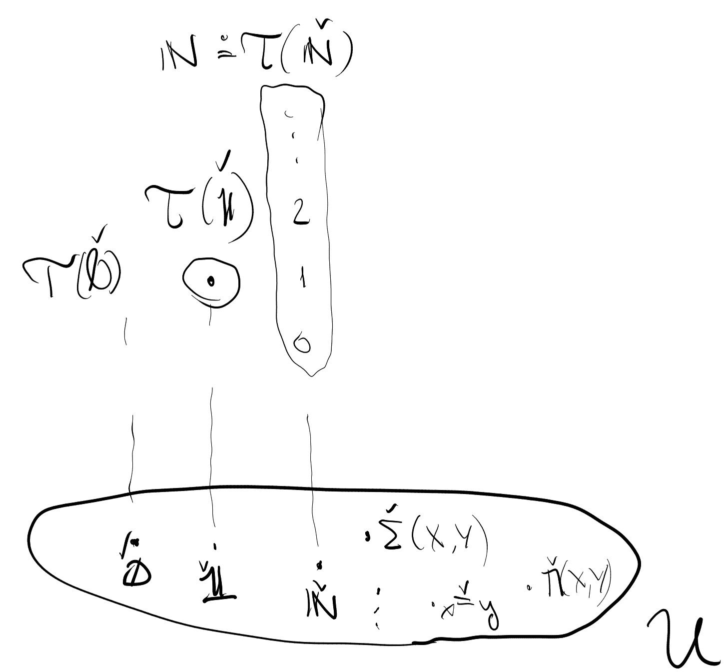
Outline

Last time: Universes $\mathcal{U}$

+ Propositions as Types



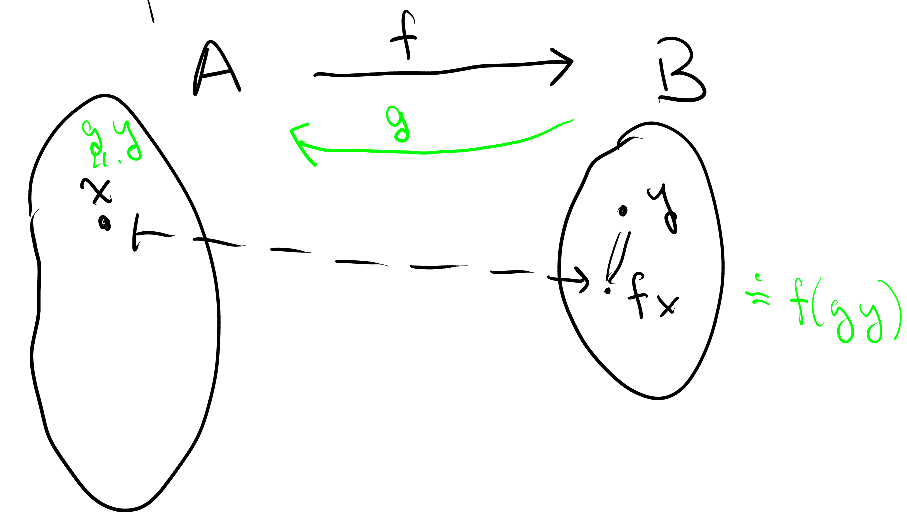
Today: Equivalences

- Homotopies
- Equivalences as bi-invertible maps
- Identifications in Σ-types
- Preview of function extensionality & univalence

Retractions & sections

- Propositions as types (Curry–Howard) interpretation of

"$f : A \to B$ is surjective"



is : $\prod_{y:B} \sum_{x:A} f\,x = y$

by type-theoretic choice, we have this

iff $\underbrace{\sum_{g:A\to B} \prod_{y:B} f\,(g\,y) = y}$ $=:$ is-split-surjective(f)

$=:$ see(f)

If $\downarrow$ holds we

say: g is a section of f

& f is a retraction of g

we have here a relation between $f \circ g$ & id_B

a priori weaker than

$$f \circ g =_{B\to B} \mathrm{id}_B$$

$\rightsquigarrow$ homotopy

Homotopies

Example $\text{neg-bool} : \text{bool} \to \text{bool}$, $\quad \text{neg-bool true} = \text{false}$
$\text{neg-bool false} = \text{true}$

Then: $\text{neg-bool}\,(\text{neg-bool true}) \doteq \text{true}$

& $\text{neg-bool}\,(\text{neg-bool false}) \doteq \text{false}$,

but $b : \text{bool} \vdash \text{neg-bool}\,(\text{neg-bool}\ b) \not\doteq b : \text{bool}$

So also $\text{neg-bool} \circ \text{neg-bool} \not\doteq \text{id}_{\text{bool}}$

However, $\prod_{b:\text{bool}} \text{neg-bool}\,(\text{neg-bool}\ b) =_{\text{bool}} b$ by ind-bool!

