2}{2}$.
L'Hôpital Algebraic Conversions:
$0 \cdot \infty \to$ rewrite as $\frac{1}{\infty^0}$ or $\frac{1}{0^\infty}$ ($\frac{f}{1/g}$ or $\frac{g}{1/f}$).
$\infty - \infty \to$ combine into a single fraction.
$1^\infty, 0^0, \infty^0 \to$ use $L = \lim f(x)^{g(x)} \implies \ln(L) = \lim[g(x)\ln(f(x))]$.

4.4.3 Constrained Extrema

To find extrema of $f(x,y)$ s.t. $g(x,y) = 0$:
1. Set up system $\nabla f = \lambda \nabla g$ and $g = 0$.
2. Solve for x, y, λ.
3. Check boundary! If region is a disk $x^2 + y^2 \leq 1$, check interior $\nabla f = 0$ AND boundary $x^2 + y^2 = 1$.

5. Integration

5.1 Theory

5.1.1 Indefinite Integrals and Techniques

Def. 151. Functions $F : I \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in (a,b)$ are called **antiderivatives** of f.

Def. 152. The family of all anti-derivatives of a function $f : I \to \mathbb{R}$ for $I \subseteq \mathbb{R}$ an open interval is called the **indefinite integral** and :=

$$\int f(x)\,dx$$

$$= \{F : I \to \mathbb{R} : F'(x) = f(x)\} = F(x) + C$$

Thm. 153. Let $f, g : I \to \mathbb{R}$ be integrable functions and let $c \in \mathbb{R}$. Then:

$$\int (f(x) \pm g(x))\,dx$$

$$= \int f(x)\,dx \pm \int g(x)\,dx$$

$$\int cf(x)\,dx = c\int f(x)\,dx$$

Thm. 154. Integration by Parts: For differentiable functions $f(x)$ and $g(x)$ we have:

$$\int f'(x)g(x)\,dx$$

$$= f(x)g(x) - \int f(x)g'(x)\,dx$$

Thm. 155. Substitution Rule: For functions $f(x)$ and $g(x)$ we have:

$$\int f(g(x))g'(x)\,dx$$

$$= \int f(u)\,du = F(g(x)) + C$$

where F is an antiderivative of f and $u = g(x)$.

5.1.2 The Definite Integral (Riemann Integral)

Def. 156. Let U and L denote the upper and lower sums of a bounded function f. Then f is called **Riemann-integrable** iff $\inf U = \sup L$. The value of the definite integral $\int_a^b f(x)\,dx$ is then this common value.

Thm. 157. Integrability Conditions: Any continuous or monotone function on $[a,b]$ is Riemann-integrable.

Thm. 158. Properties of Riemann Integral: Let f, g be integrable and $c \in \mathbb{R}$:

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

$$f(x) \leq g(x) \implies \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

Thm. 159. Mean Value Theorem for Integrals: Let $f : [a,b] \to \mathbb{R}$ be continuous. Then there exists a $c \in [a,b]$ such that:

$$\int_a^b f(x)\,dx = (b-a)f(c)$$

Thm. 160. Fundamental Theorem of Integral Calculus: Let $f : [a,b] \to \mathbb{R}$ be continuous. Then for all $C \in \mathbb{R}$, $F(x)$ is an antiderivative of f where:

$$F(x) = \int_a^x f(y)\,dy + C$$

5.1.3 Sequences of Functions and Integrals

Def. 161. Let $(f_n)_{n\in\mathbb{N}_0}$ be a sequence of functions. **Pointwise Convergence**: It converges pointwise to f if for every x, the sequence of real numbers $(f_n(x))$ converges to $f(x)$.

Def. 162. Uniform Convergence: It converges uniformly to f if for every $\epsilon > 0$, there exists an N such that for all $n \geq N$ and **for all** x simultaneously, $|f_n(x) - f(x)| < \epsilon$.

