LIMID Quality Control Models for Increasing Failure Rate Processes

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Abstract

A **Li**mited **M**emory **I**nfluence **D**iagram (LIMID) model for quality control that incorporates variable data on sample means from the output of a production process is introduced. The process operates over a finite production horizon and is *out-of-control* when the process mean for the output shifts. The probability such a shift occurs in the next time period is dependent on the elapsed time since the most recent process repair. A set of control limits that are adapted to the length the process has run without repair is selected to minimize quality control costs, and the sampling interval and sample size can be adjusted to further reduce costs if these modifications are operationally feasible. This is the first application of LIMIDs in a quality control model with an increasing rate of failure over time, and that implements variable data.

Keywords: Limited memory influence diagram; control chart; quality; statistical process control; variable data; Weibull distribution.

1. Introduction

Shewhart (1931) introduced control charts as a tool for monitoring business and production processes to ensure that output meets customer expectations for quality. These methods are designed to incorporate sample output from the process, then utilize statistics calculated from the output to identify possible *assignable* causes of variation that might negatively affect quality. For instance, a machine could become miscalibrated or worn out so that the average weight of the units produced is lower than desired. Assignable causes must be differentiated from *common* causes of variation that occur even when the process is working as intended.

Control charts and related statistical process control (SPC) methods can be classified according to whether they require attribute or variable data. *Attribute* data measure a qualitative aspect of the process output; for example, a sample unit may be categorized as defective or not defective, or the number of blemishes on a sample unit can be counted. *Variable* data is quantitative and measured on a continuous scale, such as weight or volume; for example, the mean and/or range of the weights of sample units. This paper introduces a **Li**mited **M**emory **I**nfluence **D**iagram (LIMID) model for SPC using variable data where the likelihood that an assignable cause of variation increases as the time since the last maintenance or repair lengthens.

Duncan (1956) introduced *economic design* of control charts where the user-defined parameters – the sample size, sampling interval, and control limits – are chosen to minimize relevant costs. The model suggests a decision on whether or not to stop a process and search for an assignable cause of variation that causes the mean of the process output to shift. As with many subsequent SPC methods employing economic design, the model assumes that the time between occurrences of the assignable cause follows an exponential distribution. This effectively means that the probability the assignable cause occurs in the next time interval of a given length is the same regardless of when the last maintenance or repair occurred.

Certain processes may be subject to assignable causes where the rate of failure increases as the time since the last maintenance or repair event becomes longer. This could be due to the deterioration or wear of a mechanical component, for example, and these types of processes are addressed in this paper. Prior work that considers "increasing hazard rate" processes includes models with both uniform and nonuniform sampling intervals where the time between failures follows a Weibull distribution (Banarjee and Rahim, 1988; Rahim, 1993). This idea was extended by Rahim and Costa (2000) when the process is subject to assignable causes that affect both the mean and variability of the output. Saniga (1979) studies models with discrete time between arrivals that have both constant and non-constant deterioration rates where the assignable cause affects both the mean and variance.

The previous studies primarily rely on analytical formulas for average hourly costs per time period over an infinite production horizon. Determining sample sizes, sampling intervals, and control limit policies that minimize these costs typically involves searching numerous potential solutions. The control limit policies are static over time. This paper considers a scenario where the production horizon is finite (although not necessarily short) and terminates at a scheduled maintenance interval or when a certain number of units are produced. Thus, the static control limit policies developed for the infinite horizon situation may not be the best rules to apply in this setting. The LIMID model of Lauritzen and Nilsson (2001) is implemented to determine these policies. LIMIDs have previously been used for SPC, but only for attribute data with a constant failure rate (Cobb, 2021, 2022, 2024a). Cobb (2024b) addresses nonconstant process deterioration, but only for attribute data.

The paper proceeds as follows. The next section defines the production process and the LIMID model used for SPC. This is followed by a description of the results of a baseline example problem, an interpretation of the control limits for the problem, and a sensitivity analysis of the parameters and assumptions in the model. A discussion of potential future research concludes the paper.

2. Model Description

This section describes the production process and LIMID model. The problem will be described in the context of a specific example implemented by Banarjee and Rahim (1988). The only modification of that example is the assumption of a finite production horizon.