So $\text{neg-bool} \circ \text{neg-bool}$ & id_{bool} are pointwise equal.

Def Let $f, g : \Pi_{x:A} B(x)$.

$$f \sim g \quad := \quad \Pi_{x:A}(f\,x =_{B(x)} g\,x),$$ the type of homotopies from f to g.

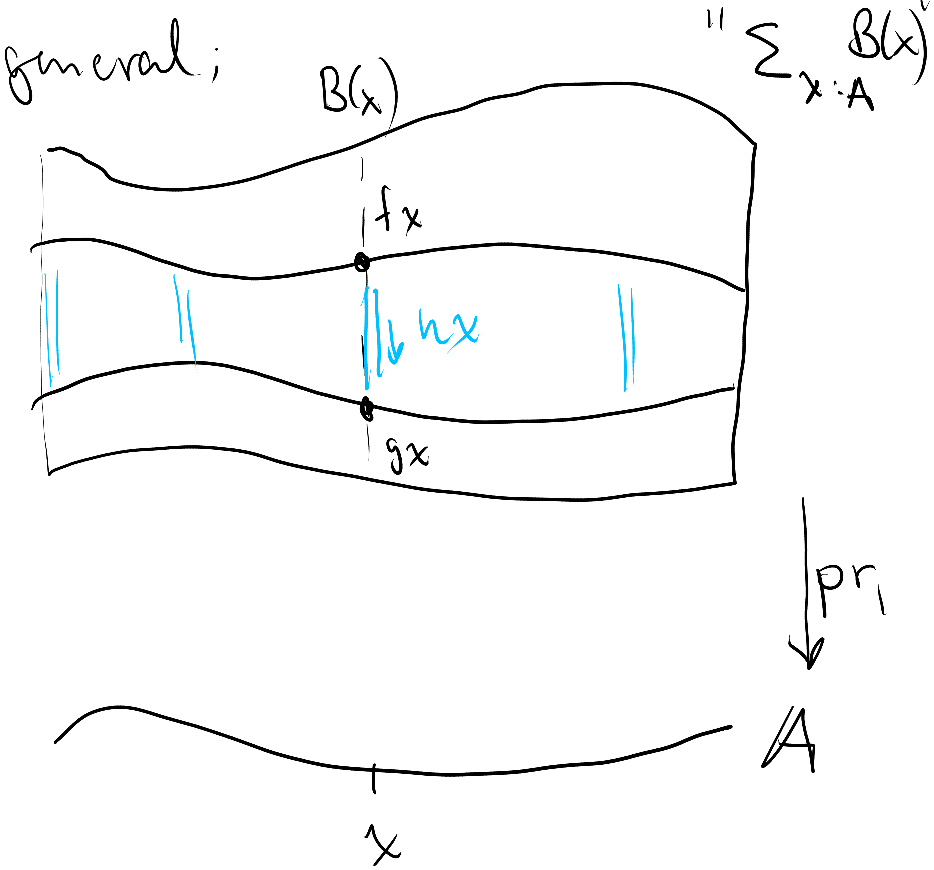
Ex neg-bool $\circ$ neg-bool $\sim$ $\mathrm{id}_{\mathrm{bool}}$

In pictures

- Special case B doesn't depend on x

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

- In general:



Commutative diagrams

to say that a diagram of types & functions commutes, we use homotopies, e.g.,

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{H}{\sim}"] & B \arrow[dl, "g"] \\ & X & \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{H}{\sim}"] & A' \arrow[d, "f'"] \\ B \arrow[r, "h"'] & B' \end{tikzcd}$$