Thm. 163. Swapping Limit and Integral: If a sequence of integrable functions (f_n) converges **uniformly** to f on $[a,b]$, then the limit function f is also integrable, and:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\int_a^b f_n(x)\,dx$$

Thm. 164. Weierstrass M-Test: If $|f_n(x)| \leq M_n$ for all $x \in D$ and the series $\sum M_n$ converges, then $\sum f_n(x)$ converges uniformly on D.

5.1.4 Definite Integration Techniques

Lem. 165. If $F : [a,b] \to \mathbb{R}$ is continuously differentiable, then:

$$F(x) = F(a) + \int_a^x F'(t)\,dt$$

$$\int_a^b F'(t)\,dt = F(b) - F(a) = \Big[F(t)\Big]_a^b$$

Lem. 166. Substitution for Definite Integrals: Let f be continuous and g continuously differentiable.

$$\int_a^b f(g(x))g'(x)\,dx$$

$$= \int_{g(a)}^{g(b)} f(u)\,du$$

Version 2: If $f'(x) \neq 0$ and f^{-1} is the inverse of f:

$$\int_a^b g(f(x))\,dx$$

$$= \int_{f(a)}^{f(b)} g(y)(f^{-1})'(y)\,dy$$

Lem. 167. Integration by Parts for Definite Integrals: Let f, g be continuously differentiable.

$$\int_a^b f'(x)g(x)\,dx$$

$$= \Big[f(x)g(x)\Big]_a^b - \int_a^b f(x)g'(x)\,dx$$

5.1.5 Geometric Applications

Area between curves: $A = \int_a^b |f(x) - g(x)|dx$.
Rotational Volume (x-axis): $V = \pi\int_a^b (f(x))^2 dx$.
Rotational Volume (y-axis): $V = 2\pi\int_a^b xf(x)dx$.
Arc Length: $L = \int_a^b \sqrt{1 + (f'(x))^2}dx$.

5.1.6 Convergence of Improper Integrals

p-test (to infinity): $\int_1^\infty \frac{1}{x^p}dx$ converges if $p > 1$ (div. $p \leq 1$).
p-test (to zero): $\int_0^1 \frac{1}{x^p}dx$ converges if $p < 1$ (div. $p \geq 1$).
Comparison Test: If $0 \leq f(x) \leq g(x)$, then $\int g$ conv $\implies \int f$ conv.

5.2 Lookup Tables

5.2.1 Common Antiderivatives

$f(x)$	$\int f(x)dx$
$x^p, p \neq -1$	$\frac{x^{p+1}}{p+1}$
$\frac{1}{x}$	$\ln\|x\|$
e^x	e^x
a^x	$\frac{a^x}{\ln a}$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$-\ln\|\cos x\|$
$\cot x$	$\ln\|\sin x\|$
$\frac{1}{\cos^2 x} = \sec^2 x$	$\tan x$
$\frac{-1}{\sin^2 x} = -\csc^2 x$	$\cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\frac{1}{1+x^2}$	$\arctan x$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$
$\frac{1}{\sqrt{x^2+1}}$	$\operatorname{arsinh} x$
$\frac{1}{\sqrt{x^2-1}}$	$\operatorname{arcosh} x$
$\ln x$	$x\ln x - x$
$\log_a x$	$\frac{x\ln x - x}{\ln a}$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\frac{x}{a}$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\|\frac{a+x}{a-x}\right\|$
$\arcsin x$	$x\arcsin x + \sqrt{1-x^2}$
$\arccos x$	$x\arccos x - \sqrt{1-x^2}$
$\arctan x$	$x\arctan x - \frac{1}{2}\ln(1+x^2)$
$\sec^2 x$	$\tan x$
$\csc^2 x$	$-\cot x$
$\sec x\tan x$	$\sec x$
$\csc x\cot x$	$-\csc x$
$\sec x$	$\ln\|\sec x + \tan x\|$
$\csc x$	$\ln\|\csc x - \cot x\|$
$\tanh x$	$\ln(\cosh x)$
$\coth x$	$\ln\|\sinh x\|$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin(\frac{x}{a})$
$\frac{1}{\sqrt{x^2+a^2}}$	$\ln(x + \sqrt{x^2+a^2})$
$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\|x + \sqrt{x^2-a^2}\|$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin(2x)}{4}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin(2x)}{4}$
$\tan^2 x$	$\tan x - x$
$\cot^2 x$	$-\cot x - x$
xe^x	$e^x(x-1)$
$x\sin x$	$\sin x - x\cos x$
$x\cos x$	$\cos x + x\sin x$
$\ln^2 x$	$x(\ln^2 x - 2\ln x + 2)$
$e^{ax}\sin bx$	$\frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$
$e^{ax}\cos bx$	$\frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$
$\sqrt{a^2-x^2}$	$\frac{x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}}{2}$
$\sqrt{x^2+a^2}$	$\frac{x\sqrt{x^2+a^2}+a^2\operatorname{arsinh}\frac{x}{a}}{2}$

