TDAB01 Probability and Statistics

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Lecture 6: Stochastic Processes

Overview

- **Definitions and classifications**
- **Markov chains**
- **Binomial processes**
- **Poisson processes**

Stochastic Processes

- **Stochastic process**:
  Sequence of random variables $X_1, X_2, \ldots$ **observed over time**
  - Example: X_t = number of "likes" for video on YouTube during day t
  - Example: X_t = temperature in certain city at time t
- **Stochastic process**:
  Random variable $X(t, \omega)$ which also depends on time, where
  - $t \in \mathcal{T}$ and $\mathcal{T}$ - set of times,
    can be discrete, e. g. $\mathcal{T} = \{1, 2, 3, \ldots\}$ or continuous, e. g. $\mathcal{T} = [0, T]$
  - $\omega \in \Omega$ - outcome of an experiment
  - Values of $X(t, \omega)$ are called **states**
  - Simplified notation $X(t) = X(t, \omega)$
- Classification of processes:
  - Discrete or continuous **states**
  - Discrete or continuous **time**
- Discrete states, continuous time: Number of "likes" on YouTube over time
- Discrete states, discrete time: Number of "likes" on YouTube during day t
- Continuous states, discrete time: Highest temperature on certain day
- Continuous states, continuous time: Temperature over time

Markov processes

- **Markov process**: The forecast for tomorrow depends only on today:

$$P(\text{future}|\text{now}, \text{history}) = P(\text{future}|\text{now})$$

- (Discrete-time) **Markov process**:
  For all times $t_1 < ... < t_n < t_{n+1}$ it holds that

  $$P(X(t_{n+1}) = x_{n+1}|X(t_1) = x_1, \ldots, X(t_n) = x_n) = P(X(t_{n+1}) = x_{n+1})|X(t_n) = x_n)$$

  i.e. $X(t_{n+1})$ is independent of $X(t_1), \ldots, X(t_{n-1})$ if $X(t_n)$ is given
- Well-developed theory, simple techniques
- But many processes are not Markov

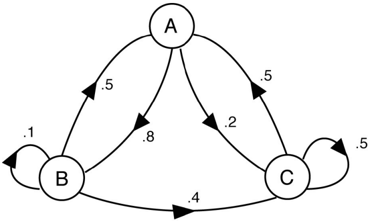
Markov chains

- **Markov chain**: Markov process with **discrete time** and **discrete states**
- Time: $\mathcal{T} = \{1, 2, 3, \dots\}$
- Enumerate states: $1, 2, \dots, n$ (or $A, B, C, \dots$)

  $n = \infty$ generally possible; not in this course
- **Transition probability (one-step)**

$$p_{ij}(t) = \boldsymbol{P}\{X(t+1) = j | X(t) = i\}$$

  $\rightarrow$ probability to move from state i to state j
- Example:



Markov chains

- **Transition probability (h-steps)**

$$p_{ij}^{(h)}(t) = \boldsymbol{P}\left\{X(t+h) = j | X(t) = i\right\}$$

$\rightarrow$ probability to move from state i to state j by means of h transitions

- **Homogeneous Markov chain**: Transition probabilities independent of time:

$$p_{ij}(t) = p_{ij}$$

- **Transition matrix**

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

- Example: $\Omega = \{\text{sunny, rainy}\}$

$P(\text{sunny tomorrow}|\text{sunny today}) = 0.9, \; P(\text{rainy tomorrow}|\text{rainy today}) = 0.3$

- Example from previous slide

$$P = \begin{pmatrix} 0 & 0.8 & 0.2 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0 & 0.5 \end{pmatrix}$$

Markov chains

- **Transition matrix** 1-step

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}$$

  Note that $p_{i1} + p_{i2} + \ldots p_{in} = 1$, for all i

- **Transition probability (h-steps)**

$$p_{ij}^{(h)}(t) = \boldsymbol{P}\left\{X(t+h) = j | X(t) = i\right\}$$

- Complicated: many paths for $i \to j$ when $h > 1$
- Example: $\Omega = \{1, 2\}$. If $h = 2$, we can make the trip $1 \to 2$ in two ways:
  - $1 \to 1 \to 2$
  - $1 \to 2 \to 2$

  $\to$ Use matrices

Markov chains

- **Transition matrix** *h*-steps

$$P^{(h)} = \begin{pmatrix} p_{11}^{(h)} & p_{12}^{(h)} & \cdots & p_{1n}^{(h)} \\ p_{21}^{(h)} & p_{22}^{(h)} & \cdots & p_{2n}^{(h)} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}^{(h)} & p_{n2}^{(h)} & \cdots & p_{nn}^{(h)} \end{pmatrix}$$

  Note that $p_{i1}^{(h)} + p_{i2}^{(h)} + \ldots p_{in}^{(h)} = 1$, for all i

- Relation between P and $P^{(h)}$

$$P^{(h)} = P \cdot P \cdots P = P^h$$

- Example: Example 6.9 in textbook

Distribution of $X(h)$

- **Initial distribution** of $X(t)$ at $t = 0$ is the row vector:

$$P_0 = (P_0(1), P_0(2), ..., P_0(n)), \quad P_0(i) = P(X(0) = i)$$

- **Probability distribution over the states** after h steps:

$$P_h = (P_h(1), P_h(2), ..., P_h(n))$$

- Computing P_h:

$$P_{\boldsymbol{h}} = P_0 P^{\boldsymbol{(h)}} = P_0 P^{\boldsymbol{h}}$$

- Example: $P_0 = (1/3, 1/3, 1/3)$ and

$$P = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.5 & 0.4 \end{pmatrix}$$

$$\Rightarrow \quad P_3 = (1/3, 1/3, 1/3) \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.5 & 0.4 \end{pmatrix}^3 = (0.333, 0.407, 0.259)$$