$H : f \sim g \circ h$ $\qquad$ $H : h \circ f \sim f' \circ g$

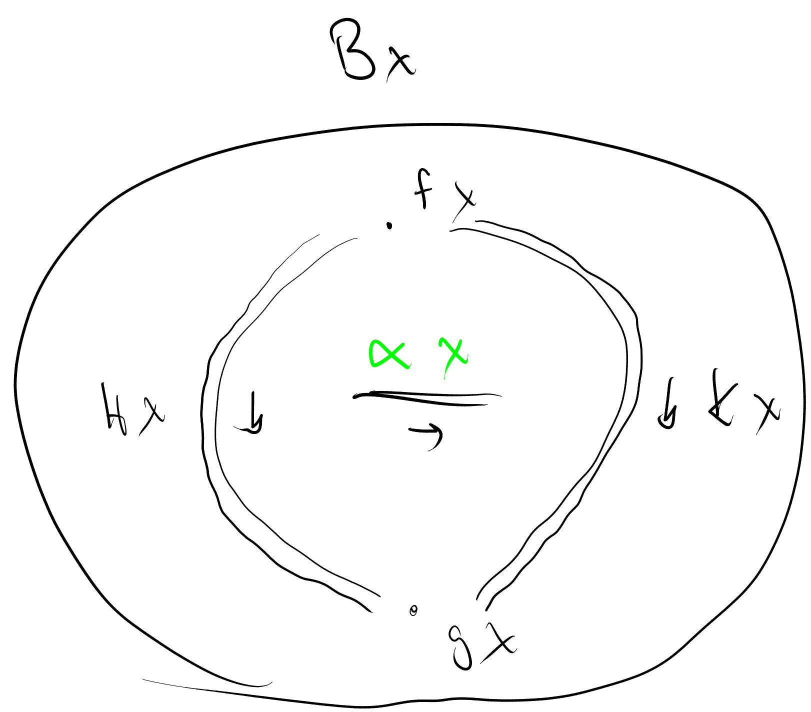
Homotopies of homotopies

If $f, g : \Pi_{x:A} B(x)$

& $H, K : (f \sim g) \doteq \Pi_{x:A}\, fx = gx$

we have $H \sim K \doteq \Pi_{x:A}\, Hx = Kx$

$x : A$



etc.

Since homotopies are pointwise identity types, we can lift laws & operations pointwise to homotopies, e.g.

$$\text{refl-htpy} : \prod_{f:P} f \sim f \qquad P \doteq \prod_{x:A} B(x)$$

$$\text{inv-htpy} : \prod_{f,g:P} (f \sim g \to g \sim f), \quad H^{-1}$$

$$\text{concat-htpy} : \prod_{f,g,h:P} (f \sim g \to g \sim h \to f \sim h), \quad H \cdot K$$

w/ groupoid laws: assoc: $(H \cdot K) \cdot L \sim H \cdot (K \cdot L)$

unit laws:

$\text{refl-htpy} \cdot H \sim H$

$H \cdot \text{refl-htpy} \sim H$

left/right inv.: $H^{-1} \cdot H \sim \text{refl-htpy}$

$H \cdot H^{-1} \sim \text{refl-htpy}$

Whiskering

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "H \Downarrow"] & B \arrow[r, "h"] & C \end{tikzcd}$$

$f, g : A \to B$

Def If $H : f \sim g$, $h : B \to C$,

then $h \cdot H : h \circ f \sim h \circ g$

$$h \cdot H :\equiv \lambda x : A,\ \mathrm{ap}_h (H x)$$

If $k : D \to A$, then

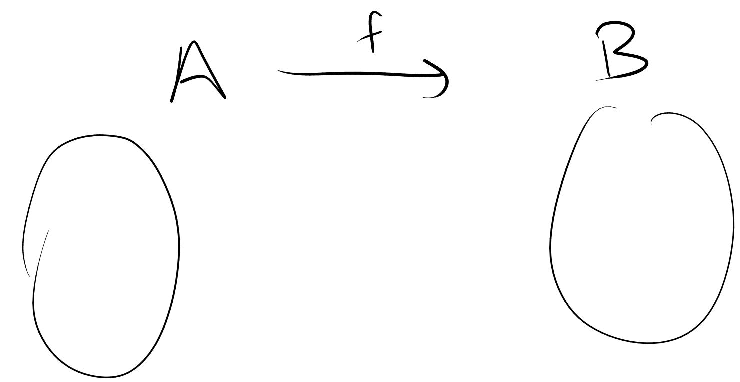
$$\begin{tikzcd} D \arrow[r, "k"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "H \Downarrow"] & B \end{tikzcd}$$

$$H \cdot k : f \circ k \sim g \circ k$$

$$H \cdot k = \lambda x : D,\ H (k x)$$

Ex: To what extent can we whisker dependent functions?

