Projective and Non-Abelian SET

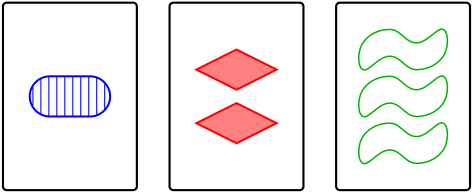
Catherine M. Hsu

University of Bristol

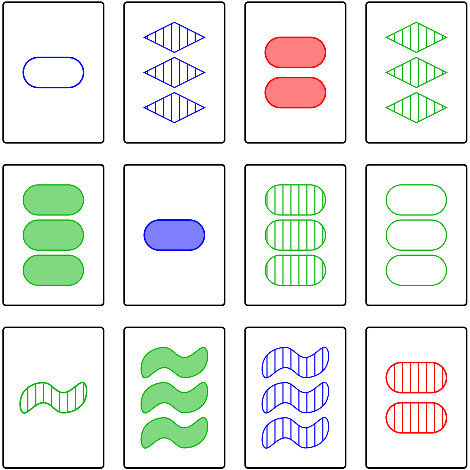
Illustrating Number Theory and Algebra
ICERM
October 24, 2019

Joint work with Jonah Ostroff and Lucas Van Meter*

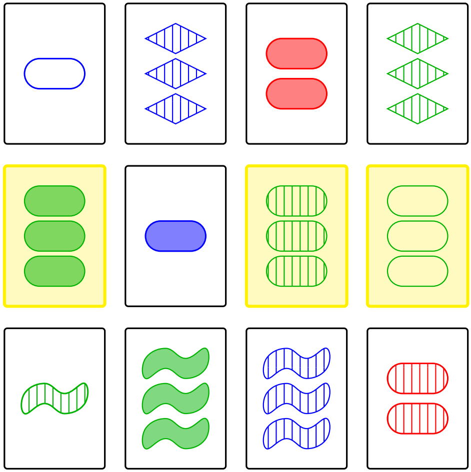
In 1974, Marsha Falco invented the card game SET:



Color	**Number**	**Shape**	**Shade**
red	one	diamond	solid
blue	two	squiggle	striped
green	three	oval	open



A **set** is a collection of three cards for which in each of the four qualities, the cards are all the same or all different.



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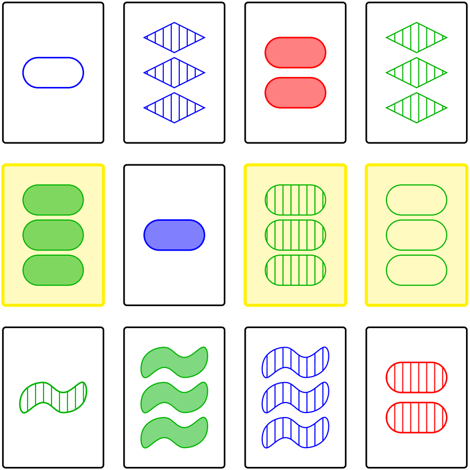
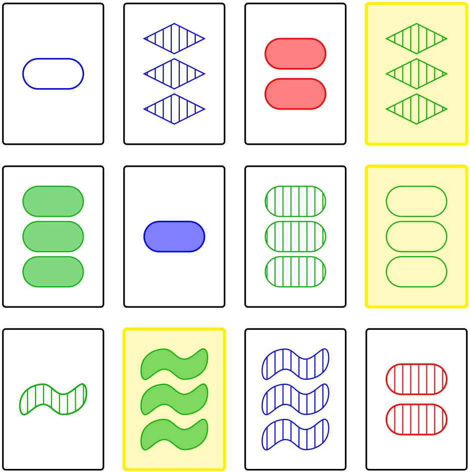


A **set** is a collection of three cards for which in each of the four qualities, the cards are all the same or all different.

Try to find another set!



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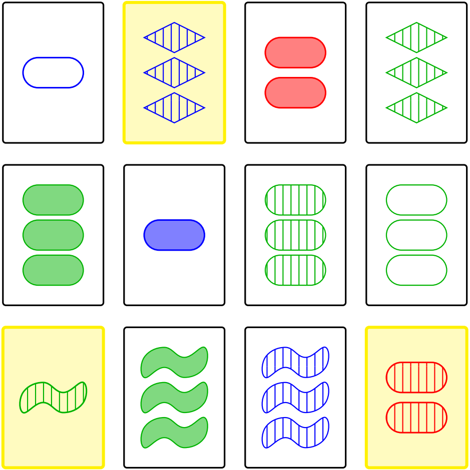

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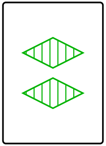
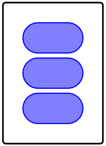
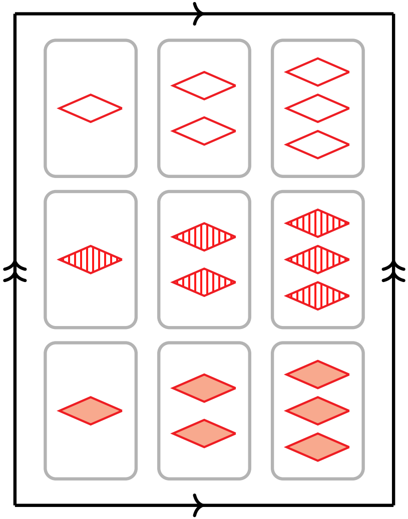
F_3	**Color**	**Number**	**Shape**	**Shade**
0	red	one	diamond	open
1	blue	two	squiggle	striped
2	green	three	oval	solid

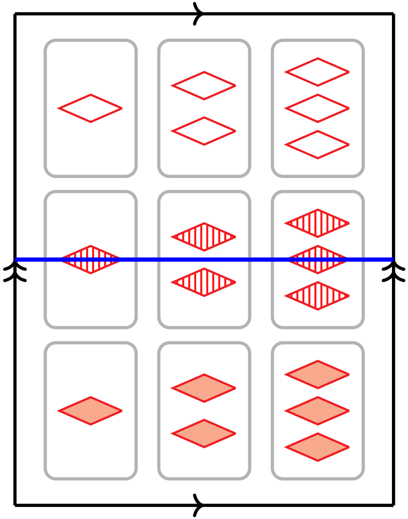


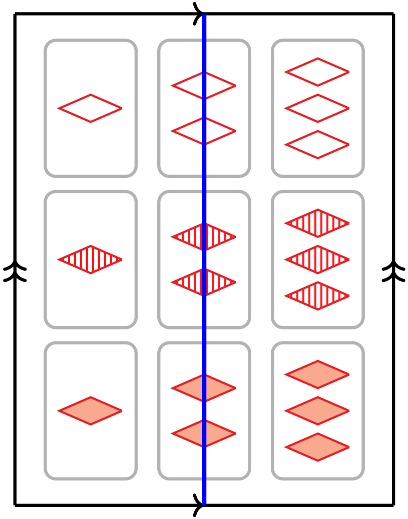
$$\longleftrightarrow \quad (2, 1, 0, 1) \in \mathbf{F}_3^4$$

