Relation

A subset of A×B is called a relation from A to B .

Number of Relation

Total number of relation from A to B is equal to the number of subsets of A×B. i.e. $|P(A\times B)| = 2^{|A\times B|}$.

Ex. $|A|=3, |B|=2 \Rightarrow$ Total number of relation from A to B is $2^{3\times 2} = 64$

Binary Relation

A subset of A×A is called a binary relation on A .

Number of binary relation on a set

If $|A|=n$, then $|A\times A| = n^2$

Total number of binary relation = 2^{n^2}.

Ex. $|A|=2 \Rightarrow$ Total number of binary relation on A = $2^{2^2} = 2^4 = 16$.

Type's of Relation

Empty Relation	Identity Relation
As ∅ is subset of every set hence ∅ ⊆ A×A is also a relation on A called empty relation. i.e. no any pair of element satisfies the given condition.	A subset I of A×A is called identity relation on A. If a∈A then (a,a) ∈ A×A . ✓Identity relation is unique for each set A. ✓On a set a relation is said to be identity if every element of A related to itself.

Reflexive Relation

A subset S of A×A is called reflexive relation. If every element of A must related to itself i.e.$\forall$ aϵA then (a,a) ϵ A×A .

<u>Properties of Reflexive Relation</u>

- ✓ Total number of reflexive relation on A = 2^{n^2-n}.where |A|= n
- ✓ A={1,2,3} , |A| = 3 $\Longrightarrow$ Total number of reflexive relation on A = $2^{3^2-3} = 2^{9-3} = 2^6$ = 64.
- ✓ Empty relation on nonempty set is never reflexive .
- ✓ Empty relation on empty set is always reflexive .
- ✓ The least cardinality of a reflexive relation on A with n elements is n.
- ✓ There are relations, which are neither reflexive nor irreflexive.

Irreflexive Relation

A subset S of A×A is called irreflexive relation. If no element of A related to itself i.e.$\forall$ aϵA then (a,a) $\notin$A×A .

<u>Properties of Irreflexive Relation</u>:-

- ✓ Total number of irreflexive relation on A = 2^{n^2-n}.where |A|= n
- ✓ A={1,2,3} , |A| = 3 $\Longrightarrow$ Total number of Irreflexive relation on A = $2^{3^2-3} = 2^{9-3} = 2^6$ = 64.
- ✓ Empty relation on nonempty set is never irreflexive .
- ✓ Empty relation on empty set is always irreflexive .
- ✓ There does not exist any non empty relation which is reflexive as well as irreflexive.

Symmetric Relation	Asymmetric Relation
Def. A subset S of A×A is called symmetric relation. If (a , b) ϵ S then (b , a) ϵ S .	Def. A subset S of A×A is called asymmetric relation. If (a , b) ϵ S then (b , a) $\notin$ S.
Properties of symmetric relation:-	**Properties of Asymmetric relation**
✓ Total number of symmetric relation on A = $2^{\sum n}$.where |A|= n , $\sum n = \frac{n(n+1)}{2}$	✓ Total number of asymmetric relation on A = $3^{\sum(n-1)}$.where |A|= n $\sum n = \frac{n(n+1)}{2}$, $\sum(n-1) = \frac{(n-1)(n-1+1)}{2} = \frac{n(n-1)}{2}$
✓ |A|=3 ⇒ Total number of symmetric relation on A $= 2^{\sum n} = 2^{\frac{n(n+1)}{2}} = 2^{\frac{3(3+1)}{2}} = 2^{\frac{3(4)}{2}} = 2^6 = 64.$	✓ |A|=3 ⇒ Total number of asymmetric relation on A $= 3^{\sum(n-1)} = 3^{\frac{n(n-1)}{2}} = 3^{\frac{3(3-1)}{2}} = 3^3 = 27$
✓ Ex. A ={1,2,3} , $\mathcal{R}$ = {(1,2),(2,1) } be a relation on A ⇒ $\mathcal{R}$ is symmetric relation on A.	Ex. A ={1,2,3} , $\mathcal{R}$ = {(1,2)(1,3),(2,3) } be a relation on A ⇒ $\mathcal{R}$ is asymmetric relation on A.

❖ Empty relation is both symmetric as well as asymmetric.

❖ A non empty relation relation cannot be both asymmetric as well as symmetric.

❖ Identity relation is symmetric relation but ther exist some relations which are symmetric but not identity.

❖ A non empty reflexive relation cannot be asymmetric.

Anti-symmetric Relation

A subset S of A×A is called anti-symmetric relation. If (a , b) ϵ S and (b , a) ϵ S then a=b.

Properties of Anti-symmetric relation

- ✓ **Total number of anti-symmetric relation on A = $2^n\, 3^{\sum(n-1)}$.where |A|= n ,**

$$\sum n = \frac{n(n+1)}{2}, \sum(n-1) = \frac{(n-1)(n-1+1)}{2} = \frac{n(n-1)}{2}$$

- ✓ Ex. A ={1,2,3} , $\mathcal{R}$ = {(1,1),(2,2) } be a relation on A $\Rightarrow$ $\mathcal{R}$ is antisymmetric relation on A. i.e. A ={1,2,3} , |A|=3 then total number of anti-symmetric relation on A = $2^n\, 3^{\sum(n-1)} = 2^n\, 3^{\frac{n(n-1)}{2}} = 2^3\, 3^{\frac{3(3-1)}{2}}$ = 8×27 = 216
- ✓ **Empty relation on empty set is always anti-symmetric .**
- ✓ **The term symmetric and anti-symmetric are not opposite.**
- ✓ **A relation may be both symmetric and anti-symmetric.**

  ex. A={1,2,3}, $\mathcal{R}$ = {(1,1),(2,2) } is both symmetric and antisymmetric.
- ✓ **A relation may not be both symmetric and anti-symmetric iff it contains some pair of the form (a , b) where a ≠ b .**

