The Mathematics of SET and Beyond

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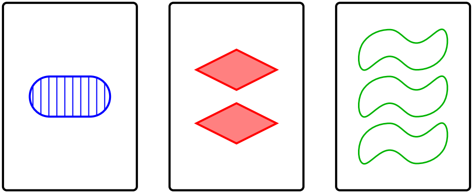
Lewis and Clark

November 7th, 2019

Joint work with Catherine Hsu and Jonah Ostroff*

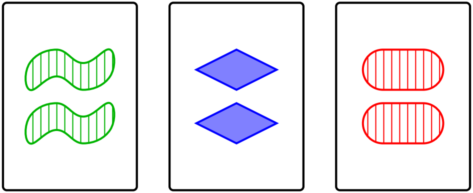
In 1974, Marsha Falco invented the card game SET:



Color	**Number**	**Shape**	**Shade**
red	one	diamond	solid
blue	two	squiggle	striped
green	three	oval	open

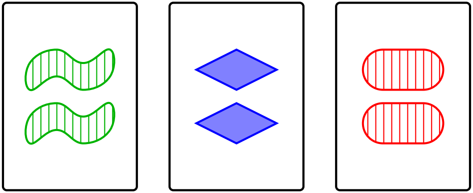
A **set** is a collection of three cards for which in each of of the four qualities the cards are all the same or all different.

Is this a set?



Card 1:	green	two	squiggle	striped
Card 2:	blue	two	diamond	solid
Card 3:	red	two	oval	striped

Is this a set?



Card 1:	green	two	squiggle	striped
Card 2:	blue	two	diamond	solid
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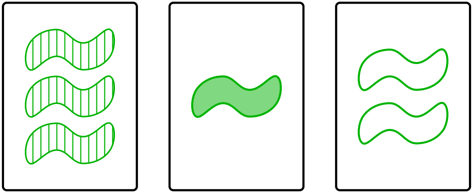
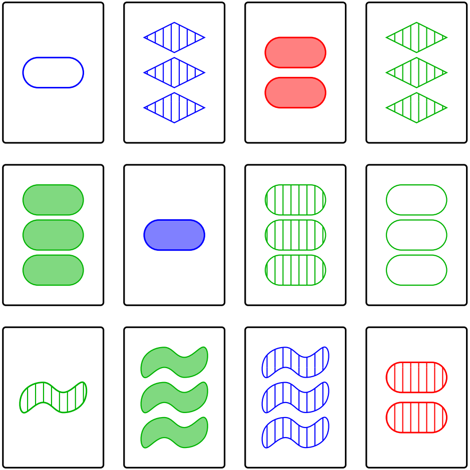


Card 1:	green	three	squiggle	striped
Card 2:	green	one	squiggle	solid
Card 3:	green	two	squiggle	open



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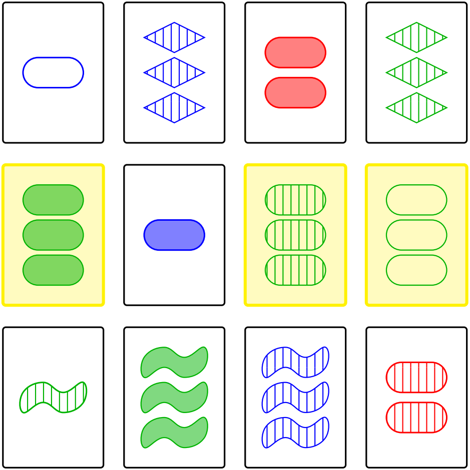

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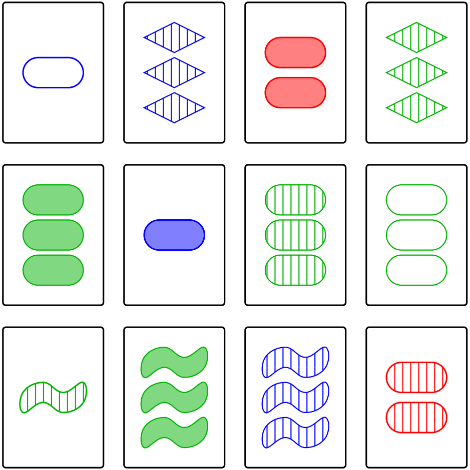
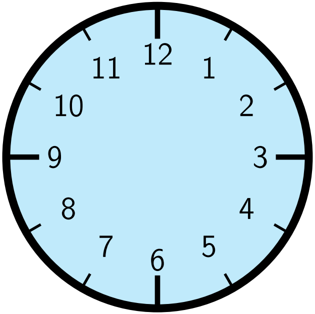


A **set** is a collection of three cards for which in each of the four qualities, the cards are all the same or all different.

Try to find another set!

To study the mathematics behind SET, we need an new idea:

Clock Arithmetic



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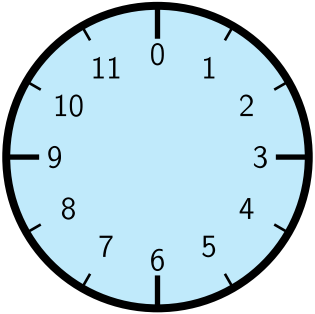

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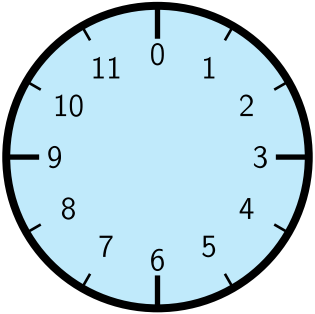
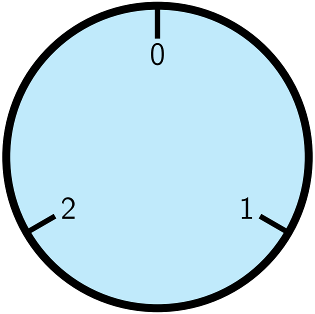


$$2 + 6 = 8$$
$$11 + 11 = 10$$
$$10 + 5 = 3$$

Let's consider clock arithmetic with 3 elements:

Clock Arithmetic C_3



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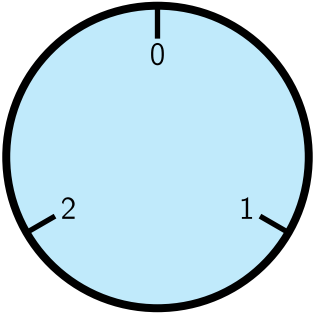
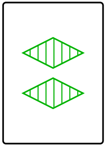