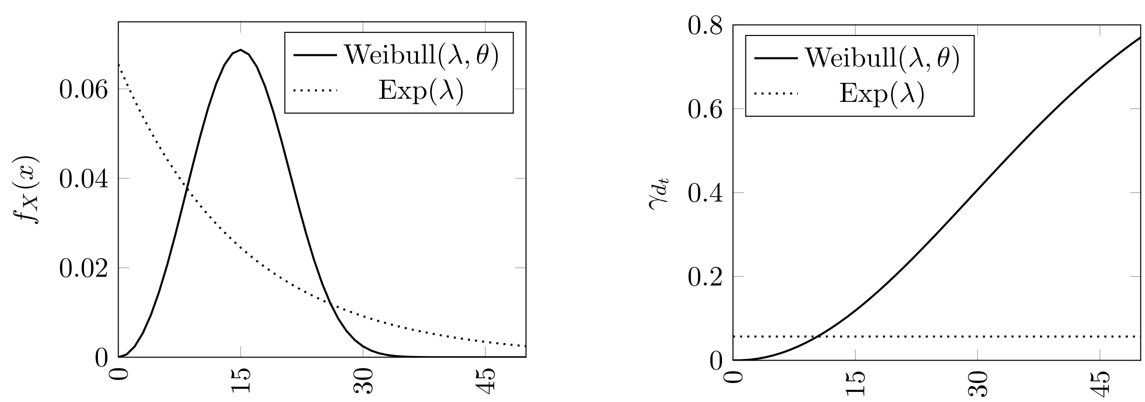
2.1. Process Overview

The production process operates over a finite horizon $H = 80$ hours that is divided into $T = 40$ intervals of length $h = H/T = 2$ hours. Sampling intervals are indexed as $t = 1, \ldots, T-1$ and a sample of size $n = 10$ units is randomly selected at each. The process begins operating in-control producing output with mean $\mu_0 = 41$ and variance $\sigma_0^2 = 1$. The assignable cause of variation shifts (with factor $\delta = 0.5$) the process mean to $\mu_1 = \mu_0 + \delta \cdot \sigma_0 = 41.5$.

The time until the assignable cause occurs can follow any distribution as long as the probability the process shifts in the next time interval conditional on the time elapsed since the last maintenance event (or the beginning of the horizon) can be calculated. The Weibull distribution has been of interest in past SPC applications, and its probability density function (PDF) is

$$f_X(x) = \frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1} e^{-(x/\lambda)^{\theta}} \tag{1}$$

for $x \geq 0$ where λ and θ are the scale and shape parameters of the distribution, respectively. For this example, $\lambda = 17.1$ and $\theta = 3$, and the mean time until failure (occurrence of the assignable cause) is 15.27 hours. The Weibull PDF is shown in Figure 1(a) along with an exponential distribution with the same mean (a special case of the Weibull distribution with $\lambda = 15.27$ and $\theta = 1$). The exponential PDF is commonly used for SPC applications, partially due to its memoryless property that dictates that the conditional probability the process fails in the next sampling period is independent of the time elapsed since maintenance.



Time (in hours) until Occurrence (X)

(a) PDF

Hours elapsed since last repair

(b) Conditional Probabilities

Figure 1: Distributions for time between occurrences of an assignable cause of variation.

2.2. Graphical Representation

Figure 2 shows a LIMID for the SPC problem with $T = 5$. The ovals in the diagram represent random variables. The variable D_t represents the current number of periods since the previous repair. This deterministic variable takes on potential values $d_t = 1, \ldots, t$. The variable S_t represents the probability that the state is out-of-control by the end of period t and assumes either s_{0t} if the process is in-control or s_{1t} if the process is out-of-control. The nodes R_t represent the sample results with value r_t representing the bin of a discrete approximation to the sampling distribution for the mean of the sample of n units.

The rectangles A_t represent the decisions made at each sampling interval. A value of a_{0t} denotes the choice to continue without investigating the process for the assignable cause,

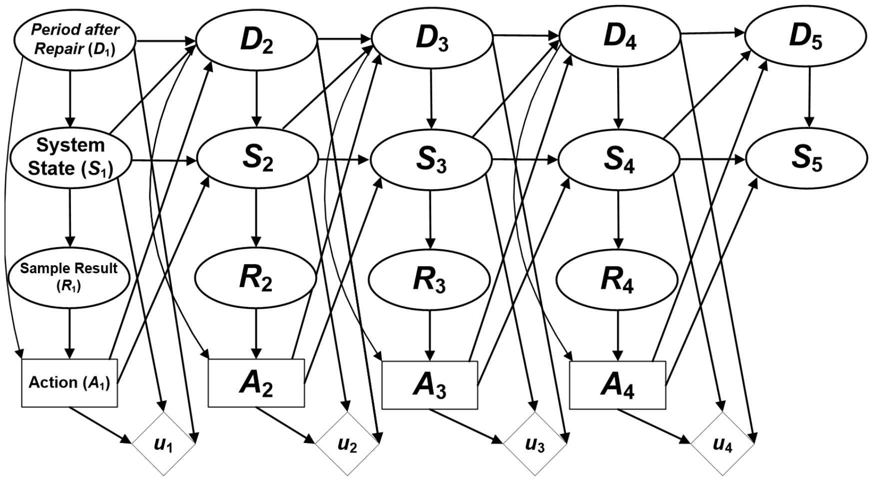


Figure 2: LIMID for the SPC model.

