Problem Set - III

Undergraduate Directed Group Reading Program 2024

Geometric Group Theory

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Instructions

- This is the third problem set—please attempt it sincerely, as it will greatly enhance your understanding over time. While it is not graded, we strongly encourage you to approach it diligently.
- If you face challenges with the readings or specific problems, feel free to reach out to either of us for guidance.
- References are provided for each problem, allowing curious readers to explore the solutions in the cited sources and delve deeper into the applications of the ideas. Download links for the referenced books can be found in the Reference section.
- **If possible**, we encourage you to write up your solutions in LaTeX and share them with both instructors via email. Effort will be duly acknowledged. If you cannot solve all the problems, simply share the LaTeX solutions for the ones you manage to solve.
- If you would like your solutions checked, please send them to me (Treanungkur) via WhatsApp. For any progress on the Chocolate Problem, kindly **email** both the instructors (with subject **"Chocolate Problem"**). Ofcourse, you will receive proper credits (even for partial solutions) and chocolates—after the semester break! Partial solutions to the Chocolate Problem are particularly encouraged. Don't hesitate to message us with any ideas for that problem, but please avoid submitting solutions copied from books or platforms like Math Stack Exchange.
- For any problem where you rely on ideas or theorems not covered in class, include a justification or a rough proof of the theorem you are using. This is especially important for the **"Chocolate Problem"**.
- Collaboration with peers is encouraged, as long as it promotes genuine learning. Best of luck, and happy problem-solving!

Problems

Problem - I

(Kernel of a Homomorphism) Recall the map defined in class from G to $\mathrm{Sym}(G/H)$, given by $\Phi : G \to \mathrm{Sym}(G/H)$ defined by:

$$\Phi(g) = \psi_g,$$

where ψ_g is the map induced by the group action, which is via "left translation" on the cosets (i.e. $\psi_g(xH) = g{\cdot}xH = gxH$). Find the closed form for the kernel this homomorphism.

Group Action via Left Translation: One of the most important actions in finite group theory is the action of G on the coset space G/H, defined by $g \cdot xH = gxH$. (verify that this defines a valid group action.) Naturally, like any group action, it induces a homomorphism from G to $\mathrm{Sym}(G/H)$, which corresponds to the map Φ defined above.

Remark: The above kernel is referred to as the normal core of the subgroup H.

Problem - II

(Gauss takes the stage!) Let p, q be distinct odd primes. Consider the group $G = (\mathbb{Z}/p\mathbb{Z})^* \times (\mathbb{Z}/q\mathbb{Z})^*$ and the subgroup $H = \{(1,1), (-1,-1)\}$.

(i) Prove that

$$S := \left\{(i,j) : 1 \le i \le p-1, 1 \le j \le \frac{q-1}{2}\right\}$$

is a set of coset representatives for H in G.

(ii) Using the isomorphism

$$\Phi : (\mathbb{Z}/p\mathbb{Z})^* \times (\mathbb{Z}/q\mathbb{Z})^* \to (\mathbb{Z}/pq\mathbb{Z})^*$$

get a set of coset representatives for $\Phi(H)$.

(iii) Use the two sets to get expressions for the product of elements of G/H and deduce the quadratic reciprocity law

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{4}}.$$

Remark: This problem is due to G. Rousseou.

Problem - III

(Connectedness of Cayley Graphs) Let G be a group, and let S be a subset of G. The Cayley graph $\Gamma(G,S)$ is connected if and only if $G = \langle S \rangle$, where $\langle S \rangle$ denotes the subgroup of G generated by S.

Hint: Consider the paths in $\Gamma(G,S)$ and how they relate to the elements of G being generated by S.

Problem - IV
(Group Presentation) Find the elements of the group given by the presentation

$$\langle x, y \mid xy^2 = y^3x,\ yx^2 = x^3y\rangle.$$

Hint: It turns out that the above gorup is just the trivial group.
Remark: There are many "exotic" presentations of the trivial group. One well-known example (which we won't verify here) is:

$$\langle a, b \mid aba^{-1}b^{-2},\ a^{-2}b^{-1}ab\rangle.$$

The famous "Andrews-Curtis Conjecture" says that any presentation of the trivial group with equal numbers of generators and relators can be simplified down to a trivial presentation using a series of basic moves. It remains open to this day. Maybe it'll be the next chocolate problem or the next reason to cry into your coffee, who knows? :)

Problem - V (Chocolate Problem)
(Quasi-Isometry of Generating Sets) If S and T are two finite generating sets for a group G, then the metric spaces $\Gamma(G, S)$ and $\Gamma(G, T)$ are quasi-isometric.

Definition: Let (X, d_X) and (Y, d_Y) be metric spaces. A map $\varphi : X \to Y$ is a "quasi-isometric embedding" if there exist constants $K > 1$ and $C > 0$ such that for all $a, b \in X$, the following inequality holds:

$$\frac{1}{K}d_X(a, b) - C \leq d_Y(\varphi(a), \varphi(b)) \leq Kd_X(a, b) + C.$$

If, in addition, every point of Y lies within a C-neighborhood of the image of φ, then φ is called a "quasi-isometry."

Hint: Consider the graph metric and try to derive bounds on the length of the largest word that can be formed using one generating set with respect to the other.