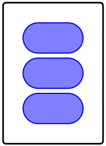
$$1 + 2 = 0$$
$$1 + 1 = 2$$
$$2 + 2 = 1$$

We can represent each SET card using clock arithmetic:

C_3	**Color**	**Number**	**Shape**	**Shade**
0	red	one	diamond	open
1	blue	two	squiggle	striped
2	green	three	oval	solid

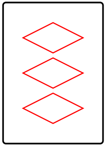


$\longleftrightarrow \quad (2,1,0,1)$ in $\mathbf{C}_3^4$

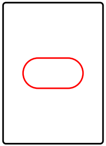


$\longleftrightarrow \quad (1,2,2,2)$ in $\mathbf{C}_3^4$

C_3	Color	Number	Shape	Shade
0	red	one	diamond	open
1	blue	two	squiggle	striped
2	green	three	oval	solid



$\longleftrightarrow \quad (?, ?, ?, ?) \text{ in } \mathbf{C}_3^4$



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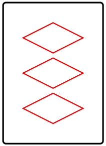
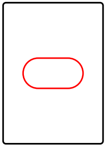


$\longleftrightarrow$ $(0, 2, 0, 0)$ in $\mathbf{C}_3^4$



$\longleftrightarrow$ $(0, 0, 2, 0)$ in $\mathbf{C}_3^4$

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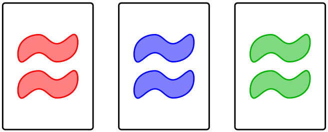

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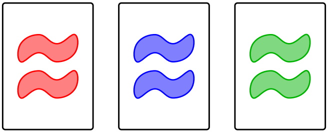

$$\Updownarrow$$

$$(0,1,1,2) + (1,1,1,2) + (2,1,1,2) = (0,0,0,0) \text{ in } \mathbf{C}_3^4$$

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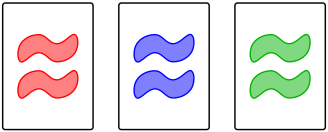

$$\Updownarrow$$

$$\underbrace{(0,1,1,2)}_{a} + \underbrace{(1,1,1,2)}_{b} + \underbrace{(2,1,1,2)}_{c} = (0,0,0,0) \text{ in } \mathbf{C}_3^4$$

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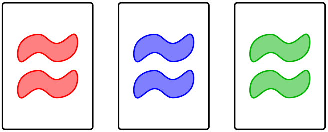

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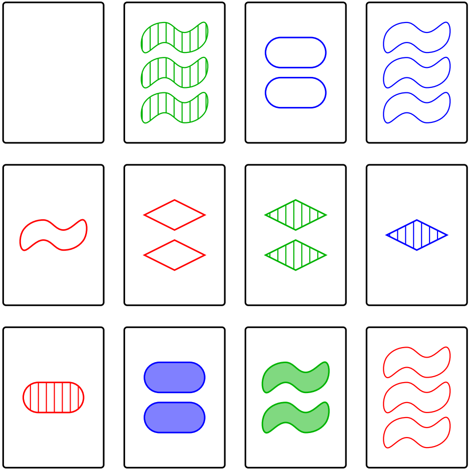
$$\Updownarrow$$

a, b, c sum to $(0,0,0,0)$ in $\mathbf{C}_3^4$

Find the missing card and the SET it goes with!



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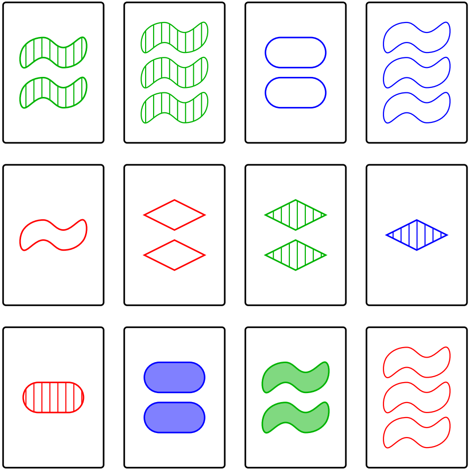
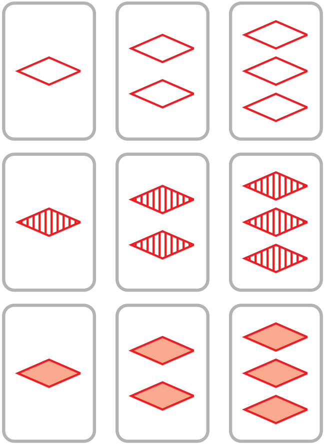
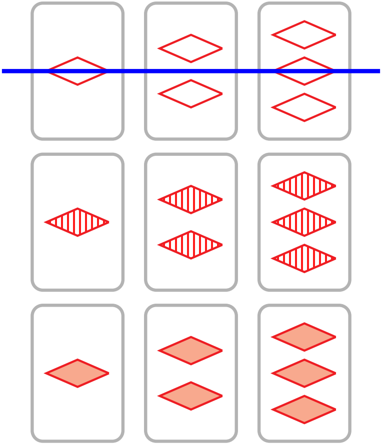
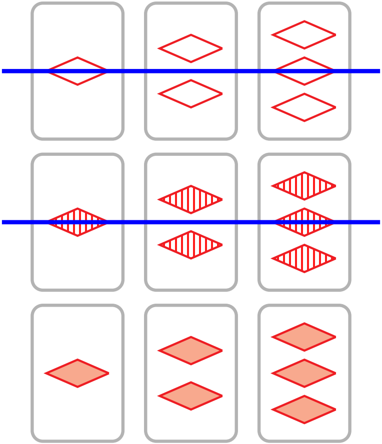
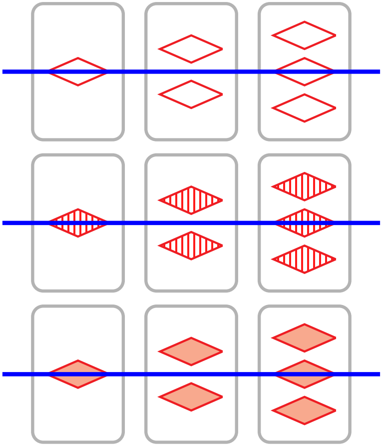
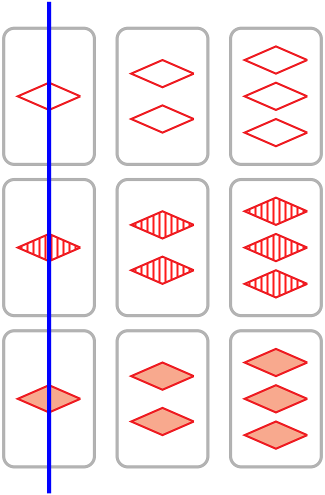


