FUNCTION OR MAPPING

Definition(1)

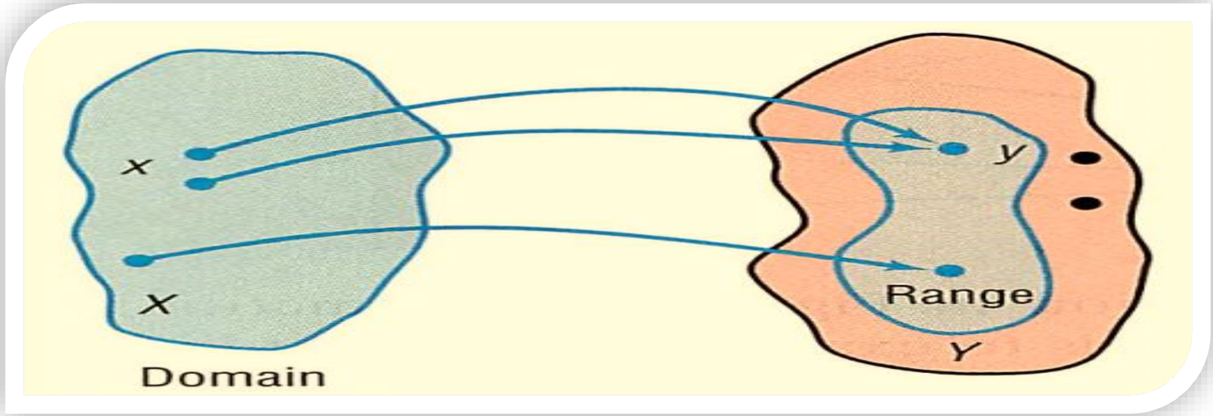
A function ϕ mapping X into Y is a relation between X and Y with the property that each $x \in X$ appears as the first member of exactly one ordered pair (x, y) in ϕ. Such a function is also called a map or mapping of X into Y .

We write $\phi : X \to Y$ and express $(x, y) \in \phi$ by $\phi(x) = y$. The domain of ϕ is the set X and the set Y is the codomain of ϕ. The range of ϕ is $\phi[X] = \{\phi(x) \mid x \in X\}$.

Definition(2)

Let X and Y be two nonempty sets. "A function from X into Y is a relation that associates with each element of X exactly one element of Y."

The set X is called the domain of the function. For each element x in X, the corresponding element y in Y is called the value of the function at x, or the image of x. The set of all images of the elements in the domain is called the range of the function.



Determine whether each relation represents a function. If it is a function, state the domain and range.

(a) { (I, 4), (2, 5), (3, 6), (4, 7)}

(b) { (1, 4), (2, 4) , (3, 5), (6, 10)}

(c) {(-3, 9), (-2, 4), (0, 0), (1,1), (-3, 8)}

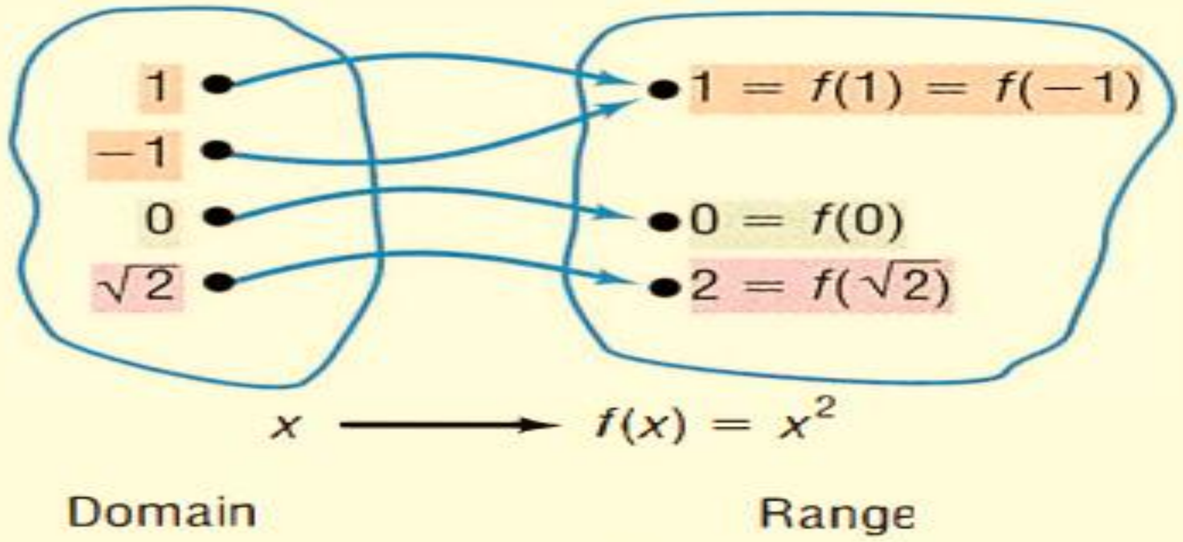
SOLUTION

(a)This relation is a function because there are no ordered pairs with the same first element and different second elements. The domain of this function is { I, 2, 3, 4}, and its range is { 4, 5, 6, 7}.

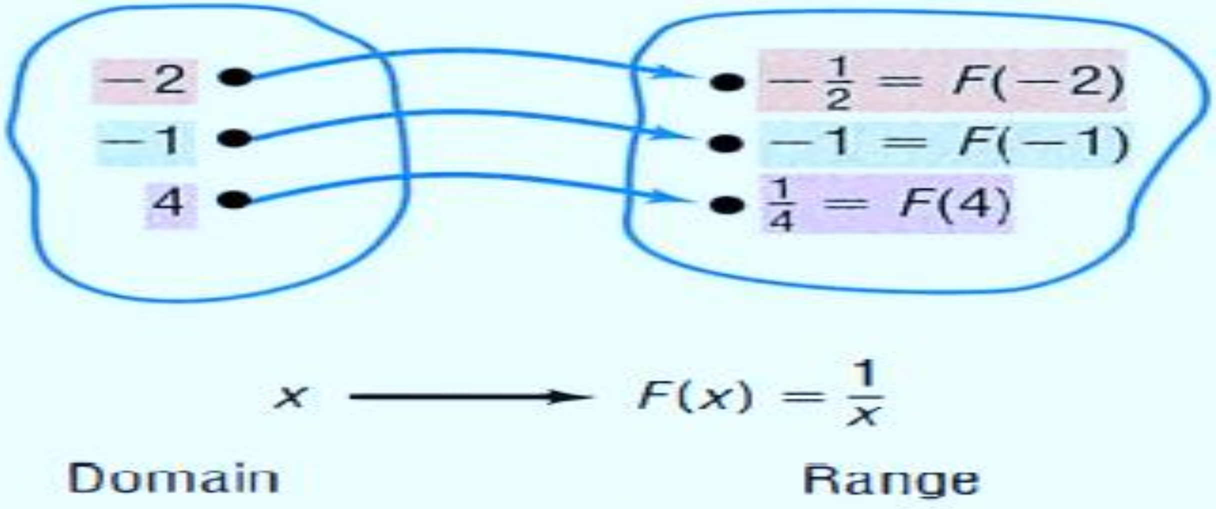
(a) This relation is a function because there are no ordered pairs with the same first element and different second elements. The domain of this function is {I, 2, 3, 6}, and its range is {4, 5, 10}.

(a) This relation is not a function because there are two ordered pairs, (-3, 9) and (-3, 8), that have the same first element and different second elements .

