Problem Set - IV

Undergraduate Directed Group Reading Program 2024

Geometric Group Theory

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Instructions

- This is the fourth problem set—please attempt it sincerely, as it will greatly enhance your understanding over time. While it is not graded, we strongly encourage you to approach it diligently.
- If you face challenges with the readings or specific problems, feel free to reach out to either of us for guidance.
- References are provided for each problem, allowing curious readers to explore the solutions in the cited sources and delve deeper into the applications of the ideas. Download links for the referenced books can be found in the Reference section.
- **If possible**, we encourage you to write up your solutions in LaTeX and share them with both instructors via email. Effort will be duly acknowledged. If you cannot solve all the problems, simply share the LaTeX solutions for the ones you manage to solve.
- If you would like your solutions checked, please send them to me (Treanungkur) via WhatsApp. For any progress on the Chocolate Problem, kindly **email** both the instructors (with subject **"Chocolate Problem"**). Ofcourse, you will receive proper credits (even for partial solutions) and chocolates—after the semester break! Partial solutions to the Chocolate Problem are particularly encouraged. Don't hesitate to message us with any ideas for that problem, but please avoid submitting solutions copied from books or platforms like Math Stack Exchange.
- For any problem where you rely on ideas or theorems not covered in class, include a justification or a rough proof of the theorem you are using. This is especially important for the **"Chocolate Problem"**.
- Collaboration with peers is encouraged, as long as it promotes genuine learning. Best of luck, and happy problem-solving!

Problems

Problem - I
(Isometry Implies Quasi-Isometry): Let X and Y be isometric metric spaces. Prove that X and Y are quasi-isometric metric spaces as well.

Hint: Recall the parameters discussed in class; you should ideally get $\lambda = 1$, $c = 0$, and $\epsilon = 0$.

Problem - II
(Quasi-Isometry Example) Show that the map $f : (\mathbb{R}, d_{\mathbb{R}}) \to (\mathbb{Z}, d_{\mathbb{Z}})$ given by $x \mapsto [x]$ is a $(1, 2)$-quasi-isometry.

Remark: Here, $d_{\mathbb{R}}$ and $d_{\mathbb{Z}}$ represents the usual metric.

Problem - III
(Constant Map as Quasi-Isometry) Show that a constant map $f : X \to X$ is a quasi-isometric embedding if and only if X has finite diameter.

Hint: Try to use the definition of quasi-isometric embedding.

Problem - IV
(Quasi-Isometry of Finite Groups) Prove that any two finite groups are quasi-isometric.

Hint: Try to see how the metric spaces corresponding to finite groups are...

Problem - V (Chocolate Problem)
(Quasi-Isometry of $\mathbb{Z}$ and $\mathbb{Z}_2 * \mathbb{Z}_2$) Show that $\mathbb{Z}$ is quasi-isometric to $\mathbb{Z}_2 * \mathbb{Z}_2$, by explicitly showing the quasi-isometry between the two.

Definition: Let G_1 and G_2 be groups. The free product $G_1 * G_2$ is the group consisting of all finite reduced words formed from elements of $G_1 \setminus \{e_1\}$ and $G_2 \setminus \{e_2\}$, where e_1 and e_2 are the identity elements of G_1 and G_2, respectively, with the group operation being concatenation followed by reduction.

Hint: Try to use the word metric and find the most natural map.