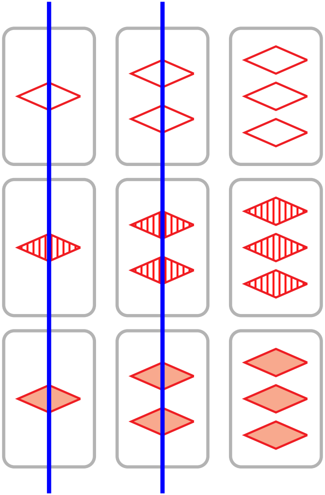


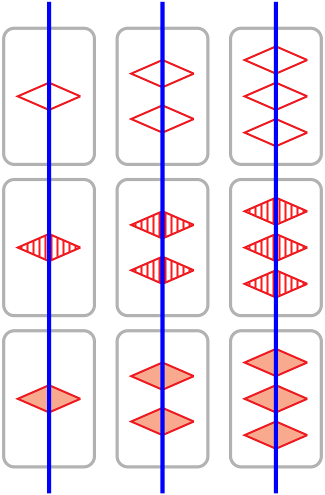


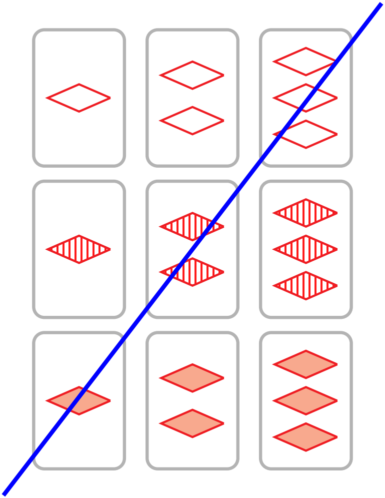


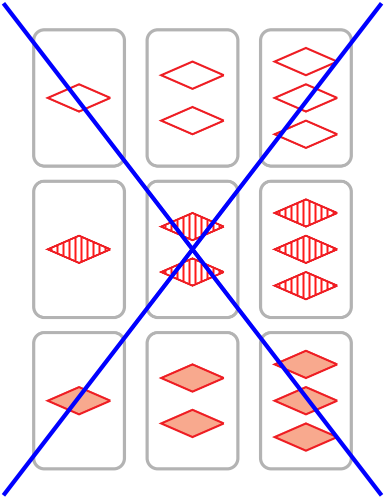


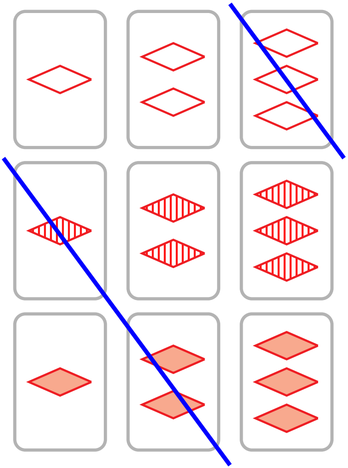


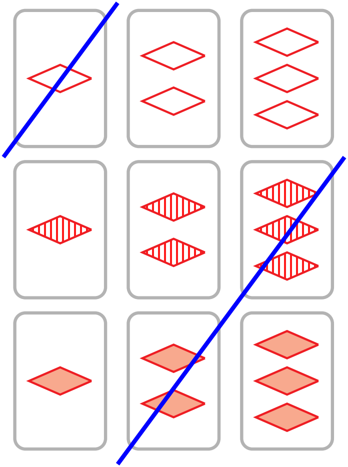


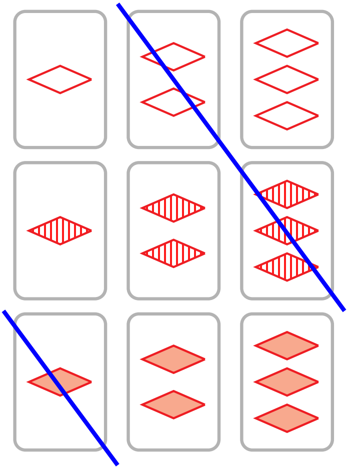


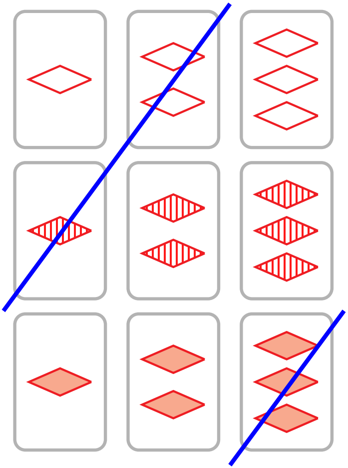


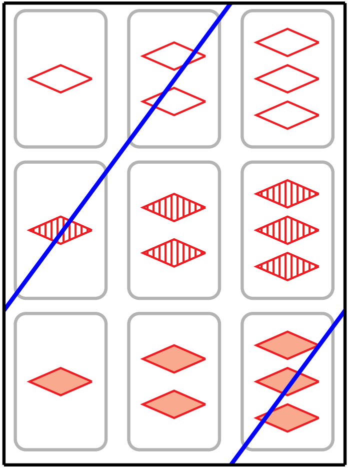


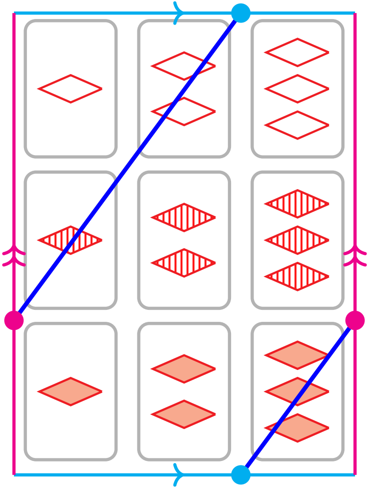










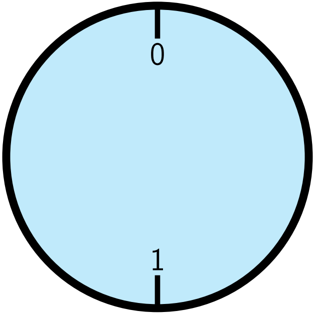


https://upload.wikimedia.org/wikipedia/commons/6/60/Torus_from_rectangle.gif

Pokémon SET with $\mathbf{C}_2$

Now, let's consider clock arithmetic with 2 hours:

Clock Arithmetic C_2



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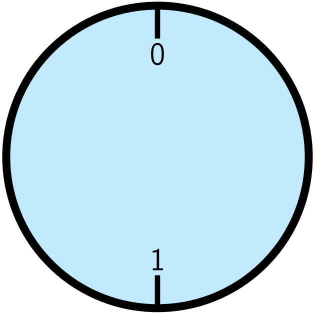
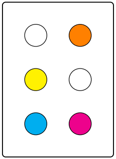


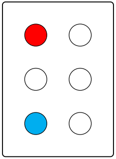
$$0 + 1 = 1$$

$$1 + 1 = 0$$

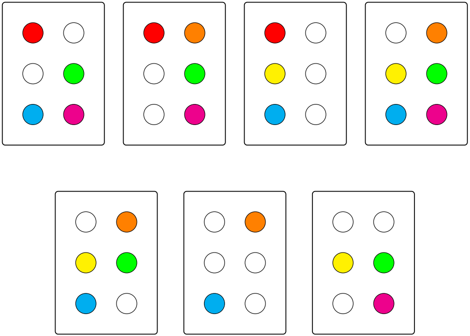
Here are some examples of cards in $\mathbf{C}_2^6$:



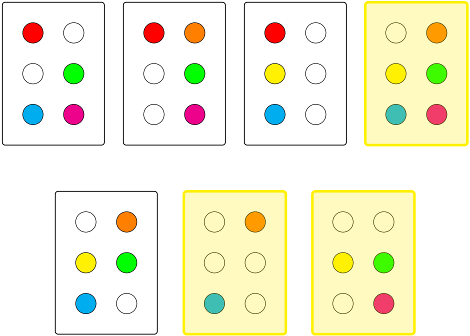
$\longleftrightarrow \quad (0,1,1,0,1,1)$ in $\mathbf{C}_2^6$



$\longleftrightarrow \quad (1,0,0,0,1,0)$ in $\mathbf{C}_2^6$



A **C_2-set** is a collection of three cards for which there's an even number of dots of each color.



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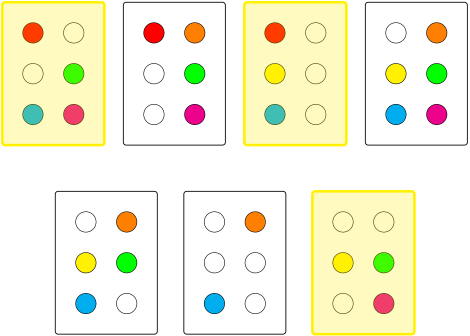

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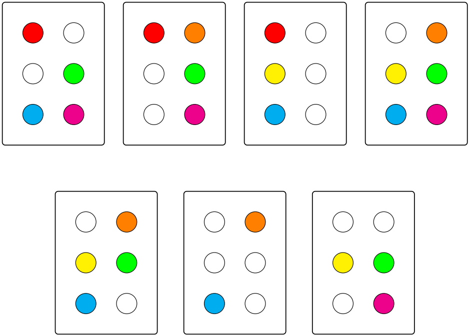
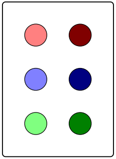


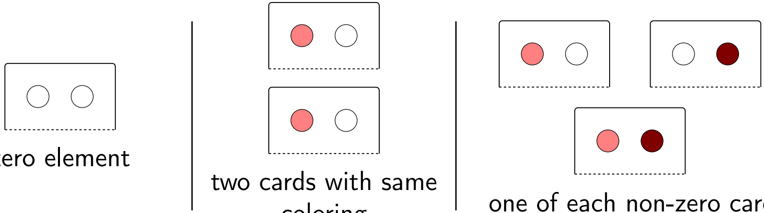
A **C_2-set** is a collection of three cards for which there's an even number of dots of each color.

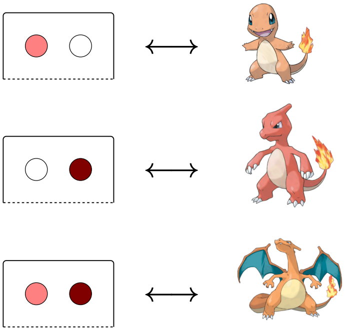
Try to find another set!



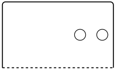
$$\longleftrightarrow \quad ((1,1),(1,1),(1,1)) \text{ in } \mathbf{C}_2^6$$

In $\mathbf{C}_2^2$, there are three ways to have an even number dots of each color:

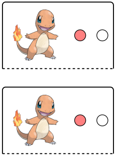




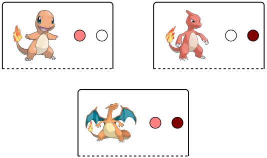
So, an even number dots can be appear as:



no Pokémon



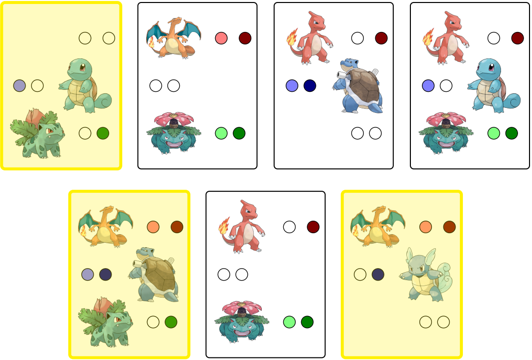
two of the same Pokémon



a full evolution of Pokémon

A **Pokémon C_2-set** is a collection of three cards for which the Pokémon can be partitioned into identical pairs or full evolutions.

Try to find a set!



A **Pokémon C_2-set** is a collection of three cards for which the Pokémon can be partitioned into identical pairs or full evolutions.

Try to find a set!

— Bonus question —

Can you find any number of cards for which the Pokémon can be partitioned into identical pairs or full evolutions?

