华中科技大学2021-2022学年第一学期

《复变函数与积分变换》A卷参考答案及评分标准

一. 单选题.(每题2分,共24分)

DBCC BABA DCAC

二. 解 (1) $u_{xx}=6x+2ay$, $u_{yy}=2bx+6y$

$\because u$ 为调和函数 $\therefore u_{xx}+u_{yy}=0$

即: $(6+2b)x+(2a+6)y=0$

$\therefore a=-3$. $b=-3$

(2) $\because u(x,y)=x^3-3x^2y-3xy^2+y^3$

$\therefore u_x=3x^2-6xy-3y^2 \cdots\cdots ①$

$u_y=-3x^2-6xy+3y^2 \cdots\cdots ②$

由 C-R方程: $u_x=v_y$

$\therefore v_y=3x^2-6xy-3y^2$

偏积分得: $v(x,y)=3x^2y-3xy^2-y^3+\varphi(x)$.

又 $u_y=-v_x$, $\therefore -3x^2-6xy+3y^2=-[6xy-3y^2+\varphi'(x)]$

$\Rightarrow \varphi'(x)=3x^2$, $\therefore \varphi(x)=x^3+c$

$\therefore v(x,y)=3x^2y-3xy^2-y^3+x^3+c$

代入 $f(z)=u(x,y)+iv(x,y)$ 即可.

三. 解：(1) $f(z)=\dfrac{1}{z^2(z-i)} = -\dfrac{1}{iz^2}\dfrac{1}{(1-\frac{z}{i})}$

$$= -\frac{1}{iz^2}\sum_{n=0}^{+\infty}\left(\frac{z}{i}\right)^n$$

$$= -\sum_{n=0}^{+\infty}\frac{1}{i^{n+1}}z^{n-2}$$

(2) $\because \dfrac{1}{z} = \dfrac{1}{z-i+i} = \dfrac{1}{z-i}\dfrac{1}{1-(-\frac{i}{z-i})}$

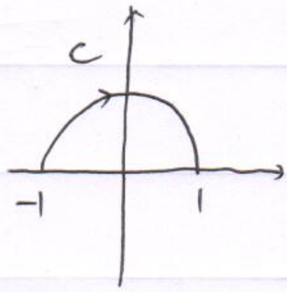
$$= \frac{1}{z-i}\sum_{n=0}^{+\infty}\left(-\frac{i}{z-i}\right)^n$$

$$= \sum_{n=0}^{+\infty}(-i)^n(z-i)^{-(n+1)}$$

$\therefore \dfrac{1}{z^2} = -\left(\dfrac{1}{z}\right)' = \sum_{n=0}^{+\infty}(-i)^n(n+1)(z-i)^{-(n+2)}$

$\therefore f(z)=\dfrac{1}{z^2(z-i)} = \sum_{n=0}^{+\infty}(-i)^n(n+1)(z-i)^{-(n+3)}$

四. 1. $\int_C \frac{3z}{\bar{z}} + \sin\frac{\pi z}{2}\,dz$



$= \int_C \frac{3z^2}{\bar{z}\cdot z} + \sin\frac{\pi z}{2}\,dz$

$= \int_C 3z^2 + \sin\frac{\pi z}{2}\,dz$

$= z^3 + (-\frac{2}{\pi}\cos\frac{\pi}{2}z)\Big|_{-1}^{1}$ (由被积函数解析性)

$= 2$

注：第一项的积分也可由曲线的参数方程化为定积分.

2. 由高阶导数定理. 原式 $= 2\pi i\,(e^{iz} - e^{-iz})'\Big|_{z=0}$

$= 2\pi i\,(i - 2i)$

$= 2\pi$.

另解：也可由留数定理计算，$z=0$是一阶极点

$\therefore$ 原式 $= 2\pi i\ \mathrm{Res}[f(z), 0]$

$= 2\pi i \lim\limits_{z\to 0} z \cdot \frac{e^{iz} - e^{i2z}}{z^2}$

$= 2\pi i \lim\limits_{z\to 0} \frac{e^{iz} - e^{i2z}}{z}$

$= 2\pi i \lim\limits_{z\to 0} (ie^{iz} - 2ie^{i2z})$

$= 2\pi i\,(-i)$

$= 2\pi$.

五. 1. 原式 $= -2\pi i \, Res[f(z), \infty]$

$$= 2\pi i \, Res[f(\tfrac{1}{z}) \cdot \tfrac{1}{z^2}, 0]$$

$$= 2\pi i \, Res[\frac{1}{(1-z^2)(1+z^8)z^5}, 0]$$

$$\because \frac{1}{(1-z^2)(1+z^8)z^5} = \frac{1}{z^5}(1+z^2+z^4+z^6+\cdots)(1-z^8+z^{16}-\cdots)$$

$$= \cdots + \frac{1}{z} + \cdots$$

$$\therefore Res[\frac{1}{(1-z^2)(1+z^8)z^5}, 0] = 1$$

$\therefore$ 原式 $= 2\pi i$

2. 原式 $= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{(x^2+3)(x^2+5)} dx$

