ALEXANDER GROTHENDIECK & THE CONCEPT OF SPACE

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50 years ago the first international category theory meeting took place in La Jolla, California. In fact, part of that meeting moved to the beach, where an inspiring talk by Jean-Louis Verdier introduced many of us to a new class of categories due to Grothendieck, writing on a blackboard that had been brought to the beach for the purpose. Jon Beck began to draw diagrams in the sand, and a lively and enthusiastic discussion started among the participants. Jean-Luis Verdier suggested that these categories embody set theory, but Erwin Engeler and I expressed doubt, because the description seemed to need a given external set theory to parameterize families for the required colimits.

There are several important threads from that meeting that are still flourishing, for example, the theory of enriched categories as presented by Eilenberg and Kelly, in particular, the role of 'cartesian-closed' categories in geometry and logic. The thread I tried to capture at La Jolla, namely, the increasing use of categories and functors as the language of abstract mathematics, has continued for the last 50 years.
The explicit formulation of the principles of category theory in my paper is still in need of improved axiomatization. I will be overjoyed when some young person responds to that need.

The recent disappearance of so many stalwarts from that epoch underlines the need for coherent and correct history as a guide to the future. I want to continue the search for such a history, focusing here on the concept of space.

There is not just one concept of space, but several categories of smoothness. (To avoid misunderstanding, I am not focussing on Riemannian space or Space Time. Those important additional structures require spaces as their domains of definition.) A common feature of spaces in these more or less smooth categories I have called COHESION, indicating that the parts of a space 'stick together' and 'hesitate' to separate (as in American slang "I'll stick around a bit, until I split").

The great dialectical geometer Hermann Grassmann discerned the two main contradictions in mathematics to be 'continuous versus discrete' and 'equality versus inequality'. Because the term 'continuous' has had a particular mathematical definition for more than a century, I will instead use 'cohesive' for this philosophical concept, but of course I will immediately try to tame it with mathematical definitions. The dialectics of inequality/equality have been rather thoroughly made explicit by mathematicians, on at least two levels: Hurewicz (1935), Kan (1955) and Moore (1955), Quillen (1967), Gabriel & Zisman (1967), Heller (1988), Grothendieck 1983 & 1989), Kan et al (2004), Maltsiniotis & Cisinski (1999 - to the present), are some of the major contributions at the level of spaces themselves.

Another level of the transformation of equality is codified in the notion of exact category introduced by Myles Tierney in our Halifax Seminar in 1969; he proved that these categories are a delinearization of Grothendieck's notion of abelian category in the sense that abelian groups in an exact category form an abelian category. This theory was expounded in the book by Barr, van Osdol, and Grillet, as well as in Barr's address to the 1970 ICM. Exact categories embody the special property of 'sheaf-theoretic images' which can be expressed in 'logical' terms: Define the image of a map X -->Y to be the smallest subobject of Y through which the map factors. That definition expresses precisely the rule of inference of existential quantification; but then to what extent does it express 'actual existence'? In other words, given a figure Q --> Y of shape Q in the codomain Y, to what extent does it 'come from' a figure in X via the map, assuming that Q --> Y lies in the image so defined? Part of the exactness property guarantees that there exists a covering P --> Q of Q with an actual figure P --> X mapping to P --> Q --> Y, to the pullback of the given figure. The other feature of exact categories is even more transparently about transformation of equality: co-equalizers come from their kernel pairs and equivalence relations all arise as kernel pairs. It is clear that the theory of exact categories has wide applicability. The book by Barr, Grillet, and Van Osdol did much to popularize it, and the work of Carboni and others very effectively used 'the exact completion' to adjoin appropriate co-equalizers to non-exact categories. Most of these works postulate the exactness properties as given conditions on a category, as does Giraud's characterization of Grothendieck toposes; part of the significance of the postulation of function spaces and power objects is that the existence of adjoint functors implies exactness without further postulation.

