Vector Boot Camp

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Preface

Vectors are a familiar concept from high school mathematics, and you likely already have a good understanding of them. However, vectors are far more important than you might realize. They are arguably one of the most crucial mathematical concepts not only in science but also in the modern society. . A vector can take on many forms: in high school math, it's often seen as two or three numbers; in university physics, it's represented as an arrow; and in university mathematics, it's an element of a vector space. What you have learned is just the beginning, and you will need to develop a deeper and more versatile understanding of vectors.

This Boot Camp is designed to bridge the gap between high school math and university physics. You will be well-prepared to tackle the challenges and opportunities of advanced physics studies. Good luck!

Remarks

This Boot Camp contains all the topics that you will need in General Physics 1 and 2 lectures. Most of the topics will be discussed during Sho's lecture, so you do not have to do all by yourself. Still, it is beneficial if you can learn these topics by yourselves, so that you can focus more on physics during the lectures.

Sho never provides you with solutions[♯1] **You students** need to make the solution. To this end,

- Share your answers to other colleagues, using LINE or Google Docs[♯2]. Compare your answers with theirs.
- Ask questions to colleagues, to the TA, or to Sho. You can utilize Sho's office hours[♯3].

> ♯**1:** Think why. Sho thinks that proding solutions is against the integrity (or a moral) as a scientist.

> ♯**2:** `https://docs.google.com/`

> ♯**3:** `https://www2.nsysu.edu.tw/iwamoto/`

B.1 Definition of Vectors

Definition B.1: Vectors (for physics)

- A vector is a physical quantity that has both **magnitude** and **direction**.
- We describe them by $\vec{a}$, $\vec{A}$, $\vec{p}_\mathrm{F}$, etc.[♯4], and their magnitudes by $|\vec{a}|$, $|\vec{A}|$, $|\vec{p}_\mathrm{F}|$, etc.
- We often describe them by arrows. The arrows' direction should match the vector's direction. The arrows' length should be *proportional to* the vector's magnitude.

♯4: Professionals use boldface, e.g., $\boldsymbol{a}$, $\boldsymbol{A}$, $\boldsymbol{p}_\mathrm{F}$, etc., but in this lecture we only use the beginners' style.

Definition B.2: Vector quantity and Scalar quantity

- Physical quantities with direction are called **vector quantity**.
  - They can be described by vectors.[♯5]
- Physical quantities without direction are called **scalar quantity**.
  - They can be described by numbers.
- Consider a vector quantity $\vec{A}$. Its magnitude $|\vec{A}|$ is a scalar quantity.

♯5: If the space considered is one-dimensional, we can describe the direction by $+$ or $-$ and a vector quantity can be described by a number. Here, however, **you** must specify which direction is positive.

For example, **mass** m and **temperature** T are scalar quantities. **Velocity** $\vec{v}$ is a vector quantity and its magnitude $|\vec{v}|$ (it has a special name "**speed**") is a scalar quantity. Acceleration $\vec{a}$ is a vector quantity.

★[A] Choose vector quantities. Choose scalar quantities.

temperature	Sho's height	time difference	resistor
air pressure	Sho's weight	time	resistance
wind speed	Sho's mass	duration	conductor
velocity	city name	distance	conductance
electric charge	position	kilometer	conductivity

B.2 How to describe directions

> **Remark:** Using natural English is nice, but it is more important, and thus we should pay greater attention, to ensure that the expression is clear and does not cause any misunderstanding.

Basic *We* need to learn how to describe directions of vectors **in English**. The easiest one is

- leftward = toward the left = to the left
- rightward = toward the right = to the right
- upward = toward the top = to the top
- downward = toward the bottom = to the bottom

If you need to describe the third dimension which is perpendicular to the textbook's page or the sheet (or the blackboard), you can use

- into the (sheet | page | blackboard) = toward the (sheet | page | blackboard)
- out of the (sheet | page | blackboard) = away from the (sheet | page | blackboard)

For 2d case, **once you specify the "north" direction,** you can use the following expressions:

- northward, southward, eastward, westward (= (to | toward) the north, etc.)
- northwestward, southwestward, etc. (= (to | toward) the northwest, etc.)

With axes Usually, we define x-axis and y-axis (and z-axis, if 3d). **If such axes are defined** (or once you have defined them), we can use

- in the positive x-direction = in the $+x$ direction
- in the positive y-direction = in the $+y$ direction
- in the negative x-direction = in the $-x$ direction
- in the negative y-direction = in the $-y$ direction

In particular, for 2d cases, we can use the angle from the positive x-axis (counterclockwise):

- 30° from the $+x$ axis (= 30° counterclockwise from the positive x-axis)
- 90° from the $+x$ axis (= in the $+y$ direction)
- 170° from the $+x$ axis (= 10° clockwise from the negative x-axis)
- $-15°$ from the $+x$ axis (= 15° clockwise from the positive x-axis)
- angle θ from the $+x$ axis, where $\tan\theta = 0.1$ and $0 < \theta < \pi/2$.

Sometimes the word "the horizontal" is useful:

- 10° above the horizontal / 30° below the horizontal

> **Remark:** It is difficult to describe general 3d directions by words. For this purpose, we usually use mathematical expressions (i.e., vectors).

Rotations To describe the direction of rotation, we use

- clockwise / counterclockwise

If you have an arrow and discuss rotations about the arrow, you can use the following expressions:

- following the right-hand rule (= counterclockwise when viewed from the arrow's direction)
- following the left-hand rule (= clockwise when viewed from the arrow's direction)

We will often use these expressions in General Physics 2.

