

THE MATHEMATICAL WORKS OF T.J. KACZYNSKI: A UNIFIED ACCOUNT AND FORMALIZATION BLUEPRINT

ABSTRACT. We give a single, self-contained account, in traditional mathematical language, of every result in the following seven sources: the doctoral dissertation *Boundary functions* (1967) [3]; the papers [1] (described only by its review MR0176080), [2], [7], [8], [9]; and the note [6] on a problem of Alan Sutcliffe. The one result known to us only through a review (Theorem 1 of [1], on homeomorphisms of the disk) is given a complete proof by the plane-topological methods of the other papers. Proofs that the originals leave to the reader (the “readily verifiable facts” of [8], the existence of special functions, the explicit Sutcliffe solutions, the extension theorems for Baire functions) are written out. Classical results from the literature that the originals cite are isolated as numbered *Facts*. The document is organised as a blueprint for a Lean 4 formalization: every numbered item names the Lean declarations that formalize it and the items it depends on.

CONTENTS

1. Introduction	2
1.1. The sources	2
1.2. How the document is organised	2
1.3. Conventions	3
2. Sutcliffe’s problem	3
3. Borel classes and Baire functions	5
3.1. Classes of sets	5
3.2. Classes of functions	6
3.3. Functions into the sphere	8
3.4. Points of continuity	9
3.5. Higher Borel and Baire classes	10
4. Arcs, crosscuts and transfer	11
4.1. Arcs and curvilinear limits	11
4.2. Crosscuts	13
4.3. Transfer	14
5. Sets of curvilinear convergence	15
The half-plane lemmas of the dissertation	17
6. Boundary functions of continuous functions and homeomorphisms	17
6.1. Homeomorphisms	19
7. Realizing sets of curvilinear convergence and boundary functions	20
7.1. Regions	20
7.2. Special families and special functions	21
7.3. The function g	22
7.4. The function h	23
7.5. The function k	24
7.6. Assembly	25
8. Boundary functions of discontinuous functions	26
8.1. Baire functions	26

8.2. Lebesgue measure	27
9. Bounded harmonic functions	30
10. A cube with a non-Borel set of curvilinear convergence	33
An elementary proof of the lemma	34
11. Wedderburn's theorem	36
Appendix A. The dissertation's special families	38
Appendix B. Concordance with the sources	39
References	40

1. INTRODUCTION

1.1. The sources. Seven documents are covered.

- [6] (1968): an elementary number-theoretic note settling a conjecture of Sutcliffe on integers that are multiplied by an integer when their digits are reversed. Section 2.
- [1] (1965), known to us only through its review. Its theorems concern boundary functions of homeomorphisms, continuous, Baire and measurable functions in a disk. Theorems 2–6 of [1] reappear, with proofs, in the dissertation [3] (which says so explicitly); Theorem 1 (on homeomorphisms) does not, and is proved here in Section 6 by the crosscut methods of [2, 3].
- [2] (1966): a short proof of McMillan's theorem that the set of curvilinear convergence of a continuous function is of type $F_{\sigma\delta}$, and a generalization to cluster sets. Section 5.
- [3] (1967): the dissertation. Its Introduction (Baire functions) is in Section 3; Chapter I in Sections 5, 6, 7; Chapter II in Section 8.
- [8] (1969): every $F_{\sigma\delta}$ set is the set of curvilinear convergence of a continuous function with a prescribed boundary function (McMillan's problem); appendix on Baire functions into the sphere. Sections 7 and 3.
- [7] (1969): boundary functions of bounded harmonic functions. Section 9.
- [9] (1969): a bounded continuous function in a cube whose set of curvilinear convergence is not Borel. Section 10.

A concordance between the numbered results of the sources and the items of this document is given in Appendix B.

1.2. How the document is organised. Several results occur in more than one source in slightly different settings (the disk $\mathbb{D}$, the upper half-plane $\mathbb{H}$, the strip-like region Q). We prove each result once, in the most convenient model, and transfer it to the others by an explicit homeomorphism (Lemma 4.18). When two sources prove the same statement by different arguments, we give the argument that is shorter or more general, and say so in the concordance.

Facts. Results from the literature that the sources use without proof are stated with their provenance and proved here (Tietze's arc theorem, the crosscut theorem, the Poisson integral, Banach's theorem, an analytic non-Borel set, Bagemihl's theorem for continuous functions). Two deep theorems that the sources rely on, the Jordan curve theorem with accessibility and Kierst's theorem on asymptotic values, are avoided: we replace the arguments that use them by direct ones (Lemmas 9.6 and 10.6), and record the original arguments in remarks.

Departures from the sources. We record here the places where our proof differs in substance from the source.

- (i) The step (B)(III) of [8] cites a theorem of Bagemihl and McMillan; we use instead the approximation theorem of the dissertation (Theorem 3.14), as [8] itself suggests.

- (ii) The proof of (F)(I) in [8] cites an invariance-of-domain corollary from [21]; we write down the inverse maps explicitly (Lemma 3.18).
- (iii) The extension theorems for Baire-one functions (cited from [15] in [3] and used in Appendix (b),(c) of [8]) are proved (Theorem 3.16).
- (iv) The “Figure 5” argument of [3] and the monotonicity of first hitting points are derived from a single imported crosscut theorem (Theorem 4.10).
- (v) Sutcliffe’s Theorems 2 and 3, used in [6], are proved with explicit solutions (Propositions 2.3, 2.4).
- (vi) In the existence of special functions (statement (XX) of [8]) we give a non-recursive construction: all disjointness constraints turn out to be independent of the other member’s size.

1.3. Conventions. $\mathbb{N} = \{1, 2, 3, \dots\}$. “Countable” includes “finite”. We identify $\mathbb{R}^2$ with $\mathbb{C}$ and write points as $z = x + iy = \langle x, y \rangle$. We use

$$\begin{aligned} \mathbb{D} &= \{z : |z| < 1\}, & C &= \{z : |z| = 1\}, & \mathbb{H} &= \{z : \operatorname{Im} z > 0\}, \\ Q &= \{\langle x, y \rangle : -1 \leq x \leq 1, 0 < y \leq 1\}, & Q^\circ &= (-1, 1) \times (0, 1), & X &= (-1, 1) \times \{0\}. \end{aligned}$$

The real axis is identified with $\mathbb{R}$; in Sections 7 we identify X with $(-1, 1)$. For $n \in \mathbb{N}$, $\mathbb{H}_n = \{0 < y < 1/n\}$, $\overline{\mathbb{H}}_n = \{0 \leq y \leq 1/n\}$ and $X_n = \{y = 1/n\}$. $\mathbb{S}^2 = \{v \in \mathbb{R}^3 : |v| = 1\}$ is the Riemann sphere with the Euclidean metric. $S(r, p)$ is the open ball of radius r about p ; $\operatorname{Cl} E$ or $\overline{E}$ is closure, E^* or $\operatorname{int} E$ interior (for subsets of C or of $\mathbb{R}$, relative to C or $\mathbb{R}$). For an interval or arc I , I^* is its relative interior; an interval or arc is *nondegenerate* if it has more than one point.

2. SUTCLIFFE’S PROBLEM

Source: [6], which uses Theorems 2 and 3 of [10]; we prove these too.

Definition 2.1 (digit solutions). For an integer $n \geq 2$ and nonnegative integers $a_h, \dots, a_0$ write $(a_h, \dots, a_1, a_0)_n = a_h n^h + \dots + a_1 n + a_0$. An $(h+1)$ -digit solution for n is a collection of integers $n, a_h, \dots, a_0, k$ with

$$n \geq 2, \quad k > 1, \quad a_h > 0, \quad a_0 > 0, \quad 0 \leq a_i \leq n-1 \quad (0 \leq i \leq h), \quad k(a_h, \dots, a_1, a_0)_n = (a_0, a_1, \dots, a_h)_n.$$

We say *there is an $(h+1)$ -digit solution for n* if such a_i, k exist. Thus a 2-digit solution is (a, c, k) with $k(an + c) = cn + a$, and a 3-digit solution is (a, b, c, k) with $k(an^2 + bn + c) = cn^2 + bn + a$, where $1 \leq a, c \leq n-1$, $0 \leq b \leq n-1$, $k \geq 2$.

Lemma 2.2 (basic inequalities). *For a 2-digit solution (a, c, k) for n , and for a 3-digit solution (a, b, c, k) for n , we have $k < n$ and $a < c$.*

Proof. In the 3-digit case $an^2 + bn + c \geq n^2$ because $a \geq 1$, and $cn^2 + bn + a \leq (n-1)(n^2 + n + 1) = n^3 - 1 < n^3$; so $kn^2 \leq k(an^2 + bn + c) < n^3$ and $k < n$. Since $k \geq 2$, $cn^2 + bn + a \geq 2(an^2 + bn + c) > an^2 + bn + c$, i.e. $(c-a)(n^2-1) > 0$, so $c > a$. The 2-digit case is the same with $an + c \geq n$, $cn + a \leq n^2 - 1$ and $(c-a)(n-1) > 0$. $\square$

Proposition 2.3 (Sutcliffe, Theorem 2, first half). *If there is a 2-digit solution for n , then $n+1$ is not prime.*

Proof. Let $k(an + c) = cn + a$. Modulo $p = n+1$ we have $n \equiv -1$, so $k(c-a) \equiv a-c$, i.e. $p \mid (k+1)(c-a)$. By Lemma 2.2, $0 < c-a < n < p$ and $3 \leq k+1 \leq n < p$. If p were prime it would divide one of the two factors, which is impossible. $\square$

Proposition 2.4 (explicit solutions). *Let $n \geq 2$ and suppose $n + 1 = de$ with integers $d \geq 3$ and $e \geq 2$. Put $a = e - 1$, $c = n - e$, $k = d - 1$. Then (a, c, k) is a 2-digit solution for n and $(a, n - 1, c, k)$ is a 3-digit solution for n .*

Proof. Digits: $a = e - 1 \geq 1$; $e \leq (n + 1)/3 \leq n - 1$ gives $c = n - e \geq 1$; clearly $a, c \leq n - 1$; and $k = d - 1 \geq 2$. Using $n + 1 = de$:

$$\begin{aligned} k(an + c) &= (d - 1)((e - 1)n + n - e) = (d - 1)e(n - 1), \\ cn + a &= (n - e)n + e - 1 = (n - 1)(n + 1 - e) = (n - 1)(de - e) = (d - 1)e(n - 1), \\ k(an^2 + (n - 1)n + c) &= (d - 1)((e - 1)n^2 + n^2 - n + n - e) = (d - 1)e(n^2 - 1), \\ cn^2 + (n - 1)n + a &= (n - e)n^2 + n^2 - n + e - 1 = (n^2 - 1)(n + 1 - e) = (d - 1)e(n^2 - 1). \quad \square \end{aligned}$$

Lemma 2.5 (factorization). *If $n \geq 4$ and $n + 1$ is not prime, then $n + 1 = de$ with $d \geq 3$ and $e \geq 2$.*

Proof. Write $n + 1 = uv$ with $2 \leq u \leq v$. Then $v \geq 3$, since $v = 2$ would force $n + 1 = 4$. Take $d = v$, $e = u$. $\square$

Theorem 2.6 (Sutcliffe, Theorem 2). *Let $n \geq 4$. There is a 2-digit solution for n if and only if $n + 1$ is not prime.*

Proof. One direction is Proposition 2.3; the other is Lemma 2.5 with Proposition 2.4. $\square$

Theorem 2.7 (Sutcliffe, Theorem 3). *If there is a 2-digit solution for n , then there is a 3-digit solution for n .*

Proof. By Lemma 2.2, $1 < k < n$, so $n \geq 3$. For $n = 3$ we would have $k = 2$ and $2(3a + c) = 3c + a$, i.e. $c = 5a \geq 5 > n - 1$; impossible. So $n \geq 4$, $n + 1$ is not prime by Proposition 2.3, and Lemma 2.5 with Proposition 2.4 gives a 3-digit solution. (In $\mathbb{N}$ -valued arithmetic the expressions $n(a - 1)$, $a - b + c$ below are to be read in $\mathbb{Z}$.) $\square$

Lemma 2.8 ([6, Lemma 1]). *There are no natural numbers a, c, n with $n \geq 2$, $a < c \leq n - 1$, $a + c \geq n + 1$, and $n(a - 1) + c \mid n(c - 1) + a$.*

Proof. Suppose there are, and write

$$k(n(a - 1) + c) = n(c - 1) + a, \quad k \geq 1. \quad (2.1)$$

(Here $a \geq 2$ because $a \geq n + 1 - c \geq 2$, so both sides are positive and $k \geq 1$.) Modulo $n - 1$ we have $n \equiv 1$, so $k(a - n + c) \equiv c - n + a$, i.e. $n - 1 \mid (k - 1)(a - n + c)$. Since $a + c \geq n + 1$, $a - n + c > 0$. If $k = 1$ then $n(a - 1) + c = n(c - 1) + a$, so $c - a = n(c - a)$ and $n = 1$, contrary to $n \geq 2$. Hence $k > 1$, $(k - 1)(a - n + c) > 0$, and so $(k - 1)(a + c - n) \geq n - 1$. Since $c - n \leq -1$, $(k - 1)(a - 1) \geq (k - 1)(a + c - n) \geq n - 1$. Therefore

$$k(n(a - 1) + c) = nk(a - 1) + kc \geq n(k - 1)(a - 1) + kc \geq n(n - 1) + kc \geq nc + kc > n(c - 1) + a,$$

using $c \leq n - 1$ and $kc > 0 \geq a - n$. This contradicts (2.1). $\square$

Lemma 2.9 ([6, Lemma 2]). *If $n = 2$ or $n = 3$, there is no 3-digit solution for n .*

Proof. By Lemma 2.2, $1 < k < n$, which is impossible for $n = 2$. For $n = 3$ we get $k = 2$ and $2(9a + 3b + c) = 9c + 3b + a$, i.e. $0 = 17a + 3b - 7c \geq 17 + 3b - 14 > 0$, a contradiction. $\square$

Theorem 2.10 (Kaczynski [6]; Sutcliffe's conjecture). *Let $n \geq 2$. If there is a 3-digit solution for n , then there is a 2-digit solution for n .*

Proof. By Lemma 2.9 we may assume $n > 3$, and by Theorem 2.6 we may assume that $p = n + 1$ is prime. Let (a, b, c, k) be a 3-digit solution:

$$k(an^2 + bn + c) = cn^2 + bn + a. \quad (2.2)$$

Modulo p , $n \equiv -1$, so $k(a - b + c) \equiv c - b + a$ and $p \mid (k - 1)(a - b + c)$. If $p \nmid a - b + c$ then $p \mid k - 1$; since $k - 1 > 0$ this gives $k - 1 \geq p$, impossible because $k < n$ (Lemma 2.2). Hence $p \mid a - b + c$. Now $-p < -n < 3 - n \leq a - b + c \leq 2n - 2 < 2p$, so $a - b + c = ep$ with $e \in \{0, 1\}$. Substituting $b = a + c - ep$,

$$an^2 + bn + c = a(n^2 + n) + c(n + 1) - epn = p((a - e)n + c), \quad cn^2 + bn + a = p((c - e)n + a),$$

so (2.2) becomes $k((a - e)n + c) = (c - e)n + a$. If $e = 1$ then $k(n(a - 1) + c) = n(c - 1) + a$; moreover $0 \leq b = a + c - p$ gives $a + c \geq n + 1$, and $a < c \leq n - 1$ by Lemma 2.2. This is impossible by Lemma 2.8. Hence $e = 0$ and $k(an + c) = cn + a$ with $1 \leq a, c \leq n - 1$ and $k > 1$: a 2-digit solution. $\square$

Corollary 2.11 ([6]). *If $n > 3$, there is a 3-digit solution for n if and only if $n + 1$ is not prime.*

Proof. If $n + 1$ is not prime, Theorem 2.6 gives a 2-digit solution and Theorem 2.7 a 3-digit one. Conversely a 3-digit solution gives a 2-digit solution (Theorem 2.10), and then $n + 1$ is not prime (Theorem 2.6). $\square$

Remark 2.12. [6] notes that a similar result was obtained independently by Prasert Na Nagara.

3. BOREL CLASSES AND BAIRE FUNCTIONS

Sources: [3, Introduction, §3], [8, Appendix]. Throughout, M is a metric space.

3.1. Classes of sets.

Definition 3.1 (F_σ , G_δ , $F_{\sigma\delta}$, ambiguous sets). A subset of a topological space is F_σ if it is a countable union of closed sets, G_δ if it is a countable intersection of open sets, $F_{\sigma\delta}$ if it is a countable intersection of F_σ sets, and *ambiguous* if it is both F_σ and G_δ . For $L \subseteq M$ we write $F_\sigma(L)$, $G_\delta(L)$, $F_{\sigma\delta}(L)$ for the corresponding classes of the subspace L .

Lemma 3.2 (closure properties). (a) *In M , open sets are F_σ , closed sets are G_δ , and countable sets are F_σ .*

(b) *F_σ is closed under countable unions and finite intersections; G_δ under countable intersections and finite unions; $F_{\sigma\delta}$ under countable intersections and finite unions. The complement of an F_σ set is G_δ and conversely.*

(c) *For $L \subseteq M$: $E \in F_\sigma(L)$ iff $E = P \cap L$ for some F_σ set P of M ; likewise for G_δ and $F_{\sigma\delta}$.*

(d) *If L is $F_{\sigma\delta}$ (resp. G_δ) in M and $E \in F_{\sigma\delta}(L)$ (resp. $G_\delta(L)$), then E is $F_{\sigma\delta}$ (resp. G_δ) in M . Every F_σ set is $F_{\sigma\delta}$, and every G_δ set is $F_{\sigma\delta}$.*

(e) *Homeomorphisms preserve all these classes.*

Proof. (a) $U = \bigcup_k \{x : \text{dist}(x, M \setminus U) \geq 1/k\}$ for $U \neq M$ open; complements give the rest, and points are closed. (b) $(\bigcup_n A_n) \cap (\bigcup_m B_m) = \bigcup_{n,m} A_n \cap B_m$, and $(\bigcap_n P_n) \cup (\bigcap_m R_m) = \bigcap_{n,m} (P_n \cup R_m)$; the rest is De Morgan. (c) Closed sets of L are traces of closed sets of M , and traces commute with countable unions and intersections. (d) $E = P \cap L$ with P in the class, and the class is closed under finite intersections; a G_δ set $\bigcap_n U_n$ is $F_{\sigma\delta}$ by (a). (e) is clear. $\square$

3.2. Classes of functions.

Definition 3.3 (function classes). Let L be a metric space and Y a metric space.

- (a) $f : L \rightarrow Y$ is of class F_σ if $f^{-1}(U) \in F_\sigma(L)$ for every open $U \subseteq Y$.
- (b) f is of Baire class ≤ 1 , written $f \in \mathcal{B}_1(L, Y)$, if there are continuous $f_n : L \rightarrow Y$ with $f_n \rightarrow f$ pointwise on L .
- (c) f is of honorary Baire class ≤ 2 if there are a countable $N \subseteq L$ and $\psi \in \mathcal{B}_1(L, Y)$ with $f = \psi$ on $L \setminus N$.
- (d) (Baire classes of all orders.) f is of Baire class 0 if it is continuous; for an ordinal $\xi \geq 1$, f is of Baire class ξ if it is the pointwise limit of a sequence f_n , each of Baire class η_n for some $\eta_n < \xi$. (So class ξ contains all classes $\eta < \xi$ when $\xi \geq 1$.)

Lemma 3.4 (reduction; [3, Lemma 1]). Let E, F be F_σ subsets of M . There are disjoint F_σ sets $A \subseteq E$, $B \subseteq F$ with $A \cup B = E \cup F$. In particular $E \setminus F \subseteq A$ and $F \setminus E \subseteq B$.

Proof. Write $E = \bigcup_n E_n$, $F = \bigcup_n F_n$ with E_n, F_n closed, and put

$$A = \bigcup_n \left(E_n \setminus \bigcup_{j < n} F_j \right), \quad B = \bigcup_n \left(F_n \setminus \bigcup_{j \leq n} E_j \right).$$

Each term is closed minus closed, i.e. closed $\cap$ open, hence F_σ (Lemma 3.2). If $x \in (E_n \setminus \bigcup_{j < n} F_j) \cap (F_m \setminus \bigcup_{j \leq m} E_j)$ then $m < n$ is impossible because $x \in F_m$, and $m \geq n$ is impossible because $x \in E_n$; so $A \cap B = \emptyset$. If $x \in E \cup F$, let n be least with $x \in E_n \cup F_n$. If $x \in E_n$ then $x \in A$; otherwise $x \in F_n \setminus \bigcup_{j \leq n} E_j \subseteq B$. $\square$

Corollary 3.5 (finite reduction). If $P_1, \dots, P_q$ are F_σ subsets of M , there are pairwise disjoint F_σ sets $R_i \subseteq P_i$ with $\bigcup_i R_i = \bigcup_i P_i$. If $\bigcup_i P_i = M$, each R_i is ambiguous.

Proof. Induction on q . Reduce P_1 against $P' = P_2 \cup \dots \cup P_q$ (Lemma 3.4), getting disjoint F_σ sets $A \subseteq P_1$, $B \subseteq P'$ with $A \cup B = P_1 \cup P'$; apply the induction hypothesis to $P_2, \dots, P_q$, getting $R'_2, \dots, R'_q$, and put $R_1 = A$, $R_i = R'_i \cap B$. If the R_i cover M , the complement of each is the finite union of the others, hence F_σ . $\square$

Theorem 3.6 (Sierpiński [22]). If $E \subseteq F \subseteq M$ with E a G_δ set and F an F_σ set, there is an ambiguous set K with $E \subseteq K \subseteq F$.

Proof. $P = M \setminus E$ and F are F_σ with $P \cup F = M$. Reduce them (Lemma 3.4): $A \subseteq P$, $B \subseteq F$ disjoint F_σ with $A \cup B = M$. Put $K = B$. Then $K \subseteq F$, $K = M \setminus A \supseteq M \setminus P = E$, and K is F_σ with F_σ complement A . $\square$

Lemma 3.7. Every $f \in \mathcal{B}_1(L, Y)$ is of class F_σ .

Proof. Let $f_n \rightarrow f$ with f_n continuous, and $U \subsetneq Y$ open. Put $V_k = \{y : \text{dist}(y, Y \setminus U) \geq 1/k\}$, a closed subset of U . Then $f^{-1}(U) = \bigcup_{k, N} \bigcap_{n \geq N} f_n^{-1}(V_k)$: if $\text{dist}(f(x), Y \setminus U) > 2/k$ then eventually $f_n(x) \in V_k$; conversely if $f_n(x) \in V_k$ for $n \geq N$ then $f(x) \in V_k \subseteq U$. $\square$

Lemma 3.8 (Banach's Hilfssatz 3). If $f : L \rightarrow Y$ is of class F_σ and takes only the values $a_1, \dots, a_q$ (distinct), then the sets $E_j = f^{-1}(a_j)$ are ambiguous in L and partition L .

Proof. With r less than half the least distance between the a_j , $E_j = f^{-1}(S(r, a_j))$ is F_σ , and its complement $\bigcup_{i \neq j} E_i$ is F_σ too. $\square$

Lemma 3.9. If $f, g : L \rightarrow \mathbb{R}$ are finitely valued and of class F_σ , so are $f + g$ and $f - g$.

Proof. By Lemma 3.8 the level sets of f and g are ambiguous; every preimage under $f \pm g$ is a finite union of sets $f^{-1}(a) \cap g^{-1}(b)$, which are F_σ . $\square$

Lemma 3.10 ([3, Lemma 2]). *Let $L \subseteq M$ and let E be ambiguous in L . There are continuous $f_n : M \rightarrow [0, 1]$ with $f_n \rightarrow \chi_E$ pointwise on L .*

Proof. Write $E = E_1 \cap L$ and $L \setminus E = F_1 \cap L$ with E_1, F_1 F_σ in M . Reduce: disjoint F_σ sets $A \subseteq E_1$, $B \subseteq F_1$ with $A \cup B = E_1 \cup F_1 \supseteq L$. Then $E = A \cap L$ and $L \setminus E = B \cap L$ (a point of E is not in F_1 , hence not in B , hence in A). Write $A = \bigcup A_n$, $B = \bigcup B_n$ with $A_n \subseteq A_{n+1}$, $B_n \subseteq B_{n+1}$ closed, and let $f_n : M \rightarrow [0, 1]$ be a Urysohn function, 1 on A_n and 0 on B_n (e.g. $\text{dist}(x, B_n)/(\text{dist}(x, A_n) + \text{dist}(x, B_n))$). For $x \in L$, eventually $x \in A_n$ or eventually $x \in B_n$. $\square$

Lemma 3.11 ([3, Lemma 3]). *If $f : L \rightarrow \mathbb{R}$ is of class F_σ and finitely valued, and $L \subseteq M$, there are continuous $f_n : M \rightarrow \mathbb{R}$ with $f_n \rightarrow f$ pointwise on L .*

Proof. $f = \sum_j a_j \chi_{E_j}$ with E_j ambiguous (Lemma 3.8); take $f_n = \sum_j a_j f_n^{(j)}$ with $f_n^{(j)} \rightarrow \chi_{E_j}$ from Lemma 3.10. $\square$

Lemma 3.12 ([3, Lemma 4], Banach's Hilfssatz 4). *If $f : L \rightarrow \mathbb{R}$ is of class F_σ with $|f| \leq C$, then for every $\varepsilon > 0$ there is a finitely valued $g : L \rightarrow \mathbb{R}$ of class F_σ with $|g - f| < \varepsilon$ and $|g| \leq C + \varepsilon$.*

Proof. Let $N \geq C/\varepsilon$ and $U_j = ((j-1)\varepsilon, (j+1)\varepsilon)$ for $j \in \mathbb{Z}$, $|j| \leq N$; these cover $[-C, C]$. The sets $f^{-1}(U_j)$ are F_σ in L and cover L ; Corollary 3.5 gives a partition of L into F_σ sets $R_j \subseteq f^{-1}(U_j)$. Put $g = j\varepsilon$ on R_j . Then $|g - f| < \varepsilon$, and $g^{-1}(W)$ is a finite union of R_j 's for every W . $\square$

Lemma 3.13 ([3, Lemma 5]). *Let $L \subseteq M$ and let $f : L \rightarrow \mathbb{R}$ be of class F_σ . There are continuous $f_j : M \rightarrow \mathbb{R}$ with $f_j \rightarrow f$ pointwise on L .*

Proof. Bounded f . Choose by Lemma 3.12 finitely valued g_n ($n \geq 0$) of class F_σ with $|g_n - f| < 2^{-n}$; put $h_0 = g_0$, $h_n = g_n - g_{n-1}$. Each h_n is finitely valued of class F_σ , $|h_n| \leq b_n := (|C|+4)2^{-n}$, and $\sum_n h_n = f$ on L . By Lemma 3.11 choose continuous $h_{n,j} \rightarrow h_n$ on L , and let $k_{n,j}$ be $h_{n,j}$ clamped to $[-b_n, b_n]$; still $k_{n,j} \rightarrow h_n$ on L . Then $f_j = \sum_n k_{n,j}$ converges uniformly, so it is continuous, and $f_j(x) \rightarrow \sum_n h_n(x) = f(x)$ for $x \in L$ by dominated convergence for series ($|k_{n,j}| \leq b_n$, $\sum b_n < \infty$).

General f . $\arctan \circ f$ is bounded of class F_σ ; take continuous $G_j \rightarrow \arctan \circ f$ on L , clamp G_j into $[-\pi/2 + \delta_j, \pi/2 - \delta_j]$, $\delta_j = 1/(j+1)$, to get H_j (still convergent on L , since the target values lie in the open interval), and put $f_j = \tan \circ H_j$. $\square$

Theorem 3.14 ([3, Lemma 6]). *Let $L \subseteq M$ and $f : L \rightarrow \mathbb{R}^m$. The following are equivalent: (i) $f \in \mathcal{B}_1(L, \mathbb{R}^m)$; (ii) f is of class F_σ ; (iii) there are continuous $f_n : M \rightarrow \mathbb{R}^m$ with $f_n \rightarrow f$ pointwise on L .*

Proof. (i) $\Rightarrow$ (ii) is Lemma 3.7; (iii) $\Rightarrow$ (i) by restriction. (ii) $\Rightarrow$ (iii): each coordinate $f_i = \pi_i \circ f$ is of class F_σ ; apply Lemma 3.13 to each. (Conversely the coordinates being of class F_σ implies f is: an open set of $\mathbb{R}^m$ is a countable union of open boxes, and $f^{-1}(\prod_i I_i) = \bigcap_i f_i^{-1}(I_i)$.) $\square$

Lemma 3.15 (uniform limits). *If $S_n : L \rightarrow Y$ are of class F_σ and $S_n \rightarrow F$ uniformly, then F is of class F_σ .*

Proof. Let $U \subseteq Y$ be open, $\varepsilon_k = 1/(k+1)$, and let U_k be the union of the balls $S(\varepsilon_k, z)$ over all z with $S(3\varepsilon_k, z) \subseteq U$, an open set. For each k choose n_k with $\text{dist}(S_{n_k}(x), F(x)) < \varepsilon_k$ for all x . Then $F^{-1}(U) = \bigcup_k S_{n_k}^{-1}(U_k)$: if $F(x) \in U$, take k with $S(3\varepsilon_k, F(x)) \subseteq U$ and $z = F(x)$; conversely, if $S_{n_k}(x) \in S(\varepsilon_k, z)$ then $F(x) \in S(2\varepsilon_k, z) \subseteq U$. $\square$

Theorem 3.16 (extension of Baire-one functions). *Let $f : L \rightarrow \mathbb{R}^m$ be of Baire class ≤ 1 , $L \subseteq M$.*

- (a) If L is G_δ in M and f is bounded, f extends to a bounded $F \in \mathcal{B}_1(M, \mathbb{R}^m)$.
 (b) If $M \setminus L$ is countable, f extends to some $F \in \mathcal{B}_1(M, \mathbb{R}^m)$.

Proof. It suffices to treat $m = 1$ (extend coordinatewise); then f is of class F_σ (Lemma 3.7).

