J.R. INSTITUTE OF MATHEMATICS

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Group Theory - 1

Def. Group : Let G be a non-empty set with a binary operation '$*$', then G is called a group w.r.t. '$*$' if following postulates are satisfied :

(i) Associativity : $(a*b)*c = a*(b*c)$ for all a, b, c G.

(ii) Existence of identity : There exists an element $e \in G$, such that $a * e = a = e * a$ for all $a \in G$.

(iii) Existence of inverse : For every $a \in G$, there exists an element $b \in G$ such that $a * b = e = b * a$.

Here , b is called inverse of a and is denoted by a^{-1}.

Def. Abelian group : A group G is said to be abelian (or commutative) if $a * b = b * a$ for all $a, b \in G$.

Remark : The word 'Abelian' is after the name of great mathematician N.H.Abel.

Def. Finite and Infinite groups : If a group G consists of a finite number of elements, then it is called a finite group, otherwise it is called an infinite group.

Def. Order of a group : The number of elements in a finite group is called the order of the group. An infinite group is said to be of infinite order. Order of a group is denoted by $o(G)$ or $|G|$.

Examples :

1. $Z_n = \{0, 1, 2, ..., n-1\}$ is an abelian group w.r.t. $+_n$ of order n.
2. $U(n) = \{m : 1 \le m < n \text{ and } \gcd(m, n) = 1\}$ is an abelian group under $\times_n$ of order $\phi(n)$.

Basic Results :

1. In a group the identity element is unique.
2. In a group every element has a unique inverse.
3. If G is a group then $\left(a^{-1}\right)^{-1} = a$ for all $a \in G$.
4. Let G be a group then $(a\,b)^{-1} = b^{-1}\,a^{-1}$ for all $a, b \in G$.
5. If $a, b, c \in G$ then $ab = ac \Rightarrow b = c$. This is called left cancellation law.
6. If $a, b, c \in G$ then $ba = ca \Rightarrow b = c$. This is called right cancellation law.
7. In the composition table of a group, each element of the group appears exactly once in each row and each column.

Def. Subgroup : A non empty subset H of a group G is called a subgroup of G , if H itself is a group w.r.t. the same binary operation as in G.

Def. Proper and Improper subgroups : For a group G, the set $\{e\}$ and G are always subgroups of G and are called improper subgroups of G. Any other subgroup [other than G and $\{e\}$] is called a proper subgroup of G.

Remark : Some authors also define a proper subgroup to be a subgroup other than G.

Results on subgroups :

1. The identity of any subgroup of a group is the same as that of the group.
2. The inverse of any element of a subgroup is the same as the inverse of that element in group.
3. A non empty subset H of a group G is a subgroup of G if and only if $ab^{-1} \in H$ for all $a, b \in H$.
4. A non empty finite subset H of a group G is a subgroup of G if and only if closure holds in H i.e. $a, b \in H \Rightarrow ab \in H$.
5. Intersection of two subgroups of a group is again a subgroup.
6. The union of two subgroups of a group is a subgroup if and only if one is contained in the other.

Def. Let H and K be two subgroups of a group G, then $HK = \{hk : h \in H, k \in K\}$. Similarly, KH may be defined.

7. Let H and K be two subgroups of G then HK is a subgroup of G if and only if $HK = KH$.
8. If H and K are finite subgroups of a group G then $o(HK) = \dfrac{o(H)\,o(K)}{o(H \cap K)}$, where $o(H)$ is the order of H.
9. Let G be a group and H be a subgroup of G then $xHx^{-1} = \left\{xhx^{-1} : x \in G, h \in H\right\}$ is also a subgroup of G.

Def. Order of an element : Let G be a group and $a \in G$ then order of 'a' is defined as the least positive integer n such that

(i) $a^n = 1$, if the operation is multiplication.

(ii) $na = 0$, if the operation is addition.

(iii) $a * a * \ldots\ldots * a\,(n\,\text{times}) = e$ if the operation is $*$.

We then write $o(a) = n$ or $|a| = n$. If there exists no such n, then a is said to be of infinite order or zero order.

Results on order of a group :

1. Let G be a group and $a \in G$ such that $o(a) = n$ then $a^m = e$ if and only if m is a multiple of n.
2. Let G be a finite group and $a \in G$ then $o(a)/o(G)$.
3. Let G be a finite group and $a \in G$ then $a^{o(G)} = e$.
4. The order of an element and its inverse are same i.e. for any $a \in G$, we have $o(a) = o(a^{-1})$
5. For any $a, x \in G$, $o(a) = o(x^{-1}ax)$
6. For any $a, b \in G$, $o(ab) = o(ba)$.
7. Let a and b be two elements of finite order of a group G. If $\gcd\big(o(a), o(b)\big) = 1$ and $ab = ba$, then $o(ab) = o(a)\; o(b)$.

8. Let a be an element of finite order of a group G and k be any positive integer then $o\left(a^k\right)=\dfrac{o(a)}{\gcd\left(k,o(a)\right)}$.

9. Lagrange's Theorem : Let G be a finite group and H be a subgroup of G then $o(H)$ divides $o(G)$.

Results on abelian groups :

1. If a group G is abelian then $(ab)^n = a^n b^n$ for all $a,b \in G$ and for all integers n.
2. A group G is abelian iff $(ab)^2 = a^2 b^2$ for all $a,b \in G$.
3. Let G be a group and $a^2 = e$ for all $a \in G$ then G is abelian.
4. A group G is abelian iff $(ab)^{-1} = a^{-1}b^{-1}$ for all $a,b \in G$.
5. Let G be a group and $(ab)^m = a^m b^m$ for all $a,b \in G$ and three consecutive integers m then G is abelian.
6. If $(ab)^m = a^m b^m$ holds for two consecutive integers m then G need not be abelian.
7. Let G be a group such that $ab = ca$ implies $b = c$, then G is abelian. In words, if cross cancellation law holds in a group then it is abelian.
8. A group is abelian iff ith row and ith column in the composition table are same for all i.
9. $\mathbb{Z}_n$ is abelian for all positive integers n.
10. $U(n)$ is abelian for all positive integers n.

Def. Cyclic group : A group G is said to be cyclic if there exists an element $a \in G$ such that every element of G is of the form a^n, where n is an integer. The element 'a' is then, called a generator of G and we write : $G = \langle a \rangle$ or (a)

Results on cyclic groups :

1. Every cyclic group is abelian.
2. Every subgroup of a cyclic group is cyclic.
3. If 'a' is a generator of a cyclic group G, then a^{-1} is also a generator of G.
4. The order of a cyclic group is equal to the order of its generator.
5. If a finite group of order n contains an element of order n then the group must be cyclic.
6. Every group of prime order is cyclic .
7. Every infinite cyclic group has exactly two generators.
8. The number of generators of a finite cyclic group of order n is $\phi(n)$, where ϕ denotes the Euler's ϕ function.

9. Let G be a finite cyclic group such that $o(G) = n$ and $G = \langle a \rangle$ then a^m will be generator of G if and only if $\gcd(m, n) = 1$.
10. Let $a \in G$ such that $o(a) = n$ then $o(a^i) = o(a^j)$ iff $\gcd(n, i) = \gcd(n, j)$.
11. A group of prime order has no proper subgroups.
12. A non trivial group G which has no proper subgroups must be a group of prime order.
13. Converse of Lagrange's theorem is true for finite cyclic group. In words, let $G = \langle a \rangle$ be a finite cyclic group such that $o(G) = n$. Let d/n then G has a subgroup of order d. Further the subgroup of order d is unique and this subgroup is given by $\langle a^{n/d} \rangle$.
14. Total number of subgroups of a finite cyclic group of order n is $\tau(n)$, the number of divisors of n.
15. Let G be cyclic group of order n and d be a positive integer which divides n, then G has $\phi(d)$ elements of order d.
16. Let G be a finite group and d be a positive integer which divides $o(G)$ then number of cyclic subgroups of order d is $\dfrac{\text{number of elements of order } d}{\phi(d)}$.
17. $\mathbb{Z}_n$ is cyclic for all positive integer n and an integer k is a generator of $\mathbb{Z}_n$ iff $\gcd(n, k) = 1$.
18. $U(n)$ is cyclic iff $n = 1$ or 2 or 4 or p^m or $2.p^m$ where p is an odd prime and m is a positive integer.
19. $U(2^n)$ is not cyclic for $n \geq 3$.

Def. Centre of a group : Let G be a group then centre of the group G is defined to be subset of all elements of G which commute with every element of G and it is denoted by $Z(G)$. In symbols, $Z(G) = \{a \in G : ax = xa \text{ for all } x \text{ in } G\}$.

Results :

1. $Z(G)$ is a subgroup of G.
2. G is abelian iff $G = Z(G)$
3. Let G be a group and $a \in Z(G)$ then in the composition table of G the row and column headed by 'a' are same.
4. As $\mathbb{Z}_n$ and $U(n)$ are abelian groups so they are centres of themselves i.e., $Z(\mathbb{Z}_n) = \mathbb{Z}_n$ and $Z(U(n)) = U(n)$.

Def. Klein's four group : A group of order four in which every element is self-inverse or every non-identity element is of order 2 is called Klein's four group. Symbolically, $K_4 = \{e, a, b, c\}$ such that $ab = ba = c,\ bc = cb = a,\ ac = ca = b$ and $a^2 = b^2 = c^2 = e$.

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Exercise 1.1

1. In the groups $\mathbb{Z}_n, 1 \le n \le 20$
   (i) Find inverse and order of each element and construct the O–D table.
   (ii) Find all subgroups and construct the subgroup table.
   (iii) Show that the group is cyclic and find all its generators.
2. In the groups $U(n)$ for $1 \le n \le 20$.
   (i) Find inverse and order of each element and construct the O–D table.
   (ii) Find all subgroups and construct the subgroup table.
   (iii) Which $U(n)$ are cyclic. If $U(n)$ is cyclic for any *n* then find all the generators.
3. How many positive integers $1 \le n \le 100$ are there such that $U(n)$ is cyclic.
4. In the Klein's four group K_4,
   (i) construct the composition table.
   (ii) Find inverse and order of each element and construct the O-D table.
   (iii) Find all subgroups and construct the subgroup table.
   (iv) Show that every proper subgroup of K_4 is cyclic.

Answers

3. 50

4. (ii)

Order	No. of elements
1	1
2	3

(iii)

Order	No. of subgroups
1	1
2	3
4	1

Exercise 1.2

1. Translate each of the following multiplicative expressions into its additive counterpart.
   (i) a^2b^3 (ii) $a^{-2}\left(b^{-1}c\right)^2$ (iii) $\left(ab^2\right)^{-3}c^2 = e$
2. For any elements *a* and *b* from a group and any integer *n*, prove that $\left(a^{-1}ba\right)^n = a^{-1}b^n a$.
3. For any integer $n > 2$, show that there are at least two elements in $U(n)$ that satisfy $x^2 = 1$.

4. An abstract algebra teacher intended to give a typist a list of nine integers that form a group under multiplication modulo 91. Instead, one of the nine integers was inadvertently left out so that the list appeared as 1, 9, 16, 22, 53, 74, 79, 81. Which integer was left out ? (This really happened!)
5. Show that the set $\{5,15,25,35\}$ is a group under multiplication modulo 40. What is the identity element of this group ?
6. The integers 5 and 15 are among a collection of 12 integers that form a group under multiplication modulo 56. List all 12.
7. Construct a Cayley table for $U(12)$.
8. Let G be a finite group. Show that the number of elements x of G such that $x^3=e$ is odd. Show that the number of elements x of G such that $x^2 \neq e$ is even.
9. In a finite group, show that the number of nonidentity elements that satisfy the equation $x^5=e$ is a multiple of 4.
10. Let x belong to a group. If $x^2 \neq e$ and $x^6=e$, prove that $x^4 \neq e$ and $x^5 \neq e$. What can we say about the order of x ?
11. For each divisor k of n, let $U_k(n)=\{x \in U(n) \mid x=1 \bmod k\}$. (For example, $U(21)=\{1,4,10,13,16,19\}$ and $U_7(21)=\{1,8\}$.) List the elements of $U_4(20), U_5(30),$ and $U_{10}(30)$. Prove that $U_k(n)$ is a subgroup of $U(n)$.
12. Suppose a group contains elements a and b such that $|a|=4, |b|=2,$ and $a^3b=ba$. Find $|ab|$.
13. Suppose G is a group that has exactly eight elements of order 3. How many subgroups of order 3 does G have ?
14. Find a cyclic subgroup of order 4 in $U(40)$.
15. Find a noncyclic subgroup of order 4 in $U(40)$.
16. Let a be an element of a group and let $|a|=15$. Compute the orders of the following elements of G.
    (i) a^3, a^6, a^9, a^{12} (ii) a^5, a^{10} (iii) a^2, a^4, a^8, a^{14}
17. Let $G=\langle a \rangle$ and let $|a|=24$. List all generators for the subgroup of order 8.
18. Suppose that $|a|=24$. Find a generator for $\langle a^{21} \rangle \cap \langle a^{10} \rangle$. In general, what is a generator for the subgroup $\langle a^m \rangle \cap \langle a^n \rangle$?
19. List the cyclic subgroups of $U(30)$.
20. Let G be a group and let a be an element of G.
    (i) If $a^{12}=e$, what can we say about the order of a ?

(ii) If $a^m = e$, what can we say about the order of a ?

(iii) Suppose that $|G| = 24$ and that G is cyclic. If $a^8 \neq e$ and $a^{12} \neq e$, show that $\langle a \rangle = G$.

21. List all the elements of order 8 in $\mathbb{Z}_{8000000}$. How do you know your list is complete ?

22. Consider the set {4, 8, 12, 16}. Show that this set is a group under multiplication modulo 20 by constructing its Cayley table. What is the identity element ? Is the group cyclic ? If so, find all of its generators.

23. List all the elements of $\mathbb{Z}_{40}$ that have order 10.

24. Let $|x| = 40$. List all the elements of $\langle x \rangle$ that have order 10.

25. Let a and b be elements of a group. If $|a| = 10$ and $|b| = 21$, show that $\langle a \rangle \cap \langle b \rangle = \{e\}$.

26. Let a and b be elements of a group. If $|a| = m$, $|b| = n$, and m and n are relatively prime, show that $\langle a \rangle \cap \langle b \rangle = \{e\}$.

27. Let a and b belong to a group. If $|a| = 24$ and $|b| = 10$, what are the possibilities for $|\langle a \rangle \cap \langle b \rangle|$?

28. If $|a^5| = 12$, what are the possibilities for $|a|$? If $|a^4| = 12$, what are the possibilities for $|a|$?

29. Suppose that $|x| = n$. Find a necessary and sufficient condition on r and s such that $\langle x^r \rangle \subseteq \langle x^s \rangle$.

Answers

1. (i) $2a + 3b$ (ii) $-2a + 2(-b + c)$ (iii) $-3(a + 2b) + 2c = 0$ 3. 1 and $n-1$ 4. 29

5. 25 6. $\{1,3,5,9,13,15,19,23,25,27,39,45\}$ 7.

	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

10. 3 or 6

11. $U_4(20) = \{1,9,13,17\}$, $U_5(30) = \{1,11\}$, $U_{10}(30) = \{1,11\}$ 12. 2 13. 4 14. $\langle 3 \rangle = \{1,3,9,27\}$

15. {1, 9, 11, 19} 16. (i) 5 (ii) 3 (iii) 15 17. $\langle a^3 \rangle, \langle a^9 \rangle, \langle a^{15} \rangle, \langle a^{21} \rangle$ 18. $a^{18}; a^{lcm(m,n)(\text{mod } 24)}$

19. $\langle 1 \rangle, \langle 7 \rangle, \langle 11 \rangle, \langle 17 \rangle, \langle 19 \rangle, \langle 29 \rangle$ 20. (i) $|a|$ divides 12 (ii) $|a|$ divides m

21. 1000000, 3000000, 5000000, 7000000. $\langle 1000000 \rangle$ is the unique subgroup of order 8 and only those on the list are generators.

22. 16; yes; $\langle 8 \rangle$ and $\langle 12 \rangle$ 23. 4, 12, 28, 36 27. 1 and 2 28. 12 or 60; 48 29. s divides r.

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Group Theory - 2

Def. Permutation : A one-one onto mapping from a set to itself is called a permutation.

Remark : If S is a finite set having n elements then clearly there are $n!$ permutations.

Def. Permutation group or Symmetric group : If a set S has n elements , then the set of all $n!$ permutations on S forms a group under composition of permutations , called Permutation group or Symmetric group of degree n and is denoted by P_n or S_n.

Example :

1. $S_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} , \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$

2. $S_3 = \left\{ \begin{matrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} , \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} , \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} , \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} , \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \end{matrix} \right\}$

Def. Cyclic permutation or cycle : A permutation which can be expressed in one row such that image of each element is next element and image of last element is first , provided fixed elements are skipped is called a cycle.

Def. Length of a cycle : Number of elements permuted by the cycle or number of elements in its one-row representation is called length of that cycle. A cycle of length k is called k-cycle.

Note : A cycle remains unaltered if its elements are changed in cyclic order.

Def. Transposition : A cycle of length two is called a transposition.

Def. Disjoint cycle : Two cycles are said to be disjoint if they have no common element in their one-row representation.

Results :

1. The symmetric groups S_2 , S_3 and S_4 , when their elements are written in one-row representation , are given as

$$S_2 = \{I , (12)\}$$

$$S_3 = \{I , (12) , (13) , (23) , (123) , (132)\}$$

$$S_4 = \left\{ \begin{matrix} I , (12) , (13) , (14) , (23) , (24) , (34) , (123) , (124) , (132) , \\ (134) , (142) , (143) , (234) , (243) , (1234) , (1243) , (1324) , \\ (1342) , (1423) , (1432) , (12)(34) , (13)(24) , (14)(23) \end{matrix} \right\}$$

2. Disjoint cycles always commute with each other.
3. Non-disjoint cycles may or may not commute.
4. Inverse of a cycle can be obtained by reverting its elements or by keeping first element unchanged and reverting remaining ones.
5. Order of a n-cycle is n.
6. If $\sigma = \sigma_1 \sigma_2 .... \sigma_k$, where $\sigma_i' s$ are disjoint cycles, then $o(\sigma)$ = lcm $\{o(\sigma_1), o(\sigma_2).....o(\sigma_k)\}$.
7. Every permutation can be expressed as a product of disjoint cycles.
8. Every cycle can be expressed as product of transpositions in infinitely many ways. However, number of transpositions in any decomposition of a cycle remains either always even or always odd.

Def. Even and odd permutations : A permutation is said to be even (odd) if it can be expressed as product of even (odd) number of transpositions.

Def. Inversion of a permutation : Let $\sigma \in S_n$ be permutation and let $\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{pmatrix}$.

Then inversion of a symbol i is denoted by $\text{inv}(i)$ and is defined as $\text{inv}(i) =$ number of symbols less than $\sigma(i)$ and right to $\sigma(i)$ and inversion of permutation σ is defined as $\text{inv}(\sigma) = \sum_{i=1}^{n-1} \text{inv}(i)$.

Example : Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 7 & 6 & 5 & 4 \end{pmatrix}$. Then $\text{inv}(1) = 1$, $\text{inv}(2) = 1$, $\text{inv}(3) = 0$, $\text{inv}(4) = 3$, $\text{inv}(5) = 2$, $\text{inv}(6) = 1$, $\text{inv}(7) = 0$ and $\text{inv}(\sigma) = \sum_{i=1}^{7} \text{inv}(i) = 1+1+0+3+2+1+0 = 8$.

Def. Signature of a permutation : Let $\sigma \in S_n$ be any permutation then signature of σ is denoted by $\text{sgn}(\sigma)$ and is defined as $\text{sgn}(\sigma) = (-1)^{\text{inv}(\sigma)}$.

In the above example, $\text{sgn}(\sigma) = (-1)^8 = 1$.

Results on signature :

1. Signature of an even permutation is always 1.
2. Signature of an odd permutation is always -1.
3. $\text{sgn}(\sigma\eta) = \text{sgn}(\sigma)\,\text{sgn}(\eta)$.
4. $\text{sgn}(\sigma^{-1}) = \text{sgn}(\sigma)$.

Results :

1. A permutation can not be both even and odd.
2. A cycle of odd length is an even permutation and a cycle of even length is an odd permutation.
3. Identity permutation is an even permutation.
4. Product of two even permutations is an even permutation.

5. Product of two odd permutations is an even permutation.
6. Product of one even and one odd permutation is an odd permutation.
7. Inverse of an even permutation is even and of odd permutation is odd.
8. In S_n , there are always $\frac{n!}{2}$ even permutations and $\frac{n!}{2}$ odd permutations.
9. Let $\sigma \in S_n$ be a permutation corresponding to the partition
$n = \underbrace{n_1 + n_1 + ... + n_1}_{\alpha_1 \text{ times}} + \underbrace{n_2 + n_2 + ... + n_2}_{\alpha_2 \text{ times}} + ... + \underbrace{n_k + n_k + ... + n_k}_{\alpha_k \text{ times}}$. Then the number of elements in S_n having same cyclic decomposition as that of $\sigma = \dfrac{n!}{n_1^{\alpha_1} n_2^{\alpha_2} ..... n_k^{\alpha_k} \alpha_1 ! \alpha_2 ! .... \alpha_k !}$.

Corollary : The number of *r*-cycles in a symmetric group S_n is given by $\frac{{}^nP_r}{r}$.

10. The set of all even permutations of S_n forms a group denoted by A_n and is called alternating group of degree *n*. e.g.
$$A_4 = \begin{Bmatrix} I, (123), (124), (132), (134), (142), (143), \\ (234), (243), (12)(34), (13)(24), (14)(23) \end{Bmatrix}$$
11. If *H* is a subgroup of S_n, then either every member of *H* is an even permutation or exactly half of them are even.
12. Let α and β belong to S_n, then $\beta\alpha\beta^{-1}$ and α are both even or both odd.
13. Every element in A_n for $n \geq 3$ can be expressed as a 3-cycle or a product of 3-cycles.
14. If σ is a *n*-cycle then σ^m is a product of *d* cycles of length $\frac{n}{d}$, where $d = \gcd(m, n)$.
15. Let σ and η be any permutations in the symmetric group S_n, then $\sigma^{-1}\eta\sigma$ has the same cyclic decomposition as that of η.
16. Centre of symmetric groups :
    (i) $Z(S_1) = S_1$, $Z(S_2) = S_2$.
    (ii) $Z(S_n) = \{I\}$ for $n \geq 3$
17. Centre of alternating groups :
    (i) $Z(A_1) = A_1$, $Z(A_2) = A_2$, $Z(A_3) = A_3$
    (ii) $Z(A_n) = \{I\}$ for $n \geq 4$

Information Table for S_6 :

S.No.	Partition	Type of the element	Name of the element	Number of elements	Order	Even/Odd
1.	$6=1+1+1+1+1+1$	I	Identity	1	1	Even
2.	$6=1+1+1+1+2$	(1 2)	Transposition	15	2	Odd
3.	$6=1+1+1+3$	(1 2 3)	$3-$cycle	40	3	Even
4.	$6=1+1+4$	(1 2 3 4)	$4-$cycle	90	4	Odd
5.	$6=1+5$	(1 2 3 4 5)	$5-$cycle	144	5	Even
6.	$6=6$	(1 2 3 4 5 6)	$6-$cycle	120	6	Odd
7.	$6=1+1+2+2$	(1 2)(3 4)	product of transpositions	45	2	Even
8.	$6=2+2+2$	(1 2)(3 4)(5 6)	product of three transpositions	15	2	Odd
9.	$6=1+2+3$	(1 2)(3 4 5)	product of a transposition & a $3-$cycle	120	6	Odd
10.	$6=3+3$	(1 2 3)(4 5 6)	product of two $3-$cycles	40	3	Even
11.	$6=2+4$	(1 2)(3 4 5 6)	product of a transposition and a $4-$cycle	90	4	Even

720

Information table for A_7

S.No.	Partition	Type of the element	Name of the element	No. of elements	Order
1.	$7=1+1+1+1+1+1+1$	I	Identity	1	1
2.	$7=1+1+1+1+3$	(1 2 3)	$3-$cycle	70	3
3.	$7=1+1+5$	(1 2 3 4 5)	$5-$cycle	504	5
4.	$7=7$	(1 2 3 4 5 6 7)	$7-$cycle	720	7
5.	$1+1+1+2+2$	(1 2)(3 4)	product of transpositions	105	2
6.	$7=2+2+3$	(1 2)(3 4)(5 6 7)	product of two transposition and a $3-$cycle	210	6
7.	$7=1+3+3$	(1 2 3)(4 5 6)	product of two $3-$cycles	280	3
8.	$7=1+2+4$	(1 2)(3 4 5 6)	product of a transposition and a $4-$cycle	630	4

2520

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Exercise 2.1

1. Construct the information table for S_n, $1 \le n \le 7$.
2. Construct the O–D table for S_n, $1 \le n \le 7$.
3. Find the number of elements satisfying the equation $x^m = e$ in the symmetric groups S_n by considering the values $m = 2,3,4,5,6$ and $n = 2,3,4,5$.
4. For what values of *n*, the symmetric group S_n is abelian and non-abelian.
5. For what values of *n*, the symmetric group S_n is cyclic and non-cyclic.
6. Find the maximum order of an element of S_n for $1 \le n \le 15$.
7. Construct the information table for A_n, $1 \le n \le 7$.
8. Construct the O–D table for A_n, $1 \le n \le 7$.
9. Find the number of elements satisfying the equation $x^m = e$ in the alternating groups A_n by considering the values $m = 2,3,4,5,6$ and $n = 2,3,4,5$.
10. For what values of *n*, the symmetric group A_n is abelian and non-abelian.
11. For what values of *n*, the symmetric group A_n is cyclic and non-cyclic.
12. Find the maximum order of an element of A_n for $1 \le n \le 15$.

Answers

4. S_n is abelian for $n = 1,2$ and non-abelian for $n \ge 3$.
5. S_n is cyclic for $n = 1,2$ and non-cyclic for $n \ge 3$.
6. 1, 2, 3, 4, 6, 6, 12, 15, 20, 30, 30, 60, 60, 70, 105.
10. A_n is abelian for $n = 1,2,3$ and non-abelian for $n \ge 4$.
11. A_n is cyclic for $n = 1,2,3$ and non-cyclic for $n \ge 4$.
12. 1, 1, 3, 3, 5, 5, 7, 15, 15, 21, 21, 35, 35, 45, 105.

Exercise 2.2

1. Find the order of each of the following permutations.

   (i) (14) (ii) (147) (iii) (14762)
2. What is the order of a *k*-cycle $(a_1, a_2....a_k)$?

3. What is the order of each of the following permutations ?

(i) (124)(357) (ii) (124)(356) (iii) (124)(3578)

4. What is the order of each of the following permutations ?

(i) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 4 & 6 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}$

5. What is the order of the product of a pair of disjoint cycles of lengths 4 and 6 ?

6. Show that A_8 contains an element of order 15.

7. Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{bmatrix}$.

Compute each of the following.

(i) α^{-1} (ii) $\beta\alpha$ (iii) $\alpha\beta$

8. Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 5 & 4 & 7 & 6 & 8 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}$.