$\Rightarrow$ $P(X(3) = 1) = 0.333$ - probability for first state after 3 transitions

- See Example 6.10 in textbook

Steady-state distribution

- Probability distribution over states after many steps - ?
- **Steady-state distribution** is the row vector

$$\pi = \lim_{h\to\infty} P_h$$

- $\lim_{h\to\infty} P_h = \lim_{h\to\infty} P_{h+1} \;\Rightarrow\; \pi P = \pi$
- Computing π:

$$\begin{cases} \pi P = \pi \\ \sum_x \pi_x = 1 \end{cases}$$

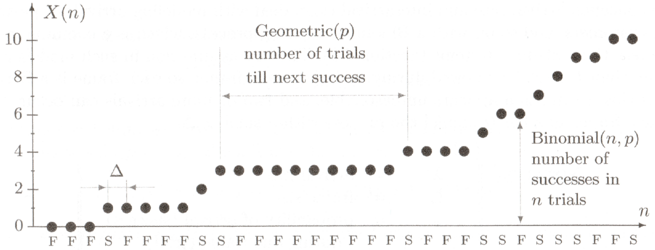
- A Markov chain is **regular** if there is h such that

$$p_{ij}^{(h)} > 0$$

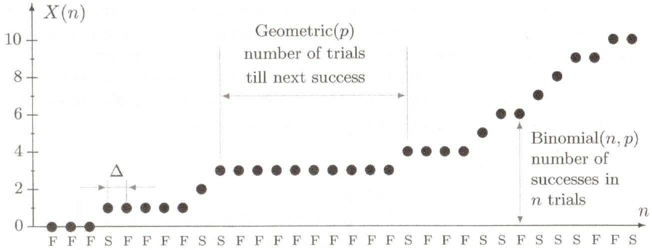
  for all i, j.
- **Any regular Markov chain has steady-state distribution**

Binomial process

- **Counting processes:** $X(t)$ - number of items counted by time t
- **Binomial process**: $X(n)$ - number of successes in first n independent Bernoulli trials
- $X(n) \sim Binomial(n, p)$, p - probability of success
- Y = number of trials between two consecutive successes
- $Y \sim Geo(p)$



Binomial process



- New Bernoulli trial every Δ seconds $\rightarrow$ Δ= **time frame**
- n trials occur during $t = n\Delta$ seconds $\rightarrow$ $n = t/\Delta$
- Process as function of time: $X(n) = X(t/\Delta)$
- **Expected number of successes during t seconds**:

$$\mathbb{E}(X(t/\Delta)) = tp/\Delta$$

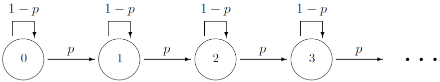
- Expected number of successes per second: $\lambda = p/\Delta$
- λ is also called **arrival rate**

Binomial process

- **Interarrival time** T - **time** between two consecutive successes
- Y = **number** of trials between two consecutive successes, $Y \sim Geo(p)$
- Interarrival time: $T = Y\Delta$
- T has rescaled geometric distribution with support $\Delta, 2\Delta, 3\Delta, \dots$

  $\rightarrow \mathbb{E}(T) = \mathbb{E}(Y\Delta) = \Delta\mathbb{E}(Y) = \Delta/p = 1/\lambda$
- Binomial process - homogeneous Markov chain with transition probabilities

$$p_{ij} = \begin{cases} p, \, j = i+1 \\ 1-p, \, j = i \\ 0, \, \textit{othervise} \end{cases}$$



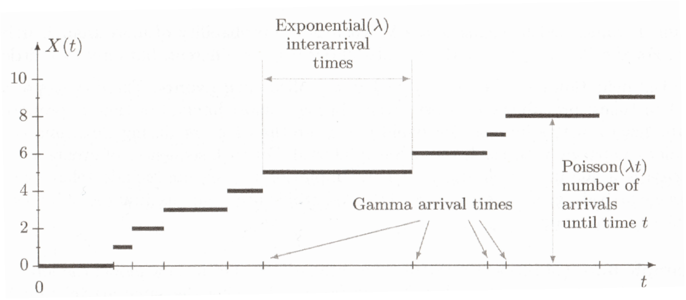
This Markov chain is non-regular as $X(n)$ non-decreasing, $p_{i,i-1}^{(h)} = 0$ for all h

$\rightarrow$ no steady-state distribution

- Example: Example 6.18 in textbook

Poisson process

- **Poisson process** - continuous-time stochastic process obtained from Binomial process by letting $\Delta \to 0$ while keeping λ constant
- From Lecture 3: $X(t) \sim Binomial(t/\Delta, p) \to Poisson(\lambda t)$ when $n = t/\Delta \to \infty$ and $p = \lambda\Delta \to 0$



- Interarrival time $T \sim Exp(\lambda)$
- Time for k-th success: $T_k \sim Gamma(k, \lambda)$ - sum of k iterarrival times
- Example: See Example 6.20 in textbook

Simulation of stochastic processes

- For simulation of stochastic processes

  see Chapter 6.4 in textbook
- See also

  `SimulateMarkovChain.R`

  for simulation Markov chains

  `SimulateBinomialProcess.R`

  for simulation of Binomial processes

  `SimulatePoissonProcess.R`

  for simulation of Poisson processes

Thank you for your attention!