设 $f(z) = \frac{1}{(z^2+3)(z^2+5)}$，则 $f(z)$ 在上半平面仅

有两个一阶极点 $\sqrt{3}i$ 和 $\sqrt{5}i$

$$\therefore Res[f(z), \sqrt{3}i] = \frac{1}{(z+\sqrt{3}i)(z^2+5)}\bigg|_{z=\sqrt{3}i} = \frac{1}{4\sqrt{3}i}$$

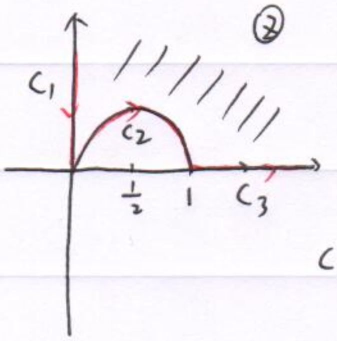
$$Res[f(z), \sqrt{5}i] = \frac{1}{(z+\sqrt{5}i)(z^2+3)}\bigg|_{z=\sqrt{5}i} = \frac{1}{-4\sqrt{5}i}$$

$$\therefore \text{原式} = \frac{1}{2} \cdot 2\pi i \left(\frac{1}{4\sqrt{3}i} + \frac{1}{-4\sqrt{5}i}\right)$$

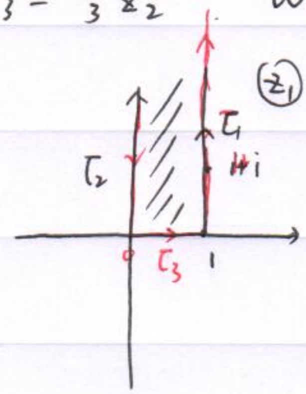
$$= \frac{\pi}{4}\left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}}\right) = \frac{\pi(\sqrt{5}-\sqrt{3})}{4\sqrt{15}}.$$

六、解 $w=e^{\frac{\pi}{3}i(1-\frac{1}{z})}$ 可分解为如下几种映射的复合

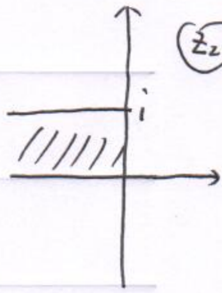
$z_1=\frac{z-1}{z}$， $z_2=iz_1$， $z_3=\frac{\pi}{3}z_2$ $w=e^{z_3}$



$z_1=\frac{z-1}{z}$



$z_2=iz_1$

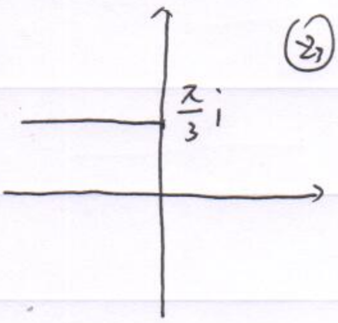


$$C_1:\begin{cases}\infty\to 1\\ i\to 1+i\\ 0\to\infty\end{cases}$$

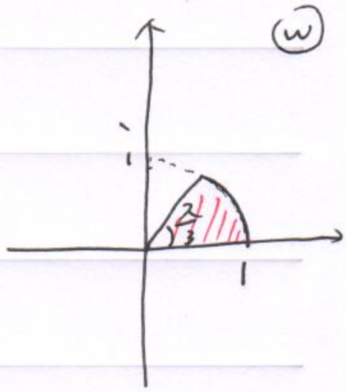
$$C_3\begin{cases}1\to 0\\ 2\to\frac{1}{2}\\ \infty\to 1\end{cases}$$

C_2 利用与 C_3 的保角性

$z_3=\frac{\pi}{3}z_2$

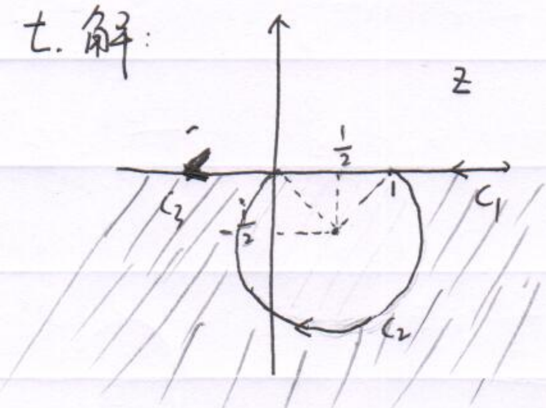


$w=e^{z_3}$

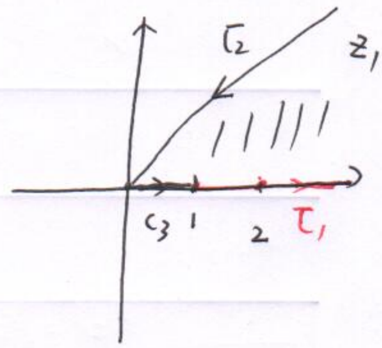


像区域为 $\{w:\ |w|<1,\ 0<\arg w<\frac{\pi}{3}\}$.