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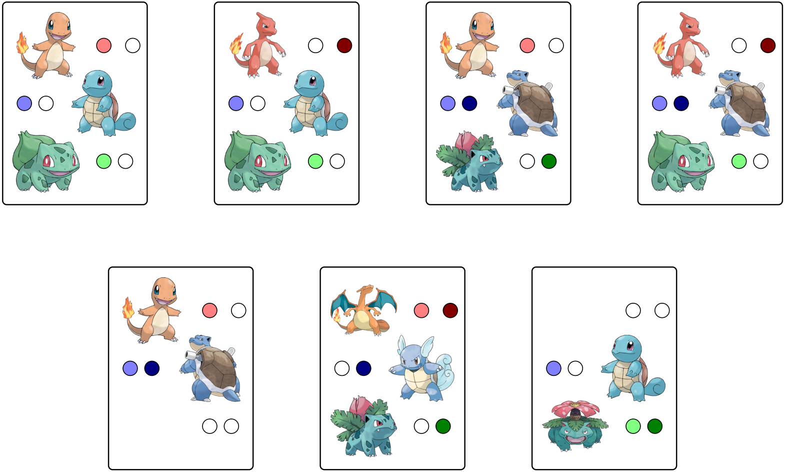
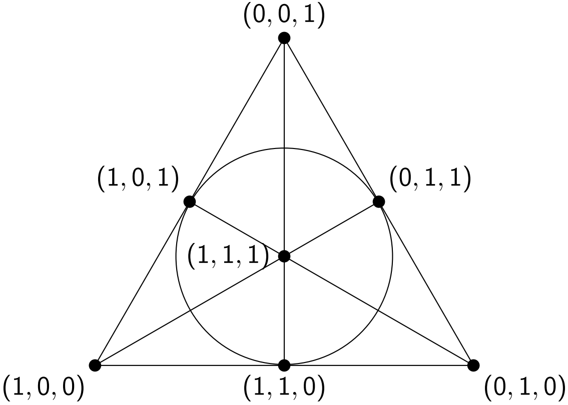
Can you find any number of cards for which the Poké-mon can be partitioned into identical pairs or full evolutions?

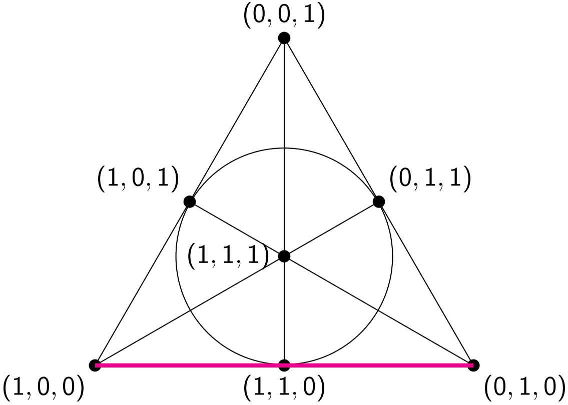


A Geometric Version of $\mathbf{C}_2$-SET

This is the Fanoplane $\mathbf{C}_2^3$.



This is the Fanoplane $\mathbf{C}_2^3$.



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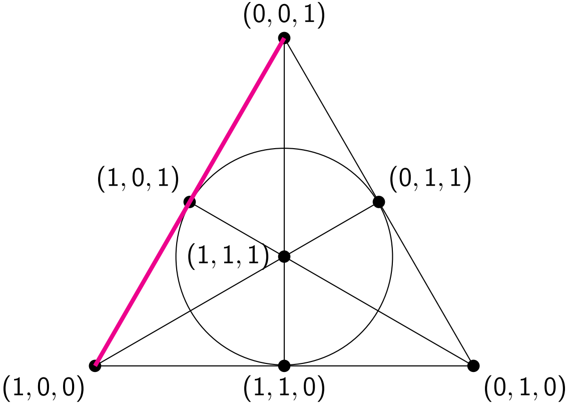

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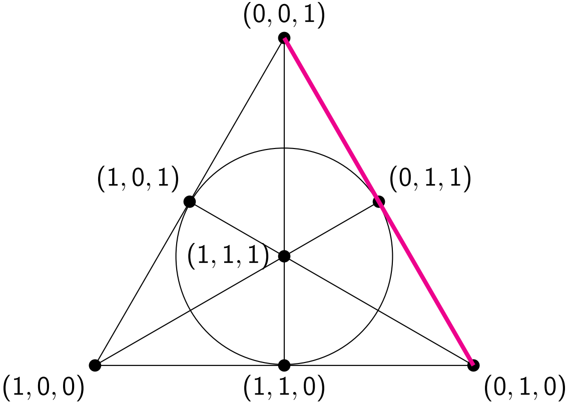

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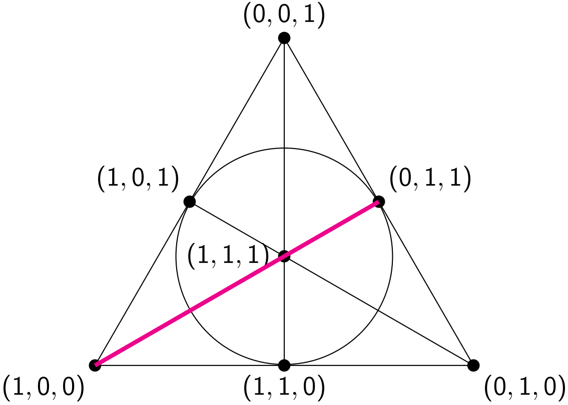
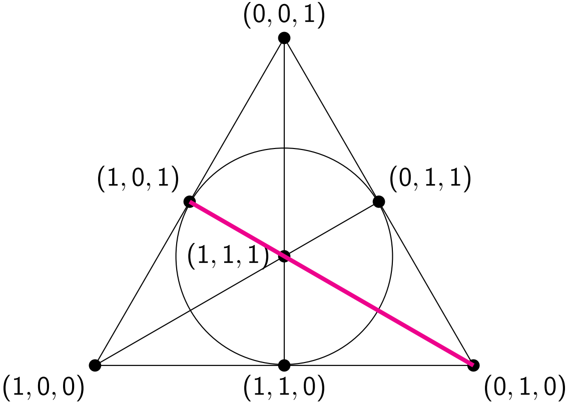


This is the Fanoplane $\mathbf{C}_2^3$.



