Review of Probability

Prob. and Stats.,
CSVTU Bhilai

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Introduction

C.S. PEIRCE

The theory of probabilities is simply the Science of logic quantitatively treated.

Probability is the measure of uncertainty of events in a random experiment. The probability of an event A can be defined as the Number of favorable outcomes of an event ($n(A)$) divided by the total number of possible outcomes of an event ($n(S)$).

Properties we know:

- $0 \leq P(A) \leq 1$
- $\sum_i A_i = 1$

Remember

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ when A and B are not mutually exclusive events.
- $P(A) + P(A') = 1$

Question !!!

If you take out a single card from a regular pack of cards, what is probability that the card is either an king or spade?

Conditional Probability

If we have two events from the same sample space, does the information about the occurrence of one of the events affect the probability of the other event ?

Definition

If E and F are two events associated with the same sample space of a random experiment, the conditional probability of the event E given that F has occurred, i.e. $P(E|F)$ is given by

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

provided $P(F) \neq 0$

Consider the experiment of tossing three fair coins. The sample space of the experiment is

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Since the coins are fair, we can assign the probability $\frac{1}{8}$ to each sample point. Let E be the event 'at least two heads appear' and F be the event 'first coin shows tail'. Then

$$E = \{HHH, HHT, HTH, THH\}$$

and $$F = \{THH, THT, TTH, TTT\}$$

Therefore $$P(E) = P(\{HHH\}) + P(\{HHT\}) + P(\{HTH\}) + P(\{THH\})$$

$$= \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2} \text{ (Why ?)}$$

and $$P(F) = P(\{THH\}) + P(\{THT\}) + P(\{TTH\}) + P(\{TTT\})$$

$$= \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}$$

Also $$E \cap F = \{THH\}$$

with $$P(E \cap F) = P(\{THH\}) = \frac{1}{8}$$

Now, suppose we are given that the first coin shows tail, i.e. F occurs, then what is the probability of occurrence of E?

$$P(E|F) = ?$$

Similarly find

$$P(F|E) = ?$$

Properties of Conditional Probability

Let E and F be events of a sample space S of an experiment, then we have

- $P(S|F) = P(F|F) = 1$
- If A and B are any two events of a sample space S and F is an event of S such that $P(F) \neq 0$, then

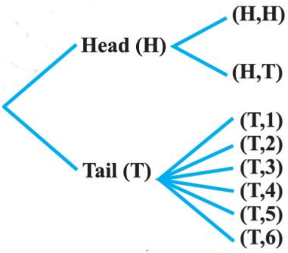
$$P((A \cup B)|F) = P(A|F) + P(B|F) – P((A \cap B)|F)$$

- $$P(E'|F) = 1 - P(E|F)$$

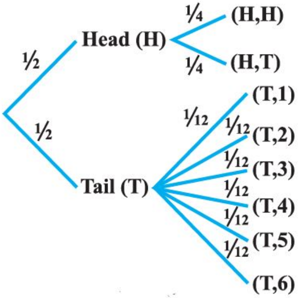
Refresh your mind...

1. A family has two children. What is the probability that both the children are boys given that at least one of them is a boy ?(Ans 1/3)
2. Ten cards numbered 1 to 10 are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than 3, what is the probability that it is an even number?(4/7)

What if the probabilities are not distributed equally?



(a) Illustration I



(b) Illustration II

Figure: Probabilities are not distributed equally.

Find the probability of the event that 'the die shows a number greater than 4' given that 'there is at least one tail'.(Ans 2/9)

Multiplication Theorem

Very often we need to find the probability of the event EF. For example, in the experiment of drawing two cards one after the other, we may be interested in finding the probability of the event 'a king and a queen'. The probability of event EF is obtained by using the conditional probability and given by:

$$P(E \cap F) = P(E) * P(F|E) = P(F) * P(E|F)$$

Let's try

1. An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that both drawn balls are black?
2. Three cards are drawn successively, without replacement from a pack of 52 well shuffled cards. What is the probability that first two cards are kings and the third card drawn is a Queen?

Independent Events

Consider the experiment of drawing a card from a deck of 52 playing cards, in which the elementary events are assumed to be equally likely. If E and F denote the events 'the card drawn is a spade' and 'the card drawn is an king' respectively, then

- $P(E) =$
- $P(F) =$
- $P(E \cap F) =$
- $P(E|F) =$
- $P(F|E) =$

Independent Events

Definition

If E and F are two events such that the probability of occurrence of one of them is not affected by occurrence of the other then they are called independent events.