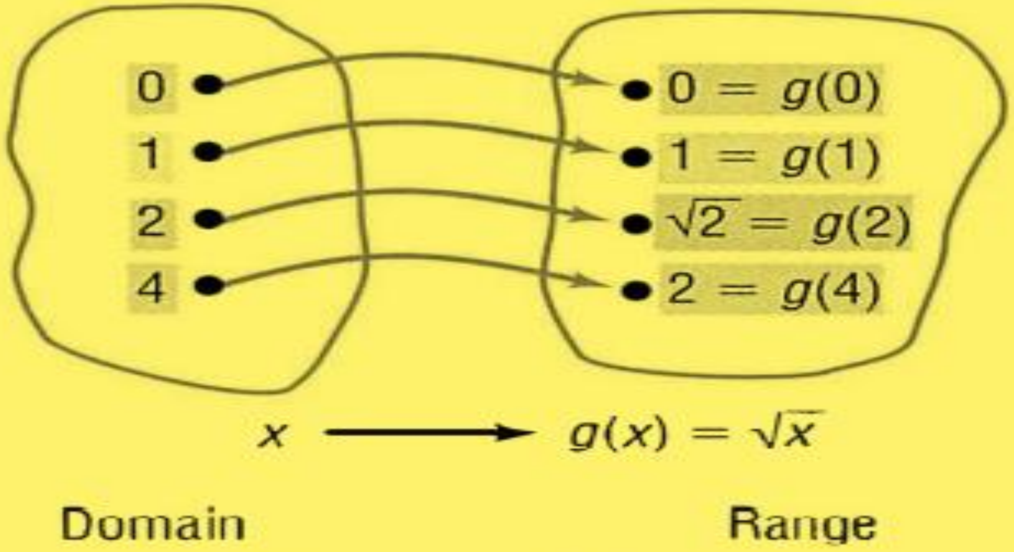
Find the Value of a Function



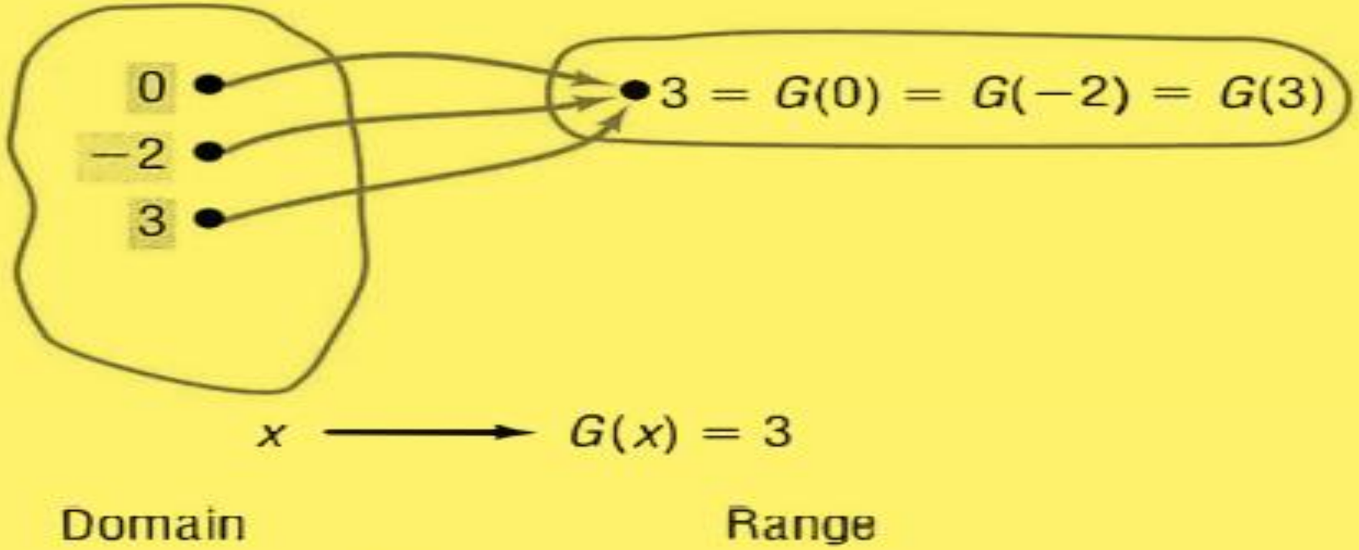
(a) $f(x) = x^2$



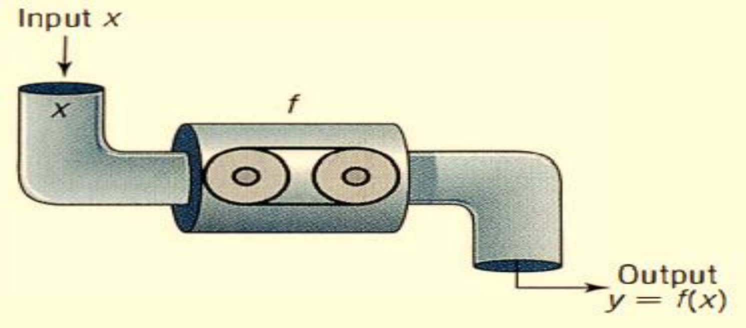
(b) $F(x) = \frac{1}{x}$



(c) $g(x) = \sqrt{x}$



(d) $G(x) = 3$

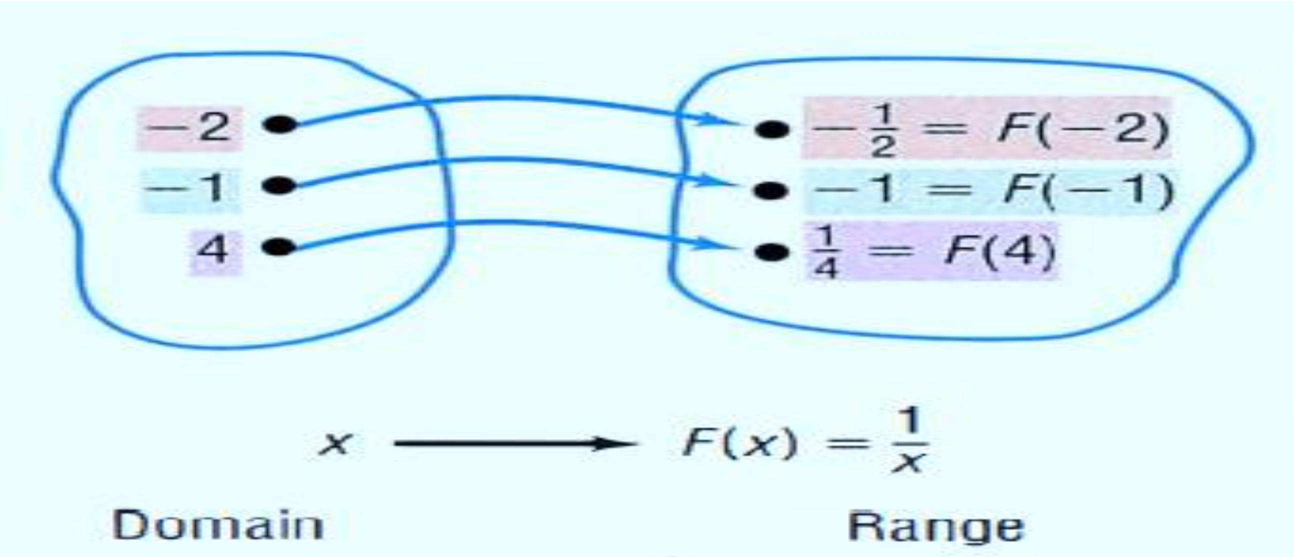


For a function y = f(x), the variable x is called the independent variable, because it can be assigned any of the permissible numbers from the domain. The variable y is called the dependent variable, because its value depends on x.