(a) *Finitely valued f of class F_σ on a G_δ set L .* By Lemma 3.8, $f = \sum_j a_j \chi_{E_j}$ with $E_1, \dots, E_q$ an ambiguous partition of L ; write $E_j = P_j \cap L$ with P_j F_σ in M . The sets $P_1, \dots, P_q, M \setminus L$ are F_σ and cover M ; let $R_1, \dots, R_q, R_0$ be the disjoint F_σ cover given by Corollary 3.5. Put $F = a_j$ on R_j and $F = 0$ on R_0 . For $x \in L \cap E_j$, $x \notin R_0$, and $x \in R_i$ forces $x \in P_i \cap L = E_i$, so $i = j$ and $F(x) = f(x)$. F is finitely valued of class F_σ and $\sup|F| \leq \max|a_j|$.

Bounded f . With h_n, b_n as in the proof of Lemma 3.13, extend each h_n to a finitely valued H_n of class F_σ with $|H_n| \leq b_n$. The partial sums $S_N = \sum_{n < N} H_n$ are finitely valued of class F_σ and converge uniformly to $F = \sum_n H_n$, which is bounded by $\sum_n b_n$, of class F_σ (Lemma 3.15), hence in $\mathcal{B}_1(M, \mathbb{R})$ (Theorem 3.14 with $L = M$); and $F|_L = \sum_n h_n = f$.

(b) Let $I = (-\pi/2, \pi/2)$. Since $M \setminus L$ is countable, it is F_σ and L is G_δ ; by (a) extend $\arctan \circ f$ to a bounded $G \in \mathcal{B}_1(M, \mathbb{R})$, of class F_σ . The set $Z = G^{-1}(\mathbb{R} \setminus I)$ lies in $M \setminus L$, so it is countable. Put $F = \tan \circ G$ off Z and $F = 0$ on Z . For open $U \subseteq \mathbb{R}$, $F^{-1}(U) = G^{-1}(I \cap \tan^{-1}(U)) \cup (Z \text{ or } \emptyset)$ is F_σ . So F is of class F_σ , hence in $\mathcal{B}_1(M, \mathbb{R})$ (Theorem 3.14), and $F|_L = \tan \circ \arctan \circ f = f$. $\square$

3.3. Functions into the sphere. We use the folding device of [8, (F)(I)]. Write points of $\mathbb{S}^2$ as $\langle x, y, z \rangle$ and let $T(z) = 4z^3 - 3z$.

Definition 3.17 (the fold). Let $s = 2^{-1/2}$ and

$$U = \{-s < z \leq 1\}, \quad V = \{-1 \leq z < s\}, \quad Z_U = \{s < z \leq 1\}, \quad Z_V = \{-1 \leq z < -s\},$$

$$\Phi\langle x, y, z \rangle = \begin{cases} \langle x(4z^2 - 1), y(4z^2 - 1), T(z) \rangle, & |z| \geq s, \\ \langle x, y, -z \rangle, & |z| \leq s, \end{cases}$$

all subsets of $\mathbb{S}^2$. Equivalently, with $\kappa(z) = \max(0, 2z^2 - 1)$, $\Phi\langle x, y, z \rangle = \langle x(1 + 2\kappa(z)), y(1 + 2\kappa(z)), z(2\kappa(z) - 1) \rangle$ (the form used in Lean). Let $p_1 = \langle 1, 0, 0 \rangle$ and let $\sigma : \mathbb{S}^2 \setminus \{p_1\} \rightarrow \mathbb{R}^2$ be stereographic projection from p_1 , a homeomorphism onto $\mathbb{R}^2$ (Mathlib's `stereographic`).

Lemma 3.18 (properties of the fold). (i) Φ is a continuous map $\mathbb{S}^2 \rightarrow \mathbb{S}^2$. (ii) There are maps $\Phi_U^{-1} : U \rightarrow Z_U$ and $\Phi_V^{-1} : V \rightarrow Z_V$, continuous on the open sets U, V , with $\Phi \circ \Phi_U^{-1} = \text{id}_U$, $\Phi \circ \Phi_V^{-1} = \text{id}_V$; in particular $\Phi(Z_U \cup Z_V) = \mathbb{S}^2$. (iii) Each fibre of Φ has at most three points. (iv) U, V are open and $U \cup V = \mathbb{S}^2$. (v) $Z_U \cup Z_V \subseteq \mathbb{S}^2 \setminus \{p_1\}$ and $\sigma(Z_U \cup Z_V)$ is bounded.

Proof. On $\mathbb{S}^2$, $(x^2 + y^2)(4z^2 - 1)^2 + T(z)^2 = (1 - z^2)(16z^4 - 8z^2 + 1) + z^2(16z^4 - 24z^2 + 9) = 1$, so Φ maps into $\mathbb{S}^2$. At $z = \pm s$ we have $4z^2 - 1 = 1$ and $T(\pm s) = \mp s$, so the two formulas agree and Φ is continuous by the pasting lemma. Since $T(\cos \theta) = \cos 3\theta$, the map $\tau(z') = \cos(\arccos(z')/3)$ is continuous on $[-1, 1]$, and for $z' \in (-s, 1]$ we have $\arccos z' < 3\pi/4$, so $\tau(z') \in (s, 1]$ and $T(\tau(z')) = z'$. Given $\langle x', y', z' \rangle \in U$, put $z = \tau(z')$, $c = 4z^2 - 1 > 1$, $x = x'/c$, $y = y'/c$; the identity above shows $\langle x, y, z \rangle \in \mathbb{S}^2$, it lies in Z_U , $\Phi\langle x, y, z \rangle = \langle x', y', z' \rangle$, and it depends continuously on $\langle x', y', z' \rangle \in U$; this is Φ_U^{-1} . Since T is odd, $\Phi_V^{-1}\langle x', y', z' \rangle$ is obtained in the same way with $z = -\tau(-z')$. (iii): if $\Phi(p) = \Phi(q)$ and p, q have the same height z , then $p = q$ (as $1 + 2\kappa(z) \geq 1$). On $\{|z| \leq s\}$ the height of Φ is $-z$; on Z_U and on Z_V it is $T(z)$, and $T(a) - T(b) = (a - b)(4(a^2 + ab + b^2) - 3)$ with $4(a^2 + ab + b^2) > 3$ when $ab > 0$, $a^2, b^2 > 1/2$. So Φ is injective on each of $Z_U, \{|z| \leq s\}, Z_V$, which cover $\mathbb{S}^2$. (iv) is clear. (v): $\{|z| \geq s\} \supseteq Z_U \cup Z_V$ is compact and misses p_1 , and σ is continuous on $\mathbb{S}^2 \setminus \{p_1\}$. $\square$

Lemma 3.19 (fold factorisation). Let $f : M \rightarrow \mathbb{S}^2$ be of class F_σ . There is $\psi : M \rightarrow Z_U \cup Z_V$ of class F_σ with $\Phi \circ \psi = f$ such that $\sigma \circ \psi : M \rightarrow \mathbb{R}^2$ is bounded and of class F_σ .

Proof. $f^{-1}(U)$ is F_σ and $M \setminus f^{-1}(V)$ is G_δ , and $M \setminus f^{-1}(V) \subseteq f^{-1}(U)$ since $U \cup V = \mathbb{S}^2$. Theorem 3.6 gives an ambiguous K with $M \setminus f^{-1}(V) \subseteq K \subseteq f^{-1}(U)$; put $L = M \setminus K$, so $f(K) \subseteq U$, $f(L) \subseteq V$. Let $\Phi_U = \Phi|_{Z_U} : Z_U \rightarrow U$ and $\Phi_V = \Phi|_{Z_V} : Z_V \rightarrow V$ and define $\psi = \Phi_U^{-1} \circ f$ on K , $\psi = \Phi_V^{-1} \circ f$ on L (Lemma 3.18(ii)). For $G \subseteq \mathbb{S}^2$ open, $\psi^{-1}(G) = [K \cap f^{-1}(U \cap (\Phi_U^{-1})^{-1}(G))] \cup [L \cap f^{-1}(V \cap (\Phi_V^{-1})^{-1}(G))]$, and $U \cap (\Phi_U^{-1})^{-1}(G)$ is open since Φ_U^{-1} is continuous on the open set U ; so ψ is of class F_σ . Since ψ avoids p_1 and σ is continuous on the open set $\mathbb{S}^2 \setminus \{p_1\}$, $\sigma \circ \psi$ is of class F_σ . Boundedness is Lemma 3.18(v). $\square$

Theorem 3.20 ([8, Appendix, Theorem (a)]; cf. [3, Theorem 1]). *A function $f : M \rightarrow \mathbb{S}^2$ is in $\mathcal{B}_1(M, \mathbb{S}^2)$ if and only if it is of class F_σ .*

Proof. One direction is Lemma 3.7. Conversely, take ψ from Lemma 3.19; by Theorem 3.14 there are continuous $\psi_n : M \rightarrow \mathbb{R}^2$ with $\psi_n \rightarrow \sigma \circ \psi$. Then $\Phi \circ \sigma^{-1} \circ \psi_n$ are continuous and converge to $\Phi \circ \psi = f$. $\square$

Theorem 3.21 ([8, Appendix, Theorem (c)]). *If E is a G_δ subset of M and $f \in \mathcal{B}_1(E, \mathbb{S}^2)$, then f extends to some $F \in \mathcal{B}_1(M, \mathbb{S}^2)$.*

Proof. By Lemmas 3.7 and 3.19, applied in the metric space E , write $f = \Phi \circ \psi$ with $\sigma \circ \psi : E \rightarrow \mathbb{R}^2$ bounded of class F_σ , hence in $\mathcal{B}_1$. By Theorem 3.16(a) it extends to a bounded $G \in \mathcal{B}_1(M, \mathbb{R}^2)$; let continuous $G_n \rightarrow G$ on M . Then $\Phi \circ \sigma^{-1} \circ G_n$ are continuous and converge on M to $F = \Phi \circ \sigma^{-1} \circ G$, and $F|_E = \Phi \circ \psi = f$. $\square$

Theorem 3.22 (honorary Baire class 2; [3, Theorem 2], [8, Appendix, Theorem (b)]). *Let Y be $\mathbb{R}^m$ or $\mathbb{S}^2$ and $f : M \rightarrow Y$. The following are equivalent.*

- (1) f is of honorary Baire class ≤ 2 .
- (2) There is a countable $N \subseteq M$ such that $f|_{M \setminus N}$ is of class F_σ .
- (3) There is a countable $N \subseteq M$ such that $f^{-1}(F) \setminus N$ is G_δ in M for every closed $F \subseteq Y$.

Proof. (1) $\Rightarrow$ (2): restrict the Baire-one function and use Lemma 3.7. (2) $\Rightarrow$ (1): $f|_{M \setminus N} \in \mathcal{B}_1$ by Theorem 3.14 or 3.20, and $M \setminus N$ is G_δ ; extend it to M by Theorem 3.16(b) or 3.21. (2) $\Leftrightarrow$ (3): $f^{-1}(F) \setminus N = (M \setminus N) \setminus (f|_{M \setminus N})^{-1}(Y \setminus F)$; a subset of the G_δ set $M \setminus N$ is G_δ in M iff it is G_δ in $M \setminus N$, and complements in $M \setminus N$ exchange $F_\sigma(M \setminus N)$ and $G_\delta(M \setminus N)$. $\square$

Lemma 3.23 (invariance of honorary Baire class 2). *If $f : M \rightarrow Y$ is of honorary Baire class ≤ 2 , so are $f|_{M'}$ for $M' \subseteq M$ and $f \circ h$ for a homeomorphism (indeed any continuous injection) $h : M'' \rightarrow M$, and $g \circ f$ for continuous $g : Y \rightarrow Y'$.*

Proof. Restrict (resp. compose with h or g) the Baire-one function, its continuous approximants and the countable set. $\square$

3.4. Points of continuity.

Theorem 3.24 (Baire, Kuratowski [18, p. 301]). *If A and Y are metric spaces and $f \in \mathcal{B}_1(A, Y)$, then the set of points of A at which f is discontinuous is of the first category in A .*

Proof. Let $f_n \rightarrow f$ be continuous. For $\varepsilon > 0$ and $N \in \mathbb{N}$ the set $F_N^\varepsilon = \{x : \text{dist}(f_n(x), f_m(x)) \leq \varepsilon \forall n, m \geq N\}$ is closed and $\bigcup_N F_N^\varepsilon = A$. Put $G^\varepsilon = \bigcup_N \text{int } F_N^\varepsilon$. Then $A \setminus G^\varepsilon \subseteq \bigcup_N (F_N^\varepsilon \setminus \text{int } F_N^\varepsilon)$, a countable union of closed nowhere dense sets. If $x \in \text{int } F_N^\varepsilon$, then $\text{dist}(f, f_N) \leq \varepsilon$ on a neighbourhood of x , and f_N is continuous, so $\text{dist}(f(x'), f(x)) \leq 3\varepsilon$ for x' near x and the oscillation of f at x is at most 6ε . Hence the discontinuity set is contained in $\bigcup_k (A \setminus G^{1/k})$. $\square$

Corollary 3.25. *If P is a nonempty complete metric space, Y a metric space and $f \in \mathcal{B}_1(P, Y)$, then f is continuous at some point of P .*

Proof. By the Baire category theorem P is not of the first category in itself. $\square$

Lemma 3.26. *A subset of $A \subseteq M$ that is of the first category in A is of the first category in M .*

Proof. If $N \subseteq A$ is nowhere dense in A and an open $O \neq \emptyset$ of M lies in $\text{Cl}_M N$, then O meets A and $O \cap A \subseteq \text{Cl}_A N$, a contradiction. $\square$

3.5. Higher Borel and Baire classes. These are needed only in Section 8.

Definition 3.27 ([3, §3]). $P^1(M)$ is the family of open subsets of M and $Q^1(M)$ that of closed subsets. For an ordinal $\xi > 1$, $P^\xi(M)$ consists of the countable unions of members of $\bigcup_{\eta < \xi} Q^\eta(M)$, and $Q^\xi(M)$ of the countable intersections of members of $\bigcup_{\eta < \xi} P^\eta(M)$. A function $f : M \rightarrow Y$ is of class P^ξ if $f^{-1}(U) \in P^\xi(M)$ for every open $U \subseteq Y$. (In Lean the classes are one inductive predicate; no set belongs to a class of index 0.)

Lemma 3.28. (a) $E \in Q^\xi$ iff $M \setminus E \in P^\xi$. (b) P^ξ and Q^ξ are closed under finite unions and intersections, P^ξ under countable unions, Q^ξ under countable intersections. (c) If $\eta < \xi$ then $P^\eta \cup Q^\eta \subseteq P^\xi \cap Q^\xi$. (d) $P^2(M)$ is the class of F_σ sets and $Q^2(M)$ that of G_δ sets. (e) Continuous preimages preserve each class. (f) For $L \subseteq M$, $P^\xi(L) = \{D \cap L : D \in P^\xi(M)\}$. (g) Every member of every class is a Borel set.

Proof. (a), (e), (f), (g): induction on the definition. (c) for $\eta = 1$ uses that in a metric space open sets are F_σ and closed sets are G_δ (Lemma 3.2(a)); for $\eta > 1$ it is immediate from the definitions. For (b), $(\bigcup A_n) \cap (\bigcup B_m) = \bigcup A_n \cap B_m$, and $A_n \in Q^{\eta_1}$, $B_m \in Q^{\eta_2}$ both lie in $Q^{\max(\eta_1, \eta_2)}$ by (c). The rest is routine. $\square$

Lemma 3.29. *If $f : M \rightarrow Y$ is of Baire class ξ and $g : K \rightarrow M$ is continuous, then $f \circ g$ is of Baire class ξ . In particular restrictions to subspaces keep the class. If $\xi \leq \xi'$, Baire class ξ is contained in Baire class ξ' .*

Proof. Induction on ξ ; for the last claim with $\xi < \xi'$ use constant sequences. $\square$

Lemma 3.30. *If $f : M \rightarrow Y$ is of Baire class ξ (Y any metric space), then f is of class $P^{\xi+1}$. In particular it is Borel measurable.*

Proof. For $\xi = 0$ preimages of open sets are open. For $\xi \geq 1$ let $f_n \rightarrow f$ with f_n of class $\eta_n < \xi$. With V_k as in Lemma 3.7, $f^{-1}(U) = \bigcup_{k,N} \bigcap_{n \geq N} f_n^{-1}(V_k)$; by induction $f_n^{-1}(V_k) \in Q^{\eta_n+1} \subseteq Q^\xi$, so the intersections lie in Q^ξ and the union in $P^{\xi+1}$. $\square$

Theorem 3.31 (Banach [14]). *Let Y be a separable, arcwise connected metric space and $\xi \geq 2$. If $f : M \rightarrow Y$ is of class $P^{\xi+1}$, then f is of Baire class ξ .*

Proof. Formalized in TJK/Baire/Banach.lean: approximate by finitely valued maps with preimages of lower class, using a dense sequence and a finite priority rule, then interpolate along a path through finitely many points of Y ; induction on ξ . For the second statement every Borel set lies in some $P^\xi \cap Q^\xi$, $\xi < \omega_1$. $\square$

Theorem 3.32 (Lebesgue–Hausdorff–Banach). *Let Y be a separable, arcwise connected metric space. Every Borel-measurable $f : M \rightarrow Y$ is of Baire class ξ for some countable ordinal ξ .*

Proof. Formalized in TJK/Baire/Banach.lean: approximate by finitely valued maps with preimages of lower class, using a dense sequence and a finite priority rule, then interpolate along a path through finitely many points of Y ; induction on ξ . For the second statement every Borel set lies in some $P^\xi \cap Q^\xi$, $\xi < \omega_1$. $\square$

Lemma 3.33 ([3, Lemma 18]). *Write $A \approx B$ if $A \setminus B$ and $B \setminus A$ are countable. This is an equivalence relation; $A \approx E$ implies $S \setminus A \approx S \setminus E$; and if $A_n \approx E_n$ for n in a countable set, then $\bigcap A_n \approx \bigcap E_n$ and $\bigcup A_n \approx \bigcup E_n$.*

Proof. $(\bigcap A_n) \setminus (\bigcap E_n) \subseteq \bigcup_n (A_n \setminus E_n)$, and similarly for the rest. $\square$

Lemma 3.34 ([3, Lemma 19], [2, Lemma 1]). *If $\mathcal{I}$ is a family of nondegenerate intervals of $\mathbb{R}$ (resp. nondegenerate closed arcs of C), then $\bigcup_{I \in \mathcal{I}} I \setminus \bigcup_{I \in \mathcal{I}} I^*$ is countable. In particular every union of nondegenerate intervals is $\approx$ an open set.*

Proof. Let $U = \bigcup_{I \in \mathcal{I}} I^*$, which is open. If x lies in some $I \in \mathcal{I}$ but not in U , pick $y \in I \setminus \{x\}$; if $x < y$ then $(x, y) \subseteq I^* \subseteq U$, so x is a point of $\mathbb{R} \setminus U$ isolated from the right in $\mathbb{R} \setminus U$; symmetrically if $y < x$. Points of a set $S \subseteq \mathbb{R}$ with $(x, y) \cap S = \emptyset$ for some $y > x$ are countably many, since the intervals (x, y) for distinct such x may be chosen pairwise disjoint. For the circle, write the arcs as $e^{i[a, b]}$, $a < b$, with relative interiors $\supseteq e^{i(a, b)}$; a point e^{it} of the first union not in $\bigcup e^{i(a, b)}$ has t in the countable real exceptional set of the family $\{[a, b]\}$. $\square$

Lemma 3.35 (two-sided limit points). *For $F \subseteq \mathbb{R}$, the set of points of $\overline{F}$ that are not two-sided limit points of F is countable. (Here x is a two-sided limit point of F if every interval $(x - r, x)$ and $(x, x + r)$, $r > 0$, meets F .)*

Proof. If $x \in \overline{F}$ and $(x, x + r) \cap F = \emptyset$, then $(x, x + r)$ misses $\overline{F}$, so x is a point of $\overline{F}$ isolated from the right in $\overline{F}$; there are countably many such points (proof of Lemma 3.34). Argue symmetrically on the left. $\square$

Lemma 3.36 ([3, Lemma 20]). *Let h be weakly increasing on $E \subseteq \mathbb{R}$ with $|x - h(x)| \leq 1$ on E . Then h extends to a weakly increasing $h_1 : \mathbb{R} \rightarrow \mathbb{R}$.*

Proof. Put $h_1(x) = \sup(\{x - 1\} \cup h((-\infty, x] \cap E))$, finite since $h(t) \leq t + 1 \leq x + 1$ there. It is weakly increasing, and for $x \in E$ the supremum is attained at $h(x)$ because $h(t) \leq h(x)$ for $t \leq x$ in E and $x - 1 \leq h(x)$. $\square$

Lemma 3.37 ([3, Lemma 21]). *If $f : \mathbb{R} \rightarrow Y$ is of Baire class ξ and $h : \mathbb{R} \rightarrow \mathbb{R}$ is weakly increasing, there is a countable $N \subseteq \mathbb{R}$ such that $(f \circ h)|_{\mathbb{R} \setminus N}$ is of Baire class ξ .*

Proof. Let N be the (countable) set of discontinuities of h and apply Lemma 3.29 to $h|_{\mathbb{R} \setminus N}$. $\square$

Lemma 3.38 ([3, Lemma 22]). *Let Y be a separable arcwise connected metric space, E a metric space, $\xi \geq 2$, and $\varphi : E \rightarrow Y$ such that for every open $U \subseteq Y$ there is $T \in P^{\xi+1}(E)$ with $\varphi^{-1}(U) \subseteq T \subseteq \varphi^{-1}(\overline{U})$. Then φ is of Baire class ξ .*

Proof. Let $\mathcal{B}$ be a countable base of Y . For open W let $\mathcal{B}(W) = \{B \in \mathcal{B} : \overline{B} \subseteq W\}$; then $W = \bigcup_{B \in \mathcal{B}(W)} B = \bigcup_{B \in \mathcal{B}(W)} \overline{B}$ (for $p \in W$ with $S(r, p) \subseteq W$ choose $p \in B \subseteq S(r/2, p)$). With $T(B)$ as in the hypothesis, $\varphi^{-1}(W) = \bigcup_{B \in \mathcal{B}(W)} T(B) \in P^{\xi+1}(E)$. Conclude by Theorem 3.31. $\square$

4. ARCS, CROSSCUTS AND TRANSFER

4.1. Arcs and curvilinear limits.

Definition 4.1 (arcs). A *path* in $\mathbb{R}^n$ is a continuous $g : [0, 1] \rightarrow \mathbb{R}^n$; a *simple arc* is the image of an injective path, and its *endpoints* are $g(0), g(1)$. Let $\Omega \subseteq \mathbb{R}^n$ be open and $p \in \partial\Omega$. An *arc at p* (in Ω) is a simple arc γ with one endpoint p and $\gamma \setminus \{p\} \subseteq \Omega$. A *crosscut* of $\mathbb{D}$ is a simple arc whose endpoints are distinct points of C and whose other points lie in $\mathbb{D}$; a crosscut of $\mathbb{H}$ is defined in the same way with $\mathbb{R}$ in place of C .

Definition 4.2 (curvilinear convergence, boundary functions, cluster sets). Let Ω be open, $B \subseteq \partial\Omega$, Y a metric space and $f : \Omega \rightarrow Y$. For an arc γ at p , $f(z) \rightarrow q$ *along* γ means $f(z) \rightarrow q$ as $z \rightarrow p$ with $z \in \gamma \setminus \{p\}$. The *set of curvilinear convergence* of f (relative to B) is

$$\text{CC}(f) = \{p \in B : \text{there are an arc } \gamma \text{ at } p \text{ and } q \in Y \text{ with } f \rightarrow q \text{ along } \gamma\}.$$

If $E \subseteq B$, a function $\varphi : E \rightarrow Y$ is a *boundary function* for f if for every $p \in E$ there is an arc γ at p with $f \rightarrow \varphi(p)$ along γ . The *cluster set* of f along γ is $C(f, \gamma) = \{q : \exists z_n \in \gamma \setminus \{p\}, z_n \rightarrow p, f(z_n) \rightarrow q\}$. The pairs (Ω, B) used are $(\mathbb{D}, C)$, $(\mathbb{H}, \mathbb{R})$ and (Q°, X) , and in Section 10 the open cube with its whole boundary.

Theorem 4.3 (Tietze [24]). *If g is a path in a Hausdorff space (e.g. in $\mathbb{R}^n$) with $g(0) \neq g(1)$, then $g([0, 1])$ contains a simple arc with endpoints $g(0)$ and $g(1)$.*

Proof. Call a closed $S \subseteq [0, 1]$ with $0, 1 \in S$ *admissible* if $g(x) = g(y)$ whenever $x < y$ lie in S and $(x, y) \cap S = \emptyset$. If S_i is a chain of admissible sets and (x, y) is a gap of $\bigcap S_i$, compactness gives for each $\varepsilon > 0$ an S_i missing $[x + \varepsilon, y - \varepsilon]$; the gap of S_i around $\frac{x+y}{2}$ has ends within ε of x and y with equal g -values, so $g(x) = g(y)$ by continuity. By Zorn's lemma there is a minimal admissible m ; minimality ($m \setminus (x, y)$ is admissible when $g(x) = g(y)$) shows that $x < y$ in m with $g(x) = g(y)$ bound a gap of m . Put $r(t) = \max(m \cap [0, t])$ and $f = g \circ r$. Then f is continuous (r is right-continuous, $t \mapsto \min(m \cap [t, 1])$ is left-continuous, and g agrees at the two), $f(0) = g(0)$, $f(1) = g(1)$, $f([0, 1]) \subseteq g([0, 1])$, and every fibre of f is an interval. For rationals $p < p'$ such that $K = f([0, p])$ and $L = f([p', 1])$ are disjoint, an Urysohn function u on the compact Hausdorff space $f([0, 1])$ with $u|_K = 0$, $u|_L = 1$ gives the monotone continuous $\varphi_{p,p'}(t) = \max_{s \leq t} u(f(s))$, constant on fibres of f ; a weighted sum h of these is continuous, monotone, and $h(s) = h(t) \iff f(s) = f(t)$. As h is a closed quotient map onto $[h(0), h(1)]$, f factors through it as an injective path from $g(0)$ to $g(1)$. $\square$

Corollary 4.4 (paths may replace arcs). *Let Ω be open, $p \in \partial\Omega$, and $g : [0, 1] \rightarrow \mathbb{R}^n$ continuous with $g(1) = p$ and $g([0, 1)) \subseteq \Omega$. Then $g([0, 1])$ contains an arc γ at p , and for every $f : \Omega \rightarrow Y$: if $f(g(t)) \rightarrow q$ as $t \rightarrow 1$ then $f \rightarrow q$ along γ ; and $C(f, \gamma)$ is contained in the set of limits of $f(g(t_n))$ with $t_n \rightarrow 1$.*

Proof. Theorem 4.3 gives a simple arc $\gamma \subseteq g([0, 1])$ from $g(0)$ to p ; since $g(t) \neq p$ for $t < 1$, $\gamma \setminus \{p\} \subseteq \Omega$. If $z_k \in \gamma \setminus \{p\}$ and $z_k \rightarrow p$, write $z_k = g(t_k)$ with $t_k < 1$; a subsequence of t_k bounded away from 1 would have a limit $t^* < 1$ with $g(t^*) = p$, which is impossible. So $t_k \rightarrow 1$. $\square$

Lemma 4.5 (subarcs and exits). (a) *If γ is an arc at p and W is a neighbourhood of p , there is an arc $\gamma' \subseteq \gamma \cap W$ at p ; limits and cluster sets along γ' and γ coincide. If $f \rightarrow q$ along γ and $V \ni q$ is open, some such γ' has $f(\gamma' \setminus \{p\}) \subseteq V$.* (b) *If g is a path, $w = g(t_0)$, and $|g(t_1) - w| \geq r > 0$, there is a closed interval J with endpoints t_0 and some t' between t_0 and t_1 such that $g(J) \subseteq \overline{S(r, w)}$ and $|g(t') - w| = r$; in particular $\text{diam } g(J) \geq r$.* (c) *If γ is an arc at p in $\mathbb{H}$ and n is large enough that γ meets X_n , then the subarc of γ from p to its first point on X_n is an arc at p contained in $\overline{\mathbb{H}}_n$ with all points other than its endpoints in $\mathbb{H}_n$.*

Proof. Parametrize and take initial segments; for (b) let t' be the first parameter between t_0 and t_1 with $|g(t') - w| \geq r$, and use continuity. $\square$

Lemma 4.6 (cluster sets). *Let K be a compact metric space, $f : \Omega \rightarrow K$ continuous and γ an arc at p . Then $C(f, \gamma) = \bigcap_{\gamma'} \overline{f(\gamma' \setminus \{p\})}$, the intersection over the subarcs γ' of γ at p ; it is nonempty, compact and connected; and $f \rightarrow q$ along γ iff $C(f, \gamma) = \{q\}$. If $\Theta : K \rightarrow K'$ is continuous into a metric space, then $C(\Theta \circ f, \gamma) = \Theta(C(f, \gamma))$.*

Proof. The sets $\overline{f(\gamma' \setminus \{p\})}$ are compact, connected (images of half-open arcs) and decrease as γ' shrinks; a decreasing intersection of nonempty compact connected sets is nonempty, compact and connected. The claim about limits is the usual compactness argument. For the last claim: $\supseteq$ by continuity of Θ ; if $\Theta(f(z_n)) \rightarrow q'$ with $z_n \rightarrow p$ along γ , a subsequence of $f(z_n)$ converges in the compact K to some $q \in C(f, \gamma)$, and $\Theta(q) = q'$. $\square$

Fact 4.7 (Tietze extension). *A bounded continuous function from a closed subset of a metric space to $\mathbb{C}$ extends to a bounded continuous function on the whole space.*

Fact 4.8 (hyperspace). *If K is a compact metric space, the space $\mathcal{C}(K)$ of nonempty closed subsets of K with the Hausdorff metric is compact.*

Fact 4.9 (Baire category theorem). *A complete metric space is not of the first category in itself; in it, a countable intersection of dense open sets is dense.*

4.2. Crosscuts.

Theorem 4.10 (crosscut theorem). *Let σ be a crosscut of $\mathbb{D}$ with endpoints $a \neq b$, and let A_1, A_2 be the two closed arcs of C with endpoints a, b . Then $\mathbb{D} \setminus \sigma$ has exactly two components U_1, U_2 , and the frontier of U_i is $\sigma \cup A_i$ ($i = 1, 2$). In particular $\overline{U_i} \cap C = A_i$ and $\sigma \subseteq \overline{U_1} \cap \overline{U_2}$.*

Proof. Formalized in `TJK/Plane/JordanCrosscut.lean` by continuous logarithms. Every component of $\mathbb{D} \setminus \sigma$ has a point of $C \setminus \{a, b\}$ in its closure, and an arc does not separate the plane; a component near each open arc of C exists; by an Eilenberg-type criterion (two points lie in the same component of the complement of a compact K iff $(w - z)/(w - z')$ has a continuous logarithm on K , proved with Tietze extension and homotopy lifting for $\exp$) the two sides are different components, since logarithms on $\sigma \cup A_1$ and $\sigma \cup A_2$ would glue to one on C ; and σ lies in the closure of each side. $\square$

Lemma 4.11 (disjoint components have disjoint boundary arcs). *Let σ, τ be crosscuts of $\mathbb{D}$, U a component of $\mathbb{D} \setminus \sigma$ and V a component of $\mathbb{D} \setminus \tau$, with $U \cap V = \emptyset$. Then the arcs $\overline{U} \cap C$ and $\overline{V} \cap C$ have disjoint relative interiors. (It suffices that σ, τ be closed and meet C only in the endpoints of these arcs.)*

Proof. Let p be interior to both arcs. Then p is not an endpoint of σ or τ , so $p \notin \sigma \cup \tau$, and there is $r > 0$ with $S(r, p) \cap (\sigma \cup \tau) = \emptyset$. The convex set $S(r, p) \cap \mathbb{D}$ lies in one component of $\mathbb{D} \setminus \sigma$; as $p \in \overline{U}$ it meets U , so it lies in U . Likewise it lies in V , contradicting $U \cap V = \emptyset$. $\square$

Definition 4.12 (charts). (a) The Cayley map $\kappa(z) = (z - i)/(z + i)$.

