Problems for the Calculus Tutorial

Ivan Savov, Minireference Co.

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Limits

P1 Calculate the following limit expressions:

$$\text{(a)} \lim_{x\to\infty} \tfrac{7}{x+4} \qquad \text{(b)} \lim_{x\to\infty} \tfrac{4x^2-7x+1}{x^2} \qquad \text{(c)} \lim_{x\to 0^-} \tfrac{1}{x}$$

1 (a) 0. (b) 4. (c) $-\infty$.

1 a) As x goes to infinity, the denominator goes to infinity, so the fraction goes to zero.

b) If we divide each term by x^2, we get the expression $\frac{4x^2-7x+1}{x^2} = 4 - \frac{7}{x} + \frac{1}{x^2}$. The limit of the first term is 4. The limits of the second and third terms are zero as x goes to infinity.

c) As x approaches 0 from the left ($x \to 0^-$), the fraction $\frac{1}{x}$ takes on larger and larger negative numbers. Therefore $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

P2 Assuming $\lim_{x\to\infty} f(x) = 2$ and $\lim_{x\to\infty} g(x) = 3$, compute

$$\text{(a)} \lim_{x\to\infty} \big(2f(x) - g(x)\big) \qquad \text{(b)} \lim_{x\to\infty} f(x)\,g(x) \qquad \text{(c)} \lim_{x\to\infty} \frac{4f(x)}{g(x)+1}.$$

2 (a) 1. (b) 6. (c) 2.

2 We're given $\lim_{x\to\infty} f(x) = 2$ and $\lim_{x\to\infty} g(x) = 3$. For a) we use the sum rule:

$$\lim_{x\to\infty} \big(2f(x) - g(x)\big) = 2 \lim_{x\to\infty} f(x) - \lim_{x\to\infty} g(x) = 2\cdot 2 - 3 = 1.$$

To solve b), we use the product rule for limits:

$$\lim_{x\to\infty} f(x)\,g(x) = \Big(\lim_{x\to\infty} f(x)\Big)\Big(\lim_{x\to\infty} g(x)\Big) = 2\cdot 3 = 6.$$

To solve c) we use the quotient rule for limits:

$$\lim_{x\to\infty} \frac{4f(x)}{g(x)+1} = \frac{4\lim_{x\to\infty} f(x)}{\lim_{x\to\infty}(g(x)+1)} = \frac{8}{\lim_{x\to\infty} g(x)+1} = \frac{8}{3+1} = 2.$$

Derivatives

P3 Find the derivative with respect to x of the functions:

$$\text{(a) } f(x) = x^{13} \qquad \text{(b) } g(x) = \sqrt[3]{x} \qquad \text{(c) } h(x) = ax^2 + bx + c.$$

3 $f'(x) = 13x^{12}$. $g'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. $h'(x) = 2ax + b$.

3 a) Use the power rule $\frac{d}{dx}x^n = nx^{n-1}$: $f'(x) = \frac{d}{dx}x^{13} = 13x^{12}$.
b) Rewrite $\sqrt[3]{x} = x^{1/3}$ and use the power rule: $g'(x) = \frac{d}{dx}x^{1/3} = \frac{1}{3}x^{-2/3}$.
c) Differentiate term-by-term: $h'(x) = \frac{d}{dx}(ax^2 + bx + c) = 2ax + b$.

P4 Calculate the derivatives of the following functions:

$$\text{(a) } p(x) = \frac{2x+3}{3x+2} \qquad \text{(b) } q(x) = \sqrt{x^2+1} \qquad \text{(c) } r(\theta) = \sin^3\theta.$$

4 $p'(x) = \frac{-5}{(3x+2)^2}$. $q'(x) = \frac{x}{\sqrt{x^2+1}}$. $r'(\theta) = 3\sin^2\theta\cos\theta$.

4 a) Use the quotient rule $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$ with $u = 2x + 3$ and $v = 3x + 2$:

$$p'(x) = \frac{2(3x+2) - (2x+3)\cdot 3}{(3x+2)^2} = \frac{6x + 4 - 6x - 9}{(3x+2)^2} = \frac{-5}{(3x+2)^2}.$$

b) Write $q(x) = (x^2+1)^{\frac{1}{2}}$ and use the chain rule:

$$q'(x) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}.$$

c) Using the chain rule fives us $r'(\theta) = 3\sin^2\theta \cdot \cos\theta$.

P5 Find the maximum and the minimum of $f(x) = x^5 - 5x$.