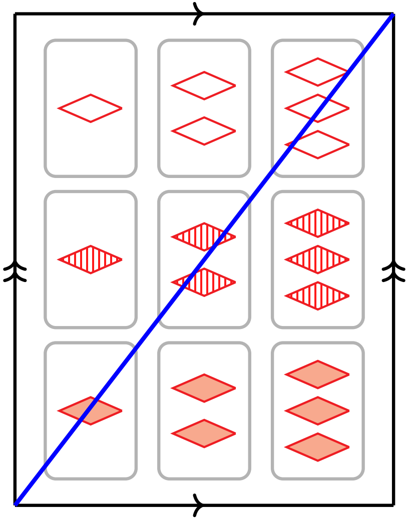


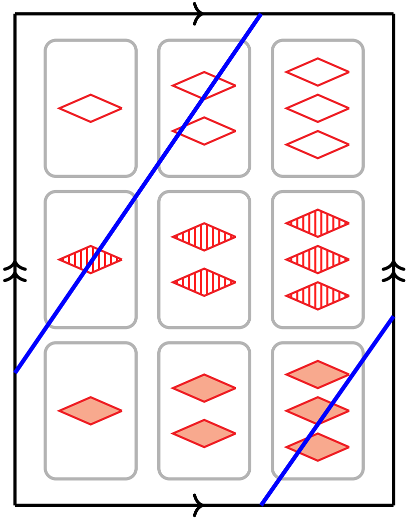
$$\longleftrightarrow \quad (1, 2, 2, 2) \in \mathbf{F}_3^4$$











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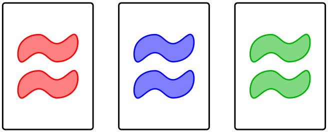

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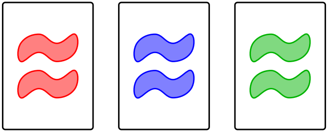

$$\Updownarrow$$

$$(0,1,1,2) + (1,1,1,2) + (2,1,1,2) = 0 \in \mathbf{F}_3^4$$

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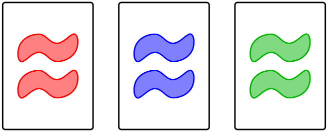

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$$\underbrace{(0,1,1,2)}_{a}+\underbrace{(1,1,1,2)}_{b}+\underbrace{(2,1,1,2)}_{c}=0\in\mathbf{F}_3^4$$

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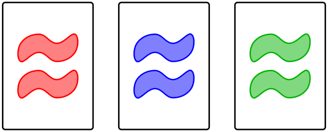

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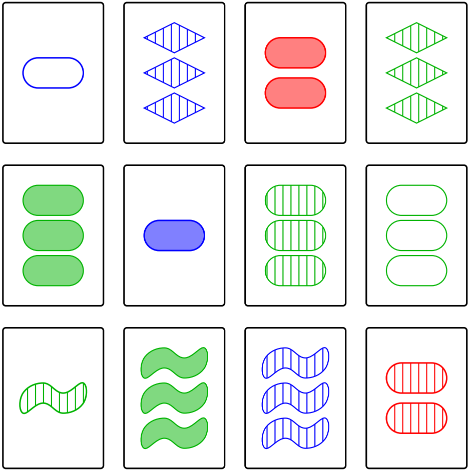
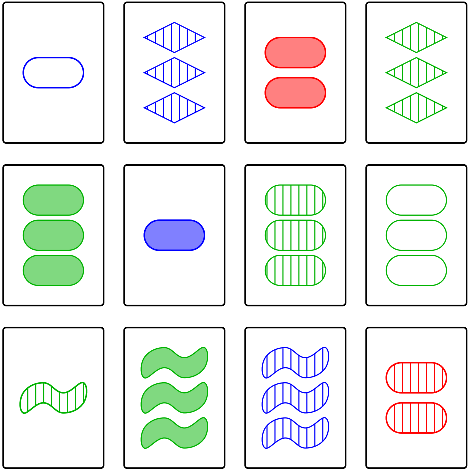
$$\Updownarrow$$

$a, b, c \in \mathbf{F}_3^4$ are collinear



New definition of a set:
a collection of four cards that sum to zero in $\mathbf{F}_3^4$.

Try to find a set!



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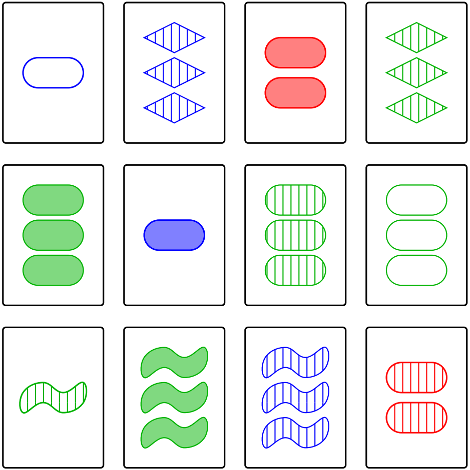
Wait! We need more info...



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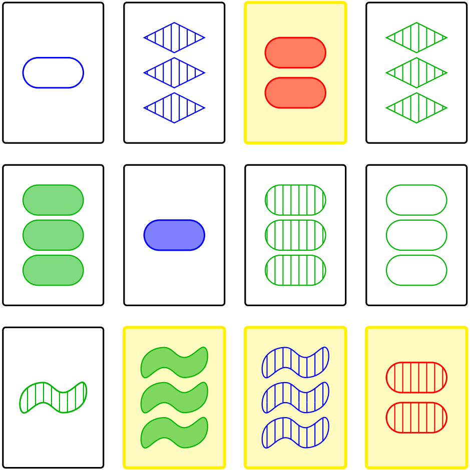

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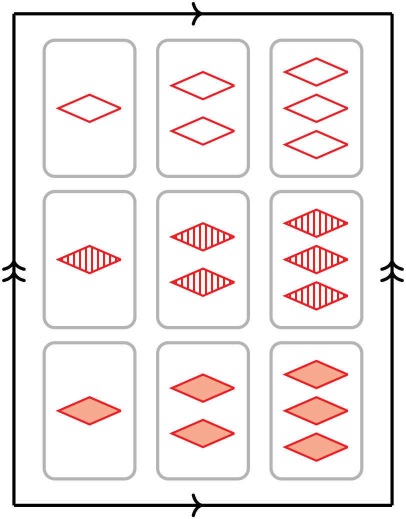