- **Cardinality :-The number of elements in a set X is the cardinality of X and is often denoted by |X|**
- **The function ϕ is onto Y if the range of ϕ is Y.**
- **Necessary condition for onto function|X|>|Y|.**
- **a one-to-one map is an injection, an onto map is a surjection, and a map that is both one to one and onto is a bijection.**
- A function f is even if, for every number x in its domain, the number -x is also in the domain and f(-x) = f(x).
- **A function f is odd if, for every number x in its domain, the number -x is also in the domain and f(-x) = -f(x).**

ONE – ONE FUNCTION:

- A function $\phi : X \rightarrow Y$ is one to one if $\phi(x) = \phi(y)$ only when $x=y$.
- Necessary condition for one – one function $|X|<|Y|$.



(b) $F(x) = \frac{1}{x}$

You are given the function $f : x \mapsto 2x + 5 ; x \in \mathbb{R}, -2 < x < 5$.

a Find f(2).

b Find the range of the function.

Solution:

a $f(2) = 2(2) + 5 = 9$

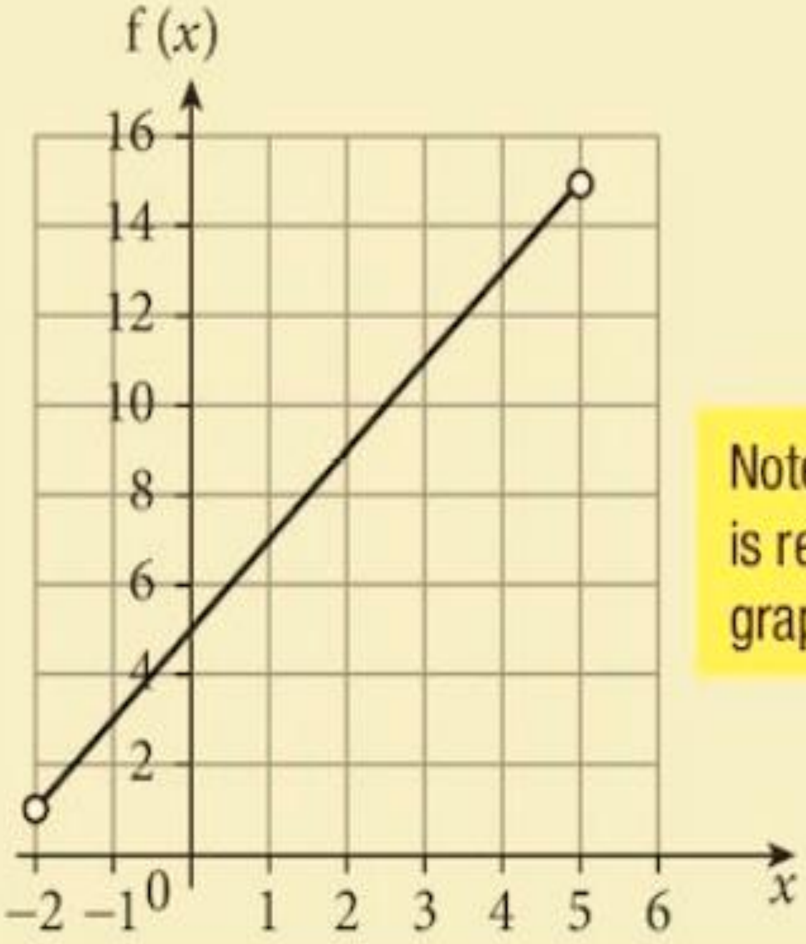
b The graph of $f(x)$ is shown.

It is linear and x can take any real value between –2 and 5.

The smallest value of $f(x)$ will be $f(-2) = 1$.

The largest value of $f(x)$ will be $f(5) = 15$.

Thus $1 < f(x) < 15$



Note that the extreme values, –2 and 5 are not in the domain so f(–2) and f(5) will not be in the range.

Note also that because the domain is restricted, there is no part of the graph beyond the end points.

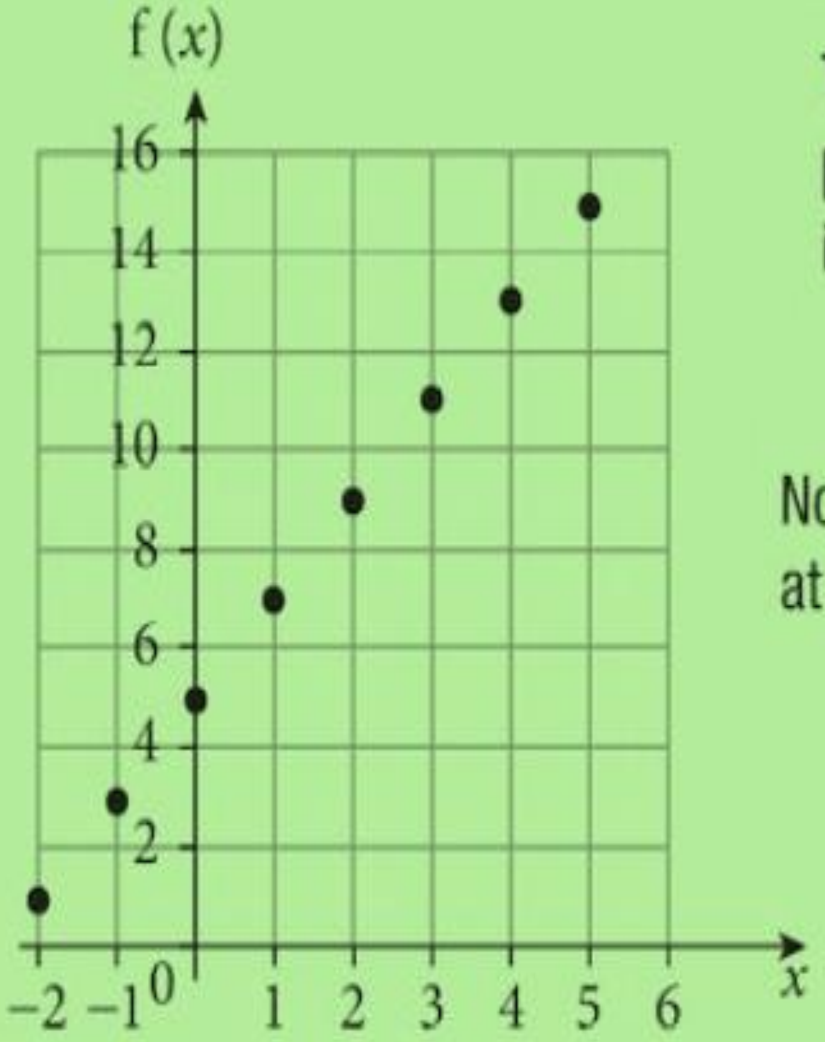
If $f : x \mapsto 2x + 5 ; x \in \mathbb{Z}, -2 \leqslant x \leqslant 5$, find the range of the function.

Solution:

The graph of $f(x)$ is shown.
It is linear but x can take only integer values between -2 and 5 (inclusive).