while a_{1t} stops the process to search for the assignable cause and perform repair when necessary. The utility nodes represent the quality control costs in the problem, including sampling costs, investigation and repair costs, stopping the process for a false alarm, and operating the process in both the in-control and out-of-control states.

The assumption of a finite time horizon is required to implement the LIMID model; however, an "infinite" production process could be interpreted as one that runs continuously until it fails and needs to be repaired. Once repair occurs, the decision rule from the model can be re-started at the first sampling period. This means that an infinite production process can be modeled with a LIMID where the time horizon H is long enough that the probability the process runs that long without failure is essentially zero. For the example in this section, $H = 80$ is sufficient even if this process is considered infinite, as $P(X > 45) = 1.39 \times 10^{-11}$ (see Figure 1(a)).

2.3. Conditional Probabilities

Conditional probabilities assigned to the deterministic variables $D_2, \ldots, D_T$ are as follows:

$$P\left(D_t = d_{t-1} + 1 | \{D_{t-1} = d_{t-1}, S_{t-1} = s_0, A_{t-1} = a_\ell\}\right) = 1$$

for $\ell = 0, 1$. When the process operated in state s_0 in the prior period, the action does not affect the number of time periods since repair. If an investigation occurred and revealed no assignable cause, the repair would not be initiated.

When the system previously operated in $S_{t-1} = s_1$,

$$P\left(D_t = d_{t-1} + 1 | \{D_{t-1} = d_{t-1}, S_{t-1} = s_1, A_{t-1} = a_0\}\right) = 1 \text{ and}$$
$$P\left(D_t = 1 | \{D_{t-1} = d_{t-1}, S_{t-1} = s_1, A_{t-1} = a_1\}\right) = 1$$

with probability 0 assigned to conditional probabilities for all other values D_t, $d_t = 1, \ldots, t$. The last scenario resets the counter for D_t after an investigation and repair. D_1 always takes on the value 1, $P(D_1 = 1) = 1$.

Next, consider the distribution for the system state variable S_t. The period since the last repair is $D_t = d_t$ with possible values $d_t = 1, \ldots, t$. Conditional probabilities for $S_t = s_1$ are determined as

$$\gamma_{d_t} = P\left(S_t = s_1 | \{S_{t-1} = s_0, D_t = d_t, A_{t-1} = a_\ell\}\right) = \frac{\displaystyle\int_{(d_t-1)\cdot h}^{d_t \cdot h} f_X(w)\, dw}{\displaystyle\int_{(d_t-1)\cdot h}^{\infty} f_X(w)\, dw}$$

for $d_t = 1, \ldots, T$ and $\ell = 0, 1$ where f_X is the PDF in Equation (1) for this example. These conditional probabilities are shown for the example in Figure 1(b), where it is apparent that the failure rate increases as the time since the last repair lengthens. After repair is pursued for an out-of-control process,

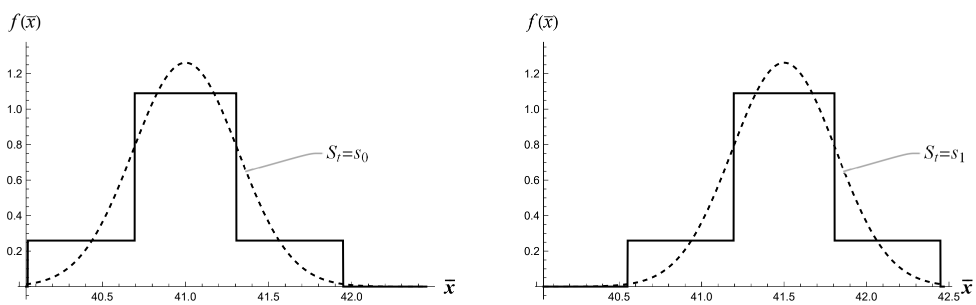
$$P\left(S_t = s_1 | \{S_{t-1} = s_1, D_t = d_t, A_{t-1} = a_1\}\right) = \gamma_1$$

for all $d_t = 1, \ldots, t$. The process is not self-correcting, so

$$P\left(S_t = s_1 | \{S_{t-1} = s_1, D_t = d_t, A_{t-1} = a_0\}\right) = 1$$

for all $d_t = 1, \ldots, t$.

