Chocolate Problem - II

Undergraduate Directed Group Reading Program 2024

Geometric Group Theory

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> **Problem - V (Chocolate Problem)**
> (A small insight to Representation Theory) Prove that every finite group G is isomorphic to a subgroup of $GL_n(\mathbb{R})$ for some n.

Hint: Think about an action of S_n on $\mathbb{R}^n$, where $n = |G|$.

Proof. Consider the map $\varphi : S_n \to K$, defined by:

$$\varphi(P) = E_{P(i),i} \quad \text{for} \quad P \in S_n,$$

where $E_{i,j}$ denotes the $n \times n$ matrix with 1 in the (i, j)-th position and 0 elsewhere.

To show that φ is a homomorphism, we need to verify that:

$$\varphi(P \circ P') = \varphi(P) \cdot \varphi(P') \quad \text{for all} \quad P, P' \in S_n.$$

Let $P, P' \in S_n$. Then:

$$\varphi(P \circ P') = E_{P \circ P'(i),i} = E_{P(P'(i)),P'(i)}.$$

On the other hand, using the definition of φ:

$$\varphi(P) \cdot \varphi(P') = E_{P(i),i} \cdot E_{P'(i),i}.$$

Thus, we have shown that $\varphi(P \circ P') = \varphi(P) \cdot \varphi(P')$, confirming that φ is a homomorphism. Next, we show that φ is bijective:

1. **Injectivity:** Suppose $\varphi(P) = \varphi(P')$, i.e., $E_{P(i),i} = E_{P'(i),i}$ for all i. This implies that the positions of the 1's in the matrices $E_{P(i),i}$ and $E_{P'(i),i}$ are the same. Therefore, the permutations P and P' must agree, so $P = P'$. Hence, φ is injective.

2. **Surjectivity:** For every permutation matrix $M \in K$, we need to find a permutation $P \in S_n$ such that $\varphi(P) = M$. By the definition of φ, $\varphi(P) = E_{P(i),i}$. Since M is a permutation matrix, it corresponds to some permutation $P \in S_n$. Therefore, for every $M \in K$, there exists a permutation $P \in S_n$ such that $\varphi(P) = M$. Hence, φ is surjective.

Since φ is both a homomorphism and bijective, we conclude that:

$$\varphi : S_n \to K \quad \text{is an isomorphism.}$$

Therefore, we have:

$$S_n \cong K \leq GL_n(\mathbb{R}).$$

Claim: The composition of two injective homomorphisms is an injective homomorphism.

Proof: Let G, H, and K be groups, and let $\varphi : G \to H$ and $\psi : H \to K$ be homomorphisms. Assume that both φ and ψ are injective.

We want to show that the composition $\psi \circ \varphi : G \to K$ is also injective. That is, if $(\psi \circ \varphi)(g_1) = (\psi \circ \varphi)(g_2)$ for some $g_1, g_2 \in G$, then we must show that $g_1 = g_2$.

$$(\psi \circ \varphi)(g_1) = (\psi \circ \varphi)(g_2) \implies \psi(\varphi(g_1)) = \psi(\varphi(g_2))$$

Since ψ is injective, we get that:

$$\varphi(g_1) = \varphi(g_2)$$

Next, since φ is injective, we get that:

$$g_1 = g_2$$

Thus, $\psi \circ \varphi$ is injective.

Now, we need to show that $\psi \circ \varphi$ is a homomorphism.

For all $g_1, g_2 \in G$, we need to verify that:

$$(\psi \circ \varphi)(g_1 g_2) = (\psi \circ \varphi)(g_1)(\psi \circ \varphi)(g_2).$$

Since φ and ψ are homomorphisms, we know that:

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2) \quad \text{and} \quad \psi(\varphi(g_1)\varphi(g_2)) = \psi(\varphi(g_1))\psi(\varphi(g_2)).$$

Thus,

$$(\psi \circ \varphi)(g_1 g_2) = \psi(\varphi(g_1 g_2)) = \psi(\varphi(g_1)\varphi(g_2)) = \psi(\varphi(g_1))\psi(\varphi(g_2)) = (\psi \circ \varphi)(g_1)(\psi \circ \varphi)(g_2).$$

Therefore, $\psi \circ \varphi$ is a homomorphism. Therefore, the composition of injective homomorphisms is an injective homomorphism.

Now using this and Cayley's theorem, we get the theorem.