Chocolate Problem - I

Undergraduate Directed Group Reading Program 2024

Geometric Group Theory

Rinkiny Ghatak (B3)
rinkiny@gmail.com

Treanungkur Mal (B2)
maltreanungkur@gmail.com

Credits: Nikhil Nagaria, BMath 3rd Year.

> **Problem - V (Chocolate Problem)**
> This problem is taken from ...surprise... (Page 101):
>
> Any non trivial subgroup of $(\mathbb{R}, +)$ is either cyclic or it is dense in $\mathbb{R}$.
>
> **Definition:** A set $S \subseteq \mathbb{R}$ is said to be *dense* in $\mathbb{R}$ if for every open interval $(a, b) \subseteq \mathbb{R}$, there exists an element $x \in S$ such that $x \in (a, b)$.

Proof. Let $H \leq \mathbb{R}$ be a subgroup that is not dense in $\mathbb{R}$ as a metric subspace. We want to show that H is cyclic.

Claim. There exists $t > 0$ such that $(0, t) \cap H = \emptyset$.
Proof. Since H is not dense, find an interval (a, b) not intersecting H. Since $0 \in H$, it must be that a and b have the same sign. Since H is a subgroup, $c \in H$ iff $-c \in H$. Thus, replacing a and b by $-a$ and $-b$ respectively if needed, we can assume that $b > a > 0$. Then, for any positive integer n, we have that $\left(\frac{a}{n}, \frac{b}{n}\right) \cap H = \emptyset$ (as if x lies in this intersection, then $nx \in H \cap (a, b) = \emptyset$, a contradiction.)
Now, find N such that $1 + 1/N < b/a$. Then, the inequality holds for all $n \geq N$ as well. Further, the inequality is equivalent to $b/(n+1) > a/n$. Thus, for $n \geq N$ and any y,

$$\left(y, \frac{b}{n+1}\right) \cup \left(\frac{a}{n}, \frac{b}{n}\right) = \left(y, \frac{b}{n}\right)$$

and so

$$\bigcup_{n=N}^{\infty} \left(\frac{a}{n}, \frac{b}{n}\right) = \bigcup_{n=N}^{\infty} \left(\frac{a}{n}, \frac{b}{N}\right) = \left(0, \frac{b}{N}\right)$$

and H does not intersect with this set. Set $t = b/N$.
Now, if $H = \{0\}$, we are done. Else, let $r = \inf\{x \in H \mid x > 0\}$. We claim that $r \in H$. If not, find $0 < s \in H$ such that $r < s < r + t$. Then, find $s' \in H$ with $r < s' < s$. Then, $0 < s - s' < t$ is in H - contradiction. Thus, $r \in H$.
Finally, claim that $\langle r \rangle = H$. Take $x \in H$ and find an integer m such that $x \in [mr, (m+1)r)$. Then, $x - mr \in H$ and $0 \leq x - mr < r$ so we must have $x - mr = 0$ or $x \in \langle r \rangle$. Thus, H is cyclic. □