The conditional probability distributions f_{R_t} for the nodes R_t are determined using discrete approximations to the normal PDF developed using the Gaussian Quadrature (GQ) method outlined by Miller and Rice (1983). Suppose that the measurement of the test statistic (for example, weight) per unit while in-control is $N(41, 1)$ and the sample size is $n = 10$. The sample mean is then distributed as $R_t | \{S_t = s_0\} \sim N(41, 1/10)$ with the discrete approximation to this PDF denoted by $f_{R_t | \{S_t = s_0\}}$. Assume that $\delta = 0.5$ so that $R_t | \{S_t = s_1\} \sim N(41.5, 1/10)$ with approximation $f_{R_t | \{S_t = s_1\}}$. The number of pieces used in the GQ approximation will be denoted by K, and the approximations are defined on the interval $[\mu_0 - 3 \cdot \sigma/\sqrt{n}, \mu_0 + \delta\sigma_0 + 3 \cdot \sigma/\sqrt{n}]$ as shown for $K = 3$ in Figure 3.



(a) $R_t \mid \{S_t = s_0\}$ (b) $R_t \mid \{S_t = s_1\}$

Figure 3: Three-piece approximations to the sampling distributions.

The conditional distributions for R_t given the two states of S_t are defined as

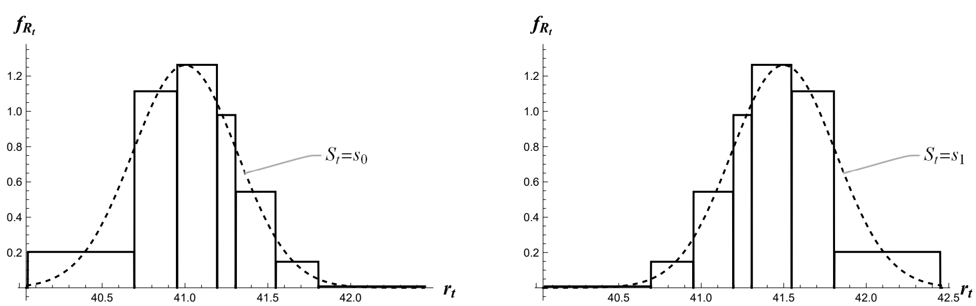
$$f_{R_t|\{S_t=s_0\}}(r_t) = \begin{cases} 0.131 & \text{if } 40.051 \leq R_t < 40.694 \\ 0.285 & \text{if } 40.694 \leq R_t < 40.952 \\ 0.303 & \text{if } 40.952 \leq R_t < 41.194 \\ 0.108 & \text{if } 41.194 \leq R_t < 41.306 \\ 0.131 & \text{if } 41.306 \leq R_t < 41.548 \\ 0.038 & \text{if } 41.548 \leq R_t < 41.806 \\ 0.005 & \text{if } 41.806 \leq R_t \leq 42.449 \end{cases} \quad \text{and} \quad f_{R_t|\{S_t=s_1\}}(r_t) = \begin{cases} 0.005 & \text{if } 40.051 \leq R_t < 40.694 \\ 0.038 & \text{if } 40.694 \leq R_t < 40.952 \\ 0.131 & \text{if } 40.952 \leq R_t < 41.194 \\ 0.108 & \text{if } 41.194 \leq R_t < 41.306 \\ 0.303 & \text{if } 41.306 \leq R_t < 41.548 \\ 0.285 & \text{if } 41.548 \leq R_t < 41.806 \\ 0.131 & \text{if } 41.806 \leq R_t \leq 42.449 \,. \end{cases} \tag{2}$$

Note that while the approximations in Figure 3 have $K = 3$ pieces or bins, the distributions in Equation 2 are defined over the combined set of endpoints—sorted in ascending order—in the domains of the two approximations, denoted by $\mathcal{R}_e$, $e = 1, \ldots, \xi$. In this case

$$\mathcal{R} = [\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5, \mathcal{R}_6, \mathcal{R}_7, \mathcal{R}_8]$$

$$\mathcal{R} = [40.051, 40.694, 40.952, 41.194, 41.306, 41.548, 41.806, 42.449]$$

This will facilitate the interpretation of the decision policies in the LIMID solution. The conditional distributions are shown in Figure 4.



(a) $f_{R_t|\{S_t=s_0\}}$ (b) $f_{R_t|\{S_t=s_1\}}$

Figure 4: Three-piece approximations to the sampling distributions.