Bi-invertible maps



Def Let $f : A \to B$

$\mathrm{sec}(f) := \sum_{g : B \to A} f \circ g \sim \mathrm{id}_B$ (f is a split surjection)

$\mathrm{retr}(f) := \sum_{h : B \to A} h \circ f \sim \mathrm{id}_A$ (f is a split monomorphism)

$\text{is-equiv}(f) := \mathrm{sec}(f) \times \mathrm{retr}(f)$ (f is bi-invertible)

Ex id_A for any A, $\text{neg-bool} : \mathrm{bool} \to \mathrm{bool}$, $\_ + k : \mathbb{Z} \to \mathbb{Z}$

$\mathrm{succ}_{\mathrm{Fin}(n)} : \mathrm{Fin}\ n \to \mathrm{Fin}\ n$ (wrap-around successor)

Def $\text{has-inverse}(f) := \sum_{g : B \to A} (f \circ g \sim \mathrm{id}_B) \times (g \circ f \sim \mathrm{id}_A)$ (f has two-sided inverse)

Lemma $\text{has-inverse}(f) \to \text{is-equiv}(f)$

<u>Discussion</u> It turns out that is-equiv(f) is much better behaved than has-inverse(f), in general, but for <u>sets</u> such as $\mathbb{N}, \mathbb{Z}$, Fin n, $\mathbb{N} \to \mathbb{N}$, etc. it makes no difference. $\longrightarrow$ We'll come back to this!

<u>Prop</u> If $f : A \to B$, then is-equiv$(f) \longrightarrow$ has-inverse(f)

<u>Proof</u> By Σ-ind, we have $g, h : B \to A$, $G : f \circ g \sim id_B$, $H : h \circ f \sim id_A$

define $K : g \sim h$ by $g \doteq id_A \circ g \overset{H^{-1} \cdot g}{\sim} (h \circ f) \circ g \doteq h \circ (f \circ g) \overset{h \cdot G}{\sim} h \circ id_B \doteq h$

now we get $H' : g \circ f \sim id_A$ by

$$g \circ f \overset{K \cdot f}{\sim} h \circ f \overset{H}{\sim} id_A \qquad \square$$

High-school algebra w/ types

Def: $A \simeq B$

$$:= \sum_{f : A \to B} \text{is-equiv}(f)$$

More examples of equivalences:

$$\mathbb{1} \times A \simeq A \simeq \mathbb{1} \times A$$

$$A \times B \simeq B \times A$$

$$(A \times B) \times C \simeq A \times (B \times C)$$

$$\emptyset \times A \simeq \emptyset$$

$$A + \emptyset \simeq A \simeq \emptyset + A$$

$$A + B \simeq B + A$$

$$(A + B) + C \simeq A + (B + C)$$

$$A \times (B + C) \simeq A \times B + A \times C$$

$$(A + B) \times C \simeq A \times C + B \times C$$

Some of these generalize to Σ-types:

$$\sum_{z : A + B} C(z) \simeq \left(\sum_{x : A} C(\mathrm{inl}\, x) \right) + \left(\sum_{y : B} C(\mathrm{inr}\, y) \right)$$

$$\sum_{x : A} \big(B(x) + C(x) \big) \simeq \left(\sum_{x : A} B(x) \right) + \left(\sum_{x : A} C(x) \right)$$

Laws for equivs

refl-equiv : $A \simeq A$

inv-equiv : $A \simeq B \longrightarrow B \simeq A$

concat-equiv : $A \simeq B \to B \simeq C \longrightarrow A \simeq C$

similar to laws for $=$, $\sim$!

have functions id-to-$\sim$: $\prod_{f,g : \prod_{x:A} B(x)} \left(f =_P g \to f \sim g \right)$

id-to-$\simeq$: $\prod_{A,B : \mathcal{U}} \left(A =_{\mathcal{U}} B \to A \simeq B \right)$

We'll soon postulate that these are ptwise equivalences getting

funext : $\prod_{f,g:P}$ is-equiv(id-to-$\sim$ f g), $\mathcal{U}A : \prod_{A,B:\mathcal{U}}$ is-equiv(id-to-$\simeq$ A B).