B.3 Vectors: Magnitude and Direction

In this Boot Camp, we forget about physics[♯6]. Focusing on mathematics, we will discuss vectors in its mathematical aspects. We will begin with a few more expressions useful for vectors.

If the angle between $\vec{a}$ and $\vec{b}$ is 90°, we say

- $\vec{a}$ is **perpendicular to** $\vec{b}$.
- $\vec{a}$ is orthogonal to $\vec{b}$.
- $\vec{a}$ is normal to $\vec{b}$.
- $\vec{a}$ and $\vec{b}$ are perpendicular to each other.
- $\vec{a}$ and $\vec{b}$ are orthogonal to each other.
- $\vec{a}$ and $\vec{b}$ are normal to each other.

(all of them are correct but the first one is the most common). If the angle is 0°, we say[♯7]

- $\vec{a}$ is **in the same direction as** $\vec{b}$.
- $\vec{a}$ and $\vec{b}$ are in the same direction.

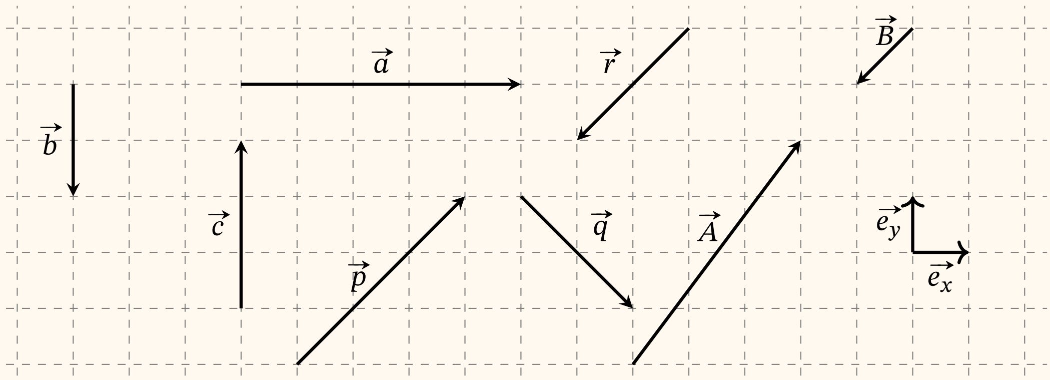
Finally, if the angle is 180°,

- $\vec{a}$ is **anti-parallel to** $\vec{b}$.
- $\vec{a}$ is opposite to $\vec{b}$.
- $\vec{a}$ and $\vec{b}$ are anti-parallel.
- $\vec{a}$ and $\vec{b}$ are in opposite directions.

> **♯6:** We forget units and significant figures, which you will learn in the beginning of General Physics 1 lectures. In other words, in the lectures, **you must not forget units and significant figures** of vector quantities.

> **♯7: Avoid** the word "parallel" because some people think it includes both 0° and 180° cases. Clarity is important in science.

★[B] Vectors are drawn on the grid, which has a spacing of 1. Answer the following questions.



(1) Describe the direction (in English words) and magnitude of each vector. Try to use multiple expressions.

(2) Describe relationship between the directions of the following vector pairs: ($\vec{a}$ and $\vec{b}$), ($\vec{b}$ and $\vec{c}$), ($\vec{p}$ and $\vec{q}$), ($\vec{p}$ and $\vec{r}$), ($\vec{e_x}$ and $\vec{e_y}$), and ($\vec{a}$ and $\vec{e_x}$). For example, "$\vec{a}$ is perpendicular to $\vec{b}$".

B.4 Vector Basics

These are a few basic things that you have learned in highschool:

- **(Zero vector)** There is a special vector $\vec{0}$. Its length is 0 (zero) and it has no direction.
- **(Scalar multiplication)** If $\vec{v}$ is a vector, $2\vec{v}$ and $-3\vec{v}$ are both vectors.
- **(Addition)** If $\vec{p}$ and $\vec{q}$ are vectors, $\vec{p}+\vec{q}$ is a vector.

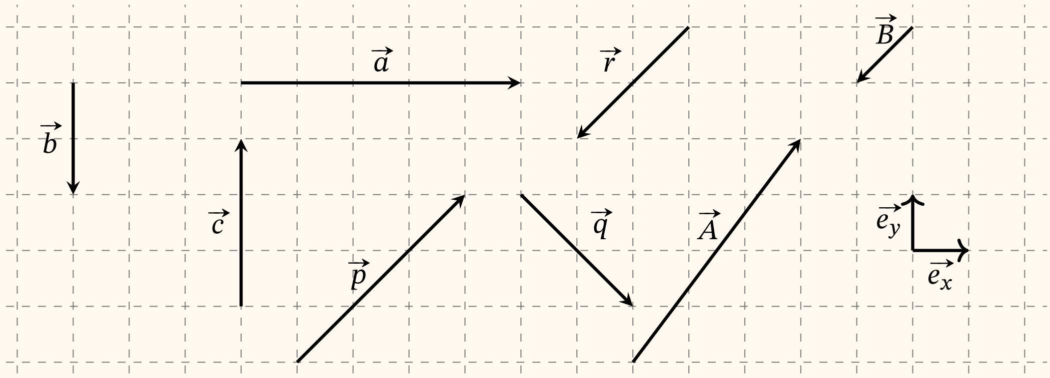
By extending the last two items, we can obtain the following statement.

- **(Linear combination)** If a and b are scalars (i.e., real numbers) and $\vec{p}$ and $\vec{q}$ are vectors,

  $$a\vec{p}+b\vec{q}$$

  is a vector. Here, a and b can be positive, zero, or negative.