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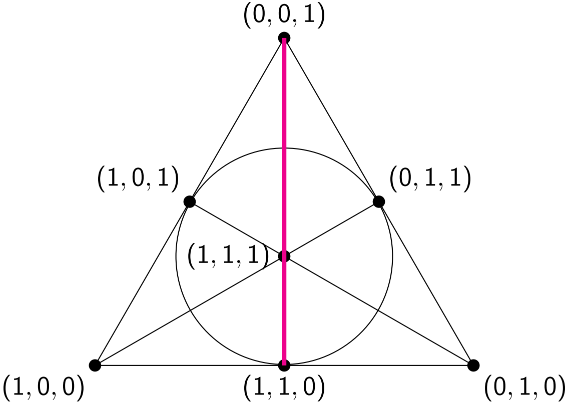

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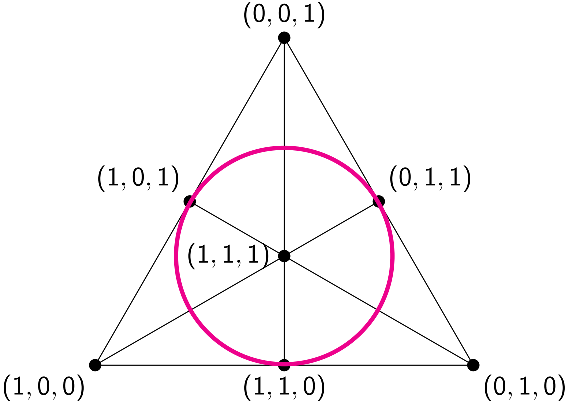
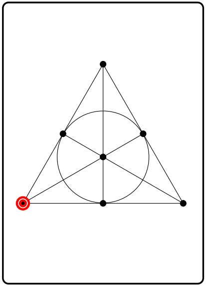
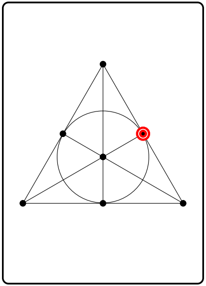


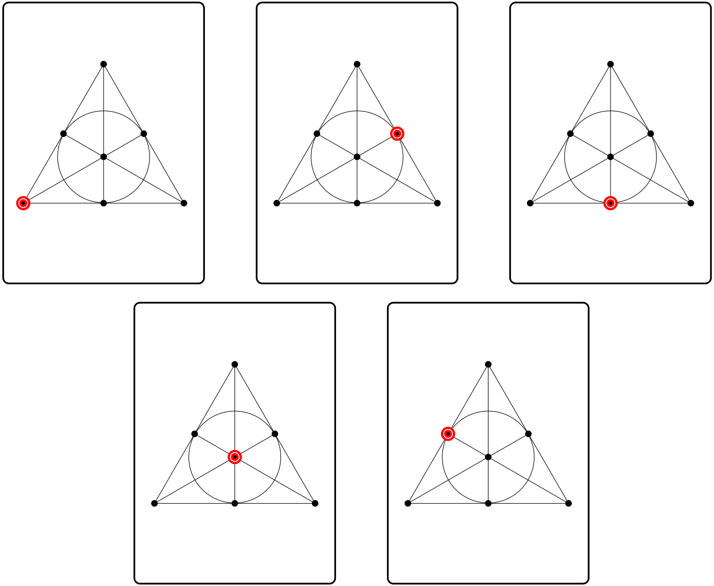


$\longleftrightarrow \quad ((1,0,0))$ in $\mathbf{C}_2^3$

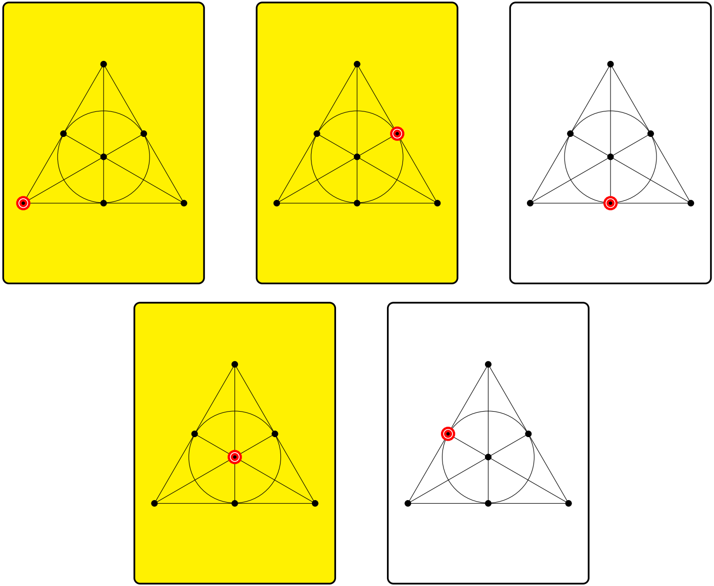


$\longleftrightarrow \quad ((0,1,1))$ in $\mathbf{C}_2^3$

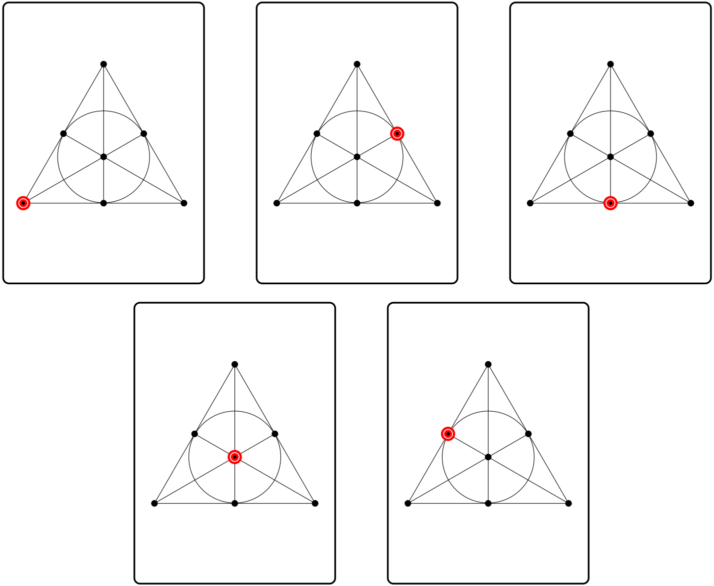
A **Fano-set** is a collection of cards with three points on a line.

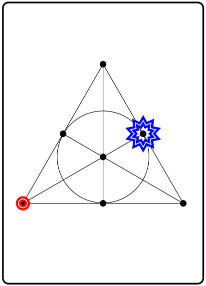


A **Fano-set** is a collection of cards with three points on a line.

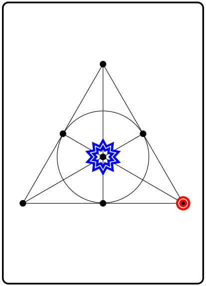


A **Fano-set** is a collection of cards with three points on a line.



