Outline

Last time:
- Structure of dependent type theory
- Π, $\rightarrow$, $\wedge/\times$ types

} Rijke §1-2

This time:
- Inductive types

} §3-4

Next time:
- Identity type

} §5

$\longrightarrow$ homotopy

Type formers

- The rules that define Π, $\to$, $\times / \wedge$, inductive types are all <u>type formers</u>.

- When we study a type theory, we choose which type formers to include.

- HoTT (as a type theory) usually means dependent type theory + all type formers introduced in this course + an axiom.

Comparison with ZF(C)

- In ZF(C), products, functions, etc have to be encoded.

  → The practice of everyday mathematics is far away from the foundations.

- In type theory, we <u>postulate</u> the existence of products, functions, etc.

  → The practice of everyday mathematics is very close to the foundations.

Inductive types

Inductive types are freely generated by canonical terms.

Ex. The booleans are freely generated by the canonical terms true, false.

Inductive types in Agda

```
data Bool : Type where
  true false : Bool
```

To define a (dependent) function out of Bool,
it suffices to define it on its canonical elements, true and false.

Inductive types in Agda

data __ : Type where

____ : __

tells Agda we are defining an inductive type

To define a (dependent) function out of an inductive type it suffices to define it on its canonical elements.

In pen-and-paper HoTT, we specify the behavior of inductive types by hand.

The booleans: bool

bool-form:

$$\frac{}{\vdash \mathrm{bool}\ \mathrm{type}}$$

bool-intro:

$$\frac{}{\vdash \mathrm{true} : \mathrm{bool}} \qquad \frac{}{\vdash \mathrm{false} : \mathrm{bool}}$$

bool-elim:

$$\frac{\Gamma, x : \mathrm{bool} \vdash D(x)\ \mathrm{type} \qquad \Gamma \vdash a : D(\mathrm{true}) \qquad \Gamma \vdash b : D(\mathrm{false})}{\Gamma, x : \mathrm{bool} \vdash \mathrm{ind}_{\mathrm{bool}}(a, b, x) : D(x)}$$

bool-comp:

$$\frac{\Gamma, x : \mathrm{bool} \vdash D(x)\ \mathrm{type} \qquad \Gamma \vdash a : D(\mathrm{true}) \qquad \Gamma \vdash b : D(\mathrm{false})}{\begin{array}{l}\Gamma, x : \mathrm{bool} \vdash \mathrm{ind}_{\mathrm{bool}}(a, b, \mathrm{true}) \doteq a : D(\mathrm{true}) \\ \Gamma, x : \mathrm{bool} \vdash \mathrm{ind}_{\mathrm{bool}}(a, b, \mathrm{false}) \doteq b : D(\mathrm{false})\end{array}}$$

Example: not : bool → bool

$\vdash$? : bool → bool

→-intro:

$$\frac{x:P \vdash q:Q}{\lambda x.q : P \to Q}$$

bool-form:

$$\frac{}{\vdash \text{bool type}}$$

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Example: $\mathrm{not} : \mathrm{bool} \to \mathrm{bool}$

$$\dfrac{\dfrac{\dfrac{}{x : \mathrm{bool} \vdash \mathrm{bool}\ \mathrm{type}} \qquad \dfrac{}{\vdash\ ?\ : \mathrm{bool}} \qquad \dfrac{}{\vdash\ ?\ : \mathrm{bool}}}{x : \mathrm{bool} \vdash\ ?\ : \mathrm{bool}}}{\vdash\ ?\ : \mathrm{bool} \to \mathrm{bool}}$$

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Example: not : bool → bool

weakened bool-intro

$$\dfrac{\dfrac{\dfrac{}{x:\mathrm{bool}\vdash \mathrm{bool}\ \mathrm{type}} \qquad \dfrac{}{\vdash \mathrm{false}:\mathrm{bool}} \qquad \dfrac{}{\vdash \mathrm{true}:\mathrm{bool}}}{x:\mathrm{bool}\vdash \ ? \ : \mathrm{bool}}}{\vdash \ ? \ : \mathrm{bool}\to\mathrm{bool}}$$

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Coproducts $+$

$+$-form:
$$\frac{\Gamma \vdash P \text{ type} \qquad \Gamma \vdash Q \text{ type}}{\Gamma \vdash P+Q \text{ type}}$$

$+$-intro:
$$\frac{\Gamma \vdash p : P}{\Gamma \vdash \mathrm{inl}(p) : P+Q} \qquad \frac{\Gamma \vdash q : Q}{\Gamma \vdash \mathrm{inr}(q) : P+Q}$$

$+$-elim:
$$\frac{\begin{array}{c}\Gamma, x : P+Q \vdash D(x) \text{ type} \\ \Gamma, p : P \vdash a : D(\mathrm{inl}\, p) \\ \Gamma, q : Q \vdash b : D(\mathrm{inr}\, q)\end{array}}{\Gamma, x : P+Q \vdash \mathrm{ind}_{P+Q}(a,b,x) : D(x)}$$

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Logical interpretation:

- We can prove $P+Q$ if we can prove P or we can prove Q.
- To prove something from $P+Q$ we do a proof by cases.
- So $+$ behaves like disjunction (or).

Set interpretation:

- $+$ behaves like $\sqcup$

Example: For any types A, B, C, there is a function $A\times B + A\times C \to A\times(B+C)$.

$\vdash \quad ? \quad : A\times B + A\times C \to A\times(B+C)$

×-elim:

$$\frac{\Gamma \vdash a : P\times Q}{\Gamma \vdash pr_1 a : P}$$

$$\frac{\Gamma \vdash a : P\times Q}{\Gamma \vdash pr_2 a : Q}$$

+-form

$$\frac{\Gamma \vdash P \text{ type} \qquad \Gamma \vdash Q \text{ type}}{\Gamma \vdash P+Q \text{ type}}$$

+-intro

$$\frac{\Gamma \vdash p : P}{\Gamma \vdash inl(p) : P+Q}$$

$$\frac{\Gamma \vdash q : Q}{\Gamma \vdash inr(q) : P+Q}$$

+-elim

$$\frac{\Gamma, x : P+Q \vdash D(x) \text{ type} \quad \Gamma, p : P \vdash a : D(inl\, p) \quad \Gamma, q : Q \vdash b : D(inr\, q)}{\Gamma, x : P+Q \vdash ind_{P+Q}(a, b, x) : D(x)}$$