★**[C]** Vectors are drawn on the grid, which has a spacing of 1.



(1) Describe $\vec{a}$ by using $\vec{e}_x$ (and a number). Describe $\vec{b}$ and $\vec{c}$ by using $\vec{e}_y$.

(2) Describe $\vec{B}$ by using $\vec{r}$. Describe $\vec{p}$ by using $\vec{r}$.

(3) Describe $\vec{A}$ and $\vec{B}$ by using $\vec{e}_x$ and $\vec{e}_y$.

(4) Describe $\vec{e}_x$ and $\vec{e}_y$ by using $\vec{A}$ and $\vec{B}$.
[Hint: See your answer of the previous problem.]

(5) Draw a vector $\vec{\beta}$ that is in the same direction as $\vec{b}$ and $|\vec{\beta}|=1$.

★**[D]** Consider a vector $\vec{s}$ whose length is 3, i.e., $|\vec{s}|=3$.

(1) Calculate the magnitude of $2\vec{s}$, $-3\vec{s}$, and $0\vec{s}$.

(2) Calculate the magnitude of $\dfrac{\vec{s}}{|\vec{s}|}$.

(3) Describe relationship between the directions of the following vector pairs:
($\vec{s}$ and $4\vec{s}$), ($0.1\vec{s}$ and $-4\vec{s}$), and ($\vec{s}$ and $\dfrac{\vec{s}}{|\vec{s}|}$).

(4) Let $\vec{S}=-3\vec{s}$ and k be a real number. Find the direction and magnitude of $\vec{s}+k\vec{S}$.

Here is one more important concept:

- **(Unit vector)** If a vector has a magnitude of one, it is called a **unit vector**.

If $\vec{a}$ is not the zero vector,

- the unit vector in the same direction as $\vec{a}$ is given by $\dfrac{\vec{a}}{|\vec{a}|}$; it is denoted by $\vec{\hat{a}}$ (hat + arrow);
- the unit vector that is anti-parallel to $\vec{a}$ is given by $-\dfrac{\vec{a}}{|\vec{a}|}$ $(= -\vec{\hat{a}})$.

> **Remark:** In the textbook, vectors are written by $\vec{\mathbf{a}}$ and unit vectors are by $\hat{\mathbf{a}}$. (Some of you will be confused by this notation in General Physics 2.) Although the notation is a bit messy, Sho will always use $\vec{a}$ and $\vec{\hat{a}}$.

***[E] Consider $\vec{s}$ and $\vec{t}$, which satisfy $|\vec{s}| = 3$ and $|\vec{t}| = 2$. Here, $k > 0$.

(1) Find the unit vector whose direction is the same as $\vec{s}$.

(2) Find the unit vector whose direction is the same as $-2\vec{t}$.

(3) Find the unit vector whose direction is the same as $4\vec{s} + 3\vec{t}$.

(4) Find the unit vector anti-parallel to $\vec{s}$.

(5) Find the vector which is anti-parallel to $\vec{s}$ and has a magnitude of 12.

(6) Find the vector which is in the same direction as $\vec{t}$ and has a magnitude of k.

You will see unit vectors many times in the following exercises.

B.5 Inner product

We use this definition, which may be different from what you learned in highschool.

Definition B.3: Inner product

For vectors $\vec{a}$ and $\vec{b}$, the **inner product** of $\vec{a}$ and $\vec{b}$ is defined by

$$\vec{a} \cdot \vec{b} \stackrel{\text{def}}{=} |\vec{a}||\vec{b}|\cos\theta, \qquad \text{where } \theta \text{ is the angle between } \vec{a} \text{ and } \vec{b}.$$

***[F] Prove the following equations **from the above definition**.

(1) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ for any vectors $\vec{a}$ and $\vec{b}$.

(2) $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b})$ for any vectors $\vec{a}$ and $\vec{b}$ and any number k.

(3) $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ for any vector $\vec{a}$.

(4) $\vec{a} \cdot \vec{b} = 0$ if $\vec{a} = \vec{0}$, $\vec{b} = \vec{0}$, or $\vec{a}$ is perpendicular to $\vec{b}$.

(5) $-|\vec{a}||\vec{b}| \le \vec{a} \cdot \vec{b} \le |\vec{a}||\vec{b}|$ for any vectors $\vec{a}$ and $\vec{b}$.

We can also prove $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$ from the above definition. However, the proof is a bit technical and thus skipped. You will anyway use this equation.

The next two problems are the most important ones in this Boot Camp.

★[G] Two vectors $\vec{x}$ and $\vec{y}$ satisfy $|\vec{x}| = |\vec{y}| = 1$ and $\vec{x} \cdot \vec{y} = 1/2$.

(1) Calculate the angle between $\vec{x}$ and $\vec{y}$.

(2) Expand $|a\vec{x} + b\vec{y}|^2$ and calculate its value. Calculate $|a\vec{x} + b\vec{y}|$.

Now, assume $\vec{s} = 3\vec{x} + 3\vec{y}$ and $\vec{t} = 2\vec{x} - \vec{y}$. Also, let $\vec{\hat{s}}$ be the unit vector whose direction is the same as $\vec{s}$.

(3) Calculate $|\vec{s}|$, $|\vec{t}|$, and $\vec{s} \cdot \vec{t}$. Find the angle between $\vec{s}$ and $\vec{t}$.

(4) Describe $\vec{\hat{s}}$. [Hint: The answer will contain $\vec{x}$ and $\vec{y}$.]

(5) Assume $\vec{u} = \vec{x} + k\vec{y}$ is perpendicular to $\vec{s}$. Find the value of k.

