TDAB01 Probability and Statistics

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Lecture 9: Hypothesis Testing

Overview

- **Hypothesis testing**
- **Z-test**
- **T-test**
- χ^2**-test**

Hypothesis testing

- Example 1. New medication affects blood pressure

$$\textbf{Null hypothesis}: \; H_0 : \mu = 0$$

$$\textbf{Alternative hypothesis}: \; H_A : \mu \neq 0$$

$\rightarrow$ *two-side* alternative

- Example 2. Proportion of defective products is higher than 4%

$$\textbf{Null hypothesis}: \; H_0 : p \leq 0.04$$

$$\textbf{Alternative hypothesis}: \; H_A : p > 0.04$$

$\rightarrow$ *one-side*, *right-tail* alternative

- Example 3. Average connection speed is worse than the provider promised

$$\textbf{Null hypothesis}: \; H_0 : \mu \geq \mu_0$$

$$\textbf{Alternative hypothesis}: \; H_A : \mu < \mu_0$$

$\rightarrow$ *one-side*, *left-tail* alternative

Hypothesis testing

- **Two-sided test**: reject H_0 if μ is greater **or** less than μ_0

$$\textbf{Null Hypothesis}: \; H_0 : \mu = \mu_0$$

$$\textbf{Alternative hypothesis}: \; H_A : \mu \neq \mu_0$$

- **One-sided tests**

$$\textbf{Null Hypothesis}: \; H_0 : \mu \leq \mu_0$$

$$\textbf{Alternative hypothesis}: \; H_A : \mu > \mu_0$$

or

$$\textbf{Null hypothesis}: \; H_0 : \mu \geq \mu_0$$

$$\textbf{Alternative hypothesis}: \; H_A : \mu < \mu_0$$

- One-sided tests are often written as

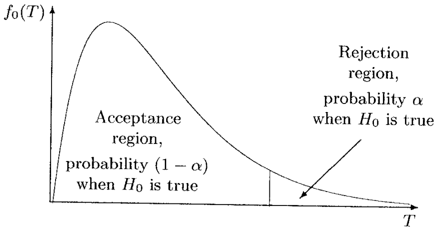
$$\textbf{Null hypothesis}: \; H_0 : \mu = \mu_0$$

$$\textbf{Alternative hypothesis}: \; H_A : \mu < \mu_0 \quad \text{or} \quad H_A : \mu > \mu_0$$

$\rightarrow$ *the same result*

Steps in hypothesis testing

1. Choose **test statistic** $T = T(X_1, \dots, X_n)$
2. Determine sampling distribution F_0 for T if H_0 is true (null distribution)
3. Determine the **rejection region** $\mathcal{R}$ such that $\boldsymbol{P}(T \in \mathcal{R} | H_0) = \alpha$
4. Reject H_0 at **significance level** α if $T_{obs} \in \mathcal{R}$, T_{obs} - **observed** value of T



On picture: f_0 - pdf of T in dependence of value of T

- H_0 is about population → H_0 is true or not with probability 1

  Accepting H_0 does **not** mean that H_0 is true with probability $1 - \alpha$

  Accepting H_0 means that evidence obtained from data is not sufficient to reject H_0

Type I and Type II errors

- **Type I error**

$$\boldsymbol{\alpha} = \boldsymbol{P}(\text{Reject } H_0 | H_0 \text{ is true})$$

α - significance level, predetermined, low

- **Type II error**

$$\beta = \boldsymbol{P}(\text{Accept } H_0 | H_A \text{ is true})$$

β depends on parameter value (set of values in H_A)

	Accept H_0	Reject H_0
H_0 true	Correct decision	Type I error
H_A true	Type II error	Correct decision

- **Power of test**: $\boldsymbol{P}(\text{Reject } H_0 | H_A \text{ is true}) = 1 - \beta$
- Formally: Do we reject H_0 or not, we never accept it

Z-test

- Sampling distribution of test statistic if H_0 is true is normal distribution
  - Case 1. $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, σ^2 known

$$\Rightarrow \quad Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

  - Case 2. Central Limit Theorem $\rightarrow$ Z appr. normally distributed
- One-sided Z-tests: $H_0 : \mu = \mu_0$ and $H_A : \mu > \mu_0$ (right-tail Z-test)

$$\begin{cases} \text{Reject } H_0 & \text{if } Z > z_{\boldsymbol{\alpha}} \\ \text{Accept } H_0 & \text{if } Z \le z_{\boldsymbol{\alpha}} \end{cases}$$