+-comp

$$\frac{\Gamma, x : P+Q \vdash D(x) \text{ type} \quad \Gamma, p : P \vdash a : D(inl\, p) \quad \Gamma, q : Q \vdash b : D(inr\, q)}{\Gamma, p : P \vdash ind_{P+Q}(a, b, inl\, p) \doteq a : D(inl\, p) \qquad \Gamma, q : Q \vdash ind_{P+Q}(a, b, inr\, q) \doteq b : D(inr\, q)}$$

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$\times$-elim:

$$\frac{\Gamma \vdash a : P\times Q}{\Gamma \vdash \mathrm{pr}_1 a : P} \qquad \frac{\Gamma \vdash a : P\times Q}{\Gamma \vdash \mathrm{pr}_2 a : Q}$$

$+$-form

$$\frac{\Gamma \vdash P \text{ type} \qquad \Gamma \vdash Q \text{ type}}{\Gamma \vdash P+Q \text{ type}}$$

$+$-intro

$$\frac{\Gamma \vdash p : P}{\Gamma \vdash \mathrm{inl}(p) : P+Q} \qquad \frac{\Gamma \vdash q : Q}{\Gamma \vdash \mathrm{inr}(q) : P+Q}$$

$+$-elim

$$\frac{\Gamma, x : P+Q \vdash D(x) \text{ type} \qquad \Gamma, p : P \vdash a : D(\mathrm{inl}\,p) \qquad \Gamma, q : Q \vdash b : D(\mathrm{inr}\,q)}{\Gamma, x : P+Q \vdash \mathrm{ind}_{P+Q}(a, b, x) : D(x)}$$

$+$-comp

$$\frac{\Gamma, x : P+Q \vdash D(x) \text{ type} \qquad \Gamma, p : P \vdash a : D(\mathrm{inl}\,p) \qquad \Gamma, q : Q \vdash b : D(\mathrm{inr}\,q)}{\begin{array}{c}\Gamma, p : P \vdash \mathrm{ind}_{P+Q}(a, b, \mathrm{inl}\,p) \doteq a : D(\mathrm{inl}\,p) \\ \Gamma, q : Q \vdash \mathrm{ind}_{P+Q}(a, b, \mathrm{inr}\,q) \doteq b : D(\mathrm{inr}\,q)\end{array}}$$

Dependent pair types Σ

(aka dependent sum types, sigma types)

Ex. $n:\mathbb{N} \vdash \mathrm{Vect}(n)$ type

We learned about $\prod_{n:\mathbb{N}} \mathrm{Vect}(n)$, the type of dependent functions.

It's also natural to take a 'union' $\sum_{n:\mathbb{N}} \mathrm{Vect}(n)$.

Ex. $n:\mathbb{N} \vdash \mathrm{isPrime}(n)$ type

There is no dependent function $\prod_{n:\mathbb{N}} \mathrm{isPrime}(n)$.

We can consider $\sum_{n:\mathbb{N}} \mathrm{isPrime}(n)$. This has many terms.

Dependent pair types Σ

Σ-form:

$$\frac{\Gamma, x:P \vdash Q(x)}{\Gamma \vdash \sum_{x:P} Q(x)}$$

Σ-intro:

$$\frac{\Gamma \vdash p:P \qquad \Gamma \vdash q : Q(p)}{\Gamma \vdash \mathrm{pair}(p,q) : \sum_{x:P} Q(p)}$$

Σ-elim:

$$\frac{\begin{array}{c}\Gamma, z : \sum_{x:P} Q(p) \vdash D(z) \\ \Gamma, x:P, y: Q(x) \vdash a : D(\mathrm{pair}(x,y))\end{array}}{\Gamma, z : \sum_{x:P} Q(p) \vdash \mathrm{ind}_\Sigma(a,z) : D(z)}$$

Σ-comp

$$\frac{\begin{array}{c}\Gamma, z : \sum_{x:P} Q(p) \vdash D(z) \\ \Gamma, x:P, y: Q(x) \vdash a : D(\mathrm{pair}(x,y))\end{array}}{\Gamma, x:P, y: Q(x) \vdash \mathrm{ind}_\Sigma(a, \mathrm{pair}(x,y)) \doteq a : D(\mathrm{pair}(x,y))}$$

Dependent pair types Σ

Σ-form:

$$\frac{\Gamma, x:P \vdash Q(x)}{\Gamma \vdash \sum_{x:P} Q(x)}$$

Σ-intro:

$$\frac{\Gamma \vdash p:P \qquad \Gamma \vdash q : Q(p)}{\Gamma \vdash \mathrm{pair}(p,q) : \sum_{x:P} Q(p)}$$

Σ-elim:

$$\frac{\begin{array}{c}\Gamma, z : \sum_{x:P} Q(p) \vdash D(z) \\ \Gamma, x:P, y: Q(x) \vdash a : D(\mathrm{pair}(x,y))\end{array}}{\Gamma, z : \sum_{x:P} Q(p) \vdash \mathrm{ind}_\Sigma(a,z) : D(z)}$$

Σ-comp

$$\frac{\begin{array}{c}\Gamma, z : \sum_{x:P} Q(p) \vdash D(z) \\ \Gamma, x:P, y: Q(x) \vdash a : D(\mathrm{pair}(x,y))\end{array}}{\Gamma, x:P, y: Q(x) \vdash \mathrm{ind}_\Sigma(a, \mathrm{pair}(x,y)) \doteq a : D(\mathrm{pair}(x,y))}$$

__Logical interpretation:__

- To prove $\sum_{x:P} Q(x)$ (thinking of P as a set and Q as a predicate), we find one element of P for which Q holds.

- Σ behaves like $\exists$.