★[H] Two vectors $\vec{e_x}$ and $\vec{e_y}$ satisfy $|\vec{e_x}| = |\vec{e_y}| = 1$ and $\vec{e_x} \cdot \vec{e_y} = 0$.

(1) Calculate the angle between $\vec{e_x}$ and $\vec{e_y}$.

(2) Expand $|a\vec{e_x} + b\vec{e_y}|^2$ and calculate its value. Calculate $|a\vec{e_x} + b\vec{e_y}|$.

Now, assume $\vec{p} = a\vec{e_x} + b\vec{e_y}$, which is not $\vec{0}$, and $\vec{q} = 3\vec{e_x} + 4\vec{e_y}$.

(3) Calculate $|\vec{p}|$, $\vec{p} \cdot \vec{e_x}$, and $\vec{p} \cdot \vec{e_y}$. Check that $\vec{p} = (\vec{p} \cdot \vec{e_x})\vec{e_x} + (\vec{p} \cdot \vec{e_y})\vec{e_y}$.

(4) Calculate $|\vec{q}|$, $\vec{q} \cdot \vec{e_x}$, $\vec{q} \cdot \vec{e_y}$, and $\vec{p} \cdot \vec{q}$.

(5) Find the unit vector whose direction is the same as $\vec{p}$.

★**[I]** Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, where $\vec{i}$, $\vec{j}$, and $\vec{k}$ are (any) vectors.

(1) Expand $|\vec{a}|^2$ and $\vec{a}\cdot\vec{b}$.

Now, assume they satisfy $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$ and $\vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{k} = \vec{k}\cdot\vec{i} = 0$.

(2) Simplify the answer of the previous question.

(3) Calculate the angle between $\vec{a}$ and $\vec{b}$.

(4) Write down $\hat{a}$ (the unit vector in $\vec{a}$'s direction) by using $a_{1,2,3}$, $\vec{i}$, $\vec{j}$, and $\vec{k}$.

***[J]** Three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ satisfy $|\vec{A}| = 2$, $|\vec{B}| = 3$, $\vec{A}\cdot\vec{B} = 3\sqrt{2}$, and $\vec{A}\cdot\vec{C} = -1$.

(1) Find the angle between $\vec{A}$ and $\vec{B}$.

(2) Calculate $|\vec{A}+\vec{B}|^2$, $|\vec{A}-\vec{B}|^2$, and $|2\vec{A}+4\vec{B}|^2$.

(3) Calculate $(\vec{A}-2\vec{B})\cdot(2\vec{A}+\vec{B}+\vec{C}) + 2\vec{B}\cdot\vec{C}$.

(4) Assume $|\vec{A}+k\vec{B}| = \sqrt{10}$. Find the value of k.

(5) Assume $\vec{B}+c\vec{C}$ is perpendicular to $\vec{A}$. Find the value of c.

(6) What do we know about the values of $|\vec{C}|$ and $|\vec{A}-\vec{C}|$?

*[K]** Two vectors $\vec{e_x}$ and $\vec{e_y}$ satisfy $|\vec{e_x}| = |\vec{e_y}| = 1$ and $\vec{e_x}\cdot\vec{e_y} = 0$. Define

$$\vec{a_x} = \frac{1}{2}\vec{e_x} + \frac{1}{2}\vec{e_y}, \qquad \vec{f_x} = \vec{e_x}\cos\theta + \vec{e_y}\sin\theta,$$
$$\vec{a_y} = \vec{e_x} + \frac{1}{2}\vec{e_y}, \qquad \vec{f_y} = -\vec{e_x}\sin\theta + \vec{e_y}\cos\theta.$$

First, consider $\vec{A} = A\vec{a_x} + B\vec{a_y}$.

(1) Calculate $\vec{a_x}\cdot\vec{a_y}$, $|\vec{a_x}|$, and $|\vec{a_y}|$.

(2) Calculate $|\vec{A}|$.

(3) Write down $\vec{A}$ by using $\vec{e_x}$, $\vec{e_y}$, A, and B.

Next, consider $\vec{P} = P\vec{f_x} + Q\vec{f_y}$ and $\vec{S} = S\vec{f_x} + T\vec{f_y}$.

(4) Calculate $\vec{f_x}\cdot\vec{f_y}$, $|\vec{f_x}|$, and $|\vec{f_y}|$.

(5) Calculate $\vec{P}\cdot\vec{S}$, $|\vec{P}|$, and $|\vec{S}|$.

(6) Write down $\vec{P}$ by using $\vec{e_x}$, $\vec{e_y}$, P, Q, and θ.

(7) Write down $\vec{e_x}$ and $\vec{e_y}$ by using $\vec{f_x}$, $\vec{f_y}$, and θ.

(8) Check the following equation:

$$\vec{P} = (\vec{P}\cdot\vec{f_x})\vec{f_x} + (\vec{P}\cdot\vec{f_y})\vec{f_y} = (\vec{P}\cdot\vec{e_x})\vec{e_x} + (\vec{P}\cdot\vec{e_y})\vec{e_y}$$

B.6 Cross product

The definition of the cross product may also be different from highschool mathematics.

Definition B.4: Cross product

For vectors $\vec{a}$ and $\vec{b}$, the **cross product** of $\vec{a}$ and $\vec{b}$ is defined as follows:

- it is a vector and denoted by $\vec{a} \times \vec{b}$;
- its magnitude $|\vec{a} \times \vec{b}|$ is given by $|\vec{a}||\vec{b}| \sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$;
- if $|\vec{a} \times \vec{b}| \neq 0$, its direction is specified so that
  - $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$;
  - The triplet $(\vec{a}, \vec{b}, \vec{a} \times \vec{b})$ satisfies the right-hand rule.