2.4. Utility Functions

Assigning functions to the utility nodes in the LIMID first requires calculating the expected time to occurrence of the assignable cause when the process begins a time interval in state s_0. These values are a function of the number of periods since repair d_t and are calculated as follows:

$$\tau_{d_t} = \left(\int_{(d_t-1)\cdot h}^{d_t \cdot h} (w - (d_t - 1) \cdot h) \cdot f_X(w)\, dw \right) \Big/ \left(\int_{(d_t-1)\cdot h}^{d_t \cdot h} f_X(w)\, dw \right)$$

for $d_t = 1, \ldots, T$. The cost parameters in the problem are as shown in Table 1 along with their values for the baseline example problem. The utility function values for $u_2, \cdots, u_{T-1}$ are then assigned as:

$$u_t(s_t, a_t, d_t) = \begin{cases} \gamma_{d_t+1}(\tau_{d_t+1} D_0 + (h - \tau_{d_t+1}) D_1) + (1 - \gamma_{d_t+1}) D_0 h + c_s \\ \qquad\qquad\qquad\qquad \text{if } S_t = s_0 \wedge A_t = a_0 \wedge D_t = d_t \\ \gamma_{d_t+1}(\tau_{d_t+1} D_0 + (h - \tau_{d_t+1}) D_1) + (1 - \gamma_{d_t+1}) D_0 h + W + c_s \\ \qquad\qquad\qquad\qquad \text{if } S_t = s_0 \wedge A_t = a_1 \wedge D_t = d_t \\ D_1 h + c_s \\ \qquad\qquad\qquad\qquad \text{if } S_t = s_1 \wedge A_t = a_0 \wedge D_t = d_t \\ \gamma_1(\tau_1 D_0 + (h - \tau_1) D_1) + (1 - \gamma_1) D_0 h + Y + c_s \\ \qquad\qquad\qquad\qquad \text{if } S_t = s_1 \wedge A_t = a_1 \wedge D_t = d_t \end{cases}$$

for $d_t = 1, \ldots, t$. If the assignable cause occurs during interval t, the cost D_0 occurs for the first part of the time period (τ_{d_t+1}) with the hourly cost D_1 expended for the remainder of the time period. This occurs with probability γ_{d_t+1}; otherwise, the hourly cost D_0 is relevant for the entire interval. Note that u_t actually captures the expected cost for time interval $t + 1$, which is entirely determined by the action and state of the system at time period t. The utility function for the first period is

$$u_1(s_1, a_1, d_1) = \begin{cases} \gamma_1(\tau_1 D_0 + (h - \tau_1) D_1) + (1 - \gamma_1) D_0 h & \\ +\gamma_2(\tau_2 D_0 + (h - \tau_2) D_1) + (1 - \gamma_2) D_0 h + c_s & \text{if } S_1 = s_0 \wedge A_1 = a_0 \wedge D_1 = 1 \\ \gamma_1(\tau_1 D_0 + (h - \tau_1) D_1) + (1 - \gamma_1) D_0 h + c_s & \\ +\gamma_2(\tau_2 D_0 + (h - \tau_2) D_1) + (1 - \gamma_2) D_0 h + W + c_s & \text{if } S_1 = s_0 \wedge A_1 = a_1 \wedge D_1 = 1 \\ \gamma_1(\tau_1 D_0 + (h - \tau_1) D_1) + (1 - \gamma_1) D_0 h + D_1 h + c_s & \text{if } S_1 = s_1 \wedge A_1 = a_0 \wedge D_1 = 1 \\ 2 \cdot (\gamma_1(\tau_1 D_0 + (h - \tau_1) D_1) + (1 - \gamma_1) D_0 h) + Y + c_s & \text{if } S_1 = s_1 \wedge A_1 = a_1 \wedge D_1 = 1\,. \end{cases}$$

The same expected cost is incurred in period 1 regardless of the action A_1, then the cost in the second period is affected by both S_1 and A_1. These costs and the parameters for the Weibull distribution in Figure 1 are consistent with those considered in examples from (Duncan, 1956; Banarjee and Rahim, 1988).

2.5. Solution

LIMIDs are solved with the Single Policy Updating (SPU) algorithm as outlined by Lauritzen and Nilsson (2001) via message passing in a junction tree (Cowell et al., 1999).

Table 1: Cost parameters in the SPC problem.