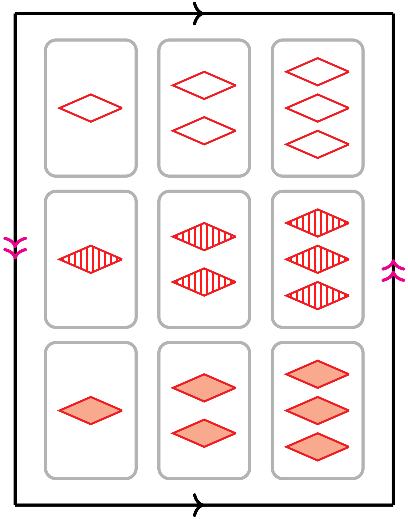

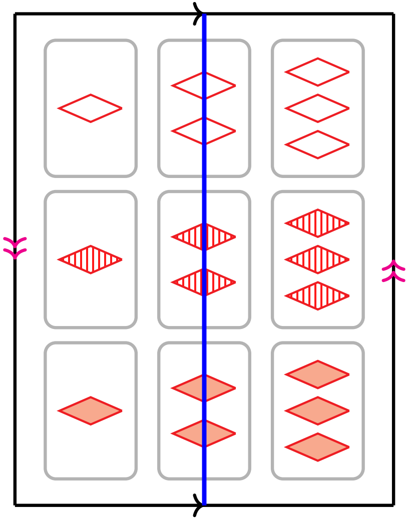

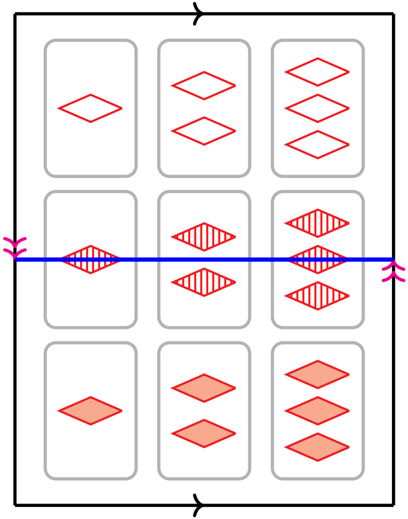

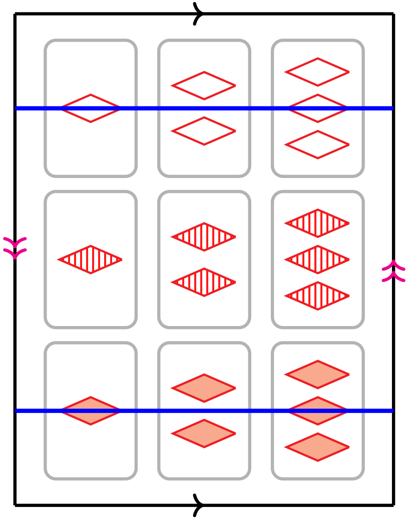

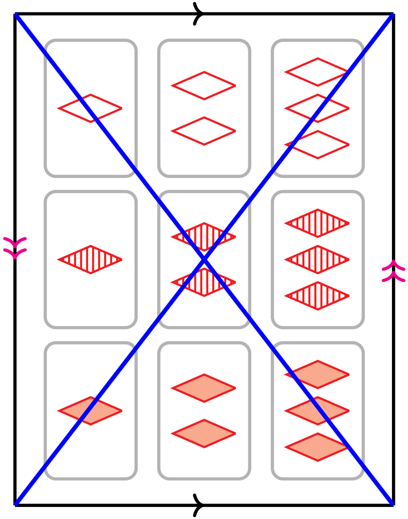

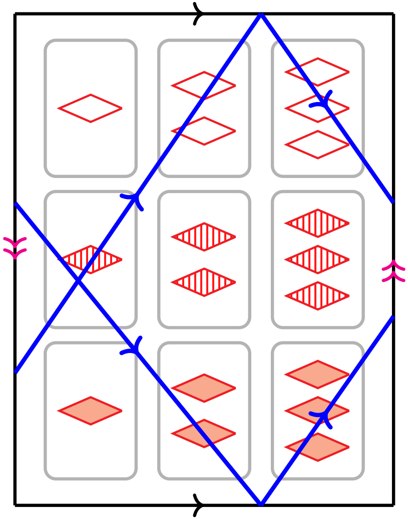

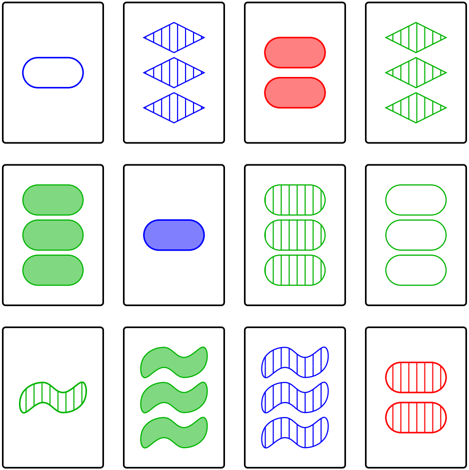

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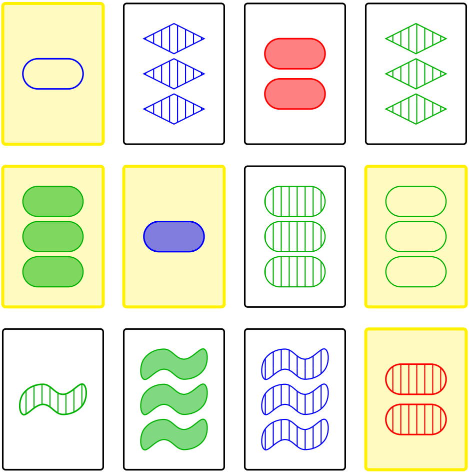
a collection of three cards that are collinear in the Möbius identification of $\mathbf{F}_3^4$.



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algebraic

a collection of elements that are collinear

geometric

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$$\boxed{\text{a collection of elements that "sum" to zero}} \xleftrightarrow{???} \boxed{\text{a collection of elements that are collinear}}$$

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Question: For what other structures can we play SET?

a collection of elements that "sum" to zero	$\xleftrightarrow{???}$	a collection of elements that are collinear
algebraic		geometric

- Our perspective focuses on practical play.

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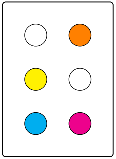
$$\boxed{\begin{array}{c}\text{a collection of elements}\\ \text{that "sum" to zero}\end{array}} \xleftrightarrow{???} \boxed{\begin{array}{c}\text{a collection of elements}\\ \text{that are collinear}\end{array}}$$

algebraic | geometric

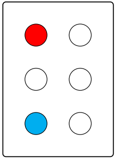
- Our perspective focuses on practical play.
- We seek interesting visual conditions for a SET.

Variation I: Projective space $\mathbf{P}^n(\mathbf{F}_2)$

Let's take $\mathbf{P}^5(\mathbf{F}_2)$ as our underlying structure:



$$\longleftrightarrow \quad (0,1,1,0,1,1) \in \mathbf{P}^5(\mathbf{F}_2)$$



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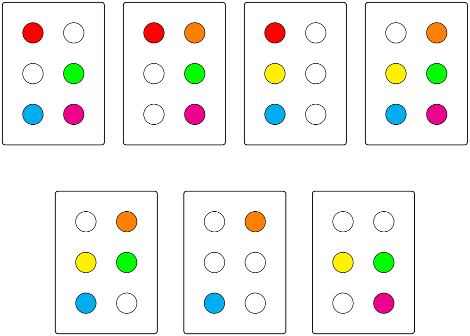
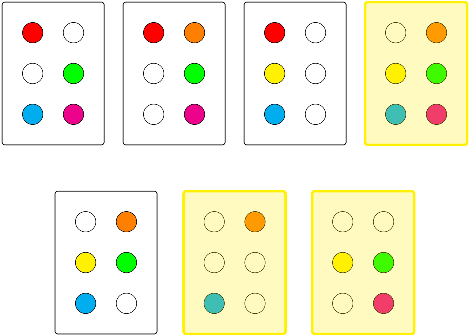


A **projective set** is a collection of three cards for which there's an even number of dots for each color.