Here, "$(\vec{p}, \vec{q}, \vec{r})$ satisfies the right-hand rule" means that if you hold your right hand so that your thumb points in the direction of $\vec{p}$ and your index finger $\vec{q}$, then your middle finger points in the direction of $\vec{r}$. Or, if you draw $\vec{p}$ upward and $\vec{q}$ leftward on a sheet, then $\vec{p} \times \vec{q}$ points out of the sheet toward you.[♯8]

♯**8:** More precisely, if $\vec{p}$ and $\vec{q}$ are on the sheet and the direction of $\vec{q}$ is θ clockwise from $\vec{p}$ with $0° < \theta < 180°$, then $\vec{p} \times \vec{q}$ is normal to the sheet and points out of the sheet. If $180° < \theta < 360°$, then $\vec{p} \times \vec{q}$ is normal to the sheet and points toward the sheet.

*****[L]** Prove the following equations **from the above definition**.

(1) $\vec{a} \times \vec{a} = \vec{0}$.

(2) $\vec{a} \times \vec{0} = \vec{0} \times \vec{a} = \vec{0}$.

(3) $\vec{a} \times \vec{b} = \vec{0}$ if $\vec{a}$ and $\vec{b}$ are parallel or anti-parallel.

(4) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

(5) $(k\vec{a}) \times \vec{b} = \vec{a} \times (k\vec{b}) = k(\vec{a} \times \vec{b})$.

(6) $0 \leq |\vec{a} \times \vec{b}| \leq |\vec{a}||\vec{b}|$.

(7) $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$.

(8) $(\vec{a} \cdot \vec{b})^2 + |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2$.

We can also prove $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$ from the above definition. However, the proof is extremely technical and thus skipped. You need this equation in the next problem.

★**[M]** Three vectors $\vec{e_x}$, $\vec{e_y}$, and $\vec{e_z}$ satisfy

$$|\vec{e_x}| = |\vec{e_y}| = |\vec{e_z}| = 1, \quad \vec{e_x} \times \vec{e_y} = \vec{e_z}, \quad \vec{e_y} \times \vec{e_z} = \vec{e_x}, \quad \vec{e_z} \times \vec{e_x} = \vec{e_y}.$$

(1) Calculate $\vec{e_x} \cdot \vec{e_y}$. Find the angles among $\vec{e_x}$, $\vec{e_y}$, and $\vec{e_z}$.

(2) Expand $(a\vec{e_x} + b\vec{e_y} + c\vec{e_z}) \times (p\vec{e_x} + q\vec{e_y} + r\vec{e_z})$ and simplify.

B.7 Operation on vectors — Introduction to the Vector/Scalar/Not game

We have seen all the operations. Namely, (only) the following operations are possible on vectors:

- **magnitude**: $|\vec{a}|$ (← scalar)
- **scalar multiplication**: $k\vec{a}$ (← vector)
- **addition**: $\vec{a}+\vec{b}$ (← vector)
- **inner product**: $\vec{a}\cdot\vec{b}$ (← scalar)
- **cross product**: $\vec{a}\times\vec{b}$ (← vector)

where k is a scalar and $\vec{a}$ and $\vec{b}$ are vectors. Any complicated symbols, such as

$$\frac{\vec{a}\times\vec{b}+(\vec{x}\cdot\vec{y})\vec{d}}{(\vec{a}\cdot\vec{b})|\vec{b}\times\vec{c}+\vec{d}|}$$

can be obtained by combining these five operations.

> **Remark:** The numerator is the sum of two vectors: the first one is $\vec{a}\times\vec{b}$, and the second one is $\vec{d}$ multiplied by a scalar (number) $\vec{x}\cdot\vec{y}$. The denominator is made of two numbers; the first number is $\vec{a}\cdot\vec{b}$, while in the second, $\vec{b}\times\vec{c}$ is a vector, so it can be added to $\vec{d}$, and then its magnitude is evaluated.

In other words, **we cannot apply any other operations** except for these five. Namely,

$$(\vec{a})^2,\quad \frac{1}{\vec{a}},\quad k+\vec{a},\quad \vec{a}\,\vec{b},\quad \sqrt{\vec{a}},\quad \frac{\vec{a}}{\vec{a}},\quad a+\vec{a},\quad \vec{a}\cdot\vec{b}+\vec{a}\times\vec{b},\quad (\vec{a}+\vec{b})\vec{c}$$

are all invalid expression; they do not express any mathematical concept.

Now, you need to distinguish **which is vector, which is scalar, and which is invalid**.

★**[N]** For each expression, answer **V** if it is vector; **S** if it is scalar, and **N** if it is an invalid expression (neither vector nor scalar).
[Notice that symbols with an arrow are vectors and symbols without an arrow are scalars.]