D_0	quality cost per hour while producing in-control in state s_0	50
D_1	quality cost per hour while producing out-of-control in state s_1	950
W	cost to investigate the process when assignable cause is not present	500
Y	cost to investigate and repair the out-of-control process	1100
c_1	fixed cost of obtaining a sample	20
c_2	per unit sampling cost	4.22
γ_{d_t}	probability the process ends the period out-of-control when process is in period d_t since last repair, $d_t = 1, \ldots, T-1$	
τ_{d_t}	expected time to occurrence of assignable cause of variation when process is in period d_t since last repair, $d_t = 1, \ldots, T-1$	
n	sample size randomly drawn at each sampling interval	10
c_s	total cost of sampling at each interval, $c_s = c_1 + c_2 \cdot n$	62.2

Initially, the strategies at each decision node are randomly selected, and the expected utility obtained from applying these strategies is calculated. The algorithm then visits each decision node to update the strategy to be optimal given the current policies at all other decision nodes. On one sequential pass through all decision nodes, the best current strategies for other decision nodes are considered, so the expected utility improves from the initial calculation.

Subsequently, each strategy is sequentially revisited again, and for a second time the strategy at each node is updated to be optimal given the current strategies at all other decision nodes. This continues until a pass through all decision nodes is made where the total expected utility is not improved. In most cases, this takes 3-4 iterations of the algorithm for this particular SPC problem.

The policies at the decision nodes resulting from the SPU procedure specify the action A_t based on the number of periods since repair D_t and the bin of the approximation $R_t = r_t$ containing the observed sample mean. These policies define a set of upper control limits UCL_{d_t} for $d_t = 1, \ldots, t$. Values of UCL_{d_t} are defined for intervals $t = d_t, \ldots, T-1$. The notation $UCL_{d_t}(t)$ represents the value of UCL_{d_t} in time interval t. The control limits are applied according to the following:

1. The current time interval t which is $t \cdot h$ hours from the start of the production horizon.

2. The current periods $D_t = d_t$ the process has operated without repair determines the control limit UCL_{d_t} to apply.

3. The current sample result $R_t = r_t$, where $\mathcal{R}_e \leq r_t \leq \mathcal{R}_{e+1}$. If $e > UCL_{d_t}(t)$, the process is stopped for investigation and repair. Thus, the UCL values will coincide with the lower endpoint of one of the bins of f_{R_t} so if the sample mean is in the next highest bin, the action $A_t = a_1$ is pursued.

Examples of these control limits are shown in the next section.

3. Results

This section interprets results to the example problem and discusses additional results when parameters under the control of managers are adjusted.

3.1. Example Problem

The minimum total cost for the example problem with $T = 40$, $n = 10$, and $K = 3$ is 19340.70. To compare solutions to problems with different sets of parameters it is often convenient to use the average hourly cost (AHC) which in this case is $AHC = 19340.70/80 = 241.76$.

Solving the LIMID for the example problem using the SPU procedure yields a set of control limits UCL_{d_t}, $t = 1, \ldots, 39$ because $T = 40$. Typically, the UCL policies for the first portion of the d_t values are the most relevant, but the limits are defined for all possible values for the number of periods in which the process operates since the last repair. Several of the UCL values for the example are shown in Figure 5.

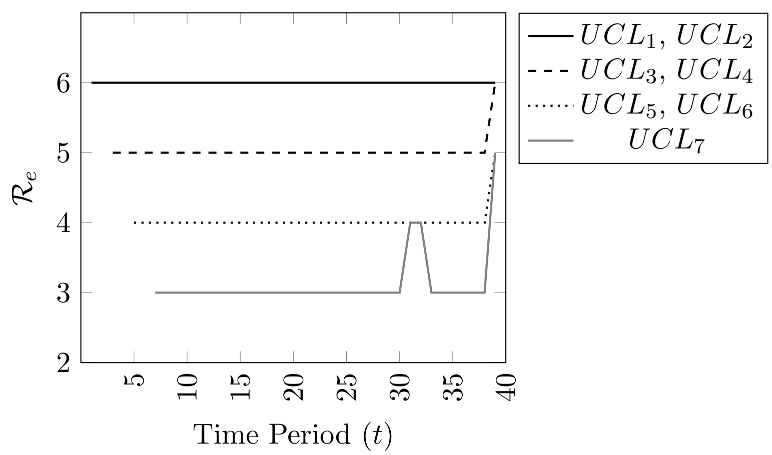


Figure 5: Control limits for the SPC model in the example problem.