(b) $\Psi_{\mathbb{H}}\langle x, y \rangle = \tan(\pi x/2) + i y/(1 - y)$ for $|x| < 1, 0 \leq y < 1$.

(c) $\Psi_{\mathbb{D}}\langle x, y \rangle = 1 - 1/w$ with $w = \frac{1}{2}(1 + y) + i \tan(\pi x/2)$, for $|x| < 1, 0 \leq y < 1$.

(d) $\mathcal{E}_h(z) = e^{\pi z/h}$ for $h > 0$.

Lemma 4.13 (charts). (a) κ is a homeomorphism of $\mathbb{H} \cup \mathbb{R}$ onto $(\mathbb{D} \cup C) \setminus \{1\}$ mapping $\mathbb{H}$ onto $\mathbb{D}$ and $\mathbb{R}$ onto $C \setminus \{1\}$; $|\kappa(z) - 1| \rightarrow 0$ iff $|z| \rightarrow \infty$. (b) $\Psi_{\mathbb{H}}$ is a homeomorphism of $Q^\circ \cup X$ onto $\mathbb{H} \cup \mathbb{R}$ mapping X onto $\mathbb{R}$ increasingly. (c) $\Psi_{\mathbb{D}}$ is a homeomorphism of $Q^\circ \cup X$ onto $G^\circ \cup (C \setminus \{1\})$, where $G^\circ = \{z \in \mathbb{D} : \operatorname{Re}(1 - z)^{-1} < 1\}$, mapping X onto $C \setminus \{1\}$. The set $G'' = \{z \in \mathbb{D} : \operatorname{Re}(1 - z)^{-1} \leq 3/4\}$ is closed in $\mathbb{D}$, contains a neighbourhood in $\mathbb{D}$ of each point of $C \setminus \{1\}$, and is disjoint from the radius $[0, 1]$. (d) $\mathcal{E}_h$ maps $\{0 \leq \operatorname{Im} z \leq h\}$ homeomorphically onto $(\mathbb{H} \cup \mathbb{R}) \setminus \{0\}$, the line $\operatorname{Im} z = 0$ increasingly onto $(0, \infty)$ and the line $\operatorname{Im} z = h$ decreasingly onto $(-\infty, 0)$.

Proof. (a) $|\kappa(z)| < 1$ iff $|z - i| < |z + i|$ iff $\operatorname{Im} z > 0$; the inverse is $w \mapsto i(1 + w)/(1 - w)$. (b) is a product of homeomorphisms $(-1, 1) \rightarrow \mathbb{R}$ and $[0, 1) \rightarrow [0, \infty)$. (c) $z \mapsto (1 - z)^{-1}$ maps $\mathbb{D}$ onto

$\{\operatorname{Re} w > \frac{1}{2}\}$ and $C \setminus \{1\}$ onto $\{\operatorname{Re} w = \frac{1}{2}\}$ (since $|1 - 1/w| < 1 \iff |w - 1| < |w|$); $\Psi_{\mathbb{D}}$ is the inverse of this map composed with a homeomorphism of $Q^\circ \cup X$ onto $\{\frac{1}{2} \leq \operatorname{Re} w < 1\}$. Near a point $x \in C \setminus \{1\}$, $\operatorname{Re}(1 - z)^{-1}$ is close to $\frac{1}{2}$; on $[0, 1]$ it is $\geq \frac{1}{2}$. (d) is elementary. $\square$

Lemma 4.14 (crosscuts of the half-plane). *Let σ be a crosscut of $\mathbb{H}$ with endpoints $a < b$. Then $\mathbb{H} \setminus \sigma$ has exactly two components, a bounded one V with $\overline{V} \cap \mathbb{R} = [a, b]$, and an unbounded one U with $\overline{U} \cap \mathbb{R} = \mathbb{R} \setminus (a, b)$; moreover $\sigma \subseteq \overline{U} \cap \overline{V}$ and $U \supseteq \{\operatorname{Im} z > \max_\sigma \operatorname{Im}\}$.*

Proof. On $\mathbb{R}$, $\kappa(x) = e^{i\vartheta(x)}$ with $\vartheta(x) = \pi + 2 \arctan x$, an increasing bijection of $\mathbb{R}$ onto $(0, 2\pi)$. So $\kappa(\sigma)$ is a crosscut of $\mathbb{D}$ from $e^{i\vartheta(a)}$ to $e^{i\vartheta(b)}$ (Lemma 4.13(a)), and κ maps $[a, b]$ onto the arc $A_1 = \{e^{it} : \vartheta(a) \leq t \leq \vartheta(b)\}$ and $\mathbb{R} \setminus (a, b)$ onto the complementary closed arc $A_2 \ni 1$. Theorem 4.10 gives the components U_1, U_2 of $\mathbb{D} \setminus \kappa(\sigma)$ with $\overline{U}_i \cap C = A_i$; put $V = \kappa^{-1}(U_1)$, $U = \kappa^{-1}(U_2)$. Since κ is a homeomorphism of $\mathbb{H} \cup \mathbb{R}$ onto $(\mathbb{D} \cup C) \setminus \{1\}$, closures correspond at points of $\mathbb{H} \cup \mathbb{R}$, which gives the traces on $\mathbb{R}$ and $\sigma \subseteq \overline{U} \cap \overline{V}$. As $1 \notin \overline{U}_1$, $|1 - w| \geq r > 0$ on U_1 , so $|\kappa^{-1}(w)| \leq 2/r$ there and V is bounded. The half-plane above σ is convex, unbounded and disjoint from σ , so it lies in U . $\square$

Lemma 4.15 (order of disjoint arcs in a strip). *Let $h > 0$, $S = \{0 < \operatorname{Im} z < h\}$, and let α, α' be disjoint simple arcs, α from $x \in \mathbb{R}$ to p with $\operatorname{Im} p = h$, α' from $x' \in \mathbb{R}$ to p' with $\operatorname{Im} p' = h$, all other points in S . If $x < x'$ then $\operatorname{Re} p < \operatorname{Re} p'$.*

Proof. By Lemma 4.13(d), $\mathcal{E}_h(\alpha)$ is a crosscut of $\mathbb{H}$ from $u = e^{\pi x/h}$ to $v = -e^{\pi \operatorname{Re} p/h}$, and $\mathcal{E}_h(\alpha')$ one from $u' > u$ to $v' = -e^{\pi \operatorname{Re} p'/h} \neq v$. By Lemma 4.14 the bounded component V of $\mathbb{H} \setminus \mathcal{E}_h(\alpha)$ has $\overline{V} \cap \mathbb{R} = [v, u]$. The connected set $\mathcal{E}_h(\alpha') \setminus \{u', v'\}$ lies in one component; its closure contains $u' \notin [v, u]$, so it lies in the unbounded component U , and then $v' \in \overline{U} \cap \mathbb{R} = \mathbb{R} \setminus (v, u)$. As $v' < 0 < u$ and $v' \neq v$, $v' < v$, i.e. $\operatorname{Re} p' > \operatorname{Re} p$. $\square$

Lemma 4.16 (geometry near a boundary point). *Let $x \in C$ and $0 < \delta < 2$. (a) $\mathbb{D} \cap S(\delta, x)$ is convex. (b) $\mathbb{D} \setminus \overline{S(\delta, x)}$ is connected and has $-x$ in its closure. (c) If $x' \in C$ and $|x' - x| < \delta$, the shorter closed arc of C from x to x' lies in $S(\delta, x)$. Any arc of C between w_1, w_2 has length at least $|w_1 - w_2|$. (d) If $A_1, \dots, A_m$ are closed arcs of C with pairwise disjoint relative interiors, their lengths add up to at most 2π . (The length of an arc is the Lebesgue measure of the corresponding set of angles in $[0, 2\pi)$.) In particular, if the endpoints of each A_i are at distance $\geq \varepsilon$, then $m\varepsilon \leq 2\pi$. In Lean a closed arc is $\{e^{it} : \theta \leq t \leq \theta + \ell\}$ with $0 \leq \ell \leq 2\pi$ (`circleArc`), its interior is the image of $(\theta, \theta + \ell)$, and its length is ℓ ; (d) is proved by pushing the open angle intervals to $\mathbb{R}/2\pi\mathbb{Z}$ with its Haar measure.*

Proof. (a) Intersection of convex sets. (b) For $z \in \mathbb{D}$ the convex function $t \mapsto |(1 - t)z - tx - x|^2$ has derivative $2(1 - |z|^2) > 0$ at $t = 0$, hence increases on $[0, 1]$; so the segment $[z, -x]$ stays in $\mathbb{D} \setminus \overline{S(\delta, x)}$ and ends in the convex set $\mathbb{D} \cap S(2 - \delta, -x)$, which lies in $\mathbb{D} \setminus \overline{S(\delta, x)}$. (c) On an arc of angle at most π the chord from an endpoint increases with the angle; chords are shorter than arcs. (d) The interiors are disjoint measurable sets of angles and the endpoints are null. $\square$

Lemma 4.17. *An open subset of $\mathbb{R}^n$ has countably many components; they are open and path-connected, and a connected subset of the open set lies in one of them.*

Proof. $\mathbb{R}^n$ is locally path-connected and second countable. $\square$

4.3. Transfer.

Lemma 4.18 (transfer). *Let Ω, Ω' be open, $B \subseteq \partial\Omega$, $B' \subseteq \partial\Omega'$, and let $\Psi : \Omega \cup B \rightarrow \Omega' \cup B'$ be a homeomorphism with $\Psi(\Omega) = \Omega'$. For $f : \Omega' \rightarrow Y$: γ is an arc at $p \in B$ iff $\Psi(\gamma)$ is an arc at $\Psi(p)$; $f \circ \Psi \rightarrow q$ along γ iff $f \rightarrow q$ along $\Psi(\gamma)$, and the cluster sets agree; hence*

$\text{CC}(f \circ \Psi) = \Psi^{-1}(\text{CC}(f))$, and φ is a boundary function for f on E' iff $\varphi \circ \Psi$ is one for $f \circ \Psi$ on $\Psi^{-1}(E')$.

Proof. Arcs at p lie in $\Omega \cup \{p\}$, where Ψ is a homeomorphism. $\square$

Lemma 4.19 (locality). *Let $p \in \partial\Omega \cap \partial\Omega'$ and suppose $W \cap \Omega = W \cap \Omega'$ for an open $W \ni p$. If $f = f'$ on $W \cap \Omega$, then f has a limit q along some arc at p in Ω iff f' has the limit q along some arc at p in Ω' .*

Proof. Lemma 4.5(a). $\square$

5. SETS OF CURVILINEAR CONVERGENCE

Sources: [2]; [3, §4, Lemmas 7–8, Theorem 3]. We work in $\mathbb{D}$ and transfer to $\mathbb{H}$.

Definition 5.1 (accessible points and crosscut arcs). For a crosscut γ of $\mathbb{D}$ with $0 \notin \gamma$, let U be the component of $\mathbb{D} \setminus \gamma$ containing 0 and $L(\gamma)^* = C \setminus \overline{U}$; by Theorem 4.10 it is the open arc of C between the endpoints of γ on the side of the other component. For a domain $\Omega \subseteq \mathbb{D} \setminus \{0\}$ let $\Gamma(\Omega)$ be the set of crosscuts γ with $\gamma \cap \mathbb{D} \subseteq \Omega$,

$$I_0(\Omega) = \bigcup_{\gamma \in \Gamma(\Omega)} L(\gamma)^*,$$

and let $\text{acc}(\Omega)$ be the set of $x \in C$ for which there is an arc at x in Ω . Note that $I_0(\Omega)$ is open in C and $\text{acc}(\Omega) \subseteq \overline{\Omega}$.

Lemma 5.2 ([2, Lemma 2]). *For every domain $\Omega \subseteq \mathbb{D} \setminus \{0\}$, $\text{acc}(\Omega) \setminus I_0(\Omega)$ is countable.*

Proof. We may assume $\text{acc}(\Omega)$ has two or more points. Let $a \in \text{acc}(\Omega)$ and $a' \neq a$ in $\text{acc}(\Omega)$, with arcs γ, γ' at a, a' in Ω , and join their other endpoints by a path in Ω . By Fact 4.3 the resulting path from a to a' contains a simple arc δ from a to a' ; it is a crosscut with $\delta \cap \mathbb{D} \subseteq \Omega$ (so $0 \notin \delta$). Write $a = e^{i\theta}$, $a' = e^{i(\theta+\ell)}$, $0 < \ell < 2\pi$. By Theorem 4.10, $L(\delta)^*$ is one of the two open arcs $e^{i(s,t)}$ with $(s, t) = (\theta, \theta + \ell)$ or $(\theta + \ell, \theta + 2\pi)$, and $a \in e^{i[s,t]}$. So $\text{acc}(\Omega)$ lies in a union of closed arcs $e^{i[s,t]}$ whose open arcs $e^{i(s,t)}$ lie in $I_0(\Omega)$, and Lemma 3.34 shows that $\text{acc}(\Omega) \setminus I_0(\Omega)$ is countable. $\square$

Lemma 5.3 ([2, Lemma 3]). *Let $\Omega_1, \Omega_2 \subseteq \mathbb{D} \setminus \{0\}$ be connected. If the sets $I_0(\Omega_1) \cap \overline{\text{acc}(\Omega_1)}$ and $I_0(\Omega_2) \cap \overline{\text{acc}(\Omega_2)}$ have a common point a , then $\Omega_1 \cap \Omega_2 \neq \emptyset$.*

Proof. Suppose $\Omega_1 \cap \Omega_2 = \emptyset$. Choose $\gamma_i \in \Gamma(\Omega_i)$ with $a \in L(\gamma_i)^*$, and let U_i, V_i be the components of $\mathbb{D} \setminus \gamma_i$, with $0 \in U_i$; so $a \notin \overline{U_i}$, while $a \in \overline{\text{acc}(\Omega_i)} \subseteq \overline{\Omega_i}$. The sets $\tau_i = \gamma_i \cap \mathbb{D}$ are nonempty, connected, contained in Ω_i and in $\overline{U_i}$ (Theorem 4.10), and $\tau_1 \cap \tau_2 = \emptyset = \tau_2 \cap \gamma_1$. Hence each τ_i lies in one component of the other complement.

If $\tau_2 \subseteq U_1$: the connected set Ω_2 misses γ_1 (as $\gamma_1 \cap \mathbb{D} \subseteq \Omega_1$), so it lies in U_1 or V_1 ; it meets U_1 (it contains τ_2), so $\Omega_2 \subseteq U_1$ and $a \in \overline{\Omega_2} \subseteq \overline{U_1}$, a contradiction. Symmetrically $\tau_1 \not\subseteq U_2$.

So $\tau_1 \subseteq V_2$ and $\tau_2 \subseteq V_1$. As $\tau_1 \subseteq \overline{U_1}$ and V_2 is open, U_1 meets V_2 ; also $0 \in U_1 \cap U_2$. The connected set U_1 meets both components of $\mathbb{D} \setminus \gamma_2$, so it meets $\gamma_2 \cap \mathbb{D} = \tau_2 \subseteq V_1$, which is absurd. $\square$

Theorem 5.4 (McMillan [20, Theorem 5]; [2, Theorem 1]). *Let K be a complete separable metric space and $f : \mathbb{D} \rightarrow K$ continuous. Then $\text{CC}(f)$ is an $F_{\sigma\delta}$ subset of C .*

Proof. Let $\{p_k\}$ be dense in K and $\{c_m\}$ dense in $\mathbb{C}$. Put $Q(n, m) = S(1/(n+1), c_m) \cap \mathbb{D} \setminus \{0\}$, and let $U(n, m, k, l)$, $l \in \mathbb{N}$, be the component of $O(n, m, k) = f^{-1}(S(2^{-n}, p_k)) \cap Q(n, m)$ containing

c_l (empty if $c_l \notin O(n, m, k)$); every component of the open set $O(n, m, k)$ is open and nonempty, so it is one of the $U(n, m, k, l)$ (Lemma 4.17). Put $A(n, m, k, l) = \text{acc } U(n, m, k, l)$ and

$$Y = \bigcap_n \bigcup_{m, k, l} \left(I_0(U(n, m, k, l)) \cap \overline{A(n, m, k, l)} \right).$$

$I_0(\cdot)$ is open in C , so Y is $F_{\sigma\delta}$.

$Y \subseteq \text{CC}(f)$. Let $y \in Y$ and choose $U_n = U(n, m[n], k[n], l[n])$ with $y \in I_0(U_n) \cap \overline{\text{acc } U_n}$. By Lemma 5.3, $U_n \cap U_{n+1}$ contains a point z_n ; join z_n to z_{n+1} by a path $\gamma_n \subseteq U_{n+1}$. Since $y \in \overline{U_{n+1}} \subseteq S(1/(n+2), c_{m[n+1]})$, γ_n lies within $2/(n+2)$ of y . So the paths γ_n , traversed on $[n, n+1]$ and reparametrized by $t \mapsto (1-t)^{-1} - 1$, followed by y at $t = 1$, form a path g with $g(1) = y$, $g([0, 1)) \subseteq \mathbb{D}$, and $g(t) \in \bigcup_{n \geq N} \gamma_n$ for t near 1. As $S(2^{-n}, p_{k[n]})$ and $S(2^{-n-1}, p_{k[n+1]})$ meet (at $f(z_n)$), $\rho(p_{k[n]}, p_{k[n+1]}) < 2^{-n+1}$, so $p_{k[n]} \rightarrow p$ for some $p \in K$; and $f(\gamma_n) \subseteq S(2^{-n-1}, p_{k[n+1]})$, so $f(g(t)) \rightarrow p$ as $t \rightarrow 1$. By Corollary 4.4, $y \in \text{CC}(f)$.

$\text{CC}(f) \subseteq \bigcap_n \bigcup_{m, k, l} A(n, m, k, l)$. Let $f \rightarrow p'$ along an arc γ_0 at x . Given n , pick k with $\rho(p', p_k) < 2^{-n}$ and m with $|x - c_m| < 1/(2n+2)$, so that $S(r, x) \cap \mathbb{D} \subseteq Q(n, m)$ for $r = \min(1/(2n+2), 1/2)$. By Lemma 4.5 a subarc γ'_0 at x has $\gamma'_0 \setminus \{x\} \subseteq O(n, m, k)$; being connected, it lies in one component, so $x \in A(n, m, k, l)$ for some l .

$\text{CC}(f) \setminus Y$ is countable. If $x \in \text{CC}(f) \setminus Y$, there is n with $x \in A(n, m, k, l)$ for some m, k, l but $x \notin I_0(U(n, m, k, l)) \cap \overline{A(n, m, k, l)}$, so $x \in A(n, m, k, l) \setminus I_0(U(n, m, k, l))$, which is countable by Lemma 5.2.

So $\text{CC}(f)$ is the union of an $F_{\sigma\delta}$ set and a countable set, hence $F_{\sigma\delta}$. (In Lean the three steps are packaged once, for any countable family of open sets $O(n, i) \subseteq \mathbb{D} \setminus \{0\}$, and reused for Theorem 5.5.) $\square$

Theorem 5.5 ([2, Theorem 2]). *Let K be a compact metric space, $\mathcal{C}(K)$ the space of nonempty closed subsets of K with the Hausdorff metric $\bar{\rho}$, $\mathcal{E} \subseteq \mathcal{C}(K)$ closed, and $f : \mathbb{D} \rightarrow K$ continuous. Then*

$$X_{\mathcal{E}} = \{x \in C : \text{there are an arc } \gamma \text{ at } x \text{ and } E \in \mathcal{E} \text{ with } C(f, \gamma) \subseteq E\}$$

is an $F_{\sigma\delta}$ set.

Proof. If $\mathcal{E} = \emptyset$ there is nothing to prove. $\mathcal{E}$ is compact (Fact 4.8), hence has a dense sequence $P(k)$. For $\varepsilon > 0$ let $\mathcal{S}(\varepsilon, E) = \{a : \text{dist}(a, E) < \varepsilon\}$, which is open, and note that $\text{dist}(a, F) \leq \text{dist}(a, E) + \bar{\rho}(E, F)$. Define $Q(n, m)$ as before, $U(n, m, k, l)$ as the components of $f^{-1}(\mathcal{S}(1/(n+1), P(k))) \cap Q(n, m)$, $A(n, m, k, l) = \text{acc } U(n, m, k, l)$, and Y by the same formula; Y is $F_{\sigma\delta}$.

$Y \subseteq X_{\mathcal{E}}$. With U_n as before, compactness gives $P \in \mathcal{E}$ and $n_1 < n_2 < \dots$ with $P(k[n_j]) \rightarrow P$. The sets U_{n_j} satisfy the hypotheses used above, so as in Theorem 5.4 we get a path g ending at y , with $g(t) \in \bigcup_{j \geq J} U_{n_j}$ for t near 1, and (Corollary 4.4) an arc γ at y whose cluster set is contained in that of $f \circ g$ at 1. Let p be such a cluster point. Given $\varepsilon > 0$, for j large $2/(n_j + 1) < \varepsilon/2$ and $\bar{\rho}(P(k[n_j]), P) < \varepsilon/2$, so $\text{dist}(f(g(t)), P) \leq \varepsilon$ for t near 1, and hence $\text{dist}(p, P) \leq \varepsilon$. As P is closed, $p \in P$. So $C(f, \gamma) \subseteq P$.

$X_{\mathcal{E}} \subseteq \bigcap_n \bigcup_{m, k, l} A(n, m, k, l)$. Let $C(f, \gamma_0) \subseteq P_0 \in \mathcal{E}$. Given n , choose k with $\bar{\rho}(P_0, P(k)) < 1/(n+1)$, so $C(f, \gamma_0) \subseteq \mathcal{S}(1/(n+1), P(k)) =: O$. Some subarc of γ_0 at x is mapped into O ; otherwise the filter of f along γ_0 meets the compact set $K \setminus O$ and has a cluster point there, which would lie in $C(f, \gamma_0) \setminus O$. Conclude as in Theorem 5.4; the countability of $X_{\mathcal{E}} \setminus Y$ is proved in the same way. $\square$

Corollary 5.6 ([2, Corollaries 1–4]). *Let $f : \mathbb{D} \rightarrow \mathbb{S}^2$ be continuous. Each of the following sets is $F_{\sigma\delta}$ ((1)–(3) hold for any compact metric target):*

- (1) (McMillan) $\{x : \exists \gamma \text{ at } x, \exists p \in E, f \rightarrow p \text{ along } \gamma\}$, for closed $E \subseteq \mathbb{S}^2$;

- (2) $\{x : \exists \gamma \text{ at } x, \text{diam } C(f, \gamma) \leq d\}$, for $d \geq 0$;
- (3) $\{x : \exists \gamma \text{ at } x, C(f, \gamma) \subseteq E\}$, for closed $E \subseteq \mathbb{S}^2$;
- (4) $\{x : \exists \gamma \text{ at } x, C(f, \gamma) \text{ is an arc of a great circle}\}$, where (since $C(f, \gamma)$ is closed and connected) an “arc” may be a point, a closed arc, or a whole great circle; that is, $C(f, \gamma)$ is contained in a great circle.

Proof. Apply Theorem 5.5 with $\mathcal{E}$ equal to: (1) $\{\{p\} : p \in E\}$, an isometric copy of the compact set E , using that $C(f, \gamma) \subseteq \{p\}$ iff $f \rightarrow p$ along γ (Lemma 4.6); (2) $\{F : \text{diam } F \leq d\}$, closed because if $\bar{\rho}(F, F') < \varepsilon$ then every two points of F are within 2ε of two points of F' , and using $E = C(f, \gamma)$ for the converse; (3) $\{E\}$ (or note the set is empty if $E = \emptyset$); (4) the sets contained in some great circle $\{v : \langle v, u \rangle = 0\}$: if $F_n \rightarrow F$ with $F_n \subseteq \{\langle \cdot, u_n \rangle = 0\}$, pass to a subsequence $u_n \rightarrow u$; each $v \in F$ is a limit of $v_n \in F_n$, so $\langle v, u \rangle = \lim \langle v_n, u_n \rangle = 0$. With $E = C(f, \gamma)$ for the converse; the connectedness of $C(f, \gamma)$ (Lemma 4.6) justifies the reading of “arc”. $\square$

Theorem 5.7 (McMillan in the half-plane; [3, Theorem 3]). *If K is a complete separable metric space and $f : \mathbb{H} \rightarrow K$ is continuous, then $\text{CC}(f) \subseteq \mathbb{R}$ is $F_{\sigma\delta}$.*

Proof. By Lemma 4.18 with $\Psi = \kappa$ (Lemma 4.13(a)), $\text{CC}(f) = \mathbb{R} \cap \kappa^{-1}(\text{CC}(f \circ \kappa^{-1}))$, and the right side is $F_{\sigma\delta}$ by Theorem 5.4 and Lemma 3.2 (preimage under the continuous map $\kappa|_{\mathbb{R}}$, then an $F_{\sigma\delta}$ subset of the closed set $\mathbb{R}$). $\square$

The half-plane lemmas of the dissertation. For a domain $\Omega \subseteq \mathbb{H}$ and a crosscut γ of $\mathbb{H}$ with endpoints $c < d$ let $L(\gamma) = [c, d]$, the trace on $\mathbb{R}$ of the closure of the bounded component of $\mathbb{H} \setminus \gamma$; let $I(\Omega) = \bigcup L(\gamma)^*$ over crosscuts with $\gamma \cap \mathbb{H} \subseteq \Omega$, and $\text{acc}(\Omega)$ the points of $\mathbb{R}$ accessible by arcs in Ω . These lemmas are the half-plane forms of Lemmas 5.2 and 5.3; they are not used elsewhere.

Lemma 5.8 ([3, Lemma 7]). *If $\text{acc}(\Omega) \neq \emptyset$, $a = \inf \text{acc}(\Omega)$ and $b = \sup \text{acc}(\Omega)$, then $I(\Omega) = (a, b)$; that is, $x \in I(\Omega)$ iff there are $c, d \in \text{acc}(\Omega)$ with $c < x < d$.*

Proof. If $x \in L(\gamma)^* = (c, d)$ then $c, d \in \text{acc}(\Omega)$ (initial halves of γ from either end are arcs at c, d in Ω), so $a \leq c < x < d \leq b$. Conversely, for $a < x < b$ pick $c', d' \in \text{acc}(\Omega)$ with $c' < x < d'$, and join them by a crosscut through Ω as in Lemma 5.2. $\square$

Lemma 5.9 ([3, Lemma 8]). *If $\Omega_1, \Omega_2 \subseteq \mathbb{H}$ are connected and $I(\Omega_1) \cap \overline{\text{acc}(\Omega_1)}$ and $I(\Omega_2) \cap \overline{\text{acc}(\Omega_2)}$ meet, then $\Omega_1 \cap \Omega_2 \neq \emptyset$.*

Proof. As in Lemma 5.3, with U_i the unbounded component of $\mathbb{H} \setminus \gamma_i$ (any two unbounded components meet, since each contains a half-plane $\{\text{Im } z > t\}$) and Lemma 4.14 in place of Theorem 4.10. $\square$

6. BOUNDARY FUNCTIONS OF CONTINUOUS FUNCTIONS AND HOMEOMORPHISMS

Sources: [3, §5, Lemma 15, Theorem 5]; [1, Theorems 1–3]; [2, remark after Corollary 4].