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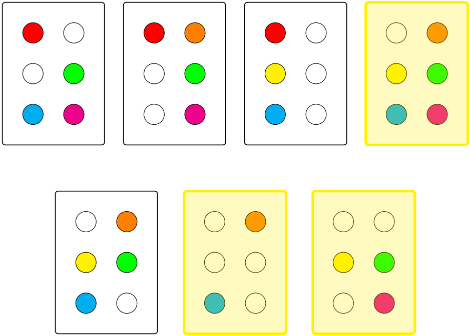

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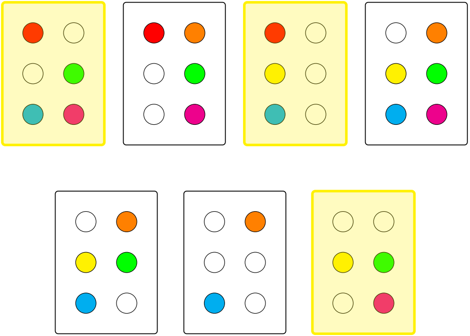

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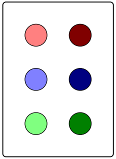
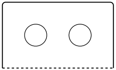
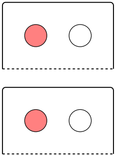


$$\longleftrightarrow \quad ((1,1),(1,1),(1,1)) \in \mathbf{P}^5(\mathbf{F}_2)$$

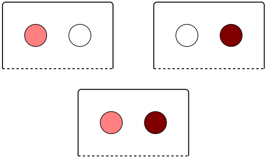
In $\mathbf{F}_2^2$, there are three ways to have an even number dots of each color:



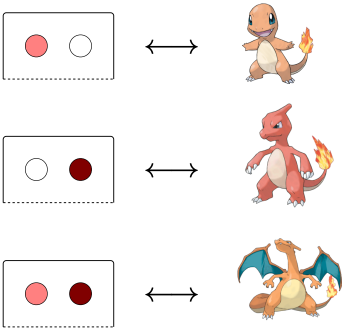
zero element



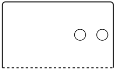
two cards with same coloring



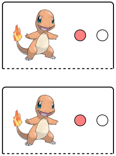
one of each non-zero card



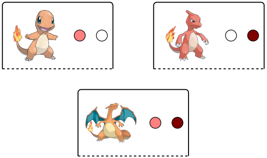
So, an even number dots can be appear as:



no Pokémon



two of the same Pokémon



a full evolution of Pokémon

A **Pokémon projective set** is a collection of three cards for which the Pokémon can be partitioned into identical pairs or full evolutions.

Try to find a set!

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$\mathbf{F}_3^4$ vs. $\mathbf{P}^5(\mathbf{F}_2)$

Deal size	**3-set in $\mathbf{F}_3^4$**	**3-set in $\mathbf{P}^5(\mathbf{F}_2)$**
3	0.01	0.02
4	0.05	0.06
5	0.12	0.16
6	0.23	0.30
7	0.39	0.48
8	0.54	0.65
9	0.71	0.80
10	0.83	0.91
11	0.92	0.96
12	0.96	0.99
13	0.99	≥ 0.99
$\vdots$	$\vdots$	$\vdots$

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7	0.39	0.48
8	0.54	0.65
9	0.71	0.80
10	0.83	0.91
11	0.92	0.96
12	0.96	0.99
13	0.99	≥ 0.99
$\vdots$	$\vdots$	$\vdots$

Let's redefine a Pokémon projective set to be a collection of three or more cards for which the Pokémon can be partitioned into identical pairs or full evolutions.

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Try to find a set of size ≥ 4.

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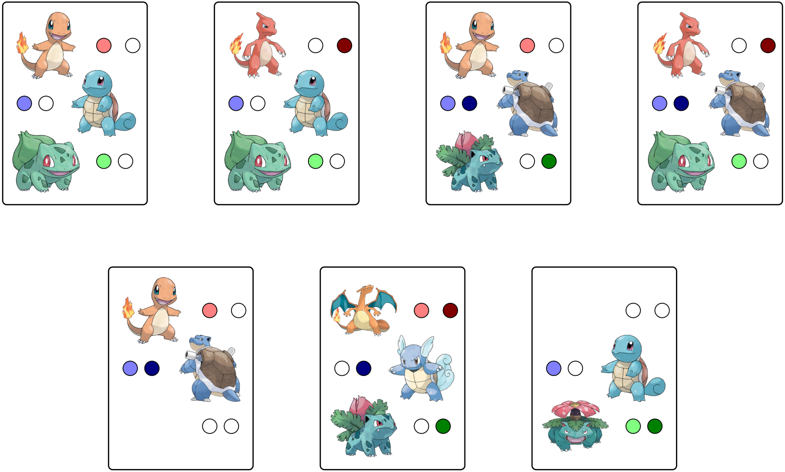
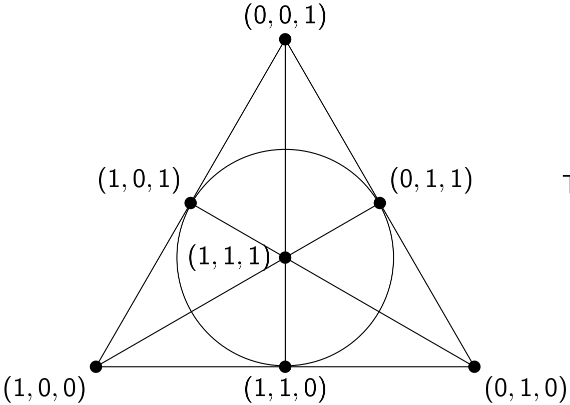
Try to find a set of size ≥ 4.

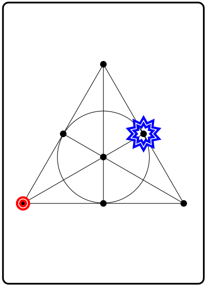
$P^5(F_2)$

Deal size	3-set	4-set	5-set	6-set	7-set
3	0.02	–	–	–	–
4	0.06	0.02	–	–	–
5	0.16	0.08	0.02	–	–
6	0.30	0.23	0.09	0.02	–
7	0.48	0.50	0.31	0.11	0.02
8	0.65	0.76	0.61	0.39	0.12
9	0.80	0.98	0.87	0.81	0.49
10	0.91	≥ 0.99	0.97	0.98	0.92
11	0.96	≥ 0.99	≥ 0.99	≥ 0.99	0.98
12	0.99	≥ 0.99	≥ 0.99	≥ 0.99	≥ 0.99
13	≥ 0.99	≥ 0.99	≥ 0.99	≥ 0.99	≥ 0.99
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

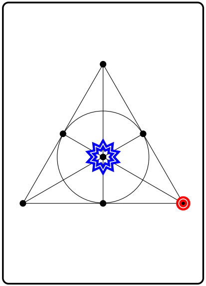




This is the Fanoplane
$\mathbf{P}^2(\mathbf{F}_2)$.