Consider the control limits UCL_1 and UCL_2, which represent the policies for a process that is in either the first or second period after the most recent repair. Since $UCL_1 = UCL_2 = 6$ for all t, the process is only stopped for investigation if the sample mean r_t is in the last bin ($r_t \geq 41.806$) of the distribution in (2). The limits UCL_3 and UCL_4 coincide and provide a tighter decision threshold when the process is in the 3rd or 4th period since repair. Note that the decision rule relaxes in the last two periods prior to the end of the production horizon since there would be limited value in expending the cost to investigate and repair the process at that point; thus, the LIMID can adjust the policy throughout the production horizon when advantageous. This is also evident in UCL_7. In most periods, a value r_t in the 4th bin of the distribution or above leads to action a_1, but in $t = 30$ and $t = 31$, the decision rule is relaxed slightly. In this case, stopping the process in those periods likely leads to one more repair before $t = 40$, so waiting to investigate for ambiguous sample results reduces cost.

3.2. Choosing Sample Size and Interval

In some situations, operational considerations may determine or at least limit the values of T and n implemented in the SPC model; however, a manager with flexibility can adjust these parameters and also the number of bins K in the discrete approximations to the sampling distribution to reduce quality control costs.

For this problem, the LIMID was solved for values $T \in [20, 80]$, $n \in [5, 30]$, and $K \in [3, 12]$ reveals that the lowest AHC of 238.96 occurs where $\{T, n, K\} = \{37, 13, 8\}$. The conditional distributions $f_{R_t|S_t=s_0}$ and $f_{R_t|S_t=s_1}$ each have 8 bins and the merged set of endpoints has $\xi = 17$ elements. The first four control limits for the problem are shown in Figure 6.

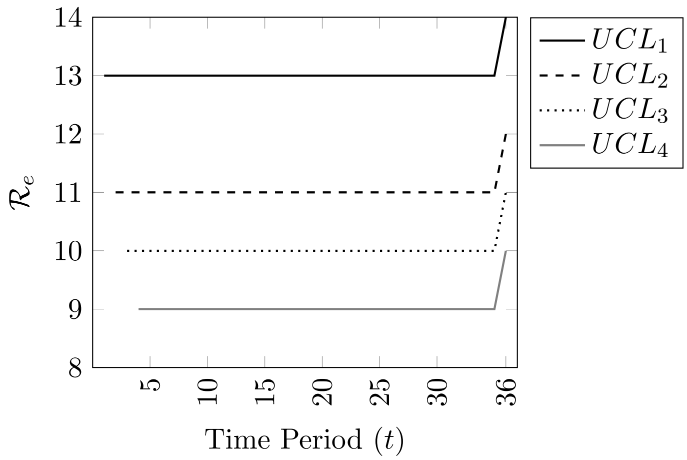


Figure 6: Control limits for the SPC model with improved parameters $T = 37$, $n = 13$, and $K = 8$.

When the process is operating in the first period after repair $D_t = 1$, UCL_1 suggests stopping the process only in bins 14 and above of f_{R_t}, which represents $r_t \geq 42.02$. The decision threshold is tightened as the process continues to operate without repair. In the fourth period after repair, UCL_4 requires the process to be stopped (a_1) when $r_t \geq 41.31$. Each of the upper control limit values relaxes somewhat in the last two sampling periods of the production horizon.

3.3. Sensitivity Analysis

To determine how the process and cost parameters affect the choices of T, n, and K selected by the manager (if operationally feasible), changes of one parameter at a time from the baseline example are considered as shown in Table 2. Potential values for the three adjustable design parameters are considered as follows:

$$T \in \{T_L, T_M, T_H\} = \{20, 30, 40\}, \;\; n \in \{n_L, n_M, n_H\} = \{5, 15, 25\},$$
$$K \in \{K_L, K_M, K_H\} = \{3, 8, 12\}$$

The choice of the number of discrete bins K in the approximation does not change the sampling plan operationally but could potentially change the effectiveness of the decision rule.

Table 2: Average hourly costs, design parameters, and computation time for sensitivity analysis scenarios.

Case	H	λ	θ	δ	D_0	D_1	W	Y	c_1	c_2	T	n	K	AHC	Comp Time (sec)
1	80	17.1	3	0.5	50	950	500	1100	20	4.22	40	15	8	239.7	122.3
2	160	17.1	3	0.5	50	950	500	1100	20	4.22	40	15	8	260.0	92.8
3	80	30	3	0.5	50	950	500	1100	20	4.22	30	15	8	178.6	56.0
4	80	17.1	1	0.5	50	950	500	1100	20	4.22	30	25	8	249.6	35.6
5	80	17.1	3	1	50	950	500	1100	20	4.22	40	15	8	239.7	72.8
6	80	17.1	3	0.5	100	950	500	1100	20	4.22	40	15	8	285.7	123.1
7	80	17.1	3	0.5	50	500	500	1100	20	4.22	20	15	12	194.7	26.5
8	80	17.1	3	0.5	50	950	1000	1100	20	4.22	30	25	12	253.1	65.2
9	80	17.1	3	0.5	50	950	500	500	20	4.22	40	15	8	207.7	119.3
10	80	17.1	3	0.5	50	950	500	1100	5	4.22	40	15	8	232.2	146.5
11	80	17.1	3	0.5	50	950	500	1100	20	1	40	25	12	207.2	137.3