The idea of an opposition between a category of cohesive spaces and a category of anti-cohesive sets also applies in particular to Cantor's description of the relation between a category of Mengen and a sub-category of Kardinalen. In fact, it appears that in general the discrete is a co-reflective subcategory of the cohesive, with the co-reflection extracting, as an Aristotelian arithmos, the Cantor 'cardinal of X' or the Hausdorff 'points of X'. (The sets of 'lauter Einsen' have isomorphisms, as do the objects in any category; the issue here, however, is not to pass to isomorphism classes, but simply to extract the underlying discrete aspect of each given space/ Menge.) The Grassmann dialectic develops further. The discrete subcategory is the negation of an identical subcategory at the opposite end, with the same functor as reflection. That is, the same category has insertions as two opposite sub-categories, the one illustrating that the 'lauter Einsen' are totally distinct, but the other demonstrating that they are nearly indistinguishable. More precisely, joint work with Matias Menni has shown that under very general assumptions the co-discrete inclusion consists of the Boolean sheaves that any topos has. However, for a category of spaces there is an additional adjoint to the sheafification. This indicates a non-trivial restriction on that cohesive category, namely the existence of that additional Cantor adjoint. Such restrictions serve as axioms for cohesion, which is our proposed characterization of 'Categories of Space'.

My use of concepts such as the Boolean sub-category reveals that I am convinced that categories of space are most effectively modeled as appropriate toposes. One of the two axioms for toposes, namely the existence of function spaces (the feature that has been called 'cartesian closed' since the Eilenberg & Kelly contribution 50 years ago) had been recognized as fundamental by Hadamard and Volterra at the time of the 1897 ICM in Zurich. That this property is essential was emphasized by Grothendieck in 1957 in his Tohoku paper. These and many other reasons point to this operation as central to all branches of mathematics. In order to achieve the function space property (in models of cohesion that are constructed as categories of structures in a discrete base), the fundamental structure needs to have the nature of figures and incidence relations, rather than of algebras of functions (which can be recovered by naturality). My 1997 Palermo paper attempted to explain this necessity. That paper, like the 1965 paper by Eilenberg & Kelly, and like publications by Steenrod, Kelley, and Brown, mentioned as an important example the k-spaces based on using compact spaces as figure types. However, none of us authors mentioned the actual origin of the k-spaces, of which I learned later on the phone from David Gale (when following up his 1950 publication in the Proceedings of the AMS). Namely, the notion of k-space was introduced by Witold Hurewicz in his Princeton lectures in the late 1940's. In fact, in the early 1940's, Hurewicz had emphasized the need, which led to the partial solution by Ralph Fox in 1945 for the case of convergent sequences as figures. Hurewicz did not speak explicitly in terms of categories, but in the exponential laws that he demanded one immediately discerns the feature of adjointness.

It is striking that Hurewicz, who in 1935 had initiated fundamental advances in the study of the Grassmann transformation of inequality into equality, made also fundamental contributions to the development of the other Grassmann transformation between continuity and discreteness. Important were his well-known contributions to dimension theory (which already made use of function spaces in 1941), but also his less cited contribution to the recognition of the fundamental role of function spaces in general. Had it not been for the temptation of the pyramid at Uxmal, he would have shown us more of the relation between the two Grassmann principles.

From functional analysis to derivateurs, Alexander Grothendieck's work has immensely illuminated that relation.

Hausdorff's great book 'Mengenlehre' was actually about topology (which is an important product of the study of cohesion), illustrating again the opposition and mutual transformation between cohesion and discreteness, as approached in his work about chaos under the pseudonym Paul Mongre'.

