Problem Set - I

Undergraduate Directed Group Reading Program 2024

Geometric Group Theory

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Instructions

- This is the first problem set—please attempt it sincerely, as it will greatly enhance your understanding over time. While it is not graded, we strongly encourage you to approach it diligently.
- If you face challenges with the readings or specific problems, feel free to reach out to either of us for guidance.
- References are provided for each problem, allowing curious readers to explore the solutions in the cited sources and delve deeper into the applications of the ideas. Download links for the referenced books can be found in the Reference section.
- **If possible**, we encourage you to write up your solutions in LaTeX and share them with both instructors via email. Effort will be duly acknowledged. If you cannot solve all the problems, simply share the LaTeX solutions for the ones you manage to solve.
- If you would like your solutions checked, please send them to me (Treanungkur) via WhatsApp. For any progress on the Chocolate Problem, kindly **email** both the instructors (with subject **"Chocolate Problem"**). Ofcourse, you will receive proper credits (even for partial solutions) and chocolates—after the semester break! Partial solutions to the Chocolate Problem are particularly encouraged. Don't hesitate to message us with any ideas for that problem, but please avoid submitting solutions copied from books or platforms like Math Stack Exchange.
- For any problem where you rely on ideas or theorems not covered in class, include a justification or a rough proof of the theorem you are using. This is especially important for the **"Chocolate Problem"**.
- Collaboration with peers is encouraged, as long as it promotes genuine learning. Best of luck, and happy problem-solving!

Problems

Problem - I
This problem is taken from Dummit and Foote (2004) (Page 175):

(Idea of building new groups out of smaller groups) Let $(A, \cdot)$ and $(B, +)$ be two groups, and let $\varphi : B \to \mathrm{Aut}(A)$ be a homomorphism, where $\mathrm{Aut}(A)$ denotes the automorphism group of A. Define the *semidirect product* of A and B with respect to φ, denoted $A \rtimes_\varphi B$, as the set $A \times B$ equipped with the binary operation $*$ defined by:

$$(a_1, b_1) * (a_2, b_2) = (a_1 \cdot \varphi(b_1)(a_2), b_1 + b_2),$$

where $a_1, a_2 \in A$ and $b_1, b_2 \in B$, and $\varphi(b_1)$ is the automorphism of A induced by $b_1 \in B$.
Is $A \rtimes_\varphi B$ a group under the operation '$*$'? Provide a rigorous justification by verifying whether the group axioms discussed in class is satisfied or not.

Hint: Begin by verifying whether the set is closed under the given operation. Next, take a guess on the identity element for the new set, $A \times B$. Proceed to determine the inverse of a typical element, say $(a, b) \in A \times B$. Finally, check if the operation satisfies the associative law (might be a bit tedious).

Problem - II
This problem is taken from MSE (2015):

Let $\mathbb{S}^1$ denote the unit circle, formally defined by:

$$\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

Prove that $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$, where $\mathbb{S}^1$ is equipped with the group operation of ... (think about it ....)

Hint: Try using the 1st Isomorphism Theorem, discussed in class. So try thinking about a map from $\mathbb{R}$ to $\mathbb{S}^1$ whose kernel is $\mathbb{Z}$

Problem - III
This problem is taken from Dummit and Foote (2004) (Page 79):

(Must try this problem, if you are uncomfortable with the idea of cosets) Let $G = \mathbb{R}^2$ and $H = \mathbb{R}e_1 \cong \mathbb{R}$ (check that H is indeed a subgroup of G). For a vector $v \in \mathbb{R}^2$, define the left H-coset of v as:

$$v + H = v + \mathbb{R}e_1 = \{v + ce_1 : c \in \mathbb{R}\}.$$

Geometrically, determine the form of these left coset. What does it signify?

Hint: It might be useful to see H as x-axis, and as discussed in class, think about what does a subgroup do to a group, when we talk about it's cosets. (It forms parti...?) Think about how x-axis can 'parti...' $\mathbb{R}^2$.

Problem - IV
This problem is taken from Rotman (2009) (Page 28):

(Solve this problem using Group Theory) Let $n \in \mathbb{Z}$ be a strictly positive integer. Then

$$\sum_{d|n} \varphi(d) = n,$$

where $\varphi(d)$ denotes Euler's totient function.

Hint: Think about the group $(\mathbb{Z}_n, +)$ defined in class, it may be useful to view the LHS as summing the number of elements of each possible order in the group.

Problem - V (Chocolate Problem)
This problem is taken from ...surprise... (Page 101):

Any non trivial subgroup of $(\mathbb{R}, +)$ is either cyclic or it is dense in $\mathbb{R}$.

Definition: A set $S \subseteq \mathbb{R}$ is said to be *dense* in $\mathbb{R}$ if for every open interval $(a, b) \subseteq \mathbb{R}$, there exists an element $x \in S$ such that $x \in (a, b)$.

Hint: Try thinking about how does an arbitrary subgroup of $\mathbb{R}$ looks like?

References

Dummit, D. S. and Foote, R. M. (2004). *Abstract Algebra*. Wiley, Hoboken, NJ, 3rd edition. Available at: https://rksmvv.ac.in/wp-content/uploads/2021/04/David_S_Dummit_Richard_M_Foote_Abstract_Algeb_230928_225848.pdf.

MSE (2015). Prove that $\mathbb{R}/\mathbb{Z}$ is isomorphic to $\mathbb{S}^1$. Available at: https://math.stackexchange.com/questions/1355730/proof-that-bbb-r-bbb-z-is-isomorphic-to-s1.

Rotman, J. J. (2009). *An Introduction to the Theory of Groups*. Springer, New York, 4th edition. Available at: https://eclass.uoa.gr/modules/document/file.php/MATH676/Rotman%20An%20introduction%20to%20the%20theory%20of%20groups.pdf.