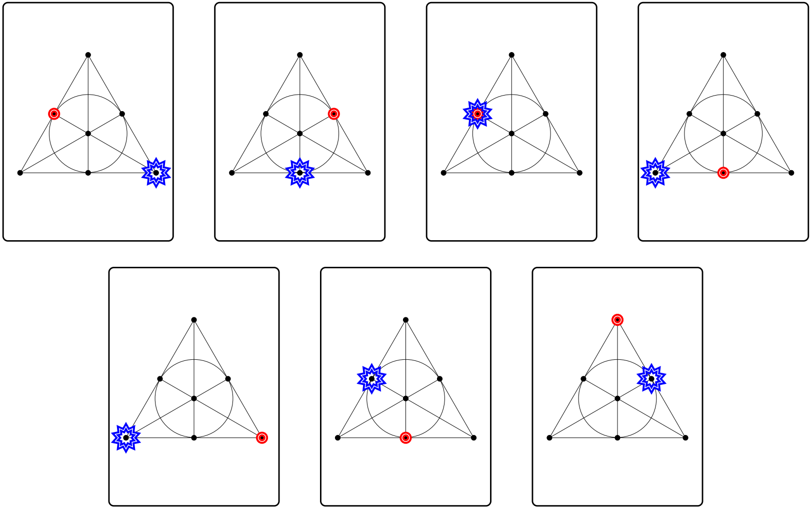


$\longleftrightarrow \quad ((1,0,0),(0,1,1))$ in $\mathbf{C}_2^6$

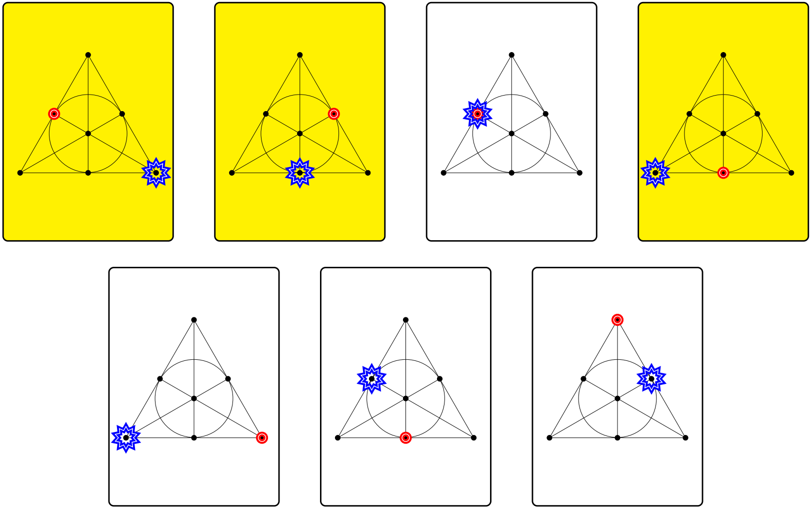


$\longleftrightarrow \quad ((0,1,0),(1,1,1))$ in $\mathbf{C}_2^6$

This game is equivalent to $\mathbf{C}_2^6$-set.



This game is equivalent to $\mathbf{C}_2^6$-set.

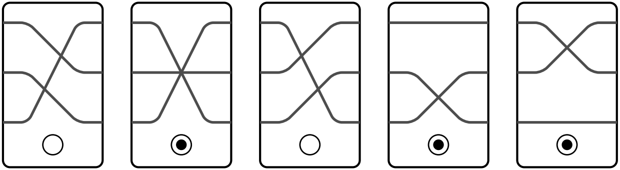


There exist geometries for $\mathbf{C}_2^4, \mathbf{C}_2^5$ and $\mathbf{C}_2^6$ too!

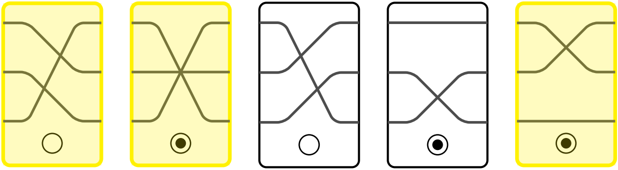
`https://upload.wikimedia.org/wikipedia/commons/b/b8/Facial_Fano_plane_within_Fano_three-space.png`

More Variations on SET

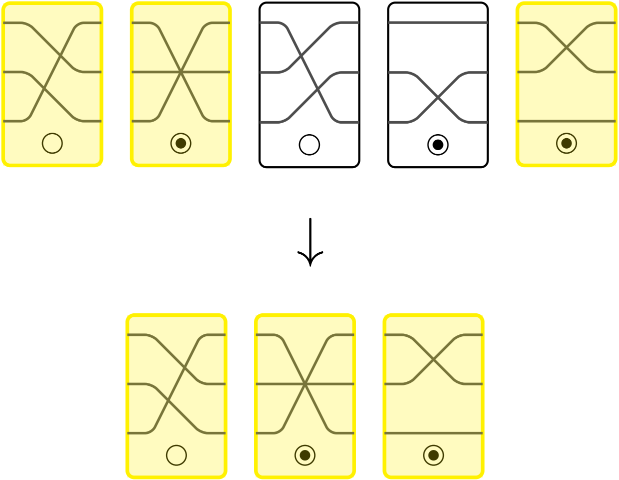
Find a subsequence of cards where all the lines return to their starting positions:



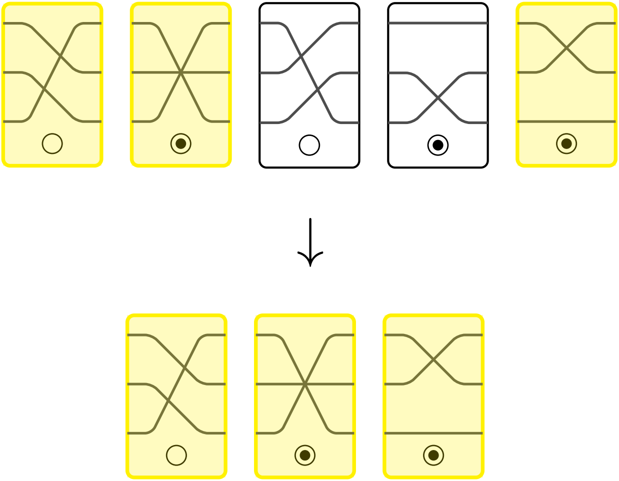
Find a subsequence of cards where all the lines return to their starting positions:



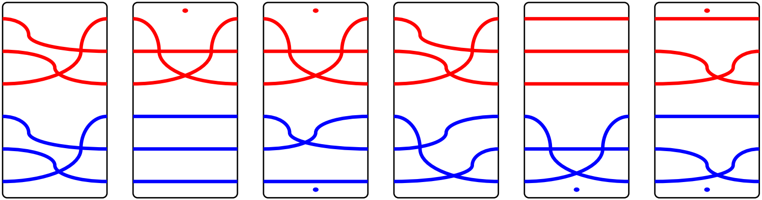
Find a subsequence of cards where all the lines return to their starting positions:

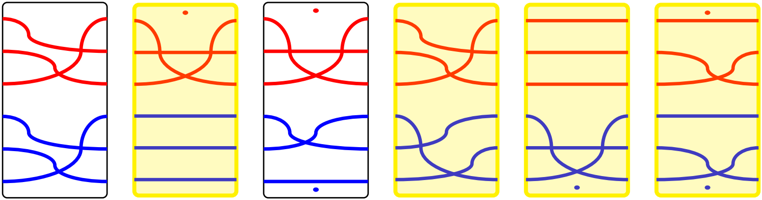


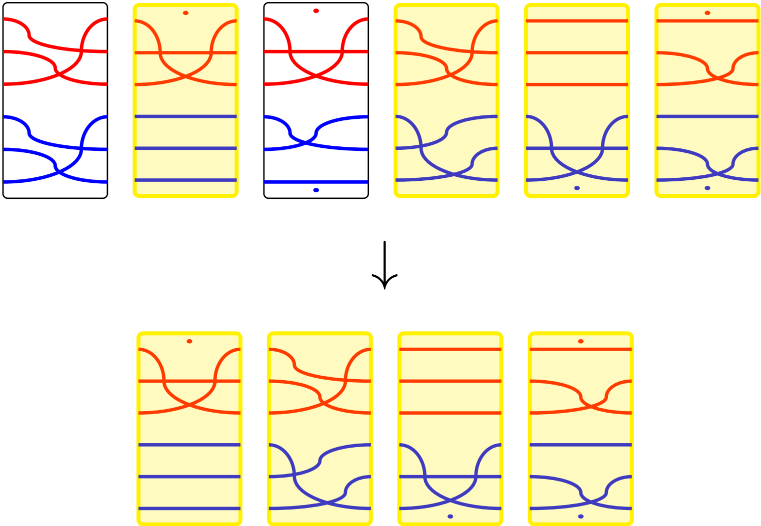
Find a subsequence of cards where all the lines return to their starting positions:



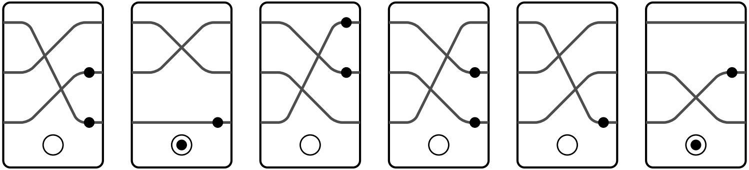
http://www.gabrieldorfsmanhopkins.com/nonabelianSet/S3/index.html



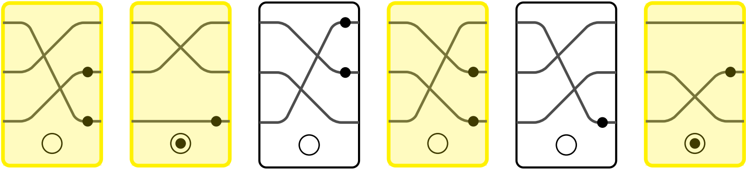




Now, each strand must also have an even number of dots:



Now, each strand must also have an even number of dots:



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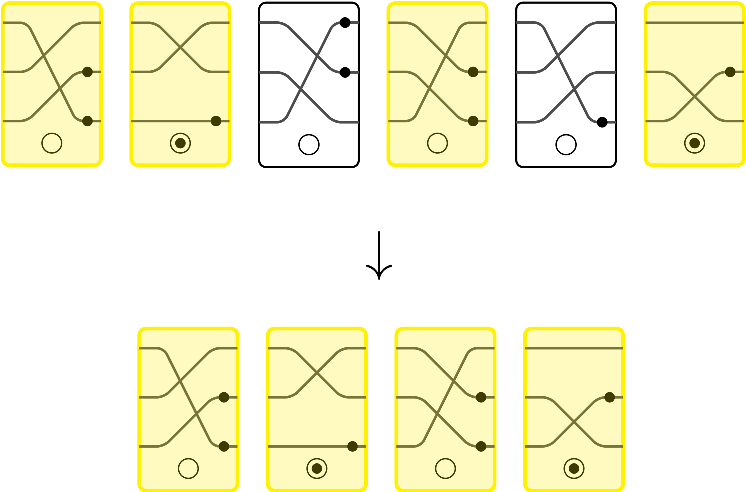

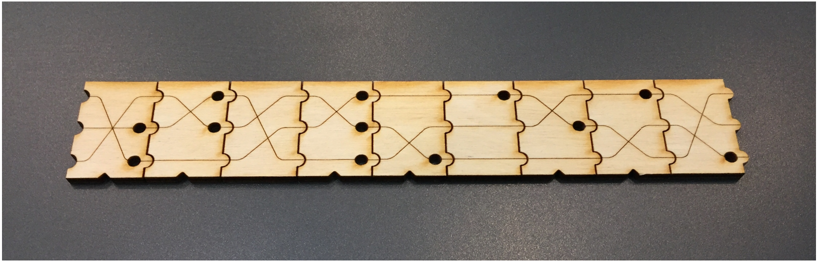

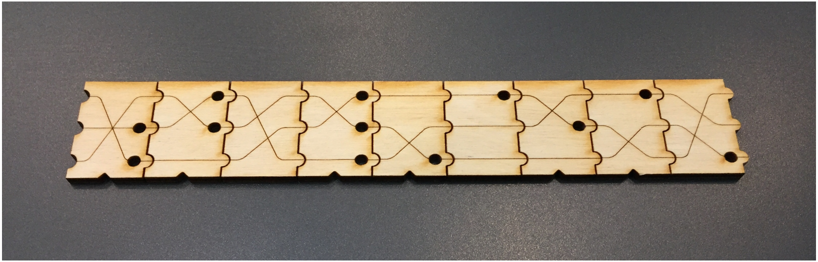

All SET decks can be found on my webpage:

https://lucasvanmeter.github.io/projects.html