Write α and β as

(i) products of disjoint cycles. (ii) products of 2-cycles.

9. Do the odd permutations in S_n form a group ? Why ?

10. Let α and β belong to S_n. Prove that $\alpha^{-1}\beta^{-1}\alpha\beta$ is an even permutation.

11. How many odd permutations of order 4 does S_6 have ?

12. Prove that (1234) is not the product of 3-cycles.

13. Let $\beta \in S_7$ and suppose $\beta^4 = (2143567)$. Find β.

14. Let $\beta = (123)(145)$. Write β^{99} in cycle form.

15. Let $H = \{\sigma \in S_n : \sigma(3) = 3\}$ then show that H is a subgroup of S_n. What is the $o(H)$?

16. Let $\beta = (1,3,5,7,9,8,6)\,(2,4,10)$. What is the smallest positive integer n for which $\beta^n = \beta^{-5}$?

17. Let $\alpha = (1,3,5,7,9)(2,4,6)(8,10)$. If α^m is a 5-cycle, what can you say about m ?

18. Let $H = \{\beta \in S_5 \mid \beta(1) = 1 \text{ and } \beta(3) = 3\}$. Prove that H is a subgroup of S_5.

19. Suppose that β is a 10-cycle. For which integers i between 2 and 10 is β^i also a 10-cycle ?

20. In S_3, find elements α and β so that $|\alpha| = 2$, $|\beta| = 2$, and $|\alpha\beta| = 3$.

21. Find group elements α and β so that $|\alpha| = 3$, $|\beta| = 3$, and $|\alpha\beta| = 5$.

22. Show that a permutation with odd order must be an even permutation.

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Answers

1. (i)2 (ii) 3 (iii) 5 2. k 3. (i) 3 (ii) 3 (iii) 12 4. (i) 6 (ii) 12 5. 12

7. (i) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 3 & 4 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 1 & 5 & 3 & 4 \end{bmatrix}$

8. (i) $\alpha = (12)(45)(67),\ \beta = (23847)(56)$ (ii) $\alpha = (12)(45)(67),\ \beta = (27)(24)(28)(23)(56)$

9. No; the identity is even. 11. 90 13. $\beta = (2457136)$ 14. (13254) 15. 2

16. 16 17. $m = 6$ 19. 3, 7, 9 20. $\alpha = (12),\ \beta = (23)$

21. Let $\alpha = (123)$ and $\beta = (145)$

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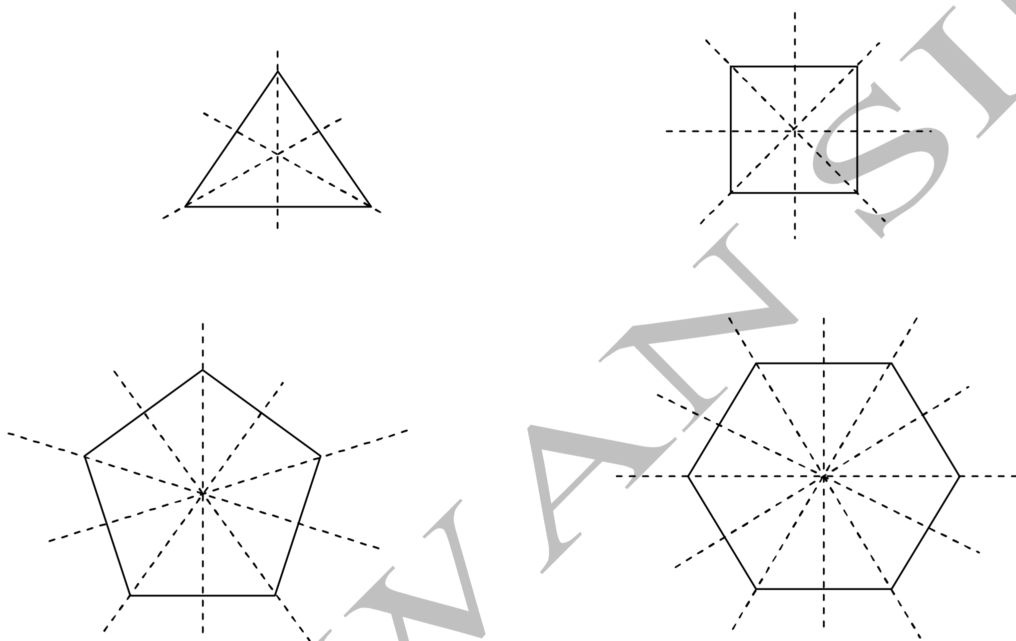
Group Theory - 3

Dihedral Groups

Def. Symmetry : An undetectable motion of an object is called its symmetry.

Def. Symmetry Line : A line along which a body is symmetric is called its symmetry line.

Result : A regular polygon with *n*-sides has *n*-symmetry lines shown below :



Def. Dihedral group : The set of all symmetries of a regular polygon of *n*-sides forms a group with respect to the composition (product) of symmetries and is known as *n*th dihedral group and is denoted by D_n.

Results :

1. The dihedral group D_n is of order $2n$ and it contains *n* rotations and *n* reflections.
2. Product of two rotations is again a rotation.
3. Product of one rotation and one reflection is a reflection.
4. Product of two reflections is a rotation.
5. Every reflection is either a transposition or a product of transpositions.
6. Order of a reflection is always 2.
7. Number of elements of order 2 in $D_n = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$
8. Number of elements of order 2 in any dihedral group is always odd.

9. Let $d(\neq 2)$ be a positive integer which divides n then D_n has $\phi(d)$ elements of order d.

10. Largest possible order of any element in D_n is n.

11. The number of subgroups of the dihedral group D_n is $\tau(n)+\sigma(n)$, where $\tau(n)$ denotes the number of positive divisors of n and $\sigma(n)$ denotes the sum of all positive divisors of n.

12. The number of cyclic subgroups of the dihedral group D_n is $\tau(n)+n$, where $\tau(n)$ denotes the number of positive divisors of n.

13. Centre of dihedral group : $Z(D_n)=\begin{cases}\{e\} & \text{, if } n \text{ is odd}\\ \left\{e,\ a^{\frac{n}{2}}\right\} & \text{, if } n \text{ is even}\end{cases}$

14. **Generator-relation form of dihedral group :** $D_n=\langle a,b : a^n=e,\ b^2=e,\ ab=ba^{-1}\rangle$

i.e., $$D_n=\begin{Bmatrix} e, & a, & a^2, & \dots & a^{n-1} \\ b, & ab, & a^2b, & \dots & a^{n-1}b \end{Bmatrix}\begin{matrix}\to \text{Rotations}\\ \to \text{Reflections}\end{matrix}$$

Composition table for D_3 :

	e	a	a^2	b	ab	a^2b
e	e	a	a^2	b	ab	a^2b
a	a	a^2	e	ab	a^2b	b
a^2	a^2	e	a	a^2b	b	ab
b	b	a^2b	ab	e	a^2	a
ab	ab	b	a^2b	a	e	a^2
a^2b	a^2	ab	b	a^2	a	e

Def. Group of quaternions Q_8 **:** The group of quaternions is given as $Q_8=\{1,-1,i,-i,j,-j,k,-k\}$ where $i^2=j^2=k^2=-1$ and $i\cdot j=k,\ j\cdot k=i,\ k\cdot i=j,\ j\cdot i=-k,\ k\cdot j=-i,\ i\cdot k=-j$

Composition table for Q_8 :

$\cdot$	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	i	$-i$	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

Remark : Some authors denote this group by Q_4.

Results :

1. Q_8 is a non-abelian group and hence non-cyclic.

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2. Every proper subgroup of Q_8 is cyclic.

3. Q_8 has six subgroups.

4. $Z(Q_8) = \{1, -1\}$

5. **Generator-relation form of** Q_8 **:** $Q_8 = \left\langle a, b : a^4 = e, a^2 = b^2 = e,\ ab = ba^{-1} \right\rangle$

Exercise 3.1

1. Construct the dihedral groups D_n for $3 \le n \le 7$. Also find inverse and order of each element and construct the O – D table.
2. Construct the O – D table for the dihedral groups $D_8, D_9, D_{20}, D_{25}, D_{30}, D_{36}$.
3. Construct the composition table for D_3 and D_4.
4. Find all subgroups of D_3, D_4, D_5. How many of them are cyclic and non-cyclic?
5. Construct the O – D table of the group of quaternions Q_8 and also find all subgroups of Q_8.
6. Show that every proper subgroup of Q_8 is cyclic.
7. Find the group of symmetries of the following figures :
   (i) Rectangle (non square) (ii) Rhombus (non square) (iii) Parallelogram (non rectangle)
8. To which groups the groups in above question are isomorphic.

Answers

2.

Order	No. of elements
1	1
2	9
4	2
8	4
	16

,

Order	No. of elements
1	1
2	9
3	2
9	6
	18

,

Order	No. of elements
1	1
2	21
4	2
5	4
10	4
20	8
	40

,

Order	No. of elements
1	1
2	25
5	4
25	20
	50

,

Order	No. of elements
1	1
2	31
3	2
5	4
6	2
10	4
15	8
30	8
	60

,

Order	No. of element
1	1
2	37
3	2
4	2
6	2
9	6
12	4
18	6
36	12
	72

4. D_3 : Total 6, cyclic 5, non-cyclic 1; D_4 : Total 10, cyclic 7, non-cyclic 3.

   D_5 : Total 8, cyclic 7, non-cyclic 1; D_6 : Total 16, cyclic 10, non-cyclic 6.

7. (i) $\{I,(13)(24),(12)(34),(14)(23)\}$ (ii) $\{I,(13)(24),(12)(34),(14)(23)\}$ (iii) $\{I,(13)(24)\}$

8. (i) K_4 (ii) K_4 (iii) $\mathbb{Z}_2$

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Group Theory - 4

Def. Direct product : Let $G_1, G_2, \ldots, G_n$ be a finite collection of groups. The direct product of $G_1, G_2, \ldots, G_n$ is denoted by $G_1 \times G_2 \times \ldots \times G_n$ and is defined as the set of all n-tuples for which the ith component is an element of G_i and the operation is componentwise.

In symbols $G_1 \times G_2 \times \ldots \times G_n = \{(g_1, g_2, \ldots, g_n) : g_i \in G_i\}$

where $(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n) = (a_1b_1, a_2b_2, \ldots, a_nb_n)$. It is easy to check that direct product of groups is itself a group.

Results :

1. The order of an element of a direct product of a finite number of finite groups is the least common multiple of the orders of the components of the element. In symbols,
$$o((g_1, g_2, \ldots, g_n)) = lcm\{o(g_1), o(g_2), \ldots, o(g_n)\}$$
2. Let G_1 and G_2 be finite cyclic groups then $G_1 \times G_2$ is cyclic if and only if $o(G_1)$ and $o(G_2)$ are co-prime.
3. Let G_1 and G_2 be two groups then $G_1 \times G_2$ is abelian iff G_1 and G_2 are abelian.
4. If H_1 is a subgroup of G_1 and H_2 is a subgroup of G_2 then $H_1 \times H_2$ is a subgroup of $G_1 \times G_2$. However every subgroup of $G_1 \times G_2$ is not of the form $H_1 \times H_2$.
5. Two finite groups are isomorphic if and only if they are of same order and have same O – D table.
6. The direct product $G_1 \times G_2 \times \ldots \times G_n$ of finite cyclic groups is cyclic if and only if $o(G_i)$ and $o(G_j)$ are co-prime for all $i \neq j$.
7. $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ if and only if $\gcd(m, n) = 1$.
8. $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_k} \cong \mathbb{Z}_{n_1 \cdot n_2 \cdots n_k}$ if and only if $\gcd(n_i, n_j) = 1 \ \forall \ i \neq j$.
9. Let m and n are co-prime positive integers then $U(mn) \cong U(m) \times U(n)$.
10. Let $n_1, n_2, \ldots, n_k$ are positive integers such that $\gcd(n_i, n_j) = 1$ for $i \neq j$ then
$$U(n_1 n_2 \ldots n_k) \cong U(n_1) \times U(n_2) \times \ldots \times U(n_k)$$
11. $U(2) \cong \mathbb{Z}_1$, $U(4) \cong \mathbb{Z}_2$, $U(2^n) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}$ for $n \geq 3$.

12. If p is an odd prime then $U\left(p^n\right) \cong \mathbb{Z}_{\phi\left(p^n\right)} = \mathbb{Z}_{p^n - p^{n-1}}$.

13. $U(n)$ is cyclic if and only if $n = 1$ or 2 or 4 or p^m or $2 \cdot p^m$ where p is an odd prime and m is a positive integer.

14. $x^{20} = 1$ for all $x \in U(100)$

Proof : $U(100) \cong U(4\times 25) \cong U(4)\times U(25) \cong Z_2 \times Z_{20}$. It can be easily seen that order of every element of $Z_2 \times Z_{20}$ is a divisor of 20. So, $x^{20} = 1$ for all $x \in U(100)$.

15. $x^{100} = 1$ for all $x \in U(1000)$.

Proof : $U(1000) \cong U(8\times 125) \cong U(8)\times U(125) \cong Z_2 \times Z_2 \times Z_{100}$. It can be easily seen that order of every element of $Z_2 \times Z_2 \times Z_{100}$ is a divisor of 100. So, $x^{100} = 1$ for all $x \in U(1000)$.

Exercise 4.1

Construct O – D table for following groups :

1. $\mathbb{Z}_2 \times \mathbb{Z}_4$
2. $\mathbb{Z}_3 \times \mathbb{Z}_6$
3. $\mathbb{Z}_4 \times \mathbb{Z}_6$
4. $\mathbb{Z}_{10} \times \mathbb{Z}_{15}$
5. $\mathbb{Z}_8 \times U(9)$
6. $\mathbb{Z}_6 \times U(15)$
7. $\mathbb{Z}_4 \times U(10)$
8. $\mathbb{Z}_3 \times S_3$
9. $U(20) \times U(25)$
10. $U(6) \times S_4$
11. $S_3 \times S_3$
12. $S_3 \times U(10)$
13. $\mathbb{Z}_3 \times U(10) \times S_3$
14. $\mathbb{Z}_{10} \times A_4$
15. $U(10) \times A_4$
16. $A_3 \times A_4$
17. $S_3 \times A_3$
18. $S_2 \times S_3 \times A_3$
19. $\mathbb{Z}_{10} \times D_3$
20. $U(12) \times D_4$
21. $S_3 \times D_3$
22. $A_3 \times D_5$
23. $D_4 \times D_5$
24. $\mathbb{Z}_2 \times U(3) \times S_2 \times A_3 \times D_3$
25. $U(10) \times A_3 \times D_3$
26. $D_4 \times Q_8$
27. $\mathbb{Z}_8 \times Q_8$
28. $U(10) \times Q_8$
29. $S_3 \times Q_8$
30. $A_4 \times Q_8$
31. $U(16)$
32. $U(32)$
33. $U(100)$
34. $U(120)$

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Answers

1.

Order	No. of elements
1	1
2	3
4	4
	8

2.

Order	No. of elements
1	1
2	1
3	8
6	8
	18

3.

Order	No. of elements
1	1
2	3
3	2
4	4
6	6
12	8
	24

4.

Order	No. of elements
1	1
2	1
3	2
5	24
6	2
10	24
15	48
30	48
	150

5.

Order	No. of elements
1	1
2	3
3	2
4	4
6	6
8	8
12	8
24	16
	48

6.

Order	No. of elements
1	1
2	7
3	2
4	8
6	14
12	16
	48

7.

Order	No. of elements
1	1
2	3
4	12
	16

8.

Order	No. of elements
1	1
2	3
3	8
6	6
	18

9.

Order	No. of elements
1	1
2	7
4	24
5	4
10	28
20	96
	160

10.

Order	No. of elements
1	1
2	19
3	8
4	12
6	8
	48

11.

Order	No. of elements
1	1
2	15
3	8
6	12
	36

12.

Order	No. of elements
1	1
2	7
3	2
4	8
6	2
12	4
	24

13.

Order	No. of elements
1	1
2	7
3	8
4	8
6	20
12	28
	72

14.

Order	No. of elements
1	1
2	7
3	8
5	4
6	8
10	28
15	32
30	32
	120

15.

Order	No. of elements
1	1
2	7
3	8
4	8
6	8
12	16
	48

16.

Order	No. of elements
1	1
2	3
3	26
6	6
	36

17.

Order	No. of elements
1	1
2	3
3	8
6	6
	18

18.

Order	No. of elements
1	1
2	7
3	8
6	20
	36

19.

Order	No. of elements
1	1
2	7
3	2
5	4
6	2
10	28
15	8
30	8
	60

20.

Order	No. of elements
1	1
2	23
4	8
	32

21.

Order	No. of elements
1	1
2	15
3	8
6	12
	36

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22.

Order	No. of elements
1	1
2	5
3	2
5	4
6	10
15	8
	30

23.

Order	No. of elements
1	1
2	35
4	12
5	4
10	20
20	8
	80

24.

Order	No. of elements
1	1
2	31
3	8
6	104
	144

25.

Order	No. of elements
1	1
2	7
3	8
4	8
6	20
12	28
	72

26.

Order	No. of elements
1	1
2	11
4	52
	64

27.

Order	No. of elements
1	1
2	3
4	28
8	32
	64

28.

Order	No. of elements
1	1
2	3
4	28
	32

29.

Order	No. of elements
1	1
2	7
3	2
4	24
6	2
12	12
	48

30.

Order	No. of elements
1	1
2	7
3	8
4	24
6	8
12	48
	96

31.

Order	No. of elements
1	1
2	3
4	4
	8

32.

Order	No. of elements
1	1
2	3
4	4
8	8
	16

33.

Order	No. of elements
1	1
2	3
4	4
5	4
10	12
20	16
	40

34.

Order	No. of elements
1	1
2	15
4	16
	32

Exercise 4.2

1. Show that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has seven subgroups of order 2.
2. Is $\mathbb{Z}_3 \times \mathbb{Z}_9$ isomorphic to $\mathbb{Z}_{27}$? Why ?
3. Is $\mathbb{Z}_3 \times \mathbb{Z}_5$ isomorphic to $\mathbb{Z}_{15}$? Why ?
4. Explain why $\mathbb{Z}_8 \times \mathbb{Z}_4$ and $\mathbb{Z}_{8000000} \times \mathbb{Z}_{4000000}$ must have the same number of elements of order 4.
5. What are the last three digits of 17^{102} ?
6. What is the order of any nonidentity element of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$?
7. How many subgroups of order 4 does $\mathbb{Z}_4 \times \mathbb{Z}_2$ have ?
8. The group $S_3 \times \mathbb{Z}_2$ is isomorphic to one of the following groups : $\mathbb{Z}_{12}, \mathbb{Z}_6 \times \mathbb{Z}_2, A_4, D_6$. Determine which one by elimination.
9. Find a subgroup of $\mathbb{Z}_4 \times \mathbb{Z}_2$ that is not of the form $H \times K$ where H is a subgroup of $\mathbb{Z}_4$ and K is a subgroup of $\mathbb{Z}_2$.
10. Find all subgroups of order 3 in $\mathbb{Z}_9 \times \mathbb{Z}_3$.
11. Find all subgroups of order 4 in $\mathbb{Z}_4 \times \mathbb{Z}_4$.
12. What is the largest order of any element in $\mathbb{Z}_{30} \times \mathbb{Z}_{20}$?
13. How many elements of order 2 are in $\mathbb{Z}_{2000000} \times \mathbb{Z}_{4000000}$?
14. Prove that $D_3 \times D_4 \ncong D_{24}$.
15. Determine the number of cyclic subgroups of order 15 in $\mathbb{Z}_{90} \times \mathbb{Z}_{36}$.
16. If a finite abelian group has exactlty 24 elements of order 6, how many cyclic subgroups of order 6 does it have ?
17. The group $\mathbb{Z}_2 \times D_3$ is isomorphic to one of the following : $\mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3, A_4, D_6$. Determine which one by elimination.
18. Determine the number of elements of order 15 and the number of cyclic subgroups of order 15 in $\mathbb{Z}_{30} \times \mathbb{Z}_{20}$.
19. Express $U(165)$ as an direct product of cyclic additive groups of the form $\mathbb{Z}_n$.

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20. Express $U(165)$ as an direct product of U–groups in three different ways.

21. Without doing any calculations in $U(27)$, decide how many subgroups $U(27)$ has.

22. What is the largest order of any element in $U(900)$?

23. Prove that $U(55)$ is isomorphic to $U(75)$.

24. Prove that $U(144)$ is isomorphic to $U(140)$.

Answers

2. No 3. Yes

4. Each of $\mathbb{Z}_8, \mathbb{Z}_4, \mathbb{Z}_{8000000}$ and $\mathbb{Z}_{4000000}$ has a unique subgroup of order 4. If $|(a,b)| = 4$ then a and b both belong to the unique subgroup of order 4. So the number of choices for a and b (actually 12 in all) is the same in either group.

5. 289 6. 3 7. 3 8. D_6 9. $\{(0,0),(3,1),(2,0),(1,1)\}$

10. $H_1 = \langle(3,0)\rangle = \{(0,0),(3,0),(6,0)\}$, $H_2 = \langle(3,1)\rangle = \{(0,0),(3,1),(6,2)\}$,
$H_3 = \langle(3,2)\rangle = \{(0,0),(3,2),(6,1)\}$, $H_4 = \langle(0,1)\rangle = \{(0,0),(0,1),(0,2)\}$

11. $H_1 = \langle(0,1)\rangle = \{(0,0),(0,1),(0,3),(0,2)\}$, $H_2 = \langle(1,0)\rangle = \{(0,0),(1,0),(2,0),(3,0)\}$,
$H_3 = \langle(1,1)\rangle = \{(0,0),(1,1),(2,2),(3,3)\}$, $H_4 = \langle(1,2)\rangle = \{(0,0),(1,2),(2,0),(3,2)\}$
$H_5 = \langle(1,3)\rangle = \{(0,0),(1,3),(2,2),(3,1)\}$, $H_6 = \langle(2,1)\rangle = \{(0,0),(2,1),(0,2),(2,3)\}$,
$H_7 = K_4 = \{(0,0),(0,2),(2,0),(2,2)\}$

12. 60 13. 3

14. $D_3 \times D_4$ has 23 elements of order 2 and D_{24} has 25 elements of order 2.

15. 4 16. 12 17. D_6 18. 48; 6 19. $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{10}$

20. $U(165) \cong U(11) \times U(15) \cong U(5) \times U(33) \cong U(3) \times U(55)$ 21. 6 22. 60

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Group Theory - 5

Survey of Groups

1. *p*-theorem : A group of prime order is always cyclic.

or

Let G be a group of order p, where p is a prime, then $G \cong \mathbb{Z}_p$.

or

If p is a prime then upto isomorphism there is only one group of order p, namely $\mathbb{Z}_p$.

2. p^2 theorem : A group of order p^2 is always abelian, where p is a prime

or

There does not exists a non-abelian groups of order p^2, where p is prime.

or

If p is a prime, then upto isomorphism there are only two groups of order p^2, namely $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p \times \mathbb{Z}_p$

3. p^3-thereom : If p is a prime, then upto isomorphism there are three abelian groups of order p^3, namely, $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ and $\mathbb{Z}_{p^3}$ and two non abelian groups of order p^3.

4. *pq*-theorem : Let p and q be two primes such that $p < q$, then

Case (i) : If p does not divide $q-1$,then up to isomorphism, there is only one group of order pq, namely $\mathbb{Z}_{pq}$ (cyclic).

Case (ii) : If p divides $q-1$, then upto isomorphism, there are only two groups of order pq. Out of these two groups, one is $\mathbb{Z}_{pq}$ (cyclic) and other is non abelian. The non abelian group is given by

$$G = \left\langle a, b: a^p = 1,\ b^q = 1,\ a^{-1}ba = b^r,\ r^p \equiv 1 \pmod p,\ r \not\equiv 1 \pmod q \right\rangle$$

5. p^2q theorem : Let G be a group of order p^2q such that p does not divide $q-1$ and q does not divide p^2-1 then the group G must be abelian.

By this theorem we conclude that the groups of order 45 and 99 are abelian.

6. Fundamental theorem for finite abelian groups : Let $n = p_1^{n_1} p_2^{n_2} \ldots . p_k^{n_k}$ where p_i's are the distinct primes, then the number of non isomorphic abelian groups of order n are $p(n_1)\, p(n_2) \ldots .. p(n_k)$, where $p(n_i)$ denotes the number of partitions of n_i.

Remark : Fundamental theorem does not depend upon the base, it depends upon the powers only.

Exercise 5.1

How many non isomorphic groups of following orders are there. Also construct them.