The range,

$f(x) = \{1, 3, 5, 7, 9, 11, 13, 15\}$



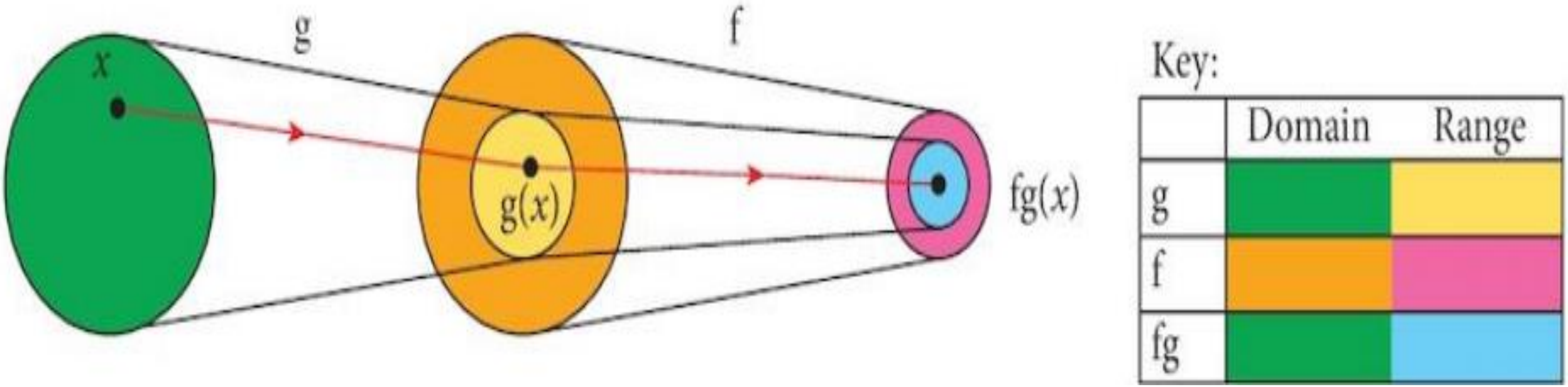
The graph consists only of the points shown since the domain is restricted to integer values.

Notice this time that the points at $x = -2$ and $x = 5$ are included.

Composite Functions

If g:X$\rightarrow Y$ and f:Y $\rightarrow Z$ defined as

g(x)=y,f(y)=z then composite function defined as fog:X$\rightarrow Z$ such that fog(x)=f(g(x))



Let f : A → B and g: B → C be functions.

- **If g ◦ f is injective, then f is injective.**
- **If g ◦ f is surjective, then g is surjective.**
- **where g ◦ f is bijective, but neither f nor g are bijective.**

The functions f and g are defined as follows:

$f : x \mapsto 2x + 5 ; x \in \mathbb{R}$ and $g : x \mapsto 3x - 2 ; x \in \mathbb{R}, -2 \leqslant x \leqslant 5$

a Form the functions **(i)** fg and **(ii)** $f^2 (= ff)$ and find the range of each function.

b Why do the functions **(i)** gf and **(ii)** g^2 not exist?

Solution:

a **(i)** $f[g(x)] = f[3x - 2]$

$= 2[3x - 2] + 5$

$f[g(x)] = 6x + 1 ; x \in \mathbb{R}, -2 \leqslant x \leqslant 5$

(ii) $f^2(x) = f[f(x)]$

$= f[2x + 5]$

$= 2[2x + 5] + 5$

$f^2(x) = 4x + 15 ; x \in \mathbb{R}$

b **(i)** $gf(1) = g(7)$

7 is not in the domain of g so this composite does not exist.

We could have chosen any value such that f(*x*) is not in the domian of g.

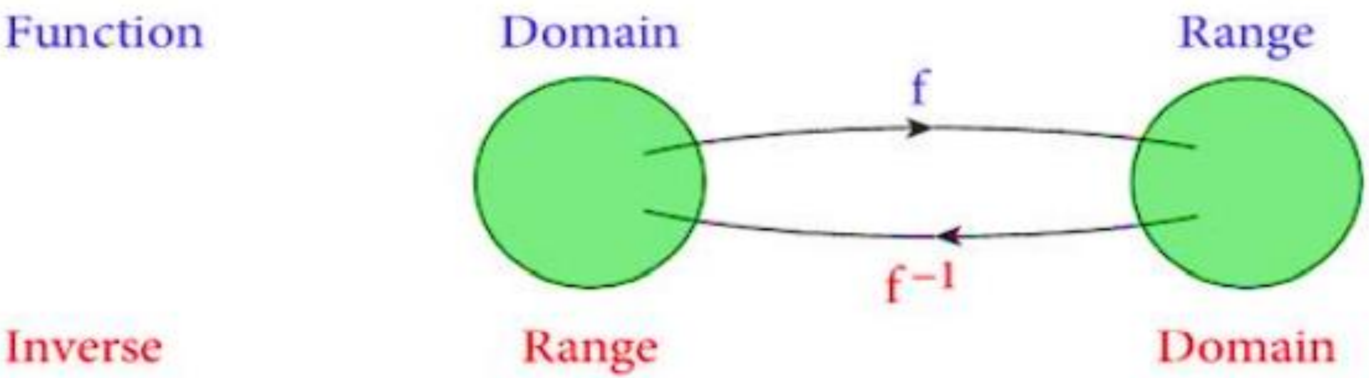
(ii) $g^2(3) = g(7)$

As before, 7 is not in the domain of g so this composite does not exist.

$g(1) = -1$.

This is in the domain of g but that does not matter.

An **inverse function** (or just **inverse**) is another function that will reverse the effect of the first function. The inverse of a function f is written f^{-1}.



> The **Domain** of the inverse is the **Range** of the function.
> The **Range** of the inverse is the **Domain** of the function.

The inverse of a one-one mapping is another one-one mapping and that is a function. However, the inverse of a many-one mapping would be one-many and that is not a function.

Thus, only one-one functions have inverses.

> Linear functions are 1:1 and so they all have inverses.

In all cases, $ff^{-1}(x) = f^{-1}f(x) = x$.

The function f is defined as follows:

$f : x \mapsto 2x + 5\ ;\ x \in \mathbb{R}$.

a Find the inverse of the function.

b Sketch the graph of the function and its inverse.

c Describe the relationship between the graph of the function and the graph of its inverse.

Solution:

a The function is 1 : 1 and so it has an inverse.

Its domain is $\mathbb{R}$ and so its range is also $\mathbb{R}$.

Writing $y = 2x + 5$ to find the inverse we make x the subject of the equation:

$$x = \frac{1}{2}(y - 5)$$

Then we swap the x and y:

$$y = \frac{1}{2}(x - 5)$$

So $f^{-1} : x \mapsto \frac{1}{2}(x - 5)$; $x \in \mathbb{R}$.

The domain of the inverse is the range of the function.

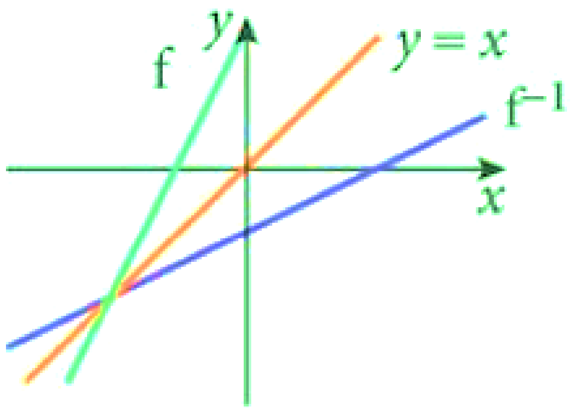
The functions

$f : x \mapsto 3x - 2$

and $g : y \mapsto 3y - 2$

are identical.