The right panel of Table 2 shows the values (of the three considered) for T, n, and K that minimize expected total costs, as well as the computational time (in seconds) required to solve the model. The examples were solved using Wolfram Mathematica 13 software on a computer with a 2.80GHz processor and 16 MB of RAM.

The most common selections for T, n, and K that minimize expected total cost are larger (H) value for sampling intervals ($T_H = 40$) and the medium (M) values for n and K. There are four scenarios where a reduction in the number of sampling intervals leads to a lower total cost, and three each where costs decrease with a larger sample size and number of discrete bins in the approximations to the sampling distributions. This implies that adjustments in the design parameters may be required for changes in the model parameters as follows:

T	λ^-	θ^+	D_1^+	W^-	n	θ^-	W^+	c_2^-	K	D_1^-	W^+	c_2^-

The signs on the model parameters denote that the design parameter (T, n, or K) changes either directly (+) or inversely (−) with the model parameter. Smaller changes in manager-controlled parameters might also be beneficial in some scenarios. For example, when a search of additional values for $\{T, n, K\}$ is pursued for the scenario where the investigation cost increases from $W = 500$ to $W = 1000$, values of $\{T, n, K\} = \{33, 20, 12\}$ provide a slightly lower AHC of 251.79 (versus 253.10).

The number of sampling intervals T is the primary factor affecting computation times. When T increases, the time to find a solution can increase substantially because the number of elements in the state spaces of the variables D_t grows. This limitation and some possibilities for addressing the issue are mentioned in Section 4. Changing the values for n and K significantly might be expected to change the computation time directly, but the number of iterations before the SPU algorithm settles on a minimum expected utility could change as well, so this relationship is not apparent in all the cases in Table 2.

3.4. Comparison

The baseline example utilized above was also solved by Banarjee and Rahim (1988) using their control chart model with both *uniform* and *non-uniform* sampling intervals. The non-uniform control chart allows the first sampling interval h_1 to be selected by the manager, then assigns the remaining sampling intervals until the next repair as

$$h_t = h_1 \cdot \left(t^{1/\theta} - (t-1)^{1/\theta} \right) .$$

The sampling intervals reset to h_1 when a repair occurs. Thus, to implement the model the manager has to monitor the sampling intervals carefully. Their non-uniform sampling scheme with control limits and sample size selected to minimize costs gives an AHC of 231.30 versus a cost for the LIMID model of 238.96. Their uniform sampling scheme is more directly comparable to the LIMID model and gives a much higher cost of $AHC = 259.85$. Thus, when static sampling intervals are employed, the ability of the LIMID to adjust control limits to the current time elapsed since repair reduces cost significantly.

4. Future Research

A **Li**mited **M**emory **I**nfluence **D**iagram was implemented for quality control in a process where the failure rate increases as the time since the last repair lengthens. The control limits are chosen to minimize costs and are adapted to the length of time since the last repair. Sampling intervals and sample sizes can be further adjusted when operationally feasible to further reduce costs. Costs are significantly reduced when uniform sampling intervals are used as compared to a model (Banarjee and Rahim, 1988) where static control limits are selected that remain the same in an infinite production horizon.

Now that the LIMID model has been implemented for increasing hazard rates in SPC models, the next step can be to suggest modifications to the model that will further reduce costs in situations where it is operationally feasible to adjust the sample size and/or sampling intervals in the LIMID model. One possibility is to establish a warning limit below the UCL that triggers a larger sample size in the subsequent sampling period. This has been implemented for attribute data on the number of defectives (Cobb, 2022), but not for variable data. It may also be possible to adjust the sampling interval in the LIMID, much as Nenes (2013) suggests in a control chart model for variable data.

The tractability of the LIMID models also needs investigation. The dynamic LIMID introduced by van Gerven et al. (2007) has been used to address infinite horizon decision problems where approximating these solutions with LIMIDs containing a long time horizon and number of sampling periods proved intractable. It's also possible that some of the UCLs generated by the LIMID solution are irrelevant because the process is almost certainly stopped for investigation before being allowed to run without repair. It may be possible to schedule maintenance at certain intervals in longer production horizons in advance to both reduce costs and computation time for the solution.

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