1. 6 2. 10 3. 14 4. 22 5. 26 6. 15 7. 21

8. 33 9. 39 10. 35 11. 55 12. 65 13. 85 14. 95

Construct all non isomorphic abelian groups of the following orders :

15. 16 16. 20 17. 80 18. 108 19. 120

Answers

1. 2 2. 2 3. 2 4. 2 5. 2 6. 1 7. 2

8. 1 9. 2 10. 1 11. 2 12. 1 13. 1 14. 1

15. $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$; $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4$; $\mathbb{Z}_2\times\mathbb{Z}_8$; $\mathbb{Z}_4\times\mathbb{Z}_4$; $\mathbb{Z}_{16}$ 16. $\mathbb{Z}_2\times\mathbb{Z}_{10}$; $\mathbb{Z}_{20}$

17. $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{10}$; $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{20}$; $\mathbb{Z}_2\times\mathbb{Z}_{40}$; $\mathbb{Z}_4\times\mathbb{Z}_{20}$; $\mathbb{Z}_{80}$

18. $\mathbb{Z}_3\times\mathbb{Z}_6\times\mathbb{Z}_6$; $\mathbb{Z}_2\times\mathbb{Z}_6\times\mathbb{Z}_9$; $\mathbb{Z}_2\times\mathbb{Z}_{54}$; $\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_{12}$; $\mathbb{Z}_9\times\mathbb{Z}_{12}$; $\mathbb{Z}_{108}$

19. $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{30}$; $\mathbb{Z}_2\times\mathbb{Z}_{60}$; $\mathbb{Z}_{120}$

Partial survery table of groups of order upto 50 :

Order of the group	No. of abelian groups	Name of abelian groups	No. of non – abelian groups	Name of non – abelian groups	Total no. of groups
1	1	$\mathbb{Z}_1$	0	–	1
2	1	$\mathbb{Z}_2$	0	–	1
3	1	$\mathbb{Z}_3$	0	–	1
4	2	$\mathbb{Z}_4,\mathbb{Z}_2\times\mathbb{Z}_2$	0	–	2
5	1	$\mathbb{Z}_5$	0	–	1
6	1	$\mathbb{Z}_6$	1	S_3	2
7	1	$\mathbb{Z}_7$	0	–	1
8	3	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2,\mathbb{Z}_2\times\mathbb{Z}_4,\mathbb{Z}_8$	2	D_4,Q_8	5
9	2	$\mathbb{Z}_9,\mathbb{Z}_3\times\mathbb{Z}_3$	0	–	2
10	1	$\mathbb{Z}_{10}$	1	D_5	2

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Order of the group	No. of abelian groups	Name of abelian groups	No. of non – abelian groups	Name of non – abelian groups	Total no. of groups
11	1	$\mathbb{Z}_{11}$	0	–	1
12	2	$\mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_6$	3	$D_6, A_4, ...$	5
13	1	$\mathbb{Z}_{13}$	0	–	1
14	1	$\mathbb{Z}_{14}$	1	D_7	2
15	1	$\mathbb{Z}_{15}$	0	–	1
16	5	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_8,$ $\mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_{16}$	9	$D_8, \mathbb{Z}_2 \times D_4,$ $\mathbb{Z}_2 \times Q_8, ...$	14
17	1	$\mathbb{Z}_{17}$	0	–	1
18	2	$\mathbb{Z}_3 \times \mathbb{Z}_6, \mathbb{Z}_{18}$	3	$D_9, \mathbb{Z}_3 \times S_3, ...$	5
19	1	$\mathbb{Z}_{19}$	0	–	1
20	2	$\mathbb{Z}_{20}, \mathbb{Z}_2 \times \mathbb{Z}_{10}$	3	$D_{10}, ...$	5

Order of the group	No. of abelian groups	Name of abelian groups	No. of non – abelian groups	Name of non – abelian groups	Total no. of groups
21	1	$\mathbb{Z}_{21}$	1	–	2
22	1	$\mathbb{Z}_{22}$	1	D_{11}	2
23	1	$\mathbb{Z}_{23}$	0	–	1
24	3	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6, \mathbb{Z}_2 \times \mathbb{Z}_{12}, \mathbb{Z}_{24}$	12	$D_{12}, S_4, \mathbb{Z}_2 \times D_6, \mathbb{Z}_2 \times A_4, \mathbb{Z}_3 \times D_4,$ $\mathbb{Z}_3 \times Q_8, \mathbb{Z}_4 \times S_3, \mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3, ...$	15
25	2	$\mathbb{Z}_{25}, \mathbb{Z}_5 \times \mathbb{Z}_5$	0	–	2
26	1	$\mathbb{Z}_{26}$	1	D_{13}	2
27	3	$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_9, \mathbb{Z}_{27}$	2	– –	5
28	2	$\mathbb{Z}_2 \times \mathbb{Z}_{14}, \mathbb{Z}_{28}$	2	$D_{14}, -$	4
29	1	$\mathbb{Z}_{29}$	0	–	1
30	1	$\mathbb{Z}_{30}$	3	$D_{15}, \mathbb{Z}_3 \times D_5, \mathbb{Z}_5 \times S_3$	4

Order of the group	No. of abelian groups	Name of abelian groups	No. of non – abelian groups	Name of non – abelian groups	Total no. of groups
31	1	$\mathbb{Z}_{31}$	0	–	1
32	7	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_8,\mathbb{Z}_2\times\mathbb{Z}_{16}$, $\mathbb{Z}_{32},\mathbb{Z}_4\times\mathbb{Z}_8,\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_4$	44	$D_{16},\mathbb{Z}_2\times\mathbb{Z}_2\times D_4$, $\mathbb{Z}_4\times D_4,\mathbb{Z}_2\times\mathbb{Z}_2\times Q_8$, $\mathbb{Z}_4\times Q_8,\mathbb{Z}_2\times Q_8,\ldots$	51
33	1	$\mathbb{Z}_{33}$	0	–	1
34	1	$\mathbb{Z}_{34}$	1	D_{17}	2
35	1	$\mathbb{Z}_{35}$	0	–	1
36	4	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_9,\mathbb{Z}_4\times\mathbb{Z}_3\times\mathbb{Z}_3$, $\mathbb{Z}_4\times\mathbb{Z}_9$	10	–	14
37	1	$\mathbb{Z}_{37}$	0	–	1
38	1	$\mathbb{Z}_{38}$	1	D_{19}	2
39	1	$\mathbb{Z}_{39}$	1	–	2
40	3	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5$, $\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_5,\mathbb{Z}_8\times\mathbb{Z}_5$	11	$D_{20},\ldots$	14

Order of the group	No. of abelian groups	Name of abelian groups	No. of non – abelian groups	Name of non – abelian groups	Total no. of groups
41	1	$\mathbb{Z}_{41}$	0	–	1
42	1	$\mathbb{Z}_{42}$	5	–	6
43	1	$\mathbb{Z}_{43}$	0	–	1
44	2	$\mathbb{Z}_4\times\mathbb{Z}_{11},\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{11}$	2	$D_{22},\mathbb{Z}_2\times D_{11}$	4
45	2	$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_5,\mathbb{Z}_9\times\mathbb{Z}_5$	0	–	2
46	1	$\mathbb{Z}_{46}$	1	D_{23}	2
47	1	$\mathbb{Z}_{47}$	0	–	1
48	5	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_3,\mathbb{Z}_2\times\mathbb{Z}_8\times\mathbb{Z}_3$, $\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_3,\mathbb{Z}_{16}\times\mathbb{Z}_3\cong\mathbb{Z}_{48}$	47	$D_{24},---$	52
49	2	$\mathbb{Z}_7\times\mathbb{Z}_7,\mathbb{Z}_{49}$	0	–	2
50	2	$\mathbb{Z}_2\times\mathbb{Z}_5\times\mathbb{Z}_5$, $\mathbb{Z}_2\times\mathbb{Z}_{25}\cong\mathbb{Z}_{50}$	3	$D_{25},---$	5

Exercise 5.2

1. What is the smallest positive integer n such that there are two nonisomorphic groups of order n ?
2. What is the smallest positive integer n such that there are three nonisomorphic abelian groups of order n?
3. What is the smallest positive integer n such that there are exactly four nonisomorphic abelian groups of order n ?
4. Show that there are two abelian groups of order 108 that have exactly one subgroup of order 3.
5. Show that there are two abelian groups of order 108 that have exactly four subgroup of order 3.
6. Show that there are two abelian groups of order 108 that have exactly 13 subgroup of order 3.
7. Suppose that G is an abelian group of order 120 and that G has exactly three elements of order 2. Determine the isomorphism class of G.
8. How many abelian groups (up to isomorphism) are there
   (i) of order 6 ?
   (ii) of order 15 ?
   (iii) of order 42 ?
   (iv) of order pq where p and q are distinct primes ?
   (v) of order pqr where p, q and r are distinct primes ?
   (vi) Generalize parts (i), (ii), (iii), (iv) and (v).
9. How does the number (up to isomorphism) of abelian groups of order n compare with the number (up to isomorphism) of abelian groups of order m where
   (i) $n=3^2$ and $m=5^2$?
   (ii) $n=2^4$ and $m=5^4$
   (iii) $n=p^r$ and $m=q^r$ where p and q are prime ?
   (iv) $n=p^r$ and $m=p^r q$ where p and q are distinct primes ?
   (v) $n=p^r$ and $m=p^r q^2$ where p and q are distinct primes ?
10. Characterize those integers n such that the only abelian groups of order n are cyclic.
11. Characterize those integers n such that any abelian group of order n belongs to one of exactly four isomorphism classes.
12. The set $G=\{1,4,11,14,16,19,26,29,31,34,41,44\}$ is a group under multiplication modulo 45. Determine the isomorphism class of this group.

13. Suppose that G is an abelian group of order 9. What is the maximum number of elements (excluding the identity) of which one needs to compute the order to determine the isomorphism class of G ?

14. Suppose that G is an abelian group of order 16, and in computing the orders of its elements, you come across an element of order 8 and two elements of order 2. Explain why no further computations are needed to determine the isomorphism class of G.

Answers

1. $n = 4;\ \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2$ 2. $n = 8;\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_2 \times \mathbb{Z}_4,\ \mathbb{Z}_8$

3. $n = 36;\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3,\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9,\ \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{Z}_9$ 4. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{27};\ \mathbb{Z}_{108}$

5. $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_9;\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_9$ 6. $\mathbb{Z}_3 \times \mathbb{Z}_6 \times \mathbb{Z}_6;\ \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{12}$ 7. $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_5$

8. (i) 1 (ii) 1 (iii) 1 (iv) 1 (v) 1

(vi) There is a unique abelian group of order n iff n is not divisible by the square of any prime.

9. (i) Equal (ii) Equal (iii) Equal (iv) Equal (v) Twice

10. $n = p_1 p_2 ....p_k$, where p_i's are distinct primes

11. $n = p_1^2 p_2^2$ or $p_1^2 p_2^2 p_3 ....p_k$, where p_i's are distinct primes.

12. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ 13. 3 14. $\mathbb{Z}_2 \times \mathbb{Z}_8$

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Group Theory - 6

Normal subgroups and Quotient groups

Coset : Let G be a group and H be its subgroup. For any element $a \in G$, the set $Ha = \{ha : h \in H\}$ is called right coset of H in G, and the set $aH = \{ah : h \in H\}$ is called left coset of H in G. Note that if the binary operation in G is additive, then the right coset of H in G is $H + a = \{h + a : h \in H\}$. Similarly , the left coset is $a + H = \{a + h : h \in H\}$.

Results on cosets :

1. Let H be any subgroup of a group G, then $Ha = aH = H$ if and only if $a \in H$.
2. Any two left (right) cosets of a subgroup are either disjoint or identical i.e. either $aH = bH$ or $aH \cap bH = \phi$.
3. If H is a subgroup of a group G, then G is equal to the union of all left (right) cosets of H in G.
4. Let H be any subgroup of a group G, then for $a, b \in G$, $Ha = Hb$ if and only if $ab^{-1} \in H$. Similarly $aH = bH$ iff $ab^{-1} \in H$.
5. Let H be a subgroup of a group G. The number of distinct left cosets of H in G is equal to number of distinct right cosets of H in G.
6. The number of distinct right (left) cosets of H in G is called the index of H in G and is denoted by $[G : H]$. For a finite group G, $[G : H] = \dfrac{o(G)}{o(H)}$.

Def. Normal Subgroup : A subgroup H of a group G is said to be a normal sub group of G if $Ha = aH$ for all $a \in G$ i.e. right and left cosets of H are same for each element of G. We denote this fact by $H \Delta G$. For a group G, G and $\{e\}$ are always normal subgroups of G and are called trivial normal subgroups. Any other normal subgroup, if exists, is called proper normal subgroups.

Def. Simple group : A group is said to be simple group if it has no proper normal subgroups. In other words, a group G is said to be simple group if its only normal subgroups are $\{e\}$ and G.

Example : The alternating group A_n, $n \neq 4$, is always a simple group.

Result : A_4 is not a simple group because it has a proper normal subgroup given by $K_4 = V_4 = \{I, (12)(34), (13)(24), (14)(23)\}$. This subgroup is called Klein's four group.

Results on normal subgroups :

1. A subgroup H of a group G is normal if and only if $g^{-1}hg \in H$ for all $h \in H, g \in G$.
2. Every subgroup of an abelian group is normal.

3. Let G be any group and H is a subgroup of G such that, $[G : H] = 2$ then H is normal subgroup of G.
4. H is a normal subgroup of G if and only if , $HaHb = Hab$ for all a , $b \in G$ i.e. product of two right cosets of H in G is again a right cosets of H in G.
5. Intersection of two normal subgroups of G is again a normal subgroup of G.
6. Let H be a normal subgroup of G and K be a subgroup of G, such that , $H \subseteq K \subseteq G$ then H is a normal subgroup of K.
7. If H and K are normal subgroup of G, then HK is also a normal subgroup of G.
8. Let H be a normal subgroup of G and K be a subgroup of G then HK is a subgroup of G.

Def. Centre of a group : Let G be a group then centre of the group G is defined to be the subset of all elements of G which commute with every element of G and it is denoted by $Z(G)$. In symbols,

$$Z(G) = \{a \in G : ax = xa \text{ for all } x \text{ in } G\}$$

Results :

1. $Z(G)$ is a normal subgroup of G.
2. G is abelian iff $G = Z(G)$

Def. Quotient Group : Let G be a group and H be a normal subgroup of G. Define $G/H = \{Ha : a \in G\}$ as the set of all right (or left) cosets of H in G. Now , let Ha , $Hb \in G/H$, then the set G/H together with the binary composition defined by, $HaHb = Hab$ is a group and is called the quotient group or factor group. The identity element of G/H is H.

Results on quotient groups :

1. Every quotient group of an abelian group is abelian . But converse may not be true.
2. Every quotient group of a cyclic group is cyclic. But converse may not be true.
3. If G is a finite group and H be a normal subgroup of G then $o(G/H) = \dfrac{o(G)}{o(H)} = [G : H]$.
4. Let H and K be two normal subgroups of a group G such that $H \subseteq K$, then K/H is a normal subgroup G/H .
5. Every subgroup of G / H is of the form K / H where K is a subgroup of G and H is a normal subgroup of K.
6. Let H be a normal subgroup of G and G / H be the quotient group, then order of aH in G / H divides the order of a in G.
7. Let G be a group and $Z(G)$ be the centre of G. If $G / Z(G)$ is cyclic then G is abelian. Contrapositive of this statement is that, if G is non abelian then $G / Z(G)$ is not cyclic. We can also express this fact as, if $G / Z(G)$ is cyclic then $G = Z(G)$ i.e., $G / Z(G)$ is trivial group.

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Exercise 6.1

1. Find all normal subgroups of S_3 and S_4. For every normal subgroup construct the corresponding quotient group. Also find the isomorphism class of the quotient group.
2. Find all normal subgroups of A_4. For every normal subgroup construct the corresponding quotient group. Also find the isomorphism class of the quotient group.
3. Find all normal subgroups of Q_8. For every normal subgroup construct the corresponding quotient group. Also find the isomorphism class of the quotient group.
4. Find all normal subgroups of D_3, D_4, D_5. For every normal subgroup construct the corresponding quotient group. Also find the isomorphism class of the quotient group.
5. Find two subgroups H and K of the dihedral group D_4 such that H is a normal subgroup of K and K is a normal subgroup of D_4 but H is not a normal subgroup of D_4.

Answers

1. Normal subgroups of S_3 are $\{I\}, A_3, S_3$ and quotient groups are $S_3/\{I\} \cong S_3$, $S_3/A_3 \cong \mathbb{Z}_2$, $S_3/S_3 \cong \{e\}$

   Normal subgroups of S_4 are $\{I\}, K_4, A_4, S_4$

   and quotient groups are $S_4/\{I\} \cong S_4$, $S_4/K_4 \cong S_3$, $S_4/A_4 \cong \mathbb{Z}_2$, $S_4/S_4 \cong \{e\}$
2. Normal subgroups of A_4 are $\{I\}, K_4, A_4$ and quotient groups are $A_4/\{I\} \cong A_4$, $A_4/K_4 \cong \mathbb{Z}_3$, $A_4/A_4 \cong \{e\}$
3. Normal subgroups of Q_8 are $\{I\}, H_1 = \{1,-1\}$, $H_2 = \{1,-1,i,-i\}$, $H_3 = \{1,-1,j,-j\}$, $H_4 = \{1,-1,k,-k\}, Q_8$

   and quotient groups are $Q_8/\{I\} \cong Q_8$, $Q_8/H_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $Q_8/H_i \cong \mathbb{Z}_2$ for i = 2, 3 and 4, $Q_8/Q_8 \cong \{e\}$
4. Normal subgroups of D_3 are $\{I\}, H = \{e, a, a^2\}, D_3$ and quotient groups are $D_3/\{I\} \cong D_3$,

   $D_3/H \cong \mathbb{Z}_2, D_3/D_3 \cong \{e\}$

   Normal subgroups of D_4 are $\{I\}, H_1 = \{e, a, a^2, a^3\}, H_2 = \{e, a^2, b, a^2b\}, H_3 = \{e, a^2, ab, a^3b\}, D_4$ and quotient groups are $D_4/\{I\} \cong D_4$, $D_4/H_i \cong \mathbb{Z}_2$ for $i = 1, 2$ and 3, $D_4/D_4 \cong \{e\}$

   Normal subgroups of D_5 are $\{I\}, H = \{e, a, a^2, a^3, a^4\}, D_5$ and quotient groups are $D_5/\{I\} \cong D_5$,

   $D_5/H \cong \mathbb{Z}_2, D_5/D_5 \cong \{e\}$
5. $H = \{e, a^2\}, K = \{e, a^2, b, a^2b\}$

Exercise 6.2

1. Let $G = \mathbb{Z}_4 \times U(4)$, $H = \langle (2,3) \rangle$ and $K = \langle (2,1) \rangle$. Determine the isomorphism class of G/H and G/K. (This shows that $H \cong K$ does not imply $G/H \cong G/K$.)
2. What is the order of element $5+\langle 6 \rangle$ in the factor group $\mathbb{Z}_{18}/\langle 6 \rangle$?
3. What is the order of the element $14+\langle 8 \rangle$ in the factor group $\mathbb{Z}_{24}/\langle 8 \rangle$?
4. What is the order of the factor group $\mathbb{Z}_{60}/\langle 15 \rangle$?
5. What is the order of the factor group $\left(\mathbb{Z}_{10} \times U(10)/\langle (2,9) \rangle\right)$?
6. Is $U(30)/U_5(30)$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$?
7. The group $(\mathbb{Z}_4 \times \mathbb{Z}_{12})/\langle (2,2) \rangle$ is isomorphic to one of $\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Which one ?
8. Let $G = U(32)$ and $H = \{1, 31\}$. The group G/H is isomorphic to one of $\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Which one ?
9. Let $G = U(16)$, $H = \{1, 15\}$ and $K = \{1, 9\}$. Are *H* and *K* isomorphic ? Are G/H and G/K isomorphic ?
10. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_4$, $H = \{(0,0), (2,0), (0,2), (2,2)\}$ and $K = \langle (1,2) \rangle$. Is G/H isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$? Is G/K isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$?
11. Show, by example, that in a factor group G/H it can happen that $aH = bH$ but $|a| \neq |b|$.
12. Let *N* be a normal subgroup of a group *G*. If *N* is cyclic, prove that every subgroup of *N* is also normal in *G*.
13. If $|G| = 30$ and $|Z(G)| = 5$, what is the structure of $G/Z(G)$?
14. If *H* is a normal subgroup of *G*, and $|H| = 2$, prove that *H* is contained in the center of *G*.
15. Let *G* be a group. If $H = \{g^2 \mid g \in G\}$ is a subgroup of *G*, prove that it is a normal subgroup of *G*.

Answers

1. $G/H \cong \mathbb{Z}_4, G/K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ 2. 6 3. 4 4. 15 5. 4

6. $\mathbb{Z}_4$ 7. $\mathbb{Z}_8$ 8. $\mathbb{Z}_8$ 9. Yes; No 10. $\mathbb{Z}_4$

11. Take $G = \mathbb{Z}_6$, $H = \{0, 3\}, a = 1, b = 4$ 13. $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$

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Group Theory - 7

Homomorphisms

Def. Homomorphism : Let $(G, *)$ and (G', o) be two groups. A mapping $f : G \to G'$ is called a homomorphism if $f(a*b) = f(a) \text{ o } f(b)$ for all $a, b \in G$.

Definitions :

1. A homomorphism which is one-one is called monomorphism.
2. A homomorphism which is onto is called epimorphism.
3. A homomorphism which is one-one and onto is called isomorphism.
4. A homomorphism from a group G to itself is called endomorphism .

Def. Kernel of a homomorphism : Let $f : G \to G'$ be a homomorphism, the kernel of f is defined by Ker $f = \{g \in G : f(g) = e'\}$ where e' is the identity of G'.

Some results : Let $f : G \to G'$ be a homomorphism then

1. $f(e) = e'$ where e and e' are identities of G and G' respectively.
2. $(f(a))^{-1} = f(a^{-1})$ for all $a \in G$.
3. $f(a^n) = (f(a))^n$ for all $a \in G$.
4. if $o(a) = n$ then $o(f(a))$ divides n.
5. If $f(a) = b$ then $f^{-1}(b) = a$ Ker f .
6. Ker f is a normal subgroup of G.
7. f to be one-one iff Ker $f = \{e\}$.
8. If H is a subgroup of G, then $f(H)$ is a subgroup of G'.
9. If H be a normal subgroup of G , then $f(H)$ is a normal subgroup of G'.
10. If H is a cyclic subgroup of G then $f(H)$ is a cyclic subgroup of G'.
11. If H is a abelian subgroup of G then $f(H)$ is a abelian subgroup of G'.
12. If $o(\text{Ker } f) = n$ then f is a n-to-1 mapping from G to $f(G)$.
13. If $o(H) = n$ then $o(f(H))$ divides n.
14. If K is a subgroup of G' then $f^{-1}(K)$ is a subgroup of G.
15. If K is a normal subgroup of G' then $f^{-1}(K)$ is a normal subgroup of G.

Theorem : The number of homomorphisms from $\mathbb{Z}_m$ to $\mathbb{Z}_n$ is $\gcd(m,n)$.

Fundamental theorem of homomorphism : Let f be a homomorphism of G to G' then $G/\text{Ker} f$ is isomorphic to $f(G)$ i.e., $G/\text{Ker} f \cong f(G)$.

or

Every homomorphic image of a group G is isomorphic to some quotient group of G.

Remark : If f is onto then $G/\text{Ker} f \cong f(G) = G'$.

Natural homomorphism / Canonical homomorphism : Let H be a normal subgroup of G. We define a mapping $f: G \to G/H$ by setting $f(g) = Hg$ for all $g \in G$, then f is a onto homomorphism and Ker $f = H$. This homomorphism is called natural homomorphism.

or

Every normal subgroup of a group G is the kernel of a homomorphism of G.

First theorem of isomorphism : Let $f : G \to G'$ is a onto homomorphism and K' is a normal subgroup of G' and let $K = f^{-1}(K')$, then $G/K \cong G'/K'$.

Second theorem of isomorphism : Let H and K are two subgroups of G s.t. $K \trianglelefteq G$, then $HK/K \cong H/H \cap K$.

Third theorem of isomorphism : Let H and K be two normal subgroups of G s.t. $H \subseteq K$, then $G/K \cong {}^{G/H}\big/_{K/H}$.

Gyley's Theorem : Every group is isomorphic to a group of permutations.

Results on number of homomorphism on finite cyclic groups :

1. The number of homomorphism from $\mathbb{Z}_m$ to $\mathbb{Z}_n$ is $\gcd(m,n)$.
2. (i) The number of homomorphism from K_4 to $\mathbb{Z}_n = \begin{cases} 4 & \text{, if } n \text{ is even} \\ 1 & \text{, if } n \text{ is odd} \end{cases}$.

   (ii) The number of homomorphism from K_4 to Q_8 is 4.

   (iii) The number of homomorphis from K_4 to $S_n = 3$ (No. of elements of order 2) + 6 (No. of non-cyclic subgroup of order 4) + 1.
3. (i) The number of homomorphism from Q_8 to $\mathbb{Z}_n = \begin{cases} 4 & \text{, if } n \text{ is even} \\ 1 & \text{, if } n \text{ is odd} \end{cases}$.

   (ii) The number of homomorphism from Q_8 to K_4 is 16.
4. (i) The number of homomorphism from S_n to K_4, $(n \geq 3)$ is 4.

   (ii) The number of homomorphis from S_n to Q_8, $(n \geq 3)$ is 2.

(iii) The number of homomorphism from S_n to $\mathbb{Z}_m$, $(n \geq 3)$ = $\begin{cases} 2 & , \text{ if } m \text{ is even} \\ 1 & , \text{ if } m \text{ is odd} \end{cases}$.

5. (i) The number of homomorphism from A_4 to $\mathbb{Z}_n = \begin{cases} 3 & , \text{ if } 3/n \\ 1 & , \text{ if } 3\chi n \end{cases}$

(ii) The number of homomorphism from A_n to $K_4, (n \geq 4)$ is 1.

(iii) The number of homomorphism from A_n to Q_8, $(n \geq 4)$ is 1.