__Set interpretation__

- Σ behaves like $\sqcup$

Example: For any $x:P \vdash Q(x)$ type, there is a projection function

$$\pi : \sum_{x:P} Q(x) \to P.$$

Σ-form:

$$\frac{\Gamma, x:P \vdash Q(x)}{\Gamma \vdash \sum_{x:P} Q(x)}$$

Σ-intro:

$$\frac{\Gamma \vdash p:P \qquad \Gamma \vdash q : Q(p)}{\Gamma \vdash \mathrm{pair}(p,q) : \sum_{x:P} Q(p)}$$

Σ-elim:

$$\frac{\begin{array}{c}\Gamma, z : \sum_{x:P} Q(p) \vdash D(z) \\ \Gamma, x:P, y: Q(x) \vdash a : D(\mathrm{pair}(x,y))\end{array}}{\Gamma, z : \sum_{x:P} Q(p) \vdash \mathrm{ind}_\Sigma(a,z) : D(z)}$$

Σ-comp:

$$\frac{\begin{array}{c}\Gamma, z : \sum_{x:P} Q(p) \vdash D(z) \\ \Gamma, x:P, y: Q(x) \vdash a : D(\mathrm{pair}(x,y))\end{array}}{\Gamma, x:P, y: Q(x) \vdash \mathrm{ind}_\Sigma(a, \mathrm{pair}(x,y)) \doteq a : D(\mathrm{pair}(x,y))}$$

Example: For any $x:P \vdash Q(x)$ type, there is a projection function

$$\pi : \sum_{x:P} Q(x) \to P.$$

$$\vdash \lambda z. \quad ? \quad : \sum_{x:P} Q(x) \to P$$

Σ-form:

$$\frac{\Gamma, x:P \vdash Q(x)}{\Gamma \vdash \sum_{x:P} Q(x)}$$

Σ-intro:

$$\frac{\Gamma \vdash p:P \qquad \Gamma \vdash q : Q(p)}{\Gamma \vdash \mathrm{pair}(p,q) : \sum_{x:P} Q(p)}$$

Σ-elim:

$$\frac{\Gamma, z: \sum_{x:P} Q(p) \vdash D(z) \qquad \Gamma, x:P, y: Q(x) \vdash a : D(\mathrm{pair}(x,y))}{\Gamma, z : \sum_{x:P} Q(p) \vdash \mathrm{ind}_\Sigma(a,z) : D(z)}$$

Σ-comp:

$$\frac{\Gamma, z: \sum_{x:P} Q(p) \vdash D(z) \qquad \Gamma, x:P, y: Q(x) \vdash a : D(\mathrm{pair}(x,y))}{\Gamma, x:P, y: Q(x) \vdash \mathrm{ind}_\Sigma(a, \mathrm{pair}(x,y)) \doteq a : D(\mathrm{pair}(x,y))}$$

Example: For any $x:P \vdash Q(x)$ type, there is a projection function

$$\pi : \sum_{x:P} Q(x) \to P.$$

$$\frac{z : \sum_{x:P} Q(x) \vdash \quad ? \quad : P}{\vdash \lambda z. \quad ? \quad : \sum_{x:P} Q(x) \to P}$$

Σ-form:

$$\frac{\Gamma, x:P \vdash Q(x)}{\Gamma \vdash \sum_{x:P} Q(x)}$$

Σ-intro:

$$\frac{\Gamma \vdash p:P \qquad \Gamma \vdash q : Q(p)}{\Gamma \vdash \mathrm{pair}(p,q) : \sum_{x:P} Q(p)}$$

Σ-elim:

$$\frac{\begin{array}{c}\Gamma, z : \sum_{x:P} Q(p) \vdash D(z) \\ \Gamma, x:P, y: Q(x) \vdash a : D(\mathrm{pair}(x,y))\end{array}}{\Gamma, z : \sum_{x:P} Q(p) \vdash \mathrm{ind}_\Sigma(a,z) : D(z)}$$

Σ-comp:

$$\frac{\begin{array}{c}\Gamma, z : \sum_{x:P} Q(p) \vdash D(z) \\ \Gamma, x:P, y: Q(x) \vdash a : D(\mathrm{pair}(x,y))\end{array}}{\Gamma, x:P, y: Q(x) \vdash \mathrm{ind}_\Sigma(a, \mathrm{pair}(x,y)) \doteq a : D(\mathrm{pair}(x,y))}$$

Example: For any $x:P \vdash Q(x)$ type, there is a projection function

$$\pi : \sum_{x:P} Q(x) \to P.$$

$$\cfrac{\cfrac{\overline{x:P,\ y:Q(x) \vdash \ ?\ : P}}{z:\sum_{x:P} Q(x) \vdash \ ?\ : P}}{\vdash \lambda z.\ \ ?\ : \sum_{x:P} Q(x) \to P}$$

Σ-form:

$$\frac{\Gamma, x:P \vdash Q(x)}{\Gamma \vdash \sum_{x:P} Q(x)}$$

Σ-intro:

$$\frac{\Gamma \vdash p:P \qquad \Gamma \vdash q : Q(p)}{\Gamma \vdash \mathrm{pair}(p,q) : \sum_{x:P} Q(p)}$$

Σ-elim:

$$\frac{\begin{array}{c}\Gamma, z:\sum_{x:P} Q(p) \vdash D(z) \\ \Gamma, x:P, y:Q(x) \vdash a : D(\mathrm{pair}(x,y))\end{array}}{\Gamma, z:\sum_{x:P} Q(p) \vdash \mathrm{ind}_\Sigma(a,z) : D(z)}$$

Σ-comp:

$$\frac{\begin{array}{c}\Gamma, z:\sum_{x:P} Q(p) \vdash D(z) \\ \Gamma, x:P, y:Q(x) \vdash a : D(\mathrm{pair}(x,y))\end{array}}{\Gamma, x:P, y:Q(x) \vdash \mathrm{ind}_\Sigma(a, \mathrm{pair}(x,y)) \doteq a : D(\mathrm{pair}(x,y))}$$

Example: For any $x:P \vdash Q(x)$ type, there is a projection function

$$\pi : \sum_{x:P} Q(x) \to P.$$

$$\dfrac{\dfrac{\overline{x:P,\ y:Q(x) \vdash x:P}}{z:\sum_{x:P} Q(x) \vdash \quad ? \quad : P}}{\vdash \lambda z. \quad ? \quad : \sum_{x:P} Q(x) \to P}$$

Σ-form:

$$\frac{\Gamma, x:P \vdash Q(x)}{\Gamma \vdash \sum_{x:P} Q(x)}$$

Σ-intro:

$$\frac{\Gamma \vdash p:P \qquad \Gamma \vdash q : Q(p)}{\Gamma \vdash pair(p,q) : \sum_{x:P} Q(p)}$$

Σ-elim:

$$\frac{\begin{array}{c}\Gamma, z : \sum_{x:P} Q(p) \vdash D(z) \\ \Gamma, x:P, y: Q(x) \vdash a : D(pair(x,y))\end{array}}{\Gamma, z: \sum_{x:P} Q(p) \vdash ind_{\Sigma}(a,z) : D(z)}$$

Σ-comp:

$$\frac{\begin{array}{c}\Gamma, z : \sum_{x:P} Q(p) \vdash D(z) \\ \Gamma, x:P, y: Q(x) \vdash a : D(pair(x,y))\end{array}}{\begin{array}{c}\Gamma, x:P, y: Q(x) \vdash ind_{\Sigma}(a, pair(x,y)) \\ \doteq a : D(pair(x,y))\end{array}}$$