七. 解:



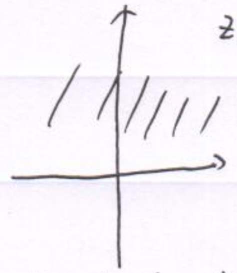
$z_1 = \frac{z}{z-1}$ $\longrightarrow$



$c_1: \begin{cases} \infty \to 1 \\ 2 \to 2 \\ 1 \to \infty \end{cases}$

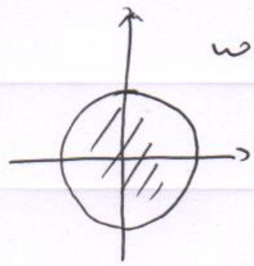
$c_3: \begin{cases} 0 \to 0 \\ -1 \to \frac{1}{2} \\ \infty \to 1 \end{cases}$

$z_2 = z_1^4$ $\longrightarrow$



c_2 利用保角性

$w = \frac{z_2 - i}{z_2 + i}$ $\longrightarrow$



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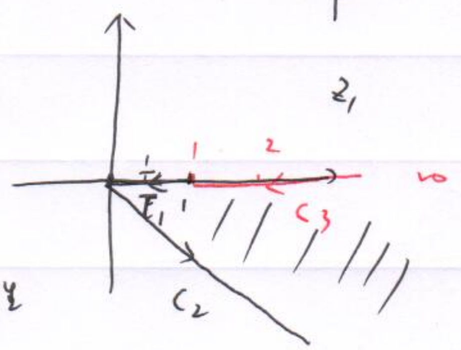
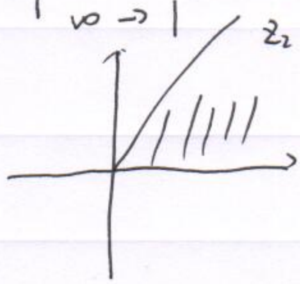
第一步还可考虑 $z_1 = \frac{z-1}{z}$ $\longrightarrow$



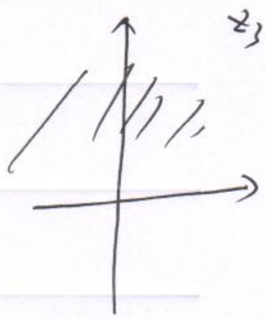
$c_1: \begin{cases} \infty \to 1 \\ 2 \to \frac{1}{2} \\ 1 \to 0 \end{cases}$ c_2 保角性

$c_3: \begin{cases} 0 \to \infty \\ -1 \to 2 \\ \infty \to 1 \end{cases}$

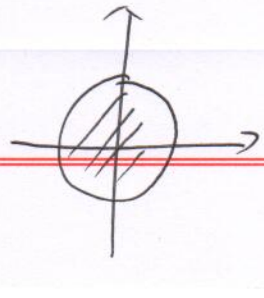
$z_2 = e^{\frac{\pi}{4} i} z_1$ $\longrightarrow$



$z_3 = z_2^4$ $\longrightarrow$



$w = \frac{z_3 - i}{z_3 + i}$ $\longrightarrow$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八. 解: 记 $X(s)=\mathcal{L}(x(t))$, $Y(s)=\mathcal{L}(y(t))$, 对方程组左右两边作 Laplace 变换得:

$$\begin{cases} sX(s)-1+Y(s)=2\cdot\dfrac{1}{s-1} \\ sY(s)-X(s)=-\dfrac{1}{s^2} \end{cases}$$

解得: $Y(s)=\dfrac{1}{(s-1)s}=\dfrac{1}{s-1}-\dfrac{1}{s}$

$$X(s)=\frac{1}{s^2}+\frac{1}{s-1}$$

$\therefore\ y(t)=\mathcal{L}^{-1}[Y(s)]=e^{t}-1$

$$x(t)=\mathcal{L}^{-1}[X(s)]=e^{t}+t$$

九：证明：∵ $f(z)$ 在 $|z|\le 2$ 上解析

设 $g(z)=f'(z)$，则 $|g(z)-1|\le|z|$，且 $g(z)$ 在 $|z|\le 2$ 上解析.

$$\therefore f''(1)=g'(1)=\frac{1}{2\pi i}\oint_{|z-1|=r}\frac{g(z)}{(z-1)^2}\,dz \qquad (r\le 1)$$

$$\therefore |f''(1)|\le\frac{1}{2\pi}\oint_{|z-1|=r}\frac{|g(z)|}{|z-1|^2}\,ds$$

$$\le\frac{1}{2\pi}\oint_{|z-1|=r}\frac{|g(z)-1|+1}{|z-1|^2}\,ds$$

$$\le\frac{1}{2\pi}\oint_{|z|=r}\frac{|z|+1}{|z-1|^2}\,ds$$

$$\le\frac{1}{2\pi}\oint_{|z|=r}\frac{|z-1|+2}{|z-1|^2}\,ds$$

$$=\frac{1}{2\pi}\frac{r+2}{r^2}\cdot 2\pi r$$

$$=1+\frac{2}{r}$$

取 $r=1$. 则 $|f''(1)|\le 3$.