$$\longleftrightarrow \quad ((1,0,0),(0,1,1))$$



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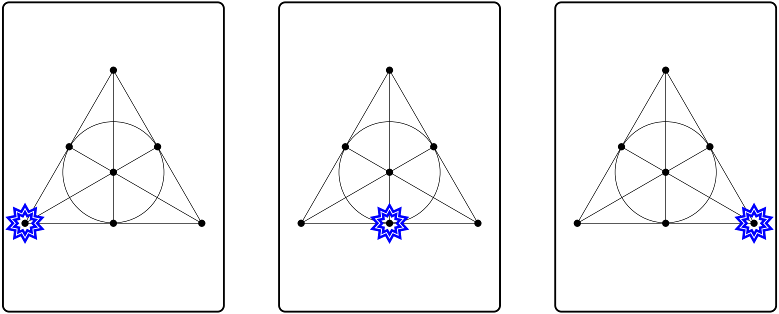
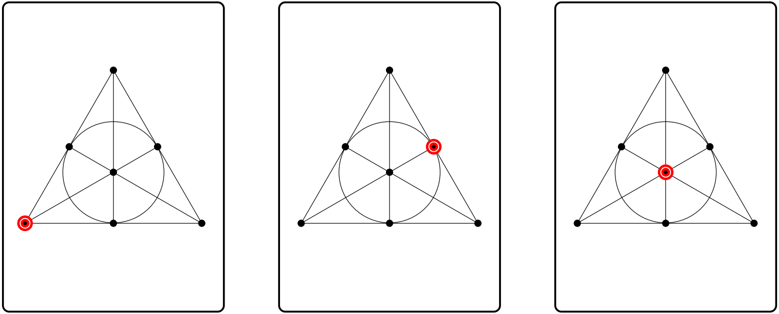
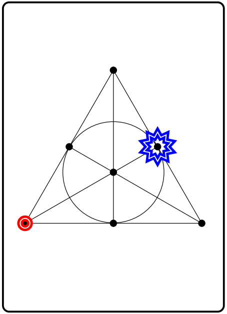
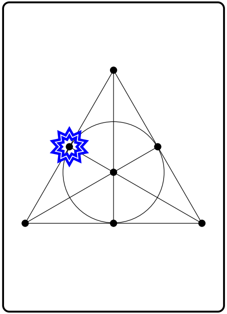
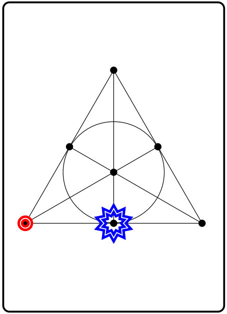
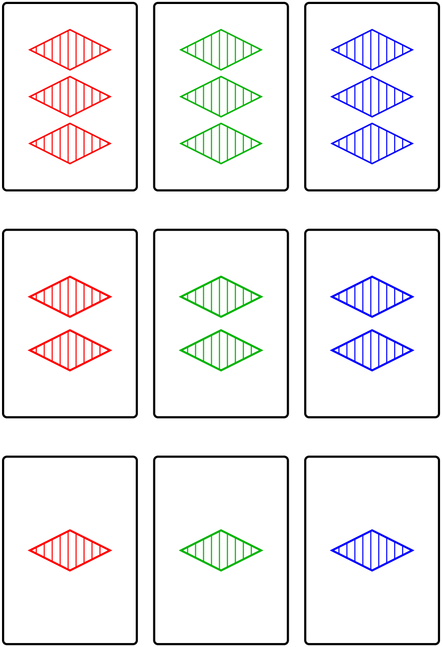
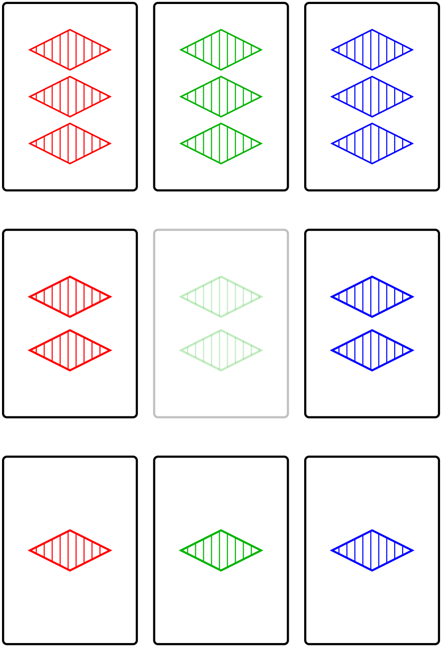


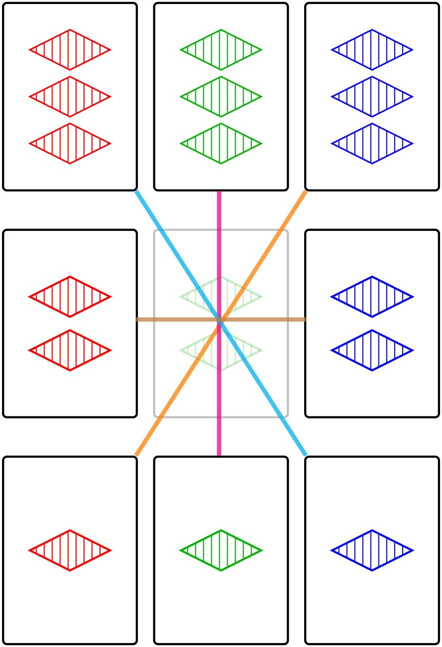


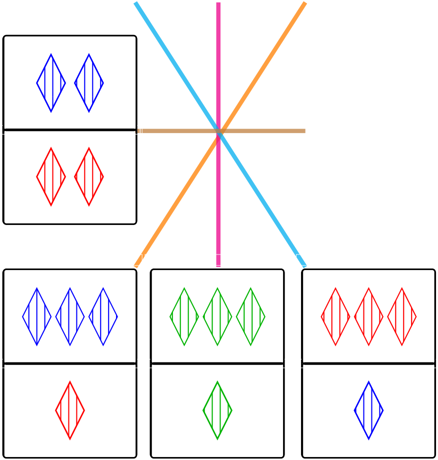




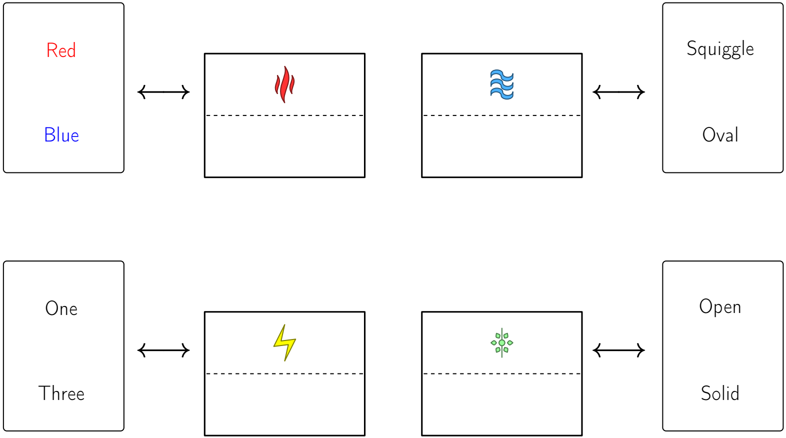
Variation II: Projectivized Normal SET $\mathbf{P}^3(\mathbf{F}_3)$