Lemma 6.1 ([3, Lemma 15]). *Let X be a topological space, $E \subseteq X$, Y a separable metric space and $\varphi : X \rightarrow Y$. Suppose that for every open $U \subseteq Y$ there are $L \in F_\sigma(X)$ and a countable $N \subseteq X$ with $E \cap \varphi^{-1}(U) \subseteq L$ and $E \cap L \subseteq \varphi^{-1}(\overline{U}) \cup N$. Then there is a countable $M \subseteq E$ such that $\varphi|_{E \setminus M}$ is of class F_σ .*

Proof. Let $\mathcal{B}$ be a countable base, choose $L(B), N(B)$ for $B \in \mathcal{B}$, put $M = E \cap \bigcup_B N(B)$, $E_0 = E \setminus M$, $\varphi_0 = \varphi|_{E_0}$. For open W let $\mathcal{B}(W) = \{B \in \mathcal{B} : \overline{B} \subseteq W\}$, so $W = \bigcup_{B \in \mathcal{B}(W)} B$ (proof of Lemma 3.38). Then

$$\varphi_0^{-1}(W) \subseteq E_0 \cap \bigcup_{B \in \mathcal{B}(W)} L(B) \subseteq E_0 \cap \bigcup_{B \in \mathcal{B}(W)} (\varphi_0^{-1}(\overline{B}) \cup N(B)) = E_0 \cap \bigcup_{B \in \mathcal{B}(W)} \varphi_0^{-1}(\overline{B}) \subseteq \varphi_0^{-1}(W),$$

so $\varphi_0^{-1}(W) = E_0 \cap \bigcup_{B(W)} L(B) \in F_\sigma(E_0)$. $\square$

Theorem 6.2 ([3, Theorem 5]). *Let Y be a separable metric space, $f : \mathbb{H} \rightarrow Y$ continuous, $E \subseteq \mathbb{R}$, and $\varphi : E \rightarrow Y$ a boundary function for f . There is a countable $M \subseteq E$ such that $\varphi|_{E \setminus M}$ is of class F_σ . The same holds for $f : \mathbb{D} \rightarrow Y$ continuous and $E \subseteq C$.*

Proof. By Lemma 6.1 it suffices, given open $U \subseteq Y$, to find $L \in F_\sigma(\mathbb{C})$ and countable N with $E \cap \varphi^{-1}(U) \subseteq L$ and $E \cap L \subseteq \varphi^{-1}(\overline{U}) \cup N$. Let $W = Y \setminus \overline{U}$, so $U \cap W = \emptyset$, and for $c > 0$

$$E_c = \{x \in \mathbb{R} : \text{some arc } \gamma \text{ at } x \text{ has a point } z \text{ with } \operatorname{Im} z \geq c, \text{ and } f(\gamma \setminus \{x\}) \subseteq U\},$$

$$K = \{x \in \mathbb{R} : \text{some arc } \gamma \text{ at } x \text{ has } f(\gamma \setminus \{x\}) \subseteq W\}.$$

If $x \in E \cap \varphi^{-1}(U)$, a subarc γ' of the arc at x along which $f \rightarrow \varphi(x)$ has $f(\gamma' \setminus \{x\}) \subseteq U$ (Lemma 4.5); its other endpoint lies in $\mathbb{H}$, so $x \in E_{1/n}$ for some n . So $E \cap \varphi^{-1}(U) \subseteq \bigcup_n E_{1/n}$, and likewise $E \cap \varphi^{-1}(W) \subseteq K$.

$K \cap \overline{E_c}$ is countable. Let $O = \{z \in \mathbb{H} : \operatorname{Im} z < c, f(z) \in W\}$, an open set. For $x \in K$ choose (Lemma 4.5) an arc γ_x from x to b_x with $\gamma_x \setminus \{x\} \subseteq O$, and let U_x be the component of O containing b_x . Let T be the set of points of $K \cap \overline{E_c}$ that are two-sided limit points of E_c . We claim $x \mapsto U_x$ is injective on T , so T is countable (Lemma 4.17). Let $x < y$ in T with $U_x = U_y$. Joining γ_x and γ_y by a path in U_x and applying Fact 4.3, we get a crosscut α' of $\mathbb{H}$ from x to y with $\alpha' \setminus \{x, y\} \subseteq O$; in particular $\operatorname{Im} < c$ on α' . Let A be the unbounded component of $\mathbb{H} \setminus \alpha'$ (Lemma 4.14); it contains every point above α' . Choose $w \in (x, y) \cap E_c$ and an arc β at w with $f(\beta \setminus \{w\}) \subseteq U$ and a point z_0 , $\operatorname{Im} z_0 \geq c$. Then $\beta \setminus \{w\}$ is connected, misses α' (as $U \cap W = \emptyset$) and contains $z_0 \in A$, so it lies in A ; hence $w \in \overline{A} \cap \mathbb{R} = \mathbb{R} \setminus (x, y)$, a contradiction. The points of $\overline{E_c}$ that are not two-sided limit points of E_c form a countable set (Lemma 3.35). Hence $K \cap \overline{E_c}$ is countable.

Put $L = \bigcup_n \overline{E_{1/n}}$, an F_σ set, and $N = \bigcup_n (K \cap \overline{E_{1/n}})$, countable. Then $E \cap \varphi^{-1}(U) \subseteq L$, and if $x \in E \cap L$ with $\varphi(x) \notin \overline{U}$, then $x \in K$, so $x \in N$.

For $\mathbb{D}$, Lemma 4.18 with κ shows that $\varphi \circ \kappa$ is a boundary function for $f \circ \kappa$ on $E' = \kappa^{-1}(E \setminus \{1\})$; if $\varphi \circ \kappa$ is of class F_σ on $E' \setminus M'$, then φ is of class F_σ on $E \setminus (\kappa(M') \cup \{1\})$, since κ^{-1} is continuous there. $\square$

Corollary 6.3 ([3, Theorem 5, Corollary]; [1, Theorems 2, 3]). *Let Y be $\mathbb{R}^m$ or $\mathbb{S}^2$. If $f : \mathbb{H} \rightarrow Y$ (resp. $f : \mathbb{D} \rightarrow Y$) is continuous, $E \subseteq \mathbb{R}$ (resp. $E \subseteq C$) and $\varphi : E \rightarrow Y$ is a boundary function for f , then φ is of honorary Baire class ≤ 2 . In particular ([1, Theorems 2, 3]) a finite-valued boundary function on C of a continuous real function in $\mathbb{D}$ is of honorary Baire class $2(C, \mathbb{R})$, and a boundary function on C of a continuous map $\mathbb{D} \rightarrow \mathbb{S}^2$ is of honorary Baire class $2(C, \mathbb{S}^2)$.*

Proof. Combine Theorem 6.2 with Theorem 3.22(2) $\Rightarrow$ (1). $\square$

Lemma 6.4 (accessible from two domains). *Let $W_1, W_2 \subseteq \mathbb{D}$ be disjoint connected open sets. At most three points of C are accessible (by a simple arc, or equivalently by a path $g : [0, 1] \rightarrow \mathbb{C}$ with $g(0) = x$, $g((0, 1]) \subseteq W_i$) from both W_1 and W_2 .*

Proof. Let S be the set of such points, $x \in S$, and suppose $y, z, w \in S \setminus \{x\}$ are distinct, numbered so that z lies on the arc of C from y to w not containing x (order them by angle measured from x). Join the paths at x and at z into W_1 by a path in the connected open set W_1 ; the resulting path from x to z has all points other than x, z in W_1 and contains a crosscut σ from x to z (Fact 4.3), $\sigma \setminus \{x, z\} \subseteq W_1$. By Theorem 4.10, $\mathbb{D} \setminus \sigma$ has two components U_1, U_2 with $\overline{U_i} \cap C = A_i$, the two closed arcs from x to z ; y and w lie in different open arcs A_1^*, A_2^* . The connected set W_2 misses σ (as $x, z \notin \mathbb{D}$ and $W_1 \cap W_2 = \emptyset$), so lies in one U_i ; but its closure contains y and w (they are accessible from W_2), which are not both in $\overline{U_i}$. So $|S \setminus \{x\}| \leq 2$, and $|S| \leq 3$. $\square$

Lemma 6.5 (Bagemihl’s ambiguous point theorem, continuous case). *For a function $f : \mathbb{D} \rightarrow \mathbb{S}^2$ continuous in $\mathbb{D}$, the set Z of $x \in C$ at which there are two arcs along which f has different limits is countable.*

Proof. Fix a countable base $\mathcal{B}$ of $\mathbb{S}^2$. Let $x \in Z$ with arcs γ_1, γ_2 and limits $p_1 \neq p_2$; choose disjoint $B_1 \ni p_1, B_2 \ni p_2$ in $\mathcal{B}$. A terminal subarc of γ_i minus x is a connected subset of the open set $f^{-1}(B_i) \cap \mathbb{D}$, so it lies in a component W_i of it, and x is accessible from W_i . The components of open subsets of $\mathbb{C}$ are countably many (each contains a point of $\mathbb{Q} + i\mathbb{Q}$), so the tuples (B_1, B_2, W_1, W_2) form a countable set; for each, $W_1 \cap W_2 = \emptyset$, and by Lemma 6.4 at most three x belong to it. $\square$

Proposition 6.6 ([2, remark after Corollary 4]). *In Corollary 5.6(1) it suffices that E be a G_δ subset of $\mathbb{S}^2$.*

Proof. $A = \text{CC}(f)$ is $F_{\sigma\delta}$ (Theorem 5.4). Choose any boundary function $\varphi : A \rightarrow \mathbb{S}^2$. By Theorem 6.2, φ is of class F_σ on $A \setminus M$ for a countable M . Off the countable set Z of Lemma 6.5 (f is continuous in $\mathbb{D}$), every arc limit of f at $x \in A$ equals $\varphi(x)$; so the set S in question satisfies $S = (P \setminus Z) \cup (S \cap (M \cup Z))$ with $P = (\varphi|_{A \setminus M})^{-1}(E)$. Writing $E = \bigcap_j O_j$ with O_j open, $P = \bigcap_j (\varphi|_{A \setminus M})^{-1}(O_j)$ is $F_{\sigma\delta}$ in $A \setminus M$, hence in C , as $A \setminus M$ is $F_{\sigma\delta}$. Countable sets are F_σ and their complements G_δ , so S is $F_{\sigma\delta}$. $\square$

6.1. Homeomorphisms. The following is Theorem 1 of [1], of which we have only the statement.

Lemma 6.7. *Let f be a homeomorphism of $\mathbb{D}$ onto $\mathbb{D}$, regarded as a map $\mathbb{D} \rightarrow \mathbb{C}$. If $f \rightarrow q \in \mathbb{C}$ along an arc at $x \in C$, then $q \in C$.*

Proof. If $q \in \mathbb{D}$ then $z = f^{-1}(f(z)) \rightarrow f^{-1}(q) \in \mathbb{D}$ as $z \rightarrow x$ along the arc, which is absurd. $\square$

Theorem 6.8 ([1, Theorem 1]). *Let f be a homeomorphism of $\mathbb{D}$ onto $\mathbb{D}$, regarded as a map $\mathbb{D} \rightarrow \mathbb{C}$, let $E \subseteq C$, and let $\varphi : E \rightarrow \mathbb{C}$ be a boundary function for f (so $\varphi(E) \subseteq C$ by Lemma 6.7). Then there is a countable $N \subseteq E$ such that $\varphi|_{E \setminus N}$ is continuous. More precisely, for every $\varepsilon > 0$ the set*

$$S_\varepsilon = \{x \in E : \text{for every } \delta > 0 \text{ there is } x' \in E \text{ with } 0 < |x' - x| < \delta, |\varphi(x') - \varphi(x)| \geq \varepsilon\}$$

has at most $2\pi/\varepsilon$ points.

Proof. Since φ is continuous (relative to E) at every point of $E \setminus \bigcup_k S_{1/k}$, the first claim follows from the second. Let $x_1, \dots, x_m$ be distinct points of S_ε . Choose $\delta \in (0, 1)$ so small that the balls $S(\delta, x_i)$ are pairwise disjoint, and for each i choose $x'_i \in E$ with $0 < |x'_i - x_i| < \delta/2$ and $|\varphi(x'_i) - \varphi(x_i)| \geq \varepsilon$. Fix i , and let γ, γ' be arcs at x_i, x'_i along which f tends to $\varphi(x_i), \varphi(x'_i)$.

A crosscut near x_i . By Lemma 4.5 choose subarcs $\beta \subseteq \gamma \cap S(\delta/2, x_i)$ at x_i and $\beta' \subseteq \gamma' \cap S(\delta/2, x'_i)$ at x'_i , with other endpoints $u, v \in \mathbb{D}$. The segment $[u, v]$ lies in the convex set $\mathbb{D} \cap S(\delta, x_i)$ (Lemma 4.16(a)). By Fact 4.3, $\beta \cup [u, v] \cup \beta'$ contains a simple arc σ_i from x_i to x'_i , which is a crosscut of $\mathbb{D}$ lying in a closed ball $\overline{S}(\delta', x_i)$ with $\delta' < \delta$. Near x_i , σ_i consists of points of β (the compact sets $[u, v]$ and β' miss x_i), so $f \rightarrow \varphi(x_i)$ along σ_i at x_i ; likewise $f \rightarrow \varphi(x'_i)$ at x'_i . Since $\varphi(x_i) \neq \varphi(x'_i)$ lie on C (Lemma 6.7), $\tau_i = \overline{f(\sigma_i \cap \mathbb{D})} \cup \{\varphi(x_i), \varphi(x'_i)\}$ is a crosscut of $\mathbb{D}$.

The small component. The connected set $\mathbb{D} \setminus \overline{S}(\delta', x_i)$ (Lemma 4.16(b)) misses σ_i , so it lies in one of the two components of $\mathbb{D} \setminus \sigma_i$ (Theorem 4.10); let R_i be the other one. Then $R_i \subseteq \overline{S}(\delta', x_i) \subseteq S(\delta, x_i)$, so $R_1, \dots, R_m$ are pairwise disjoint.

Images. f maps $\mathbb{D} \setminus \sigma_i$ homeomorphically onto $\mathbb{D} \setminus \tau_i$, so $f(R_i)$ is a component of $\mathbb{D} \setminus \tau_i$, and by Theorem 4.10 $\overline{f(R_i)} \cap C$ is a closed arc A_i with endpoints $\varphi(x_i), \varphi(x'_i)$, of length at least $|\varphi(x_i) - \varphi(x'_i)| \geq \varepsilon$ (Lemma 4.16(c)). The sets $f(R_i)$ are pairwise disjoint, so the arcs A_i have pairwise disjoint interiors (Lemma 4.11). Hence $m\varepsilon \leq 2\pi$ by Lemma 4.16(d). $\square$

7. REALIZING SETS OF CURVILINEAR CONVERGENCE AND BOUNDARY FUNCTIONS

Sources: [8, (A)–(F)]; [3, §§4–5, Lemmas 9–17, Theorems 4, 6, 7]. In this section $X = (-1, 1)$ (identified with $(-1, 1) \times \{0\}$), $Q = [-1, 1] \times (0, 1]$, $Q^\circ = (-1, 1) \times (0, 1)$, and CC refers to arcs in Q° unless said otherwise. Points of $\mathbb{H}$ are $\langle x, y \rangle$, $y > 0$.

7.1. Regions.

Definition 7.1 (Stolz angles, trapezoids, $B(K, \varepsilon, \alpha, \beta)$). For $a \in \mathbb{R}$, $\varepsilon > 0$, $0 < \theta < \pi/2$: $s(a, \varepsilon, \theta) = \{\langle x, y \rangle : 0 < y < \varepsilon, |x - a| < y \cot \theta\}$. For a nonempty interval J with endpoints $a \leq b$: $\text{Trap}(J, \varepsilon, \theta) = \{0 < y < \varepsilon, a + y \cot \theta < x < b - y \cot \theta\}$ and $\text{Tri}(J, \theta) = \{y \geq 0, a + y \cot \theta \leq x \leq b - y \cot \theta\}$. For a nonempty compact $K \subseteq \mathbb{R}$ let $J(K) = [\min K, \max K]$ and let $\mathcal{J}(K)$ be the set of components of $J(K) \setminus K$; for $0 < \beta < \alpha < \pi/2$ put

$$B(K, \varepsilon, \alpha, \beta) = \text{Trap}(J(K), \varepsilon, \alpha) \setminus \bigcup_{I \in \mathcal{J}(K)} \text{Tri}(I, \beta).$$

Lemma 7.2 (description of B). $\langle x, y \rangle \in B(K, \varepsilon, \alpha, \beta)$ iff $0 < y < \varepsilon$, $\min K + y \cot \alpha < x < \max K - y \cot \alpha$, and $\text{dist}(x, K) < y \cot \beta$.

Proof. Let $y > 0$ and $x \in J(K)$. If $x \in K$, x lies in no $\text{Tri}(I, \beta)$ (these lie over $\bar{I}$, and at an endpoint a of I we would need $a + y \cot \beta \leq a$). If $x \in I = (a, b) \in \mathcal{J}(K)$, then $x \notin \text{Tri}(I, \beta)$ iff $\min(x - a, b - x) < y \cot \beta$, i.e. iff $\text{dist}(x, K) < y \cot \beta$, and x lies over no other component. $\square$

In Lean, $\mathcal{J}(K)$ is recorded by the endpoint pairs (a, b) with $a, b \in K$, $a < b$, $(a, b) \cap K = \emptyset$ (gaps); that these intervals are exactly the components of $J(K) \setminus K$ is `exists_gap` and `connectedComponentIn_hull_diff`. Here $\mathbb{R}$ is the line $\{\Im z = 0\}$ of $\mathbb{C}$.

Lemma 7.3 ([8, (VI)–(XVIII)]; [3, (6)–(17)]). Let K, K_1, K_2 be nonempty compact subsets of $\mathbb{R}$, $0 < \beta < \alpha < \pi/2$, $0 < \theta < \pi/2$. Write $c_\theta = \cot \theta$.

- (R1) $s(a, \varepsilon, \theta)$ and $B(K, \varepsilon, \alpha, \beta)$ are open subsets of $\mathbb{H}$.
- (R2) $\text{Cl } s(a, \varepsilon, \theta) \cap \mathbb{R} = \{a\}$ and $\text{Cl } B(K, \varepsilon, \alpha, \beta) \cap \mathbb{R} \subseteq K$.
- (R3) If $\varepsilon < \varepsilon'$ and $\theta' < \theta$ then $\text{Cl } s(a, \varepsilon, \theta) \cap \mathbb{H} \subseteq s(a, \varepsilon', \theta')$.
- (R4) If $a \neq b$ and $\eta \leq |a - b|/(2c_\theta)$, then $s(a, \varepsilon, \theta) \cap s(b, \eta, \theta) = \emptyset$ for all ε .
- (R5) If K_1, K_2 split (Definition 7.4), $B(K_1, \varepsilon_1, \alpha, \beta) \cap B(K_2, \varepsilon_2, \alpha, \beta) = \emptyset$.
- (R6) If $K_1 \cap K_2 = \emptyset$ and $\eta \leq \text{dist}(K_1, K_2)/(2c_\beta)$, then $B(K_1, \varepsilon, \alpha, \beta) \cap B(K_2, \eta, \alpha, \beta) = \emptyset$ for all ε .
- (R7) If $K_1 \subseteq K$, $\varepsilon > \varepsilon_1 > 0$ and $0 < \beta < \beta_1 < \alpha_1 < \alpha < \pi/2$, then $\text{Cl } B(K_1, \varepsilon_1, \alpha_1, \beta_1) \cap \mathbb{H} \subseteq B(K, \varepsilon, \alpha, \beta)$.
- (R8) If $\alpha < \theta$ and $a \notin J(K)^*$, then $B(K, \varepsilon, \alpha, \beta) \cap s(a, \varepsilon_1, \theta) = \emptyset$.
- (R9) If $a \notin K$ and $\eta \leq \text{dist}(a, K)/(c_\theta + c_\beta)$, then $s(a, \eta, \theta) \cap B(K, \varepsilon, \alpha, \beta) = \emptyset$ and $s(a, \varepsilon, \theta) \cap B(K, \eta, \alpha, \beta) = \emptyset$ for all ε .
- (R10) If $a \in K \cap J(K)^*$, $\beta < \theta$, and $0 < \eta < \min\{\varepsilon, (a - \min K)/(c_\theta + c_\alpha), (\max K - a)/(c_\theta + c_\alpha)\}$, then $\text{Cl } s(a, \eta, \theta) \cap \mathbb{H} \subseteq B(K, \varepsilon, \alpha, \beta)$.

Proof. All follow from Lemma 7.2 and the triangle inequality. (R1) Strict inequalities between continuous functions. (R2) If $\langle x_n, y_n \rangle \rightarrow \langle x, 0 \rangle$ with $\text{dist}(x_n, K) < y_n c_\beta$ then $\text{dist}(x, K) = 0$. (R3) On the closure, $y \leq \varepsilon < \varepsilon'$ and $|x - a| \leq y c_\theta < y c_{\theta'}$. (R4) A common point has $|a - b| < 2y c_\theta < 2\eta c_\theta$. (R5) With splitting point p and $K_1 \leq p \leq K_2$, points of the first region have $x < \max K_1 \leq p$ and those of the second $x > \min K_2 \geq p$. (R6) A common point has $y < \eta$ and $\text{dist}(K_1, K_2) \leq \text{dist}(x, K_1) + \text{dist}(x, K_2) < 2y c_\beta$. (R7) On the closure $y \leq \varepsilon_1 < \varepsilon$, $\min K + y c_\alpha < \min K_1 + y c_{\alpha_1} \leq x$ (similarly on the right), and $\text{dist}(x, K) \leq \text{dist}(x, K_1) \leq y c_{\beta_1} < y c_\beta$. (R8) If $a \leq \min K$, points of the Stolz angle have $x < a + y c_\theta < \min K + y c_\alpha$; symmetrically if

$a \geq \max K$. (R9) A common point has $\text{dist}(a, K) \leq |x - a| + \text{dist}(x, K) < y(c_\theta + c_\beta)$ with $y < \eta$. (R10) On the closure, $0 < y \leq \eta < \varepsilon$, $|x - a| \leq yc_\theta$, so $\text{dist}(x, K) \leq yc_\theta < yc_\beta$ and $x \geq a - yc_\theta > \min K + yc_\alpha$. $\square$

In Lean the hypothesis of (R6) is stated pointwise ($\eta \leq |x_1 - x_2|/(2c_\beta)$ for $x_i \in K_i$, which also gives $K_1 \cap K_2 = \emptyset$ when $\eta > 0$), and the hypotheses $a \notin K$ in (R9) and $a \in J(K)^*$ in (R10), which follow from the bounds on η when $\eta > 0$, are omitted.

7.2. Special families and special functions.

Definition 7.4 (splitting, special families). Two sets $E_1, E_2 \subseteq \mathbb{R}$ *split* if there is p with $E_1 \leq p \leq E_2$ or $E_2 \leq p \leq E_1$ (pointwise). A *special family* is a countable family $\mathcal{F}$ of compact subsets of X such that any two members are equal, disjoint, or split. Let $\mathcal{F}^2$ be the members with at least two points and $E(\mathcal{F})$ the set of endpoints of the intervals $J(F)$, $F \in \mathcal{F}$, $F \neq \emptyset$.

Two distinct members of a special family meet in at most one point, and such a common point is an endpoint of both hulls. Endpoints of $J(F)$ belong to F . In particular, if $e \in E(\mathcal{F})$, $F \in \mathcal{F}$ and $e \in J(F)^*$, then $e \notin F$: e is an endpoint of some $J(F')$, so $e \in F'$ and $F' \neq F$; if $e \in F$, then F, F' split at a point p with $p = e$, and e would be an endpoint of $J(F)$ (Lean: `IsSpecialFamily.notMem_of_mem_Ioo`). (The source also asks $\mathcal{F} \neq \emptyset$; nothing uses this, and it is dropped.) In Lean, subsets of X are subsets of $\mathbb{R}$ contained in $(-1, 1)$, and “ F_σ (resp. $F_{\sigma\delta}$) in X ” is stated as “ F_σ (resp. $F_{\sigma\delta}$) in $\mathbb{R}$ ”; for subsets of X these agree since X is open, hence F_σ , in $\mathbb{R}$.

Lemma 7.5 ([8, (III)]; [3, Lemma 9]). *Every F_σ subset E of X is the union of a special family.*

Proof. Write $E = \bigcup_n C_n$ with C_n closed in $\mathbb{R}$ and put $A_n = C_1 \cup \dots \cup C_{n-1}$, so $A_1 = \emptyset$, the A_n are closed, increasing, and contained in E . Every open $U \subseteq \mathbb{R}$ is the union of countably many compact intervals any two of which split: take the maximal dyadic intervals $[k2^{-m}, (k+1)2^{-m}]$ ($m \in \mathbb{N}$, $k \in \mathbb{Z}$) contained in U . They cover U (each $x \in U$ lies in a dyadic interval inside U , and among its finitely many dyadic ancestors in U take the largest), and two distinct ones are not nested, hence have disjoint interiors, hence split. So $\mathbb{R} \setminus A_n = \bigcup_j I_{n,j}$ with compact intervals $I_{n,j}$ any two of which split. Let $\mathcal{F} = \{I_{n,j} \cap A_{n+1}\}$. Its members are compact subsets of X , and $\bigcup \mathcal{F} = \bigcup_n (A_{n+1} \setminus A_n) = E$. For distinct members $I_{n,j} \cap A_{n+1}$, $I_{n',j'} \cap A_{n'+1}$: if $n < n'$ then $n+1 \leq n'$ and the second lies in $X \setminus A_{n'} \subseteq X \setminus A_{n+1}$, so they are disjoint; if $n = n'$ then $j \neq j'$ and they split because $I_{n,j}, I_{n,j'}$ do. $\square$

Lemma 7.6 ([8, (IV)]). *If $A \subseteq X$ is $F_{\sigma\delta}$, there are special families $\mathcal{F}_n$ with $A = \bigcap_n \bigcup \mathcal{F}_n$ such that every member of $\mathcal{F}_{n+1}$ is contained in a member of $\mathcal{F}_n$.*

Proof. Write $A = \bigcap_n A_n$ with $A_n \subseteq X$ F_σ (replace A_n by $A_n \cap X$), choose special $\mathcal{S}_n$ with $A_n = \bigcup \mathcal{S}_n$, and put $\mathcal{F}_1 = \mathcal{S}_1$, $\mathcal{F}_{n+1} = \{E \cap F : E \in \mathcal{F}_n, F \in \mathcal{S}_{n+1}\}$. For two members $E \cap F$, $E' \cap F'$: if $E \neq E'$ compare E, E' , and if $E = E'$ compare F, F' ; subsets of disjoint (splitting) sets are disjoint (split). So $\mathcal{F}_{n+1}$ is special, $\bigcup \mathcal{F}_n = A_1 \cap \dots \cap A_n$, and $\bigcap_n \bigcup \mathcal{F}_n = A$. $\square$

Definition 7.7 (special functions). Let $0 < \beta < \alpha < \theta < \pi/2$ and $\mathcal{F}$ special. A *pair of special α, β, θ functions* for $\mathcal{F}$ is a pair (ε, δ) of positive functions on $E(\mathcal{F})$ and $\mathcal{F}^2$ such that: (i) for each $\eta > 0$ only finitely many $F \in \mathcal{F}^2$ have $\delta(F) \geq \eta$; (ii) for each $\eta > 0$ only finitely many $x \in E(\mathcal{F})$ have $\varepsilon(x) \geq \eta$; (iii) the sets $s(x, \varepsilon(x), \theta)$, $x \in E(\mathcal{F})$, are pairwise disjoint; (iv) the sets $B(F, \delta(F), \alpha, \beta)$, $F \in \mathcal{F}^2$, are pairwise disjoint; (v) $s(x, \varepsilon(x), \theta) \cap B(F, \delta(F), \alpha, \beta) = \emptyset$ for $x \in E(\mathcal{F})$, $F \in \mathcal{F}^2$.

Lemma 7.8 ([8, (XX)]; [3, Lemma 13]). *Every special family has a pair of special α, β, θ functions.*

Proof. If $\mathcal{F} = \emptyset$ there is nothing to prove. Otherwise enumerate $\mathcal{F} = \{F_0, F_1, \dots\}$ (repetitions allowed); for $F \in \mathcal{F}$ let $k(F)$ be its least index, and for $e \in E(\mathcal{F})$ let $k(e)$ be the least k such that $F_k \neq \emptyset$ and e is an endpoint of $J(F_k)$. Put $c = \cot \theta + \cot \beta$ and $t^+ = t$ if $t > 0$, $t^+ = 1$ otherwise; for disjoint compact K, L let $\sigma(K, L) > 0$ be a lower bound of $|x - y|$, $x \in K$, $y \in L$ (and $\sigma(K, L) = 1$ if K, L meet). Define

$$\delta(F) = \min_{j \leq k(F)} \min \left\{ \frac{1}{j+1}, \left(\frac{\sigma(F_j, F)}{2 \cot \beta} \right)^+, \left(\frac{\text{dist}(\min F_j, F)}{c} \right)^+, \left(\frac{\text{dist}(\max F_j, F)}{c} \right)^+ \right\},$$

$$\varepsilon(e) = \min_{j \leq k(e)} \min \left\{ \frac{1}{j+1}, \left(\frac{|e - \min F_j|}{2 \cot \theta} \right)^+, \left(\frac{|e - \max F_j|}{2 \cot \theta} \right)^+, \left(\frac{\text{dist}(e, F_j)}{c} \right)^+ \right\}.$$

These are minima of finitely many positive numbers. Condition (i): $\delta(F) \geq \eta$ forces $k(F) + 1 \leq 1/\eta$; (ii): $\varepsilon(e) \geq \eta$ forces $k(e) + 1 \leq 1/\eta$, and each index contributes at most two endpoints. (iii) If $e \neq e'$, say $k(e') \leq k(e)$, then e' is an endpoint of $J(F_{k(e')})$, so $\varepsilon(e) \leq |e - e'|/(2 \cot \theta)$ and (R4) applies. (iv) If $F \neq F'$ in $\mathcal{F}^2$ split, use (R5); if they are disjoint, say $k(F') < k(F)$, then $\delta(F) \leq \sigma(F', F)/(2 \cot \beta)$ and (R6) applies. (v) Let $e \in E(\mathcal{F})$ and $F \in \mathcal{F}^2$. If $e \notin J(F)^*$ use (R8) ($\beta < \alpha < \theta$). Otherwise $e \notin F$ (remark after Definition 7.4), so $\text{dist}(e, F) > 0$. If $k(F) \leq k(e)$ then $\varepsilon(e) \leq \text{dist}(e, F)/c$ and the first half of (R9) applies; if $k(F) > k(e)$ then e is an endpoint of $J(F_{k(e)})$, so $\delta(F) \leq \text{dist}(e, F)/c$ and the second half of (R9) applies. $\square$

Lemma 7.9 ([8, (XXI)]; [3, Lemma 14]). *If (ε, δ) is a special pair and $0 < \varepsilon_1 \leq \varepsilon$, $0 < \delta_1 \leq \delta$, then $(\varepsilon_1, \delta_1)$ is a special pair.*

Proof. The regions shrink as the parameters ε, δ decrease. $\square$

7.3. The function g .

Theorem 7.10 ([8, (A)]; [3, Theorem 4, proof]). *Let $A \subseteq X$ be $F_{\sigma\delta}$. There is a continuous $g : \mathbb{H} \rightarrow [0, 1]$ such that (a) for each $x \in A$, $g(z) \rightarrow 0$ as $z \rightarrow x$ through $s(x, 1, 3\pi/8)$; and (b) if $x \in \mathbb{R}$ and $g \rightarrow 0$ along some arc at x in $\mathbb{H}$, then $x \in A$.*

Proof. Take $\mathcal{F}_n$ as in Lemma 7.6 and $E_n = E(\mathcal{F}_n)$. Fix strictly monotone sequences $\beta_n \uparrow \pi/4$ in $(0, \pi/4)$, $\alpha_n \downarrow \pi/4$ in $(\pi/4, \pi/3)$, $\theta_n \uparrow 3\pi/8$ in $(\pi/3, 3\pi/8)$ (Lean: $t_n = \pi/(24(n+2))$, $\beta_n = \pi/4 - 2t_n$, $\alpha_n = \pi/4 + 2t_n$, $\theta_n = 3\pi/8 - t_n$). For a special pair (ε, δ) of $\mathcal{F}_n$ let U be the union of the pieces $s(x, \varepsilon(x), \theta_n)$, $x \in E_n$, and $B(F, \delta(F), \alpha_n, \beta_n)$, $F \in \mathcal{F}_n^2$.