$3+\lvert\vec{a}\rvert$	$\vec{a}-\vec{b}$	$\vec{a}\,\vec{b}$	$\vec{a}/\vec{b}$	$\vec{p}\times(\vec{q}\times\vec{r})$
$3\vec{a}$	$\vec{0}+1$	$\vec{0}$	$-\vec{a}$	$\vec{p}\cdot(\vec{q}\times\vec{r})$
$3\times\vec{a}$	$\vec{k}\times\vec{a}$	$x+\lvert\vec{a}\rvert$	$\lvert\vec{a}\rvert^{-1}\vec{a}$	$p\times(\vec{q}\times\vec{r})$
$\frac{1}{\vec{x}+\vec{y}}$	$\frac{x+y}{\lvert\vec{x}+\vec{y}\rvert}$	$\frac{\vec{x}+\vec{y}}{\lvert\vec{x}+\vec{y}\rvert}$	$\frac{\vec{x}+\vec{y}}{\vec{x}+\vec{y}}$	$\vec{p}(\vec{q}\times\vec{r})$
$\frac{1}{(\vec{a}\cdot\vec{b})^2}$	$\frac{\vec{a}}{(\vec{a}\cdot\vec{b})^2}$	$\frac{1}{(\vec{a})^2}$	$\frac{1}{\lvert\vec{a}\rvert^2}$	$p(\vec{q}\times\vec{r})$

B.8 Axes and Components

We have had various discussions about vectors and have covered all important concepts. Remarkably, in all our discussions so far, **we have not talked about the components of vectors!** In fact, **we haven't not even introduced x-axis or y-axis!** This demonstrates that the aforementioned concepts, including inner product and cross product, can be defined and understood without a coordinate system and without components of vectors. Namely, **vectors are independent of any coordinate system**.

This fact is related to physics, because our Universe does not have predefined axes. Axes and coordinates are artificial concepts, introduced by us, just for the sake of calculation.

Now, we will introduce three axes to discuss our three-dimensional Universe. The Cartesian coordinate system is defined by x-axis, y-axis, and z-axis, defined so that each axis is perpendicular to the other two axes.

In physics, we always use **right-handed Cartesian coordinate system**. Namely, we define z-axis so that ($+x$ axis, $+y$ axis, $+z$ axis) satisfies the right-hand rule (cf. Sec. B.6). Namely,

Definition B.5: Unit vectors in the direction of the axes

The symbol $\vec{e_x}$ denotes the unit vector in the positive x-direction; $\vec{e_y}$ and $\vec{e_z}$ are similarly defined. (In the textbook, they are denoted by $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. Please not be confused.)

and then (we can easily prove that) these unit vectors satisfy

$$|\vec{e_x}| = |\vec{e_y}| = |\vec{e_z}| = 1, \qquad \vec{e_x} \cdot \vec{e_y} = \vec{e_y} \cdot \vec{e_z} = \vec{e_z} \cdot \vec{e_x} = 0,$$
$$\vec{e_x} \times \vec{e_y} = \vec{e_z}, \qquad \vec{e_y} \times \vec{e_z} = \vec{e_x}, \qquad \vec{e_z} \times \vec{e_x} = \vec{e_y}.$$

> **Remark:** The definition of cross product uses right-hand rule because we use right-handed Cartesian coordinate system.

Now, we introduced *a notation rule*: we identify $A\vec{e_x} + B\vec{e_y} + C\vec{e_z} \overset{!!}{\longleftrightarrow} \begin{pmatrix} A \\ B \\ C \end{pmatrix}$. Then it is obvious that $\vec{e_x} \overset{!!}{\longleftrightarrow} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{e_y} \overset{!!}{\longleftrightarrow} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{e_z} \overset{!!}{\longleftrightarrow} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, etc.

> **Remark:** You probably use the horizontal notation (A, B, C) in highschool. In university, try to use this vertical notation $\left(\begin{smallmatrix} A \\ B \\ C \end{smallmatrix}\right)$ because it is consistent with the notation of matrices and thus natural when you learn linear algebra.

In practice, we simply use the equal sign. For example,

$$\vec{v} = 2\vec{e_x} + \vec{e_y} - 3\vec{e_z} \quad \overset{!!}{\longleftrightarrow} \quad \vec{v} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}.$$

However, it should be noted that the right-hand side, $\left(\begin{smallmatrix} 2 \\ 1 \\ -3 \end{smallmatrix}\right)$, is artificial and superficial; the true meaning of $\vec{v}$ is the left-hand side.

Remark: It might be interesting to observe the following transformation:

$$\begin{aligned}
\vec{x} &= (p\vec{e_x} + q\vec{e_y} + r\vec{e_z}) & \overset{!!}{\longleftrightarrow} \quad \vec{x} &= \begin{pmatrix} p \\ q \\ r \end{pmatrix} \\
&= (p\vec{e_x}) + (q\vec{e_y}) + (r\vec{e_z}) & \overset{!!}{\longleftrightarrow} \quad &= \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ q \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \\
&= p(\vec{e_x}) + q(\vec{e_y}) + r(\vec{e_z}) & \overset{!!}{\longleftrightarrow} \quad &= p\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + q\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + r\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}$$

In university mathematics, vectors are *defined* based on these equations.

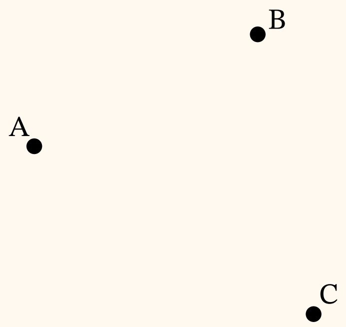
★**[0]** Let $\vec{A} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\vec{B} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$. **Prove** the following equations.

[Hint: You must have already proved them in previous sections.]

(1) $|\vec{A}| = \sqrt{a^2 + b^2 + c^2}$ **(2)** $\vec{A} \cdot \vec{B} = ap + bq + cr$ **(3)** $\vec{A} \times \vec{B} = \begin{pmatrix} br - cq \\ cp - ar \\ aq - bp \end{pmatrix}$

B.9 Position Vectors (1)

★**[P]** Three points are defined at the right.



(1) Define a point O (at any place).