Exercise 7.1

1. Construct all homomorphisms in the following groups. How may of them are one-one and how many are onto :

   (i) from $\mathbb{Z}_4$ to $\mathbb{Z}_6$ (ii) from $\mathbb{Z}_6$ to $\mathbb{Z}_9$ (iii) from $\mathbb{Z}_8$ to $\mathbb{Z}_{12}$

2. Construct all homomorphism in the following groups. How many of them are one-one and how many are onto :

   (i) from K_4 to $\mathbb{Z}_2$ (ii) from K_4 to $\mathbb{Z}_3$ (iii) from K_4 to $\mathbb{Z}_4$

   (iv) from K_4 to Q_8 (v) from K_4 to S_2 (vi) from K_4 to S_3

   (vii) from K_4 to S_4

3. Construct all homomorphisms in the following groups. How many of them are one-one and how many are onto :

   (i) from Q_8 to $\mathbb{Z}_2$ (ii) from Q_8 to $\mathbb{Z}_3$ (iii) from Q_8 to $\mathbb{Z}_4$ (iv) from Q_8 to K_4

4. Construct all homomorphisms in the following groups. How many of them are one-one and how many are onto :

   (i) from S_n to K_4 (ii) from S_n to Q_8 (iii) from S_n to $\mathbb{Z}_3$ (iv) from S_n to $\mathbb{Z}_4$

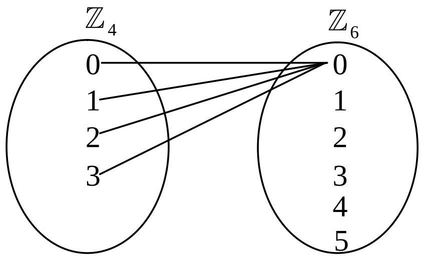
5. Construct all homomorphisms in the following groups. How many of them are one-one and how many are onto :

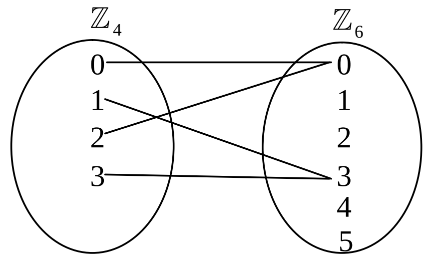
   (i) from A_4 to $\mathbb{Z}_3$ (ii) from A_4 to $\mathbb{Z}_5$ (iii) from A_4 to $\mathbb{Z}_6$ (iv) from A_4 to K_4

   (v) from A_4 to Q_8

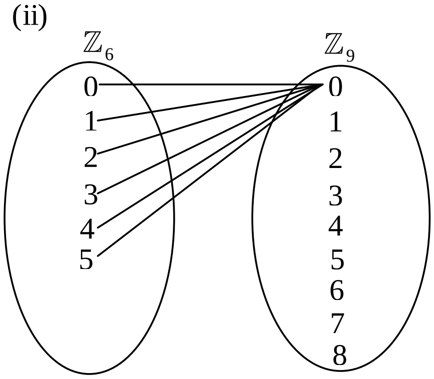
Answers

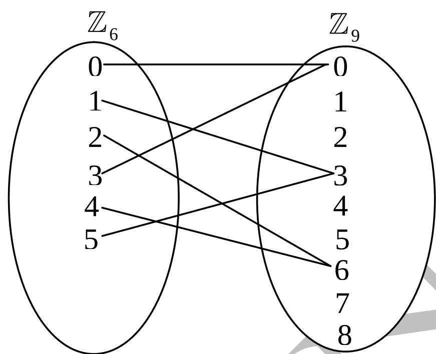
1. (i)

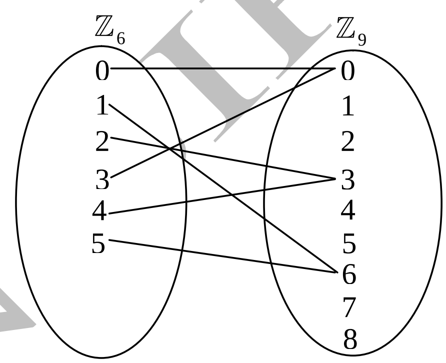




(ii)







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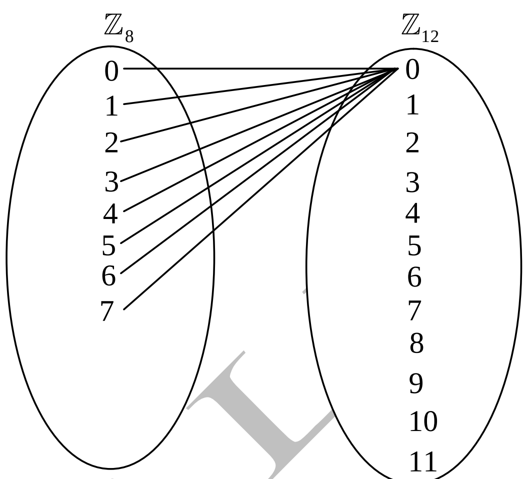
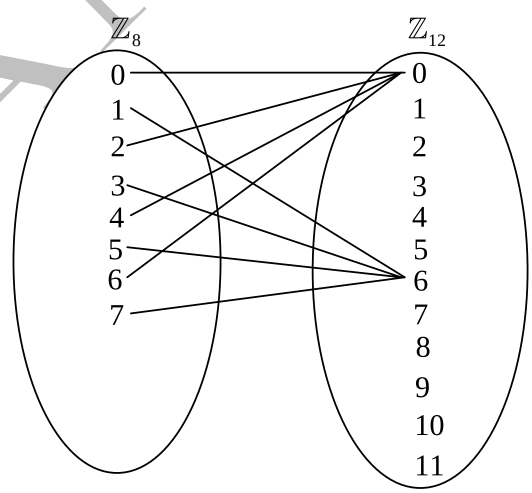
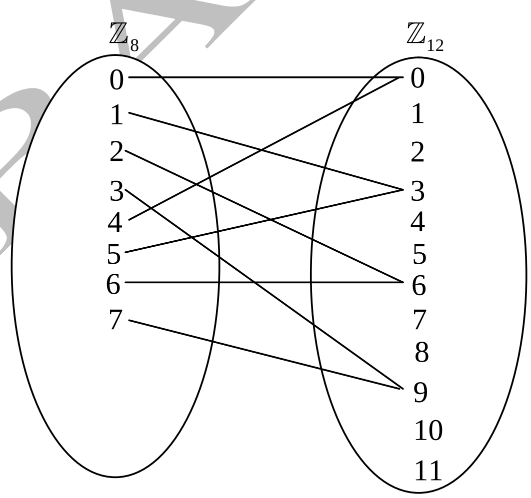
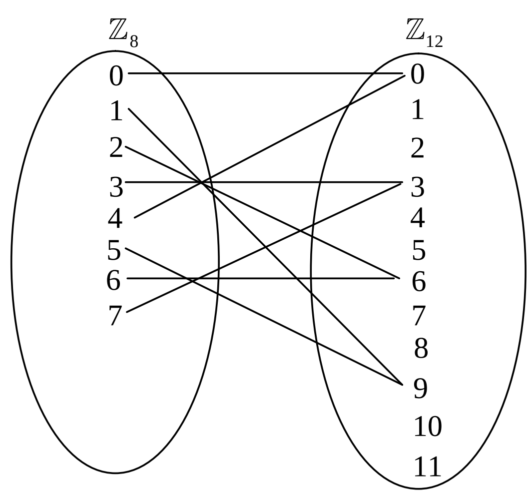
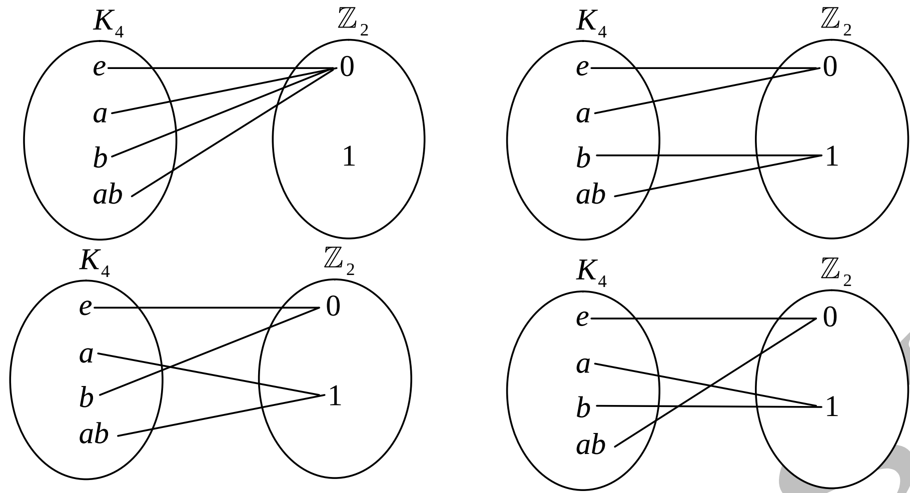






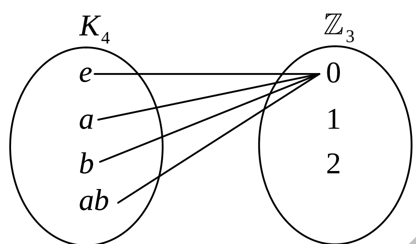


2. (i)



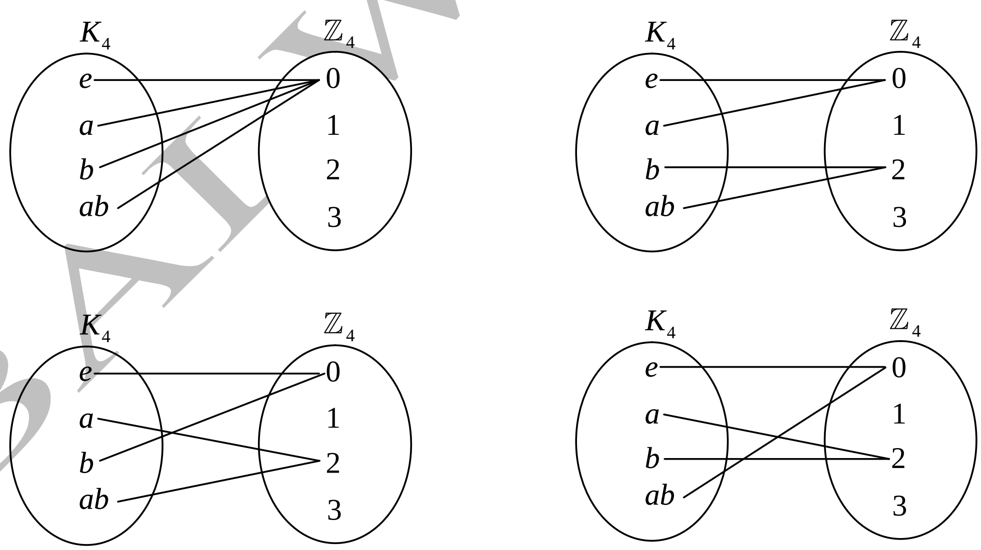
K_4
$\mathbb{Z}_2$
e
a
b
ab
0
1

(ii)



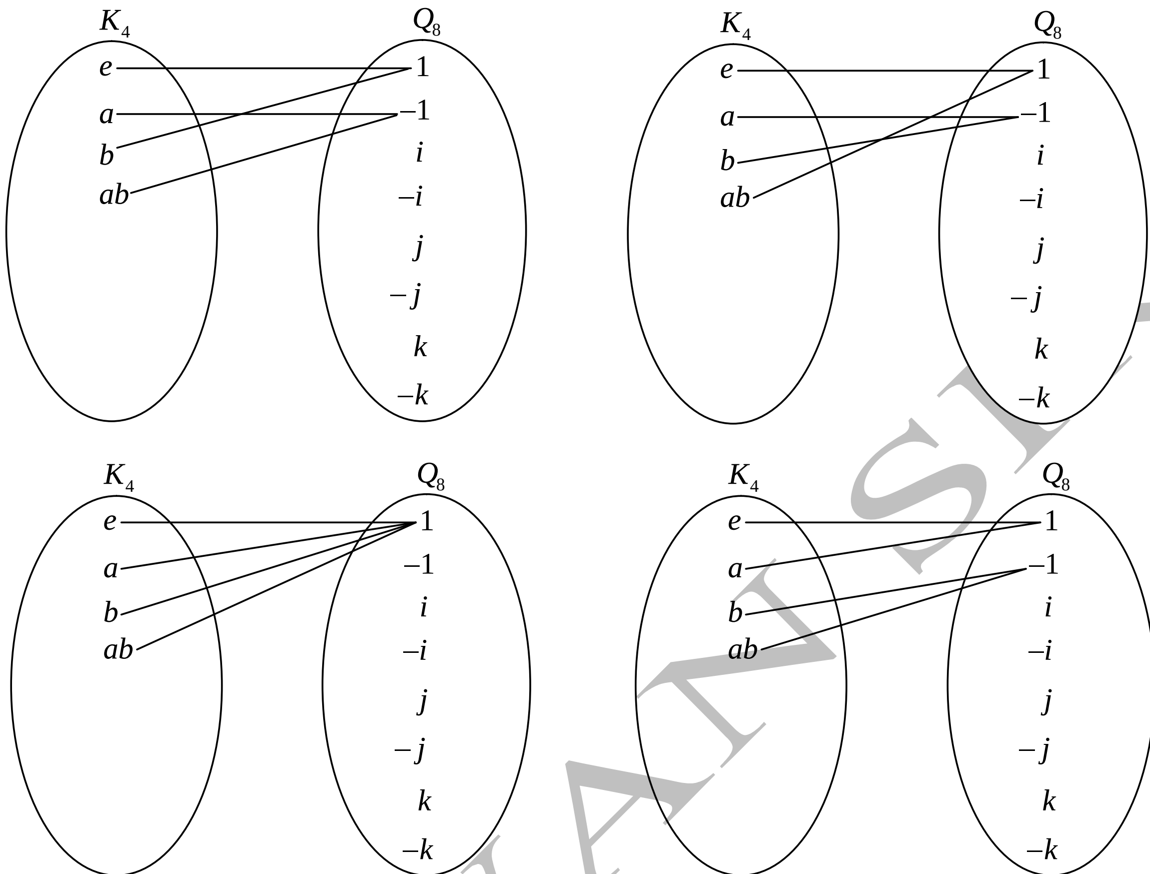
K_4
$\mathbb{Z}_3$
e
a
b
ab
0
1
2

(iii)



K_4
$\mathbb{Z}_4$
e
a
b
ab
0
1
2
3

(iv)



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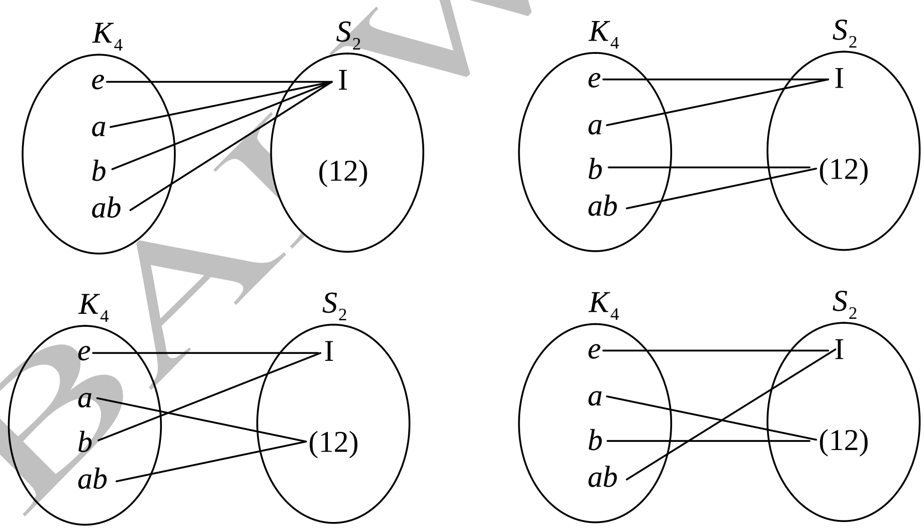

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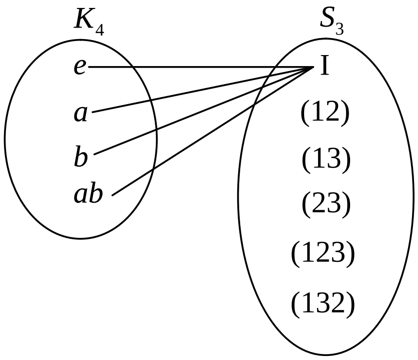
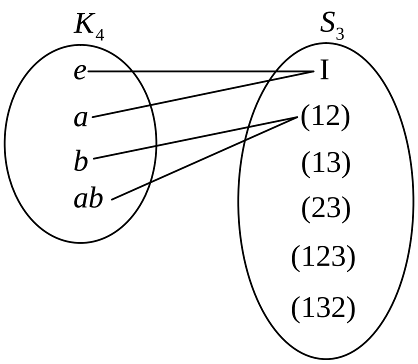
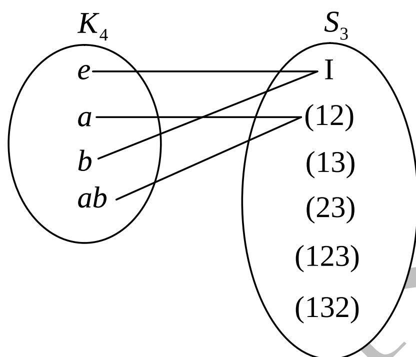
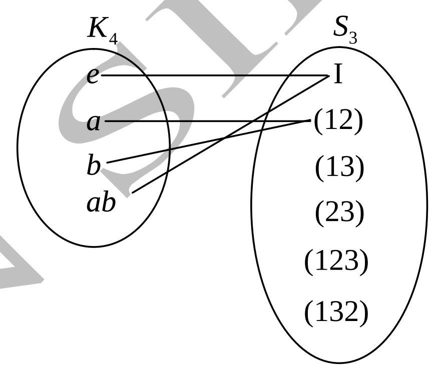
(vi)

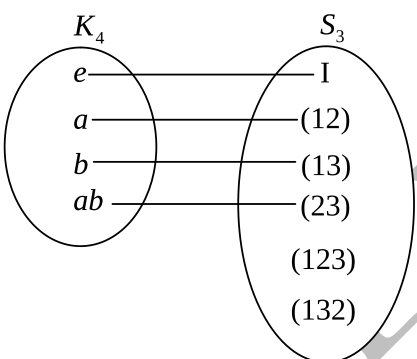


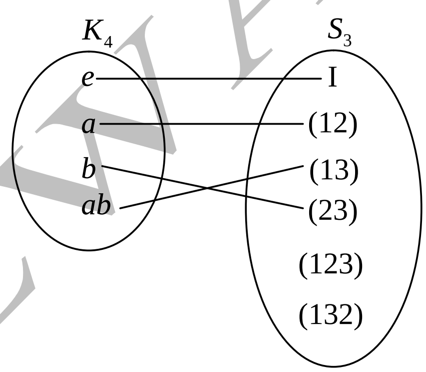


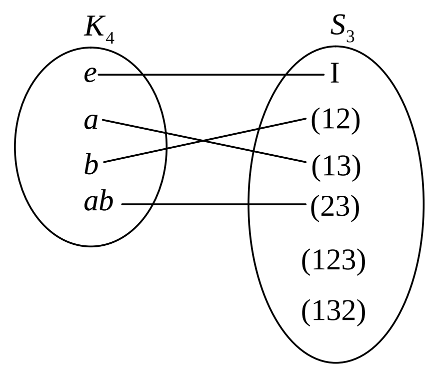


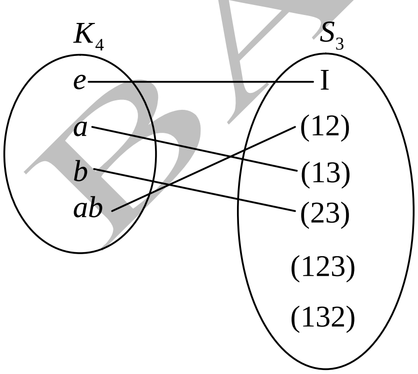


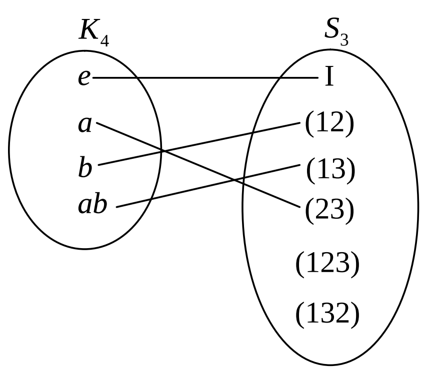
Replacing (12) by elements of order 2 in S_3, we can construct 9 such homomorphisms.

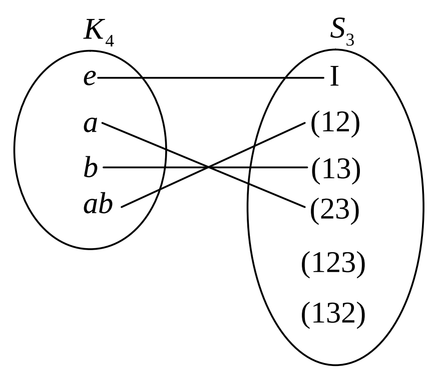












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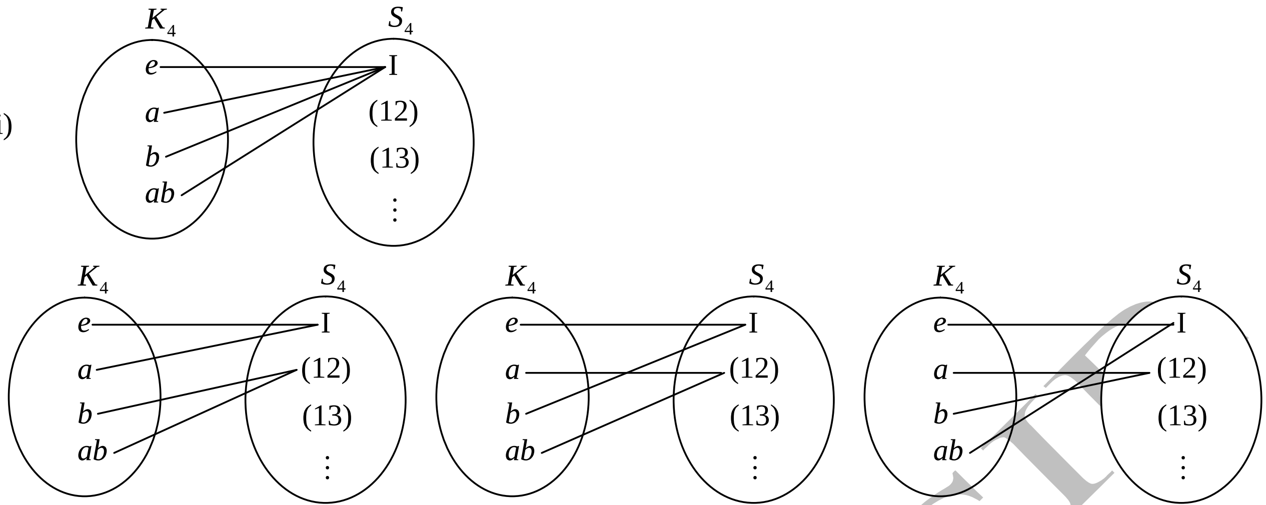
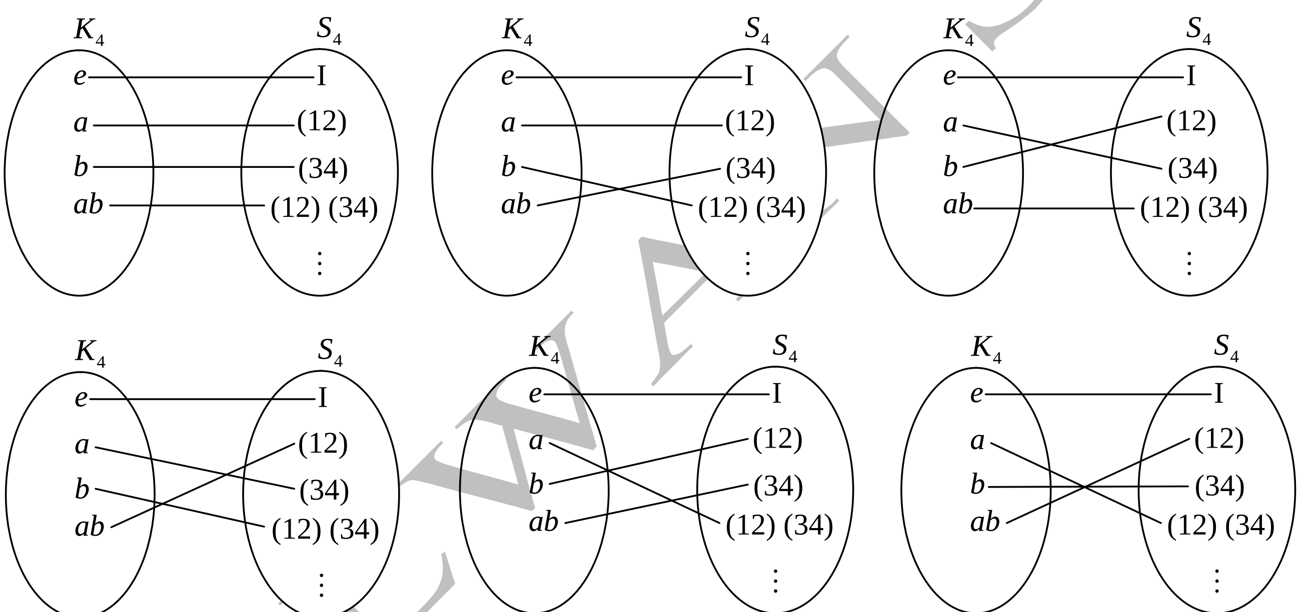


Replacing (12) by elements of order 2 in S_4 we can construct 27 such homomorphisms.



Taking all non-cyclic subgroups of order 4 in S_4, we can construct 24 such homomorphisms.

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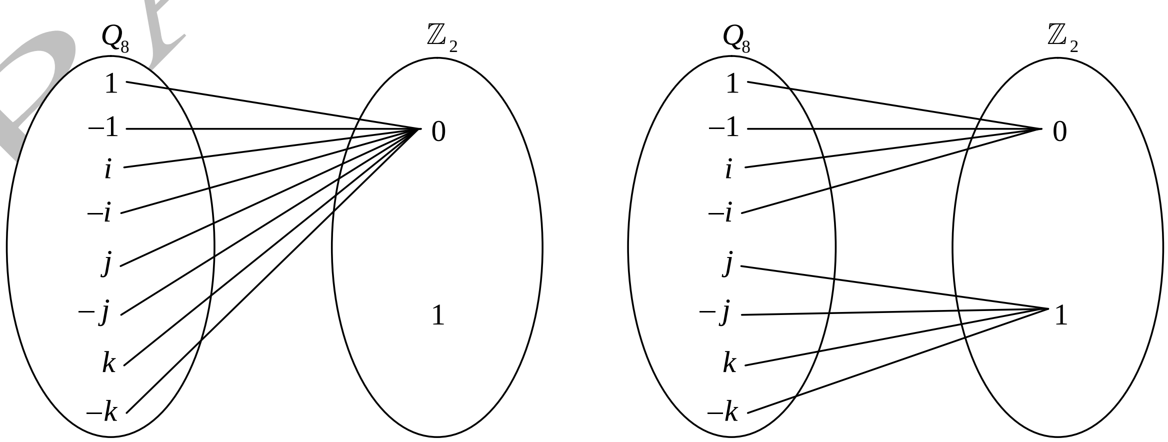
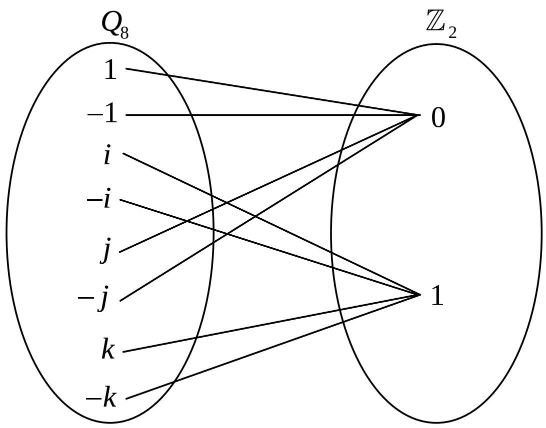
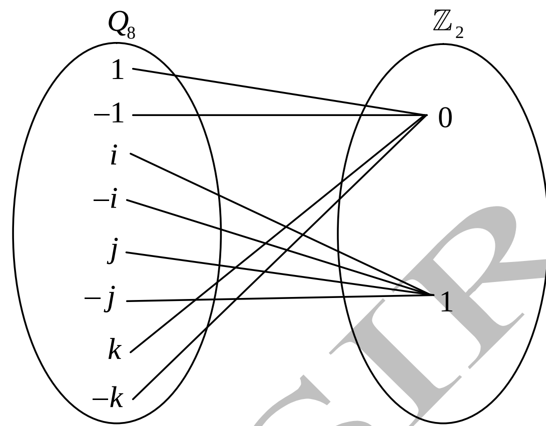
3. (i)



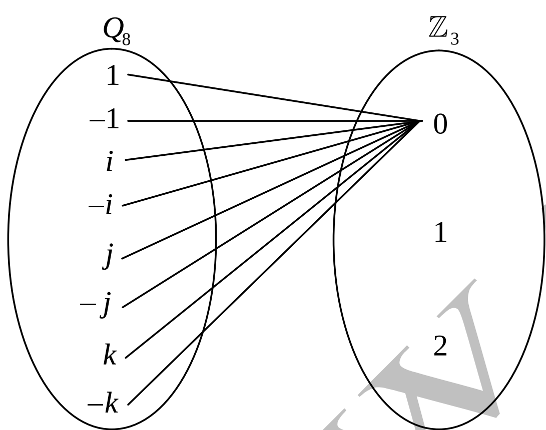


Q_8
$\mathbb{Z}_2$
1
−1
i
$-i$
j
$-j$
k
$-k$
0
1



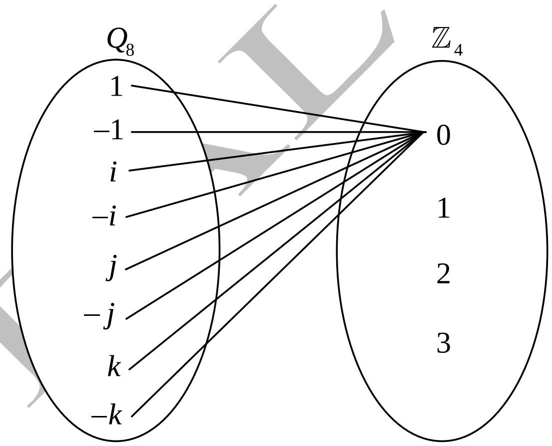
Q_8
$\mathbb{Z}_2$
1
−1
i
$-i$
j
$-j$
k
$-k$
0
1