Example: For any $x:P \vdash Q(x)$ type, there is a projection function

$$\pi : \sum_{x:P} Q(x) \to P.$$

$$\dfrac{\dfrac{\overline{x:P,\ y:Q(x) \vdash x:P}}{z:\sum_{x:P} Q(x) \vdash \mathrm{ind}_\Sigma(x,z):P}}{\vdash \lambda z.\quad ?\quad : \sum_{x:P} Q(x) \to P}$$

Σ-form:

$$\frac{\Gamma, x:P \vdash Q(x)}{\Gamma \vdash \sum_{x:P} Q(x)}$$

Σ-intro:

$$\frac{\Gamma \vdash p:P \qquad \Gamma \vdash q: Q(p)}{\Gamma \vdash \mathrm{pair}(p,q) : \sum_{x:P} Q(p)}$$

Σ-elim:

$$\frac{\Gamma, z:\sum_{x:P} Q(p) \vdash D(z) \qquad \Gamma, x:P, y:Q(x) \vdash a:D(\mathrm{pair}(x,y))}{\Gamma, z:\sum_{x:P} Q(p) \vdash \mathrm{ind}_\Sigma(a,z):D(z)}$$

Σ-comp:

$$\frac{\Gamma, z:\sum_{x:P} Q(p) \vdash D(z) \qquad \Gamma, x:P, y:Q(x) \vdash a:D(\mathrm{pair}(x,y))}{\Gamma, x:P, y:Q(x) \vdash \mathrm{ind}_\Sigma(a,\mathrm{pair}(x,y)) \doteq a:D(\mathrm{pair}(x,y))}$$

Example:

For any $x:P \vdash Q(x)$ type, there is a projection function

$$\pi : \sum_{x:P} Q(x) \to P.$$

$$\dfrac{\dfrac{\dfrac{}{x:P,\ y:Q(x) \vdash x:P}}{z:\sum_{x:P} Q(x) \vdash \mathrm{ind}_\Sigma(x,z):P}}{\vdash \lambda z.\,\mathrm{ind}_\Sigma(x,z) : \sum_{x:P} Q(x) \to P}$$

Σ-form:

$$\dfrac{\Gamma, x:P \vdash Q(x)}{\Gamma \vdash \sum_{x:P} Q(x)}$$

Σ-intro:

$$\dfrac{\Gamma \vdash p:P \qquad \Gamma \vdash q:Q(p)}{\Gamma \vdash \mathrm{pair}(p,q) : \sum_{x:P} Q(p)}$$

Σ-elim:

$$\dfrac{\begin{array}{c}\Gamma, z:\sum_{x:P} Q(p) \vdash D(z) \\ \Gamma, x:P, y:Q(x) \vdash a:D(\mathrm{pair}(x,y))\end{array}}{\Gamma, z:\sum_{x:P} Q(p) \vdash \mathrm{ind}_\Sigma(a,z):D(z)}$$

Σ-comp:

$$\dfrac{\begin{array}{c}\Gamma, z:\sum_{x:P} Q(p) \vdash D(z) \\ \Gamma, x:P, y:Q(x) \vdash a:D(\mathrm{pair}(x,y))\end{array}}{\begin{array}{c}\Gamma, x:P, y:Q(x) \vdash \mathrm{ind}_\Sigma(a,\mathrm{pair}(x,y)) \\ \doteq a:D(\mathrm{pair}(x,y))\end{array}}$$

The natural numbers $\mathbb{N}$

$\mathbb{N}$-form:

$$\frac{}{\vdash \mathbb{N}\ \mathrm{type}}$$

$\mathbb{N}$-intro:

$$\frac{}{\vdash 0 : \mathbb{N}} \qquad \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash sn : \mathbb{N}}$$

$\mathbb{N}$-elim:

$$\frac{\begin{array}{c}\Gamma, x:\mathbb{N} \vdash D(x)\ \mathrm{type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx)\end{array}}{\Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,x) : D(x)}$$

$\mathbb{N}$-comp:

$$\frac{\begin{array}{c}\Gamma, x:\mathbb{N} \vdash D(x)\ \mathrm{type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx)\end{array}}{\begin{array}{c}\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a,b,0) \doteq a : D(0) \\ \Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,sx) \doteq b\left[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y\right] : D(sx)\end{array}}$$

Example: $\iota : \mathrm{bool} \to \mathbb{N}$

$\mathbb{N}$-form:

$$\frac{}{\vdash \mathbb{N}\ \mathrm{type}}$$

$\mathbb{N}$-intro:

$$\frac{}{\vdash 0 : \mathbb{N}}$$

$$\frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash sn : \mathbb{N}}$$

$\mathbb{N}$-elim:

$$\frac{\begin{array}{r}\Gamma, x:\mathbb{N} \vdash D(x)\ \mathrm{type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx)\end{array}}{\Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,x) : D(x)}$$

$\mathbb{N}$-comp:

$$\frac{\begin{array}{r}\Gamma, x:\mathbb{N} \vdash D(x)\ \mathrm{type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx)\end{array}}{\begin{array}{c}\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a,b,0) \doteq a : D(0) \\ \Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,sx) \doteq \\ b[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y] : D(sx)\end{array}}$$

Example: $z : \mathrm{bool} \to \mathbb{N}$

$\vdash \quad ? \quad : \mathbb{N}$

$\mathbb{N}$-form:

$$\frac{}{\vdash \mathbb{N}\ \mathrm{type}}$$

$\mathbb{N}$-intro:

$$\frac{}{\vdash 0 : \mathbb{N}}$$

$$\frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash sn : \mathbb{N}}$$

$\mathbb{N}$-elim:

$$\frac{\Gamma, x:\mathbb{N} \vdash D(x)\ \mathrm{type} \quad \Gamma \vdash a : D(0) \quad \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx)}{\Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,x) : D(x)}$$

$\mathbb{N}$-comp:

$$\frac{\Gamma, x:\mathbb{N} \vdash D(x)\ \mathrm{type} \quad \Gamma \vdash a : D(0) \quad \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx)}{\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a,b,0) \doteq a : D(0)}$$

$$\Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,sx) \doteq b[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y] : D(sx)$$

Example: $z : \mathrm{bool} \to \mathbb{N}$

$$\frac{x : \mathrm{bool} \vdash \quad ? \quad : \mathbb{N}}{\vdash \quad ? \quad : \mathbb{N}}$$