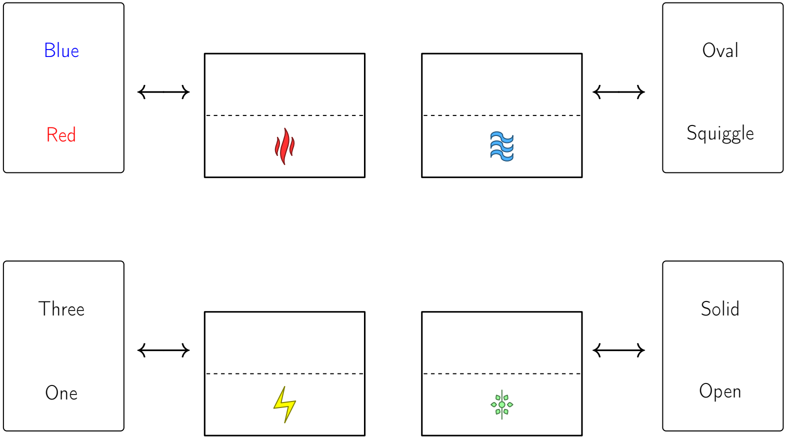




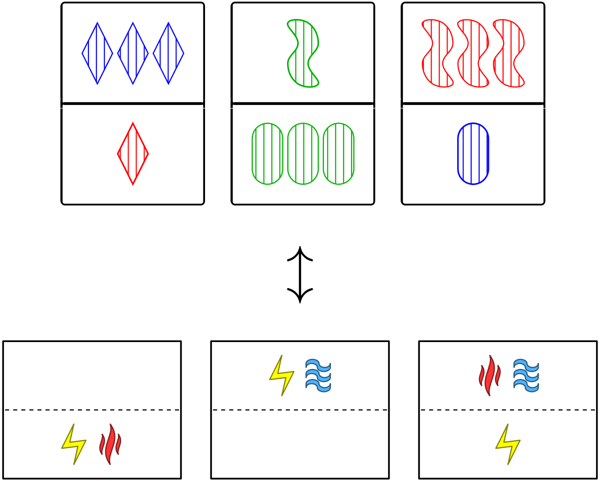





Red
Blue
Squiggle
Oval
One
Three
Open
Solid


Blue
Red
Oval
Squiggle
Three
One
Solid
Open

Three cards are collinear exactly when you can rotate one to obtain (normal) SETs on top and bottom. Equivalently, when each symbol appears on either all the same side or on all different sides.



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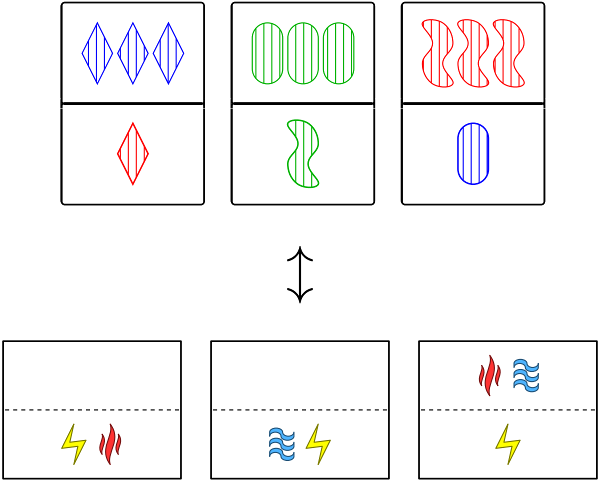
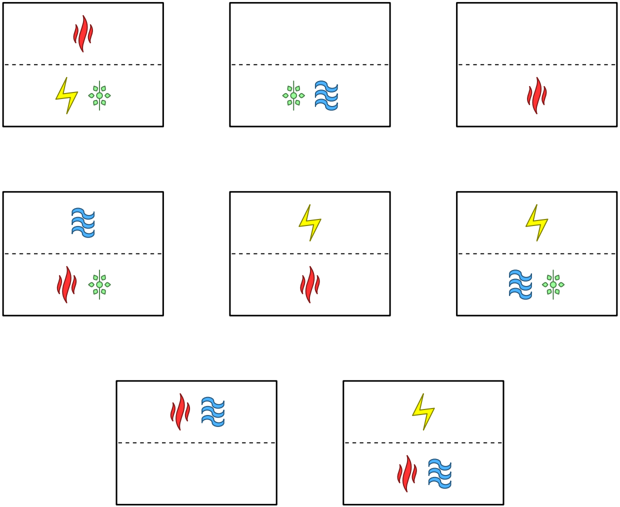
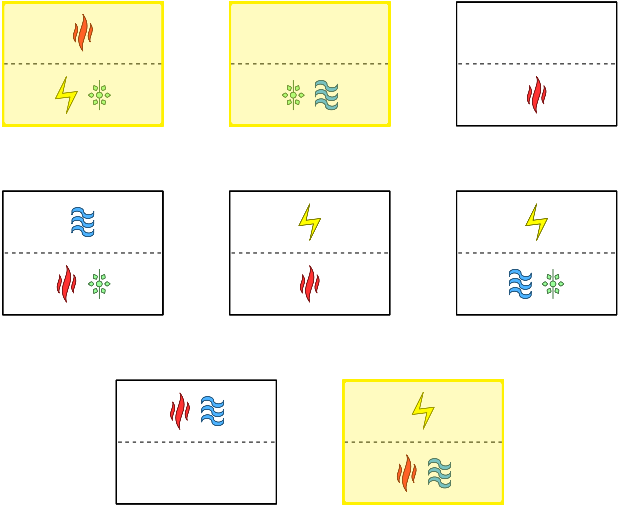
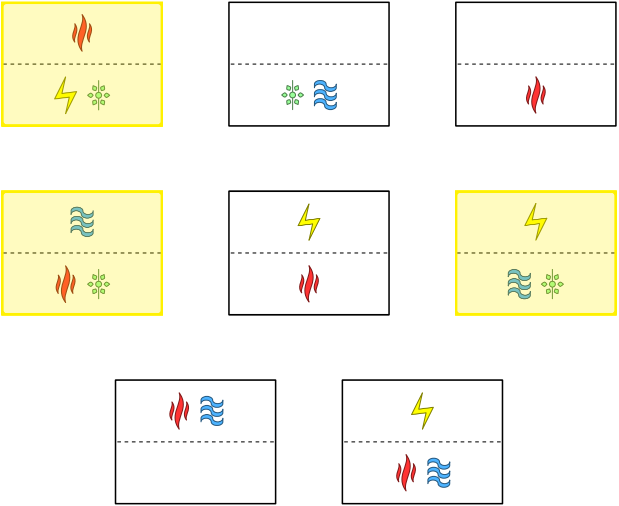
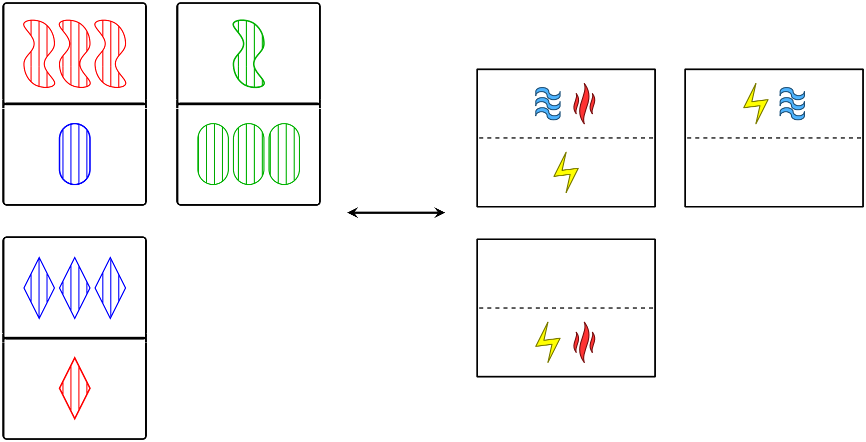