b



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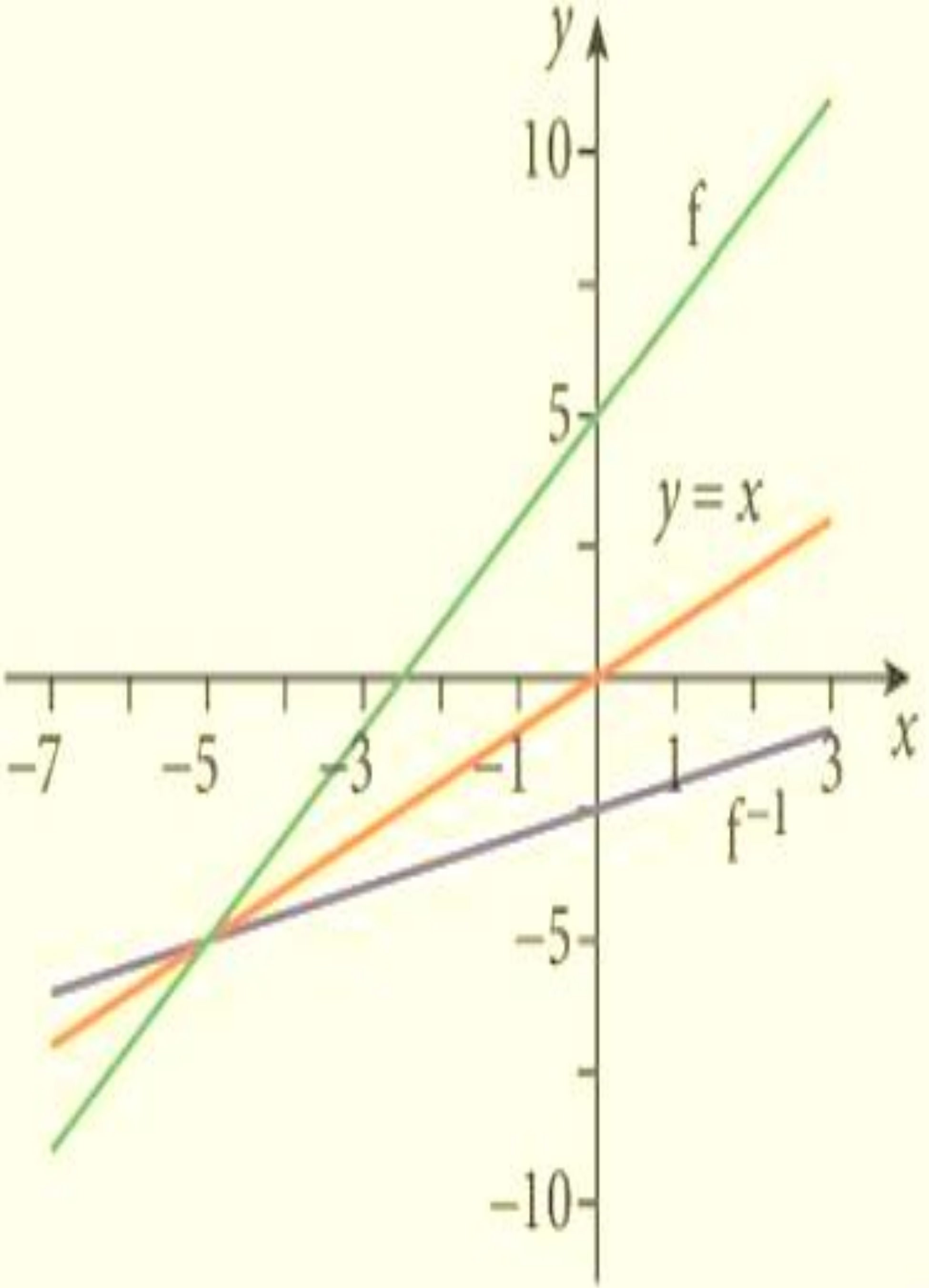
c The graph of the inverse is a reflection of the graph in the line $y = x$.

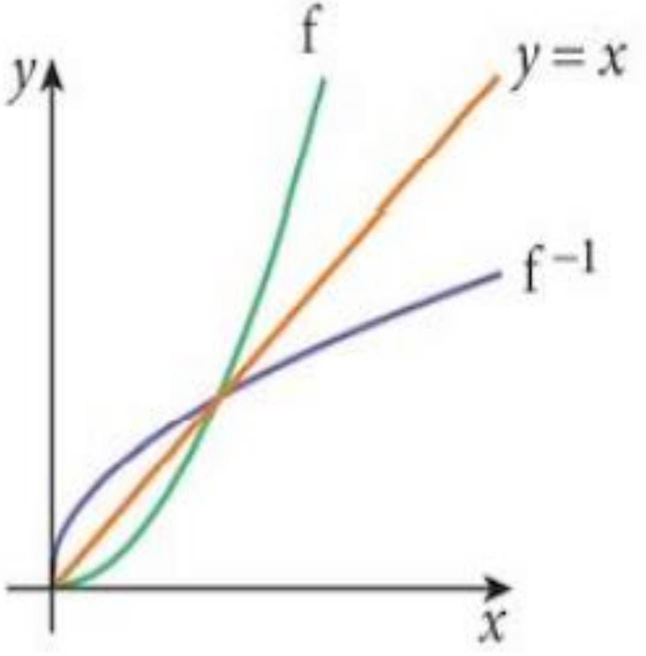
Note: When you are sketching graphs of functions and inverses, always keep the axis scales the same.
Then the relationship will be clear.

Always include the line $y = x$ drawn at 45° to aid your description.

This is the same graph with different scales and it does not look like a reflection. The line $y = x$ is not at 45°.



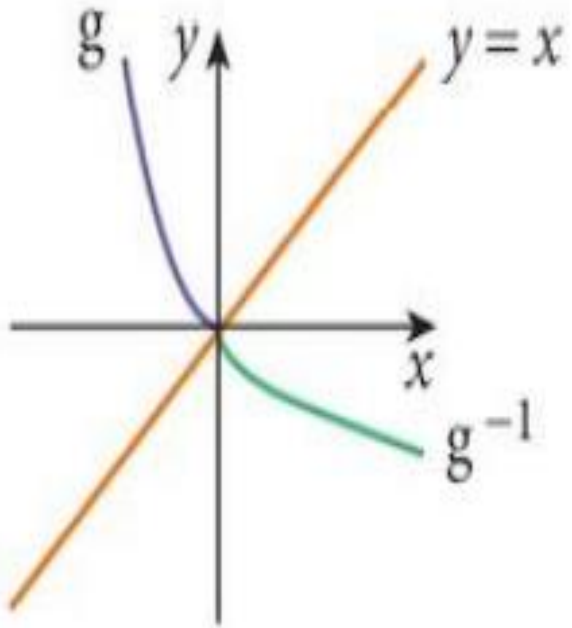
Using the standard restriction, $x \in \mathbb{R}, x \geqslant 0$, the graphs of $\mathrm{f}: x \mapsto x^2$ and its inverse, $\mathrm{f}^{-1}: x \mapsto \sqrt{x}$ are shown on the right.



Notice that the graph of f intersects the graph of f^{-1} when they both intersect with $y = x$ (in this particular case, at (0, 0) and (1, 1)). This will always happen.