Properties of U . The pieces are open and pairwise disjoint ((iii)–(v) of Definition 7.7). Each piece lies in $\{y < \varepsilon(x)\}$ or $\{y < \delta(F)\}$, so by (i), (ii) only finitely many pieces meet $\{y > y_0/2\}$; hence a point $\langle x_0, y_0 \rangle \in \text{Cl } U \cap \mathbb{H}$ lies in the closure of one piece. By (R2) the closure of a piece meets $\mathbb{R}$ only in $\bigcup \mathcal{F}_n$ (an $x \in E_n$ lies in some member of $\mathcal{F}_n$).

Small angles and regions. (1) If $x \in F \in \mathcal{F}_n$ and $\theta_n < \theta' < \pi/2$, there is $\zeta > 0$ with $\text{Cl } s(x, \zeta, \theta') \cap \mathbb{H} \subseteq U$: if x is an endpoint of $J(F)$, then $x \in E_n$ and (R3) with $\zeta = \varepsilon(x)/2$ applies; otherwise $x \in F \cap J(F)^*$, so $F \in \mathcal{F}_n^2$ and (R10) (with $\beta_n < \theta'$) applies to $B(F, \delta(F), \alpha_n, \beta_n)$. (2) If $K \in \mathcal{F}_{n+1}^2$, $K \subseteq F \in \mathcal{F}_n$, then $F \in \mathcal{F}_n^2$, and (R7) with $\eta = \delta(F)/2$ gives $\text{Cl } B(K, \eta, \alpha_{n+1}, \beta_{n+1}) \cap \mathbb{H} \subseteq U$.

Construction. Let $(\varepsilon_0, \delta_0)$ be any special $\alpha_0, \beta_0, \theta_0$ pair for $\mathcal{F}_0$. Given $(\varepsilon_n, \delta_n)$ with set U_n , take any special $\alpha_{n+1}, \beta_{n+1}, \theta_{n+1}$ pair (ε, δ) for $\mathcal{F}_{n+1}$, and let $\varepsilon_{n+1}(x) = \min\{\varepsilon(x), \zeta(x)\}$ for $x \in E_{n+1}$ with $\zeta(x)$ from (1) (for $\theta' = \theta_{n+1}$ and a member of $\mathcal{F}_n$ containing x), and $\delta_{n+1}(K) = \min\{\delta(K), \eta(K)\}$ with $\eta(K)$ from (2). By Lemma 7.9 this is a special pair, and by the properties of U_{n+1} , $\text{Cl } U_{n+1} \cap \mathbb{H} \subseteq U_n$; in particular $U_{n+1} \cap \mathbb{H} \subseteq U_n$.

The function. Let $g_n : \mathbb{H} \rightarrow [0, 1]$ be continuous with $g_n = 1$ on $\mathbb{H} \setminus U_n$ and $g_n = 0$ on $\text{Cl } U_{n+1} \cap \mathbb{H}$ (Urysohn in the metric space $\mathbb{H}$), and $g = \sum_n 2^{-n-1} g_n$. Then g is continuous with values in $[0, 1]$, and

$$g \geq 2^{-n-1} \text{ on } \mathbb{H} \setminus U_n, \quad g \leq 2^{-n-1} \text{ on } U_{n+1}, \quad (7.1)$$

the latter because $g_m = 0$ on $U_{n+1} \subseteq \text{Cl } U_{m+1}$ for $m \leq n$.

(a) Let $x_0 \in A$ and $n \in \mathbb{N}$. Then $x_0 \in F$ for some $F \in \mathcal{F}_{n+1}$, and (1) with $\theta' = 3\pi/8$ gives $\zeta > 0$ with $s(x_0, \zeta, 3\pi/8) \subseteq U_{n+1}$. So $g \leq 2^{-n-1}$ on $s(x_0, 1, 3\pi/8) \cap \{|z - x_0| < \zeta\}$.

(b) If $g \rightarrow 0$ along an arc γ at x , then for each n there is $r > 0$ with $g < 2^{-n-1}$ on $(\gamma \setminus \{x\}) \cap S(r, x)$. Parametrizing γ from x , a connected set $C = p((0, t_0])$ lies in this set and has $x \in \overline{C}$ (Lean: `TJK.IsArcAt.exists.tail`). By (7.1), $C \subseteq U_n$; a connected subset of a union of disjoint open pieces lies in one piece, so x lies in the closure of a piece, hence $x \in \bigcup \mathcal{F}_n$. Thus $x \in \bigcap_n \bigcup \mathcal{F}_n = A$. $\square$

In Lean, g is a function $\mathbb{C} \rightarrow \mathbb{R}$, continuous on $\mathbb{H}$ and mapping $\mathbb{H}$ into $[0, 1]$, and (a) is $g \rightarrow 0$ along $\mathcal{N}_{s(x, 1, 3\pi/8)}(x)$.

Theorem 7.11 ([3, Theorem 4]). *For every $F_{\sigma\delta}$ set $A \subseteq \mathbb{R}$ there is a bounded continuous $f : \mathbb{H} \rightarrow \mathbb{C}$ with $\text{CC}(f) = A$.*

Proof. Let $A' = \Psi_{\mathbb{H}}^{-1}(A) \subseteq X$, an $F_{\sigma\delta}$ set, g as in Theorem 7.10 for A' , and $f_0 = g \sin(1/y) + ig$ on Q° . If $x \in A'$, $f_0 \rightarrow 0$ along a short vertical segment at x (which lies in $Q^\circ \cap s(x, 1, 3\pi/8)$ apart from x), since $|f_0| \leq 2g$. If $f_0 \rightarrow c + di$ along an arc γ at x , then $g \rightarrow d$; if $d \neq 0$, then since $y \rightarrow 0$ continuously along γ , every tail of γ contains points with $\sin(1/y) = 1$ and with $\sin(1/y) = -1$, so $\text{Re } f_0$ has no limit. Hence $d = 0$ and $x \in A'$ by Theorem 7.10(b). So $\text{CC}(f_0) = A'$, and $f = f_0 \circ \Psi_{\mathbb{H}}^{-1}$ has $\text{CC}(f) = A$ by Lemma 4.18. $\square$

7.4. The function h .

Lemma 7.12 ([8, (B)(I)]; [3, Lemma 17]). *(a) If $f : (-1, 1) \rightarrow \mathbb{R}$ is strictly increasing, there is a continuous weakly increasing $f^* : \mathbb{R} \rightarrow [-1, 1]$ with $f^*(f(x)) = x$ for all x . (b) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing and $f(\mathbb{R})$ is unbounded above and below, there is a continuous weakly increasing $f^* : \mathbb{R} \rightarrow \mathbb{R}$ with $f^* \circ f = \text{id}$.*

Proof. (a) $f^*(t) = \sup(\{-1\} \cup \{x : f(x) \leq t\})$. It is weakly increasing, and $\{x : f(x) \leq f(x_0)\} = (-1, x_0]$ gives $f^*(f(x_0)) = x_0$. Its range contains $(-1, 1)$, so it is dense in $[-1, 1]$, and a monotone map onto a dense subset of an interval is continuous. (b) The same with $f^*(t) = \sup\{x : f(x) \leq t\}$, whose range is $\mathbb{R}$. $\square$

Lemma 7.13 ([8, (B)(II)]; [3, Lemma 16]). *Let M be a metric space and $u : M \rightarrow \mathbb{R}$. Suppose that whenever $p_n \rightarrow p$ in M and $u(p_n) \rightarrow y$ in $[-\infty, +\infty]$, we have $y \in \mathbb{R}$ and $y = u(p)$. Then u is continuous.*

Proof. If $u(p_n) \not\rightarrow u(p)$, a subsequence converges in $[-\infty, \infty]$ to some $y \neq u(p)$. $\square$

Theorem 7.14 ([8, (B)]; [3, Theorem 6]). *Let $E \subseteq X$ and let $\varphi : E \rightarrow \mathbb{R}^m$ be of honorary Baire class ≤ 2 . There is a continuous $h : \mathbb{H} \rightarrow \mathbb{R}^m$ such that for each $p \in E$ there is a segment λ_p at p , contained in $s(p, 1, 3\pi/8) \cup \{p\}$, along which $h \rightarrow \varphi(p)$. If φ is bounded, h can be taken bounded.*

Proof. Let $\varphi = \psi$ on $E \setminus N_0$ with N_0 countable and $\psi \in \mathcal{B}_1(E, \mathbb{R}^m)$. Fix an injection $e : N_0 \rightarrow \mathbb{N}$, put $s_{e(x)} = x$ for $x \in N_0$ and $s_n = 1$ for $n \notin e(N_0)$; so $s_n = s_k \in X$ implies $n = k$. Let $w(n) = 2^{-n-3}$ and

$$c(x) = \sum_{n: s_n < x} w(n) \in [0, \tfrac{1}{4}],$$

which is weakly increasing with $c(q) \geq c(s_n) + w(n)$ whenever $q > s_n$.

The function u . For $y > 0$ let $u(x, y) = \sup(\{-1\} \cup \{p \in X : p + yc(p) \leq x\})$. Since $p \mapsto p + yc(p)$ is strictly increasing, $x \mapsto u(x, y)$ is continuous and weakly increasing (proof of Lemma 7.12(a)). As $0 \leq c \leq 1$, $u(x - |y - y'|, y) \leq u(x, y') \leq u(x + |y - y'|, y)$, so u is continuous

on $\mathbb{H}$. Clearly $u(x, y) = p$ if $p \in X$, $p + yc(p) \leq x$, and $x < q + yc(q)$ for all $q \in X$ with $q > p$. Hence (*) $u = p$ along the line $x = p + ay$ whenever $p \in X$ and $c(p) \leq a \leq c(q)$ for all $q > p$.

The function h_0 . By Theorem 3.14 there are continuous $g_n : \mathbb{R} \rightarrow \mathbb{R}^m$ with $g_n \rightarrow \psi$ on E . With $\Delta(t) = (1 - |t|) \vee 0$ put

$$h_0(x, y) = \sum_{n \geq 0} \Delta(1/y - n) g_n(u(x, y)).$$

The terms vanish for $n \geq 1/y + 1$, so the sum is locally finite and h_0 is continuous on $\mathbb{H}$. For $n \leq 1/y \leq n + 1$ it is a convex combination of $g_n(u)$ and $g_{n+1}(u)$; so if $u = p \in E$ along a line at p , then $h_0 \rightarrow \psi(p)$ along it.

The corrections. Let $z_n = \varphi(s_n) - \psi(s_n)$ if $s_n \in E$ and $z_n = 0$ otherwise, and

$$\Lambda_n(x, y) = [(1 - ny) \vee 0] T_n\left(\frac{x - s_n}{y}\right) z_n, \quad T_n(\tau) = \left(1 - \frac{|2\tau - 2c(s_n) - w(n)|}{w(n)}\right) \vee 0.$$

T_n vanishes outside $(c(s_n), c(s_n) + w(n))$ and equals 1 at the midpoint; Λ_n is continuous on $\mathbb{H}$ and vanishes for $ny \geq 1$. So $h = h_0 + \sum_n \Lambda_n$ is continuous on $\mathbb{H}$.

Points of $E \setminus N_0$. Let λ be the line $x = p + c(p)y$, $y > 0$. By (*), $u = p$ on λ . There $\tau_n = (x - s_n)/y = (p - s_n)/y + c(p)$ is $> c(s_n) + w(n)$ if $s_n < p$ and $\leq c(s_n)$ if $s_n \geq p$, so every Λ_n vanishes on λ , and $h \rightarrow \psi(p) = \varphi(p)$.

Points of N_0 . Let $p = s_k$ and let λ' be the line $x = p + (c(p) + w(k)/2)y$, $y > 0$. By (*), $u = p$ on λ' . For $n \neq k$ with $z_n \neq 0$ we have $s_n \in X$, so $s_n \neq s_k$, and $\tau_n = (s_k - s_n)/y + c(s_k) + w(k)/2$ is $> c(s_n) + w(n)$ if $s_n < s_k$ and $< c(s_n)$ if $s_n > s_k$; so $\Lambda_n = 0$ on λ' , while $\Lambda_k = [(1 - ky) \vee 0] z_k \rightarrow z_k$. Thus $h \rightarrow \psi(p) + z_k = \varphi(p)$ along λ' .

The slopes lie in $[0, \frac{5}{16}]$, and $\frac{5}{16} < \sqrt{2} - 1 = \tan(\pi/8) = \cot(3\pi/8)$, so the parts of λ, λ' with $0 < y < 1$ lie in $s(p, 1, 3\pi/8)$. If $\|\varphi\| \leq b$ on E (sup norm), clamp each coordinate of h to $[-b, b]$; the limits along the segments are unchanged. $\square$

7.5. The function k .

Theorem 7.15 ([8, (C)]). *Let $d : (0, 1] \rightarrow (0, \infty)$ be weakly increasing. There is a continuous $k : \mathbb{R} \times (0, 1] \rightarrow \mathbb{C}$ with $|k| \leq \sqrt{2}$ such that for every $a \in (0, 1]$ and every connected $\gamma \subseteq [-1, 1] \times (0, a]$ containing two points at distance $\geq d(a)$, there are $p, q \in \gamma$ with $|k(p) - k(q)| \geq 2$.*

Proof. Let $p(x) = \inf_{t \in (0, 1]} (\frac{1}{2}d(t) + |x - t|)$. As an infimum of 1-Lipschitz functions, p is 1-Lipschitz. It is weakly increasing on $(0, 1]$: for $x \leq x'$ and $t \in (0, 1]$, if $t \leq x$ the term at x is at most the term at x' , and if $t > x$ then $p(x) \leq \frac{1}{2}d(x) \leq \frac{1}{2}d(t)$. For $x \in (0, 1]$ we have $0 < \min\{\frac{1}{2}d(x/2), x/2\} \leq p(x) \leq \frac{1}{2}d(x)$ (split $t \geq x/2$ and $t < x/2$). Let $m(y) = \min\{p(y), y/2, p(y/2)^2/2\}$: continuous, positive and weakly increasing on $(0, 1]$. Define

$$k_1(y) = \sin \frac{2\pi}{y m(y)}, \quad k_2(x, y) = \sin \frac{4\pi x}{p(y)}, \quad k = k_1 + ik_2;$$

k is continuous on $\mathbb{R} \times (0, 1]$ and $|k| \leq \sqrt{2}$. Let $z_1, z_2 \in \gamma$ with $|z_1 - z_2| \geq d(a)$, $z_j = \langle x_j, y_j \rangle$, $y_2 \leq y_1 \leq a$. Either

$$(12) \quad y_1 - y_2 \geq m(y_1), \quad \text{or} \quad (13) \quad y_1 - y_2 < m(y_1).$$

In case (12), using $m(y_2) \leq m(y_1)$ and $y_1 y_2 \leq 1$,

$$\frac{2\pi}{y_2 m(y_2)} - \frac{2\pi}{y_1 m(y_1)} \geq \frac{2\pi(y_1 - y_2)}{y_1 y_2 m(y_1)} \geq \frac{2\pi m(y_1)}{y_1 y_2 m(y_1)} \geq 2\pi.$$

The continuous function $2\pi/(y m(y))$ on the connected set γ therefore takes all values in an interval of length 2π , so k_1 takes the values 1 and -1 on γ . In case (13), $y_1 - y_2 < m(y_1) \leq y_1/2$,

so with $c = p(y_1/2) > 0$ we have $p(y_1), p(y_2) \geq c$ and $|p(y_1) - p(y_2)| \leq y_1 - y_2 < m(y_1) \leq c^2/2$; hence $|p(y_1)^{-1} - p(y_2)^{-1}| \leq \frac{1}{2}$. Also $d(y_1) \leq d(a)$ and $y_1 - y_2 < m(y_1) \leq p(y_1) \leq \frac{1}{2}d(y_1)$, so $|x_1 - x_2| \geq d(a) - (y_1 - y_2) \geq \frac{1}{2}d(y_1) \geq p(y_1)$. With $|x_2| \leq 1$,

$$\left| \frac{4\pi x_1}{p(y_1)} - \frac{4\pi x_2}{p(y_2)} \right| \geq 4\pi \left[\frac{|x_1 - x_2|}{p(y_1)} - \left| \frac{1}{p(y_1)} - \frac{1}{p(y_2)} \right| \right] \geq 4\pi \left[1 - \frac{1}{2} \right] = 2\pi,$$

and k_2 takes the values ± 1 on γ . In either case some $p, q \in \gamma$ have $|k(p) - k(q)| \geq 2$. $\square$

7.6. Assembly.

Theorem 7.16 ([8, (D)]). *Let $A \subseteq X$ be $F_{\sigma\delta}$ and $\varphi : A \rightarrow \mathbb{C}$ bounded and of honorary Baire class ≤ 2 . There is a bounded continuous $f : Q \rightarrow \mathbb{C}$ such that A is the set of curvilinear convergence of f (arcs in Q°) and φ is a boundary function for f .*

Proof. Let g be given by Theorem 7.10 and h (bounded, with $\mathbb{C} = \mathbb{R}^2$) by Theorem 7.14 for $E = A$, both restricted to Q . For $t \in (0, 1]$ let $D(t)$ be the supremum of the $\delta \in (0, 1]$ such that $z_1, z_2 \in [-1, 1] \times [t, 1]$, $|z_1 - z_2| < \delta$ imply $|h(z_1) - h(z_2)| \leq t$ and $|g(z_1) - g(z_2)| \leq t$; it is positive by uniform continuity on the compact set $[-1, 1] \times [t, 1]$, and weakly increasing. Let $d(t) = \min\{\frac{1}{4}D(t/2), t/4\}$, k as in Theorem 7.15, and $f = h + gk$; f is bounded since h, g, k are.

If $x \in A$, $h \rightarrow \varphi(x)$ along a segment in $s(x, 1, 3\pi/8)$, along which $gk \rightarrow 0$; a short initial piece of this segment lies in Q° (Lemma 4.5). So φ is a boundary function and $A \subseteq \text{CC}(f)$. Conversely let $f \rightarrow q$ along an arc γ at $x \in X$ in Q° ; we show $g \rightarrow 0$ along γ , so $x \in A$ by Theorem 7.10(b). Let $\varepsilon \in (0, 1]$ and $r > 0$ with $|f(z) - q| < \varepsilon/2$ for $z \in \gamma \setminus \{x\}$, $|z - x| < r$. Let $z_0 = \langle x_0, y \rangle \in \gamma \setminus \{x\}$ with $|z_0 - x| < \min\{r/2, 3\varepsilon/16\}$, and suppose $g(z_0) \geq \varepsilon$. Let $a = 4y/3 \leq 1$. As $|x - z_0| \geq y > d(a)$, Lemma 4.5(b), applied to a parametrization of γ from x to z_0 , gives a connected $\gamma' \subseteq \gamma \cap S(d(a), z_0)$ containing z_0 and a point at distance $d(a)$ from it. Since $d(a) \leq a/4 = y/3$, $\gamma' \subseteq ([-1, 1] \times [a/2, a]) \setminus \{x\}$, and Theorem 7.15 gives $p, q' \in \gamma'$ with $|k(p) - k(q')| \geq 2$. As $|p - q'| \leq 2d(a) < D(a/2)$ and $|p - z_0|, |q' - z_0| \leq d(a) < D(a/2)$, we get $|h(p) - h(q')| \leq a/2$ and $|g(p) - g(z_0)|, |g(q') - g(z_0)| \leq a/2 < \varepsilon/8$. Hence

$$|f(p) - f(q')| \geq g(z_0)|k(p) - k(q')| - |k(p)||g(p) - g(z_0)| - |k(q')||g(q') - g(z_0)| - |h(p) - h(q')| > 2\varepsilon - \frac{2\sqrt{2}+1}{8}\varepsilon > \varepsilon.$$

But $|p - x|, |q' - x| \leq y/3 + |z_0 - x| < r$, so $|f(p) - f(q')| < \varepsilon$, a contradiction. Hence $g < \varepsilon$ on $\gamma \setminus \{x\}$ near x . $\square$

Theorem 7.17 ([8, (E)]). *Let $A \subseteq C$ be $F_{\sigma\delta}$ and $\varphi : A \rightarrow \mathbb{C}$ bounded and of honorary Baire class ≤ 2 . There is a bounded continuous $f : \mathbb{D} \rightarrow \mathbb{C}$ with $\text{CC}(f) = A$ such that φ is a boundary function for f .*

Proof. If $A = \emptyset$ take $f(z) = \sin(1/(1 - |z|))$: along any arc at $x \in C$, every tail contains a connected set on which $1/(1 - |z|)$ takes all values in some $[t_0, \infty)$, so f takes the values 1 and -1 there and has no limit. Otherwise rotate so that $1 \in A$ (the rotation is a chart of $\mathbb{D} \cup C$ onto itself, Lemma 4.18). Let $A' = \Psi_{\mathbb{D}}^{-1}(A \setminus \{1\}) \subseteq X$, which is $F_{\sigma\delta}$, and apply Theorem 7.16 to A' and $\varphi \circ \Psi_{\mathbb{D}}$ (of honorary Baire class ≤ 2 by Lemma 3.23), obtaining f_1 ; put $f_0 = f_1 \circ \Psi_{\mathbb{D}}^{-1}$ on G° . On the set G'' of Lemma 4.13(c), which is closed in $\mathbb{D}$, f_0 is continuous and bounded; set $F = f_0$ on G'' and $F = \varphi(1)$ on the radius $[0, 1]$, which is closed in $\mathbb{D}$ and disjoint from G'' . By the Tietze extension theorem (Fact 4.7) F extends to a bounded continuous f on $\mathbb{D}$. Along the radius $f \rightarrow \varphi(1)$. Each $x \in C \setminus \{1\}$ has a neighbourhood W with $W \cap \mathbb{D} \subseteq G''$, so by Lemmas 4.19 and 4.18 the arc limits of f at x are those of f_1 at $\Psi_{\mathbb{D}}^{-1}(x)$. $\square$

Theorem 7.18 ([8, (F)]; McMillan's problem). *Let $A \subseteq C$ be $F_{\sigma\delta}$ and $\varphi : A \rightarrow \mathbb{S}^2$ of honorary Baire class ≤ 2 . There is a continuous $f : \mathbb{D} \rightarrow \mathbb{S}^2$ with $\text{CC}(f) = A$ such that φ is a boundary function for f .*

Proof. Let $\varphi = \psi$ off a countable N with $\psi \in \mathcal{B}_1(A, \mathbb{S}^2)$, and $A_1 = A \setminus N$. By Lemmas 3.7 and 3.19, applied in the metric space A_1 , there is $\psi_0 : A_1 \rightarrow Z_U \cup Z_V$ with $\Phi \circ \psi_0 = \varphi|_{A_1}$ and $\sigma \circ \psi_0$ of class F_σ . For $x \in N$ choose $\psi_0(x) \in Z_U \cup Z_V$ with $\Phi(\psi_0(x)) = \varphi(x)$ (possible since $\Phi(Z_U \cup Z_V) = \mathbb{S}^2$). Then $\sigma \circ \psi_0 : A \rightarrow \mathbb{R}^2$ is bounded (Lemma 3.18(v)) and of honorary Baire class ≤ 2 (Theorem 3.22). By Theorem 7.17 there is a bounded continuous $f_1 : \mathbb{D} \rightarrow \mathbb{C}$ with $\text{CC}(f_1) = A$ and boundary function $\sigma \circ \psi_0$. Put $f = \Phi \circ \sigma^{-1} \circ f_1$. Then $f \rightarrow \Phi(\psi_0(x)) = \varphi(x)$ along the arcs at $x \in A$. If $f \rightarrow q$ along an arc γ at $x \in C$, let K be a closed disk containing $f_1(\mathbb{D})$ and $\Theta = \Phi \circ \sigma^{-1}|_K$, which is at most three-to-one. By Lemma 4.6, $\Theta(C(f_1, \gamma)) = C(f, \gamma) = \{q\}$, so $C(f_1, \gamma) \subseteq \Theta^{-1}(q)$ is finite; being nonempty and connected (in a T_1 space a connected set with two points is infinite), it is a single point $\{w\}$, and $f_1 \rightarrow w$ along γ . Hence $x \in \text{CC}(f_1) = A$. $\square$

Theorem 7.19 (half-plane versions; [3, Theorems 4, 6, 7]). (a) *If $A \subseteq \mathbb{R}$ is $F_{\sigma\delta}$ and $\varphi : A \rightarrow \mathbb{C}$ is bounded and of honorary Baire class ≤ 2 , there is a bounded continuous $f : \mathbb{H} \rightarrow \mathbb{C}$ with $\text{CC}(f) = A$ and boundary function φ . The same holds with $\mathbb{S}^2$ in place of $\mathbb{C}$ (and f continuous).* (b) ([3, Theorem 6]) *If $E \subseteq \mathbb{R}$ is arbitrary and $\varphi : E \rightarrow \mathbb{R}^m$ is of honorary Baire class ≤ 2 , there is a continuous $f : \mathbb{H} \rightarrow \mathbb{R}^m$ having φ as a boundary function.* (c) ([3, Theorem 7]) *The same with $\mathbb{S}^2$ in place of $\mathbb{R}^m$.*

Proof. (a) Transfer Theorem 7.16 by $\Psi_{\mathbb{H}}$ (Lemma 4.18); no exceptional point occurs. For $\mathbb{S}^2$ argue as in Theorem 7.18. (b) Apply Theorem 7.14 to $\Psi_{\mathbb{H}}^{-1}(E)$ and $\varphi \circ \Psi_{\mathbb{H}}$; the segments have tails in Q° ; transfer. (c) Write $\varphi = \Phi \circ \sigma^{-1} \circ \chi$ with $\chi : E \rightarrow \mathbb{R}^2$ bounded and of honorary Baire class ≤ 2 , exactly as in Theorem 7.18, apply (b) to χ , and compose with $\Phi \circ \sigma^{-1}$. $\square$

8. BOUNDARY FUNCTIONS OF DISCONTINUOUS FUNCTIONS

Sources: [3, Chapter II]; [1, Theorems 4–6]. We work in $\mathbb{H}$ with boundary $\mathbb{R}$.

8.1. Baire functions.

Definition 8.1. For arcs γ_x, γ_y at $x, y \in \mathbb{R}$ and $c > 0$, γ_x *meets* γ_y *below height* c if there are subarcs $\gamma'_x \subseteq \gamma_x$ at x and $\gamma'_y \subseteq \gamma_y$ at y , contained in $\{\text{Im } z \leq c\}$, with $\gamma'_x \cap \gamma'_y \neq \emptyset$. This relation is symmetric, and if it holds for c it holds for every $c' \geq c$.