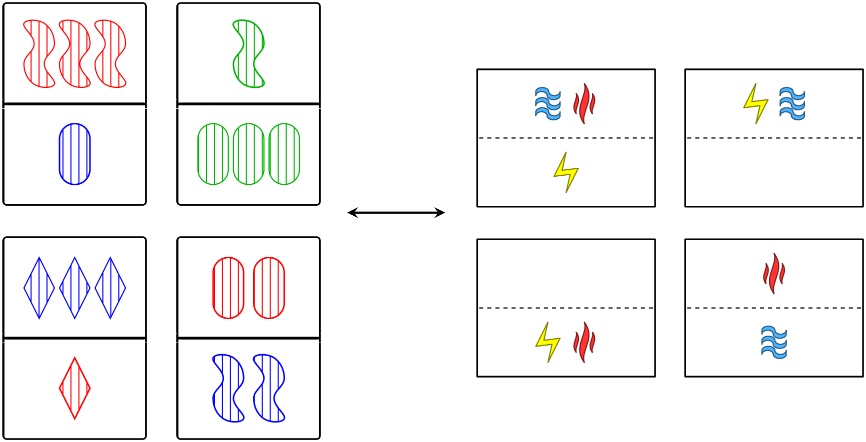








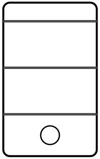
Can you figure out the missing point on this projective line?



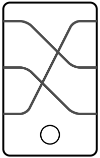
Can you figure out the missing point on this projective line?

Variation III: Non-abelian groups

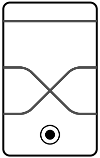
Let's start by taking the symmetric group S_3 as our underlying structure:



$\longleftrightarrow \quad \mathrm{id} \in S_3$

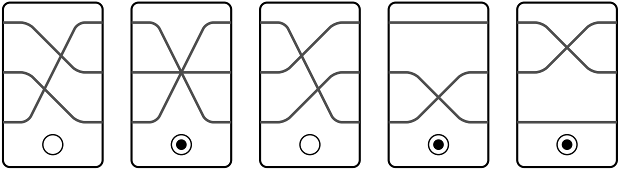


$\longleftrightarrow \quad (123) \in S_3$



$\longleftrightarrow \quad (23) \in S_3$

We deal the 5 non-trivial cards (which determines an ordering):



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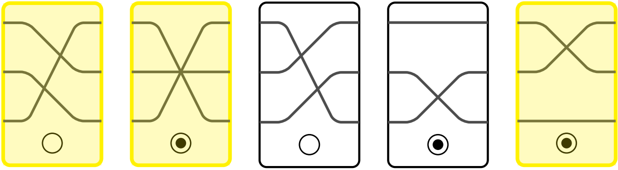

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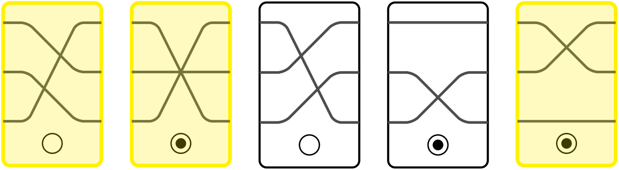
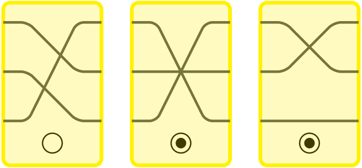
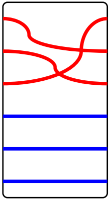
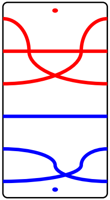




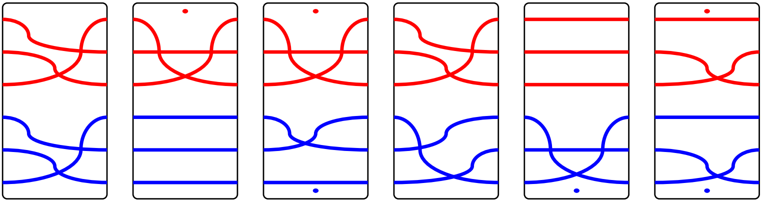
Now, let's try $S_3 \times S_3$ as our underlying structure:

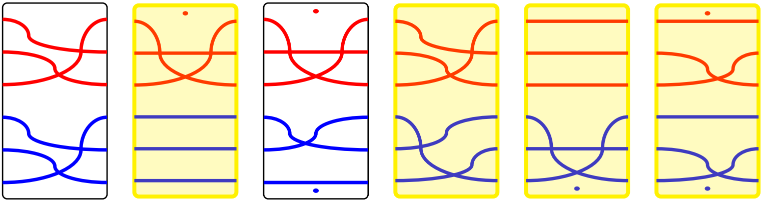


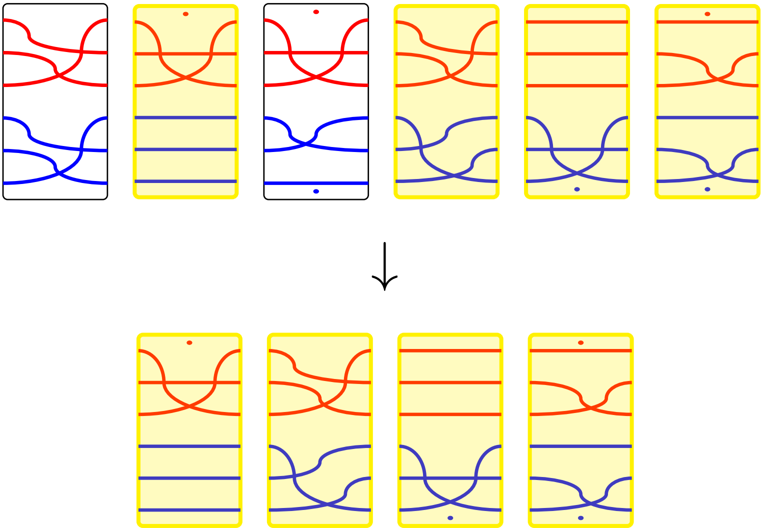
$\longleftrightarrow \quad ((123), \mathrm{id}) \in S_3 \times S_3$



$\longleftrightarrow \quad ((13), (23)) \in S_3 \times S_3$



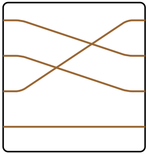




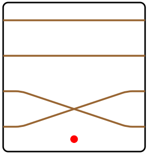
$S_3 \times S_3$

Deal size	**3-set**	**4-set**	**5-set**	**6-set**	**7-set**	**8-set**	**9-set**
3	0.03	–	–	–	–	–	–
4	0.11	0.03	–	–	–	–	-
5	0.27	0.14	0.03	–	–	–	–
6	0.49	0.38	0.15	0.03	–	–	–
7	0.69	0.67	0.46	0.19	0.03	–	–
8	0.87	0.91	0.83	0.58	0.20	0.03	–
9	0.95	0.99	0.97	0.92	0.67	0.23	0.03
10	0.99	≥ 0.99	≥ 0.99	≥ 0.99	0.96	0.76	0.25
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