Reflexive Relation	Transitive Relation
❑ Let A = {a, b, c, d} and $\mathcal{R}$ be defined as follows: $\mathcal{R}$ = {(a, a), (a, c), (b, a), (b, b), (c, c), (d, c), (d, d)}. R is a reflexive relation. ❑ Let R be a relation on a set then if R is reflexive then R^{-1} is reflexive **Proof** Let $(a, a) \in R \; \forall \; a \in A$ $\therefore \; (a, a) \in R^{-1} \;\; \forall \; a \in A$ $\therefore \; R^{-1}$ is reflexive	❑ Let $A = \{a, b, c, d\}$ and R be defined as follows: $R = \{(a, b), (a, c), (b, d), (a, d), (b, c), (d, c)\}$. Here R is transitive relation on A.
Symmetric Relation	
➢ **Exa.** equality (=) is symmetric, but strict inequality (<) is not. ➢ Let $A = \{a, b, c, d\}$ and $R = \{(a, a), (b, c), (c, b), (d, d)\}$. Show that R is symmetric. ➢ let R be a relation on a set A then R is symmetric iff $R = R^{-1}$ Sol.Assume R is a symmetric , let $(a, b) \in R \to (b, a) \in R$ $(b, a) \in R^{-1}$ and $(a, b) \in R^{-1}$ $\therefore R = R^{-1}$ let $(a, b) \in R \to (b, a) \in R^{-1}$ & $(b, a) \in R$ since $R = R^{-1}$ $\therefore$ R is symmetric	

Transitive Relation

A subset S of A×A is called transitive relation. If (a , b) ϵ S and (b , c) ϵ S then (a,c) ϵ S .

Equivalence Relation

An equivalence relation R on a set S is one that satisfies these three properties for all x, y , z ∈ S.

1. (Reflexive) $x\mathcal{R}x$, $\forall\ x\epsilon\mathcal{R}$

2. (Symmetric) If $x\mathcal{R}y$, then $y\mathcal{R}x$, $\forall\ x, y\epsilon\mathcal{R}$

3. (Transitive) If $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x\mathcal{R}z$,$\forall\ x, y, z\epsilon\mathcal{R}$

Example .

1.For any nonempty set S, the equality relation = defined by the subset {(x, x) | x ∈ S} of S × S is an equivalence relation

Inverse Relation

Given a relation R from A to B, the inverse of R, denoted R^{-1}, is the relation from B to A defined as $bR^{-1}a$

$R^{-1} = \{(b, a): (a, b) \in R\}$

For instance, if R is the relation "being a son or daughter of", then R^{-1} is the relation "being a parent of".

Example . let R = {(1, 0), (2, 0), (2, 1), (3, 0), (3, 1), (3, 2)} then

R^{-1} = {(0, 1), (0, 2), (1, 2), (0, 3), (1, 3), (2, 3}

Example . Let R and S be a relations between A and B.

i. Show that, if $R \subseteq S$ then $R^{-1} \subseteq S^{-1}$.

ii. Prove that $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

Proof (i)Let $(a, b) \in R^{-1} \rightarrow (b, a) \in R$ *definition of inverse relation*

$\therefore (b, a) \in S$ *since* $R \subseteq S$ $\therefore (a, b) \in S^{-1}$ *definition of inverse relation*

$\therefore R^{-1} \subseteq S^{-1}$ *definition of subset*

Proof (ii) (1) Let $(a,b) \in (R \cap S)^{-1}$

$\therefore (b,a) \in (R \cap S)$ *definition of inverse*

$(b,a) \in R$ *and* $(b,a) \in S$ *definition of intersection*

$(a,b) \in R^{-1}$ *and* $(b,a) \in S^{-1}$ *definition of inverse*

$(a,b) \in R^{-1} \cap S^{-1}$ *definition of intersection*

$(R \cap S)^{-1} \subseteq R^{-1} \cap S^{-1}$ *definition of subset*

(2) Let $(a,b) \in R^{-1} \cap S^{-1}$

$(a,b) \in R^{-1}$ *and* $(b,a) \in S^{-1}$ *definition of intersection*

$(b,a) \in R$ *and* $(b,a) \in S$ *definition of inverse*

$\therefore (b,a) \in (R \cap S)$ *definition of intersection*

$\therefore (a,b) \in (R \cap S)^{-1}$ *definition of inverse*

$R^{-1} \cap S^{-1} \subseteq (\mathrm{R}$

$\therefore$ **from (1)and (2)we have**

$(\mathrm{R} \cap \mathrm{S})^{-1} = R^{-1} \cap S^{-1}$

Finite set :

If a set consist of finite number of elements, it is called a finite set.

e.g., {1,2,3,..........n} , $\emptyset$ are finite set.

Def. Power set :

The power set of A denoted by P(A) , is defined as the set $\{B : B \subseteq A\}$.. Thus, P(A) is the collection of all possible subsets of A.

1. Let A be any set. Let P(A) be the power set of A, that is, the set of all subsets of A;

$P(A)=\{B:B\subseteq A\}$.Then which the following Is/ are true about the set P(A) ?

(a)$P(A) = \emptyset$ for some A

(b)P(A) is a finite set for some A

(c)P(A) is a countable set for some A

(d)P(A) is a uncounteble set for some A

Power set of any set contains at least one element so option(a) is not true Let A be finite Then P(A) is finite.

Every finite set is countable.Hence, options (b) and (c) are true.

Taking A = N ,Then, P(N) is uncountable.Hence, option (d) is true.