Lemma 8.2 (the “Figure 5” argument of [3, proof of Theorem 8]). *Let $x < t < y$ be real, let α, β be arcs at x, y in $\mathbb{H}$, contained in $\{\text{Im } z \leq c\}$, with $\alpha \cap \beta \neq \emptyset$, and let τ be an arc at t in $\mathbb{H}$ whose other endpoint p has $\text{Im } p = c$. Then τ meets $\alpha \cup \beta$.*

Proof. $\alpha \cup \beta$ is path connected, so by Fact 4.3 it contains a simple arc σ from x to y , a crosscut of $\mathbb{H}$ with $\max_\sigma \text{Im} \leq c$. Suppose $\tau \cap \sigma = \emptyset$, and let V, U be the bounded and unbounded components of $\mathbb{H} \setminus \sigma$ (Lemma 4.14). The connected set $\tau \setminus \{t\}$ lies in $U \cup V$, hence in U or in V ; and $t \in \overline{\tau \setminus \{t\}}$. If it lies in U , then $t \in \overline{U} \cap \mathbb{R} = \mathbb{R} \setminus (x, y)$, which is false. If it lies in V , then $p \in V$, so $p + i\varepsilon \in V$ for small $\varepsilon > 0$; but $p + i\varepsilon \in U$ since it lies above σ . Contradiction. $\square$

Theorem 8.3 ([3, Theorem 8]; [1, Theorem 4]). *Let Y be a separable arcwise connected metric space, $\xi \geq 1$, $f : \mathbb{H} \rightarrow Y$ of Baire class ξ , $E \subseteq \mathbb{R}$, and $\varphi : E \rightarrow Y$ a boundary function for f . Then φ is of Baire class $\xi + 1$.*

Proof. Let $U \subseteq Y$ be open, $V = Y \setminus \overline{U}$, $A = \varphi^{-1}(U)$, $B = \varphi^{-1}(V)$, $P = A \cup B$. By Lemma 3.38 (with $\xi + 1 \geq 2$) it suffices to find $T \in P^{\xi+2}(\mathbb{R})$ with $A \subseteq T$ and $T \cap B = \emptyset$: then $\varphi^{-1}(U) \subseteq T \cap E \subseteq \varphi^{-1}(\overline{U})$ and $T \cap E \in P^{\xi+2}(E)$. For $x \in P$ choose (Lemma 4.5) an arc γ_x at x with $f \rightarrow \varphi(x)$ along it, $\gamma_x \subseteq \{|z - x| \leq 1\}$, and $f(\gamma_x \setminus \{x\}) \subseteq U$ if $x \in A$, $\subseteq V$ if $x \in B$. Then

$\gamma_x \cap \gamma_y = \emptyset$ for $x \in A$, $y \in B$. Write $c_n = 1/(n+1)$ and “meets at n ” for “meets below height c_n ”. Let

$$L = \{x \in P : \forall n \exists y \in P \setminus \{x\}, \gamma_x \text{ meets } \gamma_y \text{ at } n\}, \quad M = P \setminus L,$$

and let K_n be the set of $x \in P$ such that γ_x reaches the line $\text{Im } z = c_n$ and, if $x \in M$, γ_x meets no γ_y , $y \in P \setminus \{x\}$, at n . Every $x \in P$ is in some K_n , and every $x \in M$ is in K_n for all large n (an arc at x reaches every small height; and “meets” is monotone in the height).

Step 1: $L \cap A \approx G_a \cap P$ and $L \cap B \approx G_b \cap P$ for G_δ sets G_a, G_b . Let $x \in L \cap A$, $y \in P \setminus \{x\}$ with γ_x meeting γ_y at n . We claim $[x, y] \cap K_n \subseteq L \cap A$ (with $[x, y]$ the closed interval between x and y). First, $\gamma_x \cap \gamma_y \neq \emptyset$, so $y \in A$. Let $t \in [x, y] \cap K_n$, $t \neq x$. If $t = y$, γ_t meets γ_x at n . Otherwise let τ be the subarc of γ_t from t to its first point on $\text{Im } z = c_n$; by Lemma 8.2, τ meets one of the two meeting subarcs, so γ_t meets γ_x or γ_y at n . Since $t \in K_n$, this gives $t \notin M$, so $t \in L$; and γ_t meets an arc of a point of A , so $t \in A$. Let W_n be the union of the intervals $[x, y]$ over all such pairs (x, y) . Then $L \cap A = \bigcap_n (W_n \cap P)$: “ $\subseteq$ ” since $x \in [x, y]$; and if $t \in P \cap \bigcap_n W_n$, take m with $t \in K_m$, then $t \in L \cap A$ by the claim. By Lemma 3.34 each $W_n \approx O_n$ with O_n open; let $G_a = \bigcap_n O_n$. By Lemma 3.33, $L \cap A \approx G_a \cap P$. Exchanging A and B gives G_b .

Step 2: $M \cap A \approx D \cap M$ with $D \in P^{\xi+2}(\mathbb{R})$. For $x \in K_n$ let $p_n(x)$ be the first point of $\text{Im } z = c_n$ reached along γ_x from x . If $x < x'$ in $M \cap K_n$, the subarcs from x to $p_n(x)$ and from x' to $p_n(x')$ are disjoint (they would otherwise meet at n), so $\text{Re } p_n(x) < \text{Re } p_n(x')$ by Lemma 4.15. Also $|x - \text{Re } p_n(x)| \leq 1$. By Lemma 3.36 there is a weakly increasing $h_n : \mathbb{R} \rightarrow \mathbb{R}$ equal to $\text{Re } p_n$ on $M \cap K_n$; let $g_n(x) = f(h_n(x) + ic_n)$. By Lemmas 3.29 and 3.37 there is a countable N such that each $g_n|_{M \setminus N}$ is of Baire class ξ . For $x \in M$ and n large, $g_n(x) = f(p_n(x))$ with $p_n(x) \in \gamma_x \setminus \{x\}$ and $\text{Im } p_n(x) \rightarrow 0$; as γ_x is compact and meets $\mathbb{R}$ only at x , $p_n(x) \rightarrow x$ along γ_x , so $g_n(x) \rightarrow \varphi(x)$. Hence $\varphi|_{M \setminus N}$ is of Baire class $\xi + 1$, hence of class $P^{\xi+2}$ (Lemma 3.30), so $A \cap (M \setminus N) = D \cap (M \setminus N)$ for some $D \in P^{\xi+2}(\mathbb{R})$ (Lemma 3.28(f)).

Step 3. From Step 1, $L \approx (G_a \cup G_b) \cap P$, so $M \cap A \approx D \cap M \approx D \cap (P \setminus (G_a \cup G_b))$ and

$$A = (L \cap A) \cup (M \cap A) \approx T_0 \cap P, \quad T_0 = G_a \cup (D \setminus (G_a \cup G_b)) \in P^{\xi+2}(\mathbb{R}),$$

since $G_a \in Q^2 \subseteq P^{\xi+2}$ and $\mathbb{R} \setminus (G_a \cup G_b)$ is F_σ . With the countable sets $C_1 = A \setminus T_0$ and $C_2 = (T_0 \cap P) \setminus A$, the set $T = (T_0 \setminus C_2) \cup C_1$ is in $P^{\xi+2}(\mathbb{R})$ (countable sets are F_σ), contains A , and misses B . $\square$

Corollary 8.4 ([3, Theorem 8, Corollary]; [1, Theorems 4, 5]). *Let Y be a separable, arcwise connected metric space, $f : \mathbb{H} \rightarrow Y$ Borel measurable, and $\varphi : E \rightarrow Y$ a boundary function for f . Then φ is Borel measurable. In particular ([1, Theorems 4, 5]), if f is a real function of Baire class $\xi \geq 1$ in the upper half-plane, every finite-valued boundary function of f is of Baire class $\xi + 1$, and if f is Borel measurable so is every finite-valued boundary function.*

Proof. By Fact 3.32, f is of some Baire class $\xi < \omega_1$ (if $\xi = 0$, also of class 1); by Theorem 8.3, φ is of Baire class $\xi + 1$, hence Borel measurable (Lemma 3.30). $\mathbb{R}$ is separable and arcwise connected. $\square$

8.2. Lebesgue measure. $X_0 = \mathbb{R} \times \{0\}$, $X_1 = \mathbb{R} \times \{1\}$; m is planar Lebesgue measure, m^1 linear measure on a line, and m_e , m_e^1 the outer measures. A *segment* below means a closed line segment with one endpoint $\langle a_0(s), 0 \rangle \in X_0$ and the other $\langle a_1(s), 1 \rangle \in X_1$; write $\ell_s(y) = (1-y)a_0(s) + ya_1(s)$.

Definition 8.5 (trapezoids). For $a_0 \leq b_0$, $a_1 \leq b_1$ in $[-\infty, +\infty]$ with $a_i < +\infty$ and $b_i > -\infty$ (and $0 \cdot (\pm\infty) = 0$), let $T(a_0, b_0; a_1, b_1) = \{\langle x, y \rangle : 0 \leq y \leq 1, x \in \mathbb{R}, (1-y)a_0 + ya_1 \leq x \leq (1-y)b_0 + yb_1\}$, a *closed trapezoid*; $\emptyset$ is also one. S is a *trapezoid* if $S = \emptyset$ or $T^\circ \subseteq S \subseteq T$ for a closed trapezoid T , where $T^\circ = \{\langle x, y \rangle : 0 < y < 1, (1-y)a_0 + ya_1 < x < (1-y)b_0 + yb_1\}$

is the interior of T relative to the strip $\{0 \leq y \leq 1\}$. For segments s, s' , $T(s, s')$ is the set of $\langle x, y \rangle$ with $0 \leq y \leq 1$ and x between $\ell_s(y)$ and $\ell_{s'}(y)$; for disjoint s, s' with $a_i(s) \leq a_i(s')$ this is $T(a_0(s), a_0(s'); a_1(s), a_1(s'))$, and $T(s, s) = s$. (In Lean a segment is given by its pair of abscissas $\langle a_0(s), a_1(s) \rangle$.)

A trapezoid S is Lebesgue measurable and $m(S) = \frac{1}{2}[(b_0 - a_0) + (b_1 - a_1)] = \frac{1}{2}[m^1(S \cap X_0) + m^1(S \cap X_1)]$: integrate the lengths $(1 - y)(b_0 - a_0) + y(b_1 - a_1)$ of the horizontal sections, and note that $S \cap X_i$ lies between (a_i, b_i) and $[a_i, b_i]$. (The restrictions $a_i < +\infty$, $b_i > -\infty$ exclude the degenerate sides $a_i = b_i = \pm\infty$, for which the formula fails.) Only the case of finite parameters and $S = T$ is used below (`volume_closedTrapezoid`). Two disjoint segments $s \neq s'$ satisfy either $a_i(s) < a_i(s')$ for $i = 0, 1$ or $a_i(s) > a_i(s')$ for $i = 0, 1$ (otherwise $\ell_s - \ell_{s'}$, which is affine, changes sign on $[0, 1]$), hence either $\ell_s < \ell_{s'}$ on $[0, 1]$ or $\ell_s > \ell_{s'}$ on $[0, 1]$. In particular a set Σ of pairwise disjoint segments is totally ordered by $s \preceq s' \iff a_0(s) \leq a_0(s')$ and $a_1(s) \leq a_1(s')$.

Lemma 8.6 ([3, Lemma 23]). *Let s_1, s_2, s_3, s_4 be segments, any two equal or disjoint. If $T(s_1, s_2) \cap T(s_3, s_4) \neq \emptyset$ then $T(s_1, s_3) \subseteq T(s_1, s_2) \cup T(s_3, s_4)$.*

Proof. The horizontal section of $T(s_i, s_j)$ at height y is the interval between $\ell_{s_i}(y)$ and $\ell_{s_j}(y)$. The order of the four numbers $\ell_{s_i}(y)$ (with ties exactly between equal segments) does not depend on y . Whether two such intervals meet depends only on this order; so if the two sections meet at one height they meet at every height. The union of two meeting intervals is an interval, and it contains $\ell_{s_1}(y)$ and $\ell_{s_3}(y)$. $\square$

Lemma 8.7 ([3, Lemma 24]). *If Σ is a set of pairwise disjoint segments, $\bigcup_{s, s' \in \Sigma} T(s, s')$ is a trapezoid.*

Proof. We may assume $\Sigma \neq \emptyset$. Let $A_i = \inf_s a_i(s)$, $B_i = \sup_s a_i(s)$ (in $[-\infty, +\infty]$) and $T = T(A_0, B_0; A_1, B_1)$. Since $(1 - y)A_0 + yA_1 \leq \ell_s(y) \leq (1 - y)B_0 + yB_1$ for all s , the union lies in T . Let $\langle x, y \rangle \in T^\circ$. From $(1 - y)A_0 + yA_1 < x$ we get $s_1, s_2 \in \Sigma$ with $(1 - y)a_0(s_1) + ya_1(s_2) < x$ (if $y = 1$ or $y = 0$ only one parameter matters; if $A_0 = -\infty$ and $y < 1$ take any s_2 and $a_0(s_1)$ small; symmetrically if $A_1 = -\infty$ and $y > 0$; if both are finite approximate each within $\delta = \frac{1}{2}[x - (1 - y)A_0 - yA_1]$). The $\preceq$ -smaller s of s_1, s_2 has $\ell_s(y) \leq (1 - y)a_0(s_1) + ya_1(s_2) < x$. Likewise there is $s' \in \Sigma$ with $x < \ell_{s'}(y)$, so $\langle x, y \rangle \in T(s, s')$. $\square$

Theorem 8.8 ([3, Theorem 9]). *Let Σ be a set of pairwise disjoint segments and $S = \bigcup \Sigma$. Then $m_e(S) = \frac{1}{2}[m_e^1(S \cap X_0) + m_e^1(S \cap X_1)]$.*

Proof. Let $E_i = S \cap X_i$, identified with $\{a_i(s) : s \in \Sigma\} \subseteq \mathbb{R}$. (The dissertation proves both inequalities at once using Lemmas 8.6 and 8.7; we give a shorter argument for each.)

Upper bound. It suffices to show $m_e(S) \leq \frac{1}{2}[m^1(G_0) + m^1(G_1)]$ for all open $G_i \supseteq E_i$ (outer regularity); we may assume $m^1(G_i) < \infty$, so every component of G_i is a bounded open interval. Group the segments by the pair (I, J) of components of G_0, G_1 containing $a_0(s), a_1(s)$; there are countably many classes C . For a class C let α_C, β_C (γ_C, δ_C) be the infimum and supremum of $a_0(s)$ ($a_1(s)$) over $s \in C$. Every $s \in C$ lies in the closed trapezoid $T_C = T(\alpha_C, \beta_C; \gamma_C, \delta_C)$, of area $\frac{1}{2}[(\beta_C - \alpha_C) + (\delta_C - \gamma_C)]$, and $(\alpha_C, \beta_C) \subseteq I$, $(\gamma_C, \delta_C) \subseteq J$. The intervals (α_C, β_C) are pairwise disjoint: for classes with different I this is clear; if $C \neq C'$ have the same I but different (disjoint) J, J' and x lies in both intervals, take $s_1, s_2 \in C$, $s'_1, s'_2 \in C'$ with $a_0(s_1), a_0(s'_1) < x < a_0(s_2), a_0(s'_2)$; by the order of disjoint segments, $a_1(s_1) < a_1(s'_2)$ and $a_1(s'_1) < a_1(s_2)$, so either $a_1(s_1) \in [a_1(s'_1), a_1(s'_2)] \subseteq J'$ or $a_1(s'_1) \in [a_1(s_1), a_1(s_2)] \subseteq J$, contradicting $J \cap J' = \emptyset$. The same holds for (γ_C, δ_C) . Hence $m_e(S) \leq \sum_C m(T_C) = \frac{1}{2}[\sum_C (\beta_C - \alpha_C) + \sum_C (\delta_C - \gamma_C)] \leq \frac{1}{2}[m^1(G_0) + m^1(G_1)]$.

Lower bound. First a one-dimensional fact: if $Q \subseteq \mathbb{R}^2$ is totally ordered componentwise, then $m_e^1(\pi_1 Q) + m_e^1(\pi_2 Q) \leq m_e^1(\{q_1 + q_2 : q \in Q\})$. Indeed, let G be open containing the right-hand

set and group the $q \in Q$ by the component I of G containing $q_1 + q_2$. For q, q', r, r' in one class, with $u = q \wedge r$ and $v = q' \vee r' \vee u$ (again in the class, as Q is a chain), $(q'_1 - q_1)^+ + (r'_2 - r_2)^+ \leq (v_1 + v_2) - (u_1 + u_2) \leq m^1(I)$; taking suprema and infima, $m_e^1(\pi_1 Q_I) + m_e^1(\pi_2 Q_I) \leq m^1(I)$, and summing over the countably many classes gives the claim. Now for $0 < y < 1$ the section $S^y = \{x : \langle x, y \rangle \in S\} = \{(1-y)a_0(s) + ya_1(s)\}$; applying the fact to $Q = \{\langle (1-y)a_0(s), ya_1(s) \rangle\}$, which is a chain, gives $m_e^1(S^y) \geq (1-y)m_e^1(E_0) + ym_e^1(E_1)$. Let $M \supseteq S$ be a measurable hull. By Tonelli, $m_e(S) = m(M) = \int m^1(M^y) dy \geq \int_0^1 [(1-y)m_e^1(E_0) + ym_e^1(E_1)] dy = \frac{1}{2}[m_e^1(E_0) + m_e^1(E_1)]$. $\square$

Proposition 8.9 ([3, example after Theorem 9]). *Disjointness cannot be omitted in Theorem 8.8: there are sets $M_0 \subseteq X_0$, $M_1 \subseteq X_1$ of measure zero such that every point of $\{0 < y < 1\}$ lies on a segment from M_0 to M_1 .*

Proof. Let $M_0 = M_1 = M$ be a dense G_δ subset of $\mathbb{R}$ of measure zero (e.g. the Liouville numbers, or $\bigcap_n \bigcup_k (q_k - 2^{-n-k}, q_k + 2^{-n-k})$ for an enumeration q_k of $\mathbb{Q}$). Given $\langle x_0, y_0 \rangle$ with $0 < y_0 < 1$, the segment from $\langle a, 0 \rangle$ to $\langle g(a), 1 \rangle$, $g(a) = (x_0 - (1 - y_0)a)/y_0$, passes through $\langle x_0, y_0 \rangle$. As g is an affine bijection, $g^{-1}(M)$ is a dense G_δ , so $M \cap g^{-1}(M) \neq \emptyset$ by Baire's theorem; any a in it gives the required segment. $\square$

Lemma 8.10 ([3, Lemma 25]). *For every $\varepsilon > 0$ there is a strictly increasing $h : \mathbb{R} \rightarrow \mathbb{R}$ with $m^1(h(\mathbb{R})) = 0$ and $|x - h(x)| \leq \varepsilon$ for all x .*

Proof. Put $g(t) = \sum_{i \geq 0} [2^i t] 3^{-i}$ (for $t \in [0, 1]$ this is $\frac{3}{2}$ times the point of the Cantor set whose ternary digits are twice the binary digits of t). Each term is nondecreasing in t , and if $t < t'$ then $[2^i t] < [2^i t']$ once $2^i(t' - t) \geq 1$, so g is strictly increasing. Since $2^i t - 1 < [2^i t] \leq 2^i t$, we get $3t - \frac{3}{2} \leq g(t) \leq 3t$. If $k = [2^N t]$ and $a = k/2^N$, then $[2^i t] = [2^i a]$ for $i < N$ and $[2^{N+j} t] \leq k2^j + 2^j = [2^{N+j} a] + 2^j$, so $g(a) \leq g(t) \leq g(a) + \sum_j 2^j 3^{-N-j} = g(a) + 3^{1-N}$. Hence $g([m, m+1])$ ($m \in \mathbb{Z}$) is covered by 2^N intervals of length 3^{1-N} , so it is null, and $g(\mathbb{R}) = \bigcup_m g([m, m+1])$ is null. Finally put $h(x) = \frac{\varepsilon}{3} g(x/\varepsilon)$: it is strictly increasing, $h(\mathbb{R}) = \frac{\varepsilon}{3} g(\mathbb{R})$ is null, and $0 \leq x - h(x) \leq \varepsilon/2$. $\square$

Theorem 8.11 ([3, Theorem 10]). *There is a family $(\gamma_x)_{x \in \mathbb{R}}$ of arcs, γ_x an arc at x , pairwise disjoint, whose union has planar measure zero.*

Proof. (Formalized as TJK.Discont.exists.disjoint_null_arcs'.) Choose by Lemma 8.10 strictly increasing h_n ($n \geq 0$) with $h_n(\mathbb{R})$ null and $|x - h_n(x)| \leq 2^{-n}$. Let $\rho_k(y) = \max(0, \min(1, 2^{k+1}y - 1))$, which rises from 0 at $y = 2^{-k-1}$ to 1 at $y = 2^{-k}$, and put $X_x(y) = x + \sum_{k \geq 0} (h_k(x) - h_{k+1}(x))\rho_k(y)$. The terms are bounded by $2 \cdot 2^{-k}$, so X_x is continuous, and $X_x(0) = x$. For $2^{-k-1} \leq y \leq 2^{-k}$ we have $\rho_n(y) = 0$ for $n < k$ and $\rho_n(y) = 1$ for $n > k$, and the series telescopes (as $h_n(x) \rightarrow x$) to $X_x(y) = (1 - \rho_k(y))h_{k+1}(x) + \rho_k(y)h_k(x)$: on $[2^{-k-1}, 2^{-k}]$, X_x is affine from $h_{k+1}(x)$ to $h_k(x)$. So $\gamma_x = \{\langle X_x(y), y \rangle : 0 \leq y \leq 1\}$ is an arc at x (injective in y), lying in $\mathbb{H}$ except for x . If $x < x'$ then $X_x < X_{x'}$ on $[0, 1]$, so the arcs are disjoint. Let S_k be the union over x of the segments $\{\langle X_x(y), y \rangle : 2^{-k-1} \leq y \leq 2^{-k}\}$; these are pairwise disjoint. Under the affine map $\langle u, t \rangle \mapsto \langle u, (1+t)2^{-k-1} \rangle$, S_k is the image of the union of the disjoint segments from $\langle h_{k+1}(x), 0 \rangle$ to $\langle h_k(x), 1 \rangle$, whose end sets lie in the null sets $h_{k+1}(\mathbb{R})$, $h_k(\mathbb{R})$; by Theorem 8.8 that union is null, hence so is S_k . Thus $\bigcup_x \gamma_x \subseteq X_0 \cup \bigcup_k S_k$ is null. $\square$

Corollary 8.12 ([3, Theorem 10, Corollary 1]; [1, Theorem 6]). *Let Y be a metric space with a point 0 and $\varphi : \mathbb{R} \rightarrow Y$ arbitrary. There is $f : \mathbb{H} \rightarrow Y$ with $f = 0$ outside a set of measure zero such that φ is a boundary function for f .*

Proof. Put $f = \varphi(x)$ on $\gamma_x \setminus \{x\}$ and $f = 0$ elsewhere; f is constant along each γ_x . $\square$

Lemma 8.13 (non-measurable sets; Vitali). *There is a subset of $\mathbb{R}$ that is not Lebesgue measurable.*

Proof. Let $V \subseteq [0, 1]$ contain one point of each coset of $\mathbb{Q}$ in $\mathbb{R}$. The translates $V + q$, $q \in \mathbb{Q} \cap [-1, 1]$, are pairwise disjoint, cover $[0, 1]$ and lie in $[-1, 2]$. If V were measurable they would all have measure $\lambda(V)$, so $1 \leq \sum_q \lambda(V) \leq 3$ by countable additivity, impossible whether $\lambda(V) = 0$ or $\lambda(V) > 0$. $\square$

Corollary 8.14 ([3, Theorem 10, Corollary 2]). *There is a Lebesgue-measurable $f : \mathbb{H} \rightarrow \mathbb{R}$ with a boundary function on $\mathbb{R}$ that is not Lebesgue measurable.*

Proof. Apply Corollary 8.12 to the indicator function of a non-measurable set; $f = 0$ almost everywhere. $\square$

9. BOUNDED HARMONIC FUNCTIONS

Source: [7]. A function u defined in $\mathbb{D}$ has the *asymptotic value* a at $\zeta \in C$ if $u \rightarrow a$ along some arc at ζ , and the *general limit* a at ζ if $u(z) \rightarrow a$ as $z \rightarrow \zeta$, $z \in \mathbb{D}$. $P(z, \zeta) = (1 - |z|^2)/|\zeta - z|^2$ is the Poisson kernel ($P_r(\theta - t) = P(re^{i\theta}, e^{it})$), $P[\varphi](z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{it})\varphi(e^{it}) dt$, and $\exp iS = \{e^{it} : t \in S\}$.

Theorem 9.1 (Poisson integrals [16, pp. 32–38]). (i) If φ is a bounded measurable real function on C , $P[\varphi]$ is a bounded harmonic function in $\mathbb{D}$ with $|P[\varphi]| \leq \sup|\varphi|$. (ii) If $\varphi \in L^1(C)$, $S \subseteq C$ has full measure, $\zeta_0 \in S$, and $\varphi|_S$ is continuous at ζ_0 , then $P[\varphi]$ has general limit $\varphi(\zeta_0)$ at ζ_0 . (iii) If u is harmonic in $\mathbb{D}$ and $\{\int_0^{2\pi} |u(re^{i\theta})|^p d\theta : 0 \leq r < 1\}$ is bounded for some $p > 1$, then the radial limit ψ of u exists at almost every point of C , $\psi \in L^1(C)$, and $u = P[\psi]$.

Proof. Formalized in TJK/Harmonic/Poisson.lean and Example.lean. (i): $P[\varphi]$ is the real part of a Cauchy-type integral, with $P \geq 0$ and $P[1] = 1$. (ii): an approximate-identity argument. (iii): the Poisson formula on smaller circles; uniform integrability of $u(s \cdot)$ by Hölder; a weak limit of the measures $u(s \cdot)^\pm d\theta$, absolutely continuous by uniform integrability, whose Radon–Nikodym density ψ represents $u = P[\psi]$; and radial convergence at Lebesgue points of ψ by a dyadic kernel estimate. $\square$

(i) is proved: $P(z, \zeta) = \operatorname{Re} \frac{\zeta+z}{\zeta-z}$ and $\frac{\zeta+z}{\zeta(\zeta-z)} = \frac{2}{\zeta-z} - \frac{1}{\zeta}$, so $P[\varphi] = \operatorname{Re}(2F - F(0))$ with $F(z) = \frac{1}{2\pi i} \oint \frac{\varphi(\zeta)}{\zeta-z} d\zeta$ holomorphic in $\mathbb{D}$; and $|P[\varphi]| \leq P[\sup|\varphi|] = \sup|\varphi|$ since $P \geq 0$ and $P[1] = 1$.

Remark 9.2 (Kaczynski’s Lemma). [7, Lemma] states, for general Jordan domains $\Omega_n \subseteq \mathbb{D}$ along disjoint closed arcs I_n of C with disjoint closures and $\operatorname{diam} \Omega_n \rightarrow 0$, that every point of $C \setminus \bigcup_n I_n^*$ is accessible by a simple arc in $\mathbb{D} \setminus \bigcup_n \overline{\Omega_n}$. The proof applies the Jordan curve theorem, with accessibility of boundary points, to the curve obtained from C by replacing each I_n with the rest of $\partial\Omega_n$. The Jordan curve theorem is not formalized here; for the domains actually used, Lemma 9.6 below proves the conclusion directly.

Lemma 9.3 (general limits along an arc). *Let $u : \mathbb{D} \rightarrow \mathbb{R}$ have general limit c at every point of an open arc J of C . Then u has the asymptotic value c at each endpoint of J .*

Proof. Let $\zeta_0 = e^{it_0}$ be an endpoint and $w(s) = e^{i(t_0+s\ell)} \in J$ for $0 < s \leq 1$ (suitable $\ell \neq 0$, $|\ell| < 2\pi$). Let $F = \{z \in \mathbb{D} : |z - \zeta_0| \leq |u(z) - c| \} \cup \{0\}$, $h(s) = \operatorname{dist}(w(s), F)$, $m(s) = \min(h(s), s, 1)$ and $Z(s) = (1 - m(s)/2)w(s)$. h is continuous, and $h(s) > 0$ for $s > 0$ because $u \rightarrow c$ at $w(s) \neq \zeta_0$; so Z is a simple arc from $\zeta_0 = Z(0)$ with $Z((0, 1]) \subseteq \mathbb{D}$. As $|Z(s) - w(s)| < h(s)$, $Z(s) \notin F$, i.e. $|u(Z(s)) - c| < |Z(s) - \zeta_0| \rightarrow 0$. $\square$

Lemma 9.4 (the intervals). *Let $(p_n)_{n \geq 0}$ be a sequence in $(-\pi, \pi)$. There are reals c_k , ε_k ($k \geq 0$) such that, with $\alpha(k) = c_k - \pi\varepsilon_k^2/2$ and $\beta(k) = c_k + \pi\varepsilon_k^2/2$: I. $0 < \varepsilon_k \leq 2^{-k-2}$; II. $-\pi < \alpha(k) < \beta(k) < \pi$; IV. $|e^{is} - e^{it}| \geq 4\varepsilon_k^{1/4}$ for $s \in [\alpha(k), \beta(k)]$, $t \in [\alpha(k), \beta(k)]$, $h < k$ (so III: the intervals $[\alpha(k), \beta(k)]$ are pairwise disjoint); V. each p_h lies in some $[\alpha(k), \beta(k)]$, $k \leq h$.*

Proof. Recursively: given the first q intervals, covering $p_0, \dots, p_{q-1}$, let $c_q = p_q$ if p_q is not yet covered, and otherwise $c_q = (\pi + \max_{k < q} \beta(k))/2$. Then c_q lies in no earlier interval, so $d = \min_{k < q} \text{dist}(e^{ic_q}, I_k) > 0$; choose $\varepsilon_q > 0$ with I, II and $\frac{\pi}{2}\varepsilon_q^2 + 4\varepsilon_q^{1/4} < d$. Since $|e^{ic_q} - e^{it}| \leq |c_q - t| \leq \frac{\pi}{2}\varepsilon_q^2$ for $t \in [\alpha(q), \beta(q)]$, IV holds, and V holds for $h \leq q$. $\square$

Lemma 9.5 (estimates). *With the data of Lemma 9.4, let $I_n = \text{exp}i[\alpha(n), \beta(n)]$, χ_n its indicator, $f_n = P[\chi_n]$, $v(r) = (1 - r)^{1/4}$, $H_n(s, r) = re^{i(\alpha(n) - v(r) + s(\beta(n) - \alpha(n) + 2v(r)))}$ and $\Omega_n = H_n((0, 1) \times (1 - \varepsilon_n, 1))$, so that $re^{i\theta} \in \Omega_n$ whenever $1 - \varepsilon_n < r < 1$ and $\alpha(n) - v(r) < \theta < \beta(n) + v(r)$. Then the Ω_n , with Γ_n the image under H_n of the edges $s = 0$, $r = 1 - \varepsilon_n$, $s = 1$ of the closed rectangle, are Jordan domains along the arcs I_n with pairwise disjoint closures; $\overline{\Omega_n}$ lies within $\varepsilon_n + \pi\varepsilon_n^2 + \varepsilon_n^{1/4}$ of every point of I_n (so $\text{diam } \Omega_n \rightarrow 0$); and*

- (1) $0 \leq f_n \leq \varepsilon_n$ on $\mathbb{D} \setminus \Omega_n$;
- (2) $f_n(z) \leq \varepsilon_n \sqrt{1 - |z|}$ for $z \in \mathbb{D} \setminus \Omega_n$ with $|z| > 1 - \varepsilon_n$;
- (3) $f_n \rightarrow 0$ at every point of C along $\mathbb{D} \setminus \Omega_n$.