5.2.2 Integration Tricks

Partial Fraction Decomposition: For $\int \frac{P(x)}{Q(x)}dx$. If $\deg P \geq \deg Q$, use polynomial division first. **Simple real root:** $(ax+b) \to \frac{A}{ax+b}$. **Repeated real root:** $(ax+b)^k \to \frac{A_1}{ax+b} + \cdots + \frac{A_k}{(ax+b)^k}$. **Irreducible quadratic:** $(ax^2+bx+c) \to \frac{Ax+B}{ax^2+bx+c}$.
Weierstrass Substitution: For integrals of $R(\sin x, \cos x)$, let $t = \tan(x/2)$.
Square Completion: For $\sqrt{ax^2+bx+c}$, complete the square to get $\sqrt{u^2 \pm k^2}$.
Substitution Patterns: $\sqrt{a^2-x^2} \to x = a\sin\theta$ or $x = a\cos\theta$. $\sqrt{a^2+x^2} \to x = a\tan\theta$ or $x = a\sinh u$. $\sqrt{x^2-a^2} \to x = a\sec\theta$ or $x = a\cosh u$.
Symmetry: $\int_{-a}^{a} f(x)dx = 0$ if f is odd, and $2\int_0^a f(x)dx$ if f is even.

5.3 Most Common Proofs

5.3.1 Proof Outline: Riemann Integrability

Goal: Show f is Riemann integrable on $[a,b]$.
Criterion: $\forall\epsilon > 0$, $\exists$ a partition P such that $U(f,P) - L(f,P) < \epsilon$.
Step 1: Choose an equidistant partition $P_n = \{x_0, \ldots, x_n\}$ with $\Delta x = \frac{b-a}{n}$.
Step 2: Find the supremum M_i and infimum m_i of f on each subinterval $[x_{i-1}, x_i]$.
Step 3: Express the difference: $U(f,P_n) - L(f,P_n) = \sum_{i=1}^{n}(M_i - m_i)\Delta x$.
Step 4: Bound $(M_i - m_i)$. If f is monotonically increasing, sum telescopes: $\sum_{i=1}^{n}(f(x_i) - f(x_{i-1})) = f(b) - f(a)$.
Step 5: In monotonic case, $U - L = (f(b) - f(a))\frac{b-a}{n}$. Choose n large enough so this is $< \epsilon$.

5.3.2 Proof Outline: Fundamental Theorem of Calculus

Goal: Show $\frac{d}{dx}\int_a^x f(t)dt = f(x)$ for continuous f.
Step 1: Let $F(x) = \int_a^x f(t)dt$. Evaluate difference quotient $\frac{F(x+h)-F(x)}{h}$.
Step 2: By additivity, $F(x+h) - F(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt$.
Step 3: By Mean Value Theorem for Integrals (since f is continuous), $\exists c_h \in [x, x+h]$ s.t. $\int_x^{x+h} f(t)dt = f(c_h)h$.
Step 4: Thus, the difference quotient simplifies to $\frac{f(c_h)h}{h} = f(c_h)$.
Step 5: As $h \to 0$, interval $[x, x+h]$ shrinks to x, so $c_h \to x$. By continuity of f, $\lim_{h\to 0} f(c_h) = f(x)$. Thus $F'(x) = f(x)$.