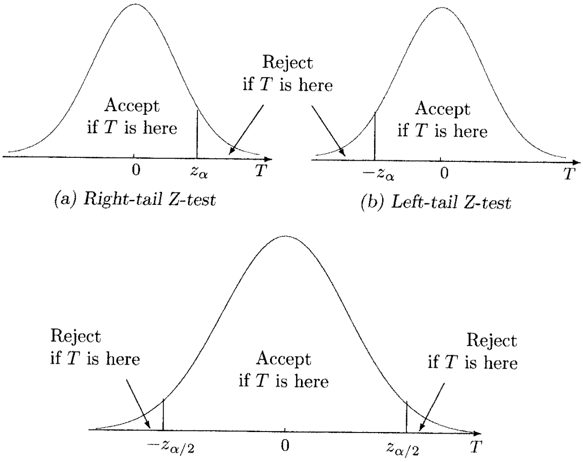
or $H_0 : \mu = \mu_0$ and $H_A : \mu < \mu_0$ (left-tail Z-test)

$$\begin{cases} \text{Reject } H_0 & \text{if } Z < -z_{\boldsymbol{\alpha}} \\ \text{Accept } H_0 & \text{if } Z \ge -z_{\boldsymbol{\alpha}} \end{cases}$$

- Two-sided Z-test: $H_0 : \mu = \mu_0$ and $H_A : \mu \neq \mu_0$

$$\begin{cases} \text{Reject } H_0 & \text{if } |Z| > z_{\boldsymbol{\alpha/2}} \\ \text{Accept } H_0 & \text{if } |Z| \le z_{\boldsymbol{\alpha/2}} \end{cases}$$

Z-test



(a) *Right-tail Z-test*

(b) *Left-tail Z-test*

(c) *Two-sided Z-test*

- Example: See Example 9.25 in textbook

Two-sample Z-test

- We can also test if **two** populations have the same expected value:

$$H_0 : \mu_X = \mu_Y$$

$$H_A : \mu_X \neq \mu_Y$$

- Example: Is the average salary the same in Stockholm and Malmö?
- $X_1, \ldots, X_n$ - random sample from $N(\mu_X, \sigma_X^2)$, σ_Y^2 known
- $Y_1, \ldots, Y_m$ - random sample from $N(\mu_Y, \sigma_Y^2)$, σ_Y^2 known
- **Test Statistics**: $\bar{X} - \bar{Y}$. Sampling distribution under H_0 - ?
  - Linear combination of normal variables is normally distributed, i.e. $\bar{X} - \bar{Y}$ is normally distributed
  - $E(\bar{X} - \bar{Y}) = \mu_X - \mu_Y = 0$ under H_0
  - $Var(\bar{X} - \bar{Y}) = Var(\bar{X}) + Var(\bar{Y}) = \sigma_X^2/n + \sigma_Y^2/m$
  - Then

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \sim N(0,1)$$

- Example: See Example 9.26 in textbook

T-test

- Z-test is used when sampling distribution of test statistic under H_0 is normal distribution
- If σ^2 **not** known but estimated with s^2

  $$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$$

  in **not** normally distributed but t-**distributed** with $n - 1$ degrees of freedom
- Use t_α instead of z_α
- t_α - $(1 - \alpha)$-quantile of t-distribution with $n - 1$ degrees of freedom
- Values for t_α in Table A5 in textbook

Duality: hypothesis test and confidence interval

A Z-test or T-test of $H_0 : \theta = \theta_0$ against $H_A : \theta \neq \theta_0$ at the significance level α

accepts the null hypothesis if and only if

θ_0 is included in a symmetric $(1 - \alpha)100\%$ confidence interval for θ.

P-value

- How do we choose α?
- α - Type I error $\rightarrow$ should be low
- Very low α requires very large / very small observed value of test statistic to reject H_0
- Idea: Test hypothesis for all α
- **P-value** = the lowest significance level α for which we can reject H_0
  = the highest significance level α for which we can accept H_0
- Alternative definition: Probability to obtain a test statistic that is as extreme or even more extreme than T_{obs}
- Example: One-sided, right-tail Z-Test:

$$P\text{-value} = \boldsymbol{P}(Z \geq Z_{obs}) = 1 - \Phi(Z_{obs})$$

- Reject H_0 at level α if $\alpha > P$-value
- Example: See Example 9.38 in textbook

χ^2-test for population variance

- Unbiased estimator of σ^2:

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$$

- **Sampling distribution** of s^2 if $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$:

$$\frac{(n-1)s^2}{\sigma^2} = \sum_{i=1}^{n}\left(\frac{X_i - \bar{X}}{\sigma}\right)^2 \sim \chi^2_{n-1}$$

- χ^2-distribution is used for hypothesis testing and confidence intervals as normal and t-distributions
- Pdf of $X \sim \chi^2_\nu$:

$$f(x) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}, \quad x > 0$$

  ν - number of degrees of freedom
- $\mathbb{E}(X) = \nu$ and $Var(X) = 2\nu$
- χ^2_ν distribution is a special case of Gamma-distribution: $Gamma(\nu/2, 1/2)$
- $\chi^2_{1-\frac{\alpha}{2}}$ and $\chi^2_{\frac{\alpha}{2}}$ quantiles of χ^2-distribution, values in Table A6
- χ^2_ν distribution is **not symmetric** $\rightarrow$ both $\chi^2_{1-\frac{\alpha}{2}}$ and $\chi^2_{\frac{\alpha}{2}}$ needed

Thank you for your attention!