(ii)

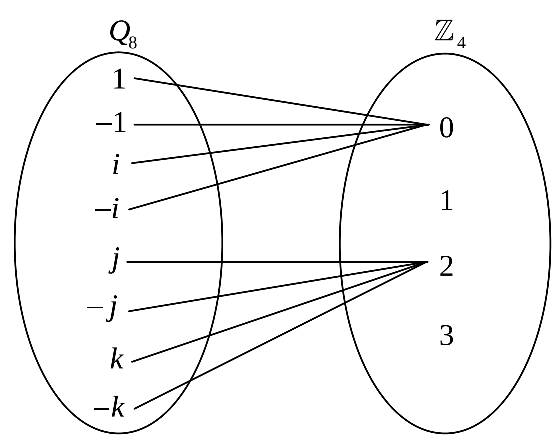


Q_8
$\mathbb{Z}_3$
1
−1
i
$-i$
j
$-j$
k
$-k$
0
1
2

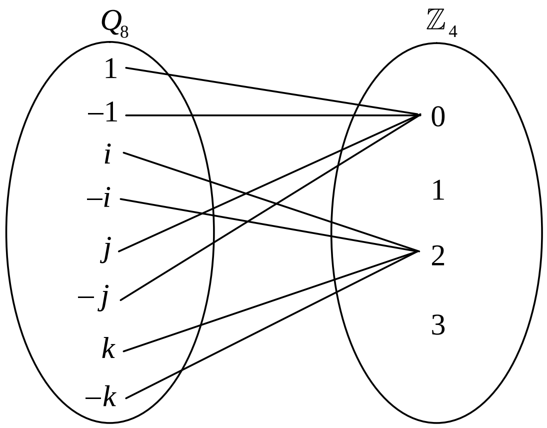
(iii)

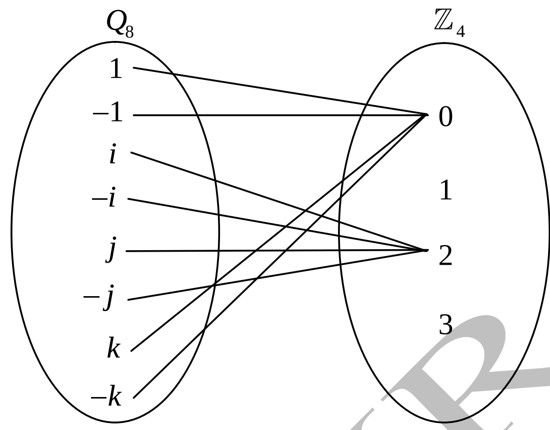


Q_8
$\mathbb{Z}_4$
1
−1
i
$-i$
j
$-j$
k
$-k$
0
1
2
3

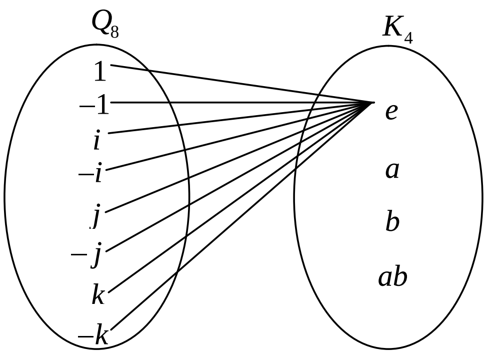


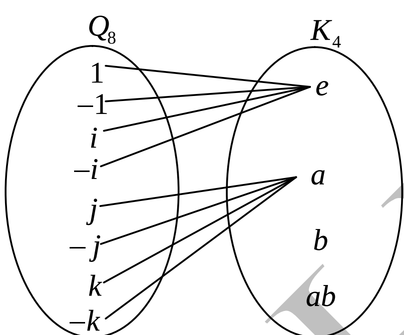
Q_8
$\mathbb{Z}_4$
1
−1
i
$-i$
j
$-j$
k
$-k$
0
1
2
3

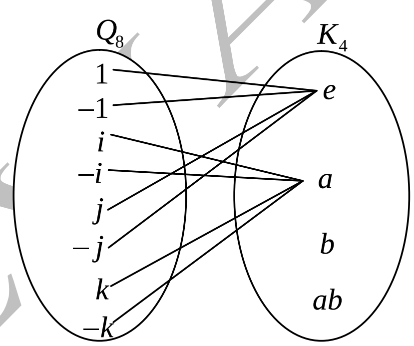


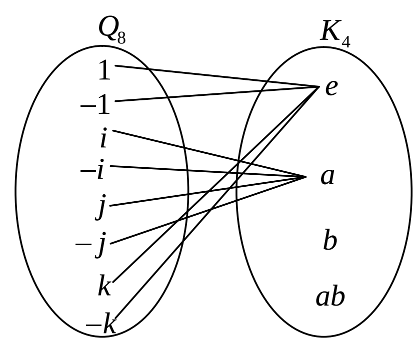


(iv)

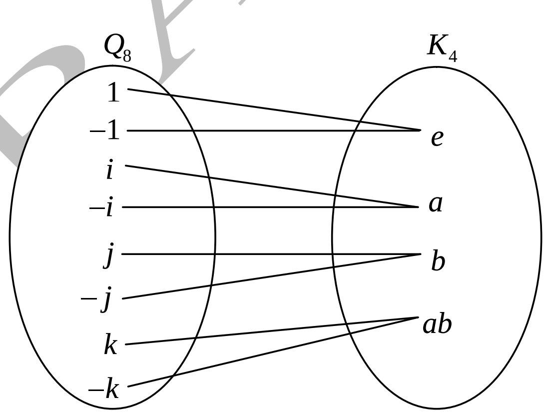


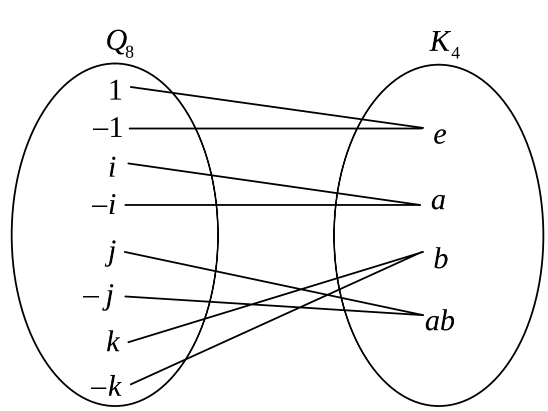


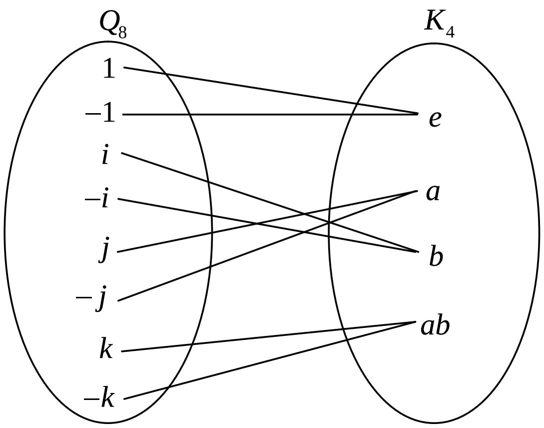




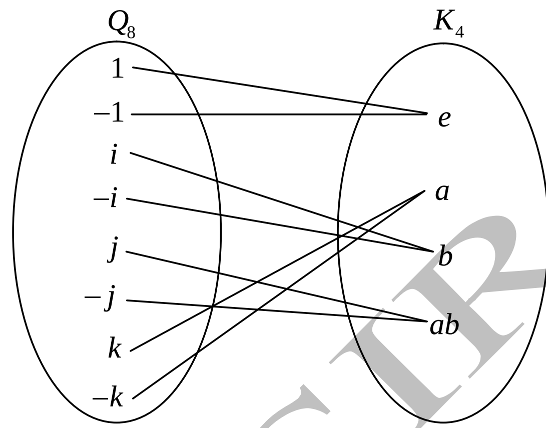
Replacing a by b and ab, we can construct 9 such homomorphisms.







Q_8
K_4
1
−1
i
−i
j
−j
k
−k
e
a
b
ab



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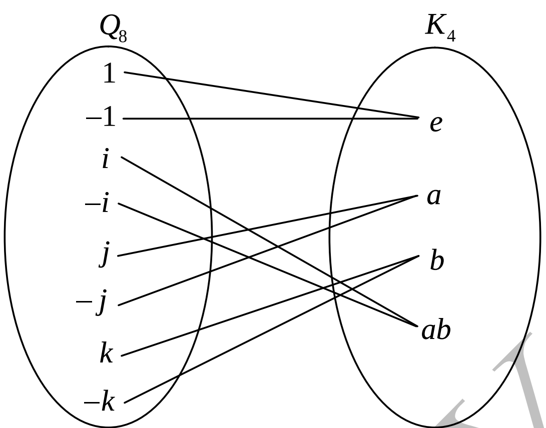

Q_8
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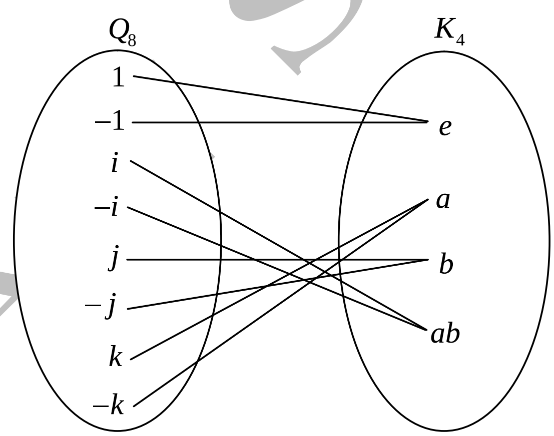

Q_8
K_4
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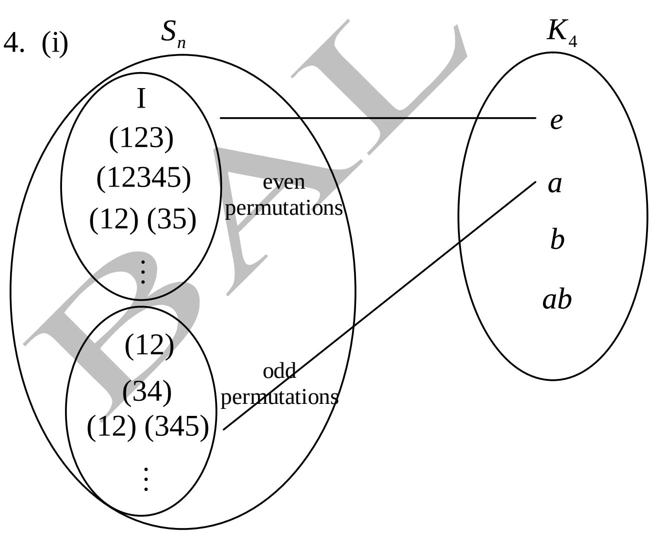
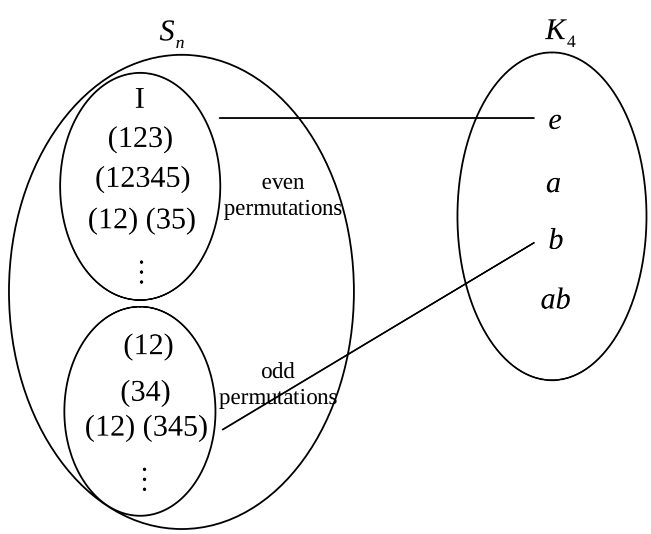
4. (i)



S_n
K_4
I
(123)
(12345)
(12) (35)
even permutations
(12)
(34)
(12) (345)
odd permutations
e
a
b
ab



S_n
K_4
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even permutations
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(12) (345)
odd permutations
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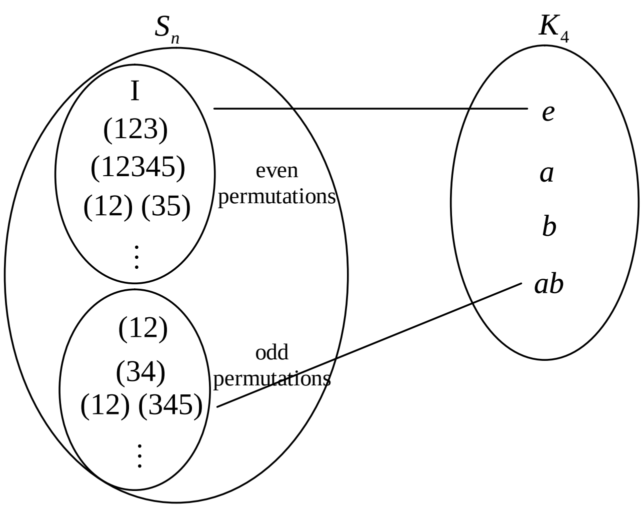
S_n
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even permutations
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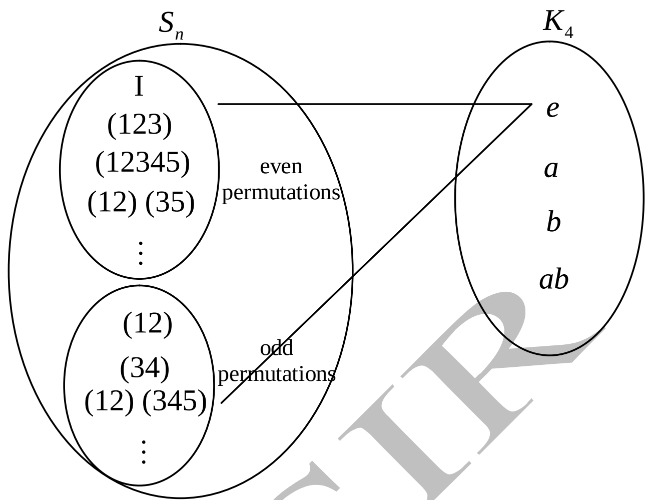
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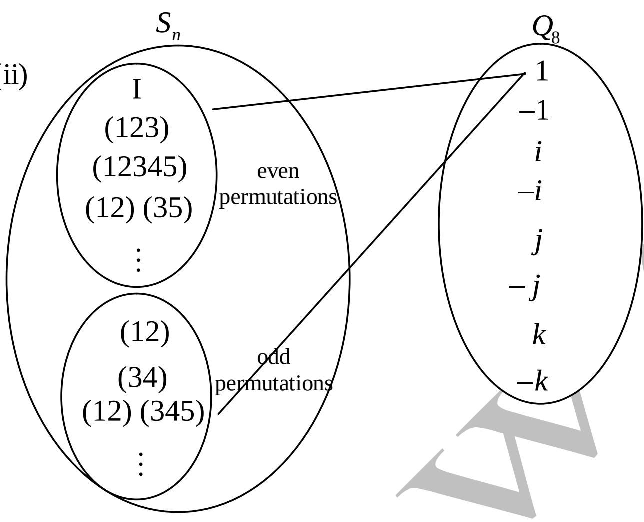
(ii)


S_n
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⋮
even permutations
(12)
(34)
(12) (345)
⋮
odd permutations
Q_8
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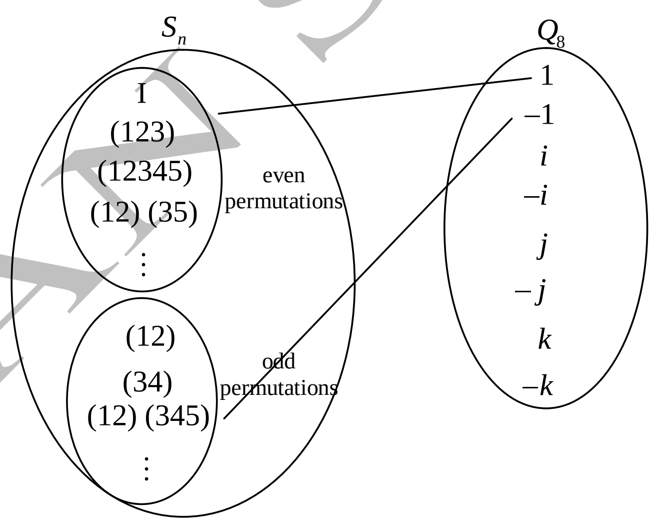
–1
i
$-i$
j
$-j$
k
$-k$


S_n
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even permutations
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odd permutations
Q_8
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k
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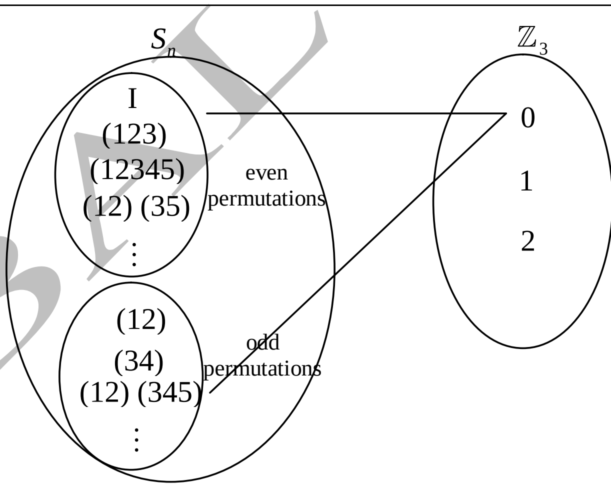
(iii)


S_n
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even permutations
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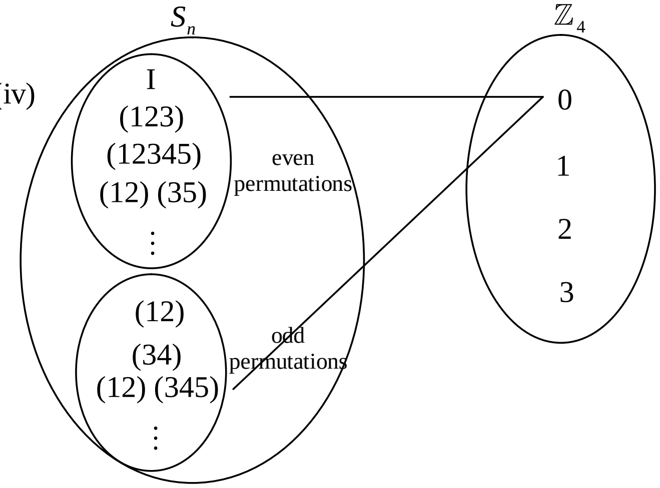
odd permutations
$\mathbb{Z}_3$
0
1
2

(iv)



S_n
$\mathbb{Z}_4$
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(123)
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even permutations
(12)
(34)
(12) (345)
odd permutations
0
1
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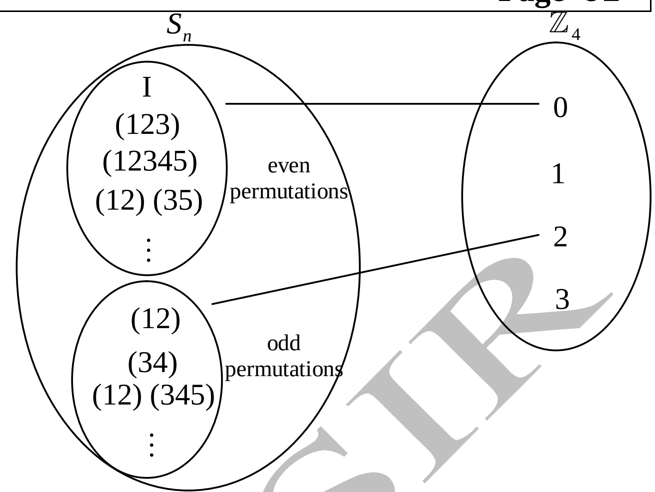
3



S_n
$\mathbb{Z}_4$
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even permutations
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odd permutations
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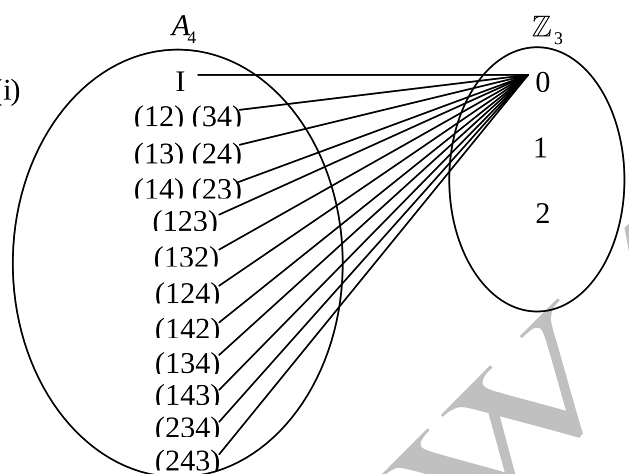
5(i)



A_4
$\mathbb{Z}_3$
I
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(13) (24)
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(123)
(132)
(124)
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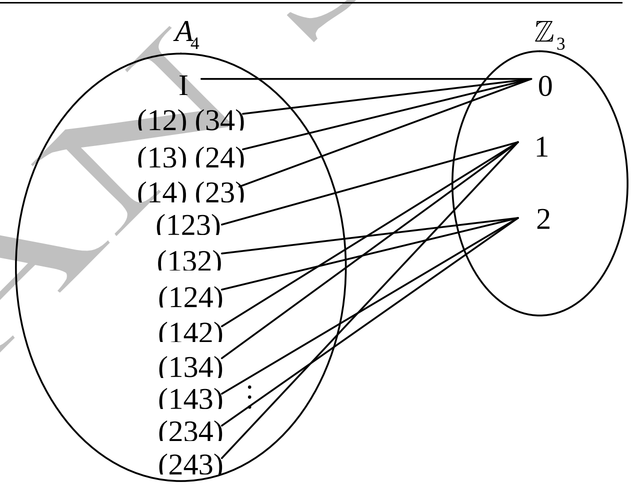
(134)
(143)
(234)
(243)
0
1
2



A_4
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I
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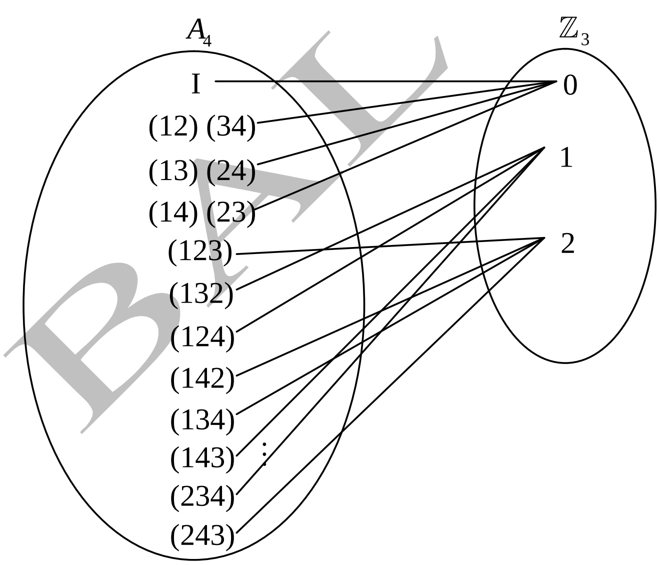

A_4
$\mathbb{Z}_3$
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(234)
(243)
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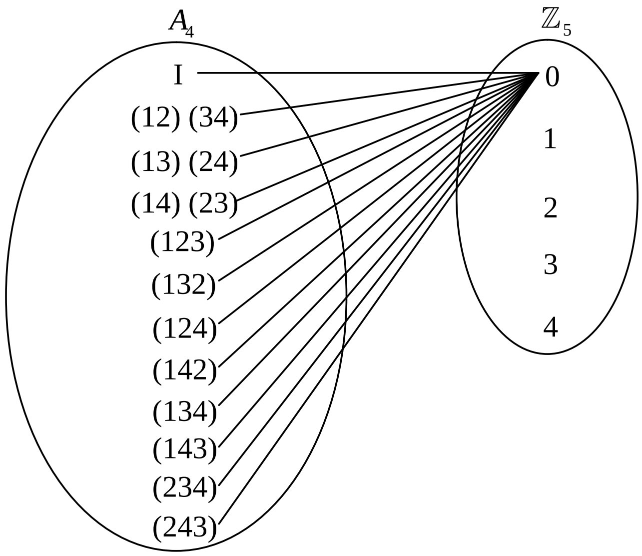
(ii)


A_4
$\mathbb{Z}_5$
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(143)
(234)
(243)
0
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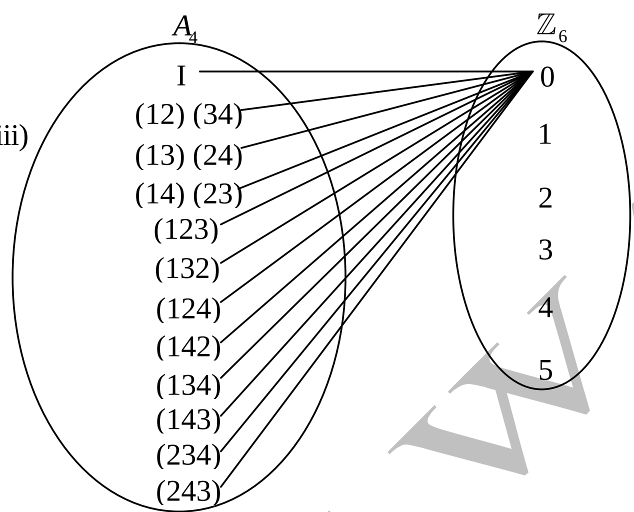
4

(iii)