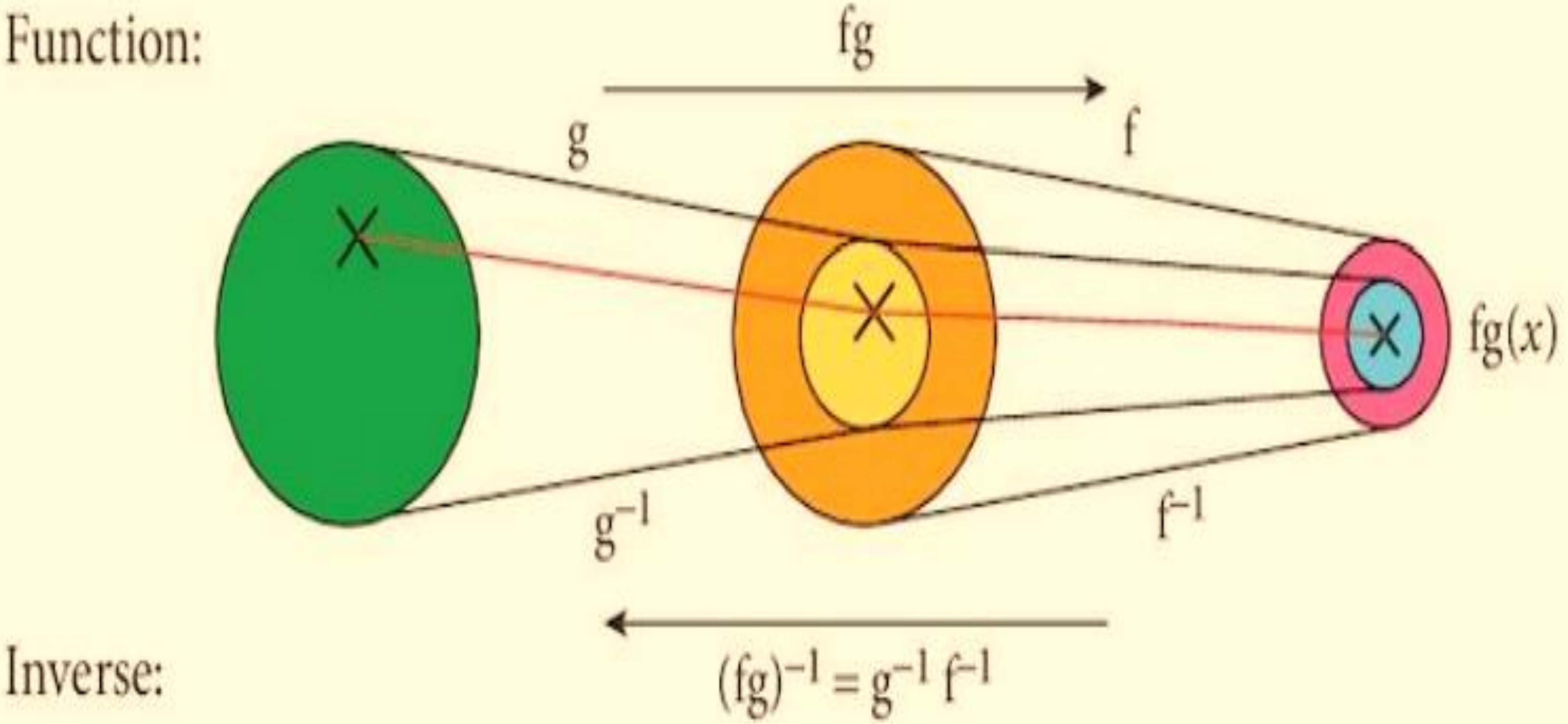
For completeness, we show

$$\mathrm{g}: x \mapsto x^2; x \in \mathbb{R}, x \leqslant 0$$
$$\mathrm{g}^{-1}: x \mapsto -\sqrt{x}\,; x \in \mathbb{R}, x \geqslant 0$$



Note that when you are finding the formula for an inverse involving a square root, you must be careful to choose the correct root, either positive or negative.

Notice that since $(g^{-1}f^{-1})(fg) : x \longmapsto x$, $(fg)^{-1} = g^{-1}f^{-1}$.

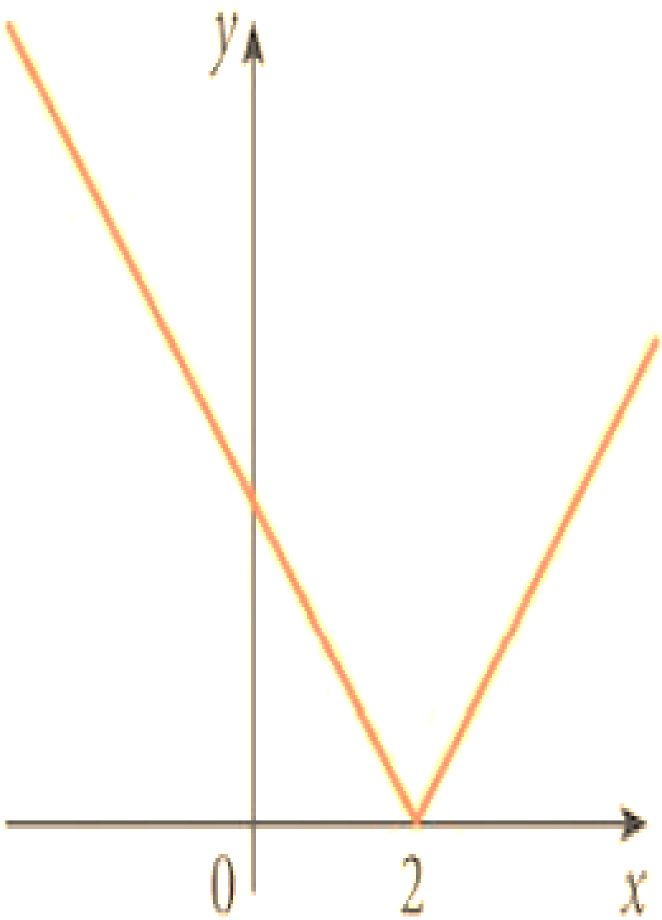


Modulus Function

The function $y = |x|$ is called the modulus function.

The graph $y = |x - 2|$ is shown. It has a vertex at (2, 0).

The part of the graph $y = x - 2$ below the x-axis has been reflected in the axis.



Examples:

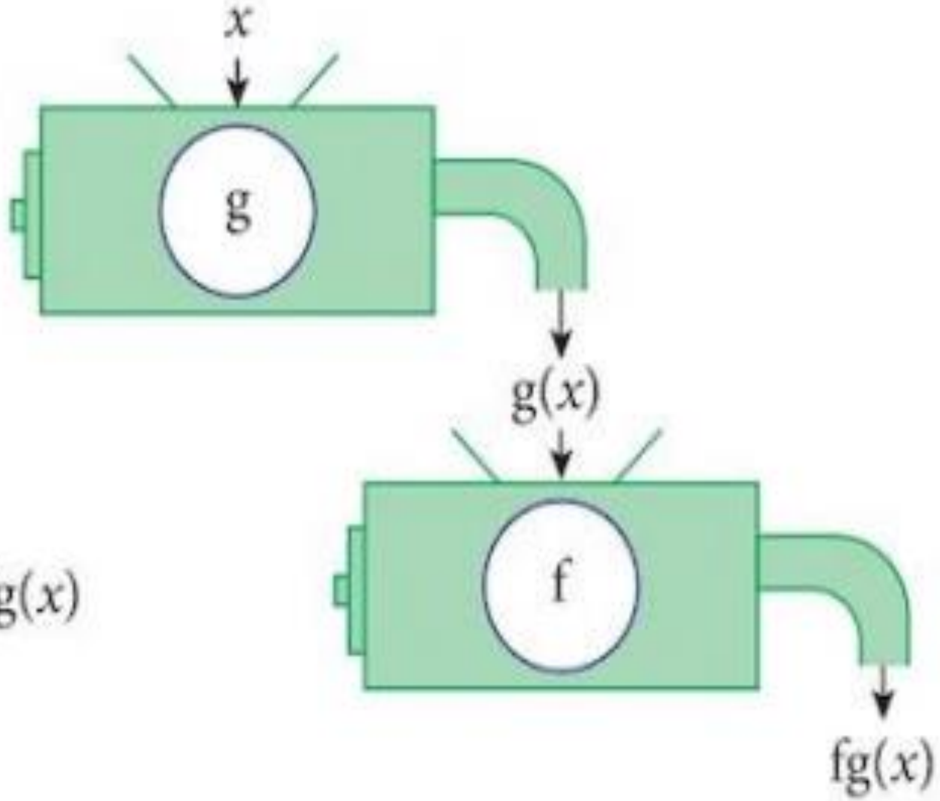
$|4| = |-4| = 4 \quad |x| = +\sqrt{(x^2)}$