$\mathbb{N}$-form:

$$\frac{}{\vdash \mathbb{N} \text{ type}}$$

$\mathbb{N}$-intro:

$$\frac{}{\vdash 0 : \mathbb{N}}$$

$$\frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash sn : \mathbb{N}}$$

$\mathbb{N}$-elim:

$$\frac{\begin{array}{r} \Gamma, x : \mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x : \mathbb{N}, y : D(x) \vdash b : D(sx) \end{array}}{\Gamma, x : \mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a, b, x) : D(x)}$$

$\mathbb{N}$-comp:

$$\frac{\begin{array}{r} \Gamma, x : \mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x : \mathbb{N}, y : D(x) \vdash b : D(sx) \end{array}}{\begin{array}{l} \Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a, b, 0) \doteq a : D(0) \\ \Gamma, x : \mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a, b, sx) \doteq \\ \qquad b[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y] : D(sx) \end{array}}$$

Example: $z : \mathrm{bool} \to \mathbb{N}$

$$\dfrac{\dfrac{\dfrac{}{\vdash\ ?\ : \mathbb{N}} \qquad \dfrac{}{\vdash\ ?\ : \mathbb{N}}}{x : \mathrm{bool} \vdash\ ?\ : \mathbb{N}}}{\vdash\ ?\ : \mathbb{N}}$$

$\mathbb{N}$-form:

$$\dfrac{}{\vdash \mathbb{N}\ \mathrm{type}}$$

$\mathbb{N}$-intro:

$$\dfrac{}{\vdash 0 : \mathbb{N}}$$

$$\dfrac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash sn : \mathbb{N}}$$

$\mathbb{N}$-elim:

$$\dfrac{\begin{array}{r} \Gamma, x : \mathbb{N} \vdash D(x)\ \mathrm{type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x : \mathbb{N}, y : D(x) \vdash b : D(sx) \end{array}}{\Gamma, x : \mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a, b, x) : D(x)}$$

$\mathbb{N}$-comp:

$$\dfrac{\begin{array}{r} \Gamma, x : \mathbb{N} \vdash D(x)\ \mathrm{type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x : \mathbb{N}, y : D(x) \vdash b : D(sx) \end{array}}{\begin{array}{c} \Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a, b, 0) \doteq a : D(0) \\ \Gamma, x : \mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a, b, sx) \doteq \\ b\left[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y\right] : D(sx) \end{array}}$$

Example: $z : \mathrm{bool} \to \mathbb{N}$

$$\dfrac{\dfrac{}{\vdash s0 : \mathbb{N}} \qquad \dfrac{}{\vdash 0 : \mathbb{N}}}{\dfrac{x : \mathrm{bool} \vdash \quad ? \quad : \mathbb{N}}{\vdash \quad ? \quad : \mathbb{N}}}$$

$\mathbb{N}$-form:

$$\dfrac{}{\vdash \mathbb{N}\ \mathrm{type}}$$

$\mathbb{N}$-intro:

$$\dfrac{}{\vdash 0 : \mathbb{N}}$$

$$\dfrac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash sn : \mathbb{N}}$$

$\mathbb{N}$-elim:

$$\dfrac{\begin{array}{r} \Gamma, x : \mathbb{N} \vdash D(x)\ \mathrm{type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x : \mathbb{N}, y : D(x) \vdash b : D(sx) \end{array}}{\Gamma, x : \mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a, b, x) : D(x)}$$

$\mathbb{N}$-comp:

$$\dfrac{\begin{array}{r} \Gamma, x : \mathbb{N} \vdash D(x)\ \mathrm{type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x : \mathbb{N}, y : D(x) \vdash b : D(sx) \end{array}}{\begin{array}{c} \Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a, b, 0) \doteq a : D(0) \\ \Gamma, x : \mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a, b, sx) \doteq \\ b[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y] : D(sx) \end{array}}$$

Example: $z : \mathrm{bool} \to \mathbb{N}$

$$\dfrac{\dfrac{\dfrac{}{\vdash s0 : \mathbb{N}} \qquad \dfrac{}{\vdash 0 : \mathbb{N}}}{x : \mathrm{bool} \vdash \mathrm{ind}_{\mathrm{bool}}(s0, 0, x) : \mathbb{N}}}{\vdash \quad ? \quad : \mathbb{N}}$$

$\mathbb{N}$-form:

$$\dfrac{}{\vdash \mathbb{N}\ \mathrm{type}}$$

$\mathbb{N}$-intro:

$$\dfrac{}{\vdash 0 : \mathbb{N}}$$

$$\dfrac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash sn : \mathbb{N}}$$

$\mathbb{N}$-elim:

$$\dfrac{\begin{array}{r} \Gamma, x : \mathbb{N} \vdash D(x)\ \mathrm{type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x : \mathbb{N}, y : D(x) \vdash b : D(sx) \end{array}}{\Gamma, x : \mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a, b, x) : D(x)}$$

$\mathbb{N}$-comp:

$$\dfrac{\begin{array}{r} \Gamma, x : \mathbb{N} \vdash D(x)\ \mathrm{type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x : \mathbb{N}, y : D(x) \vdash b : D(sx) \end{array}}{\begin{array}{c} \Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a, b, 0) \doteq a : D(0) \\ \Gamma, x : \mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a, b, sx) \doteq b[\mathrm{ind}_{\mathbb{N}}(a, b, x)/y] : D(sx) \end{array}}$$

Example: $z : \mathrm{bool} \to \mathbb{N}$

$$\dfrac{\dfrac{\overline{\vdash s0 : \mathbb{N}} \qquad \overline{\vdash 0 : \mathbb{N}}}{x : \mathrm{bool} \vdash \mathrm{ind}_{\mathrm{bool}}(s0, 0, x) : \mathbb{N}}}{\vdash \lambda x . \mathrm{ind}_{\mathrm{bool}}(s0, 0, x) : \mathbb{N}}$$

$\mathbb{N}$-form:

$$\dfrac{}{\vdash \mathbb{N} \text{ type}}$$

$\mathbb{N}$-intro:

$$\dfrac{}{\vdash 0 : \mathbb{N}}$$

$$\dfrac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash sn : \mathbb{N}}$$

$\mathbb{N}$-elim:

$$\dfrac{\begin{array}{r} \Gamma, x : \mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x : \mathbb{N}, y : D(x) \vdash b : D(sx) \end{array}}{\Gamma, x : \mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a, b, x) : D(x)}$$