5.3.3 Proof Outline: Integrability of continuous functions

Goal: Continuous f on $[a,b]$ is Riemann integrable.
Step 1: By Heine-Cantor theorem, continuous f on compact $[a,b]$ is uniformly continuous.
Step 2: Let $\epsilon > 0$. $\exists\delta > 0$ s.t. $\forall x, y \in [a,b]$, $|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{b-a}$.
Step 3: Choose partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ with mesh size $< \delta$.
Step 4: For each subinterval, supremum M_i and infimum m_i are attained at c_i, d_i (EVT).
Step 5: Since $|c_i - d_i| < \delta$, we have $M_i - m_i = f(c_i) - f(d_i) < \frac{\epsilon}{b-a}$.
Step 6: $U(f,P) - L(f,P) = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i < \frac{\epsilon}{b-a}\sum_{i=1}^{n}\Delta x_i = \epsilon$. Thus f is Riemann integrable.

5.4 Exam Strategies

5.4.1 Leibniz's Rule

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t,x)\,dt = f(b(x),x)b'(x)$$

$$-f(a(x),x)a'(x) + \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}\,dt$$

6. Differential Equations

6.1 Theory

6.1.1 Modelling and Definitions

Rem. 168. Modelling Process: To derive an ODE from a physical scenario:

1. Identify the quantity $u(t)$ and express its change over a small interval Δt: $u(t + \Delta t) = u(t) + (\text{rate}) \cdot \Delta t$
2. Rearrange and take the limit $\Delta t \to 0$: $\frac{u(t+\Delta t)-u(t)}{\Delta t} = \text{rate} \implies u'(t) = \text{rate}$

Rem. 169. Typical ODE Models:

1. Bounded Growth (Logistic): With capacity L:
$$u'(t) = ku(t)\left(1 - \frac{u(t)}{L}\right)$$
2. Bimolecular Reaction ($A + B \to C$): With concentrations a, b:
$$u'(t) = k(a - u(t))(b - u(t))$$

Def. 170. An **ordinary differential equation (ODE)** contains derivatives of a function of one independent variable. It is:

- **of n-th order**: if the n-th derivative is the highest occurring.
- **linear**: if the unknown function $u(t)$ and its derivatives appear only as a linear combination.
- **homogeneous**: if it only contains terms with $u(t)$ and its derivatives (otherwise **inhomogeneous**).

Thm. 171. Picard-Lindelöf (Existence and Uniqueness): If $f(x,y)$ is continuous and Lipschitz continuous with respect to y, then the IVP $y' = f(x,y)$ with $y(x_0) = y_0$ has a unique solution in some interval around x_0.

6.1.2 First-Order ODEs

Thm. 172. Separation of Variables: For $y' = f(x)g(y) \implies \int\frac{dy}{g(y)} = \int f(x)dx + C$.

Thm. 173. Linear First-Order ODE: $y' + p(x)y = q(x)$. **Integrating Factor Method:** 1. Calculate $F(x) = \exp\left(\int p(x)dx\right)$. 2. Multiply ODE by $F(x)$: $(yF(x))' = q(x)F(x)$. 3. Integrate: $y(x) = \frac{1}{F(x)}\left(\int q(x)F(x)dx + C\right)$.

Thm. 174. Bernoulli ODE: $y' + p(x)y = q(x)y^n$ $(n \neq 0, 1)$. **Trick:** Substitute $u = y^{1-n} \implies u' = (1-n)y^{-n}y'$. This transforms the ODE into the linear form: $\frac{u'}{1-n} + p(x)u = q(x)$.

Thm. 175. Homogeneous substitution: $y' = f(y/x)$. Substitute $u = y/x \implies y = ux, y' = u'x + u$.