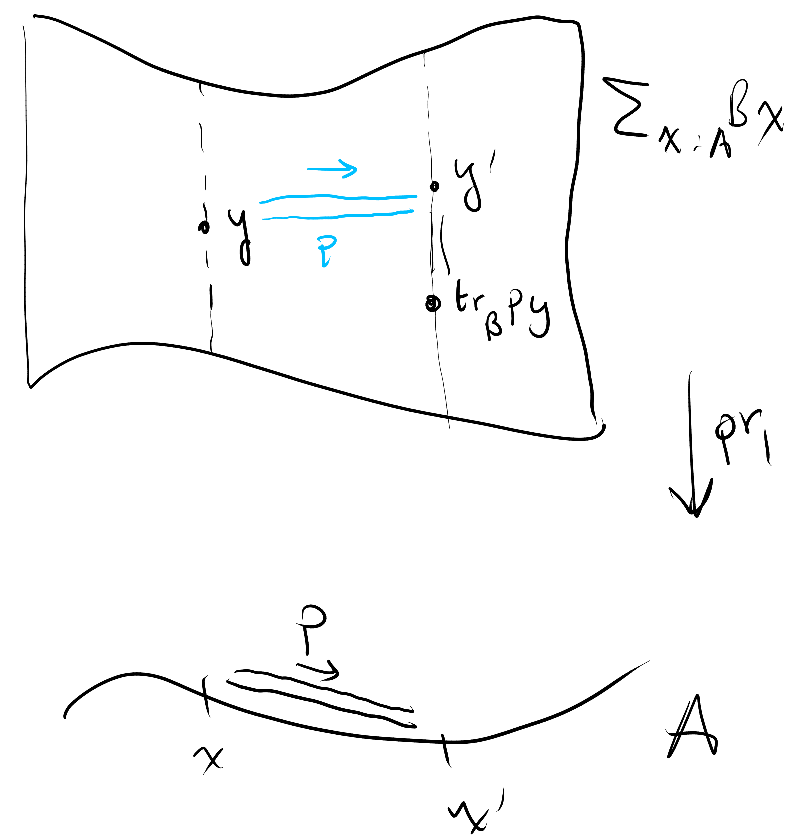
§9.3 Id's in Σ-types

Without postulates, we can give a useful description of $=$'s in Σ-types: Fix A type, $x : A \vdash B(x)$ type family

Want refl. relation R, $z, z' : S \vdash R(z, z')$ type

Intuitively, this should contain $p : pr_1\, z = pr_1\, z'$
But what about the 2nd components?

Assume $z \doteq (x, y)$, $z' \doteq (x', y')$



__Def__ $\left(y \overset{B}{\underset{p}{=}} y'\right) := \left(tr_B(p, y) =_{Bx'} y'\right)$

__Def__ $\mathrm{Eq}\text{-}\Sigma\; z\; z' := \sum_{p : pr_1 z = pr_1 z'} \left(pr_2\, z \overset{B}{\underset{p}{=}} pr_2\, z'\right)$

$\mathrm{Eq}\text{-}\Sigma\ z\ z' :\equiv \sum_{p : pr_1 z = pr_1 z'} \left(pr_2 z =^B_p pr_2 z' \right)$

$\qquad\qquad\qquad\qquad \parallel$

$\mathrm{tr}_B(p, pr_2 z) = pr_2 z'$

<u>Def</u> $\mathrm{refl}\text{-}\mathrm{Eq}\text{-}\Sigma : \prod_{z : S} \mathrm{Eq}\text{-}\Sigma\ z\ z$

$\parallel$

$\lambda z.\ (\mathrm{refl}_{pr_1 z}, \mathrm{refl}_{pr_2 z})$

<u>Def</u> $\mathrm{pair}\text{-}\mathrm{eq} : \prod_{z, z' : S} \left(z = z' \to \mathrm{Eq}\text{-}\Sigma\ z\ z' \right)$

by path ind.

<u>Def</u> $\mathrm{eq}\text{-}\mathrm{pair} : \prod_{z, z' : S} \left(\mathrm{Eq}\text{-}\Sigma\ z\ z' \longrightarrow z = z' \right)$

by Σ-ind repeated, $\left(\begin{array}{l} x, x' : A,\ y : B x \\ p : x = x',\ y' : B x' \\ q : \mathrm{tr}_B(p, y) = y' \end{array} \right)$, path ind on p

$\left(\begin{array}{l} x : A,\ y, y' : B x \\ q : y = y' \end{array} \right)$, path-ind on q: $\left(\begin{array}{l} x : A \\ y : B x,\ \text{goal: } (x, y) =_S (x, y) \end{array} \right)$

refl.

Claim: 1) $\displaystyle\prod_{z,z':S} \prod_{w : \text{Eq-}\Sigma\, z\, z'} \text{pair-eq}(\text{eq-pair } w) = w$

by Σ-ind, then repeated path induction. $\square$

2) $\displaystyle\prod_{z,z':S} \prod_{p : z = z'} \text{eq-pair}(\text{pair-eq } p) = p$

by path ind, then Σ-ind. $\square$

Cor For all $z, z' : S$, $(z = z') \simeq \text{Eq-}\Sigma\; z\; z'$