$\mathbb{N}$-comp:

$$\dfrac{\begin{array}{r} \Gamma, x : \mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x : \mathbb{N}, y : D(x) \vdash b : D(sx) \end{array}}{\begin{array}{l} \Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a, b, 0) \doteq a : D(0) \\ \Gamma, x : \mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a, b, sx) \doteq \\ \qquad b[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y] : D(sx) \end{array}}$$

Example: add: $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$

$\mathbb{N}$-form:

$$\frac{}{\vdash \mathbb{N} \text{ type}}$$

$\mathbb{N}$-intro:

$$\frac{}{\vdash 0 : \mathbb{N}}$$

$$\frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash sn : \mathbb{N}}$$

$\mathbb{N}$-elim:

$$\frac{\begin{array}{r}\Gamma, x:\mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx)\end{array}}{\Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,x) : D(x)}$$

$\mathbb{N}$-comp:

$$\frac{\begin{array}{r}\Gamma, x:\mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx)\end{array}}{\begin{array}{c}\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a,b,0) \doteq a : D(0) \\ \Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,sx) \doteq \\ b[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y] : D(sx)\end{array}}$$

Example: add: $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$

$\vdash \quad ? \quad : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})$

$\mathbb{N}$-form:

$$\frac{}{\vdash \mathbb{N} \text{ type}}$$

$\mathbb{N}$-intro:

$$\frac{}{\vdash 0 : \mathbb{N}}$$

$$\frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash sn : \mathbb{N}}$$

$\mathbb{N}$-elim:

$$\frac{\Gamma, x:\mathbb{N} \vdash D(x) \text{ type} \quad \Gamma \vdash a : D(0) \quad \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx)}{\Gamma, x:\mathbb{N} \vdash \text{ind}_{\mathbb{N}}(a,b,x) : D(x)}$$

$\mathbb{N}$-comp:

$$\frac{\Gamma, x:\mathbb{N} \vdash D(x) \text{ type} \quad \Gamma \vdash a : D(0) \quad \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx)}{\Gamma \vdash \text{ind}_{\mathbb{N}}(a,b,0) \doteq a : D(0)}$$

$$\Gamma, x:\mathbb{N} \vdash \text{ind}_{\mathbb{N}}(a,b,sx) \doteq b[\text{ind}_{\mathbb{N}}(a,b,x)/y] : D(sx)$$

Example: add: $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$

$$\frac{x:\mathbb{N} \vdash \quad ? \quad : \mathbb{N} \to \mathbb{N}}{\vdash \quad ? \quad : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})}$$

$\mathbb{N}$-form:

$$\frac{}{\vdash \mathbb{N} \text{ type}}$$

$\mathbb{N}$-intro:

$$\frac{}{\vdash 0 : \mathbb{N}}$$

$$\frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash sn : \mathbb{N}}$$

$\mathbb{N}$-elim:

$$\frac{\begin{array}{c}\Gamma, x:\mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx)\end{array}}{\Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,x) : D(x)}$$

$\mathbb{N}$-comp:

$$\frac{\begin{array}{c}\Gamma, x:\mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx)\end{array}}{\begin{array}{c}\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a,b,0) \doteq a : D(0) \\ \Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,sx) \doteq \\ b[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y] : D(sx)\end{array}}$$

Example: add: $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$

$$\dfrac{\dfrac{x:\mathbb{N}, y:\mathbb{N} \vdash \quad ? \quad : \mathbb{N}}{x:\mathbb{N} \vdash \quad ? \quad : \mathbb{N} \to \mathbb{N}}}{\vdash \quad ? \quad : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})}$$

$\mathbb{N}$-form:

$$\dfrac{}{\vdash \mathbb{N} \ \mathrm{type}}$$

$\mathbb{N}$-intro:

$$\dfrac{}{\vdash 0 : \mathbb{N}}$$

$$\dfrac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash sn : \mathbb{N}}$$

$\mathbb{N}$-elim:

$$\dfrac{\begin{array}{c} \Gamma, x:\mathbb{N} \vdash D(x) \ \mathrm{type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx) \end{array}}{\Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,x) : D(x)}$$

$\mathbb{N}$-comp:

$$\dfrac{\begin{array}{c} \Gamma, x:\mathbb{N} \vdash D(x) \ \mathrm{type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx) \end{array}}{\begin{array}{c} \Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a,b,0) \doteq a : D(0) \\ \Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,sx) \doteq \\ b[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y] : D(sx) \end{array}}$$

Example: add : $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$

$$\dfrac{\dfrac{\dfrac{x:\mathbb{N} \vdash ? : \mathbb{N} \qquad x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N} \vdash ? : \mathbb{N}}{x:\mathbb{N}, y:\mathbb{N} \vdash ? : \mathbb{N}}}{x:\mathbb{N} \vdash ? : \mathbb{N} \to \mathbb{N}}}{\vdash ? : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})}$$

$\mathbb{N}$-form:

$$\dfrac{}{\vdash \mathbb{N}\ \mathrm{type}}$$

$\mathbb{N}$-intro:

$$\dfrac{}{\vdash 0 : \mathbb{N}}$$

$$\dfrac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash sn : \mathbb{N}}$$

$\mathbb{N}$-elim:

$$\dfrac{\begin{array}{r}\Gamma, x:\mathbb{N} \vdash D(x)\ \mathrm{type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx)\end{array}}{\Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,x) : D(x)}$$

$\mathbb{N}$-comp:

$$\dfrac{\begin{array}{r}\Gamma, x:\mathbb{N} \vdash D(x)\ \mathrm{type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx)\end{array}}{\begin{array}{c}\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a,b,0) \doteq a : D(0) \\ \Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,sx) \doteq b[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y] : D(sx)\end{array}}$$

Example: add : $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$

$$
\dfrac{\dfrac{\dfrac{x:\mathbb{N} \vdash 0:\mathbb{N} \qquad x:\mathbb{N},\, y:\mathbb{N},\, z:\mathbb{N} \vdash sz:\mathbb{N}}{x:\mathbb{N},\, y:\mathbb{N} \vdash \; ? \; : \mathbb{N}}}{x:\mathbb{N} \vdash \; ? \; : \mathbb{N} \to \mathbb{N}}}{\vdash \; ? \; : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})}
$$