A remarkable aspect of the continuous/discrete dialectic is that the abstract sets of 'lauter Einsen', abstracted from the cohesion of spaces, can reciprocally act as the basis, via specific diagrams in their category, for structures constituting models of all sorts of mathematical objects, including in particular the spaces themselves. As a criterion for the adequacy of our axioms, Myles Tierney and I insisted on the proof of the Grothendieck constructions of sites and sheaves. That proof was published by Radu Diaconescu in 1975 as a necessary preliminary to his proof of change of base for toposes. That is, for a geometric morphism E-->U satisfying a boundedness condition, E can be reconstructed, by a zigzag of three geometric morphisms, from a site internal to U: the first leg is the local homeomorphism given by the slice topos over an object of U that parametrizes the objects of an internal category, the second is given by the left-exact comonad that adjoins the 'presheaf' action of the maps of that internal category, and the third is the full inclusion of sheaves for a localness operator. (Each of the three is a special case of a distinct important general closure property of the class of toposes.) For such a 'U- Topos' E, the U itself can be any elementary topos, re-inforcing Grothendieck's observation concerning the ubiquity of the powerful principle of relativization; it need not be an inaccessible universe, as in Grothendieck's original SGA4 examples; nor need it be the discrete part of a cohesive topos, as emphasized here. For each topos U there is the 2-category Top/U of U-toposes; indeed, varying U may simplify the treatment of certain problems.

The 1959 Warsaw lecture by Saunders Mac Lane in effect introduced the idea of enriched category, in its special case of 'locally small category'. As a reflection of the Bernays class/set distinction, the belief developed that categories that are not locally small are 'illegitimate'. I suggest the following alternate point of view.

Within the metacategory of categories, there are monoidal closed categories and hence other categories enriched in them. This shows the need for the existence of an actual category called the category of small sets, within the cartesian closed metacategory of all actual categories. The functor category of any two actual categories should also be actual, although of course properties like local finiteness will not be preserved. Potential categories (corresponding to subcategories of that metacategory) may or may not have actual categories that represent them up to equivalence. One of the main goals of abstract mathematics is to illustrate and use the mutual transformation between space and quantity. The spaces and quantities of primary interest are 'small', so it is reasonable to define small sets to mean those satisfying the Banach-Isbell duality and to postulate that there is an actual category U representing that notion of smallness. This postulate now seems to be one of the reasonable amendments to my 1965 La Jolla attempt to summarize in axioms the key useful features of a metacategory of categories. So functor categories of actual categories may not have small hom sets, but they are actual and thus subject to all the properties of actual categories in general.

What I have said so far has been profoundly influenced by the work of Alexander Grothendieck. Let me now touch on his contributions specific to the problem of Space as I have outlined it. It is often said that he invented toposes as domains for cohomology and that they were a 'generalization' of topological spaces. But already in 1960 he was defining and using categories in complex geometry that were toposes even if not explicity so called. His famous Me'daille de Chocolat exercise (in SGA4) is, as I told him, a key to the whole theory and application of toposes; he agreed, obviously pleased that someone had noticed. There he explains a version (in terms of sites) of the relationship between the gros topos of a space and a petit topos of the same space; the spaces in question are taken from a category of spaces which could only itself be the gros topos of a point. It is still an ongoing exercise to clarify the qualitative distinction between the kinds of toposes that appear as 'gros' or as 'petit' in this kind of situation, that is, between categories that represent a general determination of cohesion and categories that consist of variable sets as parametrized by some sort of generalized space. The generalized spaces would include the e'tale spaces discovered by Grothendieck.

What was the undesirable feature, of the earlier Dieudonne'-Grothendieck foundation of schemes, that Grothendieck so emphatically rejected in his 1973 Buffalo colloquium lecture?

The contravariant structure had already been seen by Hurewicz and others to be problematic, but in the notion of local ringed space, that structure was further dissected in two interacting components, open sets and sheaves of local rings. In hindsight, problems could have already been discerned from Serge Lang's 1960 review of EGA. There, Lang is enthusiastic about the fact that so many classical concepts can be subsumed under base change; he is also enthusiastic about Grothendieck's virtuoso proof that such fibered products actually exist. Indeed from the point of view of us less able calculators, a virtuoso was required to take the separated ingredients and re-assemble them into similar ingredients for a product scheme; in particular, the underlying topological space does not underly the product scheme.

What was the nature of Grothendieck's solution?