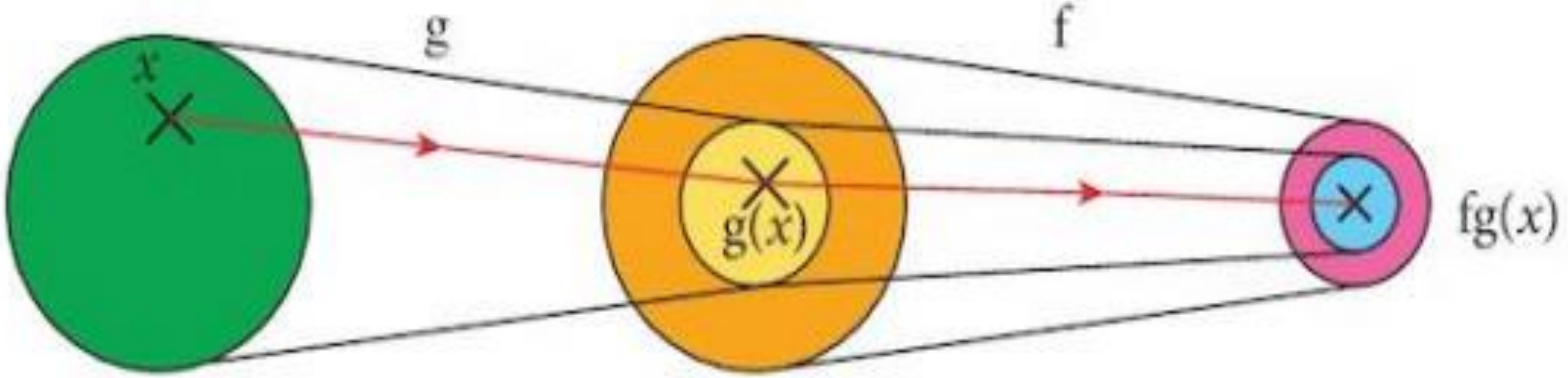
In practice, the modulus function eliminates negative values.

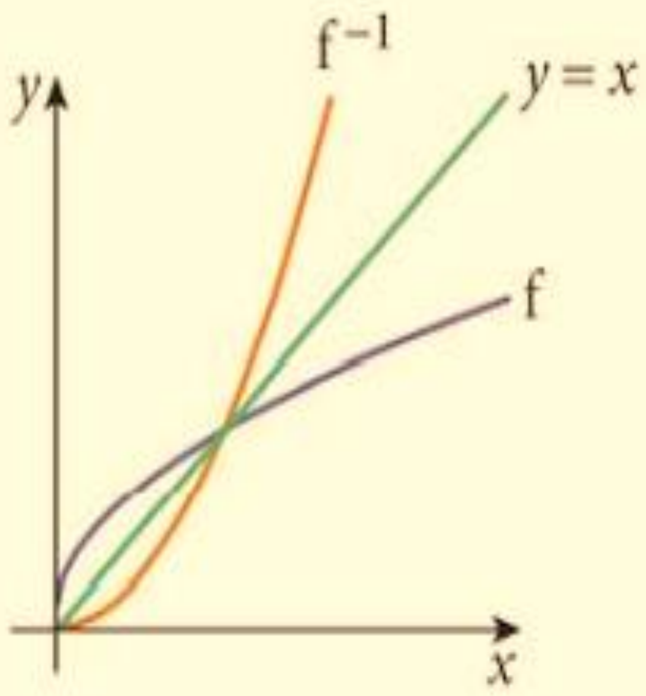
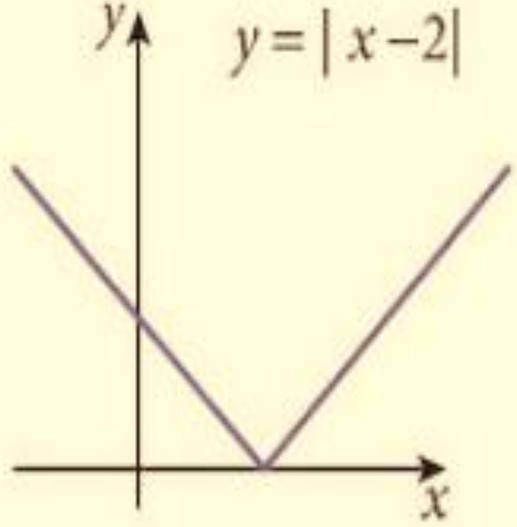
The modulus function is not 1 : 1. Hence it has no inverse.

Summary

Definition	A function is a mapping in which each input value can generate only one output value.
Domain	The set of input values.
Range	The set of output values attained.
Composite functions	fg means "g first, followed by f". The range of g must be a subset of the domain of f.





Inverse function	f^{-1} exists only if f is 1 : 1. Then $f f^{-1}(x) = f^{-1} f(x) = x$. The **domain** of the inverse is the **range** of the function. The **range** of the inverse is the **domain** of the function.	
Graph of the inverse	The graphs of the function and its inverse are reflections of each other in the line $y = x$.	
The modulus function	$\lvert x \rvert = +\sqrt{x^2}$ Makes all values positive.	

Two sets X and Y have the same cardinality if there exists a one-to-one function mapping X onto Y , that is, if there exists a one-to-one correspondence between X and Y .

We showed that $\mathbb{Z}$ and $\mathbb{Z}^+$ or $\mathbb{N}$ have the same cardinality.

- **We denote this cardinal number by $\aleph_0$, so that $|\mathbb{Z}|=|\mathbb{Z}^+|=\aleph_0$.**
- **It is fascinating that a proper subset of an infinite set may have the same number of elements as the whole set;**

➢ **an infinite set can be defined as a set having this property.**

➢ **We naturally wonder whether all infinite sets have the same cardinality as the set $\mathbb{Z}$.**

❑ **A set has cardinality $\aleph_0$ if and only if all of its elements could be listed in an infinite row, so that we could “number them” using $\mathbb{Z}^+$.**

Figure indicates that this is possible for the set $\mathbb{Q}$. $|\mathbb{Q}|=\aleph_0$

$$
\begin{array}{cccccccc}
0 & 1 & -1 & 2 & -2 & 3 & -3 & \cdots \\
\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & \frac{5}{2} & -\frac{5}{2} & \frac{7}{2} & \cdots \\
\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} & \frac{4}{3} & -\frac{4}{3} & \frac{5}{3} & \cdots \\
\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{3}{4} & \frac{5}{4} & -\frac{5}{4} & \frac{7}{4} & \cdots \\
\frac{1}{5} & -\frac{1}{5} & \frac{2}{5} & -\frac{2}{5} & \frac{3}{5} & -\frac{3}{5} & \frac{4}{5} & \cdots \\
\frac{1}{6} & -\frac{1}{6} & \frac{5}{6} & -\frac{5}{6} & \frac{7}{6} & -\frac{7}{6} & \frac{11}{6} & \cdots \\
\frac{1}{7} & -\frac{1}{7} & \frac{2}{7} & -\frac{2}{7} & \frac{3}{7} & -\frac{3}{7} & \frac{4}{7} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

- ✓ **Let A and B be sets. We say A and B have the same cardinality when there exists a bijection $f : A \to B$. We denote by $|A|$ the equivalence class of all sets with the same cardinality as A and we simply call $|A|$ the cardinality of A.**
- ✓ **Note that A has the same cardinality as the empty set if and only if A itself is the empty set. We then write $|A| := 0$.**
- ✓ **Finite set :-**

  Suppose that A has the same cardinality as $\{1,2,3,...,n\}$ for some $n \in N$. We then write $|A| := n$, and we say that A is finite. When A is the empty set, we also call A finite. , for each nonempty finite set A, there exists a unique natural number n such that there exists a bijection from A to $\{1,2,3,...,n\}$.
- ✓ **Infinite set :- We say that A is infinite or "of infinite cardinality" if A is not finite.**