A_4
$\mathbb{Z}_6$
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(14) (23)
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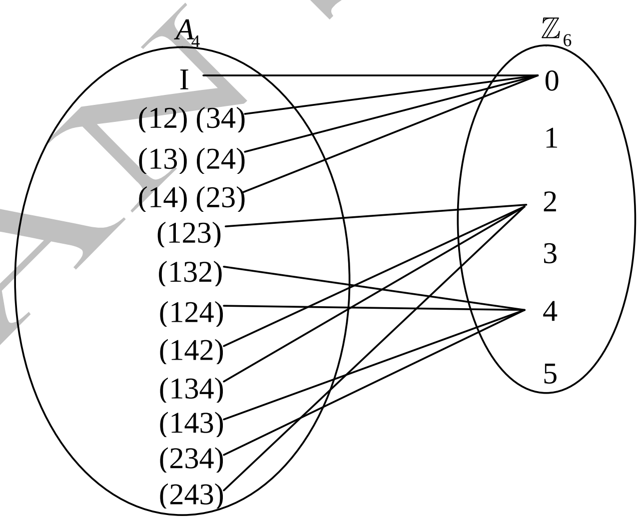
A_4
$\mathbb{Z}_6$
I
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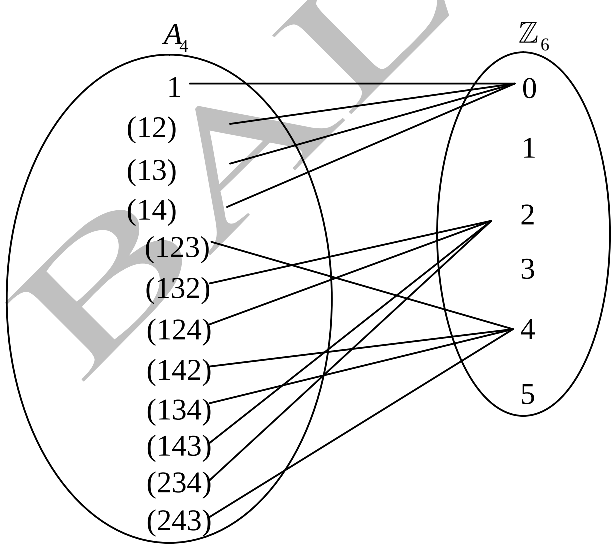
A_4
$\mathbb{Z}_6$
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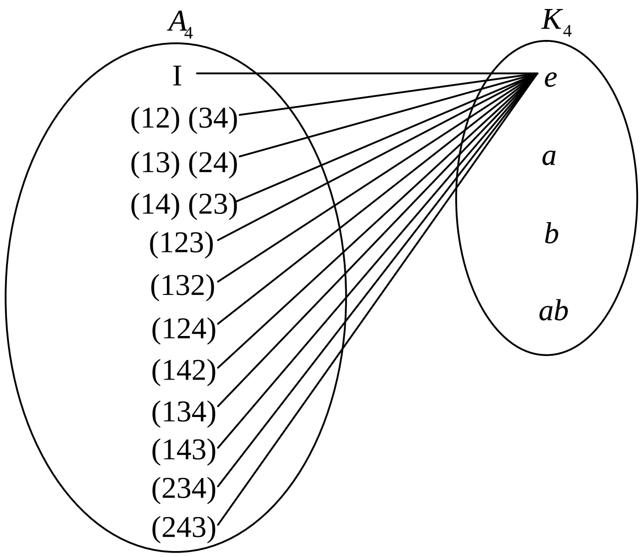
(iv)


A_4
K_4
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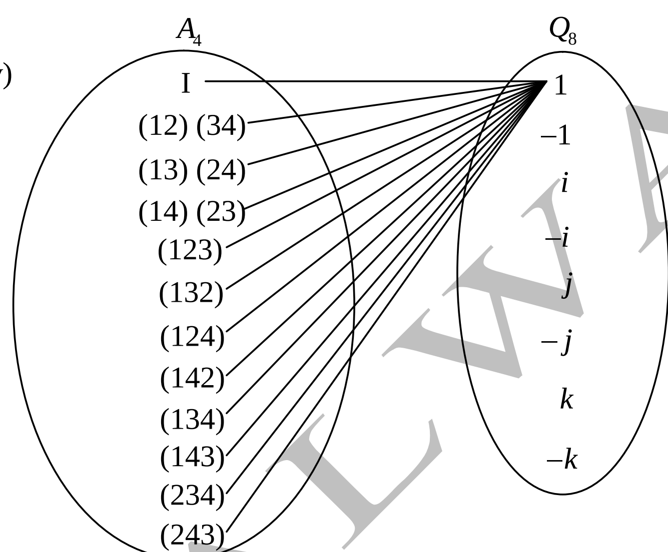
(v)


A_4
Q_8
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(134)
(143)
(234)
(243)
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–1
i
$-i$
j
$-j$
k
$-k$

Automorphisms

Def. Automorphism : A mapping $f : G \to G$ where, G is a group under the binary operation '$*$' is called an automorphism if

(i) f is a homomorphism i.e., $f(x) * y = f(x) * f(y)$, for all $x, y \in G$

(ii) f is one-one

(iii) f is onto

Remark : $Aut(G)$ denotes the set of all automorphisms of a group G i.e., $Aut(G) = \{f : f$ is an automorphism on $G\}$.

Def. Inner Automorphism : Let $a \in G$ be any fixed element. The automorphism $T_a : G \to G$ defined by

$$T_a(x) = a^{-1}xa \quad \text{for all} \quad x \in G$$

is called an inner automorphism of G corresponding to the element 'a'.

Remark : $Inn(G)$ denotes the sety of all inner automorphism of a group G.

Results :

1. The identity mapping is the only inner automorphism for an abelian group.
2. Let $f : G \to G$ be an automorphism and $a, b \in G$ be any tow elements, then

   (i) $o(a) = o\left(f(a)\right)$ (ii) $o(b^{-1}ab) = o(a)$

3. If f is an automorphism of a group G and $a \in G$ be an element, then $f\left(N(a)\right) = N\left(f(a)\right)$.
4. $Inn(G)$ is a normal subgroup of $Aut(G)$.
5. Let G be a finite group and $Z(G)$ be centre of G. Then, $G/Z(G) \cong Inn(G)$.
6. Let G be a finite group and $Z(G)$ be centre of G. Then, $Z(G) = O\left(Inn(G)\right)$.
7. If G is an abelian group such that $O(G) = n$, where n is an odd integer greater than 1, then $O\left(Aut(G)\right) > 1$.
8. If $O\left(Aut(G)\right) > 1$, then $O(G) > 2$
9. Let G be a finite cyclic group of order n, then $O\left(Aut(G)\right) = \phi(n)$.
10. Let G be an infinite cyclic group, then $O\left(Aut(G)\right) = 2$.
11. For every positive integer n. $Aut(\mathbb{Z}_n) \cong U(n)$.

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Exercise 7.1

1. Let G be a group of permutations. For each σ in G, define $\operatorname{sgn}(\sigma)=\begin{cases}+1 & \text{if } \sigma \text{ is an even permutation,}\\ -1 & \text{if } \sigma \text{ is an odd permutation.}\end{cases}$

   Prove that sgn is a homomorphism from G to the multiplicative group {+1, –1}. What is the kernel ?
2. Prove that the mapping from $G\oplus H$ to G given by $(g,h)\to g$ is a homomorphism. What is the kernel ? This mapping is called the *projection* of $G\oplus H$ onto G.
3. Let G be a subgroup of some dihedral group. For each x in G, define $\phi(x)=\begin{cases}+1 & \text{if } x \text{ is a rotation,}\\ -1 & \text{if } x \text{ is a reflection.}\end{cases}$

   Prove that ϕ is a homomorphism from G to the multiplicative group {+1, –1}. What is the kernel of ϕ ?
4. Suppose that k is a divisor of n. Prove that $\mathbb{Z}_n/\langle k\rangle \approx \mathbb{Z}_k$.
5. Explain why the correspondence $x\to 3x$ from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{10}$ is not a homomorphism.
6. Suppose that ϕ is a homomorphism from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{30}$ and $\operatorname{Ker}\phi=\{0,10,20\}$. If $\phi(23)=9$, determine all elements that map to 9.
7. Prove that there is no homomorphism from $\mathbb{Z}_8\oplus\mathbb{Z}_2$ onto $\mathbb{Z}_4\oplus\mathbb{Z}_4$.
8. Prove that there is no homomorphism from $\mathbb{Z}_{16}\oplus\mathbb{Z}_2$ onto $\mathbb{Z}_4\oplus\mathbb{Z}_4$.
9. Can there be a homomorphism form $\mathbb{Z}_4\oplus\mathbb{Z}_4$ onto $\mathbb{Z}_8$? Can there be a homomorphism from $\mathbb{Z}_{16}$ onto $\mathbb{Z}_2\oplus\mathbb{Z}_2$. Explain your answers.
10. Suppose that there is a homomorphism ϕ from $\mathbb{Z}_{17}$ to some group and that ϕ is not one to one. Determine ϕ.
11. How many homomorphisms are there from $\mathbb{Z}_{20}$ onto $\mathbb{Z}_8$? How many are there to $\mathbb{Z}_8$?
12. If ϕ is a homomorphism from $\mathbb{Z}_{30}$ onto a group of order 5, determine the kernel of ϕ.
13. Suppose that ϕ is a homomorphism from a finite group G onto $\bar{G}$ and that $\bar{G}$ has an element of order 8. Prove that G has an element of order 8. Generalize.
14. How many homomorphisms are there from $\mathbb{Z}_{20}$ onto $\mathbb{Z}_{10}$? How many are there to $\mathbb{Z}_{10}$?
15. Determine all homomorphisms from $\mathbb{Z}_4$ to $\mathbb{Z}_2\oplus\mathbb{Z}_2$.
16. Determine all homomorphisms from $\mathbb{Z}_n$ to itself.

17. Suppose that ϕ is a homomorphism from S_4 onto $\mathbb{Z}_2$. Determine Kerϕ. Determine all homomorphisms from S_4 to $\mathbb{Z}_2$.
18. Suppose that there is a homomorphism from a finite group *G* onto $\mathbb{Z}_{10}$. Prove that *G* has normal subgroups of index 2 and 5.
19. Suppose that ϕ is a homomorphism from a group *G* onto $\mathbb{Z}_6 \oplus \mathbb{Z}_2$ and that the kernel of ϕ has order 5. Explain why *G* must have normal subgroups of orders 5, 10, 15, 20, 30 and 60.
20. Suppose that ϕ is a homomorphism from $U(30)$ to $U(30)$ and that Kerϕ = {1, 11}. If $\phi(7)=7$, find all elements of $U(30)$ that map to 7.
21. Find a homomorphism ϕ from $U(30)$ to $U(30)$ with kernel {1, 11} and $\phi(7)=7$.
22. Suppose that ϕ is a homomorphism from $U(40)$ to $U(40)$ and that kerϕ ={1, 9, 17, 33}. If $\phi(11)=11$, find all elements of $U(40)$ that map to 11.
23. Find a homomorphism ϕ from $U(40)$ to $U(40)$ with kernel {1, 9, 17, 33} and $\phi(11)=11$.
24. Determine all homomorphic images of D_4 (up to isomorphism).
25. Suppose that *G* is a finite group and that $\mathbb{Z}_{10}$ is a homomorphic image of *G*. What can we say about $|G|$?
26. Suppose that $\mathbb{Z}_{10}$ and $\mathbb{Z}_{15}$ are both homomorphic images of a finite group *G*. What can be said about $|G|$?
27. Show that a homomorphism defined on a cyclic group is completely determined by its action on a generator of the group.

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Group Theory - 8

Class equation

Def. Conjugate Elements : Let a and b are two elements of group G, then a is said to be conjugate to b if there exists an element $x \in G$ such that $a = x^{-1} b x$

and we write $a \sim b$ which is read as 'a' is a conjugate to 'b'.

Result 1 : Relation of conjugacy is an equivalence relation.

Remarks :

1. Since we have proved that the relation of conjugacy is an equivalence relation, so instead of saying that a is conjugate to b or b is conjugate to a, we can say that a and b are conjugates of each other.

2. As conjugacy is an equivalence relation , so it partitions the group G into disjoint equivalence classes and these classes are known as conjugate classes.

Def. Conjugate Class : Let G be a group and $a \in G$ be any element. The collection of all conjugates of 'a' is called conjugate class of a and is denoted by $cl(a)$ or $[a]$.

Thus , $cl(a) = \{b \in G : b \sim a\} = \{x^{-1}ax : x \in G\}$.

Remark : By above definition, it is clear that to find the conjugates of an element 'a' we have to compute $x^{-1}ax$ for every element $x \in G$.

Example 1 : Partition the following groups into disjoint conjugate classes.

(a) $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$

(b) $S_3 = \{I , (1\ 2) , (1\ 3) , (2\ 3) , (1\ 2\ 3) , (1\ 3\ 2)\}$.

Solution : (a) Let us find the conjugates of 1 :

$(1)^{-1}(1)(1) = 1$ $(-1)^{-1}(1)(-1) = 1$ $(i)^{-1}(1)(i) = 1$ $(-i)^{-1}(1)(-i) = 1$

$(j)^{-1}(1)(j) = 1$ $(-j)^{-1}(1)(-j) = 1$ $(k)^{-1}(1)(k) = 1$ $(-k)^{-1}(1)(-k) = 1$

So , 1 is the only conjugate to itself.

Let us find the conjugates of -1 :

$(1)^{-1}(-1)(1) = -1$ $(-1)^{-1}(-1)(-1) = -1$ $(i)^{-1}(-1)(i) = -1$ $(-i)^{-1}(-1)(-i) = -1$

$(j)^{-1}(-1)(j) = -1$ $(-j)^{-1}(-1)(-j) = -1$ $(k)^{-1}(-1)(k) = -1$ $(-k)^{-1}(-1)(-k) = -1$

So , -1 is the only conjugate to itself.

Let us find the conjugates of i :

$(1)^{-1}(i)(1) = i$ $(-1)^{-1}(i)(-1) = i$ $(i)^{-1}(i)(i) = i$ $(-i)^{-1}(i)(-i) = i$

$(j)^{-1}(i)(j) = -i$ $(-j)^{-1}(i)(-j) = -i$ $(k)^{-1}(i)(k) = -i$ $(-k)^{-1}(i)(-k) = -i$

So , conjugates of i are i and $-i$.

Now there is no need to find conjugates of $-i$. Since conjugacy is an equivalence relation, so i and $-i$ forms one conjugate class. Similarly it can be seen that j and $-j$ forms one conjugate class and so does the k and $-k$.

Therefore, Q_8 is divided into five different conjugate classes, namely $\{1\}, \{-1\}, \{i, -i\}, \{j, -j\}, \{k, -k\}$.

(b) Let us find the conjugate of I :

$I^{-1} \; I \; I = I$ $\qquad$ $(1\ 2)^{-1} I\ (1\ 2) = I$ $\qquad$ $(1\ 3)^{-1} I\ (1\ 3) = I$

$(2\ 3)^{-1} I\ (2\ 3) = I$ $\qquad$ $(1\ 2\ 3)^{-1} I\ (1\ 2\ 3) = I$ $\qquad$ $(1\ 3\ 2)^{-1} I\ (1\ 3\ 2) = I$

So I is the only conjugate to itself.

Let us find the conjugates of (12) :

$I^{-1}\ (1\ 2)\ I = (1\ 2)$ $\qquad$ $(1\ 2)^{-1}(1\ 2)(1\ 2) = (1\ 2)$ $\qquad$ $(13\ )^{-1}(1\ 2)(1\ 3) = (2\ 3)$

$(2\ 3)^{-1}(1\ 2)(2\ 3) = (1\ 3)$ $\qquad$ $(1\ 2\ 3)^{-1}(1\ 2)(1\ 2\ 3) = (1\ 3)$ $\qquad$ $(1\ 3\ 2)^{-1}(1\ 2)(1\ 3\ 2) = (2\ 3)$

So, (1 2), (1 3) and (2 3) are conjugates of each other. Now there is no need to find the conjugates of (1 3) and (23). Similarly, it can be checked that conjugates of (1 2 3) are (1 2 3) and (1 3 2).

Therefore, S_3 is divided into three conjugate classes, namely $\{I\}$, $\{(1\ 2), (1\ 3), (2\ 3)\}$, $\{(1\ 2\ 3), (1\ 3\ 2)\}$

Result 2 : In any group G, prove that $cl(a) = \{a\}$ iff $a \in Z(G)$.

Remark : If $a \notin Z(G)$, then $cl(a)$ contains at least 2 elements.

Result 3 : If G is abelian then $cl(a) = \{a\}$ for every $a \in G$.

Result 4 : Let σ and η be any permutations in the symmetric group S_n then $\sigma^{-1}\eta\sigma$ has the same cyclic decomposition as that of η.

Result 5 : Two permutations are conjugate in the symmetric group S_n iff they have the same cyclic decomposition.

Result 6 : The number of conjugate classes in S_n is $p(n)$.

Def. Normalizer (centralizer) of an element : Let G be any group and a be any element of G, then normalizer of a in G and centralizer of a in G has the same meaning and is given by

$N(a) = C(a) = \{ x \in G : xa = ax \}$ = collection of those elements of G which commutes with a

By definition, it is clear that if $a \in Z(G)$, then $N(a) = G$.

Result 7 : Normalizer of an element is always a subgroup.

Example 2 : Find the normalizer of every element in the following groups.

(a) $\boldsymbol{Q_8 = \{1, -1, i, -i, j, -j, k, -k\}}$

(b) $\boldsymbol{S_3 = \{I, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}}$.

Remark : Normalizer of an element need not be a normal subgroup as can be seen by following example. Consider

$S_3 = \{I, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$

We see that, $\quad N((1\ 2)) = \{I, (1\ 2)\} = H$ (say)

which is not a normal subgroup because by definition of normal subgroup left and right coset for any element must be same, but we see that

$(1\ 3)\, H = (1\ 3)\ \{I, (1\ 2)\} = \{(1\ 3), (1\ 3)(1\ 2)\} = \{(1\ 3), (1\ 2\ 3)\}$

$H\,(1\ 3) = \{I, (1\ 2)\}\,(1\ 3) = \{(1\ 3), (1\ 2)(1\ 3)\} = \{(1\ 3), (1\ 3\ 2)\}$

$\Rightarrow \quad (1\ 3)\,H \neq H\,(1\ 3)$

Result 8 : Let $a \in G$ be any element, then $Z(G) \subseteq N(a)$.

Result 9 : If G is a finite group and $a \in G$, then $o\big(cl(a)\big) = \dfrac{o(\mathrm{G})}{o\big(\mathrm{N}(a)\big)}$

Result 10 : Class equation of a finite group : If G is a finite group , then $o(G) = \sum_a \dfrac{o(\mathrm{G})}{o\big(\mathrm{N}(a)\big)}$ where sum runs over 'a' taken one from each conjugate class.

Proof : We know that relation of conjugacy is an equivalence relation and it partitions the group into disjoint equivalence classes and so we can write

$G = \bigcup_{a \in G} cl(a)$ where union runs over 'a' taken one from each conjugate class.

Since the conjugate classes are disjoint , so

$\mathrm{o}(G) = \sum_a o\big(cl(a)\big)$ where sum runs over 'a' taken one from each conjugate class.

Using above result , we obtain , $\mathrm{o}(G) = \sum_a \dfrac{o(G)}{o\big(N(a)\big)}$ where sum runs over 'a' taken one from each conjugate class.

Result 11 : Second form of class – equation : If G is a finite group then $o(G) = o\big(Z(G)\big) + \sum_{a \notin Z(G)} \dfrac{o(G)}{o\big(N(a)\big)}$ where sum runs over 'a' taken one from each conjugate class.

Proof : We know that class equation is

$$\mathrm{o}(G) = \sum_{a \in G} \frac{o(G)}{o\big(N(a)\big)} = \sum_{a \in Z(G)} \frac{o(G)}{o\big(N(a)\big)} + \sum_{a \notin Z(G)} \frac{o(G)}{o\big(N(a)\big)}$$

$$= \sum_{a \in Z(G)} 1 + \sum_{a \notin Z(G)} \frac{o(G)}{o\big(N(a)\big)} \qquad \big[\text{Since } N(a) = G \text{ iff } a \in Z(G)\big]$$

$$= 1 + 1 + \ldots\ldots\ldots + 1\big(o(Z(\mathrm{G}))\text{ times}\big) + \sum_{a \notin Z(G)} \frac{o(G)}{o\big(N(a)\big)}$$

$$= \mathrm{o}\big(\mathrm{Z(G)}\big) + \sum_{a \notin Z(G)} \frac{o(G)}{o\big(N(a)\big)}$$ where sum runs over 'a' taken one from each conjugate class.

Remark : We know that $a \in \mathrm{Z}(G)$ iff $N(a) = G$ iff $cl(a) = \{a\}$. Therefore, we have $a \notin \mathrm{Z}(G)$ iff $N(a) \neq G$ iff $cl(a) \neq \{a\}$. So in class equation $\sum_{a \notin \mathrm{Z(G)}}$ can be replaced by $\sum_{N(a) \neq \mathrm{G}}$ or $\sum_{cl(a) \neq \{a\}}$

and so we can write the class equation in the following forms:

$$\mathrm{o}(G) = \mathrm{o}\big(\mathrm{Z(G)}\big) + \sum_{N(a) \neq G} \frac{o(G)}{o\big(N(a)\big)}, \qquad \mathrm{o}(G) = \mathrm{o}\big(\mathrm{Z(G)}\big) + \sum_{cl(a) \neq \{a\}} \frac{o(G)}{o\big(N(a)\big)}$$

Three important applications of class – equation :

Result 12 : If $o(G) = p^n$, where p is a prime number then $o(Z(G)) > 1$. In words , a group of prime power order always has a non – trivial centre.

Result 13 : If $o(G) = p^2$, where p is a prime number, then G is abelian.

Remark : By above theorem, we can say that groups of order 4 , 9 , 25 , 49 , 121 , etc. are all abelian.

Result 14 : If G is a non – abelian group of order p^3 , where p is a prime number , then $o(Z(G)) = p$ and G has $p^2 + p - 1$ conjugate classes.

Def. Normalizer of a subgroup : Let H be a subgroup of G , then normalizer of H ,denoted by $N(H)$, is defined to be the set of all those element of G for which left and right cosets of H are identical i.e.

$$N(H) = \{x \in G : Hx = xH\}$$

Def. Centralizer of a subgroup : Let G be any group and H be its subgroup, then centralizer of H in G is given by

$$C(H) = \{x \in G : xh = hx \ \forall \ h \in H\}$$

Result 15 : If H is a subgroup of a group G then $N(H)$ and $C(H)$ are subgroups of G .

Example 3 : Explain the difference between normalizer and centralizer of a subgroup with the help of an example.

Solution : $G = Q_8$ and $H = \{\pm 1, \pm i\}$, we find $N(H)$ and $C(H)$.

To find Normalizer : Since H is a subgroup so each element of H gives left and right cosets identical i.e. $Hh = hH = H \ \forall\, h \in H$ so $H \subseteq N(H)$. Further, we note that

$$jH = \{j, -j, k, -k\} \quad \text{and} \quad Hj = \{j, -j, k, -k\} \quad \text{i.e.,} \quad jH = Hj \ .$$

Similarly we can check that $kH = Hk$, $-jH = H(-j)$, $-kH = H(-k)$ so $\pm j, \pm k \in N(H)$

Hence $N(H) = \{\pm 1, \pm i, \pm j, \pm k\} = Q_8$

To find Centralizer : We note that only $\pm 1, \pm i$ commute with every element of H, so $C(H) = \{\pm 1, \pm i\} = H$

Remark : It should be noted that normalizer & centralizer for an element are not different things.

Result 16 :

(i) H is normal subgroup of $N(H)$.

(ii) $N(H)$ is the largest subgroup of G in which H is normal.

(iii) $H \Delta G$ iff $N(H) = G$.

Result 17 : Table of class equation of some familiar groups :

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Order	Group	Nature	Class equation
1	$\mathbb{Z}_1$	abelian	$1=1$
2	$\mathbb{Z}_2$	abelian	$2=1+1$
3	$\mathbb{Z}_3$	abelian	$3=1+1+1$
4	$\mathbb{Z}_4$	abelian	$4=1+1+1+1$
4	$\mathbb{Z}_2\times\mathbb{Z}_2$	abelian	$4=1+1+1+1$
5	$\mathbb{Z}_5$	abelian	$5=1+1+1+1+1$
6	$\mathbb{Z}_6$	abelian	$6=1+1+1+1+1+1$
6	S_3	non abelian	$6=1+2+3$
7	$\mathbb{Z}_7$	abelian	$7=1+1+1+1+1+1+1$
8	$\mathbb{Z}_8$	abelian	$8=1+1+1+1+1+1+1+1$
8	$\mathbb{Z}_2\times\mathbb{Z}_4$	abelian	$8=1+1+1+1+1+1+1+1$
8	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	abelian	$8=1+1+1+1+1+1+1+1$
8	D_4	non abelian	$8=1+1+2+2+2$
8	$\mathbb{Q}_8$	non abelian	$8=1+1+2+2+2$
9	$\mathbb{Z}_9$	abelian	$9=1+1+1+1+1+1+1+1+1$
9	$\mathbb{Z}_3\times\mathbb{Z}_3$	abelian	$9=1+1+1+1+1+1+1+1+1$
10	$\mathbb{Z}_{10}$	abelian	$10=1+1+1+1+1+1+1+1+1+1$
10	D_5	non abelian	$10=1+2+2+5$
11	$\mathbb{Z}_{11}$	abelian	$11=1+1+1+1+1+1+1+1+1+1+1$
12	D_6	non abelian	$12=1+1+3+3+2+2$
24	S_4	non abelian	$24=1+6+6+3+8$

Exercise

1. Find the order of normalizer of the following elements in the indicated group :

   (i) $(1\ 2)$ in S_4 (ii) $(1\ 2\ 3)$ in S_4 (iii) $(1\ 2)$ in S_5 (iv) $(1\ 2\ 3)$ in S_5

   (v) $(1\ 2)$ in S_6 (vi) $(1\ 2\ 3)$ in S_6 (vii) $(1\ 2\ 3)$ in A_4 (viii) $(1\ 2)(3\ 4)$ in A_4

   (ix) ab in D_4 (x) b in D_5 (xi) a^3 in D_6 (xii) a^4b in D_5

2. Write down the class equation for the groups $S_3, S_4, S_5, S_6, A_4, D_4, D_5, D_6$.

Answers

1. (i) 4 (ii) 3 (iii) 12 (iv) 6 (v) 48 (vi) 18 (vii) 3
   (viii) 4 (ix) 4 (x) 2 (xi) 12 (xii) 2

2. $6=1+2+3$, $24=1+6+6+3+8$, $120=1+10+20+30+24+15+20$, $720=1+15+40+90+144+120+45+15+120+40+90$, $12=1+8+3$, $8=1+1+2+2+2$, $10=1+2+2+5$, $12=1+1+3+3+2+2$

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Group Theory - 9

Sylow Theorems

Result 1 : Cauchy theorem for finite abelian groups : If G is a finite abelian group such that $p / o(G)$, p is a prime number , then there exists an element $a(\neq e) \in G$ such that $a^p = e$ i.e. $o(a) = p$.

Result 2 : Converse of lagrange theorem for finite abelian groups : If G is a finite abelian group and $m / o(G)$, where m is a positive integer, then G has at least one subgroup of order m.

Result 3 : Cauchy theorem for finite groups : If G is a finite group such that $p / o(G)$, p is a prime number , then there exists an element of order p in G.