6.1.3 Linear ODEs of n-th Order

Thm. 176. Superposition Principle: If y_1 and y_2 solve a linear homogeneous ODE, then for any $C_1, C_2 \in \mathbb{R}$:

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

Thm. 177. An n-th order linear homogeneous ODE has n linearly independent solutions $y_1, \ldots, y_n$. Its general solution is:

$$y(x) = \sum_{i=1}^{n} C_i y_i(x) \quad \text{with } C_i \in \mathbb{R}$$

Lem. 178. For a homogeneous ODE with constant coefficients $a_n y^{(n)} + \cdots + a_0 y = 0$, the Ansatz $y(x) = e^{\lambda x}$ yields the characteristic equation:

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0$$

with characteristic polynomial $p(\lambda)$.

Thm. 179. General Solution from Characteristic Roots: The roots of $p(\lambda)$ determine the fundamental solutions:

- **Distinct real root** $\lambda \in \mathbb{R}$
$$y(x) = e^{\lambda x}$$
- **Multiple real root** $\lambda \in \mathbb{R}$ **with multiplicity** m
$$e^{\lambda x}, xe^{\lambda x}, \ldots, x^{m-1} e^{\lambda x}$$
- **Complex pair** $\lambda = \alpha \pm i\beta \in \mathbb{C}$ **(multiplicity** m**)**
$$e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x), \ldots,$$
$$x^{m-1}e^{\alpha x}\cos(\beta x), x^{m-1}e^{\alpha x}\sin(\beta x)$$

6.1.4 Inhomogeneous Linear ODEs

Lem. 180. The general solution of $a_n y^{(n)} + \cdots + a_0 y = s(x)$ is $y(x) = y_h(x) + y_p(x)$, where y_h is the general homogeneous solution and y_p is a particular solution.

Thm. 181. Variation of Constants (First-Order Linear ODE): For $y' + p(x)y = q(x)$, the general solution is:

$$y(x) = e^{-P(x)} \left(\int q(x) e^{P(x)} \, dx + C \right)$$

where $P(x) = \int p(x)\, dx$ is an antiderivative of $p(x)$.

6.1.5 Initial Value Problems and Boundary-Value Problems

Def. 182. An **Initial Value Problem (IVP)** specifies conditions at a single point x_0 (e.g. $y(x_0) = y_0$, $y'(x_0) = y_1$). A **Boundary Value Problem (BVP)** specifies conditions at multiple points (e.g. $y(a) = y_a$, $y(b) = y_b$).

6.2 Lookup Tables

6.2.1 Method of Undetermined Coefficients (Ansatz Table)

For $ay'' + by' + cy = s(x)$, if no resonance:

Source $s(x)$	**Ansatz** $y_p(x)$
$P_n(x)$ (Polyn.)	$Q_n(x) = A_n x^n + \cdots + A_0$
Ae^{kx}	Be^{kx}
$A\sin(\omega x) + B\cos(\omega x)$	$C\sin(\omega x) + D\cos(\omega x)$
$e^{\alpha x} P_n(x)$	$e^{\alpha x} Q_n(x)$
$e^{\alpha x}\sin(\beta x)$	$e^{\alpha x}(C\cos(\beta x) + D\sin(\beta x))$

Resonance Rule: If the standard Ansatz $y_p(x)$ contains a solution to the homogeneous equation, multiply the *entire* Ansatz by x (or x^2 for double roots).

Imp. 183. Resonance: If the term e^{kx} or $e^{i\omega x}$ in $s(x)$ matches an m-fold root of the characteristic polynomial $p(\lambda)$, multiply the standard Ansatz $y_p(x)$ by x^m.

6.3 Exam Strategies

6.3.1 ODE Solver Workflow

1. Can I separate? $y' = f(x)g(y)$.
2. Is it linear 1st order? $y' + p(x)y = q(x) \to$ Integrating factor $e^{\int p}$.
3. Is it 2nd order linear with constant coeffs? $ay'' + by' + cy = s(x)$. Solve char. eq. $a\lambda^2 + b\lambda + c = 0$. Particular solution: Match $s(x)$ with Ansatz table (check for resonance!).