5 Max at $x = -1$; min at $x = 1$.

5 First we compute the derivative $f'(x) = 5x^4 - 5 = 5(x^4 - 1)$. The critical points are $x = -1$ and $x = 1$. The second derivative is $f''(x) = 20x^3$. We apply the second derivative test to each critical point:

$$f''(-1) = -20 < 0 \;\Rightarrow\; \text{local maximum at } x = -1,$$

$$f''(1) = 20 > 0 \;\Rightarrow\; \text{local minimum at } x = 1.$$

Integrals

P6 Calculate the integral function $F_0(b) = \int_0^b f(x)\,dx$ for the polynomial $f(x) = 4x^3 + 3x^2 + 2x + 1$.

6 $F_0(b) = b^4 + b^3 + b^2 + b$.

6 Integrate term-by-term using the formula $\int_0^b x^n\,dx = \frac{1}{n+1}b^{n+1}$.

P7 Find the area under $f(x) = 8 - x^3$ between $x = 0$ and $x = 2$.

7 $A_f(0,2) = 12$.

7 We compute the area using the definite integral:

$$A_f(0,2) = \int_0^2 (8 - x^3)\,dx = \left[8x - \frac{x^4}{4}\right]_0^2 = \left(16 - \frac{16}{4}\right) - 0 = 12.$$

P8 Find the area under the graph of $g(x) = \sin(x)$ from $x = 0$ to $x = \pi$.

8 $A_g(0,\pi) = 2$.

8 The area is given by the following integral:

$$A_g(0,\pi) = \int_0^\pi \sin(x)\,dx = \big(-\cos\pi\big) - \big(-\cos 0\big) = 1 - (-1) = 2.$$

P9 Compute $\int_0^1 \frac{4x}{(1+x^2)^3}\,dx$ using the substitution $u = 1 + x^2$. Check your answer numerically using the SciPy function `quad`.

9 $\frac{3}{4}$.

9 When using the change of variable $u = 1 + x^2$, we must also change the differential $du = 2x\,dx$, which conveniently contains x that appears in the numerator, which allows us to write:

$$\int_{x=0}^{x=1} \frac{4x}{(1+x^2)^3}\,dx = \int_{x=0}^{x=1} \frac{2}{u^3}\,du = \int_{x=0}^{x=1} 2u^{-3}\,du.$$

Next we must change the x-limits of integration to u-limits of integration: The lower limit $x = 0$ becomes $u = 1 + 0^2 = 1$, and the upper limit $x = 1$ becomes $u = 1 + 1^2 = 2$, which the complete substitution:

$$\int_{x=0}^{x=1} \frac{4x}{(1+x^2)^3}\,dx = 2\int_{u=1}^{u=2} u^{-3}\,du.$$

We can now proceed using the integral rule $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ to obtain

$$\begin{aligned} 2\int_{u=1}^{u=2} u^{-3}\,du &= 2\left[\frac{u^{-2}}{-2}\right]_1^2 = -\left[u^{-2}\right]_1^2 = -\left[\frac{1}{u^2}\right]_1^2 \\ &= -\left(\frac{1}{2^2} - \frac{1}{1^2}\right) = -\left(\frac{1}{4} - 1\right) = \frac{3}{4} \end{aligned}$$

Sequences and series

P10 Calculate the value of the infinite series $\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$.

10 3.

10 We can use the formula for the geometric series $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ with $r = \frac{2}{3}$, which gives us $\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = \frac{1}{1-\frac{2}{3}} = 3$.

P11 Find the Taylor series for the function $f(x) = e^{-x}$.
Hint: Use algebraic manipulations starting from a Taylor series that you know.

11 $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$.

11 Start from the known Taylor series $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Substitute $-x$ for x to get

$$f(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}.$$

Answers

P1 (a) 0. (b) 4. (c) $-\infty$. **P2** (a) 1. (b) 6. (c) 2. **P3** $f'(x) = 13x^{12}$. $g'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. $h'(x) = 2ax + b$. **P4** $p'(x) = \frac{-5}{(3x+2)^2}$. $q'(x) = \frac{x}{\sqrt{x^2+1}}$. $r'(\theta) = 3\sin^2\theta\cos\theta$. **P5** Max at $x = -1$; min at $x = 1$. **P6** $F_0(b) = b^4 + b^3 + b^2 + b$. **P7** $A_f(0,2) = 12$. **P8** $A_g(0,\pi) = 2$. **P9** $\frac{3}{4}$. **P10** 3. **P11** $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$.