- ✓ **We write $|A| \leq |B|$ if there exists an injection from A to B. We write $|A| = |B|$ if A and B have the same cardinality. We write $|A| < |B|$ if $|A| \leq |B|$, but A and B do not have the same cardinality.**
- ✓ **$|A| = |B|$ have the same cardinality if and only if $|A| \leq |B|$ and $|B| \leq |A|$. This is the so-called Cantor-Bernstein-Schroeder theorem.**
- ✓ **Countably infinite :- If $|A| = |\mathbb{N}|$, then A is said to be countably infinite. i.e. there exists a bijection between from A to $\mathbb{N}$ .then A is said to be countably infinite. Note that the cardinality of $\mathbb{N}$ is usually denoted as $\aleph_0$ (read as aleph-naught)† .**

Definition. A set A is said to be ***infinite*** if it is not finite. It is said to be ***countably infinite*** if there is a bijective correspondence

$$f : A \longrightarrow \mathbb{Z}_+.$$

EXAMPLE 1. The set $\mathbb{Z}$ of all integers is countably infinite. One checks easily that the function $f : \mathbb{Z} \to \mathbb{Z}_+$ defined by

$$f(n) = \begin{cases} 2n & \text{if } n > 0, \\ -2n+1 & \text{if } n \leq 0 \end{cases}$$

is a bijection.

Countable Set :

A set is said to be countable if it is either finite or countably infinite. A set that is not countable is said to be uncountable.

- **Let B be a nonempty set. Then the following are equivalent:**

(1) B is countable.

(2) There is a surjective function f : $\mathbb{Z}^+ \to$ B.

(3) There is an injective function g : B $\to \mathbb{Z}^+$.

- **If C is an infinite subset of $\mathbb{Z}^+$,then C is countably infinite.**
- **A subset of a countable set is countable.**

Proof. Suppose A $\subset$ B, where B is countable. There is an injection f of B into $\mathbb{Z}^+$. the restriction of f to A is an injection of A into $\mathbb{Z}^+$.

- **The set $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countably infinite**

Proof. It suffices to construct an injective map f : $\mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$.

We define f by the equation f (n, m) = $2^n 3^m$.

It is easy to check that f is injective. For suppose that $2^n 3^m = 2^p 3^q$. If $n < p$, then $3^m = 2^{p-n} 3^q$, contradicting the fact that 3^m is odd for all m. Therefore, $n = p$. As a result, $3^m = 3^q$, Then if $m < q$, it follows that $1 = 3^{q-m}$, another contradiction. Hence $m = q$. ■

❖ The set $\mathbb{Q}^+$ of positive rational numbers is countably infinite.

For we can define a surjection $g : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Q}^+$ by the equation $g(n, m) = m/n$. Because $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable, there is a surjection $f : \mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$. Then the composite $g \circ f : \mathbb{Z}^+ \to \mathbb{Q}^+$ is a surjection, so that $\mathbb{Q}^+$ is countable. And, of course, $\mathbb{Q}^+$ is infinite because it contains $\mathbb{Z}^+$.

Similarily the set $Q = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$ of all rational numbers is countably infinite $\mathbb{Z}^+$.

❖ A countable union of countable sets is countable.

❖ A finite product of countable sets is countable.

❖ Let A be a set. There is no injective map $f : P(A) \to A$, and there is no surjective map $g : A \to P(A)$.

✓ Let A be a set. The following statements about A are equivalent:

- (1) There exists an injective function $f : \mathbb{Z}^+ \to A$.
- (2) There exists a bijection of A with a proper subset of itself.
- (3) A is infinite.

(*Russell's paradox.*) Let $\mathcal{B}$ be the subset of $\mathcal{A}$ consisting of all sets that are not elements of themselves;

$$\mathcal{B} = \{A \mid A \in \mathcal{A} \text{ and } A \notin A\}.$$

- ✓ **Uncountable Set:-** If A is not countable, then A is said to be uncountable.
- ✓ The set of even natural numbers has the same cardinality as N.

Proof: Given an even natural number, write it as 2n for some $n \in N$. Then create a bijection taking 2n to n.

characterization of infinite sets: A set is infinite if and only if it is in one to one correspondence with a proper subset of itself.

Remarks:-

- ➢ $|\mathbb{R}| = c$(continuum)
- ➢ $S = \{x \in \mathbb{R} \mid 0 < x < 1\}$ has the same cardinality as $\mathbb{R}$, i.e. $|(0,1)| = c$
- ➢ If $A \subset B$ and B is countable, then A is countable. Similarly if A is uncountable, then B is uncountable.
- ➢

Results

- **Cantor theorem. $|A| < |P(A)|$. In particular, there exists no surjection from A onto P(A).**

Proof. There of course exists an injection $f : A \to P(A)$. For any $x \in A$, define $f(x) := \{x\}$. Therefore $|A| \le |P(A)|$.

To finish the proof, we have to show that no function $f : A \to P(A)$ is a surjection. Suppose that $f : A \to P(A)$ is a function. So for $x \in A$, $f(x)$ is a subset of A.

Define the set $B := \{x \in A : x \notin f(x)\}$.

We claim that B is not in the range of f and hence f is not a surjection.

Suppose that there exists an x_0 such that $f(x_0) = B$. Either $x_0 \in B$ or $x_0 \notin B$.

If $x_0 \in B$, then $x_0 \notin f(x_0) = B$, which is a contradiction. If $x_0 \notin B$, then $x_0 \in f(x_0) = B$, which is again a contradiction.

Thus such an x_0 does not exist. Therefore, B is not in the range of f , and f is not a surjection. As f was an arbitrary function, no surjection can exist.

- there do exist uncountable sets, as P(N) must be uncountable.

Determine, for each of the following sets, whether or not it is countable. Justify your answers.

(a) The set A of all functions $f : \{0, 1\} \to \mathbb{Z}_+$.

(b) The set B_n of all functions $f : \{1, \ldots, n\} \to \mathbb{Z}_+$.

(c) The set $C = \bigcup_{n \in \mathbb{Z}_+} B_n$.

(d) The set D of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$.

(e) The set E of all functions $f : \mathbb{Z}_+ \to \{0, 1\}$.

(f) The set F of all functions $f : \mathbb{Z}_+ \to \{0, 1\}$ that are "eventually zero." [We say that f is ***eventually zero*** if there is a positive integer N such that $f(n) = 0$ for all $n \geq N$.]

(g) The set G of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ that are eventually 1.

(h) The set H of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ that are eventually constant.

(i) The set I of all two-element subsets of $\mathbb{Z}_+$.

(j) The set J of all finite subsets of $\mathbb{Z}_+$.

$\mathcal{P}(\mathbb{Z}_+)$ and $\mathbb{R}$ have the same cardinality.

A real number x is said to be ***algebraic*** (over the rationals) if it satisfies some some polynomial equation of positive degree

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with rational coefficients a_i. Assuming that each polynomial equation has only finitely many roots, show that the set of algebraic numbers is countable. A real number is said to be ***transcendental*** if it is not algebraic. Assuming the reals are uncountable, show that the transcendental numbers are uncountable. (It is a somewhat surprising fact that only two transcendental numbers are familiar to us: e and π. Even proving these two numbers transcendental is highly nontrivial.)

- **A partition of a set S is a collection of nonempty subsets of S such that every element of S is in exactly one of the subsets. The subsets are the cells of the partition.**
- **(Equivalence Relations and Partitions) Let S be a nonempty set and let $\sim$ be an equivalence relation on S. Then $\sim$ yields a partition of S, where $\bar{a} = \{x \in S \mid x \sim a\}$.**
- **Also, each partition of S gives rise to an equivalence relation $\sim$ on S where $a \sim b$ if and only if a and b are in the same cell of the partition**