Result 4 : Sylow's first theorem : Let p be a prime number such that $p^m / o(G)$, where m is a positive integer, then G has a subgroup of order p^m .

Another statements of Sylow's first theorem :

(i) If any power of prime divides the order of a group G , then G has a subgroup of order equal to that power of prime.

(ii) If $o(\text{G}) = p^k q$,where p is a prime number and q is a positive integer such that gcd (p, q) = 1 then G has subgroups of orders $p, p^2, \ldots\ldots\ldots\ldots, p^k$.

Def. Sylow p – subgroup : Let p be a prime number such that p^k divides $o(\text{G})$ and p^{k+1} does not divide $o(\text{G})$. Then a subgroup of order p^k is called a Sylow p – subgroup of G.

or

If $o(\text{G}) = p^k q$ where p is a prime number and gcd(p , q) = 1 , then a subgroup of order p^k is called a Sylow p – subgroup of G.

or

Sylow p – subgroup of a group G is a subgroup whose order is p^k where k is the largest power of p such that p^k divides $o(\text{G})$.

or

A subgroup of G is called a Sylow p-subgroup if its order is equal to the maximum power of p occurring in the order of the group.

Result 5 : If H is a Sylow p – subgroup of G , then prove that $x^{-1}Hx$ is also a sylow p-subgroup of G for any $x \in \text{G}$.

Def. p group : Let p be a prime number. A group G is said to be a p – group if order of every element of G is some power of p. e.g. $\text{Q}_8 = \{1, -1, i, -i, j, -j, k, -k\}$

The group of quaternions is a 2 – group because $o(1) = 2^0$, $o(-1) = 2^1$, $o(i, -i, j, -j, k, -k) = 2^2$

i.e. order of every element of Q_8 is a some power of 2.

Result 6 : A finite group G is a p – group iff $o(G) = p^n$ for some integer n.

Result 7 : Sylow's second theorem : Any two Sylow p – subgroups of a finite group G are conjugates in G.

Result 8 : Sylow's third theorem : The number n_p of Sylow p – subgroups of a finite group G is given by

$n_p = 1 + kp$ such that $1 + kp / o(G)$ and k is a non – negative integer.

Result 9 : A Sylow p – subgroup of a finite group G is unique iff it is normal.

Def . Simple group : A group is said to be a simple group if it has no proper normal subgroups. In other words, a group G is said to be simple group if its only normal subgroups are $\{e\}$ and G.

Result 10 : The alternating group A_n , $n \neq 4$, is always a simple group.

Result 11 : A_4 is not a simple group because it has a proper normal subgroup given by

$K_4 = V_4 = \{I , (12)(34) , (13)(24) , (14)(23)\}$. This subgroup is called Klein's four group.

Result 12 : Sylow Test for Nonsimplicity : Let n be a positive integer that is not prime, and let p be a prime divisor of n. If 1 is the only divisor of n that is congruent to 1 modulo p, then there does not exist a simple group of order n.

Result 13 : 2•Odd Test : An integer of the form $2 \cdot n$, where n is an odd number greater than 1, is not the order of a simple group.

Result 14 : If G is a finite group of order ≤ 1000 and G is simple. If $o(G)$ is not a prime then we must have $o(G) = 60$ or 168 or 360 or 504 or 640.

Example 1 : Show that a group of order 28 is not simple.

or

Let $o(G) = 28$, then show that group G has a normal subgroup of order 7.

Solution : We have $o(G) = 28 = 2^2.7^1$. By Sylow first theorem , G has Sylow 2 – subgroups each of order 4 and Sylow 7 – subgroups each of order 7.

By Sylow third theorem , the number n_7 of Sylow 7 – subgroups is given by $1 + 7k$ such that

$$1 + 7k / o(G) \quad \Rightarrow \quad 1 + 7k/28$$

$\Rightarrow \quad 1 + 7k/2^2.7$

$\Rightarrow \quad 1 + 7k/4$ [Since $(1 + 7k , 7) = 1$]

$\Rightarrow \quad k = 0$

Thus , $n_7 = 1$ i.e. there is unique Sylow 7 – subgroup say H and $o(H) = 7$

But we know that "a Sylow p – subgroup is unique iff it is normal".

Thus H is a normal subgroup of order 7. Obviously H is proper. Hence G is not simple.

Example 2 : Let G be a group of order $5^2.7.11$. Then G has how many

(i) Sylow 5 – subgroups

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(ii) Sylow 7 – subgroups

(iii) Sylow 11 – subgroups.

Tell whether G is simple or not.

Solution : (i) By Sylow third theorem , the number n_5 of Sylow 5 – subgroups is given by

$1 + 5k$ such that $1+5k / o(G) \Rightarrow 1+5k / 5^2.7.11$

$\Rightarrow 1+5k / 77$ [Since $(1 + 5k , 5) = 1$]

$\Rightarrow k = 0$ or 2

Thus , $n_5 = 1 + 5.0$ or $1 + 5.2 \Rightarrow n_5 = 1$ or 11

Thus G has one or eleven Sylow 5 – subgroups.

(ii) By Sylow third theorem , the number n_7 of Sylow 7 – subgroups is given by $1 + 7k$ such that $1+7k / o(G)$

$\Rightarrow 1+7k / 5^2.7.11$

$\Rightarrow 1+7k / 275$ [Since $(1 + 7k , 7) = 1$]

$\Rightarrow k = 0$

Thus , $n_7 = 1 + 7.0 = 1$. Thus G has unique sylow 7 – subgroup.

(iii) By Sylow third theorem , the number n_{11} of Sylow 11 – subgroups is given by $1 + 11k$ such that $1+11k / o(G) \Rightarrow 1+11k / 5^2.7.11$

$\Rightarrow 1+11k / 175$ [Since $(1 + 11k , 11) = 1$]

$\Rightarrow k = 0$

Thus , $n_{11} = 1 + 11.0 = 1$

Thus G has unique sylow 11 – subgroup.

Here we see that G has a unique Sylow 7 – subgroup. Since it is unique it must be normal in G and hence G is not simple. (Here same reasoning can be given for Sylow 11 - subgroup).

Example 3 : Show that a group of order 40 is not simple.

-OR-

Show that a group of order 40 has a normal subgroup of order 5.

Solution : We have $o(G) = 40 = 2^3.5^1$. By Sylow third theorem , the number n_5 of Sylow 5 – subgroups is given by $1 + 5k$ such that $1+5k / o(G)$

$\Rightarrow 1+5k / 2^3.5$

$\Rightarrow 1+5k / 8$ [Since $(1+5k, 5) = 1$]

$\Rightarrow k = 0$

$\Rightarrow n_5 = 1 + 5 . 0 = 1$

Thus , G has a unique Sylow 5 – subgroup , say H and $o(\mathrm{H}) = 5$. Since H is unique , it must be normal and hence G is not simple.

Example 4 : Show that a group of order 20449 is not simple.

or

Show that a group of order 20449 has a normal subgroup of order 11.

Solution : We have $o(\mathrm{G}) = 20449 = 11^2.13^2$. By Sylow third theorem , the number n_{11} of Sylow 11 – subgroups is given by $1 + 11k$ such that $1+11k/o(\mathrm{G})$

$\Rightarrow \quad 1+11k/11^2.13^2$

$\Rightarrow \quad 1+11k/169$ [Since $(1+11k, 11^2) = 1$]

$\Rightarrow \quad k = 0 \quad \Rightarrow \quad n_{11} = 1 + 11.0 = 1$

Thus G has a unique Sylow 11– subgroup , say H and $o(\mathrm{H}) = 121$. Since H is unique , it must be normal and hence G is not simple.

Some typical examples on non simplcity of *g*roups : Now we consider some typical examples of non – simplicity of groups. In above examples , we have noted that a particular Sylow p – subgroup comes out be unique at very first step. But in the following examples , we require some more computations.

Example 5 : Show that a group of order 56 is not simple.

Solution : $o(\mathrm{G}) = 56 = 2^3.7^1$

By Sylow third theorem , the number n_7 of Sylow 7 – subgroups is given by $1+7k$ such that

$1+7k/o(\mathrm{G}) = 56 \quad \Rightarrow \quad 1+7k/8$ [Since $(1+7k, 7) = 1$]

$\Rightarrow \quad k = 0$ or 1

$\Rightarrow \quad n_7 = 1+7.0$ or $1+7.1$

$\Rightarrow \quad n_7 = 1$ or 8.

We consider these two cases separately.

Case (i) - $n_7 = 1$ i.e. G has a unique Sylow 7 – subgroup , say , H. Since H is unique , it must be normal and hence G is not simple.

Case (ii) - $n_7 = 8$ i.e. G has eight Sylow 7 – subgroups each of order 7. Let these are

$$\mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3, \ldots\ldots\ldots\ldots, \mathrm{H}_8 \text{ and } \quad o(\mathrm{H}_i) = 7 \text{ for } 1 \le i \le 8$$

We claim that identity is the only common element in all these $\mathrm{H}_i's$.

Now , for $i \neq j$, $\mathrm{H}_i \cap \mathrm{H}_j \subseteq \mathrm{H}_i$

By Lagrange's theorem , $o(\mathrm{H}_i \cap \mathrm{H}_j)/o(\mathrm{H}_i) = 7 \quad \Rightarrow \quad o(\mathrm{H}_i \cap \mathrm{H}_j) = 1$ or 7

If $o(\mathrm{H}_i \cap \mathrm{H}_j) = 7$ then since $o(\mathrm{H}_i) = o(\mathrm{H}_j) = 7$, we obtain $\mathrm{H}_i = \mathrm{H}_j$ i.e., $i = j$, which is not the case.

Hence $o(\mathrm{H}_i \cap \mathrm{H}_j) = 1$ for $i \neq j$ i.e. $\mathrm{H}_i \cap \mathrm{H}_j = \{e\}$ for $i \neq j$

Thus 'e' is the only common element in $\mathrm{H}_1, \mathrm{H}_2, \ldots\ldots\ldots\ldots, \mathrm{H}_8$ .Also all non – identity elements of $\mathrm{H}_i's$ are of order 7. Thus here are $8(7-1) = 48$ non – identity elements of order 7 in all $\mathrm{H}_i's$.

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As $o(\text{G}) = 56$,so remaining elements (which are not of order 7) are $56-48=8$. Further a Sylow 2 – subgroups of G is of order $2^3 = 8$ and it can not have any element of order 7. Thus , these remaining eight elements must form a unique Sylow 2 – subgroup , say K of G. Since K is unique Sylow 2 – subgroup , it must be normal. Hence G is not simple.

Example 6 : Show that a group of order 30 is not simple.

Solution : We have $o(\text{G}) = 30 = 2.3.5$ By Sylow third theorem , the number n_5 of Sylow 5 – subgroups is given by $1+5k$ such that $1+5k / o(\text{G}) = 30$

$\Rightarrow \quad 1+5k/6$ [Since $(1+5k,5)=1$]

$\Rightarrow \quad k = 0$ or 1

$\Rightarrow \quad n_5 = 1 + 5.0$ or $1 + 5.1$

$\Rightarrow \quad n_5 = 1$ or 6

Again , the number n_3 of Sylow 3 – subgroups is given by $1+3k$ such that $1+3k / o(\text{G}) = 30$

$\Rightarrow \quad 1+3k/10$ [Since $(1+3k,3)=1$]

$\Rightarrow \quad k = 0$ or 3

$\Rightarrow \quad n_3 = 1 + 3.0$ or $1 + 3.3$

$\Rightarrow \quad n_3 = 1$ or 10

Thus we have the following four possible cases

Case (i) : $n_5 = 1$ and $n_3 = 1$

Case (ii) : $n_5 = 1$ and $n_3 = 10$

Case (iii) : $n_5 = 6$ and $n_3 = 1$

Case (iv) : $n_5 = 6$ and $n_3 = 10$.

In first three cases , either a Sylow 3 – subgroup or a Sylow 5 – subgroup is unique and hence normal. Thus G is not simple in first three cases.

Now we consider the case – (iv). In this case we have six Sylow 5 – subgroups each of order 5. Let there be $\text{H}_1, \text{H}_2, \ldots\ldots\ldots\ldots, \text{H}_6$. We claim that the identity element is the only common element in all these $\text{H}_i's$.

Let $i \neq j$ and $\text{H}_i \cap \text{H}_j \subseteq \text{H}_i$

By Lagrange's theorem , $o(\text{H}_i \cap \text{H}_j) / o(\text{H}_i) = 5 \quad \Rightarrow \quad o(\text{H}_i \cap \text{H}_j) = 1$ or 5

If $o(\text{H}_i \cap \text{H}_j) = 5$ then since $o(\text{H}_i) = o(\text{H}_j) = 5$, we obtain $\text{H}_i = \text{H}_j$ i.e. $i = j$, which is not the case.

Hence $o(\text{H}_i \cap \text{H}_j) = 1$ for $i \neq j$ i.e. $\text{H}_i \cap \text{H}_j = \{e\}$ for $i \neq j$

Also , all non – identity elements of $\text{H}_i's$ are of order 5. Thus there are $6(5-1) = 24$ non – identity elements of order 5 in all $\text{H}_i's$.

Similarly , we have ten Sylow 3 – subgroups each of order 3.

Let these be $K_1, K_2, ..., K_{10}$

By the same manner as above , we can show that there are $10(3-1)$ = 20 non – identity elements of order 3.

Thus G has 24 + 20 = 44 non – identity elements , which is clearly impossible as $o(G)$ = 30. Hence case – (iv) is not possible. In the first three cases , we have already shown that G is not simple.

Example 7 : Let G be a finite group such that $o(G) = p^n$, where p is a prime. Prove that any subgroup of order p^{n-1} is a normal subgroup of G.

Example 8 : Show that no group of order 108 is simple.

or

Let G be a group of order 108. Show that G has a normal subgroup of order 27 or 9.

Solution : We have $o(G)$ = 108 = $2^2.3^3$.

By Sylow third theorem , the number n_3 of Sylow 3 – subgroups is given by 1 + 3k such that

$$1+3k/o(G)=2^2.3^3$$

$\Rightarrow \quad 1+3k/4$ [Since $(1+3k, 3^3)=1$]

$\Rightarrow \quad k$ = 0 or 1

$\Rightarrow \quad n_3$ = 1 + 3.0 or 1 + 3.1

$\Rightarrow \quad n_3$ = 1 or 4

We consider the two cases separately.

Case (i) : n_3 = 1 i.e. G has a unique Sylow 3 – subgroup , say H. Since H is unique , it must be normal and $o(H)$ = 3^3 = 27. Thus G has a normal subgroup of order 27 in this case and hence G is not simple.

Case(ii) : n_3 = 4 i.e. G has four Sylow 3 – subgroups each of order 27. Let H and K be any two distinct sylow 3 – subgroups. We claim that $o(H \cap K) = 9$ and $H \cap K$ is a normal subgroup of G.

Clearly , $H \cap K \subseteq H$, and so by Lagrange's theorem.

$$o(H \cap K)/o(H) = 27$$

$\Rightarrow \quad o(H \cap K)$ = 1 or 3 or 9 or 27.

If $o(H \cap K)$ = 27 then since $o(H)$ = $o(K)$ = 27 , we obtain H = K , which is a contradiction. Hence $o(H \cap K) \neq 27$.

If $o(H \cap K)$ = 1 or 3 then $o(HK) = \dfrac{o(H)o(K)}{o(H \cap K)} = \dfrac{27.27}{1 \text{ or } 3} > 108 = o(G)$ which is not possible.

Hence $o(H \cap K) \neq 1$, 3 and so $o(H \cap K)$ = 9.

We now show that $H \cap K$ is normal in G. For this we shall prove that $N(H \cap K)$ = G .

Now , we know a result that "if $o(H)$ = p^{n-1} and $o(G)$ = p^n then H is a normal subgroup of G."

Using this result , we see that $H \cap K$ is a normal subgroup of both H and K as $o(H \cap K)$ = 3^2 and $o(H)$ = $o(K)$ = 3^3.

Let $x \in H$ be any element , then

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$$(\mathrm{H}\cap\mathrm{K})x = x(\mathrm{H}\cap\mathrm{K}) \qquad [\text{Since } (\mathrm{H}\cap\mathrm{K}) \trianglelefteq \mathrm{H}]$$

$\Rightarrow \quad x \in \mathrm{N}(\mathrm{H}\cap\mathrm{K})$, normalizer of $\mathrm{H}\cap\mathrm{K}$.

$\Rightarrow \quad \mathrm{H} \subseteq \mathrm{N}(\mathrm{H}\cap\mathrm{K})$

Similarly , $\mathrm{K}\subseteq\mathrm{N}(\mathrm{H}\cap\mathrm{K}) \quad \Rightarrow \quad \mathrm{HK}\subseteq\mathrm{N}(\mathrm{H}\cap\mathrm{K})$

$$\Rightarrow \quad o\big(\mathrm{N}(\mathrm{H}\cap\mathrm{K})\big) \geq o(\mathrm{HK}) = \frac{o(\mathrm{H}).o(\mathrm{K})}{o(\mathrm{H}\cap\mathrm{K})} = \frac{27.27}{9} = 81$$

$$\Rightarrow \quad o\big(\mathrm{N}(\mathrm{H}\cap\mathrm{K})\big) \geq 81 \qquad ......(1)$$

On the other hand , $\mathrm{N}(\mathrm{H}\cap\mathrm{K})$ is a subgroup of G so by Lagrange's theorem

$$o\big(\mathrm{N}(\mathrm{H}\cap\mathrm{K})\big)/o(\mathrm{G}) \quad \text{i.e.} \quad o\big(\mathrm{N}(\mathrm{H}\cap\mathrm{K})\big)/108 \qquad ......(2)$$

Both (1) and (2) are possible only when

$$o\big(\mathrm{N}(\mathrm{H}\cap\mathrm{K})\big) = 108 = o(\mathrm{G})$$

$\Rightarrow \quad o(\mathrm{N}(\mathrm{H}\cap\mathrm{K})) = o(\mathrm{G})$

$\Rightarrow \quad \mathrm{N}(\mathrm{H}\cap\mathrm{K}) = \mathrm{G}$

$\Rightarrow \quad \mathrm{H}\cap\mathrm{K}$ is normal in G. $\qquad$ [Since $\mathrm{N}(H) = G$ iff $\mathrm{H} \trianglelefteq \mathrm{G}$]

Hence G is not simple.

Exercise

1. Show that a group of order 255 is not simple.
2. Show that a group of order 130 is not simple.
3. Show that there is no simple group of order 48.
4. Show that there is no simple group of order 96.
5. Let G be a finite group of order pq, where p and q are prime numbers ($p < q$). Then, G has one subgroup of order q.
6. A group G of order $2p$, where p is prime and $p > 2$ has exactly one subgroup of order p.
7. Find all subgroups of S_n for $1 \leq n \leq 4$. Also find all Sylow p–subgroups for each prime p dividing the order of S_n.
8. Find all subgroups of A_n for $1 \leq n \leq 4$. Also find all Sylow p–subgroups for each prime p dividing the order of A_n.
9. How many Sylow 5-subgroups are there in $S_5, S_6, S_7, A_5, A_6, A_7$.

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Group Theory - 10

Results :

1. The sets $\mathbb{Z},\mathbb{Q},\mathbb{R},\mathbb{C}$ are abelian groups under addition.
2. $\mathbb{Z}$ is cyclic group under addition with generators 1 and –1.
3. $\mathbb{Q},\mathbb{R},\mathbb{C}$ are not cyclic groups.
4. $\mathbb{Q}^*,\mathbb{R}^*,\mathbb{C}^*,\mathbb{Q}^+,\mathbb{R}^+$ are abelian groups w.r.t. multiplication and all these groups are non-cyclic.
5. General linear group : The set $GL(n,F)=\{[a_{ij}]_{n\times n}: a_{ij}\in F,\ \det[a_{ij}]\neq 0, F \text{ is a field}\}$ is a group w.r.t. matrix multiplication and is known as general linear group.
6. Special linear group : The set $SL(n,F)=\{[a_{ij}]_{n\times n}: a_{ij}\in F,\ \det[a_{ij}]=1, F \text{ is a field}\}$ is a group w.r.t. matrix multiplication and is known as special linear group.
7. $GL(n,F)$ and $SL(n,F)$ are non-abelian groups when $n>1$.
8. $o\left(GL(n,\mathbb{Z}_p)\right)=\left(p^n-p^{n-1}\right)\left(p^n-p^{n-2}\right)......\left(p^n-1\right)$.
9. $o\left(SL(n,\mathbb{Z}_p)\right)=\dfrac{\left(p^n-p^{n-1}\right)\left(p^n-p^{n-2}\right)......\left(p^n-1\right)}{p-1}$
10. Centre of general linear group is the set of all scalar matrices in it. i.e.,

$$Z\left(GL(n,F)\right)=\left\{\begin{bmatrix} a & 0 & ... & 0\\ 0 & a & ... & 0\\ ... & ... & ... & ...\\ 0 & 0 & ... & a\end{bmatrix} : a\neq 0,\ a\in F\right\}.$$

11. $o\left(Z\left(GL(n,\mathbb{Z}_p)\right)\right)=p-1$.
12. Centre of special linear group is the set of all scalar matrices in it. i.e.,

$$Z\left(SL(n,F)\right)=\left\{\begin{bmatrix} a & 0 & ... & 0\\ 0 & a & ... & 0\\ ... & ... & ... & ...\\ 0 & 0 & ... & a\end{bmatrix} : a^n=1, a\in F\right\}.$$

13. $o\left(Z\left(SL(n,\mathbb{C})\right)\right)=n$.
14. $o\left(Z\left(SL(n,\mathbb{R})\right)\right)=\begin{cases}1 & \text{if } n \text{ is odd}\\ 2 & \text{if } n \text{ is even}\end{cases}$.

15. $o\left(Z\left(SL\left(n,\mathbb{Z}_p\right)\right)\right)=\gcd\left(n,p-1\right)$.

16. For any group *G*, we have $Z\left(Z\left(G\right)\right)=Z\left(G\right)$.

17. $\mathbb{Q}/\mathbb{Z}$ is an infinite group with every element of finite order. Further it has elements of every finite order.

18. Number of elements of order *n* in the group $\mathbb{Q}/\mathbb{Z}$ is $\phi\left(n\right)$ and these are of the form $\frac{k}{n}+\mathbb{Z}$ where $\gcd\left(k,n\right)=1$.

19. $\mathbb{Q}/\mathbb{Z}$ has a unique subgroup of order *n* for every positive integer *n*. Further this subgroup is cyclic and is given by $H=\left\langle \frac{1}{n}+\mathbb{Z}\right\rangle$.

20. The set of all *n*th roots of unity is a multiplicative cyclic group of order *n* and therefore is isomorphic to $\mathbb{Z}_n$. In symbols $G_n=\left\{z:z^n=1\right\}$ is a multiplicative cyclic group of order *n*. e.g. $G_2=\{1,-1\}$, $G_3=\left\{1,\omega,\omega^2\right\}$, $G_4=\{1,-1,i,-i\}$ and so on.

21. The set $G=\left\{z:z^n=1,n=1,2,3,....\right\}$ is an infinite abelian group w.r.t. multiplication. This group is not cyclic. This is an infinite group with every element of finite order. Every finite subgroup of *G* is cyclic.

Exercise

1. Show that the group $GL\left(2,\mathbb{R}\right)$ is non-abelian, by exhibiting a pair of matrices *A* and *B* in $GL\left(2,\mathbb{R}\right)$ such that $AB\neq BA$.

2. Find the inverse of the element $\begin{bmatrix}2 & 6\\ 3 & 5\end{bmatrix}$ in $GL\left(2,\mathbb{Z}_{11}\right)$.

3. (From the GRE practice exam) Let *p* and *q* be distinct primes. Suppose that *H* is a proper subset of integers and *H* is a group under addition that contains exactly three elements of the set $\left\{p,p+q,pq,p^q,q^p\right\}$. Determine which of the following are the three elements in *H*.

   a. pq,p^q,q^p b. $p+q,pq,p^q$ c. $p,p+q,pq$ d. p,p^q,q^p e. p,pq,p^q

4. Prove that the set of all 2×2 matrices with entries from $\mathbb{R}$ and determinant $+1$ is a group under matrix multiplication

5. Prove that the set of all rational numbers of the form 3^m6^n, where *m* and *n* are integers, is a group under multiplication.

6. Prove that the set of all 3×3 matrices with real entries of the form $\begin{bmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{bmatrix}$ is a group.

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7. Let $G=\left\{\begin{bmatrix} a & a \\ a & a \end{bmatrix} \middle| a\in\mathbb{R}, a\neq 0\right\}$. Show that G is a group under matrix multiplication.

8. Let $\mathbb{Q}$ be the group of rational numbers under addition and let $\mathbb{Q}^*$ be the group of non zero rational numbers under multiplication. In $\mathbb{Q}$, list the elements in $\left\langle \frac{1}{2} \right\rangle$. In $\mathbb{Q}^*$, list the elements in $\left\langle \frac{1}{2} \right\rangle$.

9. Find the order of each element in $\mathbb{Q}$ and in $\mathbb{Q}^*$.

10. Suppose that H is a proper subgroup of $\mathbb{Z}$ under addition and H contains 18, 30 and 40. Determine H.

11. Consider the elements $A=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ from $SL(2,\mathbb{R})$. Find $|A|, |B|,$ and $|AB|$. Does your answer surprise you ?

12. Consider the element $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ in $SL(2,\mathbb{R})$. What is the order of A ? If we view $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ as a member of $SL(2,\mathbb{Z}_p)$ (p is a prime), what is the order of A ?

13. For any positive integer n and any angle θ, show that in the group $SL(2,\mathbb{R})$,

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Use this formula to find the order of

$$\begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} \text{ and } \begin{bmatrix} \cos\sqrt{2}^\circ & -\sin\sqrt{2}^\circ \\ \sin\sqrt{2}^\circ & \cos\sqrt{2}^\circ \end{bmatrix}.$$

14. Let G be the symmetry group of a circle. Show that G has elements of every finite order as well as elements of infinite order.

15. Let $G=\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a,b,c,d\in\mathbb{Z}\right\}$ under addition. Let $H=\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \in G,\ a+b+c+d=0\right\}$. Prove that H is a subgroup of G. What if 0 is replaced by 1 ?

16. Let $G=GL(2,\mathbb{R})$. Let $H=\{A\in G \mid \det A \text{ is a power of } 2\}$. Show that H is a subgroup of G.

17. Let H be a subgroup of $\mathbb{R}$ under addition. Let $K=\{2^a \mid a\in H\}$. Prove that K is subgroup of $\mathbb{R}^*$ under multiplication.

18. Let G be a group of functions from $\mathbb{R}$ to $\mathbb{R}^*$ under multiplication. Let $H=\{f\in G \mid f(1)=1\}$. Prove that H is a subgroup of G.

19. Let $G=GL(2,\mathbb{R})$ and $H=\left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a \text{ and } b \text{ are nonzero integers}\right\}$. Prove or disprove that H is a subgroup of G.

20. Let $H=\{a+bi \mid a,b\in\mathbb{R}, ab\geq 0\}$. Prove or disprove that H is a subgroup of $\mathbb{C}$ under mulitiplication.