Proof. H_n is continuous and injective on the closed rectangle ($|H_n| = r$, and the angles range over an interval of length $< 2\pi$), so $\overline{\Omega_n}$ is its image; its edge $r = 1$ goes onto I_n , giving $\overline{\Omega_n} \subseteq \Omega_n \cup \Gamma_n \cup I_n$, and $H_n(\frac{1}{2}, 1) = e^{ic_n} \in I_n^*$. A common point $re^{i\theta} = re^{i\theta'}$ of $\overline{\Omega_h}$ and $\overline{\Omega_k}$, $h < k$, has $1 - r \leq \varepsilon_k$, so θ, θ' are within $\varepsilon_k^{1/4}$ of points $s \in [\alpha(h), \beta(h)]$, $t \in [\alpha(k), \beta(k)]$, and $|e^{is} - e^{it}| \leq 2\varepsilon_k^{1/4}$, against IV. A point $H_n(s, r)$ is within $(1 - r) + |\theta - t| \leq \varepsilon_n + \pi\varepsilon_n^2 + \varepsilon_n^{1/4}$ of e^{it} , $t \in I_n$.

Let $z \notin \Omega_n$, $r = |z|$. If $r \leq 1 - \varepsilon_n$, $P(z, \cdot) \leq 2/(1 - r) \leq 2/\varepsilon_n$ and $f_n \leq \frac{1}{2\pi} \cdot \pi\varepsilon_n^2 \cdot \frac{2}{\varepsilon_n} = \varepsilon_n$. If $r > 1 - \varepsilon_n$, then $r > \frac{3}{4}$, and for $t \in [\alpha(n), \beta(n)]$ write $z = re^{i\theta}$ with $|\theta - t| \leq \pi$; as $z \notin \Omega_n$, $|\theta - t| \geq v = v(r)$, so $\cos(\theta - t) \leq \cos v \leq 1 - \frac{v^2}{4}$ ($v < 1$) and $|e^{it} - z|^2 = 1 + r^2 - 2r \cos(\theta - t) \geq \frac{r}{2}\sqrt{1 - r}$; hence $P(z, e^{it}) \leq \frac{16}{3}\sqrt{1 - r}$ and $f_n \leq \frac{8}{3}\varepsilon_n^2\sqrt{1 - r} \leq \varepsilon_n\sqrt{1 - r} \leq \varepsilon_n$, since $\varepsilon_n \leq \frac{1}{4}$. This proves (1) and (2); (3) follows from (2). $\square$

Lemma 9.6 (accessibility, explicit). *With the data and notation of Lemma 9.5, every $\zeta \in C \setminus \bigcup_n I_n^*$ is accessible by a simple arc in $\mathbb{D} \setminus \bigcup_n \overline{\Omega_n}$. (This replaces Kaczynski's Lemma, and the Jordan curve theorem, for the domains actually used.)*

Proof. Write $v(s) = s^{1/4}$ for $s \geq 0$ and $\zeta = e^{i\theta_0}$. By Lemma 9.5, $\overline{\Omega_k} = H_k(\text{closed rectangle})$, so if a point $(1 - s)e^{i\phi}$ with $0 < s < 1$ lies in $\overline{\Omega_k}$ then $s \leq \varepsilon_k$ and $e^{i\phi} = e^{i\psi}$ for some $\psi \in [\alpha(k) - v(s), \beta(k) + v(s)]$, and $|\psi - b| \leq v(s)$ for some $b \in [\alpha(k), \beta(k)]$. We use $|e^{ia} - e^{ib}| \leq |a - b|$ throughout, and compare fourth powers of fourth roots.

Blockers. For each n let $d_n^+ \in [0, 2\pi)$ and $d_n^- \in [0, 2\pi)$ be defined by $e^{i\theta_0} = e^{i(\beta(n) + d_n^+)} = e^{i(\alpha(n) - d_n^-)}$. Call n a *+blocker* if $d_n^+ \leq \varepsilon_n^{1/4}$ and a *-blocker* if $d_n^- \leq \varepsilon_n^{1/4}$; not both, since $d_n^+ + d_n^- + (\beta(n) - \alpha(n)) \in 2\pi\mathbb{Z}_{>0}$ while $2\varepsilon_n^{1/4} + \pi\varepsilon_n^2 < 2\pi$. For a $\pm$ blocker let $\sigma_n = \pm 1$ and $d_n = d_n^\pm$; otherwise $\sigma_n = 0$. A blocker n has an endpoint $p_n \in \{\alpha(n), \beta(n)\}$ with $|e^{ip_n} - \zeta| \leq d_n \leq \varepsilon_n^{1/4}$. Let $\rho_n(s) = \max(0, \min(1, 2 - s/\varepsilon_n))$ and $\delta_n(s) = \sigma_n \max(0, 2\rho_n(s)v(s) - d_n)$. Call n *active at s* if n is a blocker and $d_n < 2\rho_n(s)v(s)$; so $\delta_n(s) \neq 0$ only if n is active at s , and then $0 < s < 2\varepsilon_n$ and $d_n < 2v(s)$.

(a) *If $(1 - s)\zeta \in \overline{\Omega_k}$, $0 < s < 1$, then k is active at s .* Indeed $s \leq \varepsilon_k$ (so $\rho_k(s) = 1$) and $\zeta = e^{i\psi}$ with $\psi \in [\alpha(k) - v(s), \beta(k) + v(s)]$; $\psi \notin (\alpha(k), \beta(k))$ since $\zeta \notin I_k^*$; if $\psi \geq \beta(k)$ then $d_k^+ = \psi - \beta(k) \leq v(s) \leq \varepsilon_k^{1/4}$, so k is a *+blocker* with $d_k \leq v(s) < 2v(s)$; symmetrically if $\psi \leq \alpha(k)$.

(b) *At each s at most one n is active.* Let $h < k$ be active at s . By IV, $4\varepsilon_k^{1/4} \leq |e^{ip_h} - e^{ip_k}| \leq d_h + d_k < 2v(s) + \varepsilon_k^{1/4}$, so $81\varepsilon_k < 16s < 32\varepsilon_k$, which is absurd.

The arc. Each δ_n is continuous, $\delta_n(0) = 0$, and $|\delta_n| \leq 2v(2\varepsilon_n) \leq 2 \cdot 2^{-(n+1)/4}$ by I. So $\theta(s) = \theta_0 + \sum_n \delta_n(s)$ is continuous, $\theta(0) = \theta_0$, and by (b) at most one term is non-zero at each s . Let $Z(s) = (1-s)e^{i\theta(s)}$, $s \in [0, \frac{1}{2}]$. Z is continuous, injective ($|Z(s)| = 1-s$), $Z(0) = \zeta$, and $Z(s) \in \mathbb{D}$ for $s > 0$. We show $Z(s) \notin \overline{\Omega_k}$ for $0 < s \leq \frac{1}{2}$ and every k .

If no n is active at s , then $Z(s) = (1-s)\zeta$ and (a) applies. Otherwise exactly one n is active at s , say a +blocker (the other case is symmetric), and $Z(s) = (1-s)e^{i(\beta(n)+t)}$ with $t = 2\rho_n(s)v(s) \in (d_n, 2v(s)]$; $t = 2v(s)$ if $s \leq \varepsilon_n$; and $s < 2\varepsilon_n$. Suppose $Z(s) \in \overline{\Omega_k}$, with ψ, b as above; so $s \leq \varepsilon_k$.

- $k = n$: then $t = 2v(s)$, and $\beta(n)+t-\psi \in [v(s), 3v(s)+\pi\varepsilon_n^2] \subseteq (0, 2\pi)$, so $e^{i\psi} \neq e^{i(\beta(n)+t)}$.
- $k \neq n$: $|e^{i\beta(n)} - e^{ib}| \leq |e^{i\beta(n)} - e^{i(\beta(n)+t)}| + |e^{i\psi} - e^{ib}| \leq t + v(s) \leq 3v(s)$. If $k < n$, IV gives $4\varepsilon_n^{1/4} \leq 3v(s)$, so $256\varepsilon_n \leq 81s < 162\varepsilon_n$; if $k > n$, IV gives $4\varepsilon_k^{1/4} \leq 3v(s) \leq 3\varepsilon_k^{1/4}$. Both are absurd.

So $Z([0, \frac{1}{2}])$ is a simple arc at ζ with all other points in $\mathbb{D} \setminus \bigcup_k \overline{\Omega_k}$. $\square$

Theorem 9.7 ([7]; answer to [11, Problem 5]). *There are a bounded harmonic function f in $\mathbb{D}$ and a boundary function $\varphi : C \rightarrow \mathbb{R}$ for f that is not of Baire class ≤ 1 .*

Proof. Apply Lemma 9.4 to a dense sequence (p_n) in $(-\pi, \pi)$, and use the notation of Lemma 9.5. Let χ be the indicator of $\bigcup_n I_n$ and $f = P[\chi]$; f is bounded harmonic (Theorem 9.1(i)), and $f = \sum_n f_n$ in $\mathbb{D}$ (countable additivity of the integral, the I_n being disjoint).

χ is continuous with value 1 at each point of an open arc I_n^* , so f has general limit 1 there (Theorem 9.1(ii)); by Lemma 9.3 (applied to subarcs of I_n^*), f has asymptotic value 1 at every point of I_n .

Let $\zeta \in C \setminus \bigcup_n I_n^*$. By Lemma 9.6 there is an arc γ at ζ meeting no $\overline{\Omega_n}$. For $m \in \mathbb{N}$ and $z \in \gamma \setminus \{\zeta\}$, by (1) and I, $0 \leq f(z) \leq \sum_{n < m} f_n(z) + \sum_{n \geq m} \varepsilon_n \leq \sum_{n < m} f_n(z) + 2^{-m-1}$, and by (3) $\limsup f \leq 2^{-m-1}$ along γ . So f has asymptotic value 0 at ζ . Let $A = \{e^{i\alpha(n)}\}$, $B = \{e^{i\beta(n)}\}$, and $\varphi = 1$ on $\bigcup_n I_n^* \cup B$, $\varphi = 0$ on the rest of C . Then φ is a boundary function for f ; $\varphi = 0$ on A and $\varphi = 1$ on B (the closed arcs I_n being disjoint).

By V the intervals $[\alpha(k), \beta(k)]$ are dense in $(-\pi, \pi)$. Given $e^{i\alpha(n)}$ and $\delta > 0$ small, some $[\alpha(k), \beta(k)]$ meets $(\alpha(n) - \delta, \alpha(n))$; it does not contain $\alpha(n)$, so $\beta(k) \in (\alpha(n) - \delta, \alpha(n))$. Hence $A \subseteq \text{Cl } B$, and likewise $B \subseteq \text{Cl } A$. So $P = \text{Cl}(A \cup B)$ is a nonempty compact set in which both A and B are dense, and $\varphi|_P$ is continuous at no point. If φ were of Baire class ≤ 1 , so would be $\varphi|_P$ (Lemma 3.29), contradicting Corollary 3.25. $\square$

Theorem 9.8 ([7, Theorem]). *Let u be harmonic in $\mathbb{D}$ with $\{\int |u(re^{i\theta})|^p d\theta : 0 \leq r < 1\}$ bounded for some $p > 1$. Let A be the set of points of C at which u has an asymptotic value and G the set at which u has a general limit. Then $A \setminus G$ is of the first category.*

Proof. Let S be the set where the radial limit ψ exists; $S \subseteq A$, $C \setminus S$ is null, and $u = P[\psi]$ (Theorem 9.1(iii)). Extend $\psi|_S$ to a boundary function φ_1 on A (the radius is an arc). By Corollary 6.3 there is $\varphi \in \mathcal{B}_1(A, \mathbb{R})$ with $\varphi = \varphi_1$ off a countable set; extend φ by ψ off A . Countable sets are null, so $\varphi = \psi$ almost everywhere and $u = P[\varphi]$. By Theorem 3.24 the set $A \setminus K$ of discontinuity points of $\varphi|_A$ is of the first category in A , hence in C (Lemma 3.26). By Theorem 9.1(ii) (with the full-measure set $A \supseteq S$), $K \subseteq G$; so $A \setminus G \subseteq A \setminus K$. $\square$

Corollary 9.9 ([7, Corollary]). *Under the hypothesis of Theorem 9.8, if φ is a boundary function for u defined on $E \subseteq C$, then the set of points of E at which φ is discontinuous is of the first category in C .*

Proof. At a point $\zeta \in E \cap G$ with general limit L : if $|u - L| < \varepsilon$ on $\mathbb{D} \cap S(\delta, \zeta)$, then every $x' \in E \cap S(\delta, \zeta)$ has $|\varphi(x') - L| \leq \varepsilon$, since $\varphi(x')$ is a limit along an arc whose tail lies in $S(\delta, \zeta)$. So φ is continuous on $E \cap G$, and its discontinuities lie in $A \setminus G$. $\square$

Corollary 9.10 (analytic functions; [7]). *Theorem 9.8 and Corollary 9.9 hold for holomorphic F in $\mathbb{D}$ with bounded p -means, $p > 1$.*

Proof. Apply them to $\operatorname{Re} F$ and $\operatorname{Im} F$, whose p -means are bounded by those of F : if F has an asymptotic value at x so do both parts, and if both have general limits so does F . $\square$

Remark 9.11. The example of [11, Theorem 6] shows that the hypothesis on the integral means cannot be omitted.

10. A CUBE WITH A NON-BOREL SET OF CURVILINEAR CONVERGENCE

Source: [9]. For an open connected $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \rightarrow \mathbb{R}^m$, the set of curvilinear convergence $\operatorname{CC}(f)$ is the set of $p \in \partial\Omega$ for which there is an arc γ at p with f converging to a finite limit along γ .

Definition 10.1 (asymptotic values). $a \in \mathbb{R}^m$ is an *asymptotic value* of $f : \Omega \rightarrow \mathbb{R}^m$ if there is a continuous $v : [0, 1) \rightarrow \Omega$ with $\operatorname{dist}(v(t), \mathbb{R}^n \setminus \Omega) \rightarrow 0$ and $f(v(t)) \rightarrow a$ as $t \rightarrow 1$; it is a *point asymptotic value at p* if moreover $v(t) \rightarrow p \in \partial\Omega$. By Corollary 4.4, $p \in \operatorname{CC}(f)$ iff f has a (finite) point asymptotic value at p .

Theorem 10.2 (analytic non-Borel sets). *There is an analytic subset of $\mathbb{R}$ that is not a Borel set.*

Proof. Formalized in `TJK/Cube/AnalyticNonBorel.lean`: a universal open set in $(\mathbb{N} \rightarrow \{0, 1\}) \times Y$ built from a countable basis gives, by projecting closed graphs, an analytic set universal for analytic subsets of Cantor space; its diagonal is analytic with non-analytic complement, hence not Borel; transfer to $\mathbb{R}$ by a continuous injection of Cantor space. $\square$

Lemma 10.3 ([9, Lemma]). *There is a continuous $s : \mathbb{H} \rightarrow \mathbb{C}$ with $|s| < 1$ such that the set E of real asymptotic values of s lying in $(-1, 1)$ equals the set of real point asymptotic values of s lying in $(-1, 1)$, and E is not a Borel set.*

Proof. This is Lemma 10.6 below. $\square$

Remark 10.4 (Kaczynski's proof). Kaczynski derives the lemma from two classical theorems, which are not formalized here: (K) (Kierst 1936) for every analytic $A \subseteq \mathbb{R}$ there is a holomorphic h in $\mathbb{D}$ omitting i and $-i$ whose set of finite asymptotic values is $A \cup \{-i, i\}$; (N) (Lehto–Virtanen [19, p. 53]; Bagemihl–Seidel [13, Theorem 1]) if h is holomorphic in $\mathbb{D}$ and omits two finite values, every finite asymptotic value of h is a point asymptotic value. The argument: Let A be analytic and not Borel, h as in Kierst's theorem (K), and $\rho(w) = w/(1 + |w|)$, a homeomorphism of $\mathbb{C}$ onto $\mathbb{D}$ mapping $\mathbb{R}$ onto $(-1, 1)$, with inverse $u \mapsto u/(1 - |u|)$. Put $s(x, y) = \rho(h(\max\{1 - y, 0\}e^{ix}))$, continuous on $\mathbb{H}$ with $|s| < 1$. Let $b \in (-1, 1)$. If b is an asymptotic value of s along $v = \langle x(t), y(t) \rangle$, then $y(t) = \operatorname{dist}(v(t), \mathbb{C} \setminus \mathbb{H}) \rightarrow 0$, and $\rho^{-1}(b)$ is an asymptotic value of h along $w = (1 - y)e^{ix}$ (for $y < 1$, $\operatorname{dist}(w, \mathbb{C} \setminus \mathbb{D}) \leq |w - e^{ix}| = y$); as $\rho^{-1}(b)$ is real, it lies in A . If $a \in A$, by (N) a is a point asymptotic value of h at some e^{ip} , $p = \arg e^{ip}$, along a path $w(t) \rightarrow e^{ip}$. For t near 1, $w(t)e^{-ip}$ lies near 1, off the negative axis, so $\vartheta(t) = p + \arg(w(t)e^{-ip})$ is continuous, $\vartheta(t) \rightarrow p$ and $|w(t)|e^{i\vartheta(t)} = w(t)$. Hence $\rho(a)$ is a point asymptotic value of s at p along $\langle \vartheta(t), 1 - |w(t)| \rangle$. Since point asymptotic values are asymptotic values, both sets equal $\rho(A)$, which is not Borel ($A = \rho^{-1}(\rho(A))$ and ρ is continuous).

Remark 10.5. [9] notes that the lemma can also be proved without meromorphic functions, from [23, Theorem 113], a theorem of Bagemihl–McMillan, and the methods of [8], but omits the “messy construction”.

An elementary proof of the lemma. We give a proof of Lemma 10.3 that uses neither Kierst’s theorem nor normal functions: only Theorem 10.2 and plane geometry. The function s is built on a binary tree of segments in $\mathbb{H}$ (a “dyadic skeleton”); its imaginary part forces every path along which s tends to a real value to follow one branch of the tree, and its real part is read off a Souslin representation of the analytic set along the branches.

Words and the skeleton. Let W be the set of finite words $u = u_0 \dots u_{n-1}$ over $\{0, 1\}$; $|u| = n$, $u \sqsubseteq v$ means u is a prefix of v , $u \subset v$ a proper prefix, $u|k$ is the prefix of length k , and u^- is u without its last letter ($u \neq \emptyset$). The same notation is used for infinite words $\beta \in \{0, 1\}^{\mathbb{N}}$. Put $x_u = \sum_{i < |u|} u_i 2^{-i-1}$, $I_u = [x_u, x_u + 2^{-|u|}]$, $\mu_u = x_u + 2^{-|u|-1}$ (the midpoint of I_u), and $c_u = \mu_u + i 2^{-|u|} \in \mathbb{H}$. For $u \neq \emptyset$ let e_u be the segment $[c_{u^-}, c_u]$, and for $z \in \mathbb{H}$ let $d_u(z) = \text{dist}(z, e_u)/\Im z$.

- (G1) Points of e_u have heights in $[2^{-|u|}, 2^{1-|u|}]$.
- (G2) If $|v| > k$ and $p \in e_v$, then $\Re p \in I_{v|k}$ and $\text{dist}(\Re p, 2^{-k}\mathbb{Z}) \geq \Im p/4$. Indeed, let $j = |v|$. The endpoints of e_v have real parts μ_{v^-} , an odd multiple of 2^{-j} , at distance 2^{-j} from $2^{1-j}\mathbb{Z} \supseteq 2^{-k}\mathbb{Z}$, and μ_v , an odd multiple of 2^{-j-1} , at distance 2^{-j-1} from $2^{-j}\mathbb{Z} \supseteq 2^{-k}\mathbb{Z}$; both lie in $I_v \subseteq I_{v|k}$. On $I_{v|k}$ the function $x \mapsto \text{dist}(x, 2^{-k}\mathbb{Z})$ is the minimum of the distances to the two endpoints, hence concave, so along e_v (on which $\Re$ is affine) it is at least $2^{-j-1} \geq \Im p/4$ by (G1).
- (G3) If $u1 \sqsubseteq v$ and $p \in e_v$ then $\Re p \geq \mu_u$; if $u0 \sqsubseteq v$ then $\Re p \leq \mu_u$. (For $v = u1$ the endpoints of e_v have real parts $\mu_u < \mu_{u1}$; for $v \supset u1$ both endpoints lie in $I_{u1} = [\mu_u, \mu_u + 2^{-|u|-1}]$. Symmetrically for $u0$.)
- (G4) *Separation.* If $p, p' \in \mathbb{H}$, $q \in \mathbb{R}$, $\Re p \geq q$ and $\Re p' \leq q - \Im p'/4$, then $|p - p'| \geq \max\{\Im p, \Im p'\}/8$. If $\Im p' \geq \frac{7}{8}\Im p$ then $|p - p'| \geq \Re p - \Re p' \geq \Im p'/4 \geq \max\{\Im p, \Im p'\} \cdot \frac{7}{32}$; otherwise $|p - p'| \geq \Im p - \Im p' > \Im p/8$ and $\Im p = \max\{\Im p, \Im p'\}$.
- (G5) Call e_v, e_w *separated* if $|p - p'| \geq \max\{\Im p, \Im p'\}/8$ for all $p \in e_v, p' \in e_w$. This holds (a) if $|v|, |w| > k$ and $v|k \neq w|k$: if, say, $I_{v|k}$ lies to the left of $I_{w|k}$, let q be the right endpoint of $I_{v|k}$; by (G2), $\Re p \leq q - \Im p/4$ and $\Re p' \geq q$; apply (G4). (b) if $u1 \sqsubseteq v$ and $u0 \sqsubset w$: take $q = \mu_u$, the right endpoint of I_{u0} , and use (G3) for v and (G2) with $k = |u| + 1$ for w .
- (G6) *Nearness.* If e_v, e_w are separated and $d_v(z) < \frac{1}{32}$, then $d_w(z) > \frac{1}{12}$. Take $p \in e_v$ with $|z - p| < \Im z/32$; then $\Im p > \frac{31}{32}\Im z$, so every $p' \in e_w$ has $|z - p'| \geq |p - p'| - |z - p| > (\frac{31}{256} - \frac{8}{256})\Im z > \Im z/12$.
- (G7) If $\Im z < 2^{-k-1}$ and $|w| \leq k$, then $d_w(z) \geq 1$, by (G1).

Coding. Write $\mathbb{N}^{\mathbb{N}}$ for the Baire space and $N_\kappa = \{\sigma \in \mathbb{N}^{\mathbb{N}} : \kappa \sqsubseteq \sigma\}$ for $\kappa \in \mathbb{N}^{<\mathbb{N}}$. Every $u \in W$ can be written uniquely as $u = 0^{k_0}10^{k_1}1 \dots 0^{k_{j-1}}10^r$; put $\kappa(u) = (k_0, \dots, k_{j-1})$ and $\ell(u) = |u| - r$ (the length of the longest prefix of u ending in 1). Then $u \sqsubseteq v$ implies $\kappa(u) \sqsubseteq \kappa(v)$, and $\ell(u) \geq |\kappa(u)|$, the number of 1’s in u . For $\sigma \in \mathbb{N}^{\mathbb{N}}$ let $\beta_\sigma = 0^{\sigma_0}10^{\sigma_1}1 \dots$; then $\kappa(\beta_\sigma|m) \sqsubseteq \sigma$ and $|\kappa(\beta_\sigma|m)| \rightarrow \infty$. If $\beta \in \{0, 1\}^{\mathbb{N}}$ has infinitely many 1’s, then $\beta = \beta_\sigma$ for some σ (the lengths of the 0-blocks); otherwise $\beta = t0^\infty$ with t empty or ending in 1.

Lemma 10.6 (elementary form of Lemma 10.3). *The conclusion of Lemma 10.3 holds, with a proof using only Theorem 10.2.*

Proof. Let $A_0 \subseteq \mathbb{R}$ be analytic and not Borel, and $A = \varphi(A_0)$ with $\varphi(a) = a/(2(1 + |a|))$, a continuous injection of $\mathbb{R}$ into $(-\frac{1}{2}, \frac{1}{2})$. A is analytic, and not Borel because $A_0 = \varphi^{-1}(A)$. As $A \neq \emptyset$, $A = g(\mathbb{N}^{\mathbb{N}})$ for a continuous $g : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{R}$. For $u \in W$ let $r_u = g(\kappa(u)0^\infty) \in g(N_{\kappa(u)}) \subseteq$

$(-\frac{1}{2}, \frac{1}{2})$ and $\psi_u = 1/(\ell(u) + 2)$. For $z \in \mathbb{H}$ put

$$R(z) = \min\left\{\frac{1}{2}, \inf_{u \neq \emptyset} (r_u + 12 d_u(z))\right\}, \quad \Psi(z) = \min\left\{1, \inf_{u \neq \emptyset} (\psi_u + 16 d_u(z))\right\}, \quad s = R + \frac{i}{2} \Psi.$$

Continuity and bounds. On $K = \{z : \Im z \geq \eta, |z| \leq M\}$ each $z \mapsto c_u + C \operatorname{dist}(z, e_u)/\Im z$ (with $|c_u| \leq 1$, $e_u \subseteq [0, 1] \times [0, 1]$) is Lipschitz with a constant $C(1/\eta + (M+2)/\eta^2)$ independent of u , and the infimum of a family of uniformly Lipschitz functions bounded below is Lipschitz with the same constant; so R, Ψ , hence s , are continuous on $\mathbb{H}$. As $r_u > -\frac{1}{2}$, $R \in [-\frac{1}{2}, \frac{1}{2}]$; $\Psi \in [0, 1]$; so $|s| \leq \sqrt{1/4 + 1/4} < 1$.

Claim L. Let $|u| = k$, $\Im z < 2^{-k-1}$ and $d_v(z) < \frac{1}{32}$ for some $v \sqsupset u$. Then every w with $d_w(z) \leq \frac{1}{12}$ satisfies $w \sqsupset u$, and $R(z) \geq \min\{\frac{1}{2}, \inf_{w \sqsupset u} r_w\}$. Indeed if $w \not\sqsupset u$ then either $|w| \leq k$ and $d_w(z) \geq 1$ by (G7), or $|w| > k$, $w|k \neq u$, and $d_w(z) > \frac{1}{12}$ by (G5a), (G6). Then $r_w + 12d_w(z) > -\frac{1}{2} + 1 = \frac{1}{2}$.

Point asymptotic values. Let $a = g(\sigma) \in A$, $\beta = \beta_\sigma$, $p = \sum_i \beta_i 2^{-i-1} = \lim \mu_{\beta|m}$. With $\rho_m(y) = \min\{1, \max\{0, 2 - 2^m y\}\}$ let $X(y) = \mu_\emptyset + \sum_{m \geq 1} (\mu_{\beta|m} - \mu_{\beta|m-1}) \rho_m(y)$, a uniformly convergent series of continuous functions ($|\mu_{\beta|m} - \mu_{\beta|m-1}| = 2^{-m-1}$). For $y \in [2^{-m}, 2^{1-m}]$, $\rho_j(y) = 1$ for $j < m$ and $\rho_j(y) = 0$ for $j > m$, so with $\lambda = 2 - 2^m y \in [0, 1]$, $z(y) = X(y) + iy = (1 - \lambda)c_{\beta|m-1} + \lambda c_{\beta|m} \in e_{\beta|m}$. Hence $X(y) \rightarrow p$ as $y \rightarrow 0$, and $d_{\beta|m}(z(y)) = 0$, so $\Psi(z(y)) \leq \psi_{\beta|m} \rightarrow 0$ (as $\ell(\beta|m) \geq |\kappa(\beta|m)| \rightarrow \infty$) and $R(z(y)) \leq r_{\beta|m} \rightarrow a$ (as $\kappa(\beta|m) 0^\infty \rightarrow \sigma$). Given $\varepsilon > 0$ choose j with $|g - a| < \varepsilon$ on $N_{\sigma|j}$ and k with $|\kappa(\beta|k)| \geq j$; for $w \sqsupset \beta|k$, $r_w \in g(N_{\sigma|j})$, so Claim L gives $R(z(y)) \geq a - \varepsilon$ for $y < 2^{-k-1}$. So $s \rightarrow a$ along $t \mapsto z(1 - t)$, and a is a point asymptotic value of s at p .

Asymptotic values. Let $s \rightarrow b \in \mathbb{R}$ along a path $v : [0, 1) \rightarrow \mathbb{H}$ with $\Im v(t) \rightarrow 0$. Then $\Psi(v(t)) \rightarrow 0$ and $R(v(t)) \rightarrow b$. For $|u| = k$ let $\Omega_k(u) = \{z : \Im z < 2^{-k-1}, \exists v \sqsupset u, d_v(z) < \frac{1}{32}\}$, an open set. By Claim L the $\Omega_k(u)$, $|u| = k$, are pairwise disjoint, and every z with $\Im z < 2^{-k-1}$ and $\Psi(z) < \frac{1}{2}$ lies in one of them (some w has $d_w(z) < \frac{1}{32}$; then $|w| > k$ by (G7); take $u = w|k$). So for each k a tail $v([t_k, 1))$, being connected, lies in a single $\Omega_k(u_k)$; a common point of two tails and Claim L give $u_k \sqsubseteq u_{k+1}$, so $u_k = \beta|k$ for some $\beta \in \{0, 1\}^\mathbb{N}$. By (G2), points of $\Omega_k(u)$ have real part within $2^{-k-1} + \Im z/32$ of μ_u , so $v(t) \rightarrow p = \lim \mu_{\beta|k} \in \mathbb{R}$: b is a point asymptotic value of s at p . It remains to show $b \in A$.