In other words let E and F be two events associated with the same random experiment, then E and F are said to be independent if

$$P(E \cap F) = P(E) * P(F)$$

Questions

1. A die is thrown. If E is the event 'the number appearing is a multiple of 3' and F be the event 'the number appearing is even' then find whether E and F are independent ?
2. Prove that if E and F are independent events, then so are the events E and F'.
3. If A and B are two independent events, then the probability of occurrence of at least one of A and B is given by $1 - P(A') * P(B')$

Towards Bayes' Theorem

Consider that there are two bags I and II. Bag I contains 2 white and 3 red balls and Bag II contains 4 white and 5 red balls. One ball is drawn at random from one of the bags. We can find the probability of selecting any of the bags (i.e.$\frac{1}{2}$) or probability of drawing a ball of a particular colour (say white) from a particular bag (say Bag I). In other words, we can find the probability that the ball drawn is of a particular colour, if we are given the bag from which the ball is drawn.

But

can we find the probability that the ball drawn is from a particular bag (say Bag II), if the colour of the ball drawn is given?

Theorem of Total Probability

Let $E_1, E_2, ..., E_n$ be a partition of the sample space S, and suppose that each of the events $E_1, E_2, ..., E_n$ has nonzero probability of occurrence. Let A be any event associated with S, then

$$P(A) = P(E_1) * P(A|E_1) + P(E_2) * P(A|E_2) + ... + P(E_n)P(A|E_n)$$

Question

A person has undertaken a construction job. The probabilities are 0.65 that there will be strike, 0.80 that the construction job will be completed on time if there is no strike, and 0.32 that the construction job will be completed on time if there is a strike. Determine the probability that the construction job will be completed on time.

Bayes' Theorem

If $E_1, E_2, ..., E_n$ are n non empty events which constitute a partition of sample space S, and A is any event of nonzero probability, then

$$P(E_i|A) = \frac{P(E_i)P(A|E_i)}{\sum_{j=1}^{n} P(E_j)P(A|E_j)}$$

Example

Example 16 Bag I contains 3 red and 4 black balls while another Bag II contains 5 red and 6 black balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from Bag II.

Solution Let E_1 be the event of choosing the bag I, E_2 the event of choosing the bag II and A be the event of drawing a red ball.

Then $$P(E_1) = P(E_2) = \frac{1}{2}$$

Also $$P(A|E_1) = P(\text{drawing a red ball from Bag I}) = \frac{3}{7}$$

and $$P(A|E_2) = P(\text{drawing a red ball from Bag II}) = \frac{5}{11}$$

Now, the probability of drawing a ball from Bag II, being given that it is red, is $P(E_2|A)$

By using Bayes' theorem, we have

$$P(E_2|A) = \frac{P(E_2)P(A|E_2)}{P(E_1)P(A|E_1)+P(E_2)P(A|E_2)} = \frac{\frac{1}{2}\times\frac{5}{11}}{\frac{1}{2}\times\frac{3}{7}+\frac{1}{2}\times\frac{5}{11}} = \frac{35}{68}$$

Questions

1. Given three identical boxes I, II and III, each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in the box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold? (Ans: 2/3)
2. In a factory which manufactures bolts, machines A, B and C manufacture respectively 25%, 35% and 40% of the bolts. Of their outputs, 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it is manufactured by the machine B? (Ans: 28/69)

Random Variable

Definition

A random variable is a real valued function whose domain is the sample space of a random experiment.

Exa: Consider the experiment of tossing a coin two times in succession. The sample space of the experiment is $S = \{HH, HT, TH, TT\}$. Let X denotes **the number of heads obtained**, then X is a random variable and for each outcome, its value is as given below :

$$X(HH) = 2, X(HT) = 1, X(TH) = 1, X(TT) = 0.$$

Try it Yourself

- Two cards are drawn successively with replacement from a well-shuffled deck of 52 cards. Let X denotes the number of Kings. What are the values of X?
- Let X denote the number of hours you study during a randomly selected school day. The probability that X can take the values x, has the following form:

$$P(X = x) = \begin{cases} 0.1 & \text{if } x = 0 \\ 0.15x & \text{if } x = 1, 2 \\ 0.15(5 - x) & if\ x = 3 or\ 4 \\ 0 & otherwise \end{cases} \tag{1}$$

Probability Distribution Associated with Random Variables

Consider the experiment of tossing a coin two times in succession. The sample space of the experiment is $S = \{HH, HT, TH, TT\}$. Let X denotes **the number of heads obtained**, then X is a random variable and for each outcome, its value is as given below :

$$X(HH) = 2, X(HT) = 1, X(TH) = 1, X(TT) = 0.$$

and the probabilities associated with them are as follows:
$P(X = 0) = P(\{TT\}) = \frac{1}{4}$,
$P(X = 1) = P(\{HT, TH\}) = \frac{2}{4}$
$P(X = 2) = P(\{HH\}) = \frac{1}{4}$

References I

- Miller and Freund's Probability and Statistics for Engineers, Richard A. Johnson, Ninth Edition, Pearson.
- Schaum's Outline Probability and Statistics, 3rd Edition