Problem Set - II

Undergraduate Directed Group Reading Program 2024

Geometric Group Theory

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Instructions

- This is the second problem set—please attempt it sincerely, as it will greatly enhance your understanding over time. While it is not graded, we strongly encourage you to approach it diligently.
- If you face challenges with the readings or specific problems, feel free to reach out to either of us for guidance.
- References are provided for each problem, allowing curious readers to explore the solutions in the cited sources and delve deeper into the applications of the ideas. Download links for the referenced books can be found in the Reference section.
- **If possible**, we encourage you to write up your solutions in LaTeX and share them with both instructors via email. Effort will be duly acknowledged. If you cannot solve all the problems, simply share the LaTeX solutions for the ones you manage to solve.
- If you would like your solutions checked, please send them to me (Treanungkur) via WhatsApp. For any progress on the Chocolate Problem, kindly **email** both the instructors (with subject **"Chocolate Problem"**). Ofcourse, you will receive proper credits (even for partial solutions) and chocolates—after the semester break! Partial solutions to the Chocolate Problem are particularly encouraged. Don't hesitate to message us with any ideas for that problem, but please avoid submitting solutions copied from books or platforms like Math Stack Exchange.
- For any problem where you rely on ideas or theorems not covered in class, include a justification or a rough proof of the theorem you are using. This is especially important for the **"Chocolate Problem"**.
- Collaboration with peers is encouraged, as long as it promotes genuine learning. Best of luck, and happy problem-solving!

Problems

Problem - I
(A cute and simple application of Cayley's Theorem) Prove that the number of *non-isomorphic* groups of a given order n is finite.

Hint: Use Cayley's theorem to embed any group into a symmetric group, and intuitively, consider group isomorphisms as equivalence classes.

Problem - II
(Orbits form a partition of X) Let G be a group acting on a set X. Prove that the orbits of this action form a partition of the set X.

Hint: Use the definitions of equivalence relations and consider the properties of group actions.

Problem III
(Lagrange's Theorem via Group Actions) Prove that if H is a subgroup of G, then $|H| \mid |G|$.

Hint: Consider how H can act on G.

Problem IV
(Cauchy's Theorem) Prove that in a finite group G, if p divides $|G|$, then there exists an element in G of order p.

Hint: Consider the set $X = \{(x_1, x_2, \ldots, x_p) \in G^p : x_1 x_2 \cdots x_p = e\}$, and define an action of $\mathbb{Z}_p$ on X.

Problem - V (Chocolate Problem)
(A small insight to Representation Theory) Prove that every finite group G is isomorphic to a subgroup of $GL_n(\mathbb{R})$ for some n.

Hint: Think about an action of S_n on $\mathbb{R}^n$, where $n = |G|$.