$\mathbb{N}$-form:

$$\dfrac{}{\vdash \mathbb{N} \text{ type}}$$

$\mathbb{N}$-intro:

$$\dfrac{}{\vdash 0:\mathbb{N}}$$

$$\dfrac{\Gamma \vdash n:\mathbb{N}}{\Gamma \vdash sn:\mathbb{N}}$$

$\mathbb{N}$-elim:

$$\dfrac{\begin{matrix}\Gamma,\, x:\mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a : D(0) \\ \Gamma,\, x:\mathbb{N},\, y:D(x) \vdash b : D(sx)\end{matrix}}{\Gamma,\, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,x) : D(x)}$$

$\mathbb{N}$-comp:

$$\dfrac{\begin{matrix}\Gamma,\, x:\mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a : D(0) \\ \Gamma,\, x:\mathbb{N},\, y:D(x) \vdash b : D(sx)\end{matrix}}{\begin{matrix}\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a,b,0) \doteq a : D(0) \\ \Gamma,\, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,sx) \doteq \\ b[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y] : D(sx)\end{matrix}}$$

Example: add: $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$

$$\dfrac{\dfrac{\dfrac{x:\mathbb{N} \vdash 0:\mathbb{N} \qquad x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N} \vdash sz:\mathbb{N}}{x:\mathbb{N}, y:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(0, sz, y):\mathbb{N}}}{x:\mathbb{N} \vdash \quad ? \quad :\mathbb{N} \to \mathbb{N}}}{\vdash \quad ? \quad : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})}$$

$\mathbb{N}$-form:

$$\dfrac{}{\vdash \mathbb{N} \text{ type}}$$

$\mathbb{N}$-intro:

$$\dfrac{}{\vdash 0:\mathbb{N}}$$

$$\dfrac{\Gamma \vdash n:\mathbb{N}}{\Gamma \vdash sn:\mathbb{N}}$$

$\mathbb{N}$-elim:

$$\dfrac{\begin{array}{c}\Gamma, x:\mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a:D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b:D(sx)\end{array}}{\Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,x):D(x)}$$

$\mathbb{N}$-comp:

$$\dfrac{\begin{array}{c}\Gamma, x:\mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a:D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b:D(sx)\end{array}}{\begin{array}{c}\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a,b,0) \doteq a : D(0) \\ \Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,sx) \doteq \\ b[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y]:D(sx)\end{array}}$$

Example: add: $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$

$$\dfrac{\dfrac{\dfrac{x:\mathbb{N} \vdash 0:\mathbb{N} \qquad x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N} \vdash sz:\mathbb{N}}{x:\mathbb{N}, y:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(0, sz, y) : \mathbb{N}}}{x:\mathbb{N} \vdash \lambda y.\, \mathrm{ind}_{\mathbb{N}}(0, sz, y) : \mathbb{N} \to \mathbb{N}}}{\vdash \quad ? \quad : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})}$$

$\mathbb{N}$-form:

$$\dfrac{}{\vdash \mathbb{N} \text{ type}}$$

$\mathbb{N}$-intro:

$$\dfrac{}{\vdash 0 : \mathbb{N}}$$

$$\dfrac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash sn : \mathbb{N}}$$

$\mathbb{N}$-elim:

$$\dfrac{\begin{array}{c} \Gamma, x:\mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx) \end{array}}{\Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a, b, x) : D(x)}$$

$\mathbb{N}$-comp:

$$\dfrac{\begin{array}{c} \Gamma, x:\mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx) \end{array}}{\begin{array}{c} \Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a, b, 0) \doteq a : D(0) \\ \Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a, b, sx) \doteq \\ b[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y] : D(sx) \end{array}}$$

Example: add: $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$

$$\dfrac{\dfrac{x:\mathbb{N} \vdash 0:\mathbb{N} \qquad x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N} \vdash sz:\mathbb{N}}{x:\mathbb{N}, y:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(0, sz, y):\mathbb{N}}}{\dfrac{x:\mathbb{N} \vdash \lambda y.\mathrm{ind}_{\mathbb{N}}(0, sz, y):\mathbb{N} \to \mathbb{N}}{\vdash \lambda x.\lambda y.\mathrm{ind}_{\mathbb{N}}(0, sz, y):\mathbb{N} \to (\mathbb{N} \to \mathbb{N})}}$$

$\mathbb{N}$-form:

$$\dfrac{}{\vdash \mathbb{N}\ \mathrm{type}}$$

$\mathbb{N}$-intro:

$$\dfrac{}{\vdash 0:\mathbb{N}}$$

$$\dfrac{\Gamma \vdash n:\mathbb{N}}{\Gamma \vdash sn:\mathbb{N}}$$

$\mathbb{N}$-elim:

$$\dfrac{\begin{array}{c}\Gamma, x:\mathbb{N} \vdash D(x)\ \mathrm{type} \\ \Gamma \vdash a:D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b:D(sx)\end{array}}{\Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,x):D(x)}$$

$\mathbb{N}$-comp:

$$\dfrac{\begin{array}{c}\Gamma, x:\mathbb{N} \vdash D(x)\ \mathrm{type} \\ \Gamma \vdash a:D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b:D(sx)\end{array}}{\begin{array}{c}\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a,b,0) \doteq a:D(0) \\ \Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,sx) \doteq \\ b[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y]:D(sx)\end{array}}$$

Example: mult : $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$

$$\dfrac{\dfrac{\dfrac{x:\mathbb{N} \vdash ? : \mathbb{N} \qquad x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N} \vdash ? : \mathbb{N}}{x:\mathbb{N}, y:\mathbb{N} \vdash ? : \mathbb{N}}}{x:\mathbb{N} \vdash ? : \mathbb{N} \to \mathbb{N}}}{\vdash ? : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})}$$

$\mathbb{N}$-form:

$$\dfrac{}{\vdash \mathbb{N} \text{ type}}$$

$\mathbb{N}$-intro:

$$\dfrac{}{\vdash 0 : \mathbb{N}}$$

$$\dfrac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash sn : \mathbb{N}}$$

$\mathbb{N}$-elim:

$$\dfrac{\begin{array}{c} \Gamma, x:\mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx) \end{array}}{\Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,x) : D(x)}$$

$\mathbb{N}$-comp:

$$\dfrac{\begin{array}{c} \Gamma, x:\mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a : D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b : D(sx) \end{array}}{\begin{array}{c} \Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a,b,0) \doteq a : D(0) \\ \Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,sx) \doteq \\ b\left[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y\right] : D(sx) \end{array}}$$