Based on your point O, we define $\vec{a} = \overrightarrow{\mathrm{OA}}$, $\vec{b} = \overrightarrow{\mathrm{OB}}$, and $\vec{c} = \overrightarrow{\mathrm{OC}}$.

(2) Draw $\vec{a}$, $\vec{c}$, and $\vec{b} + \vec{c}$.

(3) Draw $\vec{b} - \vec{a}$ and $\vec{a} - \vec{c}$.

(4) Express $\overline{\mathrm{AB}}$ by using $\vec{a}$ and $\vec{b}$.

(5) We want to define M so that $\overrightarrow{\mathrm{OM}} = (\vec{b} + \vec{c})/2$. Draw $(\vec{b} + \vec{c})/2$. Where is M?

(6) Let G be the geometric center of the triangle ABC. Express $\overrightarrow{\mathrm{OG}}$ by $\vec{a}$, $\vec{b}$, and $\vec{c}$.

This problem describes the last topic of this Boot Camp, which is important in General Physics 2. Namely, by defining a point O, we can identify a position (in *xyz*-space) and a vector.

$$\text{point A} \overset{!!}{\longleftrightarrow} \vec{a}, \qquad \text{point B} \overset{!!}{\longleftrightarrow} \vec{b}, \qquad \text{point M} \overset{!!}{\longleftrightarrow} \vec{m}, \qquad \text{point G} \overset{!!}{\longleftrightarrow} \vec{g}, \text{ etc.,}$$

where the vectors defined by $\vec{a} = \overrightarrow{\mathrm{OA}}$ etc. are called **position vectors**; in the following, we will use the notation A($\vec{a}$) and so on. Of course, the position vectors vary depending on the choice of O.

The merit of position vector is that, if a point P is defined by other points A, B, etc., then $\vec{p} = \overrightarrow{\mathrm{OP}}$ can be expressed by $\vec{a}$, $\vec{b}$, etc. For example, if we define M($\vec{m}$) by the midpoint of B and C, then $\vec{m}$ is given by $(\vec{b} + \vec{c})/2$. Similarly, for the geometric center G, we can express $\overrightarrow{\mathrm{OG}}$ by $\vec{a}$, $\vec{b}$, and $\vec{c}$, as you did in the above problem.

The following two facts must be **firmly** memorized (for General Physics 2).

- $\overrightarrow{\mathrm{AB}} = \vec{b} - \vec{a}$ for points A($\vec{a}$) and B($\vec{b}$).
- $\overline{\mathrm{AB}} = |\vec{b} - \vec{a}|$ for points A($\vec{a}$) and B($\vec{b}$).

★**[Q]** Points A($\vec{a}$), B($\vec{b}$), and C($\vec{c}$) are drawn on the grid, which has a spacing of 1.

(1) Express $\vec{a}$ and $\vec{c}$ by using $\vec{e_x}$ and $\vec{e_y}$.

(2) Express $\vec{a}$ and $\vec{c}$ by its components, i.e., in the $\binom{x}{y}$ notation.

(3) Express $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{BC}$, $\overrightarrow{BA}$, and $\overrightarrow{CB}$ by using $\vec{a}$, $\vec{b}$, and $\vec{c}$.

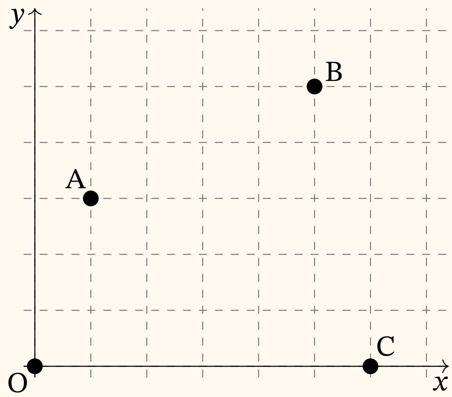
(4) Express $\overline{AB}$ by using $\vec{a}$ and $\vec{b}$.

(5) Find $\overrightarrow{AB} \cdot \overrightarrow{AC}$, $|\overrightarrow{AB}|$, and $|\overrightarrow{AC}|$.

(6) Find $\angle$BAC.

(7) Find the area of the triangle ABC.

(8) Calculate $|\overrightarrow{AB} \times \overrightarrow{AC}|/2$.



Now, we define a point P($\vec{p}$) by $\vec{p} = x\vec{e_x} + y\vec{e_y} = \begin{pmatrix} x \\ y \end{pmatrix}$.

(9) Express $\overrightarrow{AP}$ and $\overrightarrow{BP}$ by using $\vec{a}$, $\vec{b}$, and $\vec{p}$.

(10) Express $\overrightarrow{AP}$ and $\overrightarrow{BP}$ by using $\vec{e_x}$, $\vec{e_y}$, x, and y.

(11) If $\overrightarrow{AP} \cdot \overrightarrow{BP} = 0$ and $y = 2$, what is the value of x? Draw the point (x, y) on the grid.

****[R]** Let A($\vec{a}$) at $(1, 3)$ as in the previous problem. We define

$$f(x, y) = \frac{1}{\sqrt{(x-1)^2 + (y-3)^2}}.$$

We can understand that this function $f(x, y)$ is defined on the xy-plane. Namely, a number $f(x, y)$ is associated for each point P(x, y).

(1) Verify the following: if we define $\vec{p} = \begin{pmatrix} x \\ y \end{pmatrix}$, then $f(x, y) = \dfrac{1}{|\vec{p} - \vec{a}|}$.

Now, we can identify $f(x, y) \overset{!!}{\longleftrightarrow} f(\vec{p})$ from $\mathrm{P} = (x, y) \overset{!!}{\longleftrightarrow} \vec{p} = x\vec{e_x} + y\vec{e_y}$.