In a topos of set-valued functors on the category of finitely-presentable algebras, each space X has, thanks to Yoneda, an 'inside' whose objects are (in general singular) figures of representable shapes, with incidence relations given by commutative triangles. This can be viewed as a discretely-opfibered category, but such is equivalent to a set-valued functor. [I disagree with the term 'functor of points' for this, because it is a functor whose actual values include all the figures of X. Of course, 'points' of some other space associated to X may represent figures in X, but for X itself the points of it are just the restriction of X to the category of finite field extensions. That category generates the Boolean part of the big topos. The usual definition of point is unwieldy because it amounts to taking the non-exact direct limit of that restricted points functor. In general, this Boolean topos is much better suited than the category of abstract sets to serve as 'base topos' in the case of non-algebraically closed ground field. Conflating 'figures in X' with 'points of X' has a sort of science fiction air he probably did not intend. Volterra called them 'elements'.] A better version of the 'underlying topological space' is internal to the Barr-Boole-Galois topos where the actual points functor lands; this choice is also necessary for a product preserving components functor.

Between the Galois site for the Boolean part and the site for the whole category of spaces, there is the category of algebras that are finite-dimensional over the ground field; because the corresponding representable spaces are the domains of the crucial infinitesimal figures in X, we may call it a Leibniz site. The importance of these figures was emphasized by Grothendieck and his colleagues in connection with tangent bundles, e'tale maps, and so on. Two strong properties of this category in relation to the much bigger category are the following: The general figure shapes Y from the big site have the Birkhoff property relative to the inclusions L(X) --> X of the Leibniz core of any X; namely Y perceives these inclusions as epimorphisms in the sense that an infinitesimal map L(X)-->Y can be integrated in at most one way to a map X --> Y. (This means that the algebra of Y-valued functions on X is a subalgebra of a product of special very small algebras.) The other strong property (which has traces in Euler) is that any subtopos of the whole which contains the Leibniz objects will contain all the objects Y of the big (affine) site; this follows from the fact that there are enough infinitesimal function spaces to generate that big site, for example, the line is a retract of the self-exponential of the dual numbers domain. One can easily extract the subcategory of locally affine spaces i.e. algebraic schemes.

The above outline of a topos of Grothendieck algebraic spaces over a base field seems to work as well for a base rig. It was shown in detail to work for smooth geometry by Wraith, Kock, Reyes, Moerdijk, Bunge, Dubuc, Gago, Lavendhomme, and others. Some version is likely to work also for analytic geometry.

Indeed, the field of complex analysis/geometry is much advanced since 1960 and should have many topos implications and clarifications. For example, the relation between the Grauert direct image theorem as a relativization of its special case by Cartan-Serre should be clarified by explicit topos-relativization. When I proposed that to Grothendieck, he allowed that it is interesting, but pleaded insufficient expertise in logic to carry out a proof. More recently, the study of Brady-hyperbolic spaces has a very strong topos flavor that has not yet been made explicit (as far as I know).

Grothendieck made an important contribution to what he called 'tame topology'. He gave no general definition, but urged (as I had in my 1977 Milan lectures) the discovery of suitable categories that would not contain certain old pathologies that come up again in cohomology. In my view objects such as space-filling curves should have led to a 'criticism of foundations' more central than the so-called paradoxes; however, they were apparently simply tolerated for many decades, with the resignation that complication is inevitable. But Grothendieck boldly proposed using accumulated knowledge to construct less pathological categories that would still suffice for mathematical work. He arrived at a proposal involving piecewise real-analytic functions. Meanwhile, logicians including Wilkie, Pillay, MacIntyre, and van den Dries, had been pursuing a related problem of Tarski, phrased in terms of decidability. They solved it in 1986, also finding 'piecewise real analytic' to be a key ingredient, although by no means the only one. These logicians came to recognize Grothendieck's work as being related to their own. It is to be expected that conversely the work of their o-minimal school will illuminate the deeper study of cohesive space.

The work of Grothendieck illuminated and advanced the work of Cantor, Grassmann, Volterra, Hausdorff, Hurewicz, Galois, Kan, Eilenberg & Mac Lane and inspired our whole community; it will continue to inspire and guide the work of future generations.