Example: mult : $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$

$$\dfrac{\dfrac{\dfrac{x:\mathbb{N} \vdash 0:\mathbb{N} \qquad x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N} \vdash \mathrm{add}(z,x):\mathbb{N}}{x:\mathbb{N}, y:\mathbb{N} \vdash \quad ? \quad :\mathbb{N}}}{x:\mathbb{N} \vdash \quad ? \quad : \mathbb{N} \to \mathbb{N}}}{\vdash \quad ? \quad : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})}$$

$\mathbb{N}$-form:

$$\dfrac{}{\vdash \mathbb{N}\ \mathrm{type}}$$

$\mathbb{N}$-intro:

$$\dfrac{}{\vdash 0:\mathbb{N}}$$

$$\dfrac{\Gamma \vdash n:\mathbb{N}}{\Gamma \vdash sn:\mathbb{N}}$$

$\mathbb{N}$-elim:

$$\dfrac{\begin{array}{c}\Gamma, x:\mathbb{N} \vdash D(x)\ \mathrm{type} \\ \Gamma \vdash a:D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b:D(sx)\end{array}}{\Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,x):D(x)}$$

$\mathbb{N}$-comp:

$$\dfrac{\begin{array}{c}\Gamma, x:\mathbb{N} \vdash D(x)\ \mathrm{type} \\ \Gamma \vdash a:D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b:D(sx)\end{array}}{\begin{array}{c}\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a,b,0) \doteq a : D(0) \\ \Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,sx) \doteq \\ b[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y] : D(sx)\end{array}}$$

Example: $\mathrm{mult} : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})$

$$\dfrac{\dfrac{\dfrac{x:\mathbb{N} \vdash 0:\mathbb{N} \qquad x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N} \vdash \mathrm{add}(z,x):\mathbb{N}}{x:\mathbb{N}, y:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(0, \mathrm{add}(z,x), y):\mathbb{N}}}{x:\mathbb{N} \vdash \quad ? \quad : \mathbb{N}\to\mathbb{N}}}{\vdash \quad ? \quad : \mathbb{N}\to(\mathbb{N}\to\mathbb{N})}$$

$\mathbb{N}$-form:

$$\dfrac{}{\vdash \mathbb{N}\ \mathrm{type}}$$

$\mathbb{N}$-intro:

$$\dfrac{}{\vdash 0:\mathbb{N}}$$

$$\dfrac{\Gamma \vdash n:\mathbb{N}}{\Gamma \vdash sn:\mathbb{N}}$$

$\mathbb{N}$-elim:

$$\dfrac{\Gamma, x:\mathbb{N} \vdash D(x)\ \mathrm{type} \qquad \Gamma \vdash a:D(0) \qquad \Gamma, x:\mathbb{N}, y:D(x) \vdash b:D(sx)}{\Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,x):D(x)}$$

$\mathbb{N}$-comp:

$$\dfrac{\Gamma, x:\mathbb{N} \vdash D(x)\ \mathrm{type} \qquad \Gamma \vdash a:D(0) \qquad \Gamma, x:\mathbb{N}, y:D(x) \vdash b:D(sx)}{\begin{array}{c}\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a,b,0) \doteq a : D(0) \\ \Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,sx) \doteq b[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y] : D(sx)\end{array}}$$

Example: $\mathrm{mult} : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})$

$$\dfrac{\dfrac{\dfrac{x:\mathbb{N} \vdash 0:\mathbb{N} \qquad x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N} \vdash \mathrm{add}(z,x):\mathbb{N}}{x:\mathbb{N}, y:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(0, \mathrm{add}(z,x), y):\mathbb{N}}}{x:\mathbb{N} \vdash \lambda y.\, \mathrm{ind}_{\mathbb{N}}(0, \mathrm{add}(z,x), y) : \mathbb{N} \to \mathbb{N}}}{\vdash \quad ? \quad : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})}$$

$\mathbb{N}$-form:

$$\dfrac{}{\vdash \mathbb{N} \text{ type}}$$

$\mathbb{N}$-intro:

$$\dfrac{}{\vdash 0:\mathbb{N}}$$

$$\dfrac{\Gamma \vdash n:\mathbb{N}}{\Gamma \vdash sn:\mathbb{N}}$$

$\mathbb{N}$-elim:

$$\dfrac{\begin{array}{r}\Gamma, x:\mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a:D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b:D(sx)\end{array}}{\Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,x):D(x)}$$

$\mathbb{N}$-comp:

$$\dfrac{\begin{array}{r}\Gamma, x:\mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a:D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b:D(sx)\end{array}}{\begin{array}{c}\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a,b,0) \doteq a : D(0) \\ \Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,sx) \doteq \\ b\left[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y\right] : D(sx)\end{array}}$$

Example: mult : $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$

$$\dfrac{\dfrac{\dfrac{x:\mathbb{N} \vdash 0:\mathbb{N} \qquad x:\mathbb{N}, y:\mathbb{N}, z:\mathbb{N} \vdash \mathrm{add}(z,x):\mathbb{N}}{x:\mathbb{N}, y:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(0, \mathrm{add}(z,x), y):\mathbb{N}}}{x:\mathbb{N} \vdash \lambda y.\,\mathrm{ind}_{\mathbb{N}}(0,\mathrm{add}(z,x),y) : \mathbb{N} \to \mathbb{N}}}{\vdash \lambda x.\lambda y.\,\mathrm{ind}_{\mathbb{N}}(0, \mathrm{add}(z,x), y) : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})}$$

$\mathbb{N}$-form:

$$\dfrac{}{\vdash \mathbb{N} \text{ type}}$$

$\mathbb{N}$-intro:

$$\dfrac{}{\vdash 0:\mathbb{N}}$$

$$\dfrac{\Gamma \vdash n:\mathbb{N}}{\Gamma \vdash sn:\mathbb{N}}$$

$\mathbb{N}$-elim:

$$\dfrac{\begin{array}{c}\Gamma, x:\mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a:D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b:D(sx)\end{array}}{\Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,x):D(x)}$$

$\mathbb{N}$-comp:

$$\dfrac{\begin{array}{c}\Gamma, x:\mathbb{N} \vdash D(x) \text{ type} \\ \Gamma \vdash a:D(0) \\ \Gamma, x:\mathbb{N}, y:D(x) \vdash b:D(sx)\end{array}}{\begin{array}{c}\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(a,b,0) \doteq a : D(0) \\ \Gamma, x:\mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(a,b,sx) \doteq \\ b\left[\mathrm{ind}_{\mathbb{N}}(a,b,x)/y\right] : D(sx)\end{array}}$$

Next time:

Identity types!