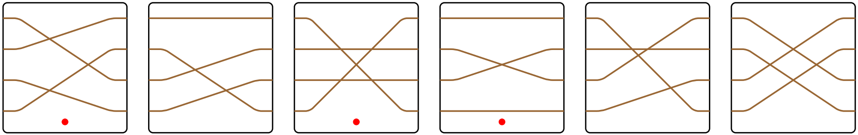
Next, we'll try S_4 as our underlying structure:

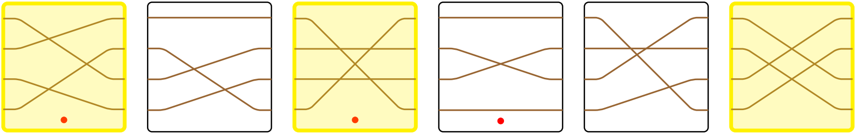


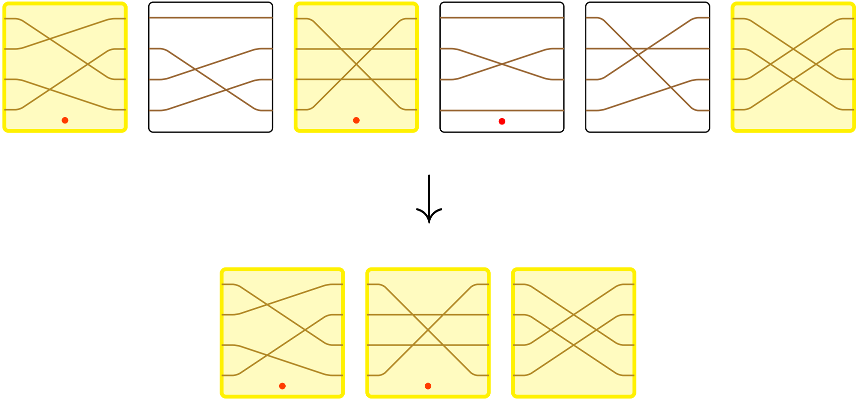
$\longleftrightarrow \quad (1\,2\,3) \in S_4$



$\longleftrightarrow \quad (3\,4) \in S_4$



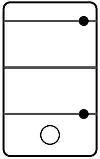




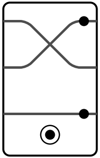
S_4

Deal size	3-set	4-set	5-set	6-set	7-set	8-set	9-set
3	0.05	–	–	–	–	–	–
4	0.17	0.04	–	–	–	–	–
5	0.40	0.21	0.04	–	–	–	–
6	0.66	0.51	0.23	0.04	–	–	–
7	0.86	0.84	0.62	0.27	0.04	–	–
8	0.97	0.98	0.92	0.74	0.30	0.03	–
9	0.99	≥ 0.99	≥ 0.99	0.98	0.82	0.32	0.04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

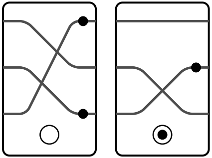
Finally, let's consider the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr S_3$:



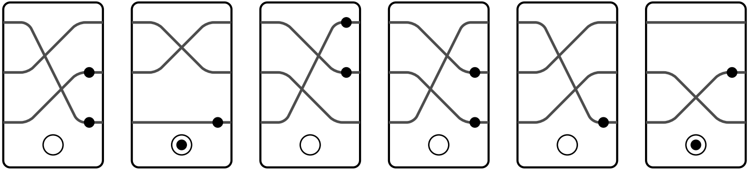
$$\longleftrightarrow \quad ((1,0,1),\mathrm{id}) \in (\mathbb{Z}/2\mathbb{Z}) \wr S_3$$

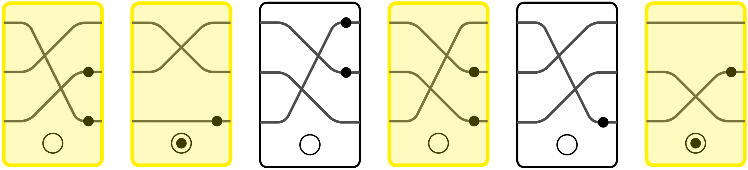


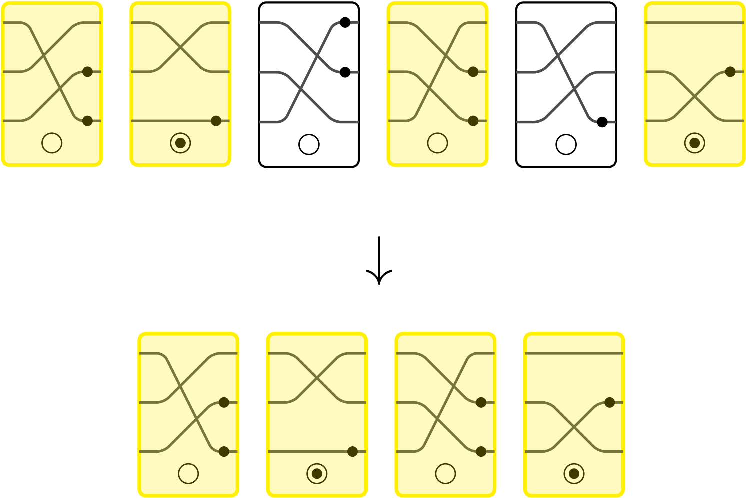
$$\longleftrightarrow \quad ((1,0,1),(12)) \in (\mathbb{Z}/2\mathbb{Z}) \wr S_3$$

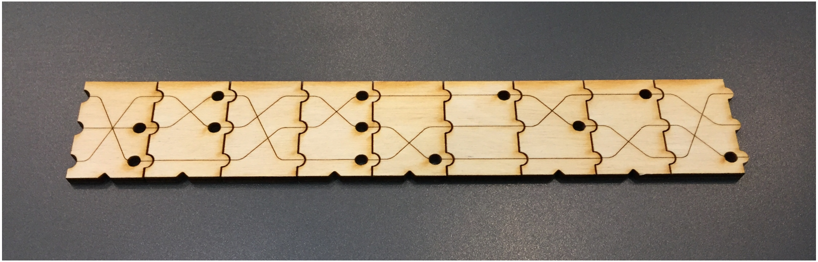


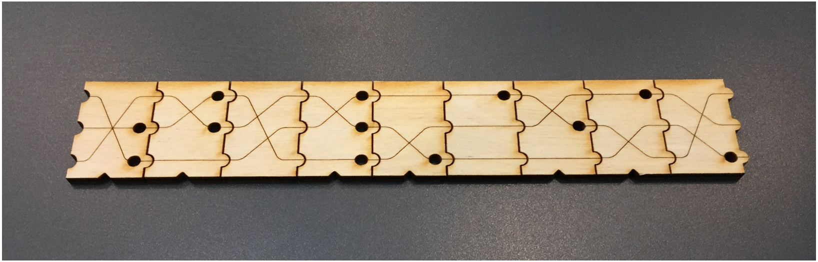
$$\longleftrightarrow \quad ((1,0,0),(13)) \in (\mathbb{Z}/2\mathbb{Z}) \wr S_3$$











All SET decks can be found on my webpage:
https://people.maths.bris.ac.uk/∼zx18363/