21. $H=\{a+bi \mid a,b\in\mathbb{R}, a^2+b^2=1\}$. Prove or disprove that H is a subgroup of $\mathbb{C}^*$ under multiplication. Describe the elements of H geometrically.

22. Find the smallest subgroup of $\mathbb{Z}$ containing

    a. 8 and 14 (the notation for this is $\langle 8,14\rangle$), b. 8 and 13, c. 6 and 15,

    d. m and n, e. 12, 18, and 45.

    In each part, find an integer k such that the subgroup is $\langle k\rangle$.

23. Let $G=GL(2,\mathbb{R})$.

    a. Find $N\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right)$. b. Find $N\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$. c. Find $Z(G)$.

24. Let $\mathbb{Z}$ denote the group of integers under addition. Is every subgroup of $\mathbb{Z}$ cyclic ? Why ? Describe all the subgroups of $\mathbb{Z}$.

25. Find all generators of $\mathbb{Z}$.

26. Let a be an element of a group and suppose that a has infinite order. How many generators does $\langle a\rangle$ have ?

27. Show that the group of positive rational numbers under multiplication is not cyclic.

28. Let m and n be elements of the group $\mathbb{Z}$. Find a generator for the group $\langle m\rangle\cap\langle n\rangle$.

29. Prove that an infinite group must have an infinite number of subgroups.

30. Prove that $H=\left\{\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n\in\mathbb{Z}\right\}$ is cyclic subgroup of $GL(2,\mathbb{R})$.

31. Let $G=\{a+b\sqrt{2}\}$, where a and b are rational numbers not both 0. Prove that G is a group under ordinary multiplication.

32. Suppose that H and K are nontrivial subgroups of $\mathbb{Q}$ under addition. Show that $H\cap K$ is a nontrivial subgroup of $\mathbb{Q}$. Is this true if $\mathbb{Q}$ is replaced by $\mathbb{R}$?

33. Let $\mathbb{R}^*$ be a group of nonzero real numbers under multiplication and let $H=\{x\in\mathbb{R}^* : x^2 \text{ is rational}\}$. Prove that H is a subgroup of $\mathbb{R}^*$.

34. Let $G=GL(2,\mathbb{R})$ and let $H=\{A\in G|\det A \text{is rational}\}$. Prove or disprove that H is a subgroup of G. What if we replace "rational" by "integer" ?

35. Prove that $\mathbb{Z}$ under addition is not isomorphic to $\mathbb{Q}$ under addition.

36. Let $\mathbb{C}$ be the complex numbers and $M=\left\{\begin{bmatrix} a & -b \\ b & a\end{bmatrix}\middle| a,b\in\mathbb{R}\right\}$.

    Prove that $\mathbb{C}$ and M are isomorphic under addition and that $\mathbb{C}^*$ and M^* the nonzero elements of M are isomorphic under multiplication.

37. Let $G=\{0,\pm 2,\pm 4,\pm 6,...\}$ and $H=\{0,\pm 3,\pm 6,\pm 9,...\}$. Show that G and H are isomorphic groups under addition.

38. Prove that $\mathbb{Q}$ under addition is not isomorphic to $\mathbb{R}^+$ under multiplication.

39. Let n be an integer greater than 1. Let $H=\{0,\pm n,\pm 2n,\pm 3n,...\}$. Find all left cosets of H in $\mathbb{Z}$. How many are there ?

40. Let M be the group of all real 2×2 matrices under addition. Let $N=\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$ under componentwise addition. Prove that M and N are isomorphic. What is the corresponding theorem for the group of $n\times n$ matrices under addition ?

41. Let $H=\left\{\begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}\middle| a,b\in\mathbb{Z}_3\right\}$. Show that H is an abelian group of order 9. Is H isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3\oplus\mathbb{Z}_3$?

42. Let $G=\{3^m 6^n \mid m,n\in\mathbb{Z}\}$ under multiplication. Prove that G is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$.

43. Let $(a_1,a_2,...,a_n)\in G_1\oplus G_2\oplus ...G_n$. Give a necessary and sufficient condition for $|(a_1,a_2,...,a_n)|=\infty$.

44. Let $G=\mathbb{Z}_3\oplus\mathbb{Z}_3\oplus\mathbb{Z}_3$ and let H be the subgroup of $SL(3,\mathbb{Z}_3)$, consisting of

    $H=\left\{\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1\end{bmatrix}\middle| a,b,c\in\mathbb{Z}_3\right\}$. Determine the number of elements of each order in G and H. Are G and H isomorphic ?

45. Let $\mathbb{R}^+$ denote he multiplicative group of positive reals and let $T=\{z\in\mathbb{C} \mid |z|=1\}$ be the multiplicative group of complex numbers of norm 1. (Recall that $|a+bi|=\sqrt{a^2+b^2}$.) Show that every element of $\mathbb{C}^*$ can be uniquely expressed in the form of *rz* where $r\in\mathbb{R}^+$ and $z\in T$.
46. Prove that $\mathbb{Q}^*$ under multiplication is not isomorphic to $\mathbb{R}^*$ under multiplication.
47. Prove that $\mathbb{Q}$ under addition is not isomorphic to $\mathbb{R}$ under addition.
48. Prove that $\mathbb{R}$ under addition is not isomorphic to $\mathbb{R}^*$ under multiplication.
49. Show that $\mathbb{Q}^+$ (the set of positive rational numbers) under multiplication is not isomorphic to $\mathbb{Q}$ under addition.
50. Give an example of a group *G* with a proper subgroup *H* such that *G* and *H* are isomorphic.
51. In $\mathbb{R}\oplus\mathbb{R}$ under componentwise addition, let $H=\{(x,3x) \mid x\in\mathbb{R}\}$. (Note that *H* is the subgroup of all points on the line $y=3x$.) Show that (2, 5) +*H* is a straight line passing through the point (2, 5) and parallel to the line $y=3x$.
52. In $\mathbb{R}\oplus\mathbb{R}$, suppose that *H* is the subgroup of all points lying on a line through the origin. Show that any left coset of *H* is just a line parallel to *H*.
53. Let $H=\left\{\begin{bmatrix} a & b \\ 0 & d\end{bmatrix} \middle| a,b,d\in\mathbb{R},\ ad\neq 0\right\}$. Is *H* a normal subgroup of $GL(2,\mathbb{R})$?
54. Prove that $SL(2,\mathbb{R})$ is a normal subgroup of $GL(2,\mathbb{R})$.
55. Viewing $\langle 3\rangle$ and $\langle 12\rangle$ as subgroups of $\mathbb{Z}$, prove that $\langle 3\rangle/\langle 12\rangle$ is isomorphic to $\mathbb{Z}_4$. Similarly, prove that $\langle 8\rangle/\langle 48\rangle$ is isomorphic to $\mathbb{Z}_6$. Generalize to arbitrary integers *k* and *n*.
56. Determine the order of $(\mathbb{Z}\oplus\mathbb{Z})/\langle(2,2)\rangle$. Is the group cyclic ?
57. Determine the order of $(\mathbb{Z}\oplus\mathbb{Z})/\langle(4,2)\rangle$. Is the group cyclic ?
58. Let $G=GL(2,\mathbb{R})$ and $H=\{A\in G \mid \det A=3^k,\ k\in\mathbb{Z}\}$. Prove that *H* is a normal subgroup of *G*.
59. Let $\mathbb{R}^*$ denote the group of all nonzero real numbers under multiplication. Let $\mathbb{R}^+$ denote the group of positive real numbers under multiplication. Prove that $\mathbb{R}^*$ is the internal direct product of $\mathbb{R}^+$ and the subgroup $\{1,-1\}$.
60. Let $\mathbb{R}^*$ be the group of nonzero real numbers under multiplication, and let *r* be a positive integer. Show that the mapping that takes *x* to x^r is a homomorphism from $\mathbb{R}^*$ to $\mathbb{R}^*$.
61. Let *G* be the group of all polynomials with real coefficients under addition. For each *f* in *G*, let $\int f$ denote the antiderivative of *f* that passes through the point (0, 0). Show that the mapping $f\to\int f$ from

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G to G is a homomorphism. What is the kernel of this mapping ? Is this mapping a homomorphism if $\int f$ denotes the antiderivative of f that passes through (0, 1) ?

62. Prove that the mapping $\phi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ given by $(a,b) \to a-b$ is a homomorphism. What is the kernel of ϕ ? Describe the set $\phi^{-1}(3)$ (that is, all elements that map to 3).
63. Suppose that there is a homomorphism ϕ from $\mathbb{Z} \oplus \mathbb{Z}$ to a group G such that $\phi(3,2)=a$ and $\phi(2,1)=b$. Determine $\phi(4,4)$ in terms of a and b. Assume that the operation of G is addition.
64. Prove that the mapping $x \to x^6$ from $\mathbb{C}^*$ to $\mathbb{C}^*$ is a homomorphism. What is the kernel ?
65. For each pair of positive integers m and n, we can define a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_m \oplus \mathbb{Z}_n$ by $x \to (x \bmod m, x \bmod n)$. What is the kernel when $(m, n) = (3, 4)$? What is the kernel when $(m, n) = (6, 4)$?
66. Let $\mathbb{Z}[x]$ be the group of polynomials in x with integer coefficients under addition. Prove that the mapping from $\mathbb{Z}[x]$ into $\mathbb{Z}$ given by $f(x) \to f(3)$ is a homomorphism. Give a geometric description of the kernel of this homomorphism. Generalize.
67. The factor group $GL(2,\mathbb{R})/SL(2,\mathbb{R})$ is isomorphic to some very familiar group. What is the group ?
68. Prove that $\mathbb{Q}/\mathbb{Z}$ under addition is an infinite group in which every element has finite order.
69. Show that $\mathbb{Q}/\mathbb{Z}$ has a unique subgroup of order n for each positive integer n.
70. Let $G = \mathbb{Z} \oplus \mathbb{Z}$ and $H = \{(x, y) \mid x \text{ and } y \text{ are even integers}\}$. Show that H is a subgroup of G. Determine the order of G/H. To which familiar group is G/H isomorphic ?

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-------------------- S C Q ------------------

1. Consider a group G. Let $Z(G)$ be its centre. i.e. $Z(G)=\{g \in G : gh = hg \text{ for all } h \in G\}$. For $n \in \mathbb{N}$, the set of positive integers, define $J_n = \{(g_1,...,g_n) \in Z(G)\times....\times Z(G) : g_1...g_n = e\}$.
   As a subset of the direct product group $G\times...\times G$ (n times direct product of the group G), J_n is
   1. not necessarily a subgroup.
   2. a subgroup but not necessarily a normal subgroup.
   3. a normal subgroup.
   4. isomorphic to the direct product $Z(G)\times...\times Z(G)$ $(n-1)$ times).

   (CSIR NET June 2011)
2. Let G be a group of order 77. Then the centre of G is isomorphic to
   1. $\mathbb{Z}_{(1)}$ 2. $\mathbb{Z}_{(7)}$ 3. $\mathbb{Z}_{(11)}$ 4. $\mathbb{Z}_{(77)}$

   (CSIR NET June 2011)
3. Let p be a prime number. The order of a p – Sylow subgroup of the group $GL_{50}(\mathbb{F}_p)$ of invertible 50×50 matrices with entries from the finite field F_p, equals :
   1. p^{50} 2. p^{125} 3. p^{1250} 4. p^{1225}

   (CSIR NET Dec 2011)
4. The number of group homomorphisms from the symmetric group S_3 to $\frac{\mathbb{Z}}{6\mathbb{Z}}$ is
   1. 1 2. 2 3. 3 4. 6

   (CSIR NET Dec 2011)
5. Consider the group $G=\mathbb{Q}/\mathbb{Z}$ where $\mathbb{Q}$ and $\mathbb{Z}$ are the groups of rational numbers and integers respectively. Let n be a positive integer. Then is there a cyclic subgroup of order n ?
   1. not necessarily.
   2. yes, a unique one.
   3. yes, but not necessarily a unique one.
   4. never.

   (CSIR NET June 2012)
6. Let G be a simple group of order 168. What is the number of subgroups of G of order 7 ?
   1. 1 2. 7 3. 8 4. 28

   (CSIR NET June 2013)
7. The number of group homomorphisms from the symmetric group S_3 to the additive group $\mathbb{Z}/6\mathbb{Z}$ is
   1. 1 2. 2 3. 3 4. 0

   (CSIR NET Dec 2013)
8. The total number of non isomorphic groups of order 122 is
   1. 2 2. 1 3. 61 4. 4

   (CSIR NET June 2014)
9. In the group of all invertible 4×4 matrices with entries in the field of 3 elements, any 3–Sylow subgroup has cardinality
   1. 3 2. 81 3. 243 4. 729

   (CSIR NET Dec 2014)
10. The number of conjugacy classes in the permutation group S_6 is
    1. 12 2. 11 3. 10 4. 6

    (CSIR NET Dec 2014)
11. Up to isomorphism, the number of abelian groups of order 108 is :
    1. 12 2. 9
    3. 6 4. 5

    (CSIR NET June 2015)
12. A group G is generated by the elements x, y with the relations $x^3 = y^2 = (xy)^2 = 1$. The order of G is
    1. 4 2. 6
    3. 8 4. 12

    (CSIR NET Dec 2015)
13. Let $G = (\mathbb{Z}/25\mathbb{Z})^*$ be the group of units (i.e. the elements that have a multiplicative inverse) in the ring $(\mathbb{Z}/25\mathbb{Z})$. Which of the following is a generator of G?
    1. 3 2. 4
    3. 5 4. 6

    (CSIR NET June 2016)
14. Let $p \geq 5$ be a prime. Then
    1. $F_p \times F_p$ has at least five subgroups of order p.

2. Every subgroup of $F_p \times F_p$ is of the form $H_1 \times H_2$ where H_1, H_2 are subgroups of F_p.
3. Every subgroup of $F_p \times F_p$ is an ideal of the ring $F_p \times F_p$.
4. The ring $F_p \times F_p$ is a field.

(CSIR NET June 2016)

15. Let S_n denote the permutation group on n symbols and A_n be the subgroup of even permutations. Which of the following is true ?
1. There exists a finite group which is not a subgroup of S_n for any $n \geq 1$.
2. Every finite group is a subgroup of A_n for some $n \geq 1$.
3. Every finite group is a quotient of A_n for some $n \geq 1$.
4. No finite abelian group is a quotient of S_n for $n > 3$.

(CSIR NET Dec 2016)

16. Let G be an open set in $\mathbb{R}^n$. Two points $x, y \in G$ are said to be equivalent if they can be joined by a continuous path completely lying inside G. Number of equivalence classes is
1. only one.
2. at most finite.
3. at most countable.
4. can be finite, countable or uncountable.

(CSIR NET Dec 2016)

17. What is the number of non-singular 3×3 matrices over $\mathbb{F}_2$, the finite field with two elements ?

1. 168 2. 384
3. 2^3 4. 3^2

(CSIR NET Dec 2016)

18. Let $f : \mathbb{Z} \to (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/6\mathbb{Z})$ be the function $f(n) = (n \bmod 4, n \bmod 6)$. Then
1. $(0 \bmod 4, 3 \bmod 6)$ is in the image of f
2. $(a \bmod 4, b \bmod 6)$ is in the image of f, for all even integers a and b
3. image of f has exactly 6 elements
4. kernel of $f = 24\mathbb{Z}$

(CSIR NET Dec 2017)

19. Let S_7 denote the group of permutations of the set $\{1,2,3,4,5,6,7\}$. Which of the following is true ?
1. There are no elements of order 6 in S_7
2. There are no elements of order 7 in S_7
3. There are no elements of order 8 in S_7
4. There are no elements of order 10 in S_7

(CSIR NET June 2018)

20. The number of group homomorphisms from $\mathbb{Z}_{10}$ to $\mathbb{Z}_{20}$ is

1. zero 2. one
3. five 4. ten

(CSIR NET June 2018)

-------------------- M C Q ----------------

1. If a non-trivial group has only one generator then the number of elements in the group is
   1. an odd prime
   2. can not be prime greater than 2
   3. even integer
   4. 2
2. Let G be a finite abelian group of odd order let $H = \{x^2 \mid x \in G\}$, then
   1. H is a subgroup of G only if G is cyclic
   2. H is a proper subsgroup of G
   3. $H = G$
   4. H may not be subgroup of G
3. Let $H = \{e, (1,2)(3,4)\}$ and $K = \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ be subgroups of S_4 where e denotes the identify element of S_4. Then
   1. H and K are normal subgroups of S_4
   2. H is normal in K and K is normal in A_4
   3. H is normal in A_4 but not normal in S_4
   4. K is normal in S_4 but H is not

(CSIR NET June 2011)

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4. Let $G = \mathbb{Z}_{10} \times \mathbb{Z}_{15}$. Then
   1. G contains exactly one element of order 2
   2. G contains exactly 5 elements of order 3
   3. G contains exactly 24 elements of order 5
   4. G contains exactly 24 elements of order 10

(CSIR NET June 2011)

5. Which of the following numbers can be orders of permutations σ of 11 symbols s.t. σ does not fix any symbol ?
   1. 18 2. 30 3. 15 4. 28

(CSIR NET Dec 2011)

6. For any group G of order 36 and any subgroup H of G order 4,
   1. $H \subset Z(G)$
   2. $H = Z(G)$
   3. H is normal in G
   4. H is an abelian group

(CSIR NET June 2012)

7. Let G denote the group $S_4 \times S_3$. Then
   1. a 2–Sylow subgroup of G is normal
   2. a 3–Sylow subgroup of G is normal
   3. G has a nontrivial normal subgroup
   4. G has a normal subgroup of order 72

(CSIR NET June 2012)

8. For a positive integer $n \geq 4$ and a prime number $p \leq n$, let $U_{p,n}$ denote the union of all p –Sylow subgroup of the alternating group A_n on n letters. Also let $K_{p,n}$ denote the subgroup of A_n generated by $U_{p,n}$, and let $|K_{p,n}|$ denote the order of $K_{p,n}$. Then
   1. $|K_{2,4}| = 12$ 2. $|K_{2,4}| = 4$
   3. $|K_{2,5}| = 60$ 4. $|K_{3,5}| = 30$

(CSIR NET Dec 2012)

9. Let σ = (1 2)(3 4 5) and τ =(1 2 3 4 5 6) be permutations in S_6, the group of permutations on six symbols. Which of the following statements are true ?
   1. The subgroups $<\sigma>$ and $<\tau>$ are isomorphic to each other
   2. σ and τ are conjugate in S_6
   3. $<\sigma> \cap <\tau>$ is the trivial group
   4. σ and τ commute

(CSIR NET June 2013)

10. Let S_n denote the symmetric group on n symbols. The group $S_3 \oplus (\mathbb{Z}/2\mathbb{Z})$ is isomorphic to which of the following groups ?
   1. $\mathbb{Z}/12\mathbb{Z}$
   2. $(\mathbb{Z}/6\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$
   3. A_4, the alternating group of order 12
   4. D_6, the dihedral group of order 12

(CSIR NET June 2013)

11. How many normal subgroups does a non abelian group G of order 21 have other than the identity subgroup $\{e\}$ and G ?
   1. 0 2. 1 3. 3 4. 7

(CSIR NET Dec 2013)

12. Determine which of the following cannot be the class equation of a group
   1. 10 = 1+1+1+2+5.
   2. 4 = 1+1+2.
   3. 8 = 1+1+3+3.
   4. 6 = 1+2+3.

(CSIR NET Dec 2013)

13. Consider the multiplicative group G of all the (complex) 2^n-th roots of unity where n = 0, 1, 2 ... Then
   1. Every proper subgroup of G is finite.
   2. G has a finite set of generators.
   3. G is cyclic.
   4. Every finite subgroup of G is cyclic.

(CSIR NET June 2014)

14. Let G be a non abelian group. Then, its order can be
   1. 25 2. 55 3. 125 4. 35

(CSIR NET Dec 2014)

15. Let G be a group of order 45. Then
   1. G has an element of order 9
   2. G has a subgroup of order 9
   3. G has a normal subgroup of order 9
   4. G has a normal subgroup of order 5

(CSIR NET Dec 2014)

16. Let $\sigma: \{1,2,3,4,5\} \to \{1,2,3,4,5\}$ be a permutation (one-to-one and onto function) such that $\sigma^{-1}(j) \leq \sigma(j) \quad \forall\, j, 1 \leq j \leq 5$.

Then which of the following are true ?

1. $\sigma\circ\sigma(j)=j$ for all j, $1\le j\le 5$
2. $\sigma^{-1}(j)=\sigma(j)$ for all j, $1\le j\le 5$
3. The set $\{k:\sigma(k)\neq k\}$ has an even number of elements.
4. The set $\{k:\sigma(k)=k\}$ has an odd number of elements.

(CSIR NET June 2015)

17. If x, y and z are elements of a group such that $xyz=1$, then

1. $yzx=1$
2. $yxz=1$
3. $zxy=1$
4. $zyx=1$

(CSIR NET June 2015)

18. Which of the following cannot be the class equation of a group of order 10 ?

1. 1+1+1+2+5 = 10
2. 1+2+3+4 = 10
3. 1+2+2+5 = 10
4. 1+1+2+2+2+2 = 10

(CSIR NET June 2015)

19. Let a_n denote the number of those permutations σ on $\{1,2,...,n\}$ such that σ is a product of exactly two disjoint cycles. Then

1. $a_5=50$
2. $a_4=14$
3. $a_5=40$
4. $a_4=11$

(CSIR NET Dec 2015)

20. Let G be a simple group of order 60. Then

1. G has six Sylow-5 subgroups.
2. G has four Sylow-3 subgroups.
3. G has a cyclic subgroup of order 6.
4. G has a unique element of order 2.

(CSIR NET Dec 2015)

21. For $n\ge 1$, let $(\mathbb{Z}/n\mathbb{Z})^*$ be the group of units of $(\mathbb{Z}/n\mathbb{Z})$. Which of the following groups are cyclic ?

1. $(\mathbb{Z}/10\mathbb{Z})^*$
2. $(\mathbb{Z}/2^3\mathbb{Z})^*$
3. $(\mathbb{Z}/100\mathbb{Z})^*$
4. $(\mathbb{Z}/163\mathbb{Z})^*$

(CSIR NET Dec 2015)

22. Let G be a finite abelian group of order n. Pick each correct statement from below.

1. If d divides n, there exists a subgroup of G of order d.
2. If d divides n, there exists an element of order d in G.
3. If every proper subgroup of G is cyclic, then G is cyclic.
4. If H is a subgroup of G, there exists a subgroup N of G such that $G/N\cong H$.

(CSIR NET June 2016)

23. Consider the symmetric group S_{20} and its subgroup A_{20} consisting of all even permutations. Let H be a 7-sylow subgroup of A_{20}. Pick each correct statement from below :

1. $|H|=49$.
2. H must be cyclic.
3. H is a normal subgroup of A_{20}.
4. Any 7-Sylow subgroup of S_{20} is a subset of A_{20}.

(CSIR NET June 2016)

24. Let p be a prime. Pick each correct statement from below. Up to isomorphism,

1. there are exactly two abelian groups of order p^2.
2. there are exactly two groups of order p^2.
3. there are exactly two commutative rings of order p^2.
4. there is exactly one integral domain of order p^2.

(CSIR NET June 2016)

25. Consider the following subsets of the group of 2×2 non-singular matrices over $\mathbb{R}$:

$$G=\left\{\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}: a,b,d\in\mathbb{R},\ ad=1\right\}$$

$$H=\left\{\begin{pmatrix} 1 & b\\ 0 & 1\end{pmatrix}: b\in\mathbb{R}\right\}$$

Which of the following statements are correct ?

1. G forms a group under matrix multiplication
2. H is a normal subgroup of G.
3. The quotient group G/H is well-defined and is Abelian.
4. The quotient group G/H is well defined and is isomorphic to the group of 2×2 diagonal matrices (over $\mathbb{R}$) with determinant 1.

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26. Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}^*$ be the group of non zero complex

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numbers under multiplication. Then which of the following are true ?

1. $\mathbb{C}^*$ is cyclic.
2. Every finite subgroup of $\mathbb{C}^*$ is cyclic.
3. $\mathbb{C}^*$ has finitely many finite subgroups.
4. Every proper subgroup of $\mathbb{C}^*$ is cyclic.

(CSIR NET Dec 2016)

27. For an integer $n \geq 2$, let S_n be the permutation group on n letters and A_n the alternating group. Let $\mathbb{C}^*$ be the group of non-zero complex numbers under multiplication. Which of the following are correct statements ?
    1. For every integer $n \geq 2$, there is a nontrivial homomorphism $\chi : S_n \to \mathbb{C}^*$
    2. For every integer $n \geq 2$, there is a unique nontrivial homomorphism $\chi : S_n \to \mathbb{C}^*$
    3. For every integer $n \geq 3$, there is a nontrivial homomorphism $\chi : A_n \to \mathbb{C}^*$
    4. For every integer $n \geq 5$, there is no nontrivial homomorphism $\chi : A_n \to \mathbb{C}^*$.

(CSIR NET June 2017)

28. Let G be a group of order 125. Which of the following statements are necessarily true ?
    1. G has a non-trivial abelian subgroup
    2. The centre of G is a proper subgroup
    3. The centre of G has order 5
    4. There is a subgroup of order 25

(CSIR NET June 2017)

29. Let F be a finite field and let K / F be a field extension of degree 6. Then the Galois group of K / F is isomorphic to
    1. the cyclic group of order 6
    2. the permutation group on $\{1, 2, 3\}$
    3. the permutation group on $\{1, 2, 3, 4, 5, 6\}$
    4. the permutation group on $\{1\}$

(CSIR NET Dec 2017)

30. Let $G = S_3$ be the permutation group of 3 symbols. Then
    1. G is isomorphic to a subgroup of a cyclic group
    2. there exists a cyclic group H such that G maps homomorphically onto H
    3. G is a product of cyclic groups
    4. there exists a nontrivial group homomorphism from G to the additive group $(\mathbb{Q}, +)$ of rational numbers

(CSIR NET June 2018)

31. Let G be a group with $|G| = 96$. Suppose H and K are subgroups of G with $|H| = 12$ and $|K| = 16$. Then
    1. $H \cap K = \{e\}$
    2. $H \cap K \neq \{e\}$
    3. $H \cap K$ is Abelian
    4. $H \cap K$ is not Abelian

(CSIR NET June 2018)

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Answer Key

SCQ

1. 3	2. 4	3. 4	4. 2	5. 2	6. 3	7. 2
8. 1	9. 4	10. 2	11. 3	12. 2	13. 1	14. 1
15. 2	16. 3	17. 1	18. 2	19. 3	20. 4	

M C Q

1. --	2. --	3. 2,4	4. 1,3,4	5. 1,2,3,4	6. 4	7. 3,4
8. 2,3	9. 1,3	10. 4	11. 2	12. 1,2,3	13. 1,4	14. 2,3
15. 2,3,4	16. 1,2,3,4	17. 1,3	18. 1,2,4	19. 1,4	20. 1	21. 1,4
22. 1,4	23. 1,4	24. 1,2,4	25. 1,2,3,4	26. 2	27. 1,2,4	28. 1,4
29. 1	30. 2,3	31. 2,3				