(2) Express the following function g and h by using x and y (and other numbers), where $\vec{b} = 5(\vec{e_x} + \vec{e_y})$:

$$g(\vec{p}) = \frac{1}{|\vec{p} - \vec{a}|^2}, \qquad h(\vec{p}) = |\vec{p} - \vec{a}| + |\vec{p} - \vec{b}|.$$

B.10 Extra Exercise

These problems are taken from 高校数学例題＆問題集 https://web.math-aquarium.jp/ thanks to the author's generosity. You may find explanations and more problems at (in Japanese)

https://web.math-aquarium.jp/rennsyuu-heimennjounobekutoru-y.pdf

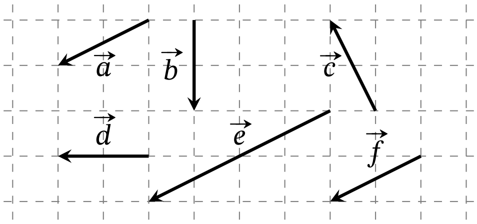
[Note: Some problems are skipped.]

***[1] Find the following from the right figure.

(1) equal vectors.

(2) vectors of equal magnitude.

(3) vectors with the same direction.



***[3] (1) A vector $\vec{x}$ satisfies $2(\vec{a}+2\vec{x})-4\vec{a}=5(\vec{x}-3\vec{b})$. Express $\vec{x}$ by $\vec{a}$ and $\vec{b}$.

(2) Vectors $\vec{x}$ and $\vec{y}$ satisfy $\vec{x}+\vec{y}=\vec{a}$ and $3\vec{x}+2\vec{y}=\vec{b}$. Express them by $\vec{a}$ and $\vec{b}$.

***[4] Consider a triangle ABC, where $\overline{\text{AB}}=1$, $\overline{\text{AC}}=\sqrt{5}$, $\overline{\text{BC}}=2$, and $\angle\text{B}=90°$. Let $\vec{e}$ be a unit vector whose direction is the same as $\overrightarrow{\text{BC}}$. Express $\vec{e}$ by $\overrightarrow{\text{AB}}$ and $\overrightarrow{\text{AC}}$.

***[5] Let $\vec{a}=\begin{pmatrix}2\\-4\end{pmatrix}$, $\vec{b}=\begin{pmatrix}5\\-3\end{pmatrix}$, and $\vec{c}=-3\vec{a}+2\vec{b}$. Find the components of $\vec{c}$ and calculate $|\vec{c}|$.

**[6] Let $\vec{a}=\begin{pmatrix}2\\-4\end{pmatrix}$, $\vec{b}=\begin{pmatrix}5\\-3\end{pmatrix}$, and $\vec{p}=\begin{pmatrix}7\\0\end{pmatrix}$. Find s and t that satisfy $\vec{p}=s\vec{a}+t\vec{b}$.

**[7] (1) Four points A$(2,-4)$, B$(5,-3)$, C$(2,1)$, and D are on xy-plane, where the quadrilateral ABCD is a parallelogram. Find the coordinate of D.

(2) Two vectors $\vec{a}=\begin{pmatrix}2\\-4\end{pmatrix}$ and $\vec{b}=\begin{pmatrix}5+t\\-3-t\end{pmatrix}$ are parallel. What is t?

***[8] (1) Let $\vec{a}$ and $\vec{b}$ satisfy $|\vec{a}|=3$ and $|\vec{b}|=2$ and form an angle 45°. Find $\vec{a}\cdot\vec{b}$.

(2) Let $\vec{a}=\begin{pmatrix}2\\-4\end{pmatrix}$ and $\vec{b}=\begin{pmatrix}5\\3\end{pmatrix}$. Find $\vec{a}\cdot\vec{b}$.

***[9] (1) Let $\vec{a}=\begin{pmatrix}3\\7\end{pmatrix}$ and $\vec{b}=\begin{pmatrix}-5\\-2\end{pmatrix}$. Find the angle formed by them.

(2) Two vectors $\vec{a}=\begin{pmatrix}2\\-4\end{pmatrix}$ and $\vec{b}=\begin{pmatrix}5+x\\3+x\end{pmatrix}$ are normal to each other. Find x.

**[10] Two vectors $\vec{a}$ and $\vec{b}$ satisfy $|\vec{a}|=3$, $|\vec{b}|=4$, and $|\vec{a}+2\vec{b}|=7$. Find the angle θ formed by them, assuming $0\le\theta\le\pi$.

**[12] (1) Consider two points A$(\vec{a})$ and B$(\vec{b})$. Two points P and M are on the line AB. They are between A and B and satisfy $\overline{\text{AP}}:\overline{\text{PB}}=5:3$ and $\overline{\text{AM}}:\overline{\text{MB}}=1:1$. Describe the position vectors of P and M by using $\vec{a}$ and $\vec{b}$.

(2) Consider a triangle ABC formed by three points A$(\vec{a})$, B$(\vec{b})$, and C$(\vec{c})$. Let G$(\vec{g})$ be its geometric center, and G$'(\vec{g}')$ be the geometric center of the triangle GBC. Describe $\vec{g}'$ by $\vec{a}$, $\vec{b}$, and $\vec{c}$.

*[15] Consider a triangle ABC and let O be its circumcenter. Consider a point H satisfying $\overrightarrow{\text{OH}}=\overrightarrow{\text{OA}}+\overrightarrow{\text{OB}}+\overrightarrow{\text{OC}}$. Prove H is the orthocenter of the triangle ABC.