Case 1: $\beta = \beta_\sigma$. Let $a = g(\sigma)$, and ε, k as above. For $t \geq t_k$ with $\Psi(v(t)) < \delta < \frac{1}{2}$ there is w with $\psi_w + 16d_w(v(t)) < \delta$; then $d_w < \frac{1}{32}$, so $w \sqsupset \beta|k$ by Claim L and $R(v(t)) \leq r_w + 12d_w < a + \varepsilon + \delta$, while $R(v(t)) \geq a - \varepsilon$. Hence $b = a \in A$.

Case 2: $\beta = t0^\infty$, t empty or ending in 1, $n = |t|$, so $\psi_{t0^i} = c_0 := 1/(n+2)$ for all i . Fix $k \geq n$. For t large ($t \geq t_k$ and $\Psi(v(t)) < c_0/2$) there is w with $\psi_w < c_0/2$ and $d_w(v(t)) < \frac{1}{32}$; so $w \sqsupset \beta|k = t0^{k-n}$ (Claim L) and w is not of the form $t0^i$; hence $t0^i 1 \sqsubseteq w$ for some $i \geq k - n$. Thus a tail of v lies in $\bigcup_i S_i$, where $S_i = \{z \in \mathbb{H} : \exists w \sqsupseteq t0^i 1, d_w(z) < \frac{1}{32}\}$ is open. For $i < i'$, $S_i \cap S_{i'} = \emptyset$ by (G5b) (with $u = t0^i$, $t0^{i+1} \sqsubset t0^{i'} 1$) and (G6). So the tail lies in a single S_i . But the tail after t_{n+i+1} also lies in $\Omega_{n+i+1}(t0^{i+1})$, and a point of both is within $\Im z/32$ of some e_w , $w \sqsupseteq t0^i 1$, and of some $e_{w'}$, $w' \sqsupseteq t0^{i+1}$, contradicting (G5b) and (G6). So Case 2 does not occur.

Thus the real asymptotic values of s are exactly the points of $A \subseteq (-\frac{1}{2}, \frac{1}{2})$, all of them point asymptotic values; and A is not Borel. $\square$

Remark 10.7. In Case 2 the imaginary part does the work: along the branch $t0^\infty$ (the only kind of branch not coding a point of $\mathbb{N}^\mathbb{N}$) Ψ stays $\geq c_0$ near the branch, and leaving it means entering one of the disjoint “side” regions S_i . Paths tending to ∞ or oscillating along $\mathbb{R}$ are excluded because $\Psi \rightarrow 0$ forces the path into the regions $\Omega_k(u)$, whose real parts are confined to I_u up to $\Im z/32$.

Theorem 10.8 ([9, Theorem]). *There is a bounded continuous $f : K^\circ \rightarrow \mathbb{C}$ on the open cube $K^\circ = (-1, 1) \times (0, 1) \times (-1, 1)$ whose set of curvilinear convergence is not a Borel set. (So McMillan's theorem does not extend to three dimensions.)*

Proof. Let s, E be as in Lemma 10.3, $g(x, y) = s(x/y, y)$ for $y > 0$, so $|g| < 1$.

Construction. For $t \in \mathbb{R}$ let $K_t = [-1, 1] \times [t, 1]$ and let $d_0(t)$ be the supremum of the $\delta \in [0, 1]$ such that $z, z' \in K_t$, $|z - z'| < \delta$ imply $|g(z) - g(z')| \leq t$. Raising t shrinks K_t and loosens the tolerance, so d_0 is weakly increasing; for $t > 0$, $d_0(t) > 0$ because g is uniformly continuous on the compact set K_t . Let $d(t) = \min\{\frac{1}{2}d_0(t/2), \frac{1}{4}t\}$, positive and weakly increasing on $(0, 1]$, and k as in Theorem 7.15 (so $|k| \leq \sqrt{2} \leq 2$). Put $f(x, y, z) = (g(x, y) - z)k(x, y)$, continuous on K° with $|f| \leq 4$. For $c, e', e'', k', k'' \in \mathbb{C}$ with $|k'|, |k''| \leq 2$,

$$|(c + e')k' - (c + e'')k''| \geq |c||k' - k''| - 2|e'| - 2|e''|, \quad (10.1)$$

because $(c + e')k' - (c + e'')k'' = c(k' - k'') + e'k' - e''k''$.

Let $F = \text{CC}(f)$. We show that for $b \in (-1, 1)$, $\langle 0, 0, b \rangle \in F$ iff $b \in E$; then F is not Borel, since otherwise $\{b \in (-1, 1) : \langle 0, 0, b \rangle \in F\} = E$ would be.

If $b \in E$, $s \rightarrow b$ along a path $\langle u(t), v(t) \rangle \rightarrow \langle p, 0 \rangle$ in $\mathbb{H}$. Along $c(t) = \langle u(t)v(t), v(t), b \rangle \rightarrow \langle 0, 0, b \rangle$, which lies in K° for t near 1, $g = s(u, v) \rightarrow b$, so $f \rightarrow 0$. By Corollary 4.4 (applied to c restricted to some $[t_0, 1)$ and closed up by $\langle 0, 0, b \rangle$, which lies in ∂K°), $\langle 0, 0, b \rangle \in F$.

Conversely let $f \rightarrow q$ along an arc $\gamma' = q_0([0, 1])$ at $q_0(0) = \langle 0, 0, b \rangle$ in K° . We claim $g(x, y) - z \rightarrow 0$ along γ' . Then $g \rightarrow b$ along γ' , so $s \rightarrow b$ along $v(t) = \langle x/y, y \rangle(q_0(1 - t))$, $t \in [0, 1)$, a path in $\mathbb{H}$ whose second coordinate tends to 0: b is an asymptotic value of s , so $b \in E$.

If the claim fails, there is $\varepsilon > 0$ such that $|g(x, y) - z| \geq \varepsilon$ at points of $\gamma' \setminus \{\langle 0, 0, b \rangle\}$ arbitrarily close to $\langle 0, 0, b \rangle$. Choose $r_1 > 0$ with $|f(P) - q| < \varepsilon/2$ for $P \in \gamma' \setminus \{\langle 0, 0, b \rangle\}$ within r_1 of $\langle 0, 0, b \rangle$, and put $r = \min\{r_1, \varepsilon/16\}$. As γ' is compact and its only point with $y = 0$ is $\langle 0, 0, b \rangle$, there is $\delta > 0$ such that points of γ' with $y \leq \delta$ lie within r of $\langle 0, 0, b \rangle$. Take $P_n = q_0(t_n) = \langle x_n, y_n, z_n \rangle$ with $|g(x_n, y_n) - z_n| \geq \varepsilon$ and $0 < y_n < \min\{3\delta/4, 3\varepsilon/32, 3/4\}$; put $a = 4y_n/3 \leq 1$, so $a \leq \delta$ and $a < \varepsilon/8$. The projection Φ of q_0 to the $\langle x, y \rangle$ -plane has $\Phi(0) = \langle 0, 0 \rangle$ at distance $\geq y_n > d(a)$ from $\Phi(t_n)$, so by Lemma 4.5(b) there is an interval $J \ni t_n$ with $\gamma^* = \Phi(J)$ in the closed disc of radius $d(a) \leq a/4$ about $\langle x_n, y_n \rangle$ and containing a point at distance $d(a)$ from it. Then $\gamma^* \subseteq [-1, 1] \times [a/2, a]$ is connected, and Theorem 7.15 gives $\tau', \tau'' \in J$ with $|k' - k''| \geq 2$ at $\Phi(\tau'), \Phi(\tau'')$. The points $P'_n = q_0(\tau')$, $P''_n = q_0(\tau'')$ have $a/2 \leq y \leq a \leq \delta$, so they differ from $\langle 0, 0, b \rangle$ and lie within r of it; hence $|f(P'_n) - f(P''_n)| < \varepsilon$, $|z'_n - z_n|, |z''_n - z_n| < 2r \leq \varepsilon/8$, and, as $\Phi(\tau'), \langle x_n, y_n \rangle \in K_{a/2}$ are closer than $d(a) < d_0(a/2)$, $|g' - g(x_n, y_n)|, |g'' - g(x_n, y_n)| \leq a/2 < \varepsilon/16$. Applying (10.1) with $c = g(x_n, y_n) - z_n$, $c + e' = g' - z'_n$, $c + e'' = g'' - z''_n$ gives $\varepsilon > |f(P'_n) - f(P''_n)| \geq 2\varepsilon - 4(\varepsilon/16 + \varepsilon/8) = \frac{5}{4}\varepsilon$, a contradiction. $\square$

Remark 10.9. [9] leaves open whether a continuous *real*-valued function in a cube can have a non-Borel set of curvilinear convergence.

11. WEDDERBURN'S THEOREM

Source: [4], a group-theoretic proof that every finite skew field (division ring) is commutative. The note cites two results from group theory: the classification of p -groups with a unique subgroup of order p ([5, p. 189]), and the solvability of finite groups whose Sylow subgroups are cyclic (Miller–Blichfeldt–Dickson). For the first we prove directly the special case that is needed (Lemma 11.2); for the second we use Mathlib's theory of Z-groups (`IsZGroup`), which proves the stronger fact that such a group has cyclic commutator subgroup. Mathlib also proves Wedderburn's theorem itself, by Witt's argument; the formalization below does not use

that proof (the instance `littleWedderburn` is switched off in `TJK/Wedderburn.lean`, and the constants on which `TJK.Wedderburn.mul_comm` depends were checked not to include it).

Lemma 11.1 ([4, Lemma]). *Let q be a prime. Then $t^2 + r^2 \equiv -1 \pmod{q}$ has a solution with $t \not\equiv 0 \pmod{q}$.*

Proof. If -1 is a square, take $r = 0$ and $t^2 = -1$. Otherwise (q is odd) every non-square is $-s^2$ for some $s \neq 0$. If some $t^2 + r^2$ is a non-square, $t^2 + r^2 = -s^2$ gives $(ts^{-1})^2 + (rs^{-1})^2 = -1$, and $t \neq 0$ or $r \neq 0$, so after exchanging t and r , $t \neq 0$. Otherwise the squares are closed under addition, so $-1 = 1 + 1 + \dots + 1$ ($q - 1$ terms) is a square, a contradiction. $\square$

Lemma 11.2 (special case of [5, Theorem 12.5.2]). *Let S be a finite p -group with exactly one subgroup of order p . If S is not cyclic, then $p = 2$ and S contains elements a, b of order 4 with $bab^{-1} = a^{-1}$.*

Proof. Induction on $|S|$. As S is not cyclic, $|S| > p$; let M be a subgroup of index p , which is normal. M has exactly one subgroup of order p ; if M is not cyclic we are done by induction. So $M = \langle x \rangle$ with $|x| = p^n$, $n \geq 1$. Choose $y \in S \setminus M$; then $y^{-1}xy = x^\rho$ and $y^p = x^\sigma$ for some integers $\rho, \sigma \geq 0$ (M is normal of index p). Since y^p commutes with y , $x^{\rho\sigma} = y^{-1}x^\sigma y = x^\sigma$, so $\sigma(\rho - 1) \equiv 0 \pmod{p^n}$. Induction on k gives $(yx^j)^k = y^k x^{j(1+\rho+\dots+\rho^{k-1})}$, so

$$(yx^j)^p = x^{\sigma+j\Sigma}, \quad \Sigma = 1 + \rho + \dots + \rho^{p-1}.$$

If $p \nmid \sigma$, then y^p generates M , so y has order $p^{n+1} = |S|$ and S is cyclic. Hence $p \mid \sigma$. If some j has $\sigma + j\Sigma \equiv 0 \pmod{p^n}$, then $yx^j \notin M$ has order p , and $\langle yx^j \rangle$ is a second subgroup of order p , as M contains one: a contradiction. It remains to see when such a j exists. It does whenever $\Sigma = p^k u$ with $p \nmid u$ and $p^k \mid \sigma$: take $j \equiv -(\sigma/p^k)u^{-1} \pmod{p^n}$.

p odd. If $\rho \not\equiv 1 \pmod{p}$, then $(\rho - 1)\Sigma = \rho^p - 1 \equiv \rho - 1 \pmod{p}$ by Fermat, so $\Sigma \equiv 1 \pmod{p}$ ($k = 0$). If $\rho = 1 + cp$, then $\rho^i \equiv 1 + icp \pmod{p^2}$, so $\Sigma \equiv p + cp \cdot \frac{p(p-1)}{2} \equiv p \pmod{p^2}$ as p is odd; thus $\Sigma = pu$ with $p \nmid u$, and $p \mid \sigma$ ($k = 1$).

$p = 2$. Here $\Sigma = 1 + \rho$. If ρ is even, Σ is odd ($k = 0$). If $\rho \equiv 1 \pmod{4}$, $\Sigma = 2u$ with u odd ($k = 1$). Let $\rho \equiv 3 \pmod{4}$. Then $\rho - 1 = 2w$ with w odd, and $\sigma(\rho - 1) \equiv 0 \pmod{2^n}$ gives $2^{n-1} \mid \sigma$. Write $1 + \rho = 2^k u$ with u odd; if $k \leq n - 1$ we are done, and if $2^n \mid \sigma$ take $j = 0$. There remains the case $\rho \equiv -1 \pmod{2^n}$, $\sigma = 2^{n-1}v$ with v odd; then $n \geq 2$ (as $2 \mid \sigma$) and $\sigma \equiv 2^{n-1} \pmod{2^n}$, so $y^2 = x^{2^{n-1}}$, while $y^{-1}xy = x^{-1}$. Put $a = x^{2^{n-2}}$ (order 4) and $b = y$: $b^2 = a^2 \neq 1$ and $b^4 = 1$, so b has order 4, and $bab^{-1} = a^{-1}$. $\square$

Lemma 11.3 ([4]). *Every Sylow subgroup of the multiplicative group F^* of a finite division ring F is cyclic.*

Proof. Let S be a Sylow p -subgroup, $S \neq 1$. Its center contains an element g of order p . If $h \in S$ generates a subgroup of order p other than $\langle g \rangle$, then g and h commute, so they generate a commutative subring of F , a commutative domain in which the p^2 elements of $\langle g, h \rangle$ are roots of $X^p - 1$: impossible. So S has a unique subgroup of order p . Suppose S is not cyclic; by Lemma 11.2, $p = 2$ and there are a, b of order 4 with $ba = a^{-1}b$. Since $(a^2 - 1)(a^2 + 1) = a^4 - 1 = 0$ and $a^2 \neq 1$, $a^2 = -1$; in particular $1 \neq -1$, so the characteristic q of F is odd. Then $a^{-1} = a^3 = -a$, so $ba = -ab$, and likewise $b^2 = -1$. Take t, r from Lemma 11.1 for q . Using $a^2 = b^2 = -1$, $ba = -ab$,

$$(t + ra + b)(r^2 + 1 + rta + tb) = r(t^2 + r^2 + 1)a + (t^2 + r^2 + 1)b = 0.$$

If $t + ra + b = 0$, then $b = -t - ra$ commutes with a . Otherwise $r^2 + 1 + rta + tb = 0$ and $t \neq 0$, so $b = -t^{-1}(r^2 + 1) - ra$ commutes with a . Either way $ab = ba = -ab$, so $2ab = 0$, which is impossible as q is odd and $ab \neq 0$. $\square$

Theorem 11.4 (Wedderburn; proof of [4]). *Every finite division ring is commutative.*

Proof. By Lemma 11.3 all Sylow subgroups of F^* are cyclic, so (Hölder, Burnside, Zassenhaus; in Mathlib `IsZGroup.isCyclic.commutator`) the commutator subgroup $A = [F^*, F^*]$ is cyclic, in particular abelian; it is normal. We show below that an abelian normal subgroup A of F^* lies in the center Z of F^* . Then $[F^*, F^*] \subseteq Z$, so F^* is nilpotent; a nilpotent group with cyclic Sylow subgroups is the direct product of them, hence cyclic, so F^* is abelian and F is commutative. (The note argues instead with a minimal normal subgroup A/Z of the solvable group F^*/Z , which is cyclic, and the fact that a group cyclic modulo a central subgroup is abelian; the key step below is the same.)

Let $x \in F^*$ and $y \in A$; we show $xy = yx$. If $x = -1$ this is clear. Otherwise $1 + x \in F^*$, and as A is normal, $xyx^{-1} \in A$ and $z = (1 + x)y(1 + x)^{-1} \in A$. From $(1 + x)y = z(1 + x)$, $y - z = zx - xy = (z - xyx^{-1})x$. If $z = xyx^{-1}$, then $y = z = xyx^{-1}$. Otherwise $x = (z - xyx^{-1})^{-1}(y - z)$; since A is abelian, z , y and xyx^{-1} commute with y , hence so does x . Thus $A \subseteq Z$. $\square$

APPENDIX A. THE DISSERTATION'S SPECIAL FAMILIES

The dissertation uses a more restrictive notion of special family than [8]. Its results are proved here for completeness; they are not used elsewhere. A *complementary interval* of a closed $F \subseteq \mathbb{R}$ is a component of $\mathbb{R} \setminus F$.

Definition A.1 ([3, §4]). A *thesis-special family* is a nonempty family $\mathcal{F}$ of bounded closed subsets of $\mathbb{R}$ with an enumeration $F_1, F_2, \dots$ (every member occurring) such that for $m > n$, either F_m lies in a complementary interval of F_n or F_m splits with F_n . (In Lean, “ F' lies in a complementary interval of F ” is: $[x, y] \cap F = \emptyset$ for all $x, y \in F'$; the enumeration starts at F_0 .)

Lemma A.2. *Any two members of a thesis-special family are equal, disjoint, or split. Hence Lemmas 7.8 and 7.9 apply to thesis-special families of subsets of X ([3, Lemmas 13, 14]).*

Proof. A set lying in a complementary interval of F_n is disjoint from F_n . $\square$

Lemma A.3 ([3, Lemma 9]). *Every F_σ set $E \subseteq \mathbb{R}$ is $\bigcup_n E_n$ for bounded closed E_n any two of which are disjoint or split.*

Proof. As in Lemma 7.5, with $\mathbb{R}$ in place of X ; enumerate the resulting countable family, padding with $\emptyset$ if it is finite (repeating a member with two points would violate the condition). $\square$

Lemma A.4 ([3, Lemma 10]). *Every F_σ set $E \subseteq \mathbb{R}$ is the union of a thesis-special family.*

Proof. Take $E_0, E_1, \dots$ from Lemma A.3 and put $A_s = \bigcup_{j < s} E_j$. For $K \subseteq \mathbb{R}$ and $x \in \mathbb{R}$ let $[x]_K = \{y : [x, y] \cap K = \emptyset\}$; this is the component of $\mathbb{R} \setminus K$ containing x if $x \notin K$, it is empty if $x \in K$, and $[x]_K = [x']_K$ whenever $[x, x'] \cap K = \emptyset$. At stage s the new sets are $N_s(x) = \overline{E_s \cap [x]_{A_s}} \subseteq E_s$, $x \in E_s$ (this is $\bar{I} \cap E_s$ for the complementary interval I of A_s containing x , up to the endpoints of I that are not limits of $E_s \cap I$; the new sets are bounded and closed, and $N_s(x) = \emptyset$ for $x \in A_s$). The closure is needed: [3] uses $I \cap E_s$, which need not be closed (for $E_0 = [-1, 0]$ and $E_1 = \{0\} \cup \{1/k : k \geq 1\}$, which split at 0, the complementary interval $I = (0, \infty)$ of $A_1 = E_0$ gives $I \cap E_1 = \{1/k : k \geq 1\}$).

Stage s has only finitely many classes $[x]_{A_s}$, $x \in E_s$. Since $[x]_{K \cup L} = [x]_K \cap [x]_L$, it suffices to treat one E_j , $j < s$. If E_j, E_s split, say at p with $E_j \leq p \leq E_s$, then $[x, x'] \cap E_j = \emptyset$ for all $x, x' \in E_s \setminus E_j$ (a point of E_j in $[x, x']$ would be $\leq p \leq \min(x, x')$, hence equal to x or x'), so all $x \in E_s \setminus E_j$ have the same class, and the others have the empty class. If $E_j \cap E_s = \emptyset$, both are compact, so $|y - x| \geq r > 0$ for $y \in E_j$, $x \in E_s$. Cover E_s by finitely many balls $B(c, r)$, $c \in E_s$; if $|x - c| < r$ no $y \in E_j$ lies in $[x, c]$ (else $|x - c| \geq 2r$), so $[x]_{E_j} = [c]_{E_j}$. (This replaces the subsequence argument of [3].)

List the finitely many nonempty sets $N_s(x)$, $x \in E_s$, as $F_{s,0}, \dots, F_{s,n_s-1}$ without repetition, put $F_{s,i} = \emptyset$ for $i \geq n_s$, and enumerate the pairs (s, i) , $i \leq n_s$, lexicographically (stage by stage, one padding $\emptyset$ at the end of each stage), giving $F_0, F_1, \dots$. Then $\bigcup_k F_k = \bigcup_s E_s$: $F_k \subseteq E_s$, and the least s with $y \in E_s$ has $y \notin A_s$, so $y \in N_s(y)$.

Let $k < l$, $F_k = F_{s,i}$, $F_l = F_{s',i'}$, so $s < s'$ or $(s = s', i < i')$. If one of them is empty, the other lies in a complementary interval of it, trivially. Otherwise $F_k = N_s(x)$, $F_l = N_{s'}(x')$. If $s = s'$ then $N_s(x) \neq N_s(x')$ forces $[x, x'] \cap A_s \neq \emptyset$, say $x < z < x'$ with $z \in A_s$; then $[x]_{A_s} < z < [x']_{A_s}$, so $N_s(x) \leq z \leq N_s(x')$ and the two split. If $s < s'$ then E_s and $E_{s'}$ are disjoint or split. If they split, so do $F_k \subseteq E_s$ and $F_l \subseteq E_{s'}$. If they are disjoint, let $u, v \in F_l$ and $z \in [u, v] \cap F_k$. Then $z \neq u, v$ since $u, v \in E_{s'}$ and $z \in E_s$. So $u < z < v$, and there are $y < z < y'$ in $E_{s'} \cap [x']_{A_{s'}}$, whence $z \in [y, y'] \subseteq [x']_{A_{s'}}$ (this set is an interval), which is impossible for $z \in F_k \subseteq E_s \subseteq A_{s'}$. Hence $[u, v] \cap F_k = \emptyset$: F_l lies in a complementary interval of F_k . Finally $\mathcal{F} = \{F_k\}$ is thesis-special with $\bigcup \mathcal{F} = E$. $\square$

Lemma A.5 ([3, Lemmas 11, 12]). *If $\mathcal{F}$ and $\mathcal{G}$ are thesis-special, so is $\mathcal{F} \wedge \mathcal{G} = \{F \cap G : F \in \mathcal{F}, G \in \mathcal{G}\}$, and $\bigcup(\mathcal{F} \wedge \mathcal{G}) = \bigcup \mathcal{F} \cap \bigcup \mathcal{G}$.*

Proof. Enumerate $\mathbb{N} \times \mathbb{N}$ as $(a(k), b(k))$ by increasing $a + b$ and, for equal sums, increasing b ; then $k < l$ implies $a(k) < a(l)$ or $b(k) < b(l)$. Put $H_k = F_{a(k)} \cap G_{b(k)}$. If $l > k$ and $a(k) < a(l)$, then $F_{a(l)}$ lies in a complementary interval of $F_{a(k)}$, which lies in a complementary interval of $H_k \subseteq F_{a(k)}$, or $F_{a(l)}$ splits with $F_{a(k)}$; in either case H_l relates to H_k in the same way. The case $b(k) < b(l)$ is symmetric. $\square$

APPENDIX B. CONCORDANCE WITH THE SOURCES

Source item	Here
[6], with [10]	
Sutcliffe Thm 2	Prop. 2.3, Prop. 2.4, Thm. 2.6
Sutcliffe Thm 3	Thm. 2.7
Lemma 1, Lemma 2	Lemma 2.8, Lemma 2.9
Theorem, Corollary	Thm. 2.10, Cor. 2.11
[1] (from the review)	
Theorem 1 (homeomorphisms)	Thm. 6.8 (new proof)
Theorems 2, 3	Cor. 6.3
Theorems 4, 5	Thm. 8.3, Cor. 8.4
$f = 0$ a.e. with given boundary function	Cor. 8.12
remarks on three dimensions	not a theorem; see Section 10
[2]	
Lemma 1	Lemma 3.34
Lemmas 2, 3	Lemmas 5.2, 5.3
Theorem 1 (McMillan)	Thm. 5.4
Theorem 2, Corollaries 1–4	Thm. 5.5, Cor. 5.6
remark (E of type G_δ)	Prop. 6.6
[3]	
Lemmas 1–6	Lemma 3.4, 3.10, 3.11, 3.12, 3.13, Thm. 3.14
Theorems 1, 2	Thm. 3.20, Thm. 3.22

Lemmas 7, 8; Theorem 3	Lemmas 5.8, 5.9; Thm. 5.7
Lemmas 9–12	Lemmas A.3, A.4, A.5
facts (6)–(17); Lemmas 13, 14	Lemma 7.3; Lemmas 7.8, 7.9, A.2
Theorem 4	Thm. 7.11 (and Thm. 7.10)
Lemma 15; Theorem 5 and Corollary	Lemma 6.1; Thm. 6.2, Cor. 6.3
Lemmas 16, 17; Theorems 6, 7	Lemma 7.13, 7.12; Thm. 7.14, 7.19
Lemmas 18–22	Lemmas 3.33, 3.34, 3.36, 3.37, 3.38
Theorem 8 and Corollary	Thm. 8.3, Cor. 8.4
Lemmas 23, 24; Theorem 9; example	Lemmas 8.6, 8.7; Thm. 8.8; Prop. 8.9
Lemma 25; Theorem 10, Corollaries	Lemma 8.10; Thm. 8.11; Cor. 8.12, 8.14
unsolved problems 1–5	open questions; not formalized
[7]	
Lemma	Remark before Lemma 9.6; Lemma 9.6
example (answer to [11, Problem 5])	Lemmas 9.4, 9.5, Thm. 9.7
Theorem, Corollary, analytic case	Thm. 9.8, Cor. 9.9, Cor. 9.10
[8]	
(A) with (I)–(XXII)	Thm. 7.10; Def. 7.4, Lemmas 7.5, 7.6; Def. 7.1, Lemma 7.3; Def. 7.7, Lemmas 7.8, 7.9
(B) with (I)–(IV)	Thm. 7.14; Lemmas 7.12, 7.13; Thm. 3.14
(C), (D), (E), (F)	Thms. 7.15, 7.16, 7.17, 7.18; Def. 3.17, Lemma 3.18
Appendix (a), (b), (c)	Thms. 3.20, 3.22, 3.21
Problems 1, 2	open questions; not formalized
[9]	
Lemma, Theorem	Lemma 10.3, Thm. 10.8
Problem	open question; not formalized
[4]	
Lemma; proof of the theorem	Lemma 11.1; Lemmas 11.2, 11.3, Thm. 11.4

REFERENCES

- [1] T. J. Kaczynski, *Boundary functions for functions defined in a disk*, J. Math. Mech. **14** (1965), 589–612. (Known here through MR0176080.)
- [2] T. J. Kaczynski, *On a boundary property of continuous functions*, Michigan Math. J. **13** (1966), 313–320.
- [3] T. J. Kaczynski, *Boundary functions*, Doctoral dissertation, University of Michigan, 1967.
- [4] T. J. Kaczynski, *Another proof of Wedderburn’s theorem*, Amer. Math. Monthly **71** (1964), 652–653.
- [5] M. Hall, *The theory of groups*, Macmillan, New York, 1959.
- [6] T. J. Kaczynski, *Note on a problem of Alan Sutcliffe*, Math. Mag. **41** (1968), 84–86.
- [7] T. J. Kaczynski, *Boundary functions for bounded harmonic functions*, Trans. Amer. Math. Soc. **137** (1969), 203–209.
- [8] T. J. Kaczynski, *Boundary functions and sets of curvilinear convergence for continuous functions*, Trans. Amer. Math. Soc. **141** (1969), 107–125.

- [9] T. J. Kaczynski, *The set of curvilinear convergence of a continuous function defined in the interior of a cube*, Proc. Amer. Math. Soc. **23** (1969), 323–327.
- [10] A. Sutcliffe, *Integers that are multiplied when their digits are reversed*, Math. Mag. **39** (1966), 282–287.
- [11] F. Bagemihl and G. Piranian, *Boundary functions for functions defined in a disk*, Michigan Math. J. **8** (1961), 201–207.
- [12] F. Bagemihl, *Curvilinear cluster sets of arbitrary functions*, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 379–382.
- [13] F. Bagemihl and W. Seidel, *Koebe arcs and Fatou points of normal functions*, Comment. Math. Helv. **36** (1961), 9–18.
- [14] S. Banach, *Über analytisch darstellbare Operationen in abstrakten Räumen*, Fund. Math. **17** (1931), 283–295.
- [15] F. Hausdorff, *Über halbstetige Funktionen und deren Verallgemeinerung*, Math. Z. **5** (1919), 292–309.
- [16] K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, 1962.
- [17] S. Kierst, *Sur l'ensemble des valeurs asymptotiques d'une fonction méromorphe dans le cercle-unité*, Fund. Math. **27** (1936), 226–233.
- [18] K. Kuratowski, *Topologie I*, 3rd ed., Warsaw, 1952.
- [19] O. Lehto and K. I. Virtanen, *Boundary behaviour and normal meromorphic functions*, Acta Math. **97** (1957), 47–65.
- [20] J. E. McMillan, *Boundary properties of functions continuous in a disc*, Michigan Math. J. **13** (1966), 299–312.
- [21] M. H. A. Newman, *Elements of the topology of plane sets of points*, 2nd ed., Cambridge, 1964.
- [22] W. Sierpiński, *Sur une propriété des ensembles ambigus*, Fund. Math. **6** (1924), 1–5.
- [23] W. Sierpiński, *General topology*, Univ. of Toronto Press, 1952.
- [24] H. Tietze, *Über stetige Kurven, Jordansche Kurvenbögen und geschlossene Jordansche Kurven